Dyadic Rods

S. Z.

January 2022

Note
While Dedekind identified a hyperrational number with every possible separation of the rationals into a left part and right part, this system identifies a hyperdyadic number with every possible initial stretch of the positive dyadics. Dedekind's metaphor was **cut** whereas this paper employs the metaphor of a **rod.**

Definition
Let $\mathscr{N}$ be the set of positive integers.

Definition
Let $\mathscr{O} = 2\mathscr{N} - 1$ be the set of positive odd integers.

Definition
For $n \in \mathscr{N}$, define $E_n = \{2^{-n}j : j \in \mathscr{N}\}$.

Proposition
$\forall n \in \mathscr{N} \;\; E_n \subset E_{n+1}$.

(1) $a \in E_n \implies a = 2^{-n}j = 2^{-(n+1)}2j \in E_{n+1}$.

Proposition
$\forall n \in \mathscr{N} \;\; \forall k \in \mathscr{N} \;\; E_n \subset E_{n+k}$.

Definition
For $n \in \mathscr{N}$, define $X_n = \{2^{-n}j : j \in \mathscr{O}\}$.

Definition
Define $X_0 = \mathscr{N}$.

Proposition
$m \neq n \implies X_m \cap X_n = \emptyset$.

Proposition
$\bigcup_{n=1}^{\infty} E_n = \bigsqcup_{n=0}^{\infty} X_n$.

(1) $x \in E_{n_0} \implies x = 2^{-n_0}j \implies \exists m \in \mathscr{N} - 1 \;\; \exists j_0 \in \mathscr{O} \;\; x = 2^{-n_0}2^m j_0$.
(2) $x \notin \mathscr{N} \implies 0 \leq m < n_0 \implies 0 < n_0 - m \in \mathscr{N} \implies x = 2^{n_0-m}j_0 \in X_{n_0-m}$.

Definition
Define $\mathscr{D} = \bigcup_{n=1}^{\infty} E_n$.

Proposition
$x, y \in \mathscr{D} \implies x + y, xy \in \mathscr{D}$.

Definition
Define $\mathscr{R} = \{A \subset \mathscr{D} : \;\; A \neq \emptyset, \;\; s < a \in A \implies s \in A, \;\; \forall a \in A \;\; \exists s \in A \;\; a < s, \;\; A \neq \mathscr{D} \;\}$.

Definition
Define $\xi : \mathscr{D} \to \mathscr{R}$ by $\xi(s) = \{a \in \mathscr{D} : a < s\}$.

Proposition
$\xi(\mathscr{D}) \subset \mathscr{R}$.

(1) $0 < 2^{-1}b < b \implies 2^{-1}b \in \xi(b)$.
(2) $a < s \in \xi(s) \implies 0 < a < s < b \implies a \in \xi(b)$.
(3) $a \in \xi(b) \implies a < b \implies a < 2^{-1}(a+b) < b \implies 2^{-1}(a+b) \in \xi(b)$.
(4) $a \in \xi(s) \implies a < s \implies s \notin \xi(b)$.

1 An Enumerating Helper Sequence

Definition
For $A \in \mathscr{R}$ and $n \in \mathscr{N}$, define $\Xi_n^A = A \cap X_n$.

Proposition
$A = \bigsqcup_n^\infty \Xi_n^A$.

Definition
Define $\mu^A = \min\{m \in \mathscr{N} : 2^{-m} \in A\}$.

Definition
For $A \subset \mathscr{D}$, define $|A|$ to be the number of elements of A, where ∞ represents countable infinity.

Proposition
$\forall n \in \mathscr{N} \;\; n \geq \mu^A \implies 0 < |\Xi_n^A| < \infty$.

Definition
Let $a \in \Omega_i, a' \in \Omega_j$. Define $a \prec a'$ if either $i < j$ or $i = j$ and $a < a'$.

Proposition
A is well-ordered by $\prec$.

Note
Each nonempty subset of a well-ordered set has a minimum.

Definition
For $A \in \mathscr{R}$, define $\omega^A : \mathscr{N} \to A$ so that ω_n^A is the nth element of A according to $\prec$.

Example
Let $A = \{a \in \mathscr{D} : a < 1\}$. Then $\omega^A = \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, ...$ and $\omega_2^A = \frac{1}{4}$.

Note
Each element of A appears exactly once as a term in the sequence ω_A.

2 A Monotonic Helper Sequence

Definition
For $A \in \mathscr{R}$ and $n \in \mathscr{N}$, define $\Omega_n^A = A \cap E_{n+\mu^A-1}$.

Proposition
$\forall n \in \mathscr{N} \;\; |\Omega_n^A| \in \mathscr{N}$.

Proposition
$A = \bigcup_n^\infty \Omega_n^A$.

Definition
For each $A \in \mathscr{R}$ define $f^A : \mathscr{N} \to A$ by $f_n^A = \max \Omega_n^A$.

Example
Let $A = \{a \in \mathscr{D} : a < 1\}$. Then $f^A = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128}, ...$

Proposition
$\forall n \in \mathscr{N} \;\; f_n^A \leq f_{n+1}^A$.

(1) $E_{n+\mu^A-1} \subset E_{n+1+\mu^A-1} \implies \Omega_n^A \subset \Omega_{n+1}^A \implies \max \Omega_n^A \leq \max \Omega_{n+1}^A \implies f_n^A \leq f_{n+1}^A$.

Proposition
$\forall n \in \mathscr{N} \;\; f_n^A \in A$.

Proposition
$\forall n \in \mathscr{N} \;\; f_n^A + 2^{-n} \notin A$.

(1)$f_n^A + 2^{-n} \in A \implies f_n^A + 2^{-\mu^A-n+1} \in A \implies \max \Omega_n^A = f_n^A < f_n^A + 2^{-\mu^A-n+1} \leq \max \Omega_n^A$, a contradiction.

Proposition
$\forall a \in A \;\; \exists n \in \mathscr{N} \;\; a \leq f_n^A$.

(1) $a \in A = \bigcup_{n=0}^{\infty} \Omega_n^A \implies \exists n_0 \in \mathscr{N} \;\; a \in \Omega_{n_0}^A \implies a \leq f_{n_0}^A = \max \Omega_{n_0}^A$.

3 A Helper Function

Definition
For $A \in \mathscr{R}$ define $l^A : A \to A$ by $l^A(a) = \omega_{m_0}^A$, where $m_0 = \min\{m \in \mathscr{N} : \omega^A(m) > a\}$.

4 An Upper Bound

Definition
For $A \in \mathscr{R}$ define $\beta^A = \min\{k \in \mathscr{D} : k \notin A\}$.

5 Multiplication

Definition
For $A, B \in \mathscr{R}$ define $AB = \{a \in \mathscr{D} : \exists a' \in A \;\; \exists b' \in B \;\; a < a'b'\}$.

Proposition
$A, B \in \mathscr{R} \implies AB \in \mathscr{R}$.

(1) $2^{-m_0} < \omega_1^A \omega_1^B \implies 2^{-m_0} \in AB$.
(2) $s < a' \in AB \implies \;\; \exists a \in A \;\; \exists b \in B \;\; s < a' < ab \implies s \in AB$.
(3) $s \in AB \implies \exists a \in A \;\; \exists b \in B \;\; s < ab < l^A(a) l^B(b) \implies ab \in AB$.
(4) $\beta^A \beta^B \in AB \implies \exists a \in A \;\; \exists b \in B \;\; \beta^A \beta^B < ab < \beta^A \beta^B$, a contradiction.

6 A Multiplicative Identity

Definition
Define $I = \{a \in \mathscr{D} : a < 1\}$.

Proposition
$\forall A \in \mathscr{R} \;\; AI = A$.

(1) $a \in AI \implies a < ai < a \implies a \in A$. Hence $AI \subset A$.
(2) $a \in A \implies a < l^A(a) \in A \implies \exists n_0 \;\; a < l^A(a)(1 - 2^{-n_0}) \implies a \in AI$. Hence $A \subset AI$.

7 A Simple Square Root

Definition
Define $\Psi : \mathscr{D} \to \mathscr{R}$ by $\Psi(s) = \{a \in \mathscr{D} : a^2 < s\}$.

Proposition
$[f_n^{\Psi(p)}]^2 \nearrow p$.

(1) $[f_n^{\Psi(p)}]^2 < p < [f_n^{\Psi(p)} + 2^{-n}]^2 = [f_n^{\Psi(p)}]^2 + 2^{1-n} f_n^{\Psi(p)} + 2^{-2n} < p + 2^{1-n}\beta^{\Psi(p)} + 2^{-2n}$.
(2) $0 < p - [f_n^{\Psi(p)}]^2 < 2^{1-n}\beta^{\Psi(p)} + 2^{-2n} \to 0$.

Proposition
$[f_n^{\Psi(p)} + 2^{-n}]^2 \searrow p$.

(1) $0 < [f_n^{\Psi(p)} + 2^{-n}]^2 - p < [f_n^{\Psi(p)2} + 2^{-n}]^2 - [f_n^{\Psi(p)}]^2 < 2^{1-n}\beta^{\Psi(p)} + 2^{-2n} \to 0$.

Proposition
$\forall s \in \mathscr{D} \;\; \Psi(s) \in \mathscr{R}$.

(1) $2^{-n_0} < \min\{x, 1\} \implies (2^{-n_0})^2 = 2^{-2n_0} < 2^{-n_0} < s \implies 2^{-n_0} \in \Psi(s)$.
(2) $a < s \in \Psi(s) \implies a^2 < s^2 < x \implies a \in \Psi(s)$.
(3) $a \in \Psi(s) \implies a^2 < s \implies (a + 2^{-n_0})^2 = a^2 + 2^{1-n_0}a + 2^{-2n_0} < s \implies (a + 2^{-n_0})^2 \in \Psi(s)$.
(4) $a = \max\{s, 1\} \implies a^2 \geq a \geq s \implies a \notin \Psi(s)$.

Proposition
$\forall s \in \mathscr{D} \;\; \Psi(s)\Psi(s) = \xi(s)$.

(1) $x \in \Psi(s)\Psi(s) \implies x < q_0 q_1 < q_{\max}^2 < s \implies x \in \xi(s)$. Hence $\Psi(s)\Psi(s) \subset \xi(s)$
(2) $p \in \xi(s)$ and $[f_n^{\Psi(p)} + 2^{-n}]^2 \searrow p \implies \exists n_0 \in \mathscr{N} \;\; p < [f_{n_0}^{\Psi(p)} + 2^{-n_0}]^2 < s$.
(3) $f_{n_0}^{\Psi(p)} + 2^{-n_0} \in \Psi(s) \implies p \in \Psi(s)\Psi(s)$. Hence $\xi(s) \subset \Psi(s)\Psi(s)$.

8 Generalizing The Simple Square Root

Definition
For all $A \subset \mathscr{D}$, define $\sqrt{A} = \{a \in \mathscr{D} : a^2 \in A\}$.

Proposition
$A \in \mathscr{R} \implies \sqrt{A} \in \mathscr{R}$.

(1) $2^{-n_0} \in P \implies 2^{-2n_0} < 2^{-n_0} \in P \implies 2^{-2n_0} \in P \implies 2^{-n_0} \in \sqrt{P}$.
(2) $a < s \in \sqrt{P} \implies a^2 < s^2 \in P \implies a^2 \in P \implies a \in \sqrt{P}$.
(3) $a \in \sqrt{P} \implies a^2 < p_0 \in P \implies (a + 2^{-n_0})^2 < p_0 \implies a + 2^{-n_0} \in \sqrt{P}$.
(4) $p < \beta_P < \beta_P^2 \implies \beta_P \notin \sqrt{P}$.

Proposition
$A \in \mathscr{R} \implies \sqrt{A}\sqrt{A} = A$.

(1) $a \in \sqrt{A}\sqrt{A} \implies \exists r_0, r_1 \in \sqrt{A} \;\; a < r_0 r_1 \leq r_{\max}^2 \in A \implies a \in A$. Hence $\sqrt{A}\sqrt{A} \subset A$.
(2) $a \in A$ and $[f_n^{\Psi(a)} + 2^{-n}]^2 \searrow a \implies \exists n_0 \in \mathscr{N} \;\; a < [f_{n_0}^{\Psi(a)} + 2^{-n_0}]^2 < l^A(a)$.
(3) $f_{n_0}^{\Psi(a)} + 2^{-n_0} \in \sqrt{A} \implies a \in \sqrt{A}\sqrt{A}$. Hence $A \subset \sqrt{A}\sqrt{A}$.

9 A Multiplicative Inverse

Definition
For $A \subset \mathscr{D}$, define $\overline{A} := \{s \in \mathscr{D} : \exists n \in \mathscr{N} \;\; \forall a \in A \; [s + 2^{-n}]a < 1\}$.

Proposition
$A \in \mathscr{R} \implies \overline{A} \in \mathscr{R}$.

(1) $[2^{-m_0} + 2^{-m_0}]\beta^A < 1 \implies [2^{-m_0} + 2^{-m_0}]a < [2^{-m_0} + 2^{-m_0}]\beta^A < 1 \implies 2^{-m_0} \in \overline{A}$.
(2) $y < x \in \overline{A} \implies \forall a \in A\ \ [y + 2^{-m_0}]a < [x + 2^{-m_0}]a < 1 \implies y \in \overline{A}$.
(3) $x \in \overline{A} \implies \forall a \in A\ \ [x + 2^{-m_0}]a = [x + 2^{-m_0-1} + 2^{-m_0-1}]a < 1 \implies x + 2^{-m_0-1} \in \overline{A}$.
(4) $2^{-n_0} < \omega_1^A \implies 1 < 2^{n_0}\omega_1^A \implies 2^{n_0} \notin \overline{A}$.

Definition
Define $\zeta_n^A = \omega_{m_n}^A$ where $m_n = \min\ M_n$, where $M_n = \{m \in \mathscr{N} : \omega_m^A[f_n^{\overline{A}} + 2^{-n}] \geq 1\}$.

Note
The set M_n above is not empty for any n, else $f_n^{\overline{A}} + 2^{-n} \in \overline{A}$, which is impossible by the definition of $f_n^{\overline{A}}$.

Proposition
$\forall n \in \mathscr{N}\ \ 0 < f_n^{\overline{A}}\zeta_n^A < 1$.

Proposition
$A \in \mathscr{R} \implies A\overline{A} = I$.

(1) $x \in A\overline{A} \implies x < a_0\overline{a}_0 < 1 \implies x \in I$. Hence $A\overline{A} \subset I$.
(2) $f_n^{\overline{A}}\zeta_n^A < 1 \leq [f_n^{\overline{A}} + 2^{1-n}]\zeta_n^A = f_n^{\overline{A}}\zeta_n^A + 2^{1-n}\zeta_n^A < 1 + 2^{1-n}\beta^A \implies 0 < 1 - f_n^{\overline{A}}\zeta_n^A < 2^{1-n}\beta^A \implies f_n^{\overline{A}}\zeta_n^A \to 1$.
(3) $p \in I \implies 0 < p < 1 \implies \exists n_0 \in \mathscr{N}\ \ 0 < p < f_{n_0}^{\overline{A}}\zeta_{n_0}^A < 1 \implies p \in A\overline{A}$. Hence $I \subset A\overline{A}$.

10 The Structure of $\mathscr{R}$

Definition
For $A \in \mathscr{R}$ and $a \in A$, define $A_a = \xi(a)$.

Note
Intuitively, A_a is an initial segment of A.

Proposition
$A \in \mathscr{R} \implies \bigcup_{a \in A} A_a = A$.

(1) $a \in A \implies a < l^A(a) \in A \implies a \in A_{l^A(a)} \subset \bigcup_{a \in A} A_a$. Hence $A \subset \bigcup_{a \in A} A_a$.
(2) $\bigcup_{s \in \mathscr{D}} A_s \implies \exists s \in \mathscr{D}\ \ a \in A_s \implies a <\in A \implies a \in A$. Hence $\bigcup_{a \in A} A_a \subset A$.

Note
A is always the union of its initial segments.

Definition
For $A, B \subset \mathscr{D}$, define $A < B$ to mean $A \subsetneqq B$.

Definition
For $A \in \mathscr{R}$, define $\Gamma(A) = \bigcup_{B \in \mathscr{R}}^{B < A} B$.

Definition
In other words, $\Gamma(A)$ is the union of all elements in $\mathscr{R}$ that are less than A.

Proposition
$A \in \mathscr{R} \implies A = \Gamma(A)$.

(1) $a \in A \implies a < l^A(a) \implies a \in \xi(l^A(a)) < A \implies a \in \Gamma(A)$. Hence $A \subset \Gamma(A)$.
(2) $\Gamma(A) \implies \exists A' \in \mathscr{R}\ \ a \in A' < A \implies a \in A$. Hence $\Gamma(A) \subset A$.

11 Hyperdyadic Rods

Define $\mathscr{R} = \{H \subset \mathscr{D} : \ H \neq \emptyset, \ A < A' \in H \implies A \in H, \ \forall A \in H \ \exists A' \in H \ A < A', \ H \neq \mathscr{D} \}$.

Definition
Define $\Phi : \mathscr{R} \to \mathscr{H}$ by $\Phi(B) = \{A \in \mathscr{R} : A < B\}$.

Proposition
$B \in \mathscr{R} \implies \Phi(B) \in \mathscr{H}$.

(1) $2^{-1}B < B \implies 2^{-1}B \in \Phi(B)$
(2) $A < A' \in \Phi(B) \implies A < A' < B \implies A \in \Phi(B)$
(3) $A \in \Phi(B) \implies \exists n_0 \in \mathscr{N} \ \ A < (1 - 2^{-n_0})B < B \implies (1 - 2^{-n_0})B \in \Phi(B)$
(4) $B < 2B \implies 2B \notin \Phi(B)$