1 Floats

Let $\mathscr{N}$ be the set of strictly positive integers.

For $n \in \mathscr{N}$, define $F_n = \{2^{-n}j : j \in \mathscr{N}\}$.

Define $\mathscr{F} = \cup_{n \in N} F_n$.

For $A \subset \mathscr{F}$, define $A_n = A \cap F_n$, so that $A = \cup A_n$.

Closure $x, y \in \mathscr{F} \implies x + y, xy \in \mathscr{F}$.

(1) $2^{-m}j + 2^{-n}k = 2^{-m-n+n}j + 2^{-n-m+m}k = 2^{-m-n}(2^n j + 2^m k)$.

(2) $2^{-m}j 2^{-n}k = 2^{-m-n}jk$.

2 The Carpenter Sequence

The elements of each ladder A are conveniently ordered. Just enumerate $A \cap F_n$ in the usual way for $n = 1, 2, 3, ...$
Let $c_A : \mathscr{N} \to A$ be this enumeration. Then $c(\mathscr{N}) = A$.
(0) $A = \cup(A \cap F_n)$ and $\forall n \in N \;\; A \cap F_n$ is finite. So

3 Floating Ladders

Intuitively, the every ladder in $\mathscr{L}$ is an initial piece or stretch of $\mathscr{F}$ with its elements (rungs) stacked vertically. Then ladders start from 0 (without containing it) and shoot up some finite distance. More formally, $A \subset \mathscr{F}$ is a ladder if it is nonempty, closed downward, has no maximum, and yet is not all of $\mathscr{F}$.

(1) $\exists x \in A$
(2) $x < y \in A \implies x \in A$.
(3) $x \in A \implies \exists y \in A \;\; x < y$
(4) $\exists x \in A^c$

4 Simple Ladders

Define $\phi : \mathscr{F} \to \mathscr{L}$ by $\phi(e) = \{d < e\} = \{d \in \mathscr{F} : d < e\}$. Then $\phi(\mathscr{F}) \subset \mathscr{L}$.

(1) $0 < 2^{-1}e < e \implies 2^{-1}e \in \phi(e)$.
(2) $d < d_1 \in \phi(e) \implies 0 < d < d_1 < e \implies d \in \phi(e)$.
(3) $d \in \phi(e) \implies d < e \implies d < 2^{-1}(d + e) < e \implies 2^{-1}(d + e) \in \phi(e)$.
(4) $d \in \phi(e) \implies d < e \implies e \notin \phi(e)$.

We call any element of $\phi(\mathscr{F})$ a **simple** ladder.

5 The Fireman Sequence

For each $A \in L$ we now construct $f_A : \mathscr{N} \to A \subset \mathscr{F}$ such that $\forall n \in N \;\; f_A(n) \in A$ and $f_A(n) + 2^{-n} \notin A$.

Define $f_A(n) = 2^{-m_0-n}j_n$, where $m_0 = \min\{m \in \mathscr{N} : 2^{-m} \in A\}$ and $j_n = \max\{k : 2^{-m_0-n}k \in A\}$

Nondecreasing $\forall n \in \mathscr{N} \quad f_A(n) \le f_A(n+1)$.

(1) $2^{-m_0-n}k_n \in A < 2^{-m_0-n-1}(k_{n+1} + 1) \notin A \implies 2k_n \le k_{n+1} \implies 2^{-m_0-n-1}2k_n = f_A(n) \le 2^{-m_0-n-1}k_{n+1} = f_A(n+1)$.

The fireman eventually climbs every rung of his assigned ladder.

Surpassing $\quad \forall a \in A \; \exists n \in \mathscr{N} \;\; f_A(n) > a$

(1) $a \in A \implies a < a_0 \in A \implies a + 2^{-n_0} < a_0 < f_A(n_0) + 2^{-n_0} \notin A \implies a < a_0 - 2^{-n_0} < f_A(n_0) \in A$.

6 The Leaper Function

Given a ladder and one of its rungs, it's convenient sometimes to generate a higher rung (to 'leap' the given rung.)

For $A \in \mathscr{L}$ define $l_A : A \to A$ by $l_A(a) = f_A(m_0)$, where $m_0 = \min\{f_A(n) > a\}$.

Then $l_A(a)$ is the first term of A's fireman sequence greater than a, which is always of course itself in A.

7 Umbrellas

For each A define the umbrella u_A of A by $u_A = \min\{k : k \notin A\}$.

Then $u_A \notin A \implies \forall n \in \mathscr{N} \;\; f_A(n) < u_A$, else $u_A \in A$, a contradiction.

8 Products

For $A, B \in \mathscr{L}$ define the product AB as $\{d < ab\} = \{d \in \mathscr{F} : \exists a \in A \;\; \exists b \in B \;\; d < ab\}$.

Closure Under Multiplication The product of ladders is always a ladder.

(1) $2^{-m_0} < ab \implies 2^{-m_0} \in AB$.
(2) $x < y \in AB \implies x < y < ab \implies x \in AB$.
(3) $x \in AB \implies x < ab < f_A(n_0)f_B(m_0) \in AB \implies ab \in AB$.
(4) $x \in AB \implies x < ab < u_A u_B \implies u_A u_B \notin AB$, else $u_A u_B < u_A u_B$, a contradiction.

9 A Multiplicative Identity

Define $I = \{d < 1\}$, then $\forall A \in \mathscr{L} \;\; AI = A$.

(1) $d \in AI \implies d < ai < a \implies d \in A$, so $AI \subset A$.
(2) $d \in A \implies d < e \in A \implies a < e(1 - 2^{-n_0}) \in AI \implies d \in AI$, hence $A \subset AI$.

10 Square Roots For Simple Ladders

Now that we have multiplication and the 'integers' like $\phi(2) \in \mathscr{L}$, we can construct that most famous of square roots.

Define $\Psi : \mathscr{F} \to \mathscr{L}$ by $\Psi(e) = \{d^2 < e\}$. Then $\Psi(\mathscr{F}) \subset \mathscr{L}$.

(1) $2^{-n_0} < \min\{x, 1\} \implies (2^{-n_0})^2 = 2^{-2n_0} < 2^{-n_0} < e \implies 2^{-n_0} \in \Psi(e)$.
(2) $d < d_1 \in \Psi(e) \implies d^2 < d_1^2 < x \implies d \in \Psi(e)$.
(3) $d \in \Psi(e) \implies d^2 < e \implies (d + 2^{-n_0})^2 = d^2 + 2^{1-n_0}d + 2^{-2n_0} < e \implies (d + 2^{-n_0})^2 \in \Psi(e)$.
(4) $d = \max\{e, 1\} \implies d^2 \geq d \geq e \implies d \notin \Psi(e)$.

Simple Square Root Theorem If $Q = \{d^2 < e\}$ and $P = \{d < e\}$, then $Q^2 = P$.

(1) $s \in Q^2 \implies \exists q_0, q_1 \in Q \;\;\; s < q_0 q_1 < \max\{p_0, p_1\}^2 < e \implies s \in P$.

(2) $p \in P \implies p < [f_{\Psi(p)}(n_0) + 2^{-n_0}]^2 < p + 2^{1-n_0}u_P + 2^{-2n_0} < e \implies f_{\Psi(p)}(n_0) + 2^{-n_0} \in Q \implies p \in Q^2$.

11 The Multiplicative Inverse

For $d \in \mathscr{F}$ and $A \subset \mathscr{F}$, define $d < A$ as $\forall a \in A \;\; d < a$ and $d > A$ as $\forall a \in A \;\; d > a$.

For $A \subset \mathscr{F}$, define $A^{-1} = \{a^{-1}\} \subset \mathscr{Q}$.

For $A \subset \mathscr{F}$, define $\overline{A} := \{\exists n \in \mathscr{N} \quad d + 2^{-n} < A^{-1}\}$.

Then $A \in \mathscr{L} \implies \overline{A} \in \mathscr{L}$.

(1) $A < u_A < 2^{m_0 - 1} \implies 2^{1-m_0} = 2^{-m_0} + 2^{-m_0} < u_A^{-1} < A^{-1} \implies 2^{-m_0} \in \overline{A}$.
(2) $y < x \in \overline{A} \implies y + 2^{m_0} < x + 2^{m_0} < A^{-1} \implies y \in \overline{A}$.
(3) $x \in \overline{A} \implies x + 2^{-m_0} = x + 2^{-m_0-1} + 2^{-m_0-1} < A^{-1} \implies x + 2^{-m_0-1} \in \overline{A}$.
(4) $2^{-n_0} < a \in A \implies a^{-1} < 2^{n_0} \implies 2^{n_0} \notin \overline{A}$.

So $A \in \mathscr{L} \implies \overline{A} \in \mathscr{L}$.

Inverse Theorem If $A \in \mathscr{L}$ and $B = \overline{A}$, then $AB = I$.

(1) Let $x \in AB$ so $x < ab \in AB$. Also $i < b + \varepsilon < a^{-1} \implies ab < 1 \implies x < ab < 1 \implies x \in I$.
(2) $p \in I, f_{\overline{A}}(n) + 2^{2-n} \notin \overline{A} \implies a_n^{-1} \le f_{\overline{A}}(n) + 2^{2-n} + 2^{2-n} \implies 1 - 2^{1-n} \le a_n f_{\overline{A}}(n) < 1 \implies p < a_{n_0} f_{\overline{A}}(n_0) \implies p \in A\overline{A}$.

12 Square Roots For All Ladders

Lemma $P \in \mathscr{L} \implies \sqrt{P} := \{d^2 \in P\} \in \mathscr{L}$.

(1) $2^{-n_0} \in P \implies 2^{-2n_0} < 2^{-n_0} \in P \implies 2^{-2n_0} \in P \implies 2^{-n_0} \in \sqrt{P}$.
(2) $d < d_1 \in \sqrt{P} \implies d^2 < d_1^2 \in P \implies d^2 \in P \implies d \in \sqrt{P}$.
(3) $d \in \sqrt{P} \implies d^2 < p_0 \in P \implies (d + 2^{-n_0})^2 < p_0 \implies d + 2^{-n_0} \in \sqrt{P}$.
(4) $\forall p \in P \quad p < \max\{u_P, 2\} \le [\max\{u_P, 2\}]^2 \implies \max\{u_P, 2\} \notin \sqrt{P}$.

General Square Root Theorem $Q = \sqrt{P} \implies Q^2 = P$.

(1) $s \in Q^2 \implies \exists q_0, q_1 \in Q \quad s < q_0 q_1 < q^2_{\max\{p_0, p_1\}} \in P \implies s \in P$.
(2) $p \in P \implies p < [f_{\Psi(p)}(n_0) + 2^{-n_0}]^2 < p + 2^{1-n_0} u_P + 2^{-2n_0} < l_P(p) \in P \implies f_{\Psi(p)}(n_0) + 2^{-n_0} \in Q \implies p \in Q^2$.