Bundles

S. Z.

January 2022

Note
The following is a construction of (dyadic) Dedekind **rods** as opposed to the famous cuts. Only positive real-like numbers are developed, though it's easy from here to use equivalence classes of pairs of rods to get the rest. Instead of thinking of the real number as a parting of the rationals, we can also think of each real number as a measuring rod. In this case each rod is a well-ordered and countably infinite set of simpler dyadic rods. The magnitude suggested by such a infinite set is the smallest stretch of space in which all of the simple rods will fit. This presentation takes some experience with limits for granted. Only the more difficult propositions are proved, and these are proved somewhat informally.

Definition
Let $\mathscr{N}$ be the set of positive integers.

Definition
For $n \in \mathscr{N}$, define $E_n = \{j/n : j \in \mathscr{N}\}$.

Definition
Define $\mathscr{P} = \bigcup_{n=1}^{\infty} E_n$.

Note
This means that $\mathscr{P}$ is just the set of positive rational numbers.

Definition
For $A, B \subset \mathscr{P}$, define $A < B$ to mean $A \subsetneq B$.

Definition
Define $\mathscr{X} = \{A \subset \mathscr{P} : \ \emptyset < A < \mathscr{P}, \ \ p < q \in A \implies p \in A, \ \ \forall q \in A \ \exists p \in A \ \ q < p\}$.

Note
The entities defined above are nonempty, proper subsets of the positive rationals which are closed downward and have no maximum.

1 Injecting $\mathscr{P}$ into $\mathscr{X}$

Definition
For $p \in \mathscr{P}$, define $(q < p) = \{q \in \mathscr{P} : q < p\}$.

Definition
Define $h : \mathscr{P} \to \mathscr{X}$ by $h(p) = (q < p)$.

Proposition
$p \in \mathscr{P} \implies h(p) \in \mathscr{X}$.

(1) $0 < p/2 < p \implies p/2 \in h(p) \implies \emptyset < h(p)$.
(2) $q \in h(p) \implies q < p \implies p \notin h(p) \implies h(p) < \mathscr{P}$.
(3) $q < q' \in h(p) \implies 0 < q < q' < p \implies q \in h(p)$.
(4) $q \in h(p) \implies q < p \implies q < (q+p)/2 < p \implies (q+p)/2 \in h(p)$.

Proposition
$\forall q, q' \in \mathscr{P} \;\; h(q) = h(q') \implies q = q'$.

2 Ordering each element of X

Definition
For $A \in \mathscr{X}$ and $n \in \mathscr{N}$, define $\Omega_n^A = A \cap E_n$.

Proposition
$\forall n \in \mathscr{N} \;\; |\Omega_n^A| \in \mathscr{N}$.

Proposition
$A = \bigcup_n^\infty \Omega_n^A$.

Definition
Let $a \in \Omega_i^A, a' \in \Omega_j^A$ and define $a \prec a'$ if (1) $i < j$ or (2) if $i = j$ and $a < a'$.

Proposition
The order $\prec$ can be used to enumerate A, with elements appearing more than once.

Definition
For $A \in \mathscr{X}$, define $\omega^A : \mathscr{N} \to A$ to be sequence of A's elements as ordered by $\prec$.

Example
Let $A = \{q \in \mathscr{P} : q < 1\}$. Then $\omega^A = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, ...$ and $\omega_2^A = \frac{1}{3}$.

Definition
Define $\mu^A = \min\{m \in \mathscr{N} : 1/m \in A\}$.

Definition
For $A \in \mathscr{X}$ and $n \in \mathscr{N}$, define $\Xi_n^A = \Omega_{n+\mu^A-1}^A$.

Proposition
$\forall n \in \mathscr{N} \;\; \Xi_n^A \neq \emptyset$.

Definition
For each $A \in \mathscr{X}$, define $f_n^A = \max \Xi_n^A$.

Example
Let $A = \{q \in \mathscr{P} : q < 1\}$. Then $\mu^A = 1$ and $f^A = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128}, ...$

Proposition
$\forall n \in \mathscr{N} \;\; \forall k \in \mathscr{N} \;\; f_n^A \leq f_{kn}^A$.

(1) $\Xi_n^A \subset \Xi_{kn}^A \implies \max \Xi_n^A \leq \max \Xi_{kn}^A$.

Proposition
$\forall n \in \mathscr{N} \;\; f_n^A \in A$.

Proposition
$\forall n \in \mathscr{N} \;\; f_n^A + 2^{-n} \notin A$.

(1)$f_n^A = \max A \cap E_{\mu^A+n} < f_n^A + 2^{-\mu^A-n} \in E_{\mu^A+n} \implies f_n^A + 2^{-\mu^A-n} \notin A \implies f_n^A + 2^{-n} \notin A$.

Proposition
$\forall a \in A \;\; \exists n \in \mathscr{N} \;\; a \leq f_n^A$.

(1) $a \in A = \bigcup (A \cap E_n) \implies a \in A \cap E_{n_0} \implies a \leq f^A_{n_0} = \max A \cap E_{n_0}$

Definition
For $A \in \mathscr{X}$ define $l^A : A \to A$ by $l^A(a) = \omega^A_{m_0}$, where $m_0 = \min\{m \in \mathscr{N} : \omega^A(m) > a\}$.

Definition
For $A \in \mathscr{X}$ define $\beta^A = \min\{k \in \mathscr{P} : k \notin A\}$.

Definition
For $A, B \in \mathscr{X}$ define $AB = \{q \in \mathscr{P} : \exists a' \in A \;\; \exists b' \in B \;\; q < a'b'\}$.

Proposition
$A, B \in \mathscr{X} \implies AB \in \mathscr{X}$.

(1) $2^{-m_0} < \omega^A_1 \omega^B_1 \implies 2^{-m_0} \in AB$.
(2) $p < q' \in AB \implies p < q' < ab \implies p \in AB$.
(3) $q \in AB \implies q < a'b' < l^A(a') l^B(b') \implies a'b' \in AB$.
(4) $\beta^A \beta^B \in AB \implies \beta^A \beta^B < a'b' < \beta^A \beta^B$, hence $\beta^A \beta^B \notin AB$.

Definition
Define $I = \{q \in \mathscr{P} : q < 1\}$.

Proposition
$\forall A \in \mathscr{X} \;\; AI = A$.

(1) $q \in AI \implies q < ai < a \implies q \in A$. Hence $AI \subset A$.
(2) $q \in A \implies q < l^A(q) \in A \implies \exists n_0 \;\; q < l^A(q)(1 - 2^{-n_0}) \implies a \in AI$. Hence $A \subset AI$.

Definition
Define $\Psi : \mathscr{P} \to \mathscr{X}$ by $\Psi(p) = \{q \in \mathscr{P} : q^2 < p\}$.

Proposition
$[f^{\Psi(p)}_n]^2 \nearrow p$.

(1) $[f^{\Psi(p)}_n]^2 < p < [f^{\Psi(p)}_n + 2^{-n}]^2 = [f^{\Psi(p)}_n]^2 + 2^{1-n} f^{\Psi(p)}_n + 2^{-2n} < p + 2^{1-n} \beta^{\Psi(p)} + 2^{-2n}$.
(2) $0 < p - [f^{\Psi(p)}_n]^2 < 2^{1-n} \beta^{\Psi(p)} + 2^{-2n} \to 0$.

Proposition
$[f^{\Psi(p)}_n + 2^{-n}]^2 \searrow p$.

(1) $0 < [f^{\Psi(p)}_n + 2^{-n}]^2 - p < [f^{\Psi(p)2}_n + 2^{-n}]^2 - [f^{\Psi(p)}_n]^2 < 2^{1-n} \beta^{\Psi(p)} + 2^{-2n} \to 0$.

Proposition
$\Psi(\mathscr{P}) \subset \mathscr{X}$.

(1) $2^{-n_0} < \min\{x, 1\} \implies (2^{-n_0})^2 = 2^{-2n_0} < 2^{-n_0} < p \implies 2^{-n_0} \in \Psi(p)$.
(2) $q < p \in \Psi(p) \implies q^2 < p^2 < x \implies q \in \Psi(p)$.
(3) $q \in \Psi(p) \implies q^2 < p \implies (q + 2^{-n_0})^2 = q^2 + 2^{1-n_0} q + 2^{-2n_0} < p \implies (q + 2^{-n_0})^2 \in \Psi(p)$.
(4) $q = \max\{p, 1\} \implies q^2 \geq q \geq p \implies q \notin \Psi(p)$.

Proposition
$\forall p \in \mathscr{P} \;\; \Psi(p)\Psi(p) = h(p)$.

(1) $x \in \Psi(p)\Psi(p) \implies x < q_0 q_1 < q^2_{\max} < p \implies x \in h(p)$. Hence $\Psi(p)\Psi(p) \subset h(p)$
(2) $p \in h(p)$ and $[f^{\Psi(p)}_n + 2^{-n}]^2 \searrow p \implies \exists n_0 \in \mathscr{N} \;\; p < [f^{\Psi(p)}_{n_0} + 2^{-n_0}]^2 < s$.
(3) $f^{\Psi(p)}_{n_0} + 2^{-n_0} \in \Psi(p) \implies p \in \Psi(p)\Psi(p)$. Hence $h(p) \subset \Psi(p)\Psi(p)$.

Definition
For all $A \subset \mathscr{P}$, define $\sqrt{A} = \{q \in \mathscr{P} : q^2 \in A\}$.

Proposition
$A \in \mathscr{X} \implies \sqrt{A} \in \mathscr{X}$.

(1) $2^{-n_0} \in P \implies 2^{-2n_0} < 2^{-n_0} \in P \implies 2^{-2n_0} \in P \implies 2^{-n_0} \in \sqrt{P}$.
(2) $q < p \in \sqrt{P} \implies q^2 < p^2 \in P \implies q^2 \in P \implies q \in \sqrt{P}$.
(3) $q \in \sqrt{P} \implies q^2 < p_0 \in P \implies (q + 2^{-n_0})^2 < p_0 \implies q + 2^{-n_0} \in \sqrt{P}$.
(4) $p < \beta_P < \beta_P^2 \implies \beta_P \notin \sqrt{P}$.

Proposition
$A \in \mathscr{X} \implies \sqrt{A}\sqrt{A} = A$.

(1) $q \in \sqrt{A}\sqrt{A} \implies \exists r_0, r_1 \in \sqrt{A} \;\; q < r_0 r_1 \leq r_{\max}^2 \in A \implies q \in A$. Hence $\sqrt{A}\sqrt{A} \subset A$.
(2) $a \in A$ and $[f_n^{\Psi(a)} + 2^{-n}]^2 \searrow a \implies \exists n_0 \in \mathscr{N} \;\; a < [f_{n_0}^{\Psi(a)} + 2^{-n_0}]^2 < l^A(a)$.
(3) $f_{n_0}^{\Psi(a)} + 2^{-n_0} \in \sqrt{A} \implies a \in \sqrt{A}\sqrt{A}$. Hence $A \subset \sqrt{A}\sqrt{A}$.

Definition
For $A \subset \mathscr{P}$, define $\overline{A} := \{q \in \mathscr{P} : \exists n \in \mathscr{N} \;\; \forall a \in A \; [q + 2^{-n}]a < 1\}$.

Proposition
$A \in \mathscr{X} \implies \overline{A} \in \mathscr{X}$.

(1) $[2^{-m_0} + 2^{-m_0}]\beta^A < 1 \implies [2^{-m_0} + 2^{-m_0}]a < [2^{-m_0} + 2^{-m_0}]\beta^A < 1 \implies 2^{-m_0} \in \overline{A}$.
(2) $y < x \in \overline{A} \implies \forall q \in A \;\; [y + 2^{-m_0}]a < [x + 2^{-m_0}]a < 1 \implies y \in \overline{A}$.
(3) $x \in \overline{A} \implies \forall q \in A \;\; [x + 2^{-m_0}]a = [x + 2^{-m_0-1} + 2^{-m_0-1}]a < 1 \implies x + 2^{-m_0-1} \in \overline{A}$.
(4) $2^{-n_0} < \omega_1^A \implies 1 < 2^{n_0}\omega_1^A \implies 2^{n_0} \notin \overline{A}$.

Definition
Define $\zeta_n^A = \omega_{m_n}^A$ where $m_n = \min \; M_n$, where $M_n = \{m \in \mathscr{N} : \omega_m^A[f_n^{\overline{A}} + 2^{-n}] \geq 1\}$.

Note
The set M_n above is not empty for any n, else $f_n^{\overline{A}} + 2^{-n} \in \overline{A}$, which is impossible by the definition of $f_n^{\overline{A}}$.

Proposition
$\forall n \in \mathscr{N} \;\; 0 < f_n^{\overline{A}}\zeta_n^A < 1$.

Proposition
$A \in \mathscr{X} \implies A\overline{A} = I$.

(1) $x \in A\overline{A} \implies x < a_0\overline{a}_0 < 1 \implies x \in I$. Hence $A\overline{A} \subset I$.
(2) $f_n^{\overline{A}}\zeta_n^A < 1 \leq [f_n^{\overline{A}} + 2^{1-n}]\zeta_n^A = f_n^{\overline{A}}\zeta_n^A + 2^{1-n}\zeta_n^A < 1 + 2^{1-n}\beta^A \implies 0 < 1 - f_n^{\overline{A}}\zeta_n^A < 2^{1-n}\beta^A \implies f_n^{\overline{A}}\zeta_n^A \to 1$.
(3) $p \in I \implies 0 < p < 1 \implies \exists n_0 \in \mathscr{N} \;\; 0 < p < f_{n_0}^{\overline{A}}\zeta_{n_0}^A < 1 \implies p \in A\overline{A}$. Hence $I \subset A\overline{A}$.

Definition
For $A \subset \mathscr{P}$ and $p \in \mathscr{P}$ define $A < q$ if $\forall a \in A \;\; a < p$.

Definition
For $A, A' \in \mathscr{X}$, define $A < A'$ iff $A' - A \neq \emptyset$.

Proposition
$[\; \exists B \in \mathscr{X} \;\; \forall n \in \mathscr{N} \;\; B > A_n \in \mathscr{X} \;\;] \implies \bigcup_{n=1}^{\infty} A_n \in \mathscr{X}$.

Proposition
$[\; A \in \mathscr{X} \;\wedge\; \forall n \in \mathscr{N} \;\; A_n = h(\omega_n^A) \;] \implies \bigcup_{n=1}^{\infty} A_n = A$.

(1) $a \in A \implies a < l^A(a) = \omega_{n_0}^A \implies a \in A_{n_0} \implies a \in \bigcup_{n=1}^{\infty} A_n$. Hence $A \subset \bigcup_{n=1}^{\infty} A_n$.
(2) $p \in A_n \implies p < \omega_n^A \in A \implies p \in A$. Hence $\bigcup_{n=1}^{\infty} A_n \subset A$.