RIBBONS

1 Definition

A function $f : \mathbb{Z} \to \mathbb{Q}$ is a **ribbon** when

(1) $z > 0 \implies f(z) < f(z+1) < f(-(z+1)) < f(-z)$

(2) $f(-z) - f(z) \longrightarrow 0.$

The structure of a ribbon is therefore

$$f(1) < f(2) < f(3) < ... < f(-3) < f(-2) < f(-1)$$

.

Denote the set of all such ribbons by $\mathbb{I}$.

2 An Example

Let $f(0) = 1$, $f(n) = n[n+1]^{-1}$ for $n > 0$, and $f(n) = [|n|+1]|n|^{-1}$ for $n < 0$. Then $f = ...\frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{2}{1}, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, ...$ is a ribbon.
As we'll see, f is a multiplicative identity.

3 Another Example : Subribbons

Let $g(n) = f(2n)$. Then $g = ...\frac{11}{10}, \frac{9}{8}, \frac{7}{6}, \frac{5}{4}, \frac{3}{2}, 1, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}, ...$ is a ribbon, which we can call a **subribbon**. As we'll see, g is equivalent to f and therefore also a multiplicative identity.

4 Order

Define $f \sqsubset g$ if $z > 0 \implies g(z) \leq f(z) < f(-z) \leq g(-z)$.

Then define $f \sim g$ if $\exists h \in \mathbb{I}$ such that $h \sqsubset f$ and $h \sqsubset g$.

Also define $f < g$ if there $\exists z > 0$ such that $f(-z) < g(z)$.

For all $f, g \in \mathbb{I}$ we have $f < g$, $f > g$, or $f \sim g$.

5 Addition and Multiplication

Define $f+g$ by $(f+g)(z)=f(z)+g(z)$.

Define fg by $(fg)(z)=f(z)g(z)$.

Then $f_0 \sim f_1, g_0 \sim g_1 \implies f_0+g_0 \sim f_1+g_1$.

Also $f_0 \sim f_1, g_0 \sim g_1 \implies f_0 g_0 \sim f_1 g_1$.

6 A Multiplicative Identity

Let $I(0)=1, I(z)=z[z+1]^{-1}$ for $z>0$, and $I(z)=[|z|+1]|z|^{-1}$ for $z<0$. Then $I = ...\frac{6}{5},\frac{5}{4},\frac{4}{3},\frac{3}{2},\frac{2}{1},1,\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6},...$ is a multiplicative identity on $\mathbb{I}$.

7 A Multiplicative Inverse

For $f \in \mathbb{I}$, define f^* by $f^*(z)=[f(-z)]^{-1}$. Then $f^* \in \mathbb{I}$ and $ff^* \sim I$.

8 Injecting rational numbers into the set of ribbons

Define $f^p : Z \to Q$ by $f^p(z)=\frac{z}{z+1}p$ if $z>0$, $f^p(z)=\frac{|z|+1}{|z|}p$ if $z<0$, and $f^p(0)=p$. Then $f^p = ...\frac{5}{4}p,\frac{4}{3}p,\frac{3}{2}p,\frac{2}{1}p,p,\frac{1}{2}p,\frac{2}{3}p,\frac{3}{4}p,\frac{4}{5}p...$ is a ribbon.

Proposition: $\neg[f<g] \wedge \neg[g<f] \implies f \approx g$.

Let $\lceil a,b \rceil$ be the maximum and $\lfloor a,b \rfloor$ be the minimum of a and b. Then $z>0 \implies \lceil f(z),g(z) \rceil < \lfloor f(-z),g(-z) \rfloor$.

Define $h(z)=\lfloor f(z),g(z) \rfloor$ for $z<0$, $h(z)=\lceil f(z),g(z) \rceil$ for $z>0$. Then $h \sqsubset f$ and $h \sqsubset g$, so $f \sim g$.

Proposition: $f \in \mathbb{I} \implies f^* \in \mathbb{I}$.

Note that $0 < f(1) \leq f(n) < f(-n)$, so that $[f(-n)]^{-1} < [f(n)]^{-1} \leq [f(1)]^{-1}$, and $f^*(-n) - f^*(n) = f(n)^{-1} - f(-n)^{-1} = [f(-n) - f(n)][f(-n)^{-1}f(n)^{-1}] < [f(-n) - f(n)][f(1)]^{-2} \to 0$. So $f^* \in I$.

9 A limit

Let $f_{n+1} \sqsubset f_n$ for all $n \in \mathbb{N}$. Then this sequence has a **center**, which is something like a limit in our realm without a distance function (because we don't have subtraction.)

Define $\mathring{f}(z) = f_{|z|}(z)$. Then $\forall n\ \mathring{f} \sqsubset f_n$, and any other ribbon that manages this is equivalent to $\mathring{f}$.