1 Definition

A function $f : \mathbb{Z} \to \mathbb{Q}$ is a **ribbon** when

(1) $f(1) < f(2) < f(3) < ... < f(-3) < f(-2) < f(-1)$

(2) $f(-n) - f(n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Denote the set of all such ribbons by $\mathbb{I}$.

2 An Example

Let $f(0) = 1, f(n) = n[n+1]^{-1}$ for $n > 0$, and $f(n) = [|n|+1]|n|^{-1}$ for $n < 0$. Then $f = ... \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{2}{1}, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, ...$ is a ribbon. As we'll see, f is a multiplicative identity.

3 Order

Define $f \sqsubset g$ if $n > 0 \implies g(n) \leq f(n) < f(-n) \leq g(-n)$.

Then define $f \sim g$ if $\exists h \in \mathbb{I}$ such that $h \sqsubset f$ and $h \sqsubset g$.

Also define $f < g$ if there $\exists n > 0$ such that $f(-n) < g(n)$.

For all $f, g \in \mathbb{I}$ we have $f < g, f > g$, or $f \sim g$.

4 Addition and Multiplication

Define $f + g$ by $(f + g)(z) = f(z) + g(z)$.

Define fg by $(fg)(z) = f(z)g(z)$.

Then $f_0 \sim f_1, g_0 \sim g_1 \implies f_0 + g_0 \sim f_1 + g_1$.

Also $f_0 \sim f_1, g_0 \sim g_1 \implies f_0 g_0 \sim f_1 g_1$.

5 A Multiplicative Identity

Let $I(0) = 1, I(z) = z[z+1]^{-1}$ for $z > 0$, and $I(z) = [|z|+1]|z|^{-1}$ for $z < 0$. Then $I = ... \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{2}{1}, 1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, ...$ is a multiplicative identity on $\mathbb{I}$.

6 A Multiplicative Inverse

For $f \in \mathbb{I}$, define f^* by $f^*(z) = [f(-z)]^{-1}$. Then $f^* \in \mathbb{I}$ and $ff^* \sim I$.

7 Injecting rational numbers into the set of ribbons

Define $f^p : Z \to Q$ by $f^p(z) = \frac{z}{z+1}p$ if $z > 0$, $f^p(z) = \frac{|z|+1}{|z|}p$ if $z < 0$, and $f^p(0) = p$. Then $f^p = ... \frac{5}{4}p, \frac{4}{3}p, \frac{3}{2}p, \frac{2}{1}p, p, \frac{1}{2}p, \frac{2}{3}p, \frac{3}{4}p, \frac{4}{5}p...$ is a ribbon.

Proposition: $\neg[f < g] \wedge \neg[g < f] \implies f \approx g$.

Let $\lceil a, b \rceil$ be the maximum and $\lfloor a, b \rfloor$ be the minimum of a and b. Then $z > 0 \implies \lceil f(z), g(z) \rceil < \lfloor f(-z), g(-z) \rfloor$.

Define $h(z) = \lfloor f(z), g(z) \rfloor$ for $z < 0$, $h(z) = \lceil f(z), g(z) \rceil$ for $z > 0$. Then $h \sqsubset f$ and $h \sqsubset g$, so $f \sim g$.

Proposition: $f \in \mathbb{I} \implies f^* \in \mathbb{I}$.

Note that $0 < f(1) \leq f(n) < f(-n)$, so that $[f(-n)]^{-1} < [f(n)]^{-1} \leq [f(1)]^{-1}$, and $f^*(-n) - f^*(n) = f(n)^{-1} - f(-n)^{-1} = [f(-n) - f(n)][f(-n)^{-1} f(n)^{-1}] < [f(-n) - f(n)][f(1)]^{-2} \to 0$. So $f^* \in I$.

8 A limit

Let $f_{n+1} \sqsubset f_n$ for all $n \in \mathbb{N}$. Then this sequence has a **center**, which is something like a limit in our realm without a distance function (because we don't have subtraction.)

Define $\mathring{f}(z) = f_{|z|}(z)$. Then $\forall n \ \mathring{f} \sqsubset f_n$, and any other ribbon that manages this is equivalent to $\mathring{f}$.

9 Identity

We check that $ff^* \in \mathbb{I}$. Let $f \in \mathbb{I}$. Then $n > 0 \implies (ff^*)(n) = \frac{f(n)}{f(-n)}$.

Also $f(n) < f(n+1)$ and $f(-n-1) < f(-n) \implies f(n)f(-n-1) < f(n+1)f(-n) \implies \frac{f(n)}{f(-n)} < \frac{f(n+1)}{f(-(n+1))}$. So ff^* is increasing on $\mathbb{N}$.

Note that $z < 0 \implies z = -n \implies (ff^*)(-n) = \frac{f(-n)}{f(n)}$.

Then $f(-n) > f(-(n+1))$ and $f(n+1) > f(n)$ give $f(-n)f(n+1) > f(-(n+1))f(n)$. So $\frac{f(-n)}{f(n)} > \frac{f(-(n+1))}{f(n+1)}$, and ff^* is decreasing on $-\mathbb{N}$.

Also $\frac{f(-n)}{f(n)} - \frac{f(n)}{f(-n)} = \frac{(f(-n)-f(n))(f(-n)+f(n))}{f(n)f(-n)} \to 0$, since f is bounded and $f \in \mathbb{I}$.