

# Euler's Identity

Published: January 3, 2025 • 4 min read

Euler's Identity is often cited as an example of mathematical beauty. It is expressed by the equation:

$$e^{i\pi} + 1 = 0$$

where:

- $e$  is Euler's number, approximately equal to 2.71828, and is the base of natural logarithms.
- $i$  is the imaginary unit, defined as  $\sqrt{-1}$ , the foundation of complex numbers.
- $\pi$  is the ratio of a circle's circumference to its diameter, approximately equal to 3.14159.
- 1 and 0 are the multiplicative and additive identities.

## Derivation from Euler's Formula

Euler's Identity can be derived from Euler's formula, which states that for any real number  $x$ :

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

By substituting  $\theta$  with  $\pi$ , we get:

$$\begin{aligned} e^{i\pi} &= \cos(\pi) + i \sin(\pi) \\ &= -1 + i \cdot 0 \\ &= -1 \end{aligned}$$

Adding 1 to both sides yields Euler's Identity:

$$e^{i\pi} + 1 = 0$$

## Geometric Interpretation

In the complex plane,  $e^{i\theta}$  traces out a unit circle as  $\theta$  varies. The point at  $\theta = \pi$  corresponds to  $-1$  on the real axis.

### Unit Circle Representation

$$\operatorname{Re}(e^{i\theta}) = \cos(\theta)$$

$$\operatorname{Im}(e^{i\theta}) = \sin(\theta)$$

## Significance

Euler's Identity is celebrated for its elegance, as it connects five fundamental mathematical constants in a single equation. It has implications in various fields, including engineering, physics, and signal processing, particularly in the analysis of waveforms and oscillations.

## Applications

Euler's Identity and Euler's formula are widely used in:

- Electrical engineering, particularly in the analysis of alternating current (AC) circuits.
- Quantum mechanics, where complex exponentials describe wave functions.
- Signal processing, for transforming signals between time and frequency domains using Fourier transforms.
- Control theory, in the design and analysis of control systems.
- Vibrations and wave analysis, where they help model oscillatory systems.
- Computer graphics, for rotations and transformations in 2D and 3D spaces.

## Related Formulas

Euler's formula leads to many other useful identities:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

These allow us to express trigonometric functions in terms of exponentials, which is important in various mathematical and engineering applications.