In dynamics, central forces lead to two-dimensional orbital motion. Let's first attempt to quantize this problem in two dimensions. The Hamiltonian is given by $H = \frac{P_r^2}{2m} + \frac{L^2}{2mr^2} - \frac{e}{r}$.

Without the azimuthal angle ϕ dimension, L is a constant of motion with quantized value l. Assuming:

$$\langle r|p_r^2 = -\frac{\hbar^2}{r}\partial_r r \partial_r \langle r| = (-\hbar^2)\left(\frac{1}{r}\partial_r + \partial_r^2\right)\langle r|$$

The radial Schrödinger equation becomes:

$$\left[\left(-\frac{\hbar^2}{2m}\right)\left(\frac{1}{r}\partial_r + \partial_r^2\right) + \frac{\hbar^2}{2m}\frac{l^2}{r^2} - \frac{e}{r} - E\right] R(r) = 0$$

As $r \to \infty$, $P_r^2 R \to 0$, and $R \to e^{-\lambda r}$; define $\lambda^2 \equiv \frac{-2mE}{\hbar^2}$ (assuming $E < 0$ for bound states).

Introducing physical scaling: $\rho = \lambda r$.

The radial equation in terms of ρ:

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{l^2}{\rho^2} + \frac{2me}{\hbar^2\lambda}\frac{1}{\rho} - 1\right) R(\rho) = 0$$

Define $\gamma = \frac{me}{\hbar^2\lambda}$.

Assuming a solution of the form $\mathcal{R}(\rho) = e^{-\rho} f(\rho)$, substitution yields the equation for $f(\rho)$:

$$f''(\rho) + \left(\frac{1}{\rho} - 2\right) f'(\rho) + \left(\frac{2\gamma - 1}{\rho} - \frac{l^2}{\rho^2}\right) f(\rho) = 0$$

Assume a series solution $f(\rho) = \sum_{m=0}^{\infty} a_m \rho^{s+m}$. Substituting gives the recurrence relation:

$$\sum_m a_m \left[(s+m)(s+m-1) + (s+m) - l^2\right] \rho^{s+m-2} - \sum_m a_m \left[2(s+m) - (2\gamma - 1)\right] \rho^{s+m-1} = 0$$

Simplifying to:

$$a_m = -\frac{2\gamma - (2l+1) - 2(m-1)}{(s+m)^2 - l^2} a_{m-1}$$

Analyzing the lowest-order coefficient: $a_0[s(s-1) + s - l^2] = 0 \Rightarrow s^2 = l^2$. For regular behavior at $r = 0$, take $s = l$ (assuming $l \geq 0$).

Assuming truncation at $m = m_{\max}$ (to obtain bound states), the termination condition requires:

$$2(s + m_{\max}) - (2\gamma - 1) = 0 \Rightarrow \gamma = s + m_{\max} + \frac{1}{2}$$

Energy expression:

$$E = -\frac{\hbar^2}{2m}\lambda^2 = -\frac{\hbar^2}{2m}\left(\frac{me}{\hbar^2}\frac{1}{\gamma}\right)^2 = -\frac{me^2}{2\hbar^2}\frac{1}{\gamma^2}$$

Possible γ values (with $s = l$ and $m_{\max} = 0, 1, 2, ...$):

$$\gamma = l + m_{\max} + \frac{1}{2}$$

1 Normalization

Assuming angular wavefunctions are normalized, the radial wavefunctions should satisfy:

$$\int_0^\infty R^2(\rho)\rho d\rho = 1$$

Table 1: Integration Results for Functions $e^{-2x}x^n$

Function	Value
$e^{-2x}x^1$	1/4
$e^{-2x}x^2$	1/4
$e^{-2x}x^3$	3/8
$e^{-2x}x^4$	3/4
$e^{-2x}x^5$	15/8

1. Ground State ($\gamma = \frac{1}{2}$, $l = 0$, $m_{\max} = 0$) Wavefunction:

$$R_{1/2}(\rho) = 2e^{-\rho}$$

where $\rho = \frac{me}{\hbar^2} \cdot 2r$, corresponding to energy:

$$E = -\frac{me^2}{2\hbar^2} \cdot 4 = -\frac{2me^2}{\hbar^2}$$

2. First Excited State ($\gamma = \frac{3}{2}$)
- Case 1: $l = 0$, $m_{\max} = 1$ Wavefunction:

$$R_{3/2}^{(0)}(\rho) = \frac{2}{\sqrt{17}} e^{-\rho}(1 - 4\rho)$$

- Case 2: $l = 1$, $m_{\max} = 0$ Wavefunction:

$$R_{3/2}^{(1)}(\rho) = \frac{2\sqrt{2}}{\sqrt{3}} e^{-\rho}\rho$$

Both cases share energy:

$$E = -\frac{me^2}{2\hbar^2} \cdot \frac{4}{9} = -\frac{2me^2}{9\hbar^2}$$

3. Second Excited State ($\gamma = \frac{5}{2}$)
- Case 1: $l = 0$, $m_{\max} = 2$ Wavefunction:

$$R_{5/2}^{(0)}(\rho) = \frac{2\sqrt{2}}{\sqrt{707}} e^{-\rho}(1 - 6\rho + 9\rho^2)$$

- Case 2: $l = 1$, $m_{\max} = 1$ Wavefunction:

$$R_{5/2}^{(1)}(\rho) = \frac{2\sqrt{6}}{\sqrt{5}} e^{-\rho}\rho\left(1 - \frac{2}{3}\rho\right)$$

- Case 3: $l = 2$, $m_{\max} = 0$ Wavefunction:

$$R_{5/2}^{(2)}(\rho) = \frac{2\sqrt{2}}{\sqrt{15}} e^{-\rho} \rho^2$$

All cases share energy:

$$E = -\frac{me^2}{2\hbar^2} \cdot \frac{4}{25} = -\frac{2me^2}{25\hbar^2}$$