Dynamically, central force leads to 2D orbiting motion.

Let's try quantizing the problem in 2D first.

$$H = \frac{p_r^2}{2m} + \frac{L^2}{2mr^2} - \frac{e}{r}$$

① no ϕ dependence $\Rightarrow$ L is constant of motion with quantized value ℓ

② assume $\langle r| p_r^2 = \frac{-\hbar^2}{r}\partial_r r \partial_r \langle r| = (-\hbar^2)\left(\frac{1}{r}\partial_r + \partial_r^2\right)\langle r|$

$$\left[\left(\frac{-\hbar^2}{2m}\right)\left(\frac{1}{r}\partial_r + \partial_r^2\right) + \frac{\hbar^2}{2m}\frac{\ell^2}{r^2} - \frac{e}{r} - E\right] R = 0$$

$r \to \infty$, $\left(\partial_r^2 + \frac{2mE}{\hbar^2}\right) R \to 0 \Rightarrow R \to e^{-\lambda r}$; $\lambda^2 \equiv \frac{-2mE}{\hbar^2}$

incorporate the physical scale : $\rho \equiv \lambda r$

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{\ell^2}{\rho^2} + \frac{2me}{\hbar^2}\frac{1}{\lambda\rho} - 1\right) R = 0 \quad ; \quad \gamma \equiv \frac{me}{\hbar^2\lambda}$$

Assume $R = e^{-\rho} f(\rho)$, $\partial_\rho R = (f'(\rho) - f(\rho)) e^{-\rho}$, $\partial_\rho^2 R = (f''(\rho) - 2f'(\rho) + f(\rho)) e^{-\rho}$

$$\left(f'' + \frac{1}{\rho}f' - \frac{\ell^2}{\rho^2} f\right) + \left((2\gamma - 1)\frac{1}{\rho} f - 2f'\right) = 0$$

Assume $f(\rho) = \sum_m a_m \rho^{s+m}$

$$\sum_m a_m \left[(s+m)(s+m-1) + (s+m) - \ell^2\right]\rho^{s+m-2} - \sum_m a_m \left[2\gamma - 1 - 2(s+m)\right]\rho^{s+m-1} = 0$$

lowest order : $s^2 - \ell^2 = 0 \Rightarrow s = \ell$

Assume $a_m = 0$ for $m > m_\ell$,

highest order : $\gamma = s + m_\ell + \frac{1}{2}$

$E = \frac{-\hbar^2}{2m}\lambda^2 = \frac{-\hbar^2}{2m}\left(\frac{me}{\hbar^2}\frac{1}{\gamma}\right)^2 = \frac{-me^2}{2\hbar^2}\cdot\frac{1}{\gamma^2}$ $\Rightarrow$ fit exp. if $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$