Burger's Equation (One – Dimension)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}$$

$u(x, t)$: velocity $[m/s]$

x : spatial coordinate $[m]$

t : time $[s]$

v : kinematic viscosity $[\mathrm{m}^2/s]$

Define Characteristic Scales

U : characteristic velocity

L : characteristic length (domain size)

$T = \frac{L}{U}$: characteristic time scale

Define Dimensionless Variables

$$x' = \frac{x}{L}$$

$$t' = \frac{t}{T}$$

$$u' = \frac{u}{U}$$

then

$$\frac{\partial u}{\partial t} = \frac{U}{T}\frac{\partial u'}{\partial t'}$$

$$u\frac{\partial u}{\partial x} = \frac{U^2}{L}u'\frac{\partial u'}{\partial x'}$$

$$v\frac{\partial^2 u}{\partial x^2} = \frac{vU}{L^2}\frac{\partial^2 u'}{\partial x'^2}$$

$$\frac{U}{T}\frac{\partial u'}{\partial t'} + \frac{U^2}{L}u'\frac{\partial u'}{\partial x'} = \frac{vU}{L^2}\frac{\partial^2 u'}{\partial x'^2}$$

divide by $\frac{U^2}{L}$

We get the new dimensionless equation as

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} = \frac{v}{U\,L} \frac{\partial^2 u'}{\partial x'^2}$$

Define the Reynolds Number $\mathrm{Re} = \dfrac{U\,L}{v}$

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} = \frac{1}{\mathrm{Re}} \frac{\partial^2 u'}{\partial x'^2}$$

Discretization

Define the domain

$x \in [0, L]$

$t \in [0, T]$

N_x = **Number of Segment in** x **grid**

N_t = **Number of Segment in** t **grid**

where

$$\Delta x = \frac{L}{N_x - 1}$$

$$\Delta t = \frac{T}{N_t}$$

using the approximation

$$u_i^n \approx u(x_i, t_n)$$

where

$x_i = i\,\Delta x$

$t_n = n\,\Delta t$

Discretization

Forward Euler

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Spatial derivative (centered)

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}{}^{n} - u_{i-1}{}^{n}}{2\,\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}{}^{n} - 2\,u_i{}^{n} + u_{i-1}{}^{n}}{\Delta x^2}$$

The Burger ' s equation becomes

$$\frac{u_i{}^{n+1} - u_i{}^{n}}{\Delta t} + u_i{}^{n}\,\frac{u_{i+1}{}^{n} - u_{i-1}{}^{n}}{2\,\Delta x} = \frac{1}{\mathrm{Re}}\,\frac{u_{i+1}{}^{n} - 2\,u_i{}^{n} + u_{i-1}{}^{n}}{\Delta x^2}$$

Update the system

we solve for $u_i{}^{n+1}$ to get

$$u_i{}^{n+1} = \left(\frac{1}{\mathrm{Re}}\,\frac{u_{i+1}{}^{n} - 2\,u_i{}^{n} + u_{i-1}{}^{n}}{\Delta x^2} - u_i{}^{n}\,\frac{u_{i+1}{}^{n} - u_{i-1}{}^{n}}{2\,\Delta x}\right)\frac{\Delta t}{-\,u_i{}^{n}}$$

$$u_i{}^{n+1} = \frac{(u_{i+1}{}^{n} - u_{i-1}{}^{n})\,\Delta t}{2\,\Delta x} - \frac{\Delta t}{\mathrm{Re}}\,\frac{(u_{i+1}{}^{n} - 2\,u_i{}^{n} + u_{i-1}{}^{n})}{u_i{}^{n}\,\Delta x^2}$$

Bouandries

Domain : $x \in [0, 1]$

Riemann step $\begin{cases} u[x, 0] = 1, & x \leq 0.5 \\ 0, & \text{otherwise} \end{cases}$

Dirichlet

$u[0, t] = u_L$ & $u[L, t] = u_R$ **for** $t > 0$