Quantum Magnetism | Problem Sheet 5

Prof. J. Knolle | Winter Term 2023/24

1/12/23

Submission deadline for the homework questions: 15/12/2023 (hand in via moodle).

Please prepare the remaining questions for the tutorials in the following week and be ready to present the questions marked with a star (*).

Exercise 5.1: Stoner in Hamiltonian Formalism (Homework question)

In this question we will derive the Stoner model without using the saddle-point approximation.

Consider the Hubbard model

$$H_{\mathrm{Hubb}} = \sum_{ij\,\sigma} t_{ij} a^\dagger_{i\sigma} a_{j\sigma} + U \sum_i n_{i\uparrow} n_{i\downarrow} \tag{1}$$

(a) Move to momentum space and show the Hamiltonian can be written as

$$H_{\mathrm{Hubb}} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + U \sum_{\mathbf{k}_1\mathbf{k}_2} \sum_{\mathbf{q}} \sum_{\sigma_1\sigma_2} a^\dagger_{\mathbf{k}_1+\mathbf{q},\sigma_1} a^\dagger_{\mathbf{k}_2-\mathbf{q},\sigma_2} a_{\mathbf{k}_2,\sigma_2} a_{\mathbf{k}_1,\sigma_1} \tag{2}$$

(b) Choose the correct decoupling of the operators around their expectation values

$$\left\langle a^\dagger_{\mathbf{p}_1,\sigma_i} a_{\mathbf{p}_2\sigma_i} \right\rangle . \tag{3}$$

You may assume that the expectation values vanish except when momenta and spin terms are equal. Therefore show that the resulting mean field interaction term is

$$-\frac{U}{2N}\sum_{\mathbf{k}_1\neq\mathbf{k}_2\,\sigma}\left[\left\langle a^\dagger_{\mathbf{k}_1,\sigma}a_{\mathbf{k}_1\sigma}\right\rangle a^\dagger_{\mathbf{k}_2,\sigma}a_{\mathbf{k}_2\sigma}+\left\langle a^\dagger_{\mathbf{k}_2,\sigma}a_{\mathbf{k}_2\sigma}\right\rangle a^\dagger_{\mathbf{k}_1,\sigma}a_{\mathbf{k}_1\sigma}-\left\langle a^\dagger_{\mathbf{k}_1,\sigma}a_{\mathbf{k}_1\sigma}\right\rangle\left\langle a^\dagger_{\mathbf{k}_2,\sigma}a_{\mathbf{k}_2\sigma}\right\rangle\right] \tag{4}$$

(c) By writing

$$N_\sigma=\sum_{\mathbf{k}}\left\langle a^\dagger_{\mathbf{k},\sigma}a_{\mathbf{k}\sigma}\right\rangle \tag{5}$$

along with $N_e = N_\uparrow + N_\downarrow$ and $M = U(N_\uparrow - N_\downarrow)/N$, show that the Stoner Hamiltonian Eq. (294) from the notes is recovered.

To do this you will need to fix half-filling $N_e = N$.

Exercise 5.2: Critical Temperature in 3D (Homework question)

This question will reproduce the phase diagram on page 118 of the lecture notes.

(a) In the $M \to 0^+$ limit show that the Stoner criterion $1 = I\chi_0(T_c)$ is recovered and implies there is a critical interaction strength I_c below which there is no order.

How is this transition different to standard transitions from statistical mechanics?

(b) Using the DoS in 3D

$$N(E) = N(0)\sqrt{(E+\mu)/\epsilon_F}, \tag{6}$$

show that one may write the Stoner condition as

$$1 = I\sqrt{\frac{T_c}{2\epsilon_F}}\int_0^\infty \mathrm{d}x\,\sqrt{x}\,\mathrm{sech}^2(x-\mu/2T_c). \tag{7}$$

Hint: simplify the exponentials in the occupation functions, possible with mathematica.

(c) Now at fixed particle number, the chemical potential will drift with T. By writing $T_c(y)$ parametrically in terms of $y = \mu/T_c$, we get

$$T_c(y) = \epsilon_{\mathrm{F}} \left[\frac{3}{2} \int_0^{\infty} \mathrm{d}x \, \sqrt{x} \left(e^{x-y} + 1 \right)^{-1} \right]^{-2/3} . \tag{8}$$

Plot a diagram of $T_c/\epsilon_{\mathrm{F}}$ against I which are solutions to these equations (7) and (8).

Hint: First substitute these equations to solve for $I(y)$, and then use mathematica ListPlot to calculate T_c and I values for a range of y.

Exercise 5.3*: Lindhard Function & Susceptibility

This question focuses on the Stoner theory in the field theory perspective, and then derives the Lindhard Function and relates it to susceptibility.

(a) Let's recap the derivation of the Stoner action from the lectures.

The Hubbard partition function to be expressed as the following Euclidian-time integral

$$Z = \int \mathcal{D}[\bar{c}, c] \exp \left\{ - \int_0^{\beta} \mathrm{d}\tau \left[\sum_{\mathbf{k}} \bar{c}(k)_\sigma (\partial_\tau + \epsilon(\mathbf{k})) c(k)_\sigma - \frac{U}{4} \sum_i (\bar{c}_{i\sigma} \sigma^z_{\sigma\sigma'} c_{i\sigma'})^2 \right] \right\} . \tag{9}$$

(i) Show that the introduction of an auxiliary magnetisation field allows this to be rewritten as

$$Z = \int \mathcal{D}[\bar{c}, c] \mathcal{D}[m] \exp \left\{ - \int_0^{\beta} \mathrm{d}\tau \left[\sum_{\mathbf{k}} \bar{c}(k)_\sigma (\partial_\tau + \epsilon(\mathbf{k})) c(k)_\sigma + \frac{U}{4} \sum_i (2\bar{c}_{i\sigma} \sigma^z_{\sigma\sigma'} c_{i\sigma'} m_i + m_i^2) \right] \right\} \tag{10}$$

(ii) Next, show that integrating over the fermion field yields

$$Z = Z_0 \int \mathcal{D}[m] \exp\left\{ +\frac{U}{4}\int_0^\beta \mathrm{d}\tau \sum_i m_i^2(\tau) + \operatorname{tr}\log[1 - \frac{U}{2} m G_0] \right\}, \tag{11}$$

where

$$\log Z_0 = \operatorname{tr}\log[G_0^{-1}], \quad G_0(k) = [i\omega_n - \epsilon(k)]^{-1}. \tag{12}$$

Find an explicit form of the interaction matrix m.

(iii) At small U there is a well-behaved diagrammatic expansion of this action. Which quantities in (11) represent the bare Fermion propagators and interaction vertices?

Write down an expression for the 'dressed' fermion propagator $G(k)$ which includes these interactions.

(b) Recovering the Lindhard functions.

(i) At some critical U_c the system undergoes a phase transition to a state where m becomes ordered. We will expand around this point where the field m acquires an expectation value and is small.

Show that such an expansion gives an effective action with a quadratic term

$$S_{\text{quad}} = \frac{1}{2}\sum_q v_2(q)|m_q|^2, \tag{13}$$

and show that

$$v_2(q) = \frac{U}{2}\left[1 - U\Pi_q\right], \qquad \Pi_q = -\frac{2T}{N}\sum_p G_{0,p}G_{0,p+q}. \tag{14}$$

Where the sum over k is a sum over momenta $\mathbf{k}$ and Matsubara frequencies ω_n.

(ii) Perform the sum to generate the Lindhard function

$$\Pi_q = -\frac{1}{N}\sum_{\mathbf{k}} \frac{n_{\mathrm{F}}(\epsilon_{\mathbf{k}}) - n_{\mathrm{F}}(\epsilon_{\mathbf{k}+\mathbf{q}})}{i\omega_n + \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}}}. \tag{15}$$

Hint: You can find how to perform Matsubara frequency summations in books like *Atland and Simons*.

(c) Susceptibility & instabilities

(i) Briefly argue why the bare susceptibility may be written as

$$\chi^{(0)}(q)_{zz} = -\frac{T}{N}\sum_k \mathrm{tr}[G(k)G(k+q)]. \tag{16}$$

Note that the difference between using G_0 and G is because the two states are expanded around different vaccua, representing m small or finite $m + \delta m$ spin wave excitations.

(ii) If there exists regions of the Brillouin zone where $\epsilon(\mathbf{q}) = \epsilon(\mathbf{q} + \mathbf{Q})$, then the sum defining $\chi^{(0)}(q)_{zz}$ will be divergent.

Repeat the derivation of the Lindhard function, except now assuming that there is nesting; show that

$$\chi(\mathbf{Q}, \omega_n) \sim N(E_F)\log(E_F/\omega_n), \tag{17}$$

and argue that as $T \to 0$ this recovers a temperature-dependent divergence

$$\chi(\mathbf{Q}) \sim N(E_F)\log(E_F/T). \tag{18}$$

(iii) Comment on why this leads to an instability of AFM phases, where there is a nesting of the Brillouin zone.

Exercise 5.4: Quantum Criticality

(a) Approximate the behaviour of the Lindhard function (15) to quadratic order at low-energy and momentum

$$|\omega_n|/|\mathbf{q}|v_\mathrm{F} \ll 1, \quad \xi|\mathbf{q}| = |\mathbf{q}|/|q_\mathrm{F}| \ll 1, \tag{19}$$

when the Fermion dispersion is given $\epsilon_\mathrm{k} = \mathbf{k}^2/2m$. Show that this leads to the following quadratic effective action

$$S_\mathrm{quad} = \frac{U^2N(0)}{4}\sum_q \left[\left(\frac{1}{UN(0)} - 1\right) + \xi^2\mathbf{q}^2 + \frac{|\omega_n|}{v|\mathbf{q}|}\right]|m_q|^2, \tag{20}$$

(b) Mean-field phase transition

  (i) Argue that the quartic contribution is governed by a coupling $u \sim U^4 N''(0)$.

  (ii) Ignoring fluctuations, predict the critical coupling U_c where there is a transition of the mean-field theory.

(c) Write the Gaussian action as

$$S[m] = \frac{T}{2} \sum_q^{\Lambda} \left[\delta + \mathbf{q}^2 + \frac{|\omega_n|^2}{\Gamma|\mathbf{q}|} \right] |m_q|^2, \tag{21}$$

where the system is regulated to a momentum $\Lambda = 1/a$ and energy $\Gamma\Lambda$.

By splitting m into slow and fast fields $\mathcal{D}[m] = \mathcal{D}[m_s]\mathcal{D}[m_f]$, where m_f with momenta greater than Λ/b are integrated out.

  (i) For the kinetic term, show that the rescaling $m'(q') = m(q)/y$ and $T' = b^z T$ keeps the action invariant, since in the Gaussian case the fluctuations decouple.

  Calculate the scaling of m.

  It will help to write out

$$\sum_q^{\Lambda} = \int^{\Lambda} \frac{\mathrm{d}^D q}{(2\pi)^D} \int^{\Gamma\Lambda} \frac{\mathrm{d}\omega}{2\pi}. \tag{22}$$

  (ii) Use that the frequency scales as $\omega_n' = b^z \omega_n$ to fix z.

  (iii) Now find the scaling dimension of δ.

  (iv) Find the scaling of the um^4 coupling to argue that this is small in the IR and the theory is well described by the Gaussian theory.

  What is the upper critical dimension of the theory?

The RG equations for this theory can shed light on how the transition changes from being classical to a quantum critical transition with new exponents.