1.1 Second Quantization

We could consider

$$|\alpha_1,\ldots,\alpha_N\rangle = \frac{1}{\sqrt{N!}}\sum_P \zeta^{\sigma(P)}\left|\alpha_{P(1)}\right\rangle \otimes \ldots \otimes \left|\alpha_{P(N)}\right\rangle,$$

where $\zeta = \pm 1$ for bosons and fermions respectively. We define creation operator via

$$a_\beta^\dagger |\alpha_1,\ldots,\alpha_N\rangle \overset{\text{def}}{=} |\beta,\alpha_1,\ldots,\alpha_N\rangle.$$

1. Adjoint a_β could be expressed as

$$a_\beta^\dagger = \sum_{\{\theta\}} |\beta,\theta_1,\ldots,\theta_M\rangle\langle\theta_1,\ldots,\theta_M|,$$
$$a_\beta = \sum_{\{\theta\}} |\theta_1,\ldots,\theta_M\rangle\langle\beta,\theta_1,\ldots,\theta_M|.$$

   Than it could be shown that

$$a_\beta |\alpha_1,\ldots,\alpha_N\rangle = \sum_k C_k |\alpha_1,\ldots,\cancel{\alpha_k},\ldots,\alpha_N\rangle,$$

   with

$$C_k = \langle\beta,\alpha_1,\ldots,\cancel{\alpha_k},\ldots,\alpha_N|\alpha_1,\ldots,\alpha_N\rangle = \frac{1}{\sqrt{N!}}\sum_P \left\langle\alpha_{P(1)}\right| \otimes \ldots \otimes \langle\beta|_{P(k)} \otimes \ldots \otimes \left\langle\alpha_{P(N)}\right| \alpha_1,\ldots,\alpha_N\rangle,$$

   where we could «move» $\langle\beta|$ to the start, by $P(k)-1$ transpositions, and due to $N!$ equal permutations we could neglect $\frac{1}{N!}$ coming to

$$C_k = \zeta^{k-1}\langle\beta|\alpha_k\rangle.$$

2. We also could find, that a_β and $a_\beta^\dagger$ fuldill the (anti)-commutation relations

$$a_\beta^\dagger a_\alpha |\theta_1,\ldots,\theta_N\rangle = a_\beta^\dagger \sum_{k=1}^N \zeta^{k-1}\langle\alpha|\theta_k\rangle \left|\theta_1,\ldots,\cancel{\theta_k},\ldots,\theta_N\right\rangle = \sum_{k=1}^N \zeta^{k-1}\langle\alpha|\theta_k\rangle \left|\beta,\theta_1,\ldots,\cancel{\theta_k},\ldots,\theta_N\right\rangle,$$

$$a_\alpha a_\beta^\dagger |\theta_1,\ldots,\theta_N\rangle = a_\alpha |\beta,\theta_1,\ldots,\theta_N\rangle = \sum_{k=1}^N \zeta^k\langle\alpha|\theta_k\rangle \left|\beta,\theta_1,\ldots,\cancel{\theta_k},\ldots,\theta_N\right\rangle + \langle\alpha|\beta\rangle |\theta_1,\ldots,\theta_N\rangle,$$

   so for bosons $\zeta = 1$ we have

$$[a_\alpha, a_\beta^\dagger] \overset{\text{def}}{=} a_\alpha a_\beta^\dagger - a_\beta^\dagger a_\alpha = \langle\alpha|\beta\rangle = \delta_{\alpha,\beta},$$

   and in the same way for fermions $\zeta = -1$ and

$$\{a_\alpha, a_\beta^\dagger\} \overset{\text{def}}{=} a_\alpha a_\beta^\dagger + a_\beta^\dagger a_\alpha = \langle\alpha|\beta\rangle = \delta_{\alpha,\beta}.$$

3. For density operator

$$\hat\rho(x) = \sum_{j=1}^N \delta(x - \hat x_j),$$

   we could find second quantized form

$$\hat\rho(x) = \sum_{\alpha\beta}\langle\alpha|\delta(x-\hat x)|\beta\rangle \hat a_\alpha^\dagger \hat a_\beta = \sum_{\alpha\beta}\int \langle\alpha|x'\rangle\langle x'|\beta\rangle\delta(x-x')\,dx'\hat a_\alpha^\dagger \hat a_\beta = \sum_{\alpha\beta}\langle\alpha|x\rangle\langle x|\beta\rangle \hat a_\alpha^\dagger \hat a_\beta,$$

   what could be reduced to the $\hat a_x^\dagger \hat a_x$ form if $|\alpha\rangle$ and $|\beta\rangle$ corresponds to the coordinates.

1.2 Mapping between Quantum and Classical Systems

We could rewrite classical 1D Ising chain partitin function as

$$\mathcal{Z}_{\mathrm{c}} = T_{s_1,s_2} \dots T_{s_{N-1},s_N} T_{s_N,s_1} = \mathrm{tr}\left(T^N\right),$$

with transfer matrix

$$T = T^a T^b = \begin{pmatrix} e^{h_{\mathrm{c}}+K_{\mathrm{c}}} & e^{h_{\mathrm{c}}-K_{\mathrm{c}}} \\ e^{-h_{\mathrm{c}}-K_{\mathrm{c}}} & e^{K_{\mathrm{c}}-h_{\mathrm{c}}} \end{pmatrix}, \qquad T^a = \begin{pmatrix} e^{h_{\mathrm{c}}} & 0 \\ 0 & e^{-h_{\mathrm{c}}} \end{pmatrix}, \quad T^b = \begin{pmatrix} e^{K_{\mathrm{c}}} & e^{-K_{\mathrm{c}}} \\ e^{-K_{\mathrm{c}}} & e^{K_{\mathrm{c}}} \end{pmatrix}.$$

There are different ways to define T,because important just eigenvalues

$$\lambda_{1,2} = \frac{1}{2} e^{-h_{\mathrm{c}}-K_{\mathrm{c}}} \left(e^{2(h_{\mathrm{c}}+K_{\mathrm{c}})} + e^{2K_{\mathrm{c}}} \pm \sqrt{e^{4K_{\mathrm{c}}}\left(e^{2h_{\mathrm{c}}}-1\right)^2 + 4e^{2h_{\mathrm{c}}}} \right).$$

For a quantum system the partitin function

$$\mathcal{Z}_{\mathrm{q}} = \mathrm{tr}\, e^{-\beta H},$$

and we want to achieve

$$\mathcal{Z}_{\mathrm{q}} = \mathcal{Z}_{\mathrm{c}} = \mathrm{tr}\left(e^{-\frac{\beta}{N} H_1} e^{-\frac{\beta}{N} H_2} \right)^N, \qquad e^{-\frac{\beta}{N} H_1} = T^a, \quad e^{-\frac{\beta}{N} H_2} = T^b.$$

Using formulas to the Pauli matrix exponents, we could find

$$H_1 = \frac{N}{-\beta} \alpha_3 \sigma_z, \qquad H_2 = \frac{N}{-\beta} (\alpha_0 \mathbb{1} - \alpha_1 \sigma_x),$$

with $\alpha_0 = \ln \sinh(2K_{\mathrm{c}}) + \ln 2$, $\alpha_1 = \ln \tanh K_{\mathrm{c}}$ and $\alpha_3 = h_{\mathrm{c}}$. I think it is possible to find other H_1 and H_2, my choice was ruled by separating K_{c} and h_{c} dependences.

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