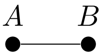
Measures of entanglement

Khoruzhii K.

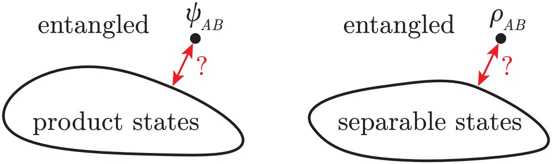
05.07.2024

States

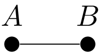


- Pure states $|\psi_{AB}\rangle$:
  - *Product state* if $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
  - A is entangled with B otherwise

- Mixed states $\hat{\rho}_{AB}$
  - *Separable state* if $\hat{\rho}_{AB} = \sum_k p_k \hat{\rho}_A^k \otimes \hat{\rho}_B^k$
  - A is entangled with B otherwise



Bipartite entanglement in pure states: SVD



Is this state entangled?

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$$

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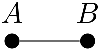

Is this state entangled?

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |01\rangle \right)$$

Of course not:

$$|\psi_{AB}\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right)$$

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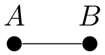

What about this?

$$|\psi_{AB}\rangle = \frac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right)$$

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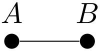

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$$|\psi_{AB}\rangle = \tfrac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right)$$

After some SVD (or **Schmidt decomposition**):

$$|\psi_{AB}\rangle = \sum_{a\in A}\sum_{b\in B}\psi_{ab}\,|a\rangle\otimes|b\rangle = \sum_{a}\sum_{b}\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}_{ab}|ab\rangle$$

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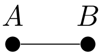

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$$U\begin{pmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\end{pmatrix}V = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & .\\ 1 & .\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & .\\ -1 & .\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$$

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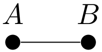

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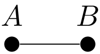
$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

We can see, that it is separable.

Bipartite entanglement in pure states: SVD



Schmidt decomposition

$$|\psi_{AB}\rangle = \sum_{a\in A}\sum_{b\in B}\psi_{ab}\,|a\rangle\otimes|b\rangle = \sum_j \sqrt{\lambda_j}\,|\psi_{A,j}\rangle\,|\psi_{B,j}\rangle$$

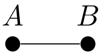
feeply connected with the reduced density matrix $\rho_{A,B}$

$$\rho_{B,A} = \mathrm{tr}_{A,B}\left(|\psi_{AB}\rangle\right) = \sum_i \lambda_i |\psi_{B,A}\rangle\langle\psi_{B,A}|.$$

If there is no entanglement, than

$$\rho_{A,B} = |\psi_{A,B}\rangle\langle\psi_{A,B}|$$

Von Neumann Entropy



Which one is more entangled?

$$\left|\psi^{I}_{AB}\right\rangle = \frac{1}{\sqrt{4}}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right),$$

$$\left|\psi^{II}_{AB}\right\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right),$$

$$\left|\psi^{III}_{AB}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

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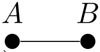

Which one is more entangled? (singular values could help):

$$\left|\psi^{I}_{AB}\right\rangle = \frac{1}{\sqrt{4}}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right), \qquad \lambda^{I}_{1,2} = \{1, 0\}$$

$$\left|\psi^{II}_{AB}\right\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right), \qquad \lambda^{II}_{1,2} = \{0.87, 0.13\}$$

$$\left|\psi^{III}_{AB}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \qquad \lambda^{III}_{1,2} = \{0.5, 0.5\}$$

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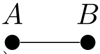

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$$\left|\psi_{AB}^{II}\right\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right), \qquad \lambda_{1,2}^{II} = \{0.87, 0.13\}$$

$$\left|\psi_{AB}^{III}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \qquad \lambda_{1,2}^{III} = \{0.5, 0.5\}$$

Measure of entanglement is fixed, if it has

- invariance under LUO $\Rightarrow E = E(\lambda)$
- continuity
- additivity $E(|\psi_{AB}\rangle \otimes |\varphi_{AB}\rangle) = E(|\psi_{AB}\rangle) + E(|\varphi_{AB}\rangle)$

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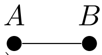

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This is *the von Neumann entropy*

$$\boxed{S(\rho_A) = S(\rho_B) = -\sum_j \lambda_j \ln \lambda_j,} \qquad S^{I} = 0,\ S^{II} = 0.6,\ S^{II} = 1.$$

Paired entanglement

The purity of the state represented by *one-tangle*

$$\tau_1[\rho_A] = 4 \det \rho_A = 1 - 4 \left(\langle S_x \rangle^2 + \langle S_y \rangle^2 + \langle S_z \rangle^2 \right)$$

for $\rho_A = \mathrm{tr}_B \, \rho_{AB}$.

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For a pure state of <u>two qubits</u> we can define *concurrence*

$$C[|\psi_{AB}\rangle] = |\langle\psi^*|\sigma_A^y \otimes \sigma_B^y|\psi\rangle| = \sqrt{\tau_1}$$

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Not all entanglement is stored in pairs:

$$\text{residual tangle} = \tau_{1,i} - \sum_{j\neq i} C_{ij}^2 \geqslant 0$$

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)

Entanglement in mixed states: EoF

The Entanglement of Formation defined throught *convex roof*:

$$E_F(\rho_{AB}) \stackrel{\text{def}}{=} \min_{\{p_j,\psi_j\}} \sum_j p_j S(\rho_{A,j}),$$

with $\rho_{AB} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ and $\rho_{A,j} = \operatorname{tr}_B |\psi_j\rangle\langle\psi_j|$.

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For two qubits:

$$E_F(\rho) = -\sum_{\sigma=\pm} \tfrac{1}{2}\sqrt{1+\sigma C^2(\rho)}\ln \tfrac{1}{2}\sqrt{1+\sigma C^2(\rho)},$$

$C(\rho) \in [0,1]$ – *concurrence*, that can be calculated from
$R = \sqrt{\rho}\tilde{\rho}\sqrt{\rho} = \sqrt{\rho}(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ with $\lambda_1^2 \geqslant \ldots \geqslant \lambda_4^2$

$$C = \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0).$$

Concurrence through two-point spin correlation

In a spin-1/2 chain

$$C_{ij} = 2\max\{0,\, C_{ij}^{I},\, C_{ij}^{II}\},$$

where

$$\begin{aligned} C_{ij}^{I} &= |g_{ij}^{xx} + g_{ij}^{yy}| - \sqrt{(1/4 + g_{ij}^{zz})^2 - M_z^2} \\ C_{ij}^{II} &= |g_{ij}^{xx} - g_{ij}^{yy}| + g_{ij}^{zz} - 1/4, \end{aligned}$$

with $g_{ij}^{\alpha\alpha} = \langle S_i^\alpha S_j^\alpha \rangle$ and $M_z = \langle S^z \rangle$.

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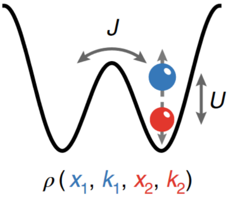
Measurable example

Consider dimer system



With Hamiltonian

$$\hat{H} = -J \sum_{\sigma} (\hat{c}_{\mathrm{L}\sigma} \hat{c}_{\mathrm{R}\sigma} + \mathrm{c.c.}) + U \sum_{j=\mathrm{L,R}} \hat{n}_{j\downarrow} \hat{n}_{j\uparrow}$$

[2] A. Bergschneider et al., Experimental Characterization of Two-Particle Entanglement through Position and Momentum Correlations, Nature Physics 15, no. 7 (2019)

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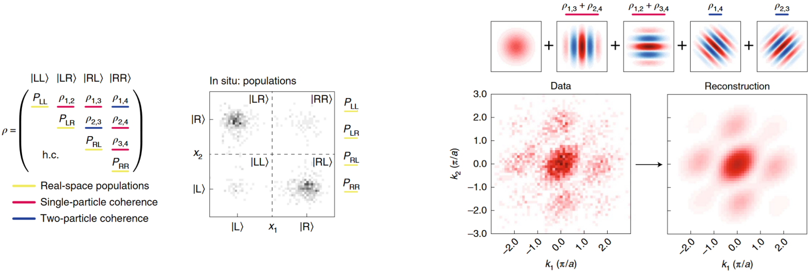
What is available for us to measure?



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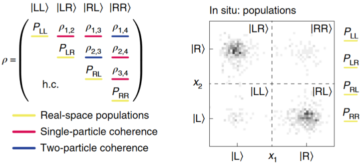

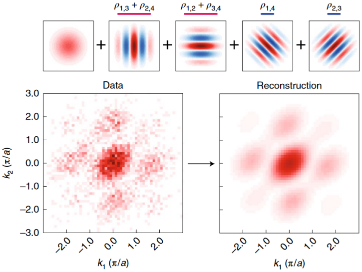

Lower bound for concurrence

$$C(\rho) \geqslant \max\left\{0,\ 2(|\rho_{1,4}| - \sqrt{P_{\mathrm{LR}}P_{\mathrm{RL}}}),\ 2(|\rho_{2,3}| - \sqrt{P_{\mathrm{LL}}P_{\mathrm{RR}}})\right\}$$

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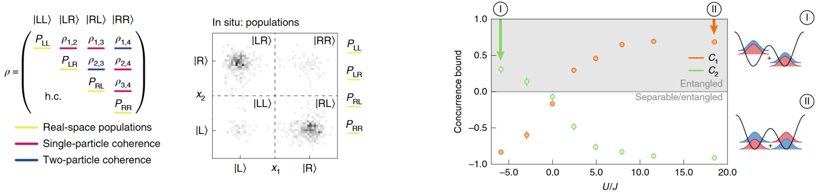

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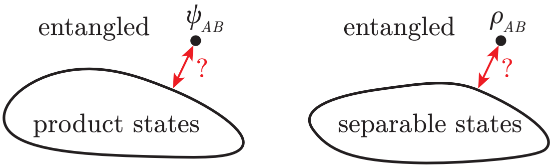
Relative entropy of entanglement



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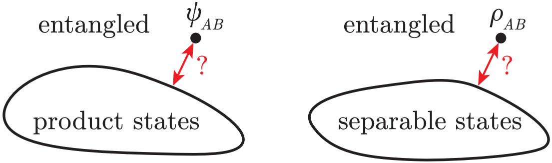

Measure of entanglement: «dinstance» from separable states

$$E(\rho) \stackrel{\text{def}}{=} \min_{\rho' \in \mathcal{D}} S(\rho||\rho'),$$

with $= S(\rho||\rho') = \operatorname{tr} \rho(\ln \rho - \ln \rho')$.

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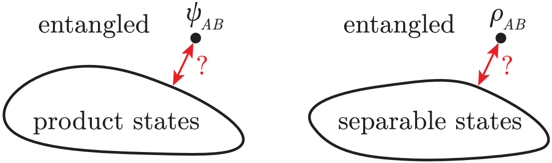

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*reduces to $S(\rho)$ in the case of pure bi-partite states.

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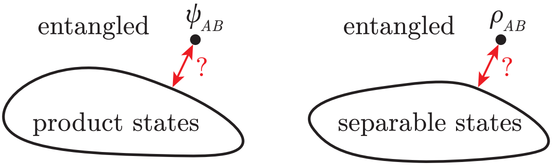
Multipartite entanglement measures



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$$E_g(\Psi) \stackrel{\text{def}}{=} -\log_2 \max_{\Phi \in \mathcal{S}} |\langle \Psi | \Phi \rangle|^2,$$

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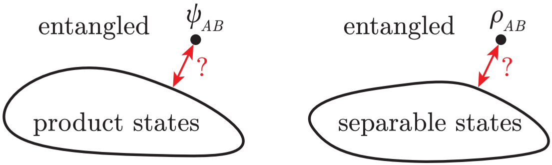

Multipartite entanglement: «dinstance» from product states

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Or average purity

$$E_{\text{gl}} \stackrel{\text{def}}{=} 2 - \frac{2}{N} \sum_{j=1}^{N} \operatorname{tr} \rho_j^2$$

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)

Indistinguishable particles: example

Consider state represented by the Slater determinant

$$|\psi\rangle^{\mathrm{SD}} = \frac{1}{\sqrt{2}} \left(|A, 0\rangle \, |B, 1\rangle - |B, 1\rangle \, |A, 0\rangle \right),$$

with A, B and $0, 1$ are the spatial and the internal degree of freedom.

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[4] A. P. Majtey et al., Indistinguishable Entangled Fermions: Basics and Future Challenges, Phil. Trans. R. Soc. A.381 (2023)

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This state results from antisymmetrizing the product state

$$|\psi\rangle^{\mathrm{prod}} = |A,0\rangle\,|B,1\rangle\,,$$

so no entanglement actually.

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
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Meanwhile reduced density matrix is inpure

$$\rho_{\mathrm{f}}^{\mathrm{SD}} = \tfrac{1}{2}|A,0\rangle\langle A,0| + \tfrac{1}{2}|B,1\rangle\langle B,1|.$$

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Indistinguishable particles: fermionic exchange correlations

In general for N identical fermions identified with a single Slater determinant reduced density matrix of M fermions

$$\mathrm{tr}\left(\rho_M^{\mathrm{SD}}\right)^2 = \binom{N}{M}^{-1}.$$

Such correlations are compatible with the possibility of assigning a complete set of properties to the individual fermions.

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Indistinguishable particles: fermionic exchange correlations

State that can not be represented by the Slater determinant

$$|\psi\rangle^{\text{SD}} = \frac{1}{2}\left(|A,0\rangle|B,1\rangle - |B,1\rangle|A,0\rangle + |A,1\rangle|B,0\rangle - |B,0\rangle|A,1\rangle\right),$$
$$|\psi\rangle^{\text{non-prod}} = \frac{1}{\sqrt{2}}\left(|A,0\rangle|B,1\rangle + |A,1\rangle|B,0\rangle\right)$$

And the purity of $\rho_{\text{f}}^{\text{non-SD}}$ is lower than the purity of $\rho_{\text{f}}^{\text{SD}}$.

This feature reveals the presence of correlations beyond exchange correlations.

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
[4] A. P. Majtey et al., Indistinguishable Entangled Fermions: Basics and Future Challenges, Phil. Trans. R. Soc. A.381 (2023)

Indistinguishable particles: fermionic exchange correlations

State that can not be represented by the Slater determinant

$$|\psi\rangle^{\text{SD}} = \frac{1}{2}\left(|A,0\rangle|B,1\rangle - |B,1\rangle|A,0\rangle + |A,1\rangle|B,0\rangle - |B,0\rangle|A,1\rangle\right),$$
$$|\psi\rangle^{\text{non-prod}} = \frac{1}{\sqrt{2}}\left(|A,0\rangle|B,1\rangle + |A,1\rangle|B,0\rangle\right)$$

And the purity of $\rho_{\text{f}}^{\text{non-SD}}$ is lower than the purity of $\rho_{\text{f}}^{\text{SD}}$.

Entanglament criteria:

$$\operatorname{tr}\rho_M^2 < \binom{N}{M}^{-1} \quad \Leftrightarrow \quad |\psi\rangle \text{ is entangled.}$$

In other words: how many Slater determinants you need to describe the state (*Slater rank*)?

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
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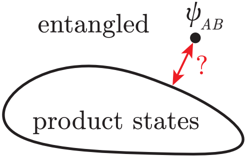
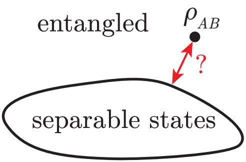
Entanglement witnesses

Some property, that differs for separable and entangled states.





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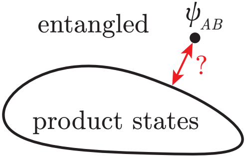

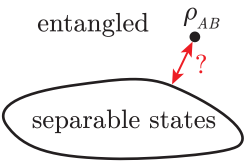

A state ρ_{AB} is entangled if and only if a positive map Λ exists:

$$(\mathbb{1}_A \otimes \Lambda_B)\,\rho_{AB} < 0.$$

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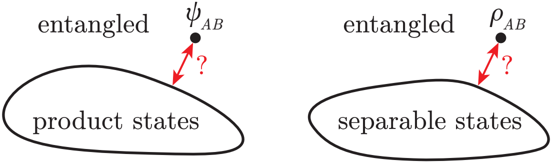

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Peres-Horodecki criterion of being separable:

$$\rho_{AB}^{T_B} \geqslant 0.$$

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Entanglement witnesses: Peres-Horodecki criterion

Peres-Horodecki criterion of being separable (sufficient):

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Necessary for:

- two qubits
- two harmonic oscillator modes

$$\rho = \frac{1}{n}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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Measure of entanglement: *the logarithmic negativity*

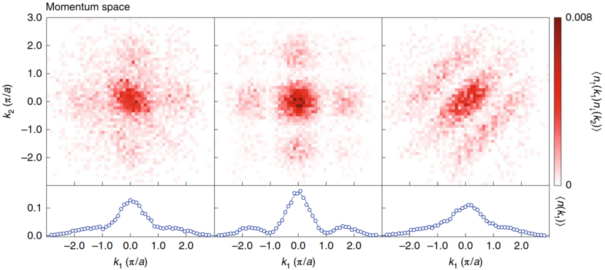
$$E_N = \log_2(2N_{AB} + 1),$$

with N_{AB} is the absolute sum of the negative eigenvalues of $\rho_{AB}^{T_B}$.

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)

Entanglement witnesses: example

We can measure correlations:



Construct the pair correlators

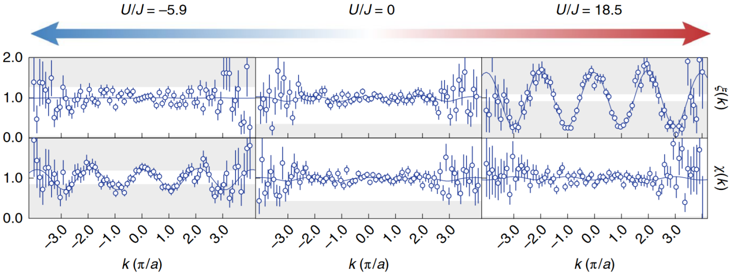
$$\xi(d = k_1 - k_2) = \frac{\int dk \ \langle n_\uparrow(k - d/2) n_\downarrow(k + d/2) \rangle}{\int dk \ \langle n_\uparrow(k - d/2) \rangle \langle n_\downarrow(k + d/2) \rangle}$$

$$\chi(s = k_1 + k_2) = \frac{\int dk \ \langle n_\uparrow(k + s/2) n_\downarrow(-k + s/2) \rangle}{\int dk \ \langle n_\uparrow(k + s/2) \rangle \langle n_\downarrow(-k + s/2) \rangle}$$

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
[2] A. Bergschneider et al., Experimental Characterization of Two-Particle Entanglement through Position and Momentum Correlations, Nature Physics 15, no. 7 (2019)

Entanglement witnesses: example

The pair correlators as entanglement witnesses:



For separable states:

$$\xi_{\min} \leqslant \xi \leqslant \xi_{\max}, \qquad \chi_{\min} \leqslant \chi \leqslant \chi_{\max}.$$

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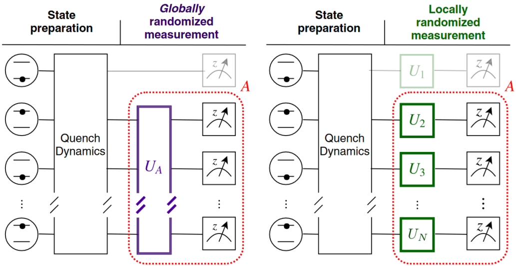
Random Unitaries

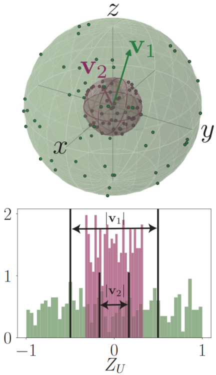
The second Rényi entropy

$$S_2(\rho_A) = -\log_2 \operatorname{tr} \rho_A^2.$$

Bipartite entanglement exists between subsystems A and B of $\mathcal{S}$ with reduced density matrices $\rho_A = \operatorname{tr}_{\mathcal{S}\setminus A}\,\rho$ and $\rho_B = \operatorname{tr}_{\mathcal{S}\setminus B}\,\rho$ **if**

$$S_2(\rho_A) > S_2(\rho_{AB}) \quad \text{or} \quad S_2(\rho_B) > S_2(\rho_{AB}).$$





[5] A. Elben et al., Statistical Correlations between Locally Randomized Measurements, Physical Review A 99, no. 5 (2019)

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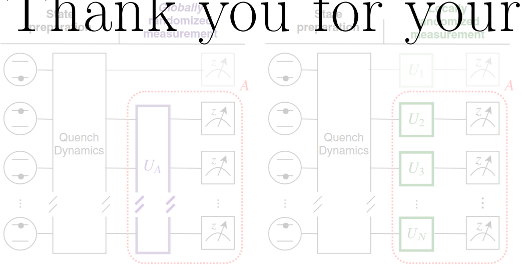
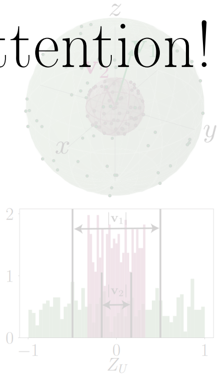
Thank you for your attention!





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