**The problem of thermalization**. Let $\{|n\rangle\}_n$ be the set of normalized eigenstates for $\hat{H}$ with energies $E_n$, then the state evolves $|\psi\rangle \to |\psi(t)\rangle$ as

$$|\Psi(t)\rangle = \sum_n e^{-iE_n t} |n\rangle \,, \qquad c_n = \langle n|\Psi\rangle.$$

Howeever for observable $\hat{O}$

$$\langle \hat{O}(t)\rangle = \langle \Psi(t)|\hat{O}|\Psi(t)\rangle = \sum_{n,n'} e^{it(E_n - E_{n'})}\bar{c}_n c_{n'} \langle n|\hat{O}|n'\rangle.$$

If a stationary value exists, this must be

$$\lim_{t\to\infty} \langle \hat{O}(t)\rangle = \lim_{t\to\infty} \frac{1}{t}\int_0^t d\tau \langle \hat{O}(\tau)\rangle = \sum_n |c_n|^2 \langle n|\hat{O}|n\rangle = \operatorname{tr}\left[\hat{\rho}_{\text{diag}}\hat{O}\right], \qquad \hat{\rho}_{\text{diag}} \propto \sum_n c_n |c_n|^2 |n\rangle\langle n|,$$

with $\operatorname{tr}\hat{\rho}_{\text{diag}} = 1$, which must be compared with the thermal enemble

$$\hat{\rho}_{\text{th}} = \frac{1}{Z_\beta} e^{-\beta \hat{H}} = \frac{1}{Z_\beta}\sum_n e^{-\beta E_n} |n\rangle\langle n|.$$

We see that a thermal ensemble is attained if $|c_n|^2 = \frac{1}{Z} e^{-\beta E_n}$. The projectors on the eigenstates $P_n = |n\rangle\langle n|$ are conserved quantities, so thermalization seems impossible.

**The importance of locality**. Consider Hamiltonian in the form

$$\hat{H} = \sum_j \hat{h}_j,$$

with $\hat{h}_j$ an operator that acts non trivially on a finite range around the lattice $j$.

We also ask all the connected correlators to vanish

$$\lim_{|j-j'|\to\infty} \left( \langle \hat{O}_j \hat{O}_{j'}\rangle - \langle \hat{O}_j\rangle\langle \hat{O}_{j'}\rangle \right),$$

this is the definition of the cluster property.

**Conservation laws and locality**. According to different conserved charges $\{\hat{H}_j\}$, the entropy maximization constrained to the knowledge of the expectation values of $\hat{H}_j$ gives a generalization of the Gibbs ensemble

$$\hat{\rho} = \frac{1}{Z} e^{-\sum_j \beta_j \hat{H}_j}.$$

To proof it we can

$$F[\hat{\rho}] = S[\hat{\rho}] + \lambda\left(\operatorname{tr}\hat{\rho} - 1\right) - \sum_j \beta_j \left(\operatorname{tr}\rho \hat{H}_j - E_j\right),$$

where $\lambda$ is a Lagrange multiplier to impose the normalization $\operatorname{tr}\hat{\rho} = 1$ and $\beta_j$ are the Lagrange multipliers associated with the charges.

$$\delta F[\hat{\rho}] = -\operatorname{tr}\left(\mathbb{1} + \ln\hat{\rho}\right)\delta\hat{\rho} + \lambda \operatorname{tr}\delta\hat{\rho} - \sum_j \beta_j \operatorname{tr}\hat{H}_j \delta\hat{\rho} \overset{(1)}{=} \operatorname{tr}\left[\left(\mathbb{1}(\lambda - 1 + \ln Z) + \textstyle\sum_j \beta_j \hat{H}_j - \sum_j \beta_j \hat{H}_j\right)\delta\hat{\rho}\right] = 0,$$

with $\overset{(1)}{=}$ we substitute $\rho = \frac{1}{Z} e^{-\sum_j \beta_j \hat{H}_j}$.