8.1 Effective action of a condensate in a double well

The following Hamiltonian is a simple model of a condensate in two wells:

$$H = -\frac{g}{2}\sum_{\langle i,j\rangle} a_i^\dagger a_j + \frac{U}{4}\sum_j n_j(n_j-1), \tag{1}$$

with $j \in \{1,2\}$. Consider a system with in total $2N$ particles. After normal ordering $[a_i, a_j^\dagger] = \delta_{ij}$

$$H(a^\dagger, a) = -\frac{g}{2}\sum_{\langle i,j\rangle} a_i^\dagger a_j + \frac{U}{4}\sum_j a_j^\dagger a_j^\dagger a_j a_j.$$

Non-interacting case. Let's start with $U = 0$ and operator canonical transformation (Fourier transform)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

which automatically satisfies the commutation relations $[a_j, a_j^\dagger] = \sin(\alpha)^2 + \cos(\alpha)^2 = 1$. Substituting into the Hamiltonian, we find the condition for diagonalization

$$\cos(\alpha)^2 - \sin(\alpha)^2 = 0, \qquad \overset{\alpha=\pi/4}{\Rightarrow} \qquad a_{1,2} = \tfrac{1}{\sqrt{2}}(b_1 \pm b_2),$$

and the Hamiltonian

$$H = -\frac{g}{2}\sum_{\langle i,j\rangle} a_i^\dagger a_j = \tfrac{g}{2} b_1^\dagger b_1 - \tfrac{g}{2} b_2^\dagger b_2, \tag{2}$$

with ground state $|0, 2N\rangle_b$. Define $|n\rangle_b \overset{\text{def}}{=} |n, 2N-n\rangle_b$. Now let's find the δN as

$$\delta N = a_2^\dagger a_2 - a_1^\dagger a_1 = -b_2^\dagger b_1 - b_1^\dagger b_2,$$
$$(\delta N)^2 = b_1^\dagger b_1 + b_2^\dagger b_2 + 2 b_2^\dagger b_1^\dagger b_1 b_2 = 2N + 4nN - 2n^2.$$

We immediately see that in the ground state

$$\langle \delta N^2 \rangle_{\text{gs}} = 2N. \tag{3}$$

Note that the temperature correction will be

$$\tfrac{1}{N}\langle \delta N^2 \rangle = 2\coth\left(\tfrac{1}{2}\beta g\right) \approx 2 + 4e^{-\beta g}.$$

To calculate this we can start with the partition function

$$Z = \sum_{n=0}^{2N} e^{-\beta E_n} = \frac{e^{\beta g(N+1)} - e^{-\beta g N}}{e^{\beta g} - 1},$$

with $E_n = -g(N-n)$, and find $\langle n \rangle$ and $\langle n^2 \rangle$ through

$$\langle N - n \rangle = \frac{1}{\beta}\frac{1}{Z}\frac{\partial Z}{\partial g} = T\partial_g \ln Z, \qquad \langle (N-n)^2 \rangle = \frac{1}{\beta^2}\frac{1}{Z}\frac{\partial^2 Z}{\partial g^2}.$$

Imaginary-time action. The imaginary-time action associated with this Hamiltonian in the coherent state representation

$$S = \int_0^\beta d\tau\ \bar\psi \partial_\tau \psi + H(\bar\psi, \psi) = \int_0^\beta d\tau\ \bar\psi\partial_\tau\psi - \frac{g}{2}\sum_{\langle i,j\rangle}\bar\psi_i\psi_j + \frac{U}{4}\sum_j \bar\psi_j\bar\psi_j\psi_j\psi_j.$$

Consider the density-phase representation given by

$$\psi_1 = \sqrt{N + \frac{\delta N}{2}}\, e^{i\varphi_1}, \qquad \psi_2 = \sqrt{N - \frac{\delta N}{2}}\, e^{i\varphi_2}.$$

The action than

$$S \overset{\text{def}}{=} \int_0^\beta d\tau\ \mathcal{L}(\varphi, \theta) = \int_0^\beta d\tau\ 2Ni\dot\theta + \frac{\delta N}{2} i\dot\varphi - g\sqrt{N^2 - \left(\frac{\delta N}{2}\right)^2}\cos\varphi + 2\frac{U}{4}\left(\frac{\delta N}{2}\right)^2 + \frac{U}{2}N^2, \tag{4}$$

with $\varphi = \varphi_1 - \varphi_2$ and $\theta = \frac{1}{2}(\varphi_1 + \varphi_2)$. We can find the physical observables that are canonical conjugates to φ and θ

$$P_\varphi = \frac{\partial \mathcal{L}}{i\partial\dot\varphi} = \frac{\delta N}{2}, \qquad P_\theta = \frac{\partial\mathcal{L}}{i\partial\dot\theta} = 2N,$$

with i factor from Wick rotation $\tau \to -it$ (it seems to me).

We can immediately see from Noether's theorem how symmetry in θ leads to conservation of $P_\theta = 2N = \text{const}$. And indeed $\mathcal{L}(\theta) = \mathcal{L}(\theta + \text{shift})$ – $U(1)$ symetry. On the other hand $\mathcal{L}(\varphi) \neq \mathcal{L}(\varphi + \text{shift})$, which corresponds to non-conservation of the $P_\varphi = \delta N$.

Effective action. Expanding the action to quadratic order in the particle number fluctuations $\delta N/N$ and the relative phase φ and neglecting constant terms

$$S_{\text{eff}}(\varphi, P_\varphi) = \int_0^\beta d\tau \; iP_\varphi \partial_\tau \varphi + \tfrac{1}{2} gN\varphi^2 + \tfrac{1}{2}(U + g/N)P_\varphi^2.$$

The fluctuations of the relative particle number between the wells $(\delta N)^2$ could be found as previous through the partition function Z

$$Z = \int D[\varphi, P_\varphi] e^{-S_{\text{eff}}(\varphi, P_\varphi)}, \qquad \langle P_\varphi^2 \rangle = \frac{1}{Z} \int D[\varphi, P_\varphi] \, P_\varphi^2 e^{-S_{\text{eff}}[\varphi, P_\varphi]} = -\frac{2}{\beta Z} \partial_U Z = -\frac{2}{\beta} \frac{\partial \ln Z}{\partial U},$$

so in what follows we only look at factors containing U. Integrating by parts

$$\int_0^\beta d\tau \; P_\varphi i \partial_\tau \varphi = P_\varphi i\varphi \Big|_0^\beta - \int_0^\beta d\tau \; \varphi i \partial_\tau P_\varphi,$$

and $D[\varphi]$ could be calculated as gaussian integral

$$Z \propto \int D[P_\varphi] \exp\left(\int_0^\beta d\tau \; \left(-\frac{(\partial_\tau P_\varphi)^2}{2gN} + \frac{1}{2}(U + g/N)P_\varphi^2 \right) \right),$$

that could be calculated in Matsubara representation $2P_\varphi = \delta N = \frac{1}{\sqrt{\beta}} \sum_k e^{i\omega_k \tau} \delta N_k$

$$Z \propto \int D[\delta N_k] \exp\left(-\frac{1}{8} \sum_k \left(\frac{\omega_k^2}{gN} + U + \frac{g}{N} \right) \delta N_k \delta N_{-k} \right).$$

Since the fluctuation δN is real, then $\delta N_{-k} = \overline{\delta N}_k$, and

$$Z \propto \prod_k \left(\frac{\omega_k^2}{gN} + U + \frac{g}{N} \right)^{-1/2} \qquad \Rightarrow \qquad \langle \delta N^2 \rangle = \frac{4}{\beta} \sum_k \left(\frac{\omega_k^2}{gN} + U + \frac{g}{N} \right)^{-1},$$

with $\omega_k = 2\pi k/\beta$. After summation as

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + x^2} = \frac{\pi}{x} \frac{1}{\coth(\pi x)}, \qquad \Rightarrow \qquad \langle \delta N^2 \rangle = 2N \frac{\coth\left(\frac{1}{2}\beta g F_U\right)}{F_U},$$

with $F_U = \sqrt{1 + NU/g}$, in full accordance with formula (3).

Low fluctuations. The expansion in $\delta N/N$ is justified with $|\delta N|/N \ll 1$ or $\coth\left(\frac{1}{2}\beta g F_U\right)/NF_U \ll 1$. Note that temperature increases fluctuations and decreases interaction. Thus we could rewrite (4) as

$$S_{\text{eff}}(\varphi, P_\varphi) = \int_0^\beta d\tau \; P_\varphi i \partial_\tau \varphi - gN\cos(\varphi) + \tfrac{1}{2} U P_\varphi^2,$$

where we neglected P_φ^2/N term.

Equations of motion. The real-time effective action is

$$S_{\text{eff}}[\varphi, P_\varphi] = i \int_0^T dt \; \mathcal{L} = i \int_0^T dt \; \left(P_\varphi \partial_t \varphi + gN\cos(\varphi) - \tfrac{1}{2} U P_\varphi^2 \right).$$

Classical equations of motion could be obtained from Euler–Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0, \qquad \Rightarrow \qquad \begin{aligned} \dot{\varphi} &= UP_\varphi, \\ \dot{P}_\varphi &= -gN\sin(\varphi) \end{aligned} \qquad \Rightarrow \qquad \partial_t^2 \varphi = -gNU\sin\varphi.$$

The current between the wells is $\partial_t \delta N/2 = \partial_t P_\varphi = -gN\sin\varphi$, limited by gN.

Oscillation frequency. With $\varphi_0 \ll 1$ we could limit $|\varphi|$ and rewrite equations as

$$\ddot{\varphi} = gNU\varphi, \qquad \Rightarrow \qquad \varphi = \varphi_0 \cos(\sqrt{gNU}t),$$

so oscillation frequency is $\sqrt{gNU}$. Fluctuations are also small as $P_\varphi = \dot{\varphi}/U$. Non-interacting bosons oscillation could be found from (2) with

$$|\psi(t)\rangle = \sum_{n=0}^{2N} \alpha_n e^{ig(N-n)t} |n, 2N-n\rangle,$$

we obtain

$$\langle \delta N(t) \rangle = \langle \psi(t)| - b_2^\dagger b_1 - b_1^\dagger b_2 |\psi(t)\rangle = \langle \psi(t)| \sum_{n=1}^{2N-1} \sqrt{n(2N-n-1)} \alpha_n e^{ig(N-n)t} |n, 2N-n\rangle e^{-igt} = \sum_n ... e^{-igt},$$

so oscillation frequency is g.

8.2 Vortex Excitation in a Superfluid