1.1 Quantum State Tomography

(a) Density matrix could be decomposed as

$$\hat{\rho} = \frac{1}{2^N} \sum_{\{\alpha_j\}} C_{\alpha_1 \ldots \alpha_N} \hat{\sigma}_{\alpha_1} \otimes \ldots \otimes \hat{\sigma}_{\alpha_N},$$

with Pauli matrix $\hat{\sigma}_j$ and $\alpha = 0, 1, 2, 3$

$$\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Coefficients $C_{\alpha_1 \ldots \alpha_N}$ could be measured using $\hat{\sigma}_j^2 = \hat{\sigma}_0$

$$C_{\alpha_1 \ldots \alpha_N} = \operatorname{tr} \left(\hat{\rho} \; \hat{\sigma}_{\alpha_1} \otimes \ldots \otimes \hat{\sigma}_{\alpha_N} \right),$$

so we need $4^N - 1$ measurements (remembering $\operatorname{tr} \rho = 1$). Doing finite number of shots we measure not average values by themselves, but their estemations, so we could observe not normalized states. For 2-qubit system a complete set of measurement operators is

$$\{\sigma_{\alpha_1} \otimes \sigma_{\alpha_2}\},$$

except $\alpha_1 = \alpha_2 = 0$.

(b) The Bloch vector can be extracted as

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi} \sin\theta/2 \end{pmatrix}, \quad \Rightarrow \quad \hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2} \begin{pmatrix} 1 + \cos\theta & e^{-i\varphi} \sin\theta \\ e^{i\varphi} \sin\theta & 1 - \cos(\theta) \end{pmatrix},$$

what could be expand as

$$\hat{\rho} = \tfrac{1}{2} \hat{\sigma}_0 + \tfrac{1}{2} \sin\theta \cos\varphi \; \hat{\sigma}_x + \tfrac{1}{2} \sin\theta \sin\varphi \; \hat{\sigma}_y + \tfrac{1}{2} \cos\theta \; \hat{\sigma}_z.$$

1.2 Semi-Classical Light–Matter Interaction

We have basic light-atom interaction

$$\hat{H} = \hat{H}_0 + \hat{V}$$

(a)