1 Operator Identity for Gaussian Theories

To compute the correlation function of the 1D weakly interacting Bose gas you made use of the operator identity

$$\langle \mathrm{e}^{i(\phi(r)-\phi(0))} \rangle = \mathrm{e}^{-\frac{1}{2}\langle(\phi(r)-\phi(0))^2\rangle}, \tag{1}$$

where $\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\phi \; O \; \mathrm{e}^{iS[\phi]}$ is the field-theoretic average. Show that this relation is indeed valid for Gaussian theories, i.e. a theory where the action $S[\phi]$ is a quadratic functional of ϕ.

2 Bogoliubov theory

Consider a microscopic Hamiltonian for bosons with weak contact interactions:

$$H = \sum_p (\epsilon_p - \mu)\, a_p^\dagger a_p + \frac{u}{2V} \sum_{p,p',q} a_{p+q}^\dagger a_{p'-q}^\dagger a_{p'} a_p, \tag{2}$$

where $\epsilon_p = \frac{p^2}{2m}$. For $u = 0$ the groundstate in a grandcanonical description is a coherent state of bosons in the zero-momentum state, i.e. all particles are Bose condensed. Finite interactions lead to scattering of bosons from the condensate into finite momentum modes and hence a depletion of the condensate fraction. However, if the interactions are weak, one can still assume that the $p = 0$ mode is macroscopically occupied, $\langle a_0^\dagger a_0 \rangle \gg 1$. As $[a_0, a_0^\dagger] = 1$ is of order 1, one can neglect it for a macroscopically occupied $p = 0$ mode and replace a_0, $a_0^\dagger$ by their expectation value $\sqrt{N_0}$, the number of bosons in the condensate. One can therefore approximate all other modes to be small $a_p \ll \sqrt{N_0}$ and therefore neglect all terms in the interaction part of above Hamiltonian which contain more than two creation/annihilation operators with $p \neq 0$.

1. Using the approximation from above, show that the mean-field Hamiltonian takes the form

$$H_{\mathrm{MF}} = -\mu N_0 + \frac{N_0^2 u}{2V} + \sum_{p>0} (\epsilon_p - \mu + 2un_0) \left(a_p^\dagger a_p + a_{-p}^\dagger a_{-p} \right) + \sum_{p>0} un_0 \left(a_p^\dagger a_{-p}^\dagger + a_{-p} a_p \right) \tag{3}$$

   where $n_0 = N_0/V$ is the condensed particle density.

2. H_{MF} can be diagonalized with a Bogoliubov transformation to a new set of creation and annihilation operators

$$\begin{aligned} a_p &= u_p \alpha_p - v_p \alpha_{-p}^\dagger \\ a_{-p}^\dagger &= u_p \alpha_{-p}^\dagger - v_p \alpha_p. \end{aligned} \tag{4}$$

The newly introduced creation α_p and annihlation operators $\alpha^\dagger_{-p}$ have to obey bosonic commutation relations. Show that this leads to the condition $|u_p|^2 - |v_p|^2 = 1$ which allows for a convenient parametrization of the form $u_p = \cosh\theta_p$, $v_p = \sinh\theta_p$ if $u, v \in \mathbb{R}$.

3. The goal of the Bogoliubov transformation is to bring the Hamiltonian to quadratic form:

$$H_{\mathrm{MF}} = -\mu N_0 + \frac{N_0^2 u}{2V} - \sum_{p>0} (\epsilon_p - \mu + 2un_0 - E_p) + \sum_{p>0} E_p \left(\alpha_p^\dagger \alpha_p + \alpha_{-p}^\dagger \alpha_{-p} \right) \quad (5)$$

   Derive the conditions on u_p and v_p which lead to this from as well as the energy E_p. Note that the ground state $|0\rangle$ of the transformed Hamiltonian (5) is simply the vacuum state of Bogoliubov quasi-particles α_p, $\alpha_p^\dagger$.

4. Determine the value of the chemical potential μ by minimizing the ground state energy Ω with respect to the number of condensed particles N_0 (with fixed N_0). Sketch the quasi-particle dispersion E_p for the thus determined value of μ. Determine the behaviour for small and large p.

   Hint: You may assume the infinite momentum sum converges (in fact it is, if you treat the potential V more carefully), and show it is $O(u^2)$.

5. Compute the zero temperature isothermal compressibility $\kappa = -\frac{1}{V}\frac{\partial^2 \Omega}{\partial \mu^2}$ from above mean-field Hamiltonian, where Ω is the ground-state energy of the system.

   Hint: Take care of the μ dependence of N_0.

6. Compute the momentum distribution function $n(k) = \langle a_k^\dagger a_k \rangle$ at $T = 0$. Make a rough sketch of this function. Find the condensate fraction N_0/N and the corresponding "quantum depletion" of the condensate $(N - N_0)/N_0$. Transform the momentum sum to an integral and perform it (in three spatial dimensions, you may use Mathematica to do so). Insert $u = \frac{4\pi a}{m}$, where a is the so-called s-wave scattering length. Insert $a = 5\text{nm}$ and $n = 10^{20}\frac{1}{m^3}$, typical values for an ultracold atom experiment with Rubidium-87, to obtain a number for the quantum depletion.