

2.1 Quantum State Tomography

(a) Density matrix could be decomposed as

$$\hat{\rho} = \frac{1}{2^N} \sum_{\{\alpha_j\}} C_{\alpha_1 \dots \alpha_N} \hat{\sigma}_{\alpha_1} \otimes \dots \otimes \hat{\sigma}_{\alpha_N},$$

with Pauli matrix $\hat{\sigma}_j$ and $\alpha = 0, 1, 2, 3$

$$\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Coefficients $C_{\alpha_1 \dots \alpha_N}$ could be measured using $\hat{\sigma}_j^2 = \hat{\sigma}_0$

$$C_{\alpha_1 \dots \alpha_N} = \text{tr}(\hat{\rho} \hat{\sigma}_{\alpha_1} \otimes \dots \otimes \hat{\sigma}_{\alpha_N}),$$

so we need $4^N - 1$ measurements (remembering $\text{tr} \rho = 1$). Doing finite number of shots we measure not average values by themselves, but their estimations, so we could observe not normalized states. For 2-qubit system a complete set of measurement operators is

$$\{\sigma_{\alpha_1} \otimes \sigma_{\alpha_2}\},$$

except $\alpha_1 = \alpha_2 = 0$.

(b) The Bloch vector can be extracted as

$$|\psi\rangle = \begin{pmatrix} \cos \theta/2 \\ e^{i\varphi} \sin \theta/2 \end{pmatrix}, \quad \Rightarrow \quad \hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2} \begin{pmatrix} 1 + \cos \theta & e^{-i\varphi} \sin \theta \\ e^{i\varphi} \sin \theta & 1 - \cos \theta \end{pmatrix},$$

what could be expand as

$$\hat{\rho} = \frac{1}{2} \hat{\sigma}_0 + \frac{1}{2} \sin \theta \cos \varphi \hat{\sigma}_x + \frac{1}{2} \sin \theta \sin \varphi \hat{\sigma}_y + \frac{1}{2} \cos \theta \hat{\sigma}_z.$$

2.2 Semi-Classical Light-Matter Interaction

We have basic light-atom interaction

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = -\frac{\hbar\omega}{2} \hat{\sigma}_z, \quad \hat{V}(t) = -\hbar\Omega \cos(\omega_L t) \hat{\sigma}_x,$$

with Rabi frequency $\Omega = dE_0/\hbar$.

(a) Using that $\exp \text{diag}(a_j) = \text{diag}(e^{a_j})$ we could simplify

$$e^{-i\theta \hat{\sigma}_z} \hat{\sigma}_x e^{i\theta \hat{\sigma}_z} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \cos(2\theta) \hat{\sigma}_x + \sin(2\theta) \hat{\sigma}_y.$$

(b) Using unitary tranform $\hat{U} = \exp(-\frac{i}{2} \omega_L t \hat{\sigma}_z)$ we could substitute $|\psi\rangle = \hat{U}^\dagger |\tilde{\psi}\rangle$ and find new $H_1 = U H U^\dagger - i\hbar U \partial_t U^\dagger$:

$$\hat{H}_1 = -\frac{\hbar}{2} \delta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2} \Omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2} \Omega \begin{pmatrix} 0 & e^{-2i\omega_L t} \\ e^{2i\omega_L t} & 0 \end{pmatrix},$$

with $\delta = \omega - \omega_L$.

(c) We can just throw away the gray term.

(d) Consider $\delta = 0$, than with $|\psi(0)\rangle = |e\rangle$ we have

$$|\tilde{\psi}(t)\rangle = \left(i \sin\left(\frac{\Omega}{2}t\right), \cos\left(\frac{\Omega}{2}t\right) \right)^T, \quad |\psi(t)\rangle = \hat{U}^\dagger |\tilde{\psi}(t)\rangle = \left(i \sin\left(\frac{\Omega}{2}t\right) e^{\frac{i\omega}{2}t}, \cos\left(\frac{\Omega}{2}t\right) e^{-\frac{i\omega}{2}t} \right)^T,$$

so that $\langle \hat{\sigma}_z \rangle$ oscillate with frequency Ω .

(e) $\nu = \frac{2eLE}{\hbar} \approx 24 \text{ MHz}$ and $T_{\pi/2} \sim 10 \text{ ns}$

(f) $\Omega = \sqrt{\frac{6\pi I c^2 \Gamma}{\hbar \omega_0^3}} \sim 0.6 \text{ ms}^{-1} \rightarrow 0.1 \text{ kHz}$ and $T_{\pi/2} \sim 2 \text{ ms}$

(g) We could simply solve Lindblad equation

$$i\hbar \partial_t \hat{\rho} = [\hat{H}, \hat{\rho}] + i\hbar \hat{L}(\hat{\rho}), \quad \hat{L}(\hat{\rho}) = -\Gamma \begin{pmatrix} -\rho_{22} & \frac{1}{2}\rho_{12} \\ \frac{1}{2}\rho_{21} & \rho_{22} \end{pmatrix}. \quad (1)$$

Considering that the detuning is zero we have evolution

$$\partial_t \hat{\rho} = \Gamma \begin{pmatrix} \rho_{22} & -\frac{\rho_{12}}{2} \\ -\frac{\rho_{21}}{2} & -\rho_{22} \end{pmatrix} - \frac{i\Omega}{2} \begin{pmatrix} \rho_{12} - \rho_{21} & \rho_{11} - \rho_{22} \\ \rho_{22} - \rho_{11} & \rho_{21} - \rho_{12} \end{pmatrix}.$$

Consider the behavior of the ρ_{22} with $\alpha = \Omega/\Gamma \ll 1$

$$\rho_{22} = \alpha^2 - \alpha^2 e^{-\Gamma t} \left(2e^{\frac{1}{2}\Gamma t} - 1 \right) + O(\alpha^4).$$

With $\alpha = \Gamma/\Omega \ll 1$ we will have Rabi oscillations, converging to the $\rho_{11} = \rho_{22} = 1/2$ with envelope $\sim e^{-\alpha t}$.

2.3 Driven System Dynamics and Rotating-Wave Approximation

Consider $H_0 = -\frac{\hbar\omega}{2}\sigma_z$ with $V = \frac{\hbar\Omega}{2}\cos(\omega t)\sigma_z - \frac{\hbar\Omega}{2}\sin(\omega t)\sigma_y$ (black dashed line) and $V_{\text{RWA}} = \hbar\Omega\cos(\omega t)\sigma_x$ (blue line) in resonant ($\omega = \omega_0$) case (fig. 1).

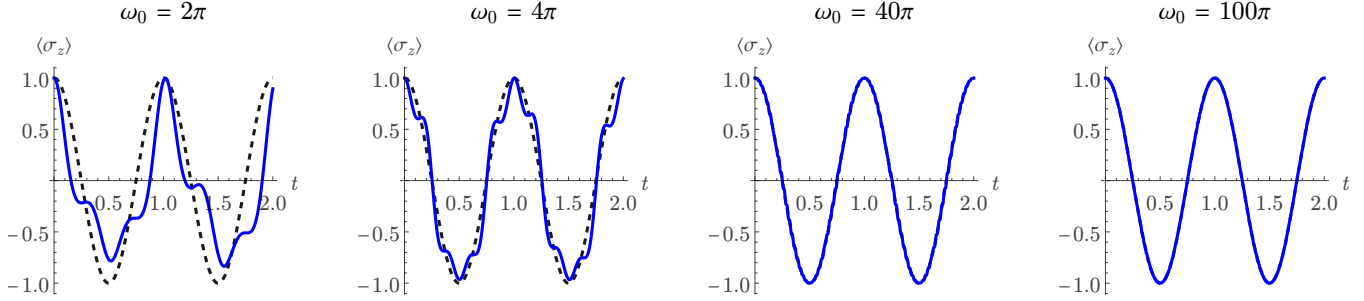


Figure 1: Comparison of the exact solution (black) and the solution in the rotating wave approximation (blue).

We could find solution in general case $\delta \neq 0$ in RWA:

$$\langle\sigma_z\rangle(t) = \cos(\Omega_\delta t) + \frac{\delta^2}{\delta^2 + \Omega^2}(1 - \cos(\Omega_\delta t)), \quad \Omega_\delta = \sqrt{\Omega^2 + \delta^2},$$

with general Rabi frequency Ω_δ . For different $\delta = 0, \Omega, 2\Omega$ and $\Omega = 2\pi$ we could compare different behaviour (fig. 2).

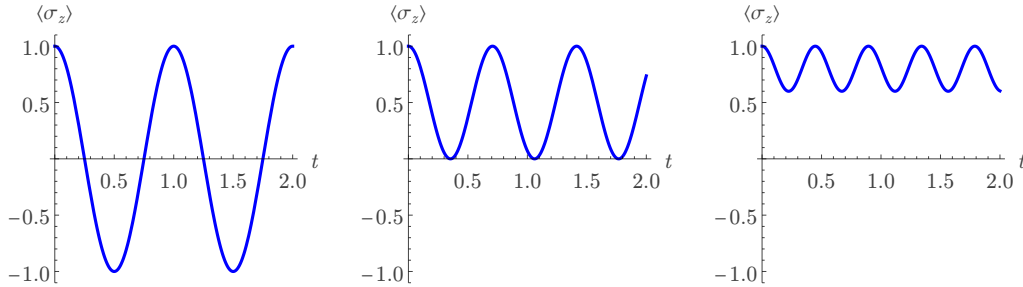


Figure 2: Rabi oscillations at different detuning values

2.4 Dephasing and Decoherence in a TLS

We could calculate $\langle\sigma\rangle$, using (1):

$$\rho(t) = \begin{pmatrix} 1 - |\beta|^2 e^{-\Gamma_1 t} & \alpha\bar{\beta}e^{-\Gamma_2 t} \\ \bar{\alpha}\beta e^{-\Gamma_2 t} & |\beta|^2 e^{-\Gamma_1 t} \end{pmatrix}, \quad \Rightarrow \quad \begin{aligned} \langle\sigma_x\rangle &= (\alpha\bar{\beta} + \bar{\alpha}\beta)e^{-\Gamma_2 t} \\ \langle\sigma_y\rangle &= i(\alpha\bar{\beta} - \bar{\alpha}\beta)e^{-\Gamma_2 t} \\ \langle\sigma_z\rangle &= 1 - 2|\beta|^2 e^{-\Gamma_1 t} \end{aligned}$$

Substituting the initial state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$$\langle\sigma_x\rangle = e^{-\Gamma_2 t}, \quad \langle\sigma_y\rangle = 0, \quad \langle\sigma_z\rangle = 1 - e^{-\Gamma_1 t},$$

thus by measuring the various components we can find the Γ_1 and Γ_2 , it's just exponential decay.