Quantum Many-Body Physics — Sheet 12
Lecturer: Prof. Dr. Michael Knap — Winter Term 2023
Tutors: Stefan Birnkammer, Julian Bösl, Gloria Isbrandt, Caterina Zerba — **Deadline: 2024-01-24 (10:00 am)**

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1 Fermions in one dimension

1. Compute the density response function $\chi_0(q, \omega)$ of a one-dimensional free Fermi gas, a.k.a. the Lindhard function. Find an analytic expression for the imaginary part $\chi_0''(q, \omega)$ at non-vanishing temperature.

2. Suppose that we want to probe the system by applying a weak time dependent potential with a wave-vector q and frequency ω to the one dimensional free Fermi gas. What is the energy absorption rate of the system from the probe? Plot the region in the q, ω - plane where the system can absorb energy at zero temperature. What is the main difference to 2D and 3D? Such a technique can be used to probe cold atom systems. A (slightly more complex) example for such an experiment would be the detection of the 'Higgs'/amplitude mode at the SF/MI transition in a 2D Bose-Hubbard system by M. Endres et al. (then in I. Bloch's group at MPQ), Nature **487**, 454 (2012).

3. Determine the sound velocity of the 1D non-interacting Fermi gas, i.e. the proportionality constant of the linear dispersion at small q.

4. Compute the width of the region in which energy can be absorbed (i.e. the width of the spectrum) $\delta\omega(q)$ at small q. Is there a sharp collective mode in the spectrum in the sense that $\delta\omega(q)/\omega(q) \to 0$ in the limit $q \to 0$? Which operators create these excitations on top of the ground-state? What is their nature? Is there a sharp mode in 2D and 3D?

5. Now we add a contact interaction u between the Fermions. Compute the density response function $\chi(q, \omega)$ of the interacting system at small q at zero temperature, using the result of the random phase approximation (RPA) from the lecture
$$\chi_{\mathrm{RPA}}(q, \omega) = \frac{\chi_0(q, \omega)}{1 - u\chi_0(q, \omega)}. \tag{1}$$
The collective sound mode is given by the pole in $\chi(q, \omega)$. Determine its dispersion at small q and find the sound velocity of the interacting system.

6. Show that the sound-mode exhausts the f-sum rule
$$\int_{-\infty}^{\infty} d\omega\ \omega\, S(q, \omega) = \frac{q^2}{2m}, \tag{2}$$
at zero temperature, i.e. all the excitations of a 1D Fermi system are phonons. This result has important implications, namely that the Hamiltonian of 1D interacting fermions can be mapped onto a Hamiltonian of non-interacting bosons (the phonons of the sound mode). This goes under the name of bosonisation, which is asymptotically exact in the low-energy, long-wavelength limit.