1 Feynman Path Integral of the Harmonic Oscillator

In the following, we will illustrate the Feynman path integral formalism with the example of the harmonic oscillator. Let us denote the quantum mechanical propagator as

$$K(q_f, T; q_i, 0) := \langle q_f | e^{-iHT} | q_i \rangle \tag{1}$$

Using Feynman's idea of writing this as a functional integral over all possible paths, we find (in the continuum limit)

$$K(q_f, T; q_i, 0) = \int \mathcal{D}q(t)\, e^{i\,S[q(t)]} \tag{2}$$

with the action for the harmonic oscillator

$$S[q(t)] = \int_0^T \mathrm{d}t \left(\frac{1}{2} m \dot{q}^2 - \frac{m\omega^2}{2} q^2 \right) \tag{3}$$

and the boundary conditions $q(0) = q_i$ and $q(T) = q_f$.

a) Writing the path as $q(t) = q_c(t) + y(t)$, where $q_c(t)$ denotes the classical trajectory and $y(t)$ some fluctuation around it, show that the exponential with the action factorizes and it holds

$$K(q_f, T; q_i, 0) = e^{iS[q_c(t)]} \int \mathcal{D}y(t)\, e^{iS[y(t)]} \,. \tag{4}$$

> Hint: The classical solution satifies the Euler-Lagrange equation.

b) Show that the action of the classical path is given as

$$S[q_c(t)] = \frac{m\omega}{2\sin(\omega T)} \left((q_i^2 + q_f^2)\cos(\omega T) - 2 q_i\, q_f \right) \,. \tag{5}$$

c) Note that the fluctuations have the boundary conditions $y(0) = y(T) = 0$. Hence, we can express $y(t)$ as a Fourier series

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\, t}{T} \right) \,. \tag{6}$$

Starting from this expansion, show that the fluctuations' contribution reads

$$F(T) = \int \mathcal{D}y(t)\, e^{iS[y(t)]} = \left(\frac{m\omega}{2\pi\, i\hbar \sin(\omega T)} \right)^{\frac{1}{2}} \,. \tag{7}$$

Hint: To facilitate the calculation, you might want to include all prefactors not depending on ω such as the Jacobian for the variable change in (6) into a constant. We can use that $F(T)\big|_{\omega\to 0} = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}}$ in the limit $\omega \to 0$ of a free particle (Bonus exercise: derive this) to obtain the correct prefactor in the end.

Hint: Euler's sine expansion might be useful: $\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right) = \frac{\sin(x)}{x}$.

d) Path integrals have been proven to be a useful tool also in statistical mechanics. To establish the connection, show that the partition function $Z = \sum_j e^{-\beta E_j}$ of a quantum mechanical system can be expressed via a propagator such as in (1) after a Wick rotation to imaginary times $T = -i\beta$.

e) Using the results from parts b), c) and d), obtain the partition function of the quantum mechanical harmonic oscillator.

2 Grassmannian Algebra (I)

In this exercise, we will investigate some of the most important properties of Grassmann numbers, which underlie the field theoretical description of fermions.

1. We will start by retracing some calculus of Grassmann numbers. For this recall that integration of a Grassmann number η is given by

$$\int \mathrm{d}\eta\, \eta = 1, \qquad \int \mathrm{d}\eta = 0. \tag{8}$$

Using this show that for a Grassmannian function $f(\eta) \equiv a + b\eta$ $(a, b \in \mathbb{C})$ the relation

$$\{\eta, \partial_\eta\} f(\eta) = \{\eta, \textstyle\int \mathrm{d}\eta\} f(\eta) = f(\eta) \tag{9}$$

holds, where we defined the shorthanded notation $\{\eta, \int \mathrm{d}\eta\} f(\eta) \equiv \eta \int \mathrm{d}\eta\ f(\eta) + \int \mathrm{d}\eta\ \eta f(\eta)$.

2. As a next step we will consider exponentials of N Grassmannian numbers $\{\eta_j\}_{j=1,\ldots,N}$ like

$$\exp\Big(\sum_{j=1}^{N} c_j \eta_j\Big) \quad \text{with } c_j \in \mathbb{C}. \tag{10}$$

Show that the series expansion of such an exponential truncates after the linear order, i.e.

$$\exp\Big(\sum_{j=1}^{N} c_j \eta_j\Big) = 1 + \sum_{j=1}^{N} c_j \eta_j \tag{11}$$

Hint: To show this first proof the algebra like structure of Grassmann numbers, i.e. show that sums as well as complex multiples of Grassmann numbers are still Grassmann-valued.

3. Expanding the exponential show that the frequently appearing integral

$$\int \mathrm{d}\bar{\eta}\mathrm{d}\eta\ \mathrm{e}^{-C\bar{\eta}\eta}, \quad C \in \mathbb{C} \tag{12}$$

simply equals C.

> Hint: Be aware of the correct order of differentials $\mathrm{d}\eta$ as they are also Grassmann-valued quantities.