Quantum Hardware
Problem Set No. 6

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1 Magic-Wavelength Dipole Trap

Optical dipole traps or optical tweezers are an essential part of experiments in many disciplines like in ultracold atoms. For an atom, we can calculate the optical trapping potential as follows

$$U_{dip}(\mathbf{r}) = -\frac{1}{2\epsilon_0 c}\mathrm{Re}(\alpha)\, I(\mathbf{r}) \tag{1}$$

As discussed in the lecture, the potential depends on the polarizability α of the state.
A simple dipole trap consists of a laser beam with a Gaussian intensity profile along the radial direction that is focused at the place of the atoms. We can express the intensity of such a Gaussian beam as

$$I(r,z) = \frac{2P}{\pi w_0}\left(\frac{w_0}{w(z)}\right)^2 \exp\left(\frac{-2r^2}{w(z)^2}\right), \tag{2}$$

where w_0 is the waist at the focus point, r the radial distance and z the distance along the beam. In the following we assume a laser beam with $\lambda = 813\,\mathrm{nm}$ ($\alpha = 286$ a.u., 1 a.u.$= 4\pi\epsilon_0 a_0^3$), a power of $7\,\mathrm{W}$ and a waist of $50\,\mu\mathrm{m}$ at the focus point.

(a) Calculate the depth of the dipole trap at the center of the beam. Express your result as a temperature. Could atoms at room temperature be trapped?

(b) The potential can be approximated by a harmonic oscillator in radial and axial directions

$$U(r,z) \approx -U_{dip}\left[1 - 2\left(\frac{r}{w_0}\right)^2 - \left(\frac{z}{z_R}\right)^2\right], \tag{3}$$

with the Rayleigh length $z_R = \pi w_0^2/\lambda$. Calculate the radial trapping frequency ω_r for the given trap parameters.

(c) Calculate the axial trapping frequency ω_z for the given trap parameters. Propose an experimental configuration to increase the confinement along this axis.

Exact calculations of the polarizability α can be very complicated since one has to take into account all main energy levels of the atom. In Fig. 1, the dependence of the polarizabilty for two states of ^{87}Sr are shown: the $^1\mathrm{S}_0$ ground state and the $^3\mathrm{P}_0$ metastable state. The transition between these two states at 698 nm is very narrow and used for atomic clocks.

(d) Draw qualitatively the potentials for a Strontium atom in a Gaussian laser beam in the $^1\mathrm{S}_0$ and the $^3\mathrm{P}_0$ state at the intersection points marked in the figure. Explain your findings and elaborated why a trap at the red dimond-shaped points could be useful for building an atomic clock.

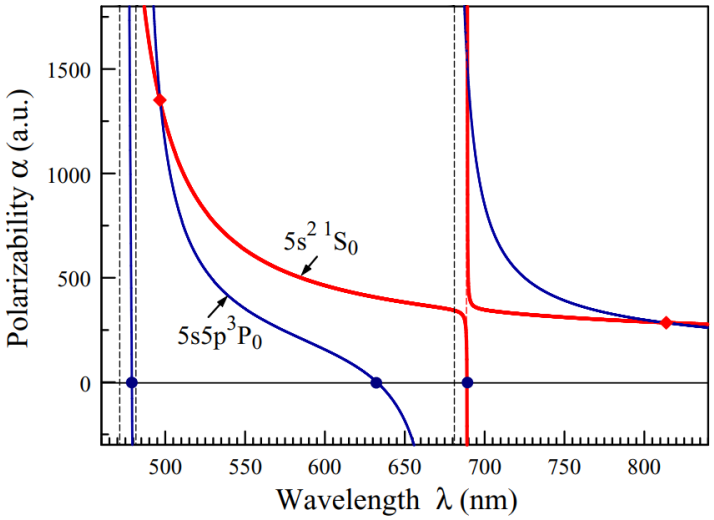

Figure 1: Polarizability of the $^1S_0$ ground state and the $^3P_0$ metastable excited state of ^{87}Sr [Phys. Rev. A **92**, 040501(R) (2015)].

2 Schrieffer-Wolff Transformation to Derive a Dispersive Hamiltonian

In this problem, we want to derive the effective Hamiltonian of a two-level system coupled to a cavity in the large detuning limit using the so-called Schrieffer-Wolff transformation.

Assuming a small perturbation λV (such as the coupling of a two-level system to a resonator) to the Hamiltonian H_0 of the unperturbed system, the idea is to identify energy corrections to H_0. The total Hamiltonian is defined as $H = H_0 + \lambda V$ and the Schrieffer-Wolff transformation is defined as

$$\hat{H}_{\text{eff}} = \mathrm{e}^{-\lambda \hat{S}} \hat{H} \mathrm{e}^{\lambda \hat{S}},$$

with $\hat{S}^\dagger = -\hat{S}$ and $\hat{S}$ chosen, such that the linear term $\lambda \hat{V}$ is eliminated

$$\mathrm{e}^{-\lambda \hat{S}} \hat{H} \mathrm{e}^{\lambda \hat{S}} = \hat{H}_0 + \mathcal{O}(\lambda^2).$$

(a) Write $\mathrm{e}^{-\lambda \hat{S}} \hat{H} \mathrm{e}^{\lambda \hat{S}}$ as a series expansion in terms of commutators using the Campbell-Baker-Hausdorff formula.

(b) Derive a condition for $\hat{S}$ using the requirement that the terms linear in λ have to vanish.

(c) Using this condition simplify the series expansion found in (a) and identify the dominant term $\mathcal{O}(\lambda^2)$ in the effective Hamiltonian.

Note: This method can be applied in situations, where we can identify decoupled low-energy and high-energy subspaces of the original unperturbed system, described by Hamiltonian $\hat{H}_0$ (with known eigenstates and eigenenergies). This means that we can identify two subspaces A (low energy) and B (high energy) that are separated by a gap Δ. Let us now assume that we add a small perturbation $\lambda \hat{V}$, which is weak compared to the energy gap Δ. The goal is to obtain an effective, simpler Hamiltonian that captures the low-energy physics of the problem, i.e., we want to project out the high-energy subspace and work in the low-energy

subspace. Note that this method can also be used to eliminate off-diagonal terms up to any order in λ, but for simplicity we will restrict the discussion here to terms of $\mathcal{O}(\lambda^2)$.

Now let's apply this formalism to the Jaynes-Cummings Hamiltonian in the large detuning limit. The Hamiltonian is defined as ($\hbar = 1$ for simplicity):

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} - \frac{\omega_{10}}{2}\hat{\sigma}_z + g\left(\hat{a}^\dagger \hat{\sigma}^- + \hat{a}\hat{\sigma}^+\right).$$

where ω is the frequency of the linear resonator and ω_{10} is the qubit frequency. The eigenstates of $\hat{H}_0 = \omega \hat{a}^\dagger \hat{a} - \frac{\omega_{10}}{2}\hat{\sigma}_z$ can be divided into two subspaces A (qubit in $|0\rangle$) and B (qubit in $|1\rangle$). The coupling term $\hat{V} = g\left(\hat{a}^\dagger \hat{\sigma}^- + \hat{a}\hat{\sigma}^+\right)$ couples the states $|n+1, 0\rangle$ and $|n, 1\rangle$.

(d) Show that the energy difference between these pairs of states is given by $\omega - \omega_{10} \equiv \Delta$

(e) To identify $\hat{S}$ make the ansatz

$$\hat{S} = \alpha \hat{a}^\dagger \hat{\sigma}^- - \alpha^* \hat{a}\hat{\sigma}^+$$

and compute α.

(f) Calculate the effective Hamiltonian taking into account terms up to second-order in λ as identified in (c)

(g) Show that the effective Hamiltonian coincides with the effective Hamiltonian in the dispersive limit $g/\Delta \ll 1$ discussed in the lecture, which is the basis for quantum non-demolition measurements in circuit QED experiments.

3 Qubit Readout

Recall that the Hamiltonian of a coupled system of a qubit and an oscillator in the dispersive regime is ($\hbar = 1$ for simplicity)

$$H = \omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) + \frac{\omega_{10}}{2}\sigma_z + \chi\sigma_z\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \tag{4}$$

Here, ω is the oscillator frequency, ω_{10} is the qubit frequency and $\chi = \frac{g}{\Delta^2}$ is the effective coupling.

(a) What are the eigenstates and eigenvalues of the Hamiltonian Eq. 4?

(b) Let us assume that the system evolves under this Hamiltonian for $t_0 = \frac{\pi}{2\chi}$. The qubit is initialized in a state $|\psi_q\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Calculate the final state of the qubit-oscillator system at $t = t_0$ when the initial state is

i. $|\psi_i\rangle = |\psi_r\rangle \otimes |\psi_q\rangle$ where $|\psi_r\rangle = |0\rangle$.

ii. $|\psi_i\rangle = |\psi_r\rangle \otimes |\psi_q\rangle$ where $|\psi_r\rangle = |1\rangle$

What measurement would you do on the oscillator to determine if the qubit is in $|0\rangle$ or $|1\rangle$?

(c) Suppose that the initial state of the qubit-oscillator system was $|\psi_i\rangle = |\psi_r\rangle \otimes |\psi_q\rangle$ where $|\psi_q\rangle = \alpha|0\rangle + \beta|1\rangle$. We assume that our resonator is in a coherent state $|\alpha\rangle$. Calculate the time evolution for

i. $|\psi_i\rangle = |\psi_r\rangle \otimes |\psi_q\rangle = |\alpha, 0\rangle$

ii. $|\psi_i\rangle = |\psi_r\rangle \otimes |\psi_q\rangle = |\alpha, 1\rangle$.

Plot both final states for $t_0 = \frac{\pi}{2\chi}$ qualitatively in the I-Q plane in the rotationg frame.

4 Cavities and Atoms

Let us consider a setup, where the cavity is initially prepared in a known state. Then, two-level atoms are prepared in the excited state and are brought to interaction with the radiation mode of a cavity one after the other (by flying through the cavity). Assume that the cavity mode is on resonance with the transition frequency of the atoms and that both the spontaneous decay of the excited state as well as the decay of photons from the cavity mode is negligible. The coupling strength g and the interaction time span t during which each atom couples to the cavity, while it is passing through the photon mode, is exactly controlled and fixed. We will model the process by a unitary time evolution followed by a partial trace.

(a) Let us assume that the cavity is in state $|n\rangle$ and an atom in state $|e\rangle$ enters the cavity at time $t = 0$. That is the combined system is in state $|\psi(0)\rangle = |n, e\rangle$. Show that after time t, the state is

$$|\psi(t)\rangle = \cos(g\sqrt{n+1}t)|n, e\rangle - i\sin(g\sqrt{n+1}t)|n+1, g\rangle$$

At time t, the atom leaves the cavity and therefore the cavity collapses to the partial trace $\rho(t) = \mathrm{Tr}_{\mathrm{atom}}|\psi(t)\rangle\langle\psi(t)|$. Compute this density matrix $\rho(t)$. Note that $\rho(0) = |n\rangle\langle n|$. The interaction of the atom converts $\rho(0)$ to $\rho(t)$.

(b) Let us now assume that the cavity is in a state $\rho = \sum_n P_n|n\rangle\langle n|$ before the interaction with the atom. After the interaction with the atom, the state evolves to $\rho' = \sum_n P'_n|n\rangle\langle n|$. Show, using part (a) that

$$P'_n = \cos^2(g\sqrt{n+1}t)P_n + \sin^2(g\sqrt{n}t)P_{n-1}$$

(c) Let us now assume that atoms pass one after the other through the cavity. The density matrix of the cavity after m atoms have passed through the cavity is then $\rho_m = \sum_n P_n(m)|n\rangle\langle n|$. Show that

$$P_n(m) = \cos^2(g\sqrt{n+1}t)P_n(m-1) + \sin^2(g\sqrt{n}t)P_{n-1}(m-1)$$

Solve for $P_n(m)$ numerically with the initial condition $P_n(0) = \delta_{0n}$, i.e., the cavity is in the vacuum state initially. Plot the distribution $P_n(m)$ as a function of n for $m = 2, 10, 100$ and 1000. Assume $gt = 1/2$.

Note: This is called a Tail recursion and simulating it can be demanding. It is recommended to use lru_catche() (or cache() if one uses Python 3.9 or higher) from functools.

5 Jaynes-Cummings Model: Bell state generation

The interaction between a Rydberg atom and a superconducting cavity is modeled by the Hamiltonian

$$H = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\omega_{10}\hat{\sigma}^+\hat{\sigma}^- + \hbar g(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger),$$

with Rydberg states transition frequency ω_{10}, cavity frequency ω and cavity-atom interaction g. We assume that the atom transition frequency is on resonance with the cavity frequency, i.e. $\delta = \omega - \omega_{10} = 0$. In this case, the time evolution operator for the states in the restricted manifold $\mathcal{S} = \{|1, g\rangle, |0, e\rangle\}$ is given by $\hat{U} = \cos(g\sqrt{n+1}t)\mathbb{1} - i\sin(g\sqrt{n+1}t)\sigma_x$. Consider two atoms (atom A, atom B) passing consecutively through the cavity. Initially atom A is prepared in its excited state $|e_A\rangle$, atom B in its ground state $|g_B\rangle$ and the cavity is initially in its ground state $|0\rangle$. Implement the following experimental sequence:
(i) Atom A passes through the cavity and interacts for a time $t_1 = \frac{\pi}{4g}$.
(ii) Subsequently, atom B passes through the cavity and interacts for a time $t_2 = \frac{\pi}{2g}$.
Show that this sequence creates a Bell state between atom A and atom B.