1 Effective action of a condensate in a double well

The following Hamiltonian is a simple model of a condensate in two wells:

$$H = -\frac{g}{2}(a_1^\dagger a_2 + a_2^\dagger a_1) + \frac{U}{4}\sum_{i=1,2} n_i(n_i - 1) \tag{1}$$

with $n_i \equiv a_i^\dagger a_i$ the number operator on well i. Consider a system with in total $2N$ particles.

(a) Write down the imaginary-time action associated with this Hamiltonian in the coherent state representation.

(b) Transform to the density-phase representation given by

$$\psi_1 = \sqrt{N + \frac{\delta N}{2}} e^{i\phi_1}, \qquad \psi_2 = \sqrt{N - \frac{\delta N}{2}} e^{i\phi_2} \tag{2}$$

for the coherent states defined by $a_i |\psi_i\rangle = \psi_i |\psi_i\rangle$, and rewrite the action in terms of the relative phase $\phi = \phi_1 - \phi_2$ and total phase $\theta = \frac{1}{2}(\phi_1 + \phi_2)$ of the two wells. What are the **physical** observables that are canonical conjugates to ϕ and θ? Is the action invariant under $U(1)$ transformations of the relative phase? What about the total phase? What are the conserved quantities associated to these transformations?

> Hint: you can give your results up to factor of imaginary unit i.

(c) Expand the action to quadratic order in the particle number fluctuations $\delta N/N$ and the relative phase ϕ. Calculate the fluctuations of the relative particle number between the wells $\sqrt{\langle \delta N^2 \rangle}$, $\delta N = N_1 - N_2$. Compare your result to the relative particle number fluctuations of the non-interacting system (i.e. $U = 0$) at zero temperature by diagonalising the quadratic Hamiltonian.

(d) Use the result of the previous step to determine under which conditions (relations between g, U, and N) the expansion in relative particle number fluctuations was justified. Use this to justify the following effective action at low energies (and temperatures), in which contrary to the previous step the relative phase fluctuations are not approximated:

$$S_{\text{eff}} = \int_0^\beta d\tau \left(i\Pi_\phi \frac{\partial \phi}{\partial \tau} - gN\cos\phi + \frac{U}{2}\Pi_\phi^2 \right) \tag{3}$$

(e) Obtain the real-time effective action from Eq. (3) and derive the classical equations of motion for ϕ and Π_ϕ. Identify the expression for the current between the wells (fluctuation of the particle number δN between the two wells). What is the largest possible current?

(f) What is the oscillation frequency of the relative particle number and of the current between the wells if the system is initialized in a state with slightly unequal particle number or with relative phase $\phi_0 \neq 0$? (hint: linearize the equation of motion stating under what conditions you can do so).

(g) Now consider a condensate of non-interacting bosons in the two wells ($U = 0$) with the same initial state as above. Use your results of diagonalizing the non-interacting Hamiltonian of part (c) to calculate the oscillation frequency in this case? (all particles are initially in the single particle state $\alpha_1|1\rangle + \alpha_2|2\rangle$).

2 Vortex Excitation in a Superfluid

The wave function of a Bose condensate is given by $\psi = \psi_0\, e^{i\vartheta}$, with the phase factor $\vartheta(\boldsymbol{r})$ depending on the 2d coordinates $\boldsymbol{r} = r(\cos(\varphi), \sin(\varphi))$. In a simple model for a vortex excitation, the phase is given by $\vartheta(\boldsymbol{r}) = \varphi$.

(a) Compute the velocity field $\boldsymbol{v}(\boldsymbol{r})$ defined by $\frac{\hat{p}}{m}\psi(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{r})\psi(\boldsymbol{r})$ and show that the curl of $\boldsymbol{v}(\boldsymbol{r})$ is given by

$$\boldsymbol{\nabla} \times \boldsymbol{v} = 2\pi \frac{\hbar}{m} \delta(\boldsymbol{r}) \hat{e}_z. \tag{4}$$

(b) Using the relation between $\boldsymbol{\nabla}\vartheta$ and the velocity field $\boldsymbol{v}$, determine the free energy associated to a single vortex excitation, given by

$$F = \int d^2\boldsymbol{r}\, \frac{\rho_s}{2m} \left(\boldsymbol{\nabla}\vartheta\right)^2. \tag{5}$$

Note that the natural cutoffs are given by the system size R and the coherence length ξ in the infrared and ultraviolet, respectively.

(c) The velocity field of a uniform rotation of a classical fluid is given by $\boldsymbol{v} = \omega r \hat{e}_\varphi$. Comparing this to the velocity field determined above, does a single vortex correspond to a uniform rotation? Show that instead, a *uniform density of vortices* in a 2d condensate of radius R mimics a rigid body rotation for a particular ω. What is the corresponding angular velocity ω in terms of the vortex density?

> Hint: consider the circulation or "vorticity" $\oint_C d\boldsymbol{l} \cdot \boldsymbol{v}$ associated to a single vortex and compare with the classical expectation.

(d) A BEC does not behave as a conventional fluid when rotated. Instead, this leads to the formation of an array of quantized vortex lines. Fig.1 shows a vortex lattice from Eric Cornell's lab. This picture is a snapshot of a uniformly rotating Bose condensate, which rotates at about $\omega = 30\, rad/s$. Assuming it is a ^{87}Rb condensate, what is the diameter of the condensate cloud, approximately?

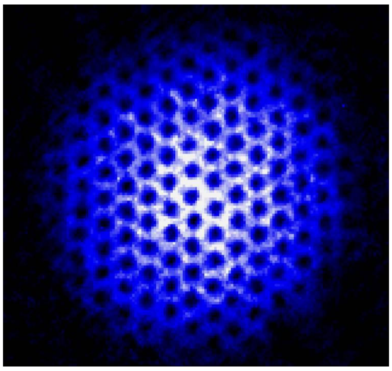

Figure 1: Rotating condensate, see http://jilawww.colorado.edu/bec/CornellGroup/gallery/index.html, from Eric Cornell's group picture gallery.