7.1 Operator Identity for Gaussian Theories

General case. To form some intuition, let's start with the proof

$$\langle e^{\sum_j b_j x_j}\rangle = e^{\frac{1}{2}\sum_{i,j} b_i\langle x_i x_j\rangle b_j}, \tag{1}$$

with averaging defined as

$$\langle f\rangle = \frac{1}{Z}\int D(\boldsymbol{x}) f(\boldsymbol{x}) e^{-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}} G^{-1}\boldsymbol{x}}, \qquad D(x) = \prod_n dx_n,$$

with $Z = \sqrt{\det(2\pi G)}$ so that $\langle 1\rangle = 1$. Both parts of the (1) could be calculated directly:

$$\langle e^{\sum_j b_j x_j}\rangle = \frac{1}{Z}\int D(x) e^{-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}} G^{-1}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{x}} = \frac{1}{Z}\int D(\boldsymbol{x}) e^{-\frac{1}{2}(\boldsymbol{x}-G\boldsymbol{b})^{\mathrm{T}} G^{-1}(\boldsymbol{x}-G\boldsymbol{b})} e^{\frac{1}{2}\boldsymbol{b}^{\mathrm{T}} G\boldsymbol{b}},$$

and with $\boldsymbol{x}' = \boldsymbol{x} - G\bar{b}$ and $D(\boldsymbol{x}') = D(\boldsymbol{x})$

$$\langle e^{\boldsymbol{b}\boldsymbol{x}}\rangle = \frac{1}{Z}\int D(\boldsymbol{x}') e^{-\frac{1}{2}\boldsymbol{x}'^{\mathrm{T}} G^{-1}\boldsymbol{x}'} e^{\frac{1}{2}\boldsymbol{b}^{\mathrm{T}} G\boldsymbol{b}} = e^{\frac{1}{2}\boldsymbol{b}^{\mathrm{T}} G\boldsymbol{b}} = e^{\frac{1}{2}\sum_{i,j} b_i\langle x_i x_j\rangle b_j},$$

with proved in the previous homework fact that $\langle x_i x_j\rangle = G_{ij}$.

Special case. We want to prove the operator identity

$$\langle e^{i(\varphi(r)-\varphi(0))}\rangle = e^{-\frac{1}{2}\langle(\varphi(r)-\varphi(0))^2\rangle}. \tag{2}$$

With $b(r') = i\delta(r'-r) - i\delta(r')$:

$$\sum_j b_j\varphi_j = \int b(r')\varphi(r')\,dr' = i(\varphi(r)-\varphi(0)),$$

and for other part $\sum_{i,j} b_i\langle x_i x_j\rangle b_j = \langle\sum_{i,j} b_i x_i x_j b_j\rangle$, so

$$\sum_{i,j} b_i x_i x_j b_j = \int b(r')\varphi(r')\varphi(r'')b(r'')\,dr'\,dr'' = \left(\int b(r')\varphi(r')\,dr'\right)^2 = -(\varphi(r)-\varphi(0))^2,$$

thus we proved (2) using (1).

Wick's theorem. Note that from (1) are convenient to obtain Wick's theorem maybe. Expanding (1) in the Taylor series we have from the LHS

$$\langle e^{\sum_j b_j x_j}\rangle = 1 + \frac{1}{2!}\sum_{i,j} b_i b_j\langle x_i x_j\rangle + \frac{1}{4!}\sum_{i,j,k,l} b_i b_j b_k b_l\langle x_i x_j x_k x_l\rangle + \dots$$

and from the RHS

$$e^{\frac{1}{2}\sum_{i,j} b_i\langle x_i x_j\rangle b_j} 1 + \frac{1}{2}\sum_{i,j} b_i b_j\langle x_i x_j\rangle + \sum_{i,j,k,l} b_i b_j b_k b_l\langle x_i x_j\rangle\langle x_k x_l\rangle + \dots,$$

so collecting terms with proper B^4 we get

$$\langle x_i x_j x_k x_l\rangle = \langle x_i x_j\rangle\langle x_i x_k\rangle\langle x_j x_l\rangle + \langle x_i x_l\rangle\langle x_j x_k\rangle.$$

This result is known as Wick's theorem.