## 1 Stoner instability

Consider a 3D Fermi gas with point-like interactions:

$$H = \sum_{\mathbf{k},\sigma} \left( \frac{\hbar^2 k^2}{2m} - \mu \right) n_{\mathbf{k}\sigma} + u \int d^3x \; \psi_\uparrow^\dagger(\mathbf{x}) \psi_\downarrow^\dagger(\mathbf{x}) \psi_\downarrow(\mathbf{x}) \psi_\uparrow(\mathbf{x}) \tag{1}$$

The chemical potential is used to fix the density $\rho$.

1. Calculate the density of states at the Fermi Energy for the non-interacting system.

2. Apply the Hartree-Fock approximation at $T = 0$ as a variational approach. Use the one parameter family of Slater-determinant states $|m\rangle$, which have a fixed magnetisation $m$ and density $n$, as trial states to find the magnetisation $m$ which minimises the energy $E(m) = \langle m|H|m\rangle$ as a function of the interaction strength for small magnetization fraction $\frac{m}{n}$.

3. What is the critical value of the dimensionless interaction strength for the gas developing a spontaneous magnetisation $m \neq 0$ within the HF approximation? Find the connection between the spontaneous magnetisation in the symmetry broken phase and the density of states at the Fermi energy.

4. What is the critical exponent $\beta$ given by

$$m \sim \left( \frac{u - u_{crit}}{u_{crit}} \right)^\beta \tag{2}$$

   in this approximation? Sketch the magnetization as a function of $u$ around the critical point $u_{crit}$. Is the phase transition of first or second order?

5. Compare this phase transition to the one in the classical Ising model as discussed in the very first lecture. Explain what quantity plays the role of $u$ there and what that tells you about the nature of the phase transition in both cases. Compare the $\beta$ found in the HF-solution of the 3D Fermi gas to the one found in the mean field solution of the Ising model (Spoiler: You should get the same. This is however a result of the mean field approximation, from which you almost always get this critical exponent.)

## 2 Bogoliubov rotation and gap equation at zero temperature

Let us consider the BCS Hamiltonian

$$H_{\mathrm{BCS}} = \sum_{\vec{k},\sigma} \xi_{\vec{k}} c^\dagger_{\vec{k}\sigma} c_{\vec{k}\sigma} + \frac{1}{\Omega} \sum_{\vec{k},\vec{k}'} V_{\vec{k}\vec{k}'} c^\dagger_{\vec{k}'\uparrow} c^\dagger_{-\vec{k}'\downarrow} c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow}, \tag{3}$$

where $\xi_{\vec{k}} = \varepsilon_{\vec{k}} - \mu$ is the single particle energy with respect to the chemical potential $\mu$, $\Omega$ the volume and $V_{\vec{k}\vec{k}'} = V_{\vec{k}'\vec{k}}$ is an effective interaction that we assume to be attractive. We introduce the creation operators for Bogoliubov quasiparticles, denoted $\gamma^\dagger_{\vec{k}\sigma}$, via the *Bogoliubov rotation*

$$\begin{pmatrix} \gamma_{\vec{k}\uparrow} \\ \gamma^\dagger_{-\vec{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \sin\theta_{\vec{k}} & -\cos\theta_{\vec{k}} \\ \cos\theta_{\vec{k}} & \sin\theta_{\vec{k}} \end{pmatrix} \cdot \begin{pmatrix} c_{\vec{k}\uparrow} \\ c^\dagger_{-\vec{k}\downarrow} \end{pmatrix} \quad \text{where } \tan\theta_{\vec{k}} = \frac{\Delta_{\vec{k}}}{E_{\vec{k}} - \xi_{\vec{k}}}, \tag{4}$$

where $\Delta_{\vec{k}} = -\frac{1}{\Omega} \sum_{\vec{k}'} V_{\vec{k}\vec{k}'} \langle c_{-\vec{k}'\downarrow} c_{\vec{k}'\uparrow} \rangle$ is the gap function and $E_{\vec{k}} = \sqrt{\Delta^2_{\vec{k}} + \xi^2_{\vec{k}}}$.

1. Show that the Bogoliubov quasiparticles satisfy the usual fermionic anti-commutation relations $\{\gamma_{\vec{k}\sigma}, \gamma_{\vec{k}'\sigma'}\} = 0$ and $\{\gamma_{\vec{k}\sigma}, \gamma^\dagger_{\vec{k}'\sigma'}\} = \delta_{\sigma\sigma'} \delta_{\vec{k}\vec{k}'}$.

2. Show that in the mean-field approximation, the BCS Hamiltonian takes the form

$$H_{\mathrm{BCS}} \approx \sum_{\vec{k},\sigma} \xi_{\vec{k}} c^\dagger_{\vec{k}\sigma} c_{\vec{k}\sigma} - \sum_{\vec{k}} \left( \Delta_{\vec{k}} c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow} + \Delta_{\vec{k}}^\star c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow} \right) + \text{const.} \tag{5}$$

> **Hint:** With the operators $A_{\vec{k}} := c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow}$ and $B_{\vec{k}} := c_{-\vec{k}\downarrow} c_{\vec{k}\uparrow}$ the mean-field approximation amounts to approximating $(A_{\vec{k}'} - \langle A_{\vec{k}'} \rangle)(B_{\vec{k}} - \langle B_{\vec{k}} \rangle) \approx 0$, i.e. neglecting fluctuations around the expectation values in quadratic order.

3. Prove that in the mean-field approximation the BCS Hamiltonian (3) describes non-interacting Bogoliubov quasiparticles. Plot their dispersion relation in the case when $\Delta_{\vec{k}} = \Delta$ is independent of $\vec{k}$.

4. Show that the BCS state, $|\mathrm{BCS}\rangle = \prod_{\vec{k}} [\sin\theta_{\vec{k}} + \cos\theta_{\vec{k}}\, c^\dagger_{\vec{k}\uparrow} c^\dagger_{-\vec{k}\downarrow}] \, |0\rangle$ has zero Bogoliubov-quasiparticles, i.e., $\gamma_{\vec{k}\sigma} |\mathrm{BCS}\rangle = 0$.

5. Evaluate the average energy $\langle \mathrm{BCS} | H_{\mathrm{BCS}} | \mathrm{BCS} \rangle$ for the full interacting Hamiltonian (3) in the BCS state[1]. Show that minimizing the energy with respect to $\theta_{\vec{k}}$ gives rise to the zero temperature gap equation

$$\Delta_{\vec{k}} = -\frac{1}{\Omega} \sum_{\vec{k}'} V_{\vec{k}\vec{k}'} \frac{\Delta_{\vec{k}'}}{2E_{\vec{k}'}}. \tag{6}$$

[1] Note that $|\mathrm{BCS}\rangle$ is *not* the ground state of $H_{\mathrm{BCS}}$, but only of its mean field approximated version.