Problem Set 2: Quantum States

Handout: Fri, October 20, 2023; Due: 23:59 CET, Fri, October 27, 2023; **(20 points)**

Notes: Problems 6b) and 7 are optional challenge problem.

Problem 1 (1+1+1=3 points)

Let A be a linear normal operator on a finite complex Hilbert space $\mathcal{H}$ with eigenvalues a_j. Prove the following implications:

(a) A hermitian $\Rightarrow \forall j : a_j \in \mathbb{R}$

(b) A unitary $\Rightarrow \forall j : a_j = \mathrm{e}^{i\varphi_j}$ for some $\varphi_j \in [0, 2\pi)$

(c) A orthogonal projector $\Rightarrow \forall j : a_j \in \{0, 1\}$

Problem 2 (3 points)

Let A be a linear operator on the Hilbert space $\mathcal{H} \cong \mathbb{C}^d$ equipped with some inner product $(\cdot, \cdot)$. Recall the properties of an inner product. Then, prove that $(\cdot, \cdot)_A$ defined as $(v, w)_A := (v, Aw)$ for all $v, w \in \mathcal{H}$ defines another inner product if and only if $A > 0$. *Hint: You can use that $A \geq 0$ implies $A = A^\dagger$.*

Problem 3 (1+1+2=4 points)

In this problem, you will explicitly calculate some useful properties of the Pauli matrices, which are a set of 2×2 matrices that are ubiquitous in quantum information theory. The Pauli-matrices, $\mathbb{1}, X, Y$ and Z (also often denoted by $\sigma_0, \sigma_x, \sigma_y$ and σ_z) are given by

$$\mathbb{1} = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(a) Verify the commutation relations $[X, Y] = 2iZ, [Y, Z] = 2iX, [Z, X] = 2iY$.

(b) Verify the anticommutation relations $\{P, Q\} = 2\delta_{P,Q}$ for $P, Q \in \{X, Y, Z\}$.

(c) Calculate the eigenvalues of the Pauli matrices and show that they all lie within the unit circle.

Problem 4 (2 points)

Consider a 2×2 matrix A:

$$A = \begin{pmatrix} (1+\alpha)/2 & \beta \\ \beta^* & (1-\alpha)/2 \end{pmatrix}, \tag{2}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$. Verify that A is Hermitian with $\mathrm{Tr}(A) = 1$ and that any Hermitian 2×2 matrix with unit trace can be expressed in this form. Show that A can be uniquely expressed as a linear combination of the Pauli matrices.

Problem 5 (2+1+2=5 points)

Consider a 2×2 Hermitian matrix ρ with unit trace given by

$$\rho = \frac{1}{2}\big(\mathbb{1} + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z\big).$$

where $n_x, n_y, n_z \in \mathbb{R}$. This expression is commonly abbreviated as $\rho = (\mathbb{1} + \vec{n} \cdot \vec{\sigma})/2$, where $\vec{n} := (n_x, n_y, n_z)$ is a 3D vector, and $\vec{\sigma} := (\sigma_x, \sigma_y, \sigma_z)$ a vector of matrices. In problem 2, you showed that any Hermitian matrix with unit trace can be written in this form.

(a) Calculate the eigenvalues of ρ — express your answer in terms of n_x, n_y and n_z.

(b) Show that $\rho \geq 0$ if and only if $|\vec{n}| := \sqrt{n_x^2 + n_y^2 + n_z^2} \leq 1$.

(c) Show that if $|\vec{n}| = 1$, then for some $v_0, v_1 \in \mathbb{C}$

$$\rho = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \begin{pmatrix} v_0^* & v_1^* \end{pmatrix}.$$

Explicitly write down an expression for v_0 and v_1 — is your answer unique? (*Hint*: Write $\vec{n}$ in spherical coordinates.)

Problem 6 (3 points)

Let $\mathcal{H} \cong \mathbb{C}^d$ be a Hilbert space and $\mathcal{D}(\mathcal{H})$ shall denote the set of density operators on $\mathcal{H}$. The set $\mathcal{D}(\mathcal{H})$ denotes the set of all hermitian operators $\rho = \rho^\dagger$ on $\mathcal{H}$ which are positive semi-definite and trace normalized to one, i.e. $\mathrm{Tr}[\rho] = 1$.

(a) Prove that $\mathcal{D}(\mathcal{H})$ is convex and bounded (e.g. with respect to $\|\cdot\|_2$) as a subset of $\mathcal{M}_{d\times d}(\mathbb{C})$. Recall that we defined positive-semidefinitness as $\rho \geq 0 :\Leftrightarrow \langle\chi|\,\rho\,|\chi\rangle \geq 0\,, \forall\,|\chi\rangle \in \mathcal{H}$.

(b) *(optional)* Prove that $\mathcal{D}(\mathcal{H})$ is closed, i.e., it contains all its limit points.

Problem 7 (optional)

For this problem, $\mathsf{U}(2) = \{U \in \mathbb{C}^{2\times 2} | U^\dagger U = UU^\dagger = I\}$, $\mathsf{SU}(2) = \{U \in \mathsf{U}(2) \mid \det(U) = 1\}$, $\mathsf{O}(3) = \{R \in \mathbb{R}^{3\times 3} | R^\mathsf{T} R = RR^\mathsf{T} = I\}$ and $\mathsf{SO}(3) = \{R \in O(3) \mid \det(R) = 1\}$.

(a) Show that $\forall\, U \in \mathsf{SU}(2)$, $\exists R \in SO(3)$ such that $\forall \vec{n} \in \mathbb{R}^3, U^\dagger(\vec{n} \cdot \vec{\sigma})U = (R\vec{n}) \cdot \vec{\sigma}$.

(b) Show that $\forall R \in \mathsf{SO}(3)$, there are two unitaries $U \in \mathsf{SU}(2)$ such that $\forall \vec{n} \in \mathbb{R}^3, (R\vec{n}) \cdot \vec{\sigma} = U^\dagger(\vec{n} \cdot \vec{\sigma})U$.

(c) Suppose $R = \mathrm{diag}(-1, -1, -1) \in O(3)$, then show that there is no $U \in \mathsf{U}(2)$ such that $\forall \vec{n} \in \mathbb{R}^3, (R\vec{n}) \cdot \vec{\sigma} = U^\dagger(\vec{n} \cdot \vec{\sigma})U$.