Question & Answer Session

Next week you will have the opportunity to ask all questions about current and previous topics of the course in a **Q&A session** replacing the lecture on **Wednesday (17.01)**.

1 Fermi Liquid Theory and Green's Functions

As we have seen on the last exercise sheet, even in the presence of interactions between fermions, the picture of particle-hole excitations around the Fermi surfance can still be valid. In this exercise, we want to understand the foundation of the celebrated Fermi Liquid Theory from the perspective of Green's functions, which gives a microscopic justification of it.

1. First of all, let us establish a connection between the retarded Green's function $G^{\mathrm{ret}}(\mathbf{k},\omega)$ and the density of states of the system $\rho(\omega)$. Show that for **free particles**,

$$\rho(\omega) = -\frac{1}{\pi}\mathrm{Im}\sum_{\mathbf{k}}\left[G^{\mathrm{ret}}(\mathbf{k},\omega)\right] \,. \tag{1}$$

   How can we express $\rho(\omega)$ in terms of the spectral function $A(\mathbf{k},\omega)$?

2. We want to get now a more general understanding of the spectral function. Starting from the Lehmann representation of the retarded Green's function

$$G^{\mathrm{ret}}(\mathbf{k},\omega) = \frac{1}{Z}\sum_{n,n'}\frac{\langle n|c_{\mathbf{k}}|n'\rangle\,\langle n'|c_{\mathbf{k}}^{\dagger}|n\rangle}{\omega + E_n - E_{n'} + i0^+}\left(e^{-\beta E_n} - \eta_{B/F}e^{-\beta E_{n'}}\right) \,, \tag{2}$$

   show that $A(\mathbf{k},\omega)$ can be interpreted as a probability distribution in ω. Give a physical interpretation of this result. What is the difference between the interacting and non-interacting case? Show that

$$\int \mathrm{d}\omega\, n_{F/B}(\omega)A(\mathbf{k},\omega) = \langle \hat{n}_{\mathbf{k}}\rangle \,. \tag{3}$$

3. In the lecture, you have derived that the effects of interactions will lead to the occurence of a finite self-energy $\Sigma^{\mathrm{ret}}(\mathbf{k},\omega)$ in the retarded Green's function. What are the consequences for the corresponding spectral function $A(\mathbf{k},\omega)$? Elaborate on the connection to the Fermi Liquid Theory.

4. Show that a finite imaginary part of $\Sigma^{\mathrm{ret}}(\mathbf{k},\omega)$ leads to a finite lifetime of the corresponding quasiparticle. To demonstrate this, calculate

$$G^{\mathrm{ret}}(\mathbf{k},t) = \int \frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}G^{\mathrm{ret}}(\mathbf{k},\omega) \,. \tag{4}$$

Assume that $\Sigma^{\text{ret}}(\mathbf{k},\omega) = \Sigma^{\text{ret}}(\mathbf{k})$. Up to which order does one need to go in perturbation theory for electrons with Coulomb interaction in order to get a finite imaginary part of the self-energy?

> Hint: $\Sigma^{\text{ret}}(\mathbf{k}) = \Sigma'_{\mathbf{k}} + i\Sigma''_{\mathbf{k}}$ with imaginary part $\Sigma'' < 0$.