Measures of entanglement

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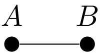
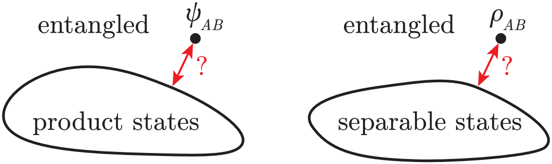
States

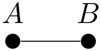


- Pure states $|\psi_{AB}\rangle$:
  - *Product state* if $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
  - A is entangled with B otherwise

- Mixed states $\hat{\rho}_{AB}$
  - *Separable state* if $\hat{\rho}_{AB} = \sum_k p_k \hat{\rho}_A^k \otimes \hat{\rho}_B^k$
  - A is entangled with B otherwise



Bipartite entanglement in pure states: SVD



Is this state entangled?

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$$

Of course not:

$$|\psi_{AB}\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

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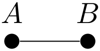

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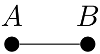

What about this?

$$|\psi_{AB}\rangle = \frac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right)$$

After some SVD (or **Schmidt decomposition**):

$$|\psi_{AB}\rangle = \sum_{a\in A}\sum_{b\in B}\psi_{ab}\,|a\rangle\otimes|b\rangle = \sum_{a}\sum_{b}\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}_{ab}|ab\rangle$$

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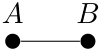

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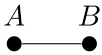

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$$U \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & . \\ 1 & . \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & . \\ -1 & . \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

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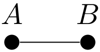

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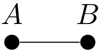
$$|\psi_{AB}\rangle = \tfrac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\otimes\tfrac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

We can see, that it is separable.

Bipartite entanglement in pure states: SVD



Schmidt decomposition

$$|\psi_{AB}\rangle = \sum_{a\in A}\sum_{b\in B}\psi_{ab}\,|a\rangle\otimes|b\rangle = \sum_{j}\sqrt{\lambda_j}\,|\psi_{A,j}\rangle\,|\psi_{B,j}\rangle$$

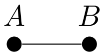
feeply connected with the reduced density matrix $\rho_{A,B}$

$$\rho_{B,A} = \mathrm{tr}_{A,B}\left(|\psi_{AB}\rangle\right) = \sum_i \lambda_i|\psi_{B,A}\rangle\langle\psi_{B,A}|.$$

If there is no entanglement, than

$$\rho_{A,B} = |\psi_{A,B}\rangle\langle\psi_{A,B}|$$

Entanglement in mixed states: EoF



The Entanglement of Formation defined throught *convex roof*:

$$E_F(\rho_{AB}) \stackrel{\text{def}}{=} \min_{\{p_j,\psi_j\}} \sum_j p_j S(\rho_{A,j}),$$

with $\rho_{AB} = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ and $\rho_{A,j} = \mathrm{tr}_B\, |\psi_j\rangle\langle\psi_j|$.