## 13.1 Stoner instability

Consider a 3D Fermi gas with point-like interactions:

$$H = T + V = \sum_{k\sigma}\left(\frac{k^2}{2m_\mathrm{e}} - \mu\right) n_{k\sigma} + u\int \psi^\dagger_\uparrow(x)\psi^\dagger_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x)\, d^3x,$$

or, completely in the momentum representation:

$$H = \sum_{k\sigma}\left(\frac{k^2}{2m_\mathrm{e}} - \mu\right) n_{k\sigma} + \frac{u}{V}\sum_{k_1,k_2,q} c^\dagger_{k_1+q,\uparrow}c^\dagger_{k_2-q,\downarrow}c_{k_2,\downarrow}c_{k_1,\uparrow},$$

after substitution $\psi_\sigma(x) = V^{-1/2}\sum_k e^{-ikx}c_{k,\sigma}$.

**The density of states**. The density of states at the Fermi Energy for the non-interacting system

$$2\int\frac{d^3k}{(2\pi)^3} = \int d\varepsilon\ D(\varepsilon), \qquad \Leftrightarrow \qquad D(\varepsilon) = 2\int\frac{d^3k}{(2\pi)^3}\delta(\varepsilon-\varepsilon_k) = \frac{\sqrt{2}}{\pi^2}m_\mathrm{e}^{3/2}\sqrt{\varepsilon},$$

with $\varepsilon_k = \frac{1}{2m_\mathrm{e}}k^2$. Considering that $n = \frac{N}{V} = \frac{2}{6\pi^2}k_\mathrm{F}^3$, we have

$$D_\mathrm{F} \overset{\mathrm{def}}{=} D(\varepsilon_\mathrm{F}) = \frac{3^{1/3}}{2\pi^{4/3}}m_\mathrm{e}\left(\frac{N}{V}\right)^{1/3}.$$

For an interacting gas, as a first approximation, we can simply replace the mass $m \to m_\mathrm{eff}$.

**The Hartree-Fock approximation**. Consider one parameter family of states $|m\rangle$:

$$m = \tfrac{1}{V}\left(N_\uparrow - N_\downarrow\right)$$
$$n = \tfrac{1}{V}\left(N_\uparrow + N_\downarrow\right)$$

which have a fixed magnetisation $m$ and density $n$, as trial states to find the magnetisation $m$ which minimises the energy $E(m) = \langle m|H|m\rangle$. For kinetic energy term[1]

$$\langle m|T|m\rangle = \sum_{k,\sigma}^{k_\mathrm{F}}\left(\frac{k^2}{2m_\mathrm{e}} - \mu\right)n_{k\sigma} = \sum_\sigma V\int\frac{d^3k}{(2\pi)^3}\left(\frac{k^2}{2m_\mathrm{e}} - \mu\right)\theta(\varepsilon_\mathrm{F}(\sigma) - \varepsilon_k) = \frac{1}{m_\mathrm{e}}\left(\frac{\pi^4}{V^2}\right)^{1/3}\left(N_\uparrow^{5/3} + N_\downarrow^{5/3}\right) + \tfrac{1}{2}\mu N,$$

with $N_{\uparrow,\downarrow} = \frac{N}{2}\left(1 \pm \frac{m}{n}\right)$. And, by Wick's theorem, for interaction energy

$$\langle m|V|m\rangle = \frac{u}{V}\sum_{k_1,k_2,q}\langle c^\dagger_{k_1+q,\uparrow}c_{k_1,\uparrow}\rangle\langle c^\dagger_{k_2-q,\downarrow}c_{k_2,\downarrow}\rangle - \frac{u}{V}\sum_{k_1,k_2,q}\langle c^\dagger_{k_1+q,\uparrow}c_{k_2,\downarrow}\rangle\langle c^\dagger_{k_2-q,\downarrow}c_{k_1,\uparrow}\rangle = \frac{u}{V}\sum_{k_1,k_2} n_{k_1\uparrow}n_{k_2\downarrow} = \frac{u}{V}N_\uparrow N_\downarrow.$$

Thus the expression for energy is

$$E(m) = \frac{1}{m_\mathrm{e}}\left(\frac{\pi^4}{V^2}\right)^{1/3}\left(\frac{N}{2}\right)^{5/3}\left(\left(1+\tfrac{m}{n}\right)^{5/3} + \left(1-\tfrac{m}{n}\right)^{5/3}\right) + \frac{uN^2}{4V}\left(1 - \frac{m^2}{n^2}\right)$$
$$D_\mathrm{F}E(m)/V = \left(\frac{9}{20} + \frac{1}{4}uD_\mathrm{F}\right)n^2 + \left(\frac{1-uD_\mathrm{F}}{4}\right)m^2 + \frac{1}{108}\frac{m^4}{n^2} + o(m^4).$$

It looks like a second-order phase transition in Landau's theory, only with interaction $u$ instead of $T$ – quantum phase transition. Magnetisation

$$m(u) = \pm\sqrt{\frac{27}{2}}\theta(uD_\mathrm{F} - 1)n\sqrt{uD_\mathrm{F} - 1}.$$

**Stoner criterion**. The critical value of the dimensionless interaction strength for the gas developing a spontaneous magnetisation $m \neq 0$ within the HF approximation

$$uD_\mathrm{F} > 1, \qquad \Rightarrow \qquad u_\mathrm{crit} = \frac{1}{D_\mathrm{F}}.$$

The critical exponent

$$m \sim \theta(u - u_\mathrm{crit})\left(\frac{u - u_\mathrm{crit}}{u_\mathrm{crit}}\right)^\beta,$$

corresponds to the $\beta = 1/2$, as in the classical Ising model – second order phase transition.

[1] Here I don't write $5^{-1}3^{5/3}2^{-1/3} \approx 0.99 \approx 1$, but take into account in calculations.

## 13.2 Bogoliubov rotation and gap equation at zero temperature

Let us consider the BCS Hamiltonian

$$H = \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} + \frac{1}{\Omega} \sum_{k,k'} V_{kk'} c^\dagger_{k'\uparrow} c^\dagger_{-k'\downarrow} c_{-k\downarrow} c_{k\uparrow},$$

where $\xi_k = \varepsilon_k - \mu$ is the single particle energy with respect to the chemical potential $\mu$. We introduce the creation operators for Bogoliubov quasiparticles, denoted $\gamma^\dagger_{k\sigma}$, via the Bogoliubov rotation

$$\begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^\dagger_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} \sin\theta_k & -\cos\theta_k \\ \cos\theta_k & \sin\theta_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow} \end{pmatrix}, \qquad \tan\theta_k = \frac{\Delta_k}{E_k - \xi_k},$$

where $\Delta_k = -\frac{1}{\Omega}\sum_k V_{kk'} \langle c_{-k\downarrow} c_{k'\uparrow}\rangle$ is the gap function and $E_k = \sqrt{\Delta_k^2 + \xi_k^2}$. It's usefull to have

$$\begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} \sin\theta_k & \cos\theta_k \\ -\cos\theta_k & \sin\theta_k \end{pmatrix} \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^\dagger_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} u_p & v_p \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^\dagger_{-k\downarrow} \end{pmatrix}$$

The Bogoliubov quasiparticles satisfy the usual fermionic anti-commutation relations **(1)**:

$$\{\gamma_{k\uparrow}, \gamma_{k'\downarrow}\} = \sin\theta_k \cos\theta_{-k'} \delta_{k,-k'} - \cos\theta_k \sin\theta_{-k'} \delta_{k,-k'} = 0,$$

$$\{\gamma_{k\sigma}, \gamma_{k'\sigma'}\} = \left\{ \sin\theta_k c_{k\uparrow} - \cos\theta_k c^\dagger_{-k\downarrow},\ \sin\theta_{k'} c_{k'\uparrow} - \cos\theta_{k'} c^\dagger_{-k'\downarrow} \right\} = 0,$$

$$\left\{ \gamma_{k\uparrow},\ \gamma^\dagger_{k'\uparrow} \right\} = \sin\theta_k \sin\theta_{k'} \delta_{kk'} + \cos\theta_k \cos\theta_{k'} \delta_{kk'} = \delta_{kk'}.$$

**Hamiltonian**. In the mean-field approximation, the BCS Hamiltonian takes the form

$$H \approx \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} - \sum_k \left( \Delta_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \bar{\Delta}_k c_{-k\downarrow} c_{k\uparrow} + \text{const} \right),$$

introducing operators of the form

$$A_k \overset{\text{def}}{=} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}, \qquad B_k \overset{\text{def}}{=} c_{-k\downarrow} c_{k\uparrow},$$

the mean-field approximation amounts to approximating

$$\left( A_{k'} - \langle A_{k'} \rangle \right) \left( B_k - \langle B_k \rangle \right) \approx 0,$$

i.e. neglecting fluctuations around the expectation values in quadratic order. The interaction **(2)**

$$\begin{aligned} V = \frac{1}{\Omega} \sum_{kk'} V_{kk'} A_{k'} B_k &= \frac{1}{\Omega} \sum_{kk'} A_{k'} \langle B_k \rangle + \frac{1}{\Omega} \sum_{kk'} V_{kk'} \langle A'_k \rangle B_k - \frac{1}{\Omega} \sum_{kk'} \langle A_{k'} \rangle \langle B_k \rangle \\ &= -\sum_{k'} \Delta_{k'} A_{k'} - \sum_k \bar{\Delta}_k B_k + \text{const}. \end{aligned}$$

Moving on to quasiparticles, we find **(3)**

$$\begin{aligned} H &= \sum_k \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \begin{pmatrix} \xi_k & -\Delta_k \\ -\Delta_k & \xi_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} + \text{const} \\ &= \sum_k \begin{pmatrix} \gamma^\dagger_{k\uparrow} & \gamma_{-k\downarrow} \end{pmatrix} \begin{pmatrix} \sin\theta_k & \cos\theta_k \\ -\cos\theta_k & \sin\theta_k \end{pmatrix} \begin{pmatrix} \xi_k & -\Delta_k \\ -\Delta_k & \xi_k \end{pmatrix} \begin{pmatrix} \sin\theta_k & -\cos\theta_k \\ \cos\theta_k & \sin\theta_k \end{pmatrix} \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^\dagger_{-k\downarrow} \end{pmatrix} + \text{const} \\ &\overset{?}{=} \sum_k \begin{pmatrix} \gamma^\dagger_{k\uparrow} & \gamma_{-k\downarrow} \end{pmatrix} \begin{pmatrix} \tilde{E}_k & 0 \\ 0 & -\tilde{E}_k \end{pmatrix} \begin{pmatrix} \gamma_{k\uparrow} \\ \gamma^\dagger_{-k\downarrow} \end{pmatrix}, \end{aligned}$$

with

$$\tilde{E}_k = \xi_k - 2\xi_k \cos(\theta_k)^2 + 2\Delta_k \sin\theta_k \cos\theta_k = \frac{\xi_k^2 + \Delta_k^2}{E_k} = E_k.$$

With $\Delta_k = \Delta$ we could plot the dispersion relation (fig. 1).

**Ground state**. The BCS state

$$|\text{gs}\rangle = \prod_k \left( \sin\theta_k + \cos\theta_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right) |0\rangle,$$

has zero Bogoliubov quasiparticles **(4)**:

$$\begin{aligned} \gamma_{k\uparrow} |\text{gs}\rangle &= \left( \sin\theta_k c_{k\uparrow} - \cos\theta_k c^\dagger_{-k\downarrow} \right) \prod_k \left( \sin\theta_k + \cos\theta_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \right) |0\rangle \\ &= \sin\theta_k c_{k\uparrow} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} |0\rangle - \cos\theta_k \sin\theta_k c^\dagger_{-k\downarrow} |0\rangle = 0, \end{aligned}$$

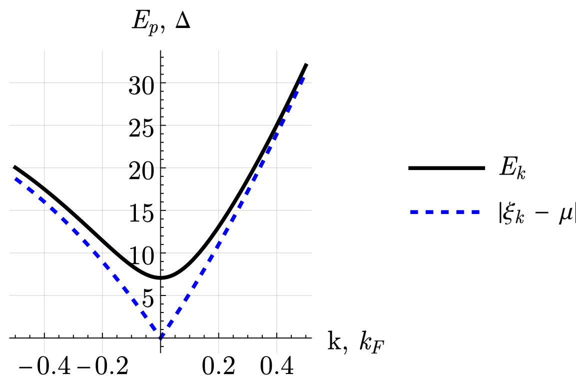


Figure 1: The dispersion relation

so $|\text{gs}\rangle$ is ground state of the mean field approximation of $H$. Actually the $|\text{gs}\rangle$ could be observed by

$$|\text{gs}\rangle \overset{\text{n}}{=} \prod_k \gamma_{-k\downarrow}\gamma_{k\downarrow} |0\rangle .$$

In this state

$$E(\theta_k) = \langle \text{gs}|H|\text{gs}\rangle = \sum_k 2\xi_k \cos^2\theta_k + \frac{1}{4\Omega}\sum_{kk'} V_{kk'} \sin(2\theta_k)\sin(2\theta_{k'}),$$

and we could find $\theta_k$ just by minimizing the energy

$$\frac{\partial}{\partial\theta_q} E(\theta_k) = \frac{1}{\Omega}\cos(2\theta_q)\sum_k V_{qk}\sin(2\theta_k) - 2\xi_q \sin(2\theta_q) = 0.$$

Using this relation we could simplify expression **(5)**

$$\Delta_k = -\frac{1}{\Omega}\sum_{k'} V_{kk'}\langle c_{-k'\downarrow}c_{k'\uparrow}\rangle_{\text{gs}} = -\frac{1}{2\Omega}\sum_{k'} V_{kk'}\sin(2\theta_{k'}) = -\frac{1}{\Omega}\sum_{k'} V_{kk'}\frac{\Delta_{k'}}{2E_{k'}},$$

the zero temperature gap equation.