1.1 Second Quantization

We could consider

$$|\alpha_1, \ldots, \alpha_N\rangle = \frac{1}{\sqrt{N!}} \sum_P \zeta^{\sigma(P)} |\alpha_{P(1)}\rangle \otimes \ldots \otimes |\alpha_{P(N)}\rangle,$$

where $\zeta = \pm 1$ for bosons and fermions respectively. We define creation operator via

$$a_\beta^\dagger |\alpha_1, \ldots, \alpha_N\rangle \overset{\text{def}}{=} |\beta, \alpha_1, \ldots, \alpha_N\rangle.$$

1. Adjoint a_β could be expressed as

$$a_\beta^\dagger = \sum_{\{\theta\}} |\beta, \theta_1, \ldots, \theta_M\rangle\langle\theta_1, \ldots, \theta_M|,$$

$$a_\beta = \sum_{\{\theta\}} |\theta_1, \ldots, \theta_M\rangle\langle\beta, \theta_1, \ldots, \theta_M|.$$

Than it could be shown that

$$a_\beta |\alpha_1, \ldots, \alpha_N\rangle = \sum_k C_k |\alpha_1, \ldots, \cancel{\alpha_k}, \ldots, \alpha_N\rangle,$$

with

$$C_k = \langle\beta, \alpha_1, \ldots, \cancel{\alpha_k}, \ldots, \alpha_N | \alpha_1, \ldots, \alpha_N\rangle = \frac{1}{\sqrt{N!}} \sum_P \langle\alpha_{P(1)}| \otimes \ldots \otimes \langle\beta|_{P(k)} \otimes \ldots \otimes \langle\alpha_{P(N)}| \alpha_1, \ldots, \alpha_N\rangle,$$

where we could «move» $\langle\beta|$ to the start, by $P(k) - 1$ transpositions, and due to $N!$ equal permutations we could neglect $\frac{1}{N!}$ coming to

$$C_k = \zeta^{k-1}\langle\beta|\alpha_k\rangle.$$

2. We also could find, that a_β and $a_\beta^\dagger$ fuldill the (anti)-commutation relations

$$a_\beta^\dagger a_\alpha |\theta_1, \ldots, \theta_N\rangle = a_\beta^\dagger \sum_{k=1}^N \zeta^{k-1}\langle\alpha|\theta_k\rangle |\theta_1, \ldots, \cancel{\theta_k}, \ldots, \theta_N\rangle = \sum_{k=1}^N \zeta^{k-1}\langle\alpha|\theta_k\rangle |\beta, \theta_1, \ldots, \cancel{\theta_k}, \ldots, \theta_N\rangle,$$

$$a_\beta a_\alpha^\dagger |\theta_1, \ldots, \theta_N\rangle = a_\beta |\beta, \theta_1, \ldots, \theta_N\rangle = \sum_{k=1}^N \zeta^k \langle\alpha|\theta_k\rangle |\beta, \theta_1, \ldots, \cancel{\theta_k}, \ldots, \theta_N\rangle + \langle\alpha|\beta\rangle |\theta_1, \ldots, \theta_N\rangle,$$

so for bosons $\zeta = 1$ we have

$$[a_\alpha, a_\beta^\dagger] \overset{\text{def}}{=} a_\alpha a_\beta^\dagger - a_\beta^\dagger a_\alpha = \langle\alpha|\beta\rangle = \delta_{\alpha,\beta},$$

and in the same way for fermions $\zeta = -1$ and

$$\{a_\alpha, a_\beta^\dagger\} \overset{\text{def}}{=} a_\alpha a_\beta^\dagger + a_\beta^\dagger a_\alpha = \langle\alpha|\beta\rangle = \delta_{\alpha,\beta}.$$

3. For density operator

$$\hat\rho(x) = \sum_{j=1}^N \delta(x - \hat x_j),$$

we could find second quantized form

$$\hat\rho(x) = \sum_{\alpha\beta} \langle\alpha|\delta(x - \hat x)|\beta\rangle \hat a_\alpha^\dagger \hat a_\beta = \sum_{\alpha\beta} \int \langle\alpha|x'\rangle\langle x'|\beta\rangle \delta(x - x')\, dx'\, \hat a_\alpha^\dagger \hat a_\beta = \sum_{\alpha\beta} \langle\alpha|x\rangle\langle x|\beta\rangle \hat a_a^\dagger \hat a_\beta,$$

what could be reduced to the $\hat a_x^\dagger \hat a_x$ form if $|\alpha\rangle$ and $|\beta\rangle$ corresponds to the coordinates.

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1.2 Mapping between Quantum and Classical Systems

We could rewrite classical 1D Ising chain partitin function as

$$\mathcal{Z}_{c} = T_{s_1,s_2} \ldots T_{s_{N-1},s_N} T_{s_N,s_1} = \operatorname{tr}\left(T^N\right),$$

with transfer matrix

$$T = T^a T^b = \begin{pmatrix} e^{h_c+K_c} & e^{h_c-K_c} \\ e^{-h_c-K_c} & e^{K_c-h_c} \end{pmatrix}, \qquad T^a = \begin{pmatrix} e^{h_c} & 0 \\ 0 & e^{-h_c} \end{pmatrix}, \quad T^b = \begin{pmatrix} e^{K_c} & e^{-K_c} \\ e^{-K_c} & e^{K_c} \end{pmatrix}.$$

There are different ways to define T, because important just eigenvalues

$$\lambda_{1,2} = \frac{1}{2} e^{-h_c-K_c} \left(e^{2(h_c+K_c)} + e^{2K_c} \pm \sqrt{e^{4K_c}\left(e^{2h_c}-1\right)^2 + 4e^{2h_c}} \right).$$

For a quantum system the partitin function

$$\mathcal{Z}_{q} = \operatorname{tr} e^{-\beta H},$$

and we want to achieve

$$\mathcal{Z}_{q} = \mathcal{Z}_{c} = \operatorname{tr}\left(e^{-\frac{\beta}{N}H_1} e^{-\frac{\beta}{N}H_2} \right)^N, \qquad e^{-\frac{\beta}{N}H_1} = T^a, \qquad e^{-\frac{\beta}{N}H_2} = T^b.$$

Using formulas to the Pauli matrix exponents, we could find

$$H_1 = \frac{N}{-\beta}\alpha_3 \sigma_z, \qquad H_2 = \frac{N}{-\beta}(\alpha_0 \mathbb{1} - \alpha_1 \sigma_x),$$

with $\alpha_0 = \ln\sinh(2K_c) + \ln 2$, $\alpha_1 = \ln\tanh K_c$ and $\alpha_3 = h_c$. I think it is possible to find other H_1 and H_2, my choice was ruled by separating K_c and h_c dependences.

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