Quantum Hardware
Problem Set No. 2

Stefan Filipp
Gleb Krylov, Niklas Glaser
(Due: November 2, 12:00, submission)

October 26, 2023

1 Quantum State Tomography

To determine the state of an N-qubit system a specific number of measurements have to be performed on identically prepared systems. From the results of such a complete set of measurements, the initial state can then be fully characterized.

(a) How many measurements do you need to determine the quantum state of the system and how is the number of required measurements related to the normalization of the state? What does it mean if, after performing state tomography on a real system with a finite number of shots, you find that the state is not normalized? Write down a complete set of measurement operators for a 2-qubit system.

(b) Show how the Bloch vector can be extracted from the set of measurements for a single qubit. Assume that only projective measurements on the z-axis are available.

2 Semi-Classical Light–Matter Interaction

Let us repeat some basics about light–matter interaction. We consider a simple two-level atom with ground and excited states $|g\rangle$ and $|e\rangle$. The Hamiltonian describing this atom can be expressed as

$$\hat{H}_a = -\frac{\hbar\omega}{2}\hat{\sigma}_z,$$

where $\hat{\sigma}_z$ is the Pauli-z matrix. Now we introduce an additional classical electric field $E(t) = E_0\cos(\omega_L t)$, whose interaction with the two-level atom is described by the light-atom interaction (in dipole approximation, where $E(r,t) = E(t)$)

$$\hat{V}(t) = -d\cdot E(t)\hat{\sigma}_x.$$

For simplicity, we assume that the dipole-matrix element $d = \langle\hat{d}\rangle$ is real. $\hat{\sigma}_x$ is the Pauli-x matrix. The combined Hamiltonian is then: $\hat{H}(t) = \hat{H}_a + \hat{V}(t)$.

(a) Show that $\mathrm{e}^{-\mathrm{i}\theta\hat{\sigma}_z}\hat{\sigma}_x\mathrm{e}^{\mathrm{i}\theta\hat{\sigma}_z} = \cos(2\theta)\hat{\sigma}_x + \sin(2\theta)\hat{\sigma}_y$.

(b) Compute the transformation into the rotating frame of the electric field using the unitary operator $\hat{U} = \mathrm{e}^{-\mathrm{i}\frac{\omega_L t}{2}\cdot\hat{\sigma}_z}$. The transformed Hamiltonian $\hat{H}_I$ is given by

$$\hat{H}_I(t) = \hat{U}\hat{H}\hat{U}^\dagger - \mathrm{i}\hbar\hat{U}\partial_t\hat{U}^\dagger$$

(c) Apply the rotating-wave approximation to $\hat{H}_I(t)$ by neglecting rapidly oscillating terms at frequency $2\omega_L$.

(d) Now consider a resonant interaction, where the detuning $\omega - \omega_L = 0$. Compute the time evolution $|\psi(t)\rangle$ for the initial state $|e\rangle$. What is the characteristic oscillation frequency Ω?

(e) Calculate the Rabi frequency and the time required to fully excite the qubit for a superconducting circuit. Here a single Cooper pair oscillates between two charge islands separated by a distance $L = 50\ \mu$m. The system therefore has a dipole moment of $d = 2eL$. The circuit is typically driven by microwave fields with a field strength of 1 mV/m.

(f) With a semiclassical approach, the spontaneous decay rate of an atom is given by:

$$\Gamma = \frac{\omega_0^3}{3\pi\epsilon_0\hbar c^3}|\langle e|\,\mu\,|g\rangle|^2, \tag{1}$$

where $d_{ge} = |\langle e|\,\mu\,|g\rangle|$ represents the dipole matrix element. When a laser matches the transition frequency of the atom, we can drive the transition between the states $|g\rangle$ and $|e\rangle$ on resonance. The coupling strength can be calculated using the Rabi frequency:

$$\hbar\Omega = d_{ge} \cdot E. \tag{2}$$

Strontium (Sr) is a suitable atom for quantum technologies due to its extremely narrow transition between the $1S_0 \longrightarrow 3P_0$ states with a lifetime of 118 s for the excited state at a frequency of 429 THz. For a typical laser intensity of 10 μW/μm^2 calculate the Rabi frequency and the minimal time required for a maximum population of the excited state $|e\rangle$. Compare the Rabi frequency and the shortest time necessary for a maximum population of the excited state between superconducting circuits and the atom-laser system.

(g) Consider the dynamics from part (d) while accounting for spontaneous emission at a rate Γ. Explain the qualitative change of the dynamics and discuss the limiting cases $\Omega \gg \Gamma$ and $\Omega \ll \Gamma$.

3 Driven System Dynamics and Rotating-Wave Approximation

Note: For this exercise, you have to use the Python package qutip*. You can find a template for the numerics on Moodle.*

Consider a two-level system described by $H_0 = -\frac{\hbar\omega_0}{2}\sigma_z$, where the two levels are influenced by a resonant ($\omega = \omega_0$) time-dependent driving field

$$V(t) = \hbar\frac{\Omega}{2}\left[\cos(\omega t)\,\sigma_x - \sin(\omega t)\,\sigma_y\right]. \tag{3}$$

(a) Calculate numerically the time-evolution of a state $|\psi_{\text{init}}\rangle$ evolving in the time-dependent Hamiltonian $H(t) = H_0 + V(t)$. Analyze the system for the exemplary values $\Omega = 2\pi$, $\omega_0 = 2\pi$, and $t \in [0, 2]$. Start in the ground state $|\psi_{\text{init}}\rangle = |\psi(t_1)\rangle = |0\rangle$. Plot the $\langle\sigma_z\rangle$ expectation values as a function of time.

(b) Repeat the calculation in (a) using the rotating wave-approximation, with $V_{\text{RWA}}(t) = \Omega\cos(\omega t)\,\sigma_x$ and $H(t) = H_0 + V_{RWA}(t)$.

(c) Compare and plot the probabilities in (a) and (b) for the three values $\omega_0 = 2\pi, 40\pi$ and 100π. In which limit does (b) approach (a)?

(d) Consider now a general detuned Rabi oscillation in the rotating frame, described by the Hamiltonian

$$H = -\hbar\frac{\delta\omega}{2}\sigma_z + \hbar\frac{\Omega_o}{2}\sigma_x, \tag{4}$$

with the detuning $\delta\omega = \omega - \omega_0$, and the drive strength Ω_0. Calculate the expression for $\langle\sigma_z\rangle$(t), starting in the ground state $|\psi(0)\rangle = |0\rangle$ under this Hamiltonian. You may introduce a generalized Rabi frequency Ω.

(e) Plot the expectation value $\langle\sigma_z\rangle$ for the detunings $\delta\omega = 0, \Omega_0, 2\Omega_0$. How do the amplitude and the oscillation frequency change, compared to the resonant case?

4 Dephasing and Decoherence in a TLS

In the lecture, the Bloch-Redfield model of decoherence has been discussed. If the longitudinal relaxation rate is Γ_1 and the transverse relaxation rate is Γ_2, the density matrix at time t of a state starting at $\alpha|0\rangle + \beta|1\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$ is

$$\rho(t) = \begin{pmatrix} 1 - |\beta|^2 e^{-\Gamma_1 t} & \alpha\beta^* e^{-\Gamma_2 t} \\ \alpha^*\beta e^{-\Gamma_2 t} & |\beta|^2 e^{-\Gamma_1 t} \end{pmatrix} \tag{5}$$

The relaxation rates are related via $\Gamma_2 = \frac{\Gamma_1}{2} + \Gamma_\phi$, where Γ_ϕ accounts for pure dephasing noise.

(a) Assume $\Gamma_\phi = 0$, i.e., no extra phase noise. Calculate $\langle\sigma_z\rangle$ and $\langle\sigma_x\rangle$ and plot the observables as a function of time. Use $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ as a starting state, discuss the result.

(b) Now assume $\Gamma_\phi = 10\Gamma_1$, i.e., strong dephasing. Will this affect the population measured by $\langle\sigma_z\rangle$. Calculate and plot again $\langle\sigma_z\rangle$ and $\langle\sigma_x\rangle$ and discuss the results. Calculate $\langle\sigma_y\rangle$. Does this provide more information? When is a measurement of $\langle\sigma_y\rangle$ and $\langle\sigma_x\rangle$ useful?

(c) How can you experimentally measure Γ_ϕ, starting with an initial state $|0\rangle$, using only single qubit rotations and measurements in the z-basis?