1 Introduction to Landau's Fermi Liquid Theory

Landau's Fermi Liquid Theory states that even in the presence of interactions between fermionic particles, if the ground state of the interacting system is adiabatically connected to the Fermi sea of the free particles, the low-lying excitations can still be described by single-particle excitations around the Fermi surface. These states are referred to as quasi-particle states.

1.1 Phase Space Argument for the Life Time of Quasi-particles

In this first part, we are going to show that the quasi-particles have a asymptotically diverging life-time as $\varepsilon \to \varepsilon_F$ and are thus stable excitations.

1. Consider the Coulomb interaction in second quantization between electrons:

$$\hat{V} = \frac{1}{2\mathcal{V}} \sum_{\sigma,\sigma'} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} V(\mathbf{q}) c^\dagger_{\mathbf{k}+\mathbf{q},\sigma} c^\dagger_{\mathbf{k}'-\mathbf{q},\sigma'} c_{\mathbf{k}',\sigma'} c_{\mathbf{k},\sigma} \tag{1}$$

   $\mathcal{V}$ is the volume. Taking into account only scattering processes that involve two particles, derive an expression for the inverse life time $\frac{1}{\tau_\mathbf{k}}$ of the state $|\mathbf{k},\sigma\rangle = c^\dagger_{\mathbf{k},\sigma} |\Omega\rangle$, where $|\Omega\rangle$ denotes the state, where all states below the Fermi surface are occupied.

2. By anticipating the result for the screening of the Coulomb interaction, we can assume that $V(\mathbf{q}) \approx V(0) = \text{const}$. We consider the system at $T = 0$. Use an argument about the available phase space for the decay of $|\mathbf{k},\sigma\rangle$ to rewrite the formula for $\frac{1}{\tau_\mathbf{k}}$ and express the occuring momentum sums in the continuum limit as integrals over energy relative to ε_F. Show that

$$\frac{1}{\tau_\mathbf{k}} \propto (\varepsilon_\mathbf{k} - \varepsilon_F)^2 \,. \tag{2}$$

   > Hint: You don't need to calculate the angle-dependent part explicitly which would take care of the delta-function properly. Instead, use phase space arguments and the delta-function to get restrictions on the two integrations.

1.2 Microscopic Basis of the Fermi-liquid Theory

As mentioned above, Fermi liquid theory only holds if the ground state of the interacting system is connected adiabatically to the non-interacting Fermi sea. One can treat this as turning on the interactions adiabatically. The ground state $|\phi\rangle$ of the full system and the excitation state $\left|\widetilde{\mathbf{k}\sigma}\right\rangle$ can then be written as

$$|\phi\rangle = U|\Omega\rangle \qquad\qquad \left|\widetilde{\mathbf{k}\sigma}\right\rangle = U|\mathbf{k}\sigma\rangle \,,$$

where $|\mathbf{k}\sigma\rangle = c^\dagger_{\mathbf{k}\sigma}|\Omega\rangle$. The time evolution operator in the interaction picture can be expressed as a time-ordered exponential

$$U = T\left\{e^{-i\int_{-\infty}^{0}\hat{V}(t)\mathrm{d}t}\right\} \tag{3}$$

3. The state $\left|\widetilde{\mathbf{k}\sigma}\right\rangle = a^\dagger_{\mathbf{k}\sigma}|\phi\rangle$ describes a quasiparticle above the ground state $|\phi\rangle$. Express the quasi-particle creation operator $a^\dagger_{\mathbf{k}\sigma}$ in terms of the bare creation operator $c^\dagger_{\mathbf{k}\sigma}$.

4. For Fermi liquid theory to be valid, what requirement does one need to impose on the wavefunction renormalization constant

$$Z_{\mathbf{k}} = |\langle\widetilde{\mathbf{k}\sigma}|c^\dagger_{\mathbf{k}\sigma}|\phi\rangle|^2 \quad ? \tag{4}$$

   Express $c^\dagger_{\mathbf{k}\sigma}$ as a series in the quasi-particle operators.

5. **Spectral Function** Argue why adding a particle to the ground state excites a continuum of states in the presence of interactions. Show that the spectral function

$$A(\mathbf{k},\omega) = -\frac{1}{\pi}\,\mathrm{Im}G^{ret}(\mathbf{k},\omega) = \sum_\lambda |M_\lambda|^2\delta(\omega-\xi_\lambda) \tag{5}$$

   with $\xi_\lambda = \varepsilon_\lambda - \varepsilon_F$ and the overlap $|M_\lambda|^2 = |\langle\lambda|c^\dagger_{\mathbf{k}\sigma}|\phi\rangle|^2$ exhibits a sharp quasi-particle peak at $\xi_{\mathbf{k}}$. Does the sum rule for the spectral function hold? Make a sketch of $A(\mathbf{k},\omega)$ for fixed momentum.

6. **Momentum distribution** Using your result from the 5., show that the momentum distribution $\langle\hat{n}_{\mathbf{k}\sigma}\rangle$ has a jump at the Fermi momentum. Sketch $\langle\hat{n}_{\mathbf{k}\sigma}\rangle$.