12 Fermions in one dimension

Consider 1D non-interacting electrons described by the Hamiltonian

$$\hat{H} = \sum_k \xi_k \hat{c}_k^\dagger \hat{c}_k, \qquad \xi_k = \varepsilon_k - \mu.$$

We want to compute the Linchard function χ_0 is the correlation function associated with the response to a change of the chemical potential.

1. Lindhard function. The density response function $\chi_0(q,\omega)$ of a one-dimensional Fermi gas

$$\chi_0(q,t) = -\frac{i}{\hbar}\theta(t)\langle[\hat{\rho}_q(t), \hat{\rho}_{-q}(0)]\rangle,$$

Thus we find for the Fourier transform of the Lindhard function

$$\chi_0(q,\omega) = -\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\ e^{i\omega t}\theta(t)\langle[\hat{\rho}_q(t), \hat{\rho}_{-q}(0)]\rangle = -\frac{i}{\hbar V}\sum_k \int_0^\infty dt\ e^{i(\omega-(\xi_k-\xi_{k+q}))t}\left[n_{k+q}(1-n_k) - n_k(1-n_{k+q})\right],$$

where we took advantage of the quadratic Hamiltonian of free fermions.

$$\chi_0(q,\omega) = \frac{1}{\hbar V}\sum_k \frac{n_{k+q}-n_k}{\omega - (\xi_k - \xi_{k+q}) + 0i}.$$

Moving on to integration

$$\chi_0(q,\omega) = \frac{1}{V\hbar}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{n_{k+q}-n_k}{\omega-(\xi_k-\xi_{k+q})+0i} = \mathcal{P}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{n_{k+q}-n_k}{\omega-(\xi_k-\xi_{k+q})} - i\pi\int_{-\infty}^{\infty}\frac{dk}{2\pi}\left(n_{k+q}-n_k\right)\delta(\omega-(\xi_{k+q}-\xi_k)),$$

we get

$$\chi_0''(q,\omega) = \operatorname{Im}\chi_0(q,\omega) = \frac{1}{2}\int_{-\infty}^{\infty} dk\ \left(\delta(\omega-(\xi_{k+q}-\xi_k)) - \delta(\omega-(\xi_k-\xi_{k-1}))\right) n_k$$

$$= \frac{m}{2|q|}\int_{-\infty}^{\infty} n_k(\delta(k-k_-)-\delta(k-k_+))\,dk = \frac{m}{2|q|}\left(n_{k_-}-n_{k_+}\right),$$

with defined zeros

$$\delta(\omega-(\xi_{k+q}-\xi_k)) = \frac{m}{|q|}\delta(k-k_\pm), \qquad k_\pm = \frac{2m\omega \pm q^2}{2q}.$$

2. Perturbation. The energy absorption rate $\propto \chi_0(q,\omega)$ from the perturbation (q,ω). At zero temperature $\chi_0(T=0)''$ could be rewritten as

$$\chi_0''(q,\omega) = \frac{m}{2|q|}\left(\theta(-\xi_{k_-}) - \theta(-\xi_{k_+})\right), \qquad \xi_k = \frac{\hbar^2k^2}{2m} - \frac{\hbar^2 k_\mathrm{F}^2}{2m}, \qquad k_\pm = \frac{2m\omega+q^2}{2q},$$

which completely defines regions with non-zero absorption. Let's find conditions for the boundaries (in units of k_F and ε_F)

$$\begin{cases} k_\mathrm{F}^2 - k_-^2 > 0 \\ k_\mathrm{F}^2 - k_+^2 < 0 \end{cases} \quad\Rightarrow\quad \begin{cases} q - \frac{1}{2}q^2 < \omega < q + \frac{1}{2}q^2, & 0 < q < 2 \\ -q + \frac{1}{2}q^2 < \omega < q + \frac{1}{2}q^2, & q > 2 \end{cases}$$

, which, taking into account symmetry $\chi_0''(q) = \chi_0''(-q)$, leads to (fig. 1). Here we see well defined sharp $\omega(q)$ at $q \ll k_\mathrm{F}$ (in contrast to 2D/3D case with macroscopic Fermi surface).

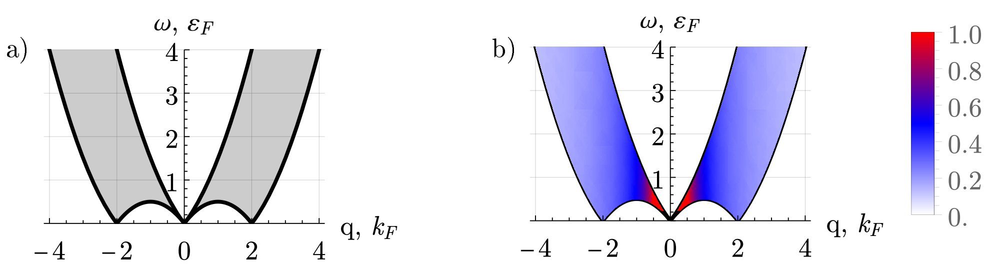

Figure 1: a) Nonzero energy absorption rate regions. b) The density response function χ of the weak interacting system.

3. The sound velocity. The sound velocity of the 1D non-interacting Fermi gas, i.e. the proportionality constant of the linear dispersion at small q

$$v_{\text{sound}} = \frac{\hbar k_{\text{F}}}{m}\frac{\partial}{\partial q}\left(q - \tfrac{1}{2}q^2\right)|_{q=0} = \frac{\hbar k_{\text{F}}}{m} = v_{\text{F}}.$$

4. Width. The width of the region in which energy can be absorbed $\delta\omega(q \ll k_{\text{F}})$

$$\delta\omega(q \ll k_{\text{F}}) \approx \varepsilon_{\text{F}}(q/k_{\text{F}})^2,$$

so there is a sharp collective mode in the spectrum in the sense that

$$\lim_{q\to 0}\frac{\delta\omega(q)}{\omega(q)} = \frac{\varepsilon_{\text{F}}}{k_{\text{F}}^2}\lim_{q\to 0}\frac{q^2}{v_{\text{F}}q} = 0,$$

and (remembering Luttinger's liquid) operator that create these excitations on top of the ground-state $\hat{\rho}_q$, which is specific to one-dimensional systems.

5. RPA. Now we add a contact interaction u between the Fermions. The density response function $\chi(q,\omega)$ of the interacting system at small q at zero temperature, using the result of the random phase approximation (RPA)

$$\chi(q,\omega) = \frac{\chi_0(q,\omega)}{1 - u\chi_0(q,\omega)}.$$

To find the pole we need

$$\begin{aligned}
\chi_0'(q,\omega) &= \operatorname{Re}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{n_{k+q}-n_k}{\omega - (\xi_{k+q}-\xi_k) + 0i} \\
&= \operatorname{Re}\int_{-k_{\text{F}}-q}^{k_{\text{F}}-q}\frac{dk}{2\pi}\frac{1}{\omega - \frac{1}{2m}(2kq+q^2) + i0} - \operatorname{Re}\int_{-k_{\text{F}}}^{k_{\text{F}}}\frac{dk}{2\pi}\frac{1}{\omega - \frac{1}{2m}(2kq+q^2) + i0} \\
&\approx \frac{1}{\pi}\frac{q^2 v_{\text{F}}}{\omega^2 + v_{\text{F}}^2 q^2}.
\end{aligned}$$

Thus ξ has a pole at $1 - u\chi_0 = 0$

$$\omega = \sqrt{1 + \tfrac{1}{\pi v_{\text{F}}}}\,v_{\text{F}} = \tilde{v}_{\text{sound}}q, \qquad \tilde{v}_{\text{sound}} = v_{\text{F}}\sqrt{1 + \frac{u}{\pi v_{\text{F}}}},$$

with almost the same the sound velocity.

6. On the way to bosonisation. The sound-mode exhausts the f-sum rule

$$\int_{-\infty}^{\infty}\omega S(q,\omega)\,d\omega = \frac{q^2}{2m},$$

at zero temperature, i.e. all the excitations of a 1D Fermi system are phonons. The imaginary part

$$\chi''(q,\omega) = \frac{\chi_0''}{(1-u\chi_0')^2 + u^2(\chi_0'')^2} \overset{\chi''\neq 0}{=} \frac{m}{2|q|}\left(\left(\frac{um}{2q}\right)^2 + \left(1 - \frac{u}{\pi}\frac{q^2 v_{\text{F}}}{q^2 v_{\text{F}}^2 + \omega^2}\right)^2\right)^{-1}.$$

Thus sum rule

$$\int_{-\infty}^{\infty}\omega S(q,\omega)\,d\omega = 2\int_{-\infty}^{\infty}\omega\chi''(q,\omega)\theta(\omega)\,d\omega = \frac{1}{\pi}\int_0^{\infty}\omega\chi''(q,\omega)\,d\omega.$$

We could for example assume $q \ll k_{\text{F}}$ and expand in series of ω

$$\omega\chi''(q,\omega) \approx \frac{m\omega}{2q\left(\frac{m^2u^2}{4q^2} + \left(1 - \frac{u}{\pi v_{\text{F}}}\right)^2\right)},$$

and

$$\int_{-\infty}^{\infty}\omega S(q,\omega) \approx \int_{q-q^2/2}^{q+q^2/2}\frac{\omega}{2q}\left(\frac{u^2}{4q^2} + (1-\frac{u}{\pi})^2\right) \approx q^2\left(\frac{u^2}{4q^2} + \frac{u^2}{\pi^2} - \frac{2u}{\pi} + 1\right)^{-1},$$

with $m = 1$, $v_{\text{F}} = 1$. In general, something like a quadratic.