Quantum Hardware
Problem Set No. 5

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1 Rotating-Wave Approximation in the Jaynes-Cummings Model

The coupling between a resonator with frequency ω_r and a two-level system with frequency ω_{10} is described via the interaction

$$H_{\mathrm{int}} = \hbar g\,(\sigma^+ + \sigma^-)\,(\hat{a}^\dagger + \hat{a}),$$

The total Hamiltonian H is composed of the resonator–qubit term, referred to as $H_0 = \hbar\omega a^\dagger a - \hbar\omega_{10}\sigma_z/2$, and the interaction term H_{int}.

(a) Transform the system to the co-rotating frame of resonator and qubit in which only the interaction term H_{int} remains. To this end, use the transformation $U = e^{\mathrm{i}H_0 t/\hbar}$.

(b) Apply the rotating-wave approximation. In which limit is it a good approximation?

(c) Use the approximated Hamiltonian and perform the transformation back to the original frame. You should get the well-known, simplified interaction action term.

(d) After understanding the limitations of the simplified interaction term, transfer this Hamiltonian to the rotating frame of the resonator, instead of the co-rotating frame. Show that in this frame the static Hamiltonian can be written as a sum of a diagonal σ_z term and a photon-number dependent off-diagonal term.

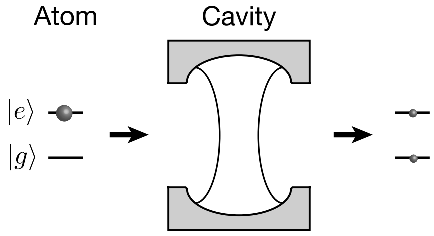
(e) Imagine an experiment in which the resonator is filled with photons, one by one. Use instantaneous σ_x-operations on the qubit, (i.e. qubit operations that are much faster than all other experimental timescales and their duration can therefore be neglected). Apply a sequence of these σ_x-operations at well defined times followed by a JC-interaction of qubit and resonator. At which times do you need to apply σ_x to fill the resonator with n photons?

2 Jaynes-Cummings Hamiltonian

The Hamiltonian of the Jaynes-Cummings (JC) Model is given by

$$\hat{H}_{\mathrm{JC}} = \hbar\omega_{10}\hat{\sigma}^\dagger\hat{\sigma} + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g\left(\hat{a}\hat{\sigma}^\dagger + \hat{a}^\dagger\hat{\sigma}\right),$$

where the uncoupled states of the atomic two-level-system are given by $|e\rangle$ and $|g\rangle$, while $|n\rangle$ denotes the Fock-states of the (single-mode, single-frequency) light field. ($|n, e\rangle = |n\rangle \otimes |e\rangle$)



2.1 Part A

(a) What is the effect of the atomic operators $\hat{\sigma}$ and $\hat{\sigma}^\dagger$ on the atomic states $|g\rangle$ and $|e\rangle$?

(b) What are the eigenstates and eigenenergies of the system in the uncoupled case $g = 0$?

(c) Show that the interaction term in $\hat{H}$ does couples only the following pairs of states: $|n, e\rangle \leftrightarrow |n+1, g\rangle$ and $|n-1, e\rangle \leftrightarrow |n, g\rangle$.

(d) Calculate the matrix elements of $\hat{H}$ in the basis $\{|n, e\rangle, |n+1, g\rangle\}$. How does the matrix element scale with the mode volume of the light field?

2.2 Part B

For the following discussion assume that the light field and the atom are on resonance.

(e) As depicted in the figure, assume that an atom is initially prepared in the excited state $|e\rangle$ while the cavity is empty $n = 0$. Now the atom is sent through the cavity, such that the interaction time is equivalent to a $\pi/2$-rotation for the coupled system that evolves under $\hat{H}$. Write down the final state.

(f) After the atom leaves the cavity it evolves freely under the effect of spontaneous emission (no additional dephasing). Calculate the density matrix of the atom by tracing out the field. How does the density matrix evolve after leaving the cavity according to the Bloch-Redfield model? Sketch the trajectory on the Bloch sphere.

(g) How is the trajectory on the Bloch sphere modified for an initial state $(|g\rangle + |e\rangle)/\sqrt{2}$? Sketch the trajectory and explain the dynamics in a few sentences.

3 Collapse and Revival in the Jaynes-Cummings model

Consider the interaction for a single-mode cavity field and a stationary two-level atom, described by the Jaynes-Cummings Hamiltonian $H_{JC} = H_0 + H_{int}$, with $H_{int} = \hbar g(\sigma^+ a + a^\dagger a^-)$.

(a) Define $U(t) = e^{-iH_{int}t/\hbar}$. Show that it can be expressed as

$$U(t) = \cos(gt\sqrt{a^\dagger a + 1})|e\rangle\langle e| + \cos(gt\sqrt{a^\dagger a})|g\rangle\langle g| \\ - i\frac{\sin(gt\sqrt{a^\dagger a + 1})}{\sqrt{a^\dagger a + 1}}a|e\rangle\langle g| - ia^\dagger\frac{\sin(gt\sqrt{a^\dagger a + 1})}{\sqrt{a^\dagger a + 1}}|g\rangle\langle e|$$

Consider the state of the cavity to be a coherent state $|\alpha\rangle$, described as a coherent superposition of Fock states $|n\rangle$

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

(b) In the interaction picture, the evolution of the joint state is described by $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. It is it is completely described by the previously defined operator $U(t)$ and the initial conditions. In our case, we start with the two-level system in the excited state and the cavity in a coherent state $|\psi(0)\rangle = |\alpha\rangle|e\rangle$. Find $|\psi(t)\rangle$.

(c) To obtain the probability to find the two-level system in the excited state, project the previous state $|\psi(t)\rangle$ to the state $|n, e\rangle$. Be careful, because we need to take into account the contribution from all the cavity states $n \in \{0, 1, 2, ...\infty\}$.

(d) Assuming that $\overline{n} \gg 1$, we can take Ω_n to be a distributed between $[\overline{n} - \Delta n, \overline{n} + \Delta n]$, where in our case $\Delta n = \sqrt{\overline{n}}$. Use the fact that now Ω_n is distributed to estimate the collapse time t_c and the revival time t_r of the oscillations.

Hint 1: $\Omega_n \propto \sqrt{n}$.

Hint 2: A phase shift of π *between two waves produces a destructive interference.*

Hint 3: For a revival, we want the oscillations corresponding to consecutive photons to interfere constructively.