6.2 Schrieffer-Wolff Transformation

Consider Hamiltonian $H = H_0 + \lambda V$ and the Schrieffer-Wolff transformation is defined as

$$H = e^{\lambda S} \hat{H} e^{-\lambda S},$$

with $S^\dagger = -S$ and S chosen, such that the linear term λV is eliminated

$$e^{\lambda \hat{S}} \hat{H} e^{-\lambda \hat{S}} = \hat{H}_0 + \mathcal{O}(\lambda^2).$$

Using the Campbell-Baker-Hausdorff formula **(a)**

$$\hat{H}' = e^{\lambda \hat{S}} \hat{H} e^{-\lambda \hat{S}} = \hat{H} + [\lambda \hat{S}, \hat{H}] + \tfrac{1}{2}[\lambda \hat{S}, [\lambda \hat{S}, \hat{H}]] + \dots,$$

substituting the Hamiltonian

$$\hat{H} = \hat{H}_0 + \lambda \hat{V} + [\lambda \hat{S}, \hat{H}_0 + \lambda \hat{V}] = \dots + \lambda \left(\hat{V} + \hat{S}\hat{H}_0 - \hat{H}_0 \hat{S} \right),$$

we get the condition **(b)**

$$[\hat{S}, \hat{H}_0] + \hat{V} = 0. \tag{1}$$

Using this condition we can simplify the series expansion **(c)**

$$\hat{H}' = \hat{H}_0 + \tfrac{\lambda^2}{2}[\hat{S}, \hat{V}] + \mathcal{O}(\lambda^3). \tag{2}$$

Now let's apply this formalism to the Jaynes-Cummings Hamiltonian in the large detuning limit:

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} - \frac{\omega_0}{2} \hat{\sigma}_z + g \left(\hat{a}^\dagger \hat{\sigma}^- + \hat{a} \hat{\sigma}^+ \right).$$

The coupling term $\hat{V} = g(\hat{a}^\dagger \hat{\sigma}^- + \hat{a}\hat{\sigma}^+)$ couples the states $|g, n+1\rangle$ and $|e, n\rangle$ with **(d)**

$$\langle g, n+1 | \hat{H} | g, n+1 \rangle = \omega n + \omega, \qquad \langle e, n | \hat{H} | e, n \rangle = \omega n + \omega_0, \qquad \Rightarrow \qquad \Delta = \omega - \omega_0.$$

To find $\hat{S}$ we could start with ansatz $\hat{S} = \alpha \hat{a}^\dagger \hat{\sigma}^- - \bar{\alpha} \hat{a} \hat{\sigma}^+$ and find **(e)**

$$\alpha = -\frac{1}{\Delta}.$$

That leads to the effective Hamiltonian **(f)**

$$\hat{H}' = \left(\omega + \tfrac{g^2}{\Delta} \hat{\sigma}_z \right) \hat{a}^\dagger \hat{a} - \tfrac{1}{2} \left(\omega_0 + \tfrac{g^2}{\Delta} \right) \hat{\sigma}_z.$$

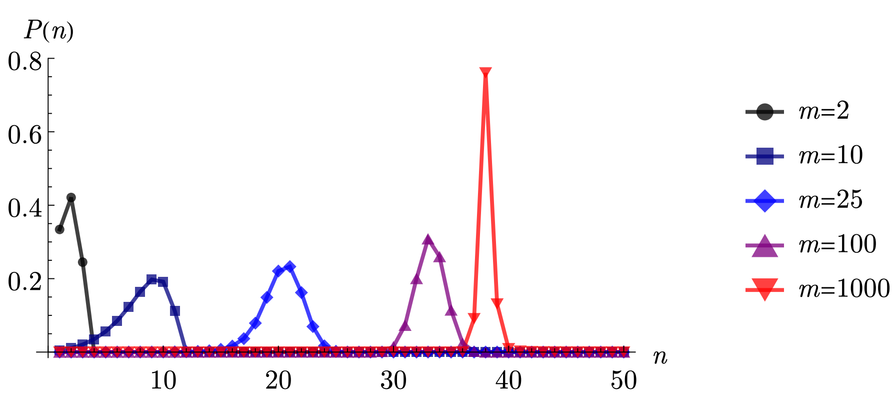

Figure 1: **(6.4.c)** The distribution $P_n(m)$ as a function of n for $gt = \frac{1}{2}$

6.3 Qubit Readout

Consider the Hamiltonian

$$\hat{H} = \omega \left(\hat{a}^\dagger \hat{a} + \tfrac{1}{2} \right) + \tfrac{1}{2} \omega_0 \hat{\sigma}_z + \chi \hat{\sigma}_z \left(\hat{a}^\dagger \hat{a} + \tfrac{1}{2} \right),$$

with $\chi = g^2/\Delta$. Eigenavalues could be expressed as **(a)**

$$E_n^g = \omega(n + \tfrac{1}{2}) - \frac{\omega_0}{2} - \chi(n + \tfrac{1}{2}), \qquad E_n^e = \omega(n + \tfrac{1}{2}) + \frac{\omega_0}{2} + \chi(n + \tfrac{1}{2})$$

corresponding eigenstates $|g, n\rangle$ and $|e, n\rangle$.

Evoultion is just **(b)**

$$\hat{U}(t_0)(|g,0\rangle + |e,0\rangle) = |g,0\rangle - ie^{-i\omega_0 t_0}|e,0\rangle,$$
$$\hat{U}(t_0)(|g,1\rangle + |e,1\rangle) = |g,0\rangle + ie^{-i\omega_0 t_0}|e,0\rangle,$$

with $t_0 = \frac{\pi}{2}/\chi$. For coherent state **(c)**

$$\hat{U}(t_0)|g,0\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty} e^{-iE_n^g t}\frac{\alpha^n |g,n\rangle}{\sqrt{n!}}, \qquad \hat{U}(t_0)|e,0\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty} e^{-iE_n^e t}\frac{\alpha^n |e,n\rangle}{\sqrt{n!}}.$$

6.4 Cavities and Atoms

Solving equations

$$\begin{cases} \dot{c}^g_{n+1}(t) = -ig\sqrt{n+1}c^e_n(t) \\ \dot{c}^e_n(t) = -ig\sqrt{n+1}c^g_{n+1}(t) \end{cases}$$

We get for $c^e_n(0) = 1,\ c^g_{n+1}(0) = 0$

$$|\psi(t)\rangle = \cos(g\sqrt{n+1}t)|e,n\rangle - i\sin(g\sqrt{n+1}t)|g,n+1\rangle,$$

and density matrix

$$\rho(t) = \cos(g\sqrt{n+1}t)^2|e,n\rangle\langle e,n| - i\sin(g\sqrt{n+1}t)\cos(g\sqrt{n+1}t)|g,n+1\rangle\langle e,n| +$$
$$+ i\sin(g\sqrt{n+1}t)\cos(g\sqrt{n+1}t)|e,n\rangle\langle g,n+1| + \sin(g\sqrt{n+1}t)^2|g,n+1\rangle\langle g,n+1|$$

Taking a partial trace **(a)**

$$\rho(t) = \cos(g\sqrt{n+1}t)^2|n\rangle\langle n| + \sin(g\sqrt{n+1}t)^2|n+1\rangle\langle n+1|$$

Let us now assume that the cavity is in a state $\rho = \sum_n P_n|n\rangle\langle n|$ before the interaction with the atom.decomposed After the interaction with the atom, the state evolves to

$$\rho(t) = \sum_n P_n|n(t)\rangle\langle n(t)|$$

we could do this because the Hilbert space could be decomposed to pairs $|g,n\rangle$ and $|e,n\rangle$. Substituting $|n(t)\rangle = \cos(gt\sqrt{n+1})|e,n\rangle - i\sin(gt\sqrt{n+1})|g,n+1\rangle$ we have **(b)**

$$\rho(t) = \sum P'_n|n\rangle\langle n|, \qquad P'_n = \cos(gt\sqrt{n+1})^2P_n + \sin(gt\sqrt{n})^2P_{n-1}. \tag{3}$$

Let us now assume that atoms pass one after the other through the cavity. At each step the probability $P_n(m)$ could be expressed through the previous as in (3)

$$P_n(m) = \cos(gt\sqrt{n+1})^2P_n(m-1) + \sin(gt\sqrt{n})^2P_{n-1}(m-1), \quad \Leftrightarrow \quad \boldsymbol{P}(m) = \hat{M}\boldsymbol{P}(m-1) = \hat{M}^m\boldsymbol{P}(0),$$

with some matrix $\hat{M}$. So we could **(c)** calculate figure 1.