1 Specific heat of a BCS superconductor

As we have shown in the previous exercise, in the mean field approximation the BCS Hamiltonian describes non-interacting fermionic Bogoliubov quasiparticles with dispersion $E_{\mathbf{k}}$. Consequently, their average occupation at inverse temperature $\beta = 1/k_B T$ is given by the Fermi-Dirac distribution $\langle \gamma^\dagger_{\mathbf{k}\sigma}\gamma_{\mathbf{k}\sigma}\rangle = \frac{1}{\mathrm{e}^{\beta E_{\mathbf{k}}+1}} \equiv f_{\mathbf{k}}$. We have taken the chemical potential to be zero in this expression because the number of quasiparticles is not conserved. Note that, in contrast to the standard situation in thermodynamics, the energy levels $E_{\mathbf{k}}$ are themselves temperature-dependent because the superconducting order parameter $\Delta_{\mathbf{k}} = \Delta_{\mathbf{k}}(\beta)$ is a function of the temperature.

1. Starting from the expression (the factor 2 is due to spin degeneracy)

$$S = -2k_B \sum_{\mathbf{k}} [(1-f_{\mathbf{k}})\log(1-f_{\mathbf{k}}) + f_{\mathbf{k}} \log f_{\mathbf{k}}] \tag{1}$$

   for the entropy of a gas of non-interacting fermions, show that the specific heat of a BCS superconductor is given by

$$C_{\mathrm{BCS}} = 2\beta k_B \sum_{\mathbf{k}} \left(-\frac{\partial f_{\mathbf{k}}}{\partial E_{\mathbf{k}}}\right)\left(E^2_{\mathbf{k}} + \frac{\beta}{2}\frac{\mathrm{d}\Delta^2_{\mathbf{k}}}{\mathrm{d}\beta}\right). \tag{2}$$

2. Show that the specific heat vanishes exponentially for temperatures $T \ll T_c$, where T_c is the critical temperature at which superconductivity sets in. Determine the dimensionless value $\beta\Delta_0$, at which $C_{\mathrm{BCS}}(T)$ is just 1% of the specific heat $C_n(T) = \Omega\, N(0)2\pi^2/3 \cdot k_B^2 T$ in the normal state.

   > **Hint:** The sum over momenta can be approximated by an integral $\sum_{\mathbf{k}} \to \Omega\, N(0) \int \mathrm{d}\xi$ over energy relative to the Fermi level, with $N(0)$ the corresponding density of states per spin in the normal state. This amounts to assuming that only energies very close to the Fermi level contribute to the sum/integral, and thus the density of states is roughly constant.

3. Determine the value $\Delta C = C_{\mathrm{BCS}}(T_c^-) - C_n(T_c)$ of the jump in the specific heat just below T_c which is associated with the jump in $d\Delta^2/d\beta$ from zero above T_c to the finite value $8\pi^2(k_B T_c)^3/(7\zeta(3))$ just below T_c. Show that the resulting *relative* jump $\Delta C/C_n(T_c)$ in the specific heat of a BCS superconductor has a universal value $12/7\zeta(3) \simeq 1.43$, independent of microscopic details like the concrete value of T_c.

4. Consider now the BCS wavefunction at $T = 0$. The BCS ground-state energy is then

$$\Omega_g = \sum_k (\xi_k - \sqrt{\xi_k^2 + |\Delta_k|^2} + \frac{|\Delta_k|^2}{2E_k}) \tag{3}$$

Compute the zero-temperature isothermal compressibility $\kappa = -\frac{1}{V}\frac{\partial^2 \Omega_g}{\partial \mu^2}$ of the system. You may assume the gap to be momentum independent $\Delta_k = \Delta\, \theta(\hbar\omega_D - |\xi_k|)$ and $\Delta \ll \hbar\omega_D$.

Compare the compressibility with that of the ideal fermi gas, i.e. $\Delta = 0$.