14.1 Specific heat of a BCS superconductor

As we have shown in the previous exercise, in the mean field approximation the BCS Hamilto- nian describes non-interacting fermionic Bogoliubov quasiparticles with dispersion E_k. Conse- quently, their average occupation at inverse temperature β is given by the Fermi-Dirac distribution

$$\langle \hat{\gamma}_{k\sigma}^\dagger \hat{\gamma}_{k\sigma} \rangle = f_k = \frac{1}{e^{\beta E_k(\beta)}+1}, \qquad E_k = \sqrt{\xi_k^2 + \Delta_k^2(\beta)}.$$

Next we assume that $k_{\rm B} = 1$.

1. The specific heat. Starting from the expression

$$S = -2\sum_k \left((1-f_k)\ln(1-f_k) + f_k \ln f_k\right),$$

for the entropy of a gas of non-interacting fermions, we could find

$$C = T\frac{\partial S}{\partial T}\bigg|_V = -\beta\frac{\partial S}{\partial \beta} = 2\beta\sum_k \ln\left(\frac{f_k}{1-f_k}\right)\frac{\partial f_k}{\partial \beta}.$$

All that remains is to find

$$\frac{\partial f_k}{\partial \beta} = \left(\frac{1}{2E_k}\frac{d\Delta_k^2}{\beta} + \frac{E_k}{\beta}\right)\frac{\partial f_k}{\partial E_k},$$

thus

$$C = 2\beta\sum_k \left(-\frac{\partial f_k}{\partial E_k}\right)\left(E_k^2 + \frac{\beta}{2}\frac{d\Delta + k^2}{d\beta}\right).$$

2. Nernst's theorem. We can find low-temperature asymptotics to $C(\beta \to \infty)$

$$C = -2\beta V N(0)\int \frac{\partial f_k}{\partial E_k}\left(E_k^2 + \frac{\beta}{2}\frac{d\Delta^2}{d\beta}\right)d\xi.$$

For low temperatures we can neglect the temperature dependence for the $\Delta_k(\beta\to\infty) \approx 1.76 T_c$ and it is convenient to use the saddle-point method

$$C \approx 2VN(0)\Delta^2 e^{-\beta\Delta}\int e^{-(\beta\xi)^2/2\beta\Delta}\,d(\beta\xi) = 2\sqrt{2\pi}VN(0)(\beta\Delta)^{5/2}\frac{e^{-\beta\Delta}}{\beta},$$

which corresponds to the desired exponential decay.

$$\frac{C}{C_n} \approx (\beta\Delta)^{5/2}e^{-\beta\Delta}.$$

So $C/C_n = 0.01$ at $\beta\Delta \approx 10$ or $T \approx 0.2T_c$.

3. The specific heat jump. In a second order phase transition the jump in the specific heat just below T_c

$$\Delta C = -\beta^2\frac{d\Delta^2}{d\beta}\sum_k\frac{\partial f_k}{\partial E_k}\bigg|_{T_c-0} = 2VN(0)\beta^2\frac{d\Delta^2}{d\beta}\bigg|_{T_c-0}\int\frac{\partial f_k}{\partial E_k}\,d\xi = \frac{8\pi^2}{7\zeta(3)}N(0)VT_c,$$

with $d\Delta^2/d\beta|_{T_c-0} = 8\pi^2(T_c)^3/(7\zeta(3))$. The universal ratio is

$$\frac{\Delta C}{C_n(T_c)} = \frac{12}{7\zeta(3)} \approx 1.43.$$

4. Compressibility. Consider now the BCS wavefunction at $T=0$. The BCS ground-state energy is then

$$\Omega_{\rm g} = \sum_k\left(\xi_k - \sqrt{\xi_k^2 + |\Delta_k|^2} + \frac{|\Delta_k|^2}{2E_k}\right), \qquad \Delta_k = \Delta\theta(\hbar\omega_{\rm D} - |\xi_k|),$$

with Debye temperature $T_D = \hbar\omega_D \gg \Delta$. The zero-temperature isothermal compressibility

$$\kappa = -\frac{1}{V}\frac{\partial^2\Omega_{\rm g}}{\partial\mu^2}.$$

More precisely, the difference from the ideal Fermi gas

$$\Delta\kappa = -\frac{1}{V}\frac{\partial^2}{\partial\mu^2}\left(\Omega_{\rm g}^{\rm BCS} - \Omega_{\rm g}^{\rm FG}\right) = -\frac{1}{2V}\frac{\partial^2}{\partial\mu^2}\left(-N(\varepsilon_{\rm F})\Delta^2\right),$$

with $\Delta|_{T=0} \approx 2\hbar\omega_D\exp\left(-\frac{1}{\alpha N(\varepsilon_{\rm F})}\right)$. Due to we live in 3D $N(E) = \frac{\sqrt{2}}{\pi^2}m_{\rm e}^{3/2}\sqrt{E}$. Thus

$$\Delta\kappa = \frac{\Delta^2}{2V}\left(\frac{1}{\frac{\sqrt{2}}{\pi^2}m_{\rm e}^{3/2}\alpha^2\varepsilon_{\rm F}^{5/2}} - \frac{1}{2\alpha\varepsilon_{\rm F}^2} - \frac{\frac{\sqrt{2}}{\pi^2}m_{\rm e}^{3/2}}{2\varepsilon_{\rm F}^{3/2}}\right) > 0,$$

apparently because of Cooper pairs.