4.1 Feynman Path Integral of the Harmonic Oscillator

Consider propagator as

$$K \overset{\text{def}}{=} \langle q_f | e^{-i\hat{H}T} | q_i \rangle,$$

that we could rewrite in terms of the Feinman's integral

$$K = \int e^{iS[q(t)]} \mathcal{D}q(t),$$

with in particular action for the harmonic oscillator

$$S[q(t)] = \int_0^T \frac{m}{2} \left(\dot{q}^2 - \omega^2 q^2\right) \tag{1}$$

with boundary conditions $q(0) = q_i$ and $q(T) = q_f$.

(a) Writing the path as $q(t) = q_c(t) = y(t)$, due to $\delta S[q_c(t)] = 0$ we could rewrite S as

$$S[q(t)] = \int_0^T \frac{m}{2} \left((\dot{q}_c + \dot{y})^2 - \omega^2 (q_c + y)^2\right) dt = S[q_c(t)] + S[y(t)] + \int_0^T m \left(\dot{q}_c \, dy - \omega^2 q_c y \, dt\right) = S[q_c(t)] + S[y(t)].$$

It was used that Euler-Lagrange equation $\partial_x L - \frac{d}{dt} \partial_{\dot{x}} L = 0$ leads to classical equation of motion $\ddot{q}_c = -m\omega^2 q_c(t)$:

$$\int_0^T m \dot{q}_c \, dy = y(t)\dot{q}_c \Big|_0^T - \int_0^T m y \ddot{q}_c \, dt = \int_0^T m\omega^2 y q_c \, dt.$$

Thus we could factorise K

$$K = e^{iS[q_c(t)]} F(T), \qquad F(T) = \int e^{iS[y(t)]} \mathcal{D}y(t).$$

(b) Solving $\ddot{q}_c = -m\omega^2 q_c(t)$ with boundary conditions $q(0) = q_i$ and $q(T) = q_f$ we get

$$q_c(t) = A\cos(\omega t) + B\sin(t), \quad \Rightarrow \quad \begin{cases} q_i = B \\ q_f = A\sin(\omega T) + B\cos(\omega T) \end{cases} \quad \Rightarrow \quad A = \frac{q_f - q_i \cos(\omega T)}{\sin(\omega T)}, \quad B = q_i,$$

and substituting into the action (1)

$$S[q_c(t)] = \frac{m\omega}{2\sin(\omega T)} \left((q_i^2 + q_f^2)\cos(\omega T) - 2q_i q_f\right).$$

(c) The fluctuations can be expressed as a Fourier series

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t}{T}\right),$$

we go to the integration over $\prod_n a_n$. It is useful to calculate

$$\int_0^T \dot{y}^2 \, dt = \frac{T}{2} \sum_{n=1}^{\infty} \left(\frac{\pi n}{T}\right)^2 a_n^2, \qquad \int_0^T y^2 \, dt = \frac{T}{2} \sum_{n=1}^{\infty} a_n^2.$$

So we could find F as

$$F(T) \propto \int \exp\left(-\sum_{n=1}^{\infty} \alpha_n a_n^2\right) \prod_n da_n = \prod_{n=1}^{\infty} \sqrt{\frac{\pi}{\alpha_n}}, \qquad \alpha_n = \frac{m}{2i\hbar} \frac{T}{2} \left(\frac{\pi n}{T}\right)^2 \left(1 - \frac{\omega^2 T^2}{n^2 \pi^2}\right).$$

Ignoring all factors without ω, we have

$$F(T) = C \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2 T^2}{n^2 \pi^2}\right)^{-1/2} = C\sqrt{\frac{\omega T}{\sin(\omega T)}},$$

with some constant C that could be find from the free particle case $\omega \to 0$

$$\lim_{\omega \to 0} F(T) = C = \sqrt{\frac{m}{2\pi i\hbar T}}, \qquad \Rightarrow \qquad F(T) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega T)}}.$$

(d, e) Now we could calculate the partition function $Z = \operatorname{tr} e^{-\beta \hat{H}}$ after a Wick rotation to imaginary times $T = -i\beta$

$$Z = \int \langle x | e^{-\beta \hat{H}} | x \rangle \, dx = \int e^{iS[q_c(-i\beta)]} F(-i\beta) \, dx \overset{(1)}{=} \frac{1}{2\sinh\left(\frac{1}{2}\omega\beta\right)},$$

where in $\overset{(1)}{=}$ we calculated Gaussian integral

$$\int \exp(-\alpha x^2) \, dx = \sqrt{\frac{\pi}{\alpha}}, \qquad \alpha = \frac{im\omega\,(1 - \cos(\omega T))}{\sin(\omega T)}.$$

4.2 Grassmannian Algebra (I)

We know, that for the Grassmann number η

$$\int \eta\, d\eta = 1, \qquad \int d\eta = 0.$$

1. Interesting to note, that for $f(\eta) = a + b\eta$ $(a,\, b \in \mathbb{C})$ we have

   $$\{\eta,\, \partial_\eta\} f(\eta) = \{\eta,\, \textstyle\int d\eta\} f(\eta) = f(\eta).$$

   Enough to calculate

   $$\eta \textstyle\int d\eta f(\eta) = b\eta, \qquad \eta\partial_\eta f(\eta) = b\eta, \qquad \int d\eta\eta f(\eta) = a, \qquad \partial_\eta \eta f(\eta) = a.$$

2. As a next step we calculate

   $$\exp\left(\sum_j c_j\eta_j\right) = 1 + \sum_j c_j\eta_j + \sum_{j,k} c_j c_k \eta_j \eta_k = 1 + \sum_j c_j\eta_j + \sum_{j>k} c_j c_k (\eta_j\eta_k + \eta_k\eta_j) = 1 + \sum_j c_j\eta_j,$$

   with $c_j \in \mathbb{C}$. Actually it is the same as proof that $\sum_j c_j\eta_j$ is still Grassmann number by calculating anticommutative relations.

3. Finally, we could find integral

   $$\int d\bar{\eta} d\eta\; e^{-C\bar{\eta}\eta} = \int d\bar{\eta} d\eta \left(1 - C\bar{\eta}\eta + \tfrac{C^2}{2}\bar{\eta}\eta\bar{\eta}\eta + \ldots\right) = C \int d\bar{\eta} \left(\int d\eta\; \eta\bar{\eta}\right) = C,$$

   with $C \in \mathbb{C}$. It was used that $\bar{\eta}\eta\bar{\eta}\eta = -\bar{\eta}^2\eta^2 = 0$.