Quantum Many-Body Physics — Sheet 2
Lecturer: Prof. Dr. Michael Knap — Winter Term 2023
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1 Review on Second Quantization

Let us consider

$$|\alpha_1, \alpha_2, \dots, \alpha_N\rangle \equiv \frac{1}{\sqrt{N!}} \sum_{\text{perm } P \in S_N} \zeta^{\sigma(P)} \left|\alpha_{P(1)}\right\rangle \otimes \left|\alpha_{P(2)}\right\rangle \otimes \dots \otimes \left|\alpha_{P(N)}\right\rangle , \tag{1}$$

the N-particle completely (anti)-symmetric wavefunction, $\zeta = +1$ for bosons and $\zeta = -1$ for fermions respectively, single particle states $|\beta\rangle, |\alpha_k\rangle$ and where $\sigma(P)$ is the signature of the permutation P, defined as $\sigma(P) = 0, 1$ for even/odd permutations respectively. We define the creation operator via

$$a^\dagger_\beta |\alpha_1, \alpha_2, \dots, \alpha_N\rangle \equiv |\beta, \alpha_1, \alpha_2, \dots, \alpha_N\rangle . \tag{2}$$

1. Show that the adjoint of $a^\dagger_\beta$ is given by

$$a_\beta |\alpha_1, \alpha_2, \dots, \alpha_N\rangle = \sum_k \zeta^{k-1} \langle\beta|\alpha_k\rangle |\alpha_1, \dots, \alpha_{k-1}, \cancel{\alpha_k}, \alpha_{k+1}, \dots, \alpha_N\rangle , \tag{3}$$

> Hint: determine the inner-product of two (anti)-symmetric wave functions $|\alpha_1, \alpha_2, \dots, \alpha_N\rangle$. You might want to use that there are $N!$ permutations of N elements, and that the following relation holds between the signature σ of two different permutations P and Q: $\zeta^{\sigma(P)}\zeta^{\sigma(Q)} = \zeta^{\sigma(P^{-1})}\zeta^{\sigma(Q)} = \zeta^{\sigma(QP^{-1})}$. Use the definition of the adjoint of an operator to prove the claim for any (anti)-symmetric state.

2. Show that in particular, a_β and $a^\dagger_\beta$ fulfill the (anti)-commutation relations introduced in the lecture

$$\begin{aligned} [a_\alpha, a^\dagger_\beta] &\equiv a_\alpha a^\dagger_\beta - a^\dagger_\beta a_\alpha = \langle\alpha|\beta\rangle = \delta_{\alpha,\beta} \text{ for bosons,} \\ \{a_\alpha, a^\dagger_\beta\} &\equiv a_\alpha a^\dagger_\beta + a^\dagger_\beta a_\alpha = \langle\alpha|\beta\rangle = \delta_{\alpha,\beta} \text{ for fermions,} \end{aligned} \tag{4}$$

   if $\{|\alpha\rangle\}$ form an orthonormal basis of the Hilbert space of one particle.

> Hint: apply both $a_\alpha a^\dagger_\beta$ and $a^\dagger_\beta a_\alpha$ to an arbitrary many-body state $|\alpha_1, \alpha_2, \dots, \alpha_N\rangle$ and compare the expressions.

3. In the lecture we have showed that single particle operators of the form $\hat{V}^{(N)} = \sum_{i=1}^{N} \hat{V}^{(1)}$, where $\hat{V}^{(1)}$ acts on each particle i separately, can be expressed in terms of creation and annihilation operators as

$$\hat{V} = \sum_{\alpha,\beta} \langle\alpha|V^{(1)}|\beta\rangle a^\dagger_\alpha a_\beta \tag{5}$$

where α, β run over a complete set of quantum numbers. Derive the second quantized form of the density operator

$$\hat{\rho}(\mathbf{x}) = \sum_{\mathbf{i=1}}^{\mathbf{N}} \delta(\mathbf{x} - \hat{\mathbf{x}}_{\mathbf{i}}), \tag{6}$$

where $\mathbf{x}_i$ denotes the position of the i^{th} particle, in real space.

> Hint: In this case α, β do not take discrete but continuous values. Thus, we should replace the sums by integrals.

2 Mapping between Quantum and Classical Systems

For many physical systems there exists a mapping between a d dimensional quantum mechanical system and a corresponding classical partition function in $d+1$ spatial dimensions, as you have discussed for the path-integral formulation in the lecture. We want to establish this connection here by the example of the Ising model.
We start by discussing the 1 dimensional *classical* Ising model. This is defined by the partition function

$$\mathcal{Z}_{\mathrm{c}} = \sum_{\{s_j\}} e^{-H_{\mathrm{c}}} = \sum_{\{s_j\}} e^{K_{\mathrm{c}} \sum_j s_j s_{j+1} + h_{\mathrm{c}} \sum_j s_j}, \tag{7}$$

which is a sum over spin configurations $\{s_j\}$ on a 1D chain which we take to be periodic, $s_{N+1} = s_1$. The Ising interaction K_{c} and the magnetic field h_{c} contain implicitly the temperature.

1. We would like to rewrite this by introducing a *transfer matrix*

   $$T_{s_j, s_{j+1}} = T^a_{s_j} T^b_{s_j, s_{j+1}} = e^{K_{\mathrm{c}} s_j s_{j+1}} e^{h_{\mathrm{c}} s_j}. \tag{8}$$

   so that the partition function becomes:

   $$\mathcal{Z}_{\mathrm{c}} = T_{s_1, s_2} T_{s_2, s_3} \dots T_{s_{N-1}, s_N} T_{s_N, s_1} = \mathrm{Tr}\left(T^N\right). \tag{9}$$

   What is the form of the transfer matrix?

   > We can think of the transfer matrices T^a and T^b as 2×2 matrices where the indices corresponds to $s_j = \pm 1$ and $s_{j+1} = \pm 1$, respectively.

2. Our goal is now to reinterpret Eq. (9) as the partition function of a quantum system. It is a trace of a 2×2 matrix, thus we are looking for a system with two states, a single spin 1/2 particle. Thus, it is a system with one less spatial dimension (0 in this case), as promised.

   For a quantum system the partition function takes the form $\mathcal{Z}_{\mathrm{q}} = \mathrm{Tr}\, e^{-\beta H}$, where H is the Hamiltonian and β is the inverse temperature (unrelated to the temperature of the classical system!). To find the quantity that corresponds to the transfer matrix we need to take the n-th root: $\mathcal{Z}_{\mathrm{q}} = \mathrm{Tr}\left(e^{-\frac{\beta}{N} H}\right)^N$, where N is now an arbitrary integer. Let us assume that the Hamiltonian is the sum of two non-commuting matrices $H = H_1 + H_2$. If N is sufficiently large then we can expand the exponential as $e^{-\frac{\beta}{N}H} = e^{-\frac{\beta}{N}H_1} e^{-\frac{\beta}{N}H_2} + O(1/N)$. Therefore we are looking for 2×2 Hamiltonians H_1 and H_2 such that $e^{-\frac{\beta}{N}H_1} = T^a$ and $e^{-\frac{\beta}{N}H_2} = T^b$.

All 2×2 hermitian matrices can be written as linear combinations of the identity $\mathbb{1}$ and the three Pauli matrices σ^x, σ^y and σ^z. Find H_1 and H_2 corresponding to the classical Ising model.

Note the following two equalities:

$$e^{a\sigma^x} = \begin{pmatrix} \cosh a & \sinh a \\ \sinh a & \cosh a \end{pmatrix}. \qquad e^{b\sigma^z} = \begin{pmatrix} e^b & 0 \\ 0 & e^{-b}. \end{pmatrix} \tag{10}$$

The construction above can be generalized to higher dimensions to relate a d dimensional quantum system to a $d+1$ dimensional classical one. For example in $d = 1$ the quantum Hamiltonian might be a $2^L \times 2^L$ matrix acting on L spins. The corresponding transfer matrix in the classical system also has size $2^L \times 2^L$ and it acts on adjacent rows of a two-dimensional classical spin configuration.