5.1 Rotating-Wave Approximation in the Jaynes-Cummings Model

Consider a resonator with frequency ω and a two-level system with frequency ω_0 Hamiltonian

$$H = H_0 + H_{\text{int}}, \qquad H_0 = \hbar\omega\hat{a}^\dagger\hat{a} - \hbar\omega_0\tfrac{1}{2}\sigma_z, \qquad H_{\text{int}} = \hbar g(\hat{\sigma}^+ + \hat{\sigma}^-)(\hat{a}^\dagger + \hat{a}).$$

After interaction transformation $|\psi\rangle = \hat{U}^\dagger \left|\tilde{\psi}\right\rangle$ with $\hat{U} = e^{i\hat{H}_0 t/\hbar}$ we have (**a**)

$$H_{\text{I}} = \hat{U}H_{\text{int}}\hat{U}^\dagger = \hbar g\left(\hat{a}\hat{\sigma}^+ e^{i(\omega_0-\omega)t} + \text{h.c.}\right) + \hbar g\left(\hat{a}^\dagger\hat{\sigma}^+ e^{i(\omega_0+\omega)t} + \text{h.c.}\right),$$

but after rotating-wave approximation (**b**), which is goog with $\omega_0 \sim \omega \gg g$,

$$H_{\text{I}} = \hat{U}H_{\text{int}}\hat{U}^\dagger = \hbar g\left(\hat{a}\hat{\sigma}^+ e^{i(\omega_0-\omega)t} + \text{h.c.}\right)$$

Using the approximated Hamiltonian we could perform the inverse transformation (**c**) as

$$\hat{H} = \hat{H}_0 + \hat{U}^\dagger H_{\text{I}}\hat{U} = H_0 + \hbar g(\hat{a}\hat{\sigma}^+ + \text{h.c.}).$$

Transfering this Hamiltonian to the rotating frame of the resonator (**d**) we have

$$\hat{U}_{\text{R}} = e^{i\omega t\hat{a}^\dagger\hat{a}}, \qquad \Rightarrow \qquad H_{\text{I,R}} = -\hbar\omega_0\tfrac{1}{2}\sigma_z + \hbar g\left(\hat{a}\hat{\alpha}^+ e^{-i\omega t} + \text{h.c.}\right) = \hbar\omega\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\end{pmatrix} + \hbar g\begin{pmatrix} 0 & \hat{a}^\dagger e^{i\omega t} \\ \hat{a}e^{-i\omega t} & 0\end{pmatrix}.$$

In resonance case (the non-resonant case is more non-trivial) we could fill the resonator with n photons (**e**) by applying $\hat{\sigma}_x$. We could start with $|\text{e}, n\rangle$ and consider reduced two-dimensional Hamiltonian that evol $|\text{e}, n\rangle$ to the $|\text{g}, n+1\rangle$ by Rabi oscillations:

$$|\psi(t)\rangle = c_{g,n+1}(t)\,|\text{g}, n+1\rangle + c_{e,n}(t)\,|\text{e}, n\rangle,$$

with $\delta = \omega_0 - \omega$

$$\tfrac{d}{dt}c_{g,n+1}(t) = -ig\sqrt{n+1}e^{i\delta t}c_{e,n}(t),$$
$$\tfrac{d}{dt}c_{e,n}(t) = -ig\sqrt{n+1}e^{-i\delta t}c_{g,n+1}(t).$$

Thus we have Rabi oscillations with

$$\Omega_{\text{R, n}} = \sqrt{g^2(n+1) + \left(\tfrac{\delta}{2}\right)^2} \overset{\delta=0}{=} g\sqrt{n+1}.$$

To pump n photons we need do π-pulse (apply $\hat{\sigma}_x$) after T_n, with

$$T_n = \frac{\pi}{g\sqrt{n+1}}.$$

5.2 Jaynes-Cummings Hamiltonian

Consider the Hamiltonian of the Jaynes-Cummings

$$\hat{H} = \begin{pmatrix} 0 & 0 \\ 0 & \hbar\omega_0\end{pmatrix} + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g\begin{pmatrix} 0 & \hat{a}^\dagger \\ \hat{a} & 0\end{pmatrix}.$$

Note that (**a**)

$$\hat{\sigma}^+\,|g\rangle = |e\rangle, \qquad \hat{\sigma}^-\,|e\rangle = |g\rangle.$$

Eigenstates and eigenvalues could be expressed as $|g, n\rangle$ with energy $\hbar\omega n$ and $|e, n\rangle$ with energy $\hbar\omega n + \hbar\omega_0$. As it was noticed in (5.1.e) the nonzero matrix elements of the Hamiltonian coulples (**c**) only $|g, n+1\rangle$ and $|e, n\rangle$. The Hamiltonian $\hat{H}$ could be expressed as (**d**)

$$\langle \text{g}, n+1|\hat{H}|\text{g}, n+1\rangle = \hbar\omega n + \hbar\omega, \qquad \langle \text{e}, n|\hat{H}|\text{e}, n\rangle = \hbar\omega n + \hbar\omega_0, \qquad \langle \text{e}, n|\hat{H}|\text{g}, n+1\rangle = \hbar g\sqrt{n+1}.$$

In resonance (**e**) we have (in the interaction picture)

$$\tfrac{d}{dt}c_{g,1}(t) = -igc_{e,0}(t),$$
$$\tfrac{d}{dt}c_{e,0}(t) = -igc_{g,1}(t),$$

so evolution could be expressed as

$$|\psi(t)\rangle = c_{g,1}(t)\,|\text{g}, 1\rangle + c_{e,0}(t)\,|\text{e}, 0\rangle,$$

with coefficients

$$c_{g,1}(t) = e^{-i(\omega t - \pi/2)}\sin(gt), \qquad c_{e,0}(t) = e^{-i\omega t}\cos(gt),$$

in the moment $t_{\pi/2} = \frac{\pi/2}{g}$ we have $|c_{e,0}(t_{\pi/2})|^2 = |c_{g,1}(t_{\pi/2})|^2 = 1/2$.

5.3 Collapse and Revival in the Jaynes-Cummings model

Consider $H_{\text{JC}} = H_0 + H_{\text{int}}$ with $H_{\text{int}} = \hbar g\left(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^\dagger\right)$. In the interaction picture we have (5.1), that in the resonance case could be written as

$$\hat{H}_{\text{I}} = \hbar g \begin{pmatrix} 0 & \hat{a}^\dagger \\ \hat{a} & 0 \end{pmatrix}.$$

We could calculate state evolution by $\hat{U}(t) = e^{-iH_{\text{I}}t/\hbar}$. Thus we need

$$\hat{h} = \begin{pmatrix} 0 & \hat{a}^\dagger \\ \hat{a} & 0 \end{pmatrix}, \quad \hat{h}^2 = \begin{pmatrix} \hat{n} & 0 \\ 0 & \hat{n}+1 \end{pmatrix}, \quad \Rightarrow \quad \hat{h}^{2k} = \begin{pmatrix} \hat{n}^k & 0 \\ 0 & (\hat{n}+1)^k \end{pmatrix}, \quad \hat{h}^{2k+1} = \begin{pmatrix} 0 & \hat{a}^\dagger(\hat{n}+1)^k \\ (\hat{n}+1)^k a & 0 \end{pmatrix},$$

with $\hat{a}^\dagger\hat{a} = n$. Matrix elements could be expressed as

$$\langle \mathrm{g}|\hat{U}(t)|\mathrm{g}\rangle = \sum_{k=0}^{\infty}(gt)^{2k}\frac{(-i)^{2k}}{(2k)!}n^k = \sum_{k=0}^{\infty}(gt)^{2k}\frac{(-1)^k}{(2k)!}(\sqrt{n})^{2k} = \cos(gt\sqrt{n}),$$

$$\langle \mathrm{e}|\hat{U}(t)|\mathrm{e}\rangle = \sum_{k=0}^{\infty}(gt)^{2k}\frac{(-i)^{2k}}{(2k)!}(n+1)^k = \ldots = \cos(gt\sqrt{n+1}),$$

$$\langle \mathrm{g}|\hat{U}(t)|\mathrm{e}\rangle = -i\hat{a}^\dagger\sum_{k=0}^{\infty}(gt)^{2k+1}\frac{(-i)^{2k}}{(2k+1)!}(\sqrt{n+1})^{2k} = -i\hat{a}^\dagger\frac{\sin(gt\sqrt{n+1})}{\sqrt{n+1}}.$$

Starting with the exited state and the cavity in a coherent state

$$|\psi(t)\rangle = \hat{U}(t)\,|\mathrm{e},\alpha\rangle = \sum_{n=1}^{\infty} c_{g,n}\,|\mathrm{g},n\rangle + \sum_{n=0}^{\infty} c_{e,n}\,|\mathrm{e},n\rangle\,,$$

with

$$c_{g,n+1} = -i\frac{e^{-|\alpha|^2/2}\alpha^n}{\sqrt{n!}}\sin(gt\sqrt{n+1}), \qquad c_{e,n} = \frac{e^{-|\alpha|^2/2}\alpha^n}{\sqrt{n!}}\cos(gt\sqrt{n+1}),$$

so $|c_{g,n+1}|^2 + |c_{e,n}|^2 = \frac{1}{n!}e^{-|\alpha|^2}\alpha^{2n}$.

The probability to find the two-level system in the excited state could be expressed as

$$P_e = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty}|c_{e,n}|^2 = \sum_{n=0}^{\infty}\frac{e^{-|\alpha|^2}\alpha^{2n}}{n!}\cos(gt\sqrt{n+1})^2.$$

Assumin $\bar{n} = |\alpha|^2 \gg 1$ we can estimate Rabi-frequency oscillations as $\Omega_{\text{R}} = g\sqrt{n}$. We can take $\Omega_n = g\sqrt{n+1}$ to be a distributed between $[|\alpha|^2 - \alpha, |\alpha|^2 + \alpha]$ ($\Delta n = \sqrt{\bar{n}}$), thus collapse time could be estimated from decoherence time

$$t_{\text{c}} \sim \frac{\pi}{g}\left(\sqrt{\bar{n} + \sqrt{\bar{n}}} - \sqrt{\bar{n} - \sqrt{\bar{n}}}\right)^{-1} \sim \frac{\pi}{g}.$$

To calculate revival time we need states to interfere constructively

$$t_{\text{r}} \sim m\frac{\pi}{g}\left(\sqrt{\bar{n}} - \sqrt{\bar{n}-1}\right)^{-1} \sim m\frac{2\pi\sqrt{\alpha}}{g}, \qquad m = 1, 2, \ldots$$

but by numerical calculations I have $t_{\text{r}} \sim 4\pi\frac{\sqrt{\alpha}}{g}$, so maybe I lost factor 2 somewhere.