8.1 Effective action of a condensate in a double well

The following Hamiltonian is a simple model of a condensate in two wells:

$$H = -\frac{g}{2}\sum_{\langle i,j\rangle} a_i^\dagger a_j + \frac{U}{4}\sum_j n_j(n_j-1), \tag{1}$$

with $j \in \{1,2\}$. Consider a system with in total $2N$ particles. After normal ordering $[a_i, a_j^\dagger] = \delta_{ij}$

$$H(a^\dagger, a) = -\frac{g}{2}\sum_{\langle i,j\rangle} a_i^\dagger a_j + \frac{U}{4}\sum_j a_j^\dagger a_j^\dagger a_j a_j.$$

Non-interacting case. Let's start with $U=0$ and operator canonical transformation (Fourier transform)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

which automatically satisfies the commutation relations $[a_j, a_j^\dagger] = \sin(\alpha)^2 + \cos(\alpha)^2 = 1$. Substituting into the Hamiltonian, we find the condition for diagonalization

$$\cos(\alpha)^2 - \sin(\alpha)^2 = 0, \qquad \overset{\alpha=\pi/4}{\Rightarrow} \qquad a_{1,2} = \tfrac{1}{\sqrt{2}}(b_1 \pm b_2),$$

and the Hamiltonian

$$H = -\frac{g}{2}\sum_{\langle i,j\rangle} a_i^\dagger a_j = \tfrac{g}{2} b_1^\dagger b_1 - \tfrac{g}{2} b_2^\dagger b_2,$$

with ground state $|0, 2N\rangle_b$. Define $|n\rangle_b \overset{\text{def}}{=} |n, 2N-n\rangle_b$. Now let's find the δN as

$$\delta N = a_2^\dagger a_2 - a_1^\dagger a_1 = -b_2^\dagger b_1 - b_1^\dagger b_2,$$
$$(\delta N)^2 = b_1^\dagger b_1 + b_2^\dagger b_2 + 2 b_2^\dagger b_1^\dagger b_1 b_2 = 2N + 4nN - 2n^2.$$

We immediately see that in the ground state

$$\langle \delta N^2 \rangle_{\text{gs}} = 2N. \tag{2}$$

Note that in the limit of large N the temperature correction will be

$$\tfrac{1}{N}\langle \delta N^2 \rangle = 2 + 4e^{-\beta g},$$

regardless of N. To calculate this we can start with the partition function

$$Z = \sum_{n=0}^{2N} e^{-\beta E_n} = \left(e^{g\beta/2} + e^{-g\beta/2}\right)^{2N} = \frac{e^{\beta g(N+1)} - e^{-\beta g N}}{e^{\beta g} - 1},$$

with $E_n = -g(N-n)$, and find $\langle n \rangle$ and $\langle n^2 \rangle$ through

$$\langle N - n \rangle = \frac{1}{\beta}\frac{1}{Z}\frac{\partial Z}{\partial g} = T\partial_g \ln Z, \qquad \langle (N-n)^2 \rangle = \frac{1}{\beta^2}\frac{1}{Z}\frac{\partial^2 Z}{\partial g^2}.$$

Imaginary-time action. The imaginary-time action associated with this Hamiltonian in the coherent state representation

$$S = \int_0^\beta d\tau\ \bar\psi \partial_\tau \psi + H(\bar\psi, \psi) = \int_0^\beta d\tau\ \bar\psi\partial_\tau\psi - \frac{g}{2}\sum_{\langle i,j\rangle}\bar\psi_i\psi_j + \frac{U}{4}\sum_j \bar\psi_j\bar\psi_j\psi_j\psi_j.$$

Consider the density-phase representation given by

$$\psi_1 = \sqrt{N + \frac{\delta N}{2}} e^{i\varphi_1}, \qquad \psi_2 = \sqrt{N - \frac{\delta N}{2}} e^{i\varphi_2}.$$

The action than

$$S \overset{\text{def}}{=} \int_0^\beta d\tau\ \mathcal{L}(\varphi, \theta) = \int_0^\beta d\tau\ 2Ni\dot\theta + \frac{\delta N}{2} i\dot\varphi - g\sqrt{N^2 - \left(\frac{\delta N}{2}\right)^2}\cos\varphi + 2\frac{U}{4}\left(\frac{\delta N}{2}\right)^2 + \frac{U}{2}N^2,$$

with $\varphi = \varphi_1 - \varphi_2$ and $\theta = \frac{1}{2}(\varphi_1 + \varphi_2)$. We can find the physical observables that are canonical conjugates to φ and θ

$$P_\varphi = \frac{\partial \mathcal{L}}{i\partial\dot\varphi} = \frac{\delta N}{2}, \qquad P_\theta = \frac{\partial\mathcal{L}}{i\partial\dot\theta} = 2N,$$

with i factor from Wick rotation $\tau \to -it$ (it seems to me).

We can immediately see from Noether's theorem how symmetry in θ leads to conservation of $P_\theta = 2N = \text{const}$. And indeed $\mathcal{L}(\theta) = \mathcal{L}(\theta + \text{shift}) - U(1)$ symetry. On the other hand $\mathcal{L}(\varphi) \neq \mathcal{L}(\varphi + \text{shift})$, which corresponds to non-conservation of the $P_\varphi = \delta N$.

Effective action. Expanding the action to quadratic order in the particle number fluctuations $\delta N/N$ and the relative phase φ and neglecting constant terms

$$S_{\text{eff}}(\varphi, P_\varphi) = \int_0^\beta d\tau \; iP_\varphi \partial_\tau \varphi + \tfrac{1}{2} gN\varphi^2 + \tfrac{1}{4}(U + g/N)P_\varphi^2.$$

The fluctuations of the relative particle number between the wells $(\delta N)^2$ could be found as previous through the partition function

$$Z = \int D[\varphi, \delta N] e^{-S}.$$