10.1 Phase Space Argument for the Life Time of Quasi-particles

1. Exact expression. Consider the Coulomb interaction in second quantization between electrons:

$$\hat{V} = \frac{1}{2\mathcal{V}} \sum_{\sigma\sigma'} \sum_{kk'q} V(q) c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} c_{k,\sigma},$$

$\mathcal{V}$ is the volume. Taking into account only scattering processes that involve two particles, we could derive an expression for the inverse life time $1/\tau_k$ of the state[1] $|i\rangle = |k_1,\sigma_1\rangle \overset{\text{n}}{=} c^\dagger_{k_1,\sigma_1} |\Omega\rangle$, where $|\Omega\rangle$ denotes the state, where all states below Fermi surface are occupied.

With Fermi's Golden Rule (fig. 1b)

$$\frac{1}{\tau_{k_1}} = 2\pi \sum_f |\langle f|\hat{V}|k,\sigma\rangle|^2 \delta(\varepsilon_i - \varepsilon_f), \qquad |f\rangle \overset{\text{n}}{=} c^\dagger_{k_1-Q,\sigma_1} c^\dagger_{k_2+Q,\sigma_2} c_{k_2,\sigma_2} |\Omega\rangle .$$

Thus life time could be expressed from the matrix elements

$$\langle i|\hat{V}|f\rangle = \frac{1}{2\mathcal{V}} \sum_{\sigma\sigma'} \sum_{k'q} \mathcal{N}_{i,f} V(q) \langle \Omega| c_{k_1,\sigma_1} c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k',\sigma'} c_{k,\sigma} c^\dagger_{k_1-Q,\sigma_1} c^\dagger_{k_2+Q,\sigma_2} c_{k_2,\sigma_2} |\Omega\rangle,$$

with normalizing factor

$$\mathcal{N}_{i,f} = \left((1-n_{k_1,\sigma_1})(1-n_{k_1-Q,\sigma_1})(n_{k_2,\sigma_2})(1-n_{k_2+Q,\sigma_2})\right)^{-1/2},$$

since $\langle c^\dagger c\rangle = n$ and $\langle cc^\dagger\rangle = 1-n$.

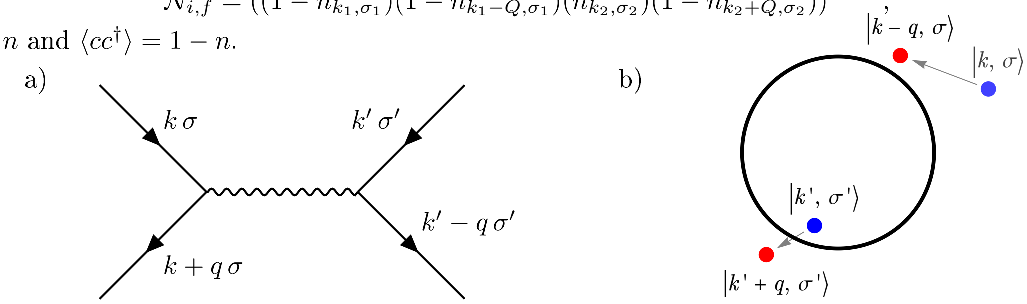

Figure 1: Scattering process

To calculate this matrix element we need just to decide how to distribute $|k_1,\sigma_1\rangle$, $|k_2,\sigma_2\rangle$, $|k_1-Q,\sigma_1\rangle$, $|k_2+Q,\sigma_2\rangle$ over the $|k,\sigma\rangle$, $|k',\sigma'\rangle$, $|k'-q,\sigma'\rangle$, $|k+q,\sigma\rangle$ (fig. 1a). There are only two different ways to do this (sign could be found as in the Wick's theorem):

$$\begin{aligned}\langle i|\hat{V}|f\rangle_1 &= \frac{\mathcal{N}_{i,f}}{2\mathcal{V}} \sum_{k,k',q} \sum_{\sigma_1,\sigma_2} V(q) \delta_{\sigma_1,\sigma} \delta_{\sigma_2,\sigma'} \delta(k_1-k-q)\delta(Q-q)(1-n_{k_1})n_{k_2}(1-n_{k_1-Q})(1-n_{k_2+Q}) \\ &= \frac{\mathcal{N}_{i,f}}{2\mathcal{V}} V(Q)(1-n_{k_1,\sigma_1})n_{k_2,\sigma_2}(1-n_{k_2+Q,\sigma_2})(1-n_{k_1-Q,\sigma_1}),\end{aligned}$$

and

$$\langle i|\hat{V}|f\rangle_2 = \ldots = -\frac{\mathcal{N}_{i,f}}{2\mathcal{V}} V(k_1-k_2-Q)\delta_{\sigma_1,\sigma_2}(1-n_{k_1,\sigma_1})n_{k_2,\sigma_2}(1-n_{k_2+Q,\sigma_2})(1-n_{k_1-Q,\sigma_1}).$$

Due to symmetry, each term will appear twice, which means

$$\langle i|\hat{V}|f\rangle = \frac{\mathcal{N}_{i,f}}{\mathcal{V}} V(Q)(1-\delta_{\sigma_1,\sigma_2})(1-n_{k_1,\sigma_1})(1-n_{k_1-Q,\sigma_1})(n_{k_2,\sigma_2})(1-n_{k_2+Q,\sigma_2}) = \frac{1-\delta_{\sigma_1,\sigma_2}}{\mathcal{V}\mathcal{N}_{i,f}} V(Q).$$

As expected, we found that only particles with different spins are scattered.

Let's move on to integration

$$\frac{1}{\tau_{k_1}} = 2\pi \sum_f |\langle f|\hat{V}|k,\sigma\rangle|^2 \delta(\varepsilon_i-\varepsilon_f) = \frac{2\pi}{\mathcal{V}^2} \int \frac{d^3k_2\, d^3Q}{(2\pi)^6} |V(Q)|^2 \mathcal{N}^{-2}_{i,f} \delta(\varepsilon_i-\varepsilon_f) \sum_{\sigma_2} (1-\delta_{\sigma_1,\sigma_2}),$$

with $\varepsilon_i - \varepsilon_f = \varepsilon_{k_1} - \varepsilon_{k_1-Q} - \varepsilon_{k_2+Q} + \varepsilon_{k_2}$.

2. Approximate calculation. By anticipating the result for the screening of the Coulomb interaction, we can assume that $V(Q) \approx V(0) = \text{const}$. It is convenient to use (with $\xi = \varepsilon - \varepsilon_F$ and $d\boldsymbol{k} = d\cos\theta\, d\varphi$)

$$\int \frac{d^3k}{(2\pi)^3} = \int \frac{d\boldsymbol{k}}{4\pi} \int d\xi\, N(\xi) = N(0) \int \frac{d\boldsymbol{k}}{4\pi} \int d\xi.$$

We consider system at $T=0$:

$$(1-n_{k_1,\sigma_1})(1-n_{k_1-Q,\sigma_1})(n_{k_2,\sigma_2})(1-n_{k_2+Q,\sigma_2}) = \theta(\xi_{k_1})\theta(-\xi_{k_2})\theta(\xi_{k_1-Q})\theta(\xi_{k_2+Q}).$$

[1] Here and further $\overset{\text{n}}{=}$ means that we ignore normalization factor, that appears in $\mathcal{N}_{i,f}$.

Substituting this into the expression for the lifetime, we find

$$\frac{1}{\tau_{k_1}} \approx \frac{2\pi}{\mathcal{V}^2}|N(0)|^2 V(0)^2 \int d\xi_{k_2} \int \frac{d\boldsymbol{k}_{k_2}}{4\pi} \int d\xi_Q \int \frac{d\boldsymbol{k}_Q}{4\pi} \theta(\xi_{k_1})\theta(-\xi_{k_2})\theta(\xi_{k_1-Q})\theta(\xi_{k_2+Q})\delta(\varepsilon_{k_1} - \varepsilon_{k_1-Q} - \varepsilon_{k_2+Q} + \varepsilon_{k_2}).$$

Let's define $k_f = k_1 - Q$. As in the fig. 1b we take $\xi_{k_1} > 0$ and $\xi_{k_2+Q} > 0$

$$\delta(\varepsilon_{k_1} - \varepsilon_{k_f} - \varepsilon_{k_1+k_2-k_f} + \varepsilon_{k_2}) = \frac{1}{2\varepsilon_F}\delta\left(1 + \tilde{\boldsymbol{k}}_1 \cdot \tilde{\boldsymbol{k}}_2 - \tilde{\boldsymbol{k}}_1 \cdot \tilde{\boldsymbol{k}}_f - \tilde{\boldsymbol{k}}_2 \cdot \tilde{\boldsymbol{k}}_f\right),$$

with $\tilde{\boldsymbol{k}} = \boldsymbol{k}/k$. Rewriting the integral for the last time

$$\frac{1}{\tau_{k_1}} \approx \frac{2\pi}{\mathcal{V}^2}|N(0)|^2 V(0)^2 \int_{-\xi_{k_1}}^{0} d\xi_{k_2} \int_0^{\xi_{k_1}-\xi_{k_2}} d\xi_{k_f} \int \frac{d\boldsymbol{k}_{k_2}}{4\pi} \int \frac{d\boldsymbol{k}_Q}{4\pi} \frac{1}{2\varepsilon_F}\delta\left(1 + \tilde{\boldsymbol{k}}_1 \cdot \tilde{\boldsymbol{k}}_2 - \tilde{\boldsymbol{k}}_1 \cdot \tilde{\boldsymbol{k}}_f - \tilde{\boldsymbol{k}}_2 \cdot \tilde{\boldsymbol{k}}_f\right),$$

so finally

$$\frac{1}{\tau_{k_1}} \propto (\varepsilon_{k_1} - \varepsilon_F)^2.$$

10.2 Microscopic Basis of the Fermi-liquid Theory

Fermi liquid theory only holds if the ground state of the interacting system is connected adiabatically to the non-interacting Fermi sea. One can treat this as turning on the interactions adiabatically. The ground state $|\text{gs}\rangle$ of the full system and the excitation state $|\boldsymbol{k},\sigma)$ can then be written as

$$|\text{gs}\rangle = U|\Omega\rangle, \qquad |\boldsymbol{k},\sigma) = U|\boldsymbol{k},\sigma\rangle = U c^\dagger_{\boldsymbol{k},\sigma}|\Omega\rangle.$$

The time evolution operator in the interaction picture can be expressed as a time-ordered exponential

$$U = T\left\{e^{-i\int_{-\infty}^0 \hat{V}(t)\,dt}\right\}.$$

The quasi-particle creation operator (№3) could be expressed from the

$$|\boldsymbol{k},\sigma) = a^\dagger_{\boldsymbol{k},\sigma}|\text{gs}\rangle = U c^\dagger_{\boldsymbol{k},\sigma}|\Omega\rangle = U c^\dagger_{\boldsymbol{k},\sigma}U^{-1}|\varphi\rangle, \qquad \Rightarrow \qquad a^\dagger_{\boldsymbol{k},\sigma} = U c^\dagger_{\boldsymbol{k},\sigma}U^{-1}.$$

For Fermi liquid theory to be valid, one need to add requirement on the wavefunction renormalization constant

$$Z_k = |(k\sigma|c^\dagger_{k\sigma}|\text{gs}\rangle|^2 = |\langle\text{gs}|a_{k\sigma}c^\dagger_{k\sigma}|\text{gs}\rangle|^2.$$

Expressing $c^\dagger_{k\sigma}$ as a series in the quasi-particle operators

$$\begin{aligned} c^\dagger_{k\sigma} &= U^\dagger a^\dagger_{k\sigma}U \approx \left(1 + i\int_{-\infty}^0 \hat{V}(t)\,dt\right)a^\dagger_{k\sigma}\left(1 - i\int_{-\infty}^0 \hat{V}(t)\,dt\right) = a^\dagger_{k\sigma} + i\int_{-\infty}^0 [\hat{V}(t), a^\dagger_{k\sigma}]\,dt + O(V^2) \\ &= \sqrt{Z_k}\,a^\dagger_{k\sigma} + \text{higher order} \end{aligned}$$

If we want to $c^\dagger_{k\sigma} \approx a^\dagger_{k\sigma}$, then (№4) $0 < \sqrt{Z_k} < 1$.

The spectral function (fig. 2)

$$A(k,\omega) = -\frac{1}{\pi}\operatorname{Im}G^{\text{ret}}(k,\omega) = \sum_\lambda |M_\lambda|^2\delta(\omega - \xi_\lambda), \qquad |M_\lambda|^2 = |\langle\lambda|c^\dagger_{k,\sigma}|\text{gs}\rangle|^2$$

exhibits a sharp quasiparticle peak at ξ_k

$$A(k,\omega) = Z_k\delta(\omega - \xi_k) + \ldots$$

with $Z_k > 0$. Thus momentum distribution $\langle\hat{n}_{k\sigma}\rangle$ has a jump at the Fermi momentum

$$\langle\hat{n}_{k\sigma}\rangle = \langle\text{gs}|c^\dagger_{k\sigma}c_{k\sigma}|\text{gs}\rangle = \int A(k,\omega)n_f(k,\omega)\,d\omega = \int Z_k\delta(\omega - \varepsilon_k)\theta(\varepsilon_f - \theta)\,d\omega + \ldots = Z_k\theta(\varepsilon_F - \varepsilon_k) + \ldots$$

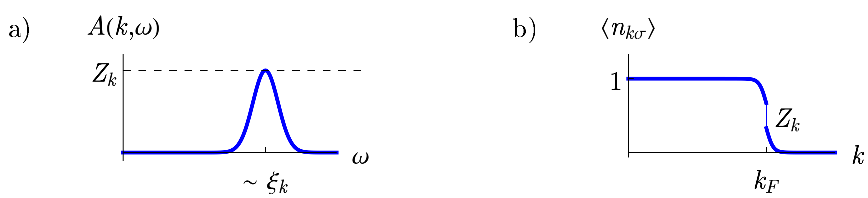

Figure 2: a) The spectral function. b) The momentum distribution.