1 The Transverse Field Ising Model

Aim of this exercise is to introduce the Transverse Field Ising Model (TFIM) and the Jordan Wigner transformations, which allows to solve it exactly.
Let us consider the Hamiltonian

$$\hat{H} = -J\sum_{i=1}^{L} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + h\sum_{i=1}^{L} \hat{\sigma}_i^z \tag{1}$$

where the parameters are $J, h > 0$. σ_i^a are the spin$-\frac{1}{2}$ operators in the a direction, i the site index and L the total number of sites of the underlying 1D lattice. We impose periodic boundary condition (PBC) by defining $\hat{\sigma}_L^x \hat{\sigma}_{L+1}^x = \hat{\sigma}_L^x \hat{\sigma}_1^x$.

1. Let us consider the mapping of spins to bosonic operators

$$\hat{\sigma_i^x} = \hat{b}_i + \hat{b}_i^\dagger \qquad \hat{\sigma_i^y} = i(\hat{b}_i - \hat{b}_i^\dagger) \qquad \hat{\sigma_i^z} = (1 - 2\hat{b}_i^\dagger \hat{b}_i). \tag{2}$$

   (a) Bosons as defined above are an example of "hard-core bosons". Compute their commutation and anticommutation relations. Why are they peculiar?

   (b) The previous result suggests that the most natural transformation is fermionic. We introduce the fermions $\hat{c}$

$$\hat{b}_i = \hat{K}_i \hat{c}_i \qquad \hat{K}_i = \prod_{j=1}^{i-1}(1 - 2\hat{n}_i) \tag{3}$$

   Show that if $\hat{c}$ are fermions then $\hat{b}$ satisfies the commutation and anticommutation relations as in the previous point.

   (c) Find the corresponding transformation from spins to fermions and rewrite the Hamiltonian in terms of the fermions. This is the *Jordan Wigner transformation of the TFIM*. Is the number of fermions conserved? Why?

   > **Hint**: Mind the PBC term: after the transformation in fermions, you should obtain antiperiodic boundary conditions if there is an even number of fermions, periodic boundary conditions if there is an odd number of fermions.

2. Diagonalization of the Hamiltonian.

   (a) Perform a Fourier Transform of the fermionic operators

$$\hat{c}_k = \frac{1}{\sqrt{L}}\sum_r e^{ikr}\hat{c}_r \qquad \hat{c}_r = \frac{1}{\sqrt{L}}\sum_r e^{-ikr}\hat{c}_k. \tag{4}$$

Hint: The parity of the fermions is a conserved quantity (namely, if there is an even or odd number of fermions). You can define two distinct sectors of the Hilbert space, the even and odd sector, and define two different sets of k momenta for the Fourier transform such that the APBC and PBC of the two sectors are fulfilled.

(b) By symmetrizing the sum over all momenta, write the obtained Hamiltonian as

$$\hat{H} = \sum_{k>0} \Psi_k^\dagger H_k \Psi_k \tag{5}$$

where $\Psi_k^\dagger = (\hat{c}_k^\dagger, \hat{c}_{-k})$ and diagonalize it by means of a unitary transformation U such as

$$\hat{H} = \sum_{k>0} \Psi_k^\dagger U^\dagger U H_k U^\dagger U \Psi_k = \sum_{k>0} \Phi_k^\dagger D_k \Phi_k \tag{6}$$

with D_k a diagonal matrix, $\Phi_k^\dagger = (\hat{\gamma}_{1k}^\dagger, \hat{\gamma}_{2k}^\dagger)$. Compute the commutation and anticommutation relations of the operator $\gamma_{1,2}$. Are they bosons or fermions? Sketch the two bands of the Hamiltonian in a regime of parameters of your choice.

3. Ground state analysis.

   (a) Write the ground state of the Hamiltonian in terms of the $\hat{c}$ fermions and their vacuum. How can the excitations on top of the ground state be written?

   (b) What is the energy gap between the ground state and the lowest excitation? Is there a condition of the parameters h and J such that the system becomes gapless?

4. In quantum systems, the presence of a gapless point as the parameters are changed is a hallmark of a second order quantum phase transition. What are the two phases? Second order phase transitions are characterized by divergences of thermodynamic quantities at the critical point, such as correlation length and relaxation time. The divergences have a powerlaw nature and the associated exponents are universal for phase transitions belonging to the same universality class. Knowing that in the proximity of the critical point h_c the thermalization time scales as

$$\tau_c \propto \frac{1}{E_{gap}} \propto \xi^z \propto (h - h_c)^{-\nu z} \tag{7}$$

with z and ν critical exponents. What is the value of $z\nu$ for the universality class of the TFI model?