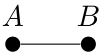
Entanglement

Khoruzhii K.

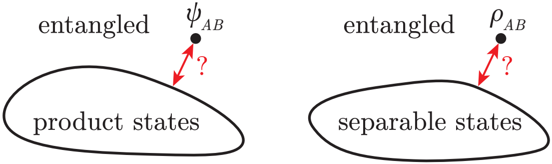
05.07.2024

States

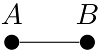


- Pure states $|\psi_{AB}\rangle$:
  - *Product state* if $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
  - A is entangled with B otherwise

- Mixed states $\hat{\rho}_{AB}$
  - *Separable state* if $\hat{\rho}_{AB} = \sum_k p_k \hat{\rho}_A^k \otimes \hat{\rho}_B^k$
  - A is entangled with B otherwise



Bipartite entanglement in pure states: SVD



Is this state entangled?

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |01\rangle\right)$$

Bipartite entanglement in pure states: SVD

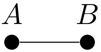
Is this state entangled?

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |01\rangle\right)$$

Of course not:

$$|\psi_{AB}\rangle = |0\rangle \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right)$$

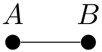
Bipartite entanglement in pure states: SVD



What about this?

$$|\psi_{AB}\rangle = \frac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right)$$

Bipartite entanglement in pure states: SVD



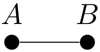
What about this?

$$|\psi_{AB}\rangle = \tfrac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right)$$

After some SVD (or **Schmidt decomposition**):

$$|\psi_{AB}\rangle = \sum_{a\in A}\sum_{b\in B}\psi_{ab}\,|a\rangle\otimes|b\rangle = \sum_{a}\sum_{b}\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}_{ab}|ab\rangle$$

Bipartite entanglement in pure states: SVD



What about this?

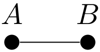
$$|\psi_{AB}\rangle = \tfrac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right)$$

After some SVD (or **Schmidt decomposition**):

$$|\psi_{AB}\rangle = \sum_{a\in A}\sum_{b\in B}\psi_{ab}\,|a\rangle\otimes|b\rangle = \sum_{a}\sum_{b}\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}_{ab}|ab\rangle$$

$$U\begin{pmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\end{pmatrix}V = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & .\\ 1 & .\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & .\\ -1 & .\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$$

Bipartite entanglement in pure states: SVD



What about this?

$$|\psi_{AB}\rangle = \tfrac{1}{2}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right)$$

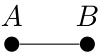
After some SVD (or **Schmidt decomposition**):

$$|\psi_{AB}\rangle = \sum_{a\in A}\sum_{b\in B}\psi_{ab}\,|a\rangle\otimes|b\rangle = \sum_{a}\sum_{b}\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}_{ab}|ab\rangle$$

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

We can see, that it is separable.

Bipartite entanglement in pure states: SVD



Schmidt decomposition

$$|\psi_{AB}\rangle = \sum_{a\in A}\sum_{b\in B}\psi_{ab}\,|a\rangle\otimes|b\rangle = \sum_j \sqrt{\lambda_j}\,|\psi_{A,j}\rangle\,|\psi_{B,j}\rangle$$

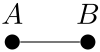
feeply connected with the reduced density matrix $\rho_{A,B}$

$$\rho_{B,A} = \mathrm{tr}_{A,B}\left(|\psi_{AB}\rangle\right) = \sum_i \lambda_i|\psi_{B,A}\rangle\langle\psi_{B,A}|.$$

If there is no entanglement, than

$$\rho_{A,B} = |\psi_{A,B}\rangle\langle\psi_{A,B}|$$

Von Neumann Entropy



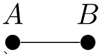
Which one is more entangled?

$$\left|\psi^{I}_{AB}\right\rangle = \frac{1}{\sqrt{4}}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right),$$

$$\left|\psi^{II}_{AB}\right\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right),$$

$$\left|\psi^{III}_{AB}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

Von Neumann Entropy



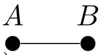
Which one is more entangled? (singular values could help):

$$\left|\psi^{I}_{AB}\right\rangle = \frac{1}{\sqrt{4}}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right), \qquad \lambda^{I}_{1,2} = \{1, 0\}$$

$$\left|\psi^{II}_{AB}\right\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right), \qquad \lambda^{II}_{1,2} = \{0.87, 0.13\}$$

$$\left|\psi^{III}_{AB}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \qquad \lambda^{III}_{1,2} = \{0.5, 0.5\}$$

Von Neumann Entropy



Which one is more entangled? (singular values could help):

$$\left|\psi^{I}_{AB}\right\rangle = \frac{1}{\sqrt{4}}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right), \qquad \lambda^{I}_{1,2} = \{1, 0\}$$

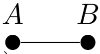
$$\left|\psi^{II}_{AB}\right\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right), \qquad \lambda^{II}_{1,2} = \{0.87, 0.13\}$$

$$\left|\psi^{III}_{AB}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \qquad \lambda^{III}_{1,2} = \{0.5, 0.5\}$$

Measure of entanglement is fixed, if it has

- invariance under (Local Unitary Operations) LUO: $E = E(\lambda)$
- continuity
- additivity $E(|\psi_{AB}\rangle \otimes |\varphi_{AB}\rangle) = E(|\psi_{AB}\rangle) + E(|\varphi_{AB}\rangle)$

Von Neumann Entropy



Which one is more entangled? (singular values could help):

$$\left|\psi^{I}_{AB}\right\rangle = \frac{1}{\sqrt{4}}\left(|00\rangle + |10\rangle - |01\rangle - |11\rangle\right), \qquad \lambda^{I}_{1,2} = \{1, 0\}$$

$$\left|\psi^{II}_{AB}\right\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right), \qquad \lambda^{II}_{1,2} = \{0.87, 0.13\}$$

$$\left|\psi^{III}_{AB}\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \qquad \lambda^{III}_{1,2} = \{0.5, 0.5\}$$

Measure of entanglement is fixed, if it has

- invariance under (Local Unitary Operations) LUO: $E = E(\lambda)$
- continuity
- additivity $E(|\psi_{AB}\rangle \otimes |\varphi_{AB}\rangle) = E(|\psi_{AB}\rangle) + E(|\varphi_{AB}\rangle)$

This is *the von Neumann entropy*

$$\boxed{S(\rho_A) = S(\rho_B) = -\sum_j \lambda_j \ln \lambda_j,} \qquad S^{I} = 0,\ S^{II} = 0.6,\ S^{III} = 1.$$

Example of measurement

Sufficient to show:

$$\mathrm{tr}(\rho_A^2) < \mathrm{tr}(\rho_{AB}^2)$$
$$\mathrm{tr}(\rho_B^2) < \mathrm{tr}(\rho_{AB}^2)$$

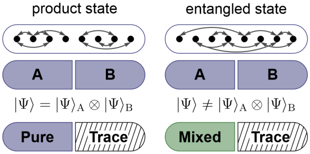


Figure 1: Bipartite entanglement and partial measurements.

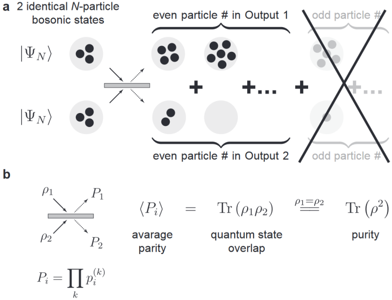


Figure 2: Measurement of quantum purity with many-body bosonic interference of quantum twins

[2] Rajibul Islam et al., Measuring entanglement entropy through the interference of quantum many-body twins Nature, 528 (2015)

Example of measurement

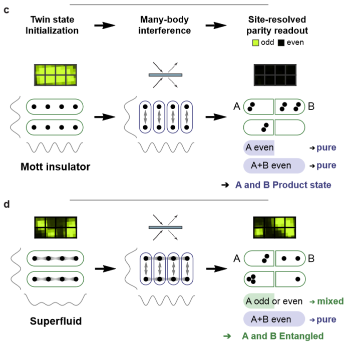


Figure 1: Many-body interference to probe entanglement in optical lattices

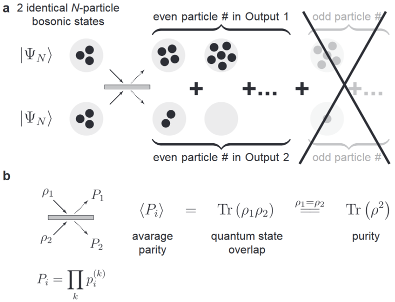
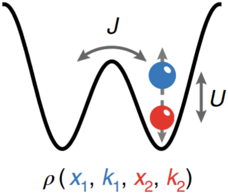


Figure 2: Measurement of quantum purity with many-body bosonic interference of quantum twins

[2] Rajibul Islam et al., Measuring entanglement entropy through the interference of quantum many-body twins Nature, 528 (2015)

Measurable example

Consider dimer system



With Hamiltonian

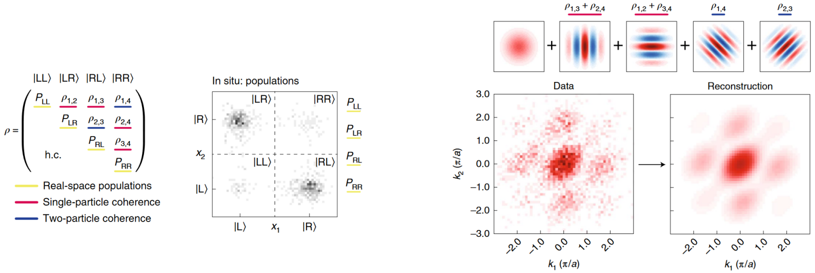
$$\hat{H} = -J \sum_{\sigma} (\hat{c}_{\mathrm{L}\sigma} \hat{c}_{\mathrm{R}\sigma} + \mathrm{c.c.}) + U \sum_{j=\mathrm{L,R}} \hat{n}_{j\downarrow} \hat{n}_{j\uparrow}$$

[2] A. Bergschneider et al., Experimental Characterization of Two-Particle Entanglement through Position and Momentum Correlations, Nature Physics 15, no. 7 (2019)

Measurable example

$$\hat{H} = -J \sum_{\sigma} (\hat{c}_{\mathrm{L}\sigma} \hat{c}_{\mathrm{R}\sigma} + \mathrm{c.c.}) + U \sum_{j=\mathrm{L,R}} \hat{n}_{j\downarrow} \hat{n}_{j\uparrow}$$

What is available for us to measure?

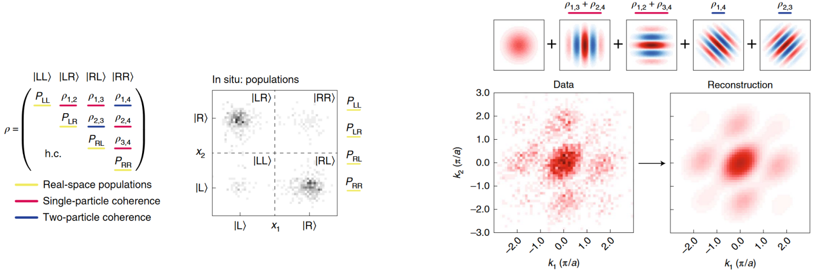


[2] A. Bergschneider et al., Experimental Characterization of Two-Particle Entanglement through Position and Momentum Correlations, Nature Physics 15, no. 7 (2019)

Measurable example

$$\hat{H} = -J \sum_{\sigma} (\hat{c}_{\mathrm{L}\sigma} \hat{c}_{\mathrm{R}\sigma} + \mathrm{c.c.}) + U \sum_{j=\mathrm{L,R}} \hat{n}_{j\downarrow} \hat{n}_{j\uparrow}$$

What is available for us to measure?



Lower bound for concurrence

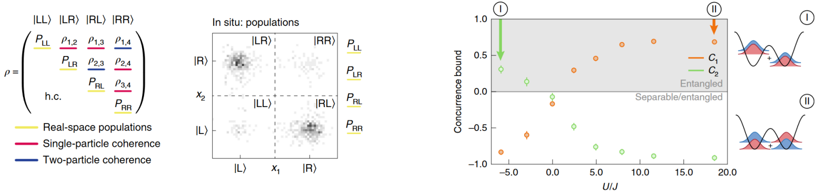
$$C(\rho) \geqslant \max\left\{0,\ 2(|\rho_{1,4}| - \sqrt{P_{\mathrm{LR}} P_{\mathrm{RL}}}),\ 2(|\rho_{2,3}| - \sqrt{P_{\mathrm{LL}} P_{\mathrm{RR}}})\right\}$$

[2] A. Bergschneider et al., Experimental Characterization of Two-Particle Entanglement through Position and Momentum Correlations, Nature Physics 15, no. 7 (2019)
[3] M. Jafarpour et al., A Useful Strong Lower Bound on Two-Qubit Concurrence, Quantum Information Processing 11, no. 6 (2012)

Measurable example

$$\hat{H} = -J \sum_{\sigma} (\hat{c}_{L\sigma}\hat{c}_{R\sigma} + \text{c.c.}) + U \sum_{j=L,R} \hat{n}_{j\downarrow}\hat{n}_{j\uparrow}$$

What is available for us to measure?



Lower bound for concurrence

$$C(\rho) \geqslant \max\left\{0,\ 2(|\rho_{1,4}| - \sqrt{P_{LR}P_{RL}}),\ 2(|\rho_{2,3}| - \sqrt{P_{LL}P_{RR}})\right\}$$

[2] A. Bergschneider et al., Experimental Characterization of Two-Particle Entanglement through Position and Momentum Correlations, Nature Physics 15, no. 7 (2019)
[3] M. Jafarpour et al., A Useful Strong Lower Bound on Two-Qubit Concurrence, Quantum Information Processing 11, no. 6 (2012)

Indistinguishable particles: example

Consider state represented by the Slater determinant

$$|\psi\rangle^{\mathrm{SD}} = \frac{1}{\sqrt{2}} \left(|A, 0\rangle |B, 1\rangle - |B, 1\rangle |A, 0\rangle \right),$$

with A, B and $0, 1$ are the spatial and the internal degree of freedom.

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
[4] A. P. Majtey et al., Indistinguishable Entangled Fermions: Basics and Future Challenges, Phil. Trans. R. Soc. A.381 (2023)

Indistinguishable particles: example

Consider state represented by the Slater determinant

$$|\psi\rangle^{\mathrm{SD}} = \frac{1}{\sqrt{2}} \left(|A,0\rangle \, |B,1\rangle - |B,1\rangle \, |A,0\rangle \right),$$

with A, B and $0, 1$ are the spatial and the internal degree of freedom.

This state results from antisymmetrizing the product state

$$|\psi\rangle^{\mathrm{prod}} = |A,0\rangle \, |B,1\rangle \,,$$

so no entanglement actually.

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
[4] A. P. Majtey et al., Indistinguishable Entangled Fermions: Basics and Future Challenges, Phil. Trans. R. Soc. A.381 (2023)

Indistinguishable particles: example

Consider state represented by the Slater determinant

$$|\psi\rangle^{\mathrm{SD}} = \frac{1}{\sqrt{2}} \left(|A,0\rangle\, |B,1\rangle - |B,1\rangle\, |A,0\rangle \right),$$

with A, B and $0, 1$ are the spatial and the internal degree of freedom.

Meanwhile reduced density matrix is inpure

$$\rho_{\mathrm{f}}^{\mathrm{SD}} = \tfrac{1}{2}|A,0\rangle\langle A,0| + \tfrac{1}{2}|B,1\rangle\langle B,1|.$$

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
[4] A. P. Majtey et al., Indistinguishable Entangled Fermions: Basics and Future Challenges, Phil. Trans. R. Soc. A.381 (2023)

Indistinguishable particles: fermionic exchange correlations

In general for N identical fermions identified with a single Slater determinant reduced density matrix of M fermions

$$\mathrm{tr}\left(\rho_M^{\mathrm{SD}}\right)^2 = \binom{N}{M}^{-1}.$$

Such correlations are compatible with the possibility of assigning a complete set of properties to the individual fermions.

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
[4] A. P. Majtey et al., Indistinguishable Entangled Fermions: Basics and Future Challenges, Phil. Trans. R. Soc. A.381 (2023)

Indistinguishable particles: fermionic exchange correlations

State that can not be represented by the Slater determinant

$$|\psi\rangle^{\text{SD}} = \frac{1}{2}\left(|A,0\rangle|B,1\rangle - |B,1\rangle|A,0\rangle + |A,1\rangle|B,0\rangle - |B,0\rangle|A,1\rangle\right),$$
$$|\psi\rangle^{\text{non-prod}} = \frac{1}{\sqrt{2}}\left(|A,0\rangle|B,1\rangle + |A,1\rangle|B,0\rangle\right)$$

And the purity of $\rho_{\mathrm{f}}^{\text{non-SD}}$ is lower than the purity of $\rho_{\mathrm{f}}^{\text{SD}}$.

This feature reveals the presence of correlations beyond exchange correlations.

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
[4] A. P. Majtey et al., Indistinguishable Entangled Fermions: Basics and Future Challenges, Phil. Trans. R. Soc. A.381 (2023)

Indistinguishable particles: fermionic exchange correlations

State that can not be represented by the Slater determinant

$$|\psi\rangle^{\mathrm{SD}} = \frac{1}{2}\left(|A,0\rangle\,|B,1\rangle - |B,1\rangle\,|A,0\rangle + |A,1\rangle\,|B,0\rangle - |B,0\rangle\,|A,1\rangle\right),$$
$$|\psi\rangle^{\text{non-prod}} = \frac{1}{\sqrt{2}}\left(|A,0\rangle\,|B,1\rangle + |A,1\rangle\,|B,0\rangle\right)$$

And the purity of $\rho_{\mathrm{f}}^{\text{non-SD}}$ is lower than the purity of $\rho_{\mathrm{f}}^{\mathrm{SD}}$.

Entanglament criteria:

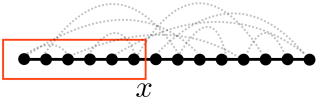
$$\operatorname{tr}\rho_M^2 < \binom{N}{M}^{-1} \quad \Leftrightarrow \quad |\psi\rangle \text{ is entangled.}$$

In other words: how many Slater determinants you need to describe the state (*Slater rank*)?

[1] L. Amico et al., Entanglement in Many-Body Systems, Reviews of Modern Physics 80, no. 2 (2008)
[4] A. P. Majtey et al., Indistinguishable Entangled Fermions: Basics and Future Challenges, Phil. Trans. R. Soc. A.381 (2023)

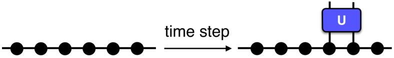
Non-local Unitary Operations

Renyi entropy:

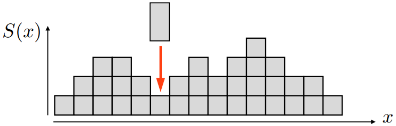


$$S_n(x) = \frac{1}{1-n} \ln(\operatorname{tr} \rho_x^n), \qquad |S(x+1) - S(x)| \leqslant 1.$$

With random updates



$$S_0(x, t+1) = \min[S_0(x-1, t), S_0(x+1, t)] + 1$$



[4] Adam Nahum et al., Quantum Entanglement Growth Under Random Unitary Dynamics, Physical Review X, 7.3 (2017s)