6.1 Thermal Green's functions

The thermal Green's function is defined as

$$G_{ij}(\tau) = -\langle \mathrm{T}_\tau \ \psi_i(\tau)\psi_j^\dagger(0)\rangle$$

The path integral formulation of the Green's function of non-interacting particles is

$$G_{ij}(\tau) = -\frac{1}{Z}\int D(\bar\psi,\psi)\psi_i(\tau)\bar\psi_j(0)e^{-S[\bar\psi,\psi]}, \qquad S = \sum_j \int_0^\beta d\tau\, \bar\psi_j(\partial_\tau + \varepsilon_j - \mu)\psi_j = \sum_j s[\bar\psi_j,\psi_j]. \tag{1}$$

1. Time ordering. The path integral automatically takes care of the time ordering:

$$G_{ij}(\tau>0) = -\langle \psi_i(\tau)\psi_j^\dagger(0)\rangle = \operatorname{tr}\left(e^{-(\beta-\tau)H}\psi_i e^{-H\tau}\psi_j^\dagger\right),$$

and than we could repeat the construction of the path integral and get (1). In other case

$$\begin{aligned}G_{ij}(\tau<0) &= \zeta\langle \psi_j^\dagger(0)\psi_i(\tau)\rangle = \operatorname{tr}\left(\psi_j^\dagger e^{-(-\tau)H}\psi_i e^{-(\beta+\tau)H}\right)\\ &= -\frac{\zeta}{Z}\int D(\bar\psi,\psi)\bar\psi_j(0)\psi_i(\tau)e^{-S(\bar\psi,\psi)} = -\frac{\zeta^2}{Z}\int D(\bar\psi,\psi)\psi_i(\tau)\bar\psi_j(0)e^{-S(\bar\psi,\psi)},\end{aligned}$$

thus we come to the same (1).

2. Green's function as Matsubara sum. After the Fourier transform (unitary)

$$\psi_j(\tau) = \frac{1}{\sqrt\beta}\sum_n e^{-i\omega_n\tau}\psi_{jn}, \qquad \omega_n = \frac{\pi}{\beta}\begin{cases}2n+1, & \text{fermions},\\ 2n, & \text{bosons},\end{cases} \tag{2}$$

we get

$$G_{ij}(\tau) = -\frac{1}{Z}\frac{1}{\beta}\sum_{n,m}\int D(\bar\psi,\psi)e^{-i\omega_n\tau}\psi_{in}\bar\psi_{jm}e^{-S[\bar\psi,\psi]}, \qquad S = \sum_{j,n}\bar\psi_{jn}(-i\omega_n + \varepsilon_j - \mu)\psi_{jn}. \tag{3}$$

We could simplify calculations noticing that due to the sign symetry of the action

$$\int d(\bar\psi_j,\psi_j)\psi_{jn}e^{-s[\bar\psi_{jn},\psi_{jn}]} = 0, \qquad \Rightarrow \qquad G_{i,j\neq i}(\tau) = 0.$$

To the next simplification in $G_{jj}(\tau)$ we could factor

$$I^j_{nm} = \int d(\bar\psi_{jn},\psi_{jn})\, d(\bar\psi_{jm},\psi_{jm})\, \psi_{jn}\bar\psi_{jm}e^{-s[\bar\psi_{jn},\psi_{jn}]}e^{-s[\bar\psi_{jm},\psi_{jm}]} \propto \delta_{nm}$$

again due to the symmetry. It is useful to rewrite I^j_{nn} as

$$I^j_{nn} = \int d(\bar\psi_{jn},\psi_{jn})\psi_{jn}\bar\psi_{jn}e^{-s[\bar\psi_{jn},\psi_{jn}]}\int d(\bar\psi_{jn},\psi_{jn})e^{-s[\bar\psi_{jn},\psi_{jn}]} = \frac{1}{-i\omega_n+\varepsilon_j-\mu}\int d(\bar\psi_{jn},\psi_{jn})e^{-s[\bar\psi_{jn},\psi_{jn}]}.$$

It remains to note that «blue»-term helps us to factorize partition function in the (3)

$$Z = \left(\prod_{k\neq j}\int d(\bar\psi_k,\psi_k)e^{-\sum_{k\neq j}s[\bar\psi_k,\psi_k]}\right)\cdot\left(\prod_{m\neq n}\int d(\bar\psi_{jm},\psi_{jm})e^{-s[\bar\psi_{jm},\psi_{jm}]}\right)\cdot\left(\int d(\bar\psi_{jn},\psi_{jn})e^{-s[\bar\psi_{jn},\psi_{jn}]}\right)$$

Finally $G_{ij}(\tau)$ could be expressed as

$$G_{ij}(\tau) = \frac{\delta_{ij}}{\beta}\sum_n e^{-i\omega_n\tau}G_0(j,i\omega_n), \qquad G_0(j,i\omega_n) \overset{\text{def}}{=} \frac{1}{i\omega_n - \varepsilon_j + \mu}. \tag{4}$$

Substituting ω_n from (2) as usual $G_{ij}(\tau)$ could be rewritten as

$$G_{jj}(\tau>0) = -\zeta\oint\frac{dz}{2\pi i}\frac{e^{-z\tau}}{z-\xi_j}n_{\text{BF}}(-z), \tag{5}$$

$$G_{jj}(\tau<0) = -\zeta\oint\frac{dz}{2\pi i}\frac{e^{z\tau}}{-z+\xi_j}n_{\text{BF}}(z), \tag{6}$$

with $\xi_j \overset{\text{def}}{=} \varepsilon_j - \mu$ and $n_{\text{BF}}(z) = (e^{\beta z}-\zeta)^{-1}$. Sign of z was chosen to provide convergence. Summing over the outer pole we get

$$G_{jj}(\tau>0) = \zeta n_{\text{BF}}(-\xi_j)e^{-\xi_j\tau},$$
$$G_{jj}(\tau<0) = -\zeta n_{\text{BF}}(\xi_j)e^{-\xi_j\tau}.$$

Combining all this happiness into one expression

$$\boxed{G_{ij}(\tau) = -\delta_{ij}\left(\theta(\tau) + \zeta n_{\text{BF}}(\xi_j)\right)e^{-\xi_j\tau}}. \tag{7}$$

In general, it is quite logical to obtain the theta function due to T-ordering, since $\hat a\hat a^\dagger = 1 + \zeta\hat a^\dagger\hat a$.

3. The occupation number. The occupation number in a single particle state j is in general given by

$$n_j = \langle \psi_j^\dagger(0)\psi_j(0)\rangle = \zeta \lim_{\tau\to 0^-} \langle \mathrm{T}_\tau \psi_i(\tau)\psi_j^\dagger \rangle = -\zeta \lim_{\tau\to 0^-} G_{jj}(\tau),$$

$$= \zeta \lim_{\tau\to 0^+} \langle \mathrm{T}_\tau \psi_i(\tau)\psi_j^\dagger - 1\rangle = \zeta \lim_{\tau\to 0^+} (-G_{jj}(\tau) - 1)$$

Expanding (7) we get

$$n_j = -\zeta \lim_{\tau\to 0^-} G_{jj}(\tau) = \zeta^2 n_{\text{BF}}(\xi_j) \lim_{\tau\to 0^-} e^{-\xi_j \tau} = n_{\text{BF}}(\xi_j).$$

4. The generating functional. The generating functional for correlation functions is defined as

$$\mathcal{Z}[\bar{J}, J] = \int D\left(\bar{\psi}, \psi\right) \, \exp\left(-S[\bar{\psi}, \psi] - \sum_j \int_0^\beta d\tau \left(\bar{J}_j \psi_j + \bar{\psi}_j J_j\right)\right).$$

These can be obtained as functional derivatives of $\mathcal{Z}[\bar{J}, J]$, where the source fields are set to zero after the evaluation:

$$\langle \mathrm{T}_\tau \psi_{in}\psi_{jm}\rangle = \frac{\zeta}{\mathcal{Z}[0,0]} \frac{\delta^2 \mathcal{Z}[\bar{J}, J]}{\delta \bar{J}_{in}\delta J_{jm}}\bigg|_{J,\bar{J}=0}.$$

It remains to calculate

$$\mathcal{Z}[\bar{J}, J] = \int D(\bar{\psi}, \psi) \, \exp\left(-\sum_{j,n} \bar{\psi}_{jn} \left(-G_0^{-1}(j, i\omega_n)\right) \psi_{jn} + \sum_{j,n} \left(\bar{J}_{jn}\psi_{jn} + \bar{\psi}_{jn} J_{jn}\right)\right)$$
$$= \mathcal{Z}[0,0] \exp\left(-\sum_{j,n} \bar{J}_{jn} G_0(j, i\omega_n) J_{jn}\right).$$

Thus for the Green's function $G_{ij}(i\omega_n)$ in Matsubara space

$$G_{ij}(i\omega_n) = -\langle \mathrm{T}_\tau \psi_{in}\psi_{jm}\rangle = -\zeta \frac{\delta^2}{\delta \bar{J}_{in}\delta J_{jm}} \exp\left(-\sum_{j,n} \bar{J}_{jn} G_0(j, i\omega_n) J_{jn}\right)\bigg|_{\bar{J}=J=0} = \delta_{ij}\delta_{nm} G_0(j, i\omega_n),$$

corresponding to (4).

6.2 Nambu-Goldstone Modes in the Heisenberg Ferromagnet

We consider an isotropic Heisenberg ferromagnet with spin 1/2-particles fixed to the sites of a lattice:

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j,$$

with $J_{ij} > 0$. Let us label the ground states by their orientation in space:

$$|0_{\boldsymbol{n}}\rangle = \bigotimes_{i=1}^{N} |i, \boldsymbol{n}\rangle\,,$$

with the single site states satisfying $\boldsymbol{n}\cdot\hat{\boldsymbol{S}}_j\,|j,\boldsymbol{n}\rangle = -\frac{1}{2}\,|j,\boldsymbol{n}\rangle$.

1. Orthogonal states. In spherical coordinates the single site state could be found from

$$\boldsymbol{n}\cdot\hat{\boldsymbol{S}} = \frac{1}{2}\begin{pmatrix} \cos(\theta) & e^{-i\varphi}\sin(\theta) \\ e^{i\varphi}\sin(\theta) & -\cos(\theta) \end{pmatrix},$$

with eigenstate

$$|0_{\boldsymbol{n}}\rangle = \frac{\cos(\theta)-1}{\sqrt{2-2\cos(\theta)}}\,|\uparrow\rangle + \frac{e^{i\varphi}\sin\theta}{\sqrt{2-2\cos(\theta)}}\,|\downarrow\rangle\,.$$

We need projection to the $|\downarrow\rangle$, that could be simplified to the

$$|\langle\downarrow|0_{\boldsymbol{n}}\rangle| = \left|\frac{2\sin\left(\frac{1}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right)}{2\sin(\frac{1}{2}\theta)}\right| = |\cos(\tfrac{1}{2}\theta_{\downarrow\boldsymbol{n}})|$$

In the thermodynamic limit $N\to\infty$

$$|\langle 0_{\boldsymbol{n}_1}|0_{\boldsymbol{n}_2}\rangle| = \lim_{N\to\infty}|\cos(\tfrac{1}{2}\theta_{\boldsymbol{n}_1\boldsymbol{n}_1})|^N = 0.$$

2. Hamiltonian. We could substitute

$$\hat{S}^x = \tfrac{1}{2}\left(\hat{S}^+ + \hat{S}^-\right), \qquad \hat{S}^y = \tfrac{1}{2i}\left(\hat{S}^+ - \hat{S}^-\right), \qquad \hat{S}^z = \hat{S}^+\hat{S}^- - \tfrac{1}{2},$$

that leads to terms as

$$\begin{aligned}
\hat{S}_i^x\hat{S}_j^x &= \tfrac{1}{4}\left(\hat{S}_i^+\hat{S}_j^+ + \hat{S}_i^+\hat{S}_j^- + \hat{S}_i^-\hat{S}_j^+ + \hat{S}_i^-\hat{S}_j^-\right)\\
\hat{S}_i^y\hat{S}_j^y &= \tfrac{-1}{4}\left(\hat{S}_i^+\hat{S}_j^+ - \hat{S}_i^+\hat{S}_j^- - \hat{S}_i^-\hat{S}_j^+ + \hat{S}_i^-\hat{S}_j^-\right)\\
\hat{S}_i^z\hat{S}_j^z &= \hat{S}_i^+\hat{S}_i^-\hat{S}_j^+\hat{S}_j^- - \tfrac{1}{2}\hat{S}_i^+\hat{S}_i^- - \tfrac{1}{2}\hat{S}_j^+\hat{S}_j^- + \tfrac{1}{4},
\end{aligned}$$

so the Hamiltonian

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\left(-\tfrac{1}{2}\left(\hat{S}_i^+ - \hat{S}_J^+\right)\left(\hat{S}_i^- - \hat{S}_j^-\right) + \hat{S}_i^+\hat{S}_i^-\hat{S}_j^+\hat{S}_j^- + \tfrac{1}{4}\right).$$

3. One-particle Hamiltonian. We can reduce the Hilbert space to the one-particle states $|j\rangle = \hat{S}_j^+\,|0_\uparrow\rangle$. Neglecting constant terms, reduced Hamiltonian $\hat{H}'$

$$H'|i\rangle = -\frac{1}{2}\sum_{kj} J_{kj}\left(-\frac{1}{2}\left(\delta_{ki}|k\rangle - \delta_{ji}|k\rangle - \delta_{ki}|j\rangle + \delta_{ji}|j\rangle\right) + \delta_{ji}\delta_{kj}|k\rangle\right) = \frac{1}{2}\sum_j J_{ij}(|i\rangle - |j\rangle)$$

with matrix elements

$$\langle i|\hat{H}|i\rangle = \frac{1}{2}\sum_j J_{ij}, \qquad \langle j|\hat{H}|i\rangle = -\frac{1}{2}\sum_j J_{ij}.$$

Assuming $J_{ij} = J(|\boldsymbol{x}_i - \boldsymbol{x}_j|)$ consider the plane wave state $|k\rangle = \sum_j e^{i\boldsymbol{k}\boldsymbol{x}_j}\,|j\rangle$:

$$\hat{H}'\,|k\rangle = \sum_i e^{i\boldsymbol{k}\boldsymbol{x}_i}\,\tfrac{1}{2}\sum_j J_{ij}\left(|i\rangle - |j\rangle\right).$$

To simplify calculations consider

$$\langle m|\hat{H}'|k\rangle = \sum_i e^{i\boldsymbol{k}\boldsymbol{x}_i}\,\tfrac{1}{2}\sum_j J_{ij}\left(\delta_{mi} - \delta_{mj}\right) = e^{i\boldsymbol{k}\boldsymbol{x}_m}\,\tfrac{1}{2}\sum_j J_{mj} - \sum_j e^{i\boldsymbol{k}\boldsymbol{x}_j}\,\tfrac{1}{2}J_{jm} = \tfrac{1}{2}\sum_j J_{jm}\left(1 - e^{i\boldsymbol{k}(\boldsymbol{x}_j - \boldsymbol{x}_m)}\right)e^{i\boldsymbol{k}\boldsymbol{x}_m}.$$

We could sum over $\boldsymbol{x}_n = \boldsymbol{x}_j - \boldsymbol{x}_m$ and notice that $J_{jm} = J(|\boldsymbol{x}_j - \boldsymbol{x}_m|) = J(|\boldsymbol{x}_n|)$

$$\langle m|\hat{H}'|k\rangle = \frac{1}{2}\left(\sum_n J_{n0}\left(1 - e^{i\boldsymbol{k}\boldsymbol{x}_n}\right)\right)e^{i\boldsymbol{k}\boldsymbol{x}_m} = E_{\boldsymbol{k}}\langle m|k\rangle,$$

thus we have proven that $|k\rangle$ is eigenstate with energy $E_{\boldsymbol{k}}$

$$E_{\boldsymbol{k}} = \frac{J_0 - J_k}{2}, \qquad J_k = \sum_j J(|x_j|) e^{-i\boldsymbol{k}\boldsymbol{x}_j}.$$

For a constant nearest neighbour interaction on a square lattice $E_{\boldsymbol{k}}$ could be calculated explicitly:

$$J_k = \sum_{\boldsymbol{x}_j = \pm \boldsymbol{e}_{x,y}} J(|\boldsymbol{x}_j|) e^{-i\boldsymbol{k}\boldsymbol{x}_j} = 2J\left(\cos k_x + \cos k_y\right),$$

and energy

$$E_k = J\left(2 - \cos k_x - \cos k_y\right) \overset{k\to 0}{\approx} \frac{J}{2}\left(k_x^2 + k_y^2\right).$$