5.1 Complex analysis

Consider Gaussian integral

$$I_1 = \int_{-\infty}^{\infty} e^{-iax^2}\, dx,$$

with $\alpha \in \mathbb{R}^+$. We could use that

$$I_2 = \int_{-\infty}^{\infty} e^{-bx^2}\, dx = \sqrt{\frac{\pi}{a}}.$$

Define $\mathcal{C}_1 = \{z = e^{i\frac{\pi}{4}x} \colon x \in \mathbb{R}\}$, $\mathcal{C}_2 = \{z = x \colon x \in \mathbb{R}\}$ and $\mathcal{C}_R^{\pm} = \{z = \pm Re^{i\varphi} \colon \varphi \in [0, \pi/4]\}$. Applying Cauchy integral theorem for holomorphic functions to the $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_R^+ \cup \mathcal{C}_R^-$ we have

$$-I[\mathcal{C}_1] + I[\mathcal{C}_2] + I[\mathcal{C}_R^+] + I[\mathcal{C}_R^-] = 0,$$

with $I[\mathcal{C}_1]e^{-i\frac{\pi}{4}} = I_1$, $I[\mathcal{C}_2] = I_2$.

The $I[\mathcal{C}_R^{\pm}]$ could be estimated as

$$|I[\mathcal{C}_{R\to\infty}^{\pm}]| \leqslant \lim_{R\to\infty} \int_0^{\pi/2} e^{-aR^2\varphi} r\, d\varphi = \lim_{R\to\infty} \frac{1}{aR}\left(1 - e^{-\frac{1}{2}a\pi r^2}\right) = 0,$$

thus we have

$$I_1 = I[\mathcal{C}_1]e^{-i\frac{\pi}{4}} = I[\mathcal{C}_2]e^{-i\frac{\pi}{4}} = e^{-i\frac{\pi}{4}}\sqrt{\frac{\pi}{a}},$$

that could be generalized as

$$\int_{-\infty}^{\infty} e^{\pm iax^2}\, dx = e^{\pm i\frac{\pi}{4}}\sqrt{\frac{\pi}{a}}.$$

5.2 Effective action of coupled harmonic oscillators

We could derive the low-energy effective action for a system of two coupled harmonic oscillators, formally described by the classical partition function

$$Z = \int DxDX\, \exp\left(i\int dt\; L(x, X, \dot{x}, \dot{X})\right),$$

with

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2 + \frac{1}{2}M\dot{X}^2 - \frac{1}{2}M\Omega^2X^2 - gXx.$$

It could be expanded as

$$Z = \int Dx\; e^{iS_0 + iS_{\text{int}}}, \qquad e^{iS_{\text{int}}} = \int DX\; \exp\left(i\int dt\, \left[\tfrac{1}{2}M\dot{X}^2 - \tfrac{1}{2}M\Omega^2X^2 - gXx\right]\right).$$

Integrating by parts we have Gaussian integral that coul be calculated directly

$$e^{iS_{\text{int}}} = \int DX\; \exp\left(i\int dt\, \left[\tfrac{1}{2}MX(\partial_t^2 + \Omega^2)X - gXx\right]\right) = \mathcal{N}\exp\left(i\int dt\frac{g^2}{2M}x\left(\partial_t^2 + \Omega^2\right)^{-1}x\right)$$

with $\mathcal{N}$ as some irrelevant normalizing factor. That leads to some L_{eff}

$$L_{\text{eff}} = \tfrac{1}{2}m\ddot{x}^2 - \frac{1}{2}mx^2\omega_{\text{eff}}^2, \qquad \omega_{\text{eff}} = \omega\sqrt{1 - \alpha^2\frac{m}{M}\left(\frac{\omega}{\Omega}\right)^2},$$

with $g = \alpha m\omega^2$ and, apparently, $m_{\text{eff}} = m$.