1 Complex analysis - a short example

Appropriate use of Cauchy's integral theorem as well as of residual theorem can help solving various integrals exactly. It will be used frequently later in the course to evalute for e.g. Matsubara summations, arising from the previously derived path integral expressions.

1. In the lectures you have seen some examples of Wick rotations, which represent typical operations in field theory. These rotations of integration contours in the complex plane are ultimately justified by Cauchy's theorem.
   We want to make use of such a rotation to compute the complex continuation of a Gaussian integral
   $$I(a) = \int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-\mathrm{i}ax^2} \quad , a \in \mathbb{R}^+. \tag{1}$$
   Despite the integrand of $I(a)$ is oscillating the value of the integral can be related to the well-known result for an ordinary Gaussian integral
   $$I_2(b) = \int_{-\infty}^{\infty} \mathrm{d}x\ \mathrm{e}^{-bx^2} = \sqrt{\frac{\pi}{b}} \quad , b \in \mathbb{R}^+. \tag{2}$$
   considering the complex continuation $b = \mathrm{i}a$. Compute the value of $I(a)$ by relating it to $I_2(b)$ using Cauchy's theorem.

   > Hint: Express $I(a)$ and $I_2(a)$ as special cases of integration in the complex plane, i.e. of
   > $$\int_{\mathcal{C}} \mathrm{d}z\ \mathrm{e}^{-az^2} \quad , z \in \mathbb{C}. \tag{3}$$
   > You may choose two different paths $\mathcal{C}_1 = \{z = e^{i\frac{\pi}{4}}x, x \in \mathbb{R}\}$ and $\mathcal{C}_2 = \{z = x, x \in \mathbb{R}\}$ to recover $I(a)$ and $I_2(a)$. Connect both of these contours by additional contours $\mathcal{B}_1, \mathcal{B}_2$ to get an overall contour integration and apply the Cauchy integral theorem for holomorphic functions.

2. In eq. (1) we restricted the possible choices of a to positive values. Argue that this constraint can be lifted to allow $a \in \mathbb{R}$. Which modifications have to be made in your computations of part (1.) .

2 Effective action of coupled harmonic oscillators

In this exercise you are asked to derive the low-energy effective action for a system of two coupled harmonic oscillators, formally described by the classical partition function

$$Z = \int DxDXe^{i\int \mathrm{d}tL(x,X,\dot{x},\dot{X})} \tag{4}$$

where the Lagrangian takes the form

$$L(x, X, \dot{x}, \dot{X}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + \frac{1}{2}M\dot{X}^2 - \frac{1}{2}M\Omega^2 X^2 - gXx. \tag{5}$$

In order to derive the low-energy theory you may assume $\Omega \gg \omega$, while $g \sim m\omega^2,\ m \sim M$. This promotes X to a high-energy degree of freedom oscillating around its classical mean value. Derive the low-energy Lagrangian L_{eff} by integrating out the high-energy degree of freedom. Include corrections up to leading order $\frac{1}{\Omega}$ and determine the effective mass m^* and the effective spring constant $m^*(\omega^*)^2$ of the soft degrees of freedom.