2.1 Quantum State Tomography

(a) Density matrix could be decomposed as

$$\hat{\rho} = \frac{1}{2^N} \sum_{\{\alpha_j\}} C_{\alpha_1 \ldots \alpha_N} \hat{\sigma}_{\alpha_1} \otimes \ldots \otimes \hat{\sigma}_{\alpha_N},$$

with Pauli matrix $\hat{\sigma}_j$ and $\alpha = 0, 1, 2, 3$

$$\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Coefficients $C_{\alpha_1 \ldots \alpha_N}$ could be measured using $\hat{\sigma}_j^2 = \hat{\sigma}_0$

$$C_{\alpha_1 \ldots \alpha_N} = \operatorname{tr}\left(\hat{\rho}\ \hat{\sigma}_{\alpha_1} \otimes \ldots \otimes \hat{\sigma}_{\alpha_N}\right),$$

so we need $4^N - 1$ measurements (remembering $\operatorname{tr} \rho = 1$). Doing finite number of shots we measure not average values by themselves, but their estemations, so we could observe not normalized states. For 2-qubit system a complete set of measurement operators is

$$\{\sigma_{\alpha_1} \otimes \sigma_{\alpha_2}\},$$

except $\alpha_1 = \alpha_2 = 0$.

(b) The Bloch vector can be extracted as

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix}, \qquad \Rightarrow \qquad \hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-\cos(\theta) \end{pmatrix},$$

what could be expand as

$$\hat{\rho} = \tfrac{1}{2}\hat{\sigma}_0 + \tfrac{1}{2}\sin\theta\cos\varphi\ \hat{\sigma}_x + \tfrac{1}{2}\sin\theta\sin\varphi\ \hat{\sigma}_y + \tfrac{1}{2}\cos\theta\ \hat{\sigma}_z.$$

2.2 Semi-Classical Light–Matter Interaction

We have basic light-atom interaction

$$\hat{H} = \hat{H}_0 + \hat{V}, \qquad \hat{H}_0 = -\frac{\hbar\omega}{2}\hat{\sigma}_z, \quad \hat{V}(t) = -\hbar\Omega\cos(\omega_L t)\hat{\sigma}_x,$$

with Rabi frequency $\Omega = dE_0/\hbar$.

(a) Using that $\exp \operatorname{diag}(a_j) = \operatorname{diag}(e^{a_j})$ we could simplify

$$e^{-i\theta\hat{\sigma}_z}\hat{\sigma}_x e^{i\theta\hat{\sigma}_z} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \cos(2\theta)\hat{\sigma}_x + \sin(2\theta)\hat{\sigma}_y.$$

(b) Using unitary tranform $\hat{U} = \exp\left(-\frac{i}{2}\omega t\hat{\sigma}_z\right)$ we could substitute $|\psi\rangle = \hat{U}^\dagger|\tilde{\psi}\rangle$ and find new $H_{\mathrm{I}} = UHU^\dagger - i\hbar U\partial_t U^\dagger$:

$$\hat{H}_{\mathrm{I}} = -\frac{\hbar}{2}\delta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{\hbar}{2}\Omega\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{\hbar}{2}\Omega\begin{pmatrix} 0 & e^{-2i\omega_L t} \\ e^{2i\omega_L t} & 0 \end{pmatrix},$$

with $\delta = \omega - \omega_L$.

(c) We can just throw away the gray term.

(d) Consider $\delta = 0$, than with $|\psi(0)\rangle = |e\rangle$ we have

$$|\tilde{\psi}(t)\rangle = \left(i\sin(\tfrac{\Omega}{2}t),\ \cos(\tfrac{\Omega}{2}t)\right)^{\mathrm{T}}, \qquad |\psi(t)\rangle = \hat{U}^\dagger|\tilde{\psi}(t)\rangle = \left(i\sin(\tfrac{\Omega}{2}t)e^{\frac{i\omega}{2}t},\ \cos(\tfrac{\Omega}{2}t)e^{-\frac{i\omega}{2}t}\right)^{\mathrm{T}},$$

so that $\langle\hat{\sigma}_z\rangle$ oscillate with frequency Ω.

(e) $\nu = \frac{2eLE}{h} \approx 24\,\mathrm{MHz}$ and $T_{\pi/2} \sim 10\,\mathrm{ns}$

(f) $\Omega = \sqrt{\frac{6\pi I c^2\Gamma}{\hbar\omega_0^3}} \sim 0.6\,\mathrm{ms}^{-1} \to 0.1\,\mathrm{kHz}$ and $T_{\pi/2} \sim 2\,\mathrm{ms}$

(g) We could simply solve Lindblad equation

$$i\hbar\partial_t\hat{\rho} = \left[\hat{H}, \hat{\rho}\right] + i\hbar\hat{L}(\hat{\rho}), \qquad \hat{L}(\hat{\rho}) = -\Gamma\begin{pmatrix} -\rho_{22} & \frac{1}{2}\rho_{12} \\ \frac{1}{2}\rho_{21} & \rho_{22} \end{pmatrix}. \tag{1}$$

Considering that the detuning is zero we have evolution

$$\partial_t\hat{\rho} = \Gamma\begin{pmatrix} \rho_{22} & -\frac{\rho_{12}}{2} \\ -\frac{\rho_{21}}{2} & -\rho_{22} \end{pmatrix} - \frac{i\Omega}{2}\begin{pmatrix} \rho_{12} - \rho_{21} & \rho_{11} - \rho_{22} \\ \rho_{22} - \rho_{11} & \rho_{21} - \rho_{12} \end{pmatrix}.$$

Consider the behavior of the ρ_{22} with $\alpha = \Omega/\Gamma \ll 1$

$$\rho_{22} = \alpha^2 - \alpha^2 e^{-\Gamma t}\left(2e^{\frac{1}{2}\Gamma t} - 1\right) + O(\alpha^4).$$

With $\alpha = \Gamma/\Omega \ll 1$ we will have Rabi oscillations, converging to the $\rho_{11} = \rho_{22} = 1/2$ with envelope $\sim e^{-\alpha t}$.

2.3 Driven System Dynamics and Rotating-Wave Approximation

Consider $H_0 = -\frac{\hbar\omega}{2}\sigma_z$ with $V = \frac{\hbar\Omega}{2}\cos(\omega t)\sigma_z - \frac{\hbar\Omega}{2}\sin(\omega t)\sigma_y$ (black dashed line) and $V_{\text{RWA}} = \hbar\Omega\cos(\omega t)\sigma_x$ (blue line) in resonant ($\omega = \omega_0$) case (fig. 1).

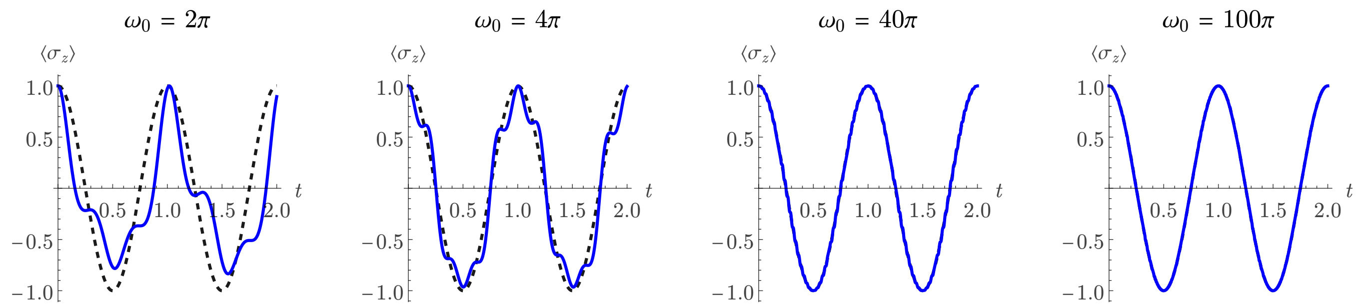

Figure 1: Comparison of the exact solution (black) and the solution in the rotating wave approximation (blue).

We could find solution in general case $\delta \neq 0$ in RWA:

$$\langle\sigma_z\rangle(t) = \cos(\Omega_\delta t) + \frac{\delta^2}{\delta^2 + \Omega^2}(1 - \cos(\Omega_\delta t)), \qquad \Omega_\delta = \sqrt{\Omega^2 + \delta^2},$$

with general Rabi frequency Ω_δ. For different $\delta = 0, \Omega, 2\Omega$ and $\Omega = 2\pi$ we could compare different behaviour (fig. 2).

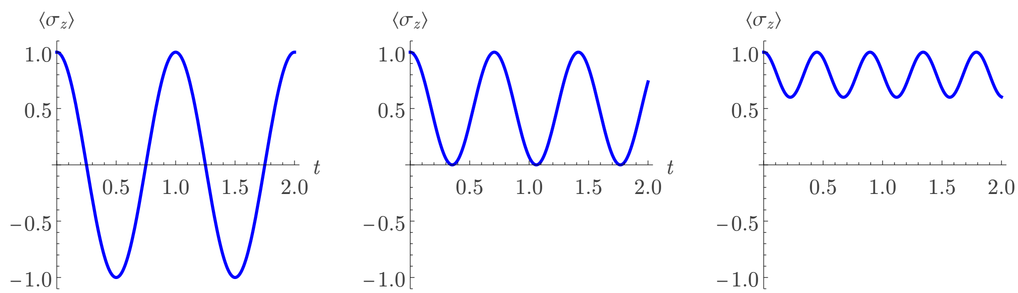

Figure 2: Rabi oscillations at different detuning values

2.4 Dephasing and Decoherence in a TLS

We could calculate $\langle\boldsymbol{\sigma}\rangle$, using (1):

$$\rho(t) = \begin{pmatrix} 1 - |\beta|^2 e^{-\Gamma_1 t} & \alpha\bar{\beta}e^{-\Gamma_2 t} \\ \bar{\alpha}\beta e^{-\Gamma_2 t} & |\beta|^2 e^{-\Gamma_1 t} \end{pmatrix}, \quad \Rightarrow \quad \begin{aligned} \langle\sigma_x\rangle &= (\alpha\bar{\beta} + \bar{\alpha}\beta)e^{-\Gamma_2 t} \\ \langle\sigma_y\rangle &= i(\alpha\bar{\beta} - \bar{\alpha}\beta)e^{-\Gamma_2 t} \\ \langle\sigma_z\rangle &= 1 - 2|\beta|^2 e^{-t\Gamma_1} \end{aligned}$$

Substituting the initial state $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$$\langle\sigma_x\rangle = 1 - e^{-\Gamma_1 t}, \qquad \langle\sigma_y\rangle = 0, \qquad \langle\sigma_z\rangle = e^{-\Gamma_2 t},$$

thus by measuring the various components we can find the Γ_1 and Γ_2, it's just exponential decay.

October 30, 2023, Khoruzhii Kirill