Ultracold Atoms and Quantum Gases
Problem Set no 1

Simon Fölling (simon.foelling@lmu.de), WS 23/24

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This first problem is related to some of the things we did in the lecture as well as to remind you of some basic concepts of atomic physics and the density matrix formalism.

1 de-Broglie-Wavelength

In the lecture we learned about an approximate criterion for a quantum gas: The de-Broglie-wavelength should be approximately the same as the inter-particle distance.

(a) At which temperature does the thermal de-Broglie-wavelength $\lambda_{\mathrm{dB}} = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}$ of a Rubidium atom (^{87}Rb, $m = 87u$) coincide with the wavelength of the above Nd:YAG laser? What is a typical velocity at this temperature? (The typical kinetic energy of an atom is given by $E_{\mathrm{kin}} = 3/2\, k_BT$.)

(b) At what critical density n_c (typical units: cm^{-3}) would atoms with this de-Broglie-wavelength start to condense if you assume the approximate relationship $n_c \cdot \lambda^3 \approx 1$?

(c) A typical ultracold atomic cloud consists of 10^6 atoms. If these atoms would form a homogeneous sphere at the above critical density, what would be its radius?

(d) How many atoms would occupy the above volume at room temperature and (i) at atmospheric pressure or (ii) the typical pressure of a good vacuum chamber ($p_{\mathrm{atm}} = 1013\,\mathrm{mbar}$, $p_{\mathrm{UHV}} = 10^{-11}\,\mathrm{mbar}$)?

(e) What is the thermal de-Broglie wavelength for room temperature electrons, and how does it compare to the density of particles in a solid Rb crystal ($\rho = 1.532\,\mathrm{g/cm^3}$)? Are the conduction electrons in the metal degenerate?

2 Hamiltonian with rotating wave approximation

This problem revisits the atom-light field hamiltonian and Rabi oscillations, from a different angle, using eigenstates instead of differential equation approach. You might find some of it basic. Particularly the results of the final two parts however will be relevant for the lecture fairly soon.

We consider a two-level-atom, where the two levels are coupled by a monochromatic laser field with Rabi frequency Ω_0 and detuning δ.

We start with the differential equations derived in the lecture:

$$\dot{c}_1 = i\frac{\Omega_0}{2}e^{i\delta t}c_2 \qquad \dot{c}_2 = i\frac{\Omega_0}{2}e^{-i\delta t}c_1$$

(a) Show that, under the basis change $|\tilde{1}\rangle = e^{i\frac{\delta}{2}t}\,|1\rangle$ and $|\tilde{2}\rangle = e^{-i\frac{\delta}{2}t}\,|2\rangle$, the above equations transform into:

$$\frac{d}{dt}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix} = \frac{i}{2}\begin{pmatrix}-\delta & \Omega_0\\ \Omega_0 & \delta\end{pmatrix}\begin{pmatrix}\tilde{c}_1\\ \tilde{c}_2\end{pmatrix}$$

Note: Start by relating $\tilde{c}_1$ to c_1 and then take the time derivative of this. Only then insert the above differential equation.
This is effectively a Schrödinger equation for the $\tilde{c}_{1,2}$-coefficients, which corresponds to the Hamiltonian:

$$\hat{H} = \frac{-\hbar}{2} \begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & \delta \end{pmatrix}$$

Here $|g\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ correspond to the ground and excited states of the atom, respectively.

(b) Calculate the eigenstates $|S_+\rangle$, $|S_-\rangle$ (for $\delta = 0$, but general Ω_0) of this Hamiltonian by diagonalizing it. The eigenstates are also referred to as *dressed states* and represent the eigenstates of the atom in the presence of the light field.

In the lecture we have derived the Rabi oscillations between the ground $|g\rangle$ and excited $|e\rangle$ states of a two-level atom in a light field by explicitly solving the differential equation. In the following we will derive the same results using the dressed state approach. Assume that the atom is initially ($t \leq 0$) in the ground state $|\Phi(t \leq 0)\rangle = |g\rangle$. At time $t = 0$ we switch on a light field with Rabi frequency Ω_0. For simplicity, we will initially assume only resonant excitation, so $\delta = 0$.

(c) Since the dressed states form a basis of the Hilbert state, one can express any state also in this basis. Express the state of the atom at time $t = 0$ in the dressed state basis, i.e. calculate the coefficients c_+, c_-.

$$|\Phi(t = 0)\rangle = |g\rangle = c_+ |S_+\rangle + c_- |S_-\rangle$$

This assumes that the light is already on, but did not have the time to change the state of the atom yet, so that the atom is still in state $|g\rangle$. This is known as the *sudden approximation.*

(d) For any Hamiltonian, the time evolution is easiest for an eigenstate. Give a general expression for the time evolution of an eigenstate $|\Psi_0\rangle$ with eigenenergy E_0 of a general Hamiltonian, i.e. give a general formula for $|\Psi_0(t)\rangle$.

(e) Give an expression for the state of the atom $|\Phi(t)\rangle$ at later times $t > 0$!
(Use the fact that $|S_+\rangle$ and $|S_-\rangle$ are eigenstates.)

(f) Calculate the population $|\langle e|\Phi(t)\rangle|^2$ of the excited state $|e\rangle$ as a function of time!

(g) Compute the eigenstates and eigenvalues for the more general case $\delta \neq 0$

(h) Sketch the eigenenergies of the two eigenstates as a function of δ for $\Omega_0 = \gamma$ and for $\Omega_0 = 0$ in one graph. Here, γ is just an arbitrary fixed parameter. It fixes the axes of the graph to units of γ for the frequency axis and $\hbar\gamma$ for the energy axis. Choose $\delta = -3\gamma ... + 3\gamma$ for example.

3 Mixed states, entropy, and density matrices reminder

Density matrices are a convenient tool to express not only pure states (including superposition states), but also mixed states like e.g. thermal states. If you are not familiar with density matrices you can find information on them in every quantum mechanics

textbook. This problem is touching some typical uses and contexts of density matrices such as they will be needed in this lecture and many others.
In the following we will assume an ensemble of two-level atoms with the two internal states $|1\rangle$ (ground state) and $|2\rangle$ (excited state).

(a) Assume that you have a an ensemble of atoms that are all in the state $|\Psi\rangle = \sqrt{3}/2\,|1\rangle - \frac{i}{2}\,|2\rangle$. What is the density matrix of an atom in this ensemble in the basis $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

(b) We create an ensemble mixing 75% ground state atoms and 25% excited state atoms. If you pick one atom at random out of this ensemble, what is its density matrix?

(c) Assume that somebody gave you an ensemble of atoms. How could you experimentally distinguish between the above two situation?

(d) The entropy of a density matrix is defined as $S = -k_B\,\mathrm{Tr}(\hat{\rho}\log(\hat{\rho}))$. Calculate the entropy of the above two density matrices!

(e) What is the density matrix of a thermal state at $T = 0$ and $T = \infty$ of a two-level system, and what is its entropy? A thermal density matrix is defined as $\hat{\rho} = \frac{1}{Z}\sum_i e^{-\frac{E_i}{k_B T}}\,|i\rangle\langle i|$, with Z the normalization constant, ensuring that the sum of the diagonal elements is 1.

(f) Show that a density matrix that has diagonal shape with more than one non-zero element necessarily represents a non-pure ensemble!