Submission deadline for the homework questions: 20/10/2023 (hand in via moodle).
Please prepare the remaining questions for the tutorials in the following week and be ready to present the questions marked with a star ($\star$).

Exercise 1.1*: Classical Dipoles

(a) For a classical magnetic dipole, define its magnetic moment $\vec{\mu}$ in terms of its angular momentum and a constant.

(b) Discuss how these may be interpreted if the source of the moment is a current loop.

(c) What is the energy of this dipole configuration when an external field $\mathbf{B}$ is applied?

Discuss the effect of this magnetic interaction on the orientation and position of the dipole.

(d) By once again considering the source of the dipole as being an infinitesimal element of a current carrying wire, argue that the force this part of the dipole $\vec{\mu}$ experiences in the field B is

$$\mathrm{d}\mathbf{F} = I\,\mathrm{d}\mathbf{r} \times \mathbf{B}. \tag{1}$$

Therefore, by integrating over the loop of wire, derive the expression for the torque $\vec{\tau}$ that the dipole experiences.

(e) By equating the torque to the rate of change of angular momentum, and using part (a), show that the magnetic moment precesses with a frequency of $\omega_L = \gamma B$.

Exercise 1.2: Angular Momentum & Spin (Homework Question)

(a) Argue semiclassically that the magnetic moment of an electron in the ground state of a hydrogen atom is given by $\mu = -\mu_B = -e\hbar/2m_e$, where μ_B is the Bohr magneton.

To do this, consider the orbit as a single-electron current with radius r_e.

What are the gyromagnetic ratio and Lamor frequency for this single-electron system?

(b) This question will discuss the group structure of angular momentum in a quantum mechanical context.

In 2D the group of rotations on the plane is $\mathrm{Spin}(2) = SO(2)$, with elements that are the rotation matrices. Write a quantum mechanical operator $U_L(\theta)$ which acts on the coordinates in this way

$$U_L(\theta) \begin{pmatrix} x \\ y \end{pmatrix} U_L^\dagger(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2}$$

(i) Now argue that one may write this generator of rotations in terms of the Hermitian angular momentum operator

$$U_L(\theta) = \exp(iL\theta/\hbar), \tag{3}$$

by showing that for infinitesimal θ, (2) gives the correct commutation relation for L.

(ii) Use the fact that the coordinates are unchanged by a 2π rotation to show that L has quantised eigenvalues.

We may define the spin operator S as being a new operator which generates the same group as L but without modifying the coordinate eigenstates. Does S have quantised eigenvalues in 2D?

(c) In 3D the rotation group is $SO(3)$, generated by rotations in three directions. The operators which generate the quantum spin group are S_i satisfying

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \tag{4}$$

(i) Show that the Pauli matrices $S_i = (\hbar/2)\sigma_i$ satisfy this commutation relation (4).

This is the spin-1/2 representation of the $\mathrm{Spin}(3) = SU(2)$ algebra (or the 2×2 traceless Hermitian matrices).

Show that the general such matrix can be written as $\vec{\sigma} \cdot \mathbf{d}$, where $\mathbf{d} = (d_x, d_y, d_z)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

Furthermore show that the eigenvalues of such a matrix are $\pm|\mathbf{d}|$.

(ii) Now we will investigate spin-s reps of $SU(2)$.

What does the non-commuting nature of the spin operators S_i mean for the possibility of simultaneous measurement of the components?

Therefore argue why we must label states by $|s, s_z\rangle$ where s is the total spin eigenvalue and s_z the S_z eigenvalue.

(iii) Show that we may define raising and lowering operators

$$S_{\pm} = S_x \pm iS_y \tag{5}$$

and, through finding the commutation relations with S_z and S^2, show that these operators move between s_z states in a space of constant-s states.

(iv) Write S^2 in terms of $S_{\pm}$ and S_z.

Given that there exists a highest weight state $|s, s\rangle$ which is annihilated by S_+, show that $S^2 |s, s\rangle = \hbar^2 s(s+1) |s, s\rangle$, and generalise to all s_z.

(v) By considering $\langle s, s_z| S_{\mp}S_{\pm} |s, s_z\rangle$, show that

$$S_{\pm} |s, s_z\rangle = \sqrt{(s \mp s_z)(s \pm s_z + 1)}\, |s, s_z \pm 1\rangle\,. \tag{6}$$

Show that there exists a lowest weight (s_z) representation, and therefore find the degeneracy of a spin-s state.

(vi) What are the operators $S_{\pm}$, S^2 in the spin-1/2 Pauli basis?

(d) (i) The Dirac equation of electrons can be approximated in the non-relativistic setting to give

$$\left[\left[\frac{1}{2m_e}(p + eA)^2 - e\phi\right]\mathbf{1} + \frac{e}{m_e}\mathbf{S}\cdot\mathbf{B}\right]|\psi\rangle = E\,|\psi\rangle\,. \tag{7}$$

Where $|\psi\rangle$ is a 2-component spinor. What does this equation predict as the magnetic moment g?

Discuss why the kinetic term has this form.

(ii) Argue why (7) implies that the total spin $s = \frac{\hbar}{2}$ for the electron. Using the fact that rotations around the z-axis are generated by $U_S(\theta) = \exp(i\sigma_z\theta/2)$, show that $U_S(2\pi)\,|\psi\rangle = -\,|\psi\rangle$, and discuss its interpretation.

(iii) With $\mathbf{B} = B\mathbf{z}$, calculate evolution of spin eigenstates under the time-dependent Schrödinger equation to show the electrons undergo Lamor precession (take the electron to not be moving).

Exercise 1.3: Paramagnetism (Homework Question)

(a) Let us calculate the paramagnetic behaviour of non-interacting classical spins at a finite temperature.

(i) Apply a magnetic field B along the z-direction, and parameterise the spins with polar angle θ.

Writing the partition function of a single spin at temperature T as a sum over θ states, show that

$$Z = \int_0^\pi \exp(\mu B \cos\theta / k_B T) \sin\theta \, \mathrm{d}\theta \,, \tag{8}$$

up to a constant coefficient.

(ii) Therefore show that the magnetisation is

$$M = n \langle \mu_z \rangle = n\mu \left[\coth\left(\frac{\mu B}{k_B T} \right) - \frac{k_B T}{\mu B} \right] , \tag{9}$$

where n is the number density of spins.

Sketch this function and discuss its behaviour in low and high field B.

(iii) For low field, show that the susceptibility obeys Curie's law.

(b) Now we would like to investigate the paramagnetic response of a spin-1/2 system (with $g = 2$) in a magnetic field.

Show the energies of the states are $\pm\mu_B B$.

Hence show $Z = 2\cosh(\mu_B B/k_B T)$ and by calculating $F = -nk_B T \log Z$, show

$$M = -\left(\frac{\partial F}{\partial B} \right)_T = M_s \tanh\left(\frac{\mu B}{k_B T} \right), \tag{10}$$

and find the saturation magnetisaiton M_s.

(c) Show that for a quantum system with spin-J, the partition function is

$$Z = \sum_{m_J=-J}^{J} \exp(m_J g_J \mu_B B / k_B T). \tag{11}$$

Therefore find the magnetisation is given in terms of the Brillouin function, and confirm the results from (a) and (b) are recovered in the appropriate limits.

Exercise 1.4: Coupled Spins (Homework Question)

(a) Consider coupling two spin-1/2 particles a, b with a Heisenberg exchange term

$$H = J \mathbf{S}_a \cdot \mathbf{S}_b. \tag{12}$$

The total spin is given by $\mathbf{S}_{\text{tot}} = \mathbf{S}_a + \mathbf{S}_b$.

(i) Write down all states in the combined Hilbert space, in the simple tensor product basis.

Are these states eigenvalues of $\mathbf{S}_{\text{tot}}^2$ and $\mathbf{S}_{\text{tot},z}$?

(ii) Find a linear combination of these states which are eigenvalues of $\mathbf{S}_a \cdot \mathbf{S}_b$. Show that these are the singlet ($S = 0$) and triplet ($S = 1$) states, and write down the energies.

(iii) What is a perturbation which could be added to H that could split the triplet state?

(b) Generally, when taking tensor products of spin states by decomposing the spins under the eigenvalues of the combined spin operator, the product of a spin-S with spin-1/2 space, may be decomposed as follows

$$\mathcal{H}_S \otimes \mathcal{H}_{1/2} = \mathcal{H}_{S-1/2} \oplus \mathcal{H}_{S+1/2}. \tag{13}$$

Use this to show

$$\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} = \mathcal{H}_{3/2} \oplus \mathcal{H}_{1/2} \oplus \mathcal{H}_{1/2}. \tag{14}$$

Confirm that the dimensions of the product space is the same as the sum of dimensions of the three spaces we decompose into.

(c) Define the chiral spin operator

$$\chi = \mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3), \tag{15}$$

which commutes with the total spin operator $\mathbf{S}$. We may therefore categorise states as $|S, M_S; \chi\rangle$.

(i) Show $\left|\frac{3}{2}, \frac{3}{2}; 0\right\rangle = |{\uparrow\uparrow\uparrow}\rangle$, and find an expression for all $\left|\frac{3}{2}, M; 0\right\rangle$ in terms of this, using the lowering operator S_-.

(ii) Argue that $\chi\left|\frac{3}{2}, M; 0\right\rangle = 0$ by thinking about the symmetry of the eigenstate under exchanging pairs of spins.

(iii) Show that

$$\begin{aligned}\chi = \frac{i}{2}\big[- S_{-,1}S_{+,2}S_{z,3} + S_{+,1}S_{-,2}S_{z,3} \\ + S_{-,1}S_{z,2}S_{+,3} - S_{+,1}S_{z,2}S_{-,3} \\ - S_{z,1}S_{-,2}S_{+,3} + S_{z,1}S_{+,2}S_{-,3}\big]\end{aligned} \tag{16}$$

(iv) Now there are two spaces of spin-1/2 subspaces in the three-particle product space. We will distinguish these by their chiral eigenvalue.

Show that the following state is an eigenstate of χ and evaluate its eigenvalue

$$\left|\frac{1}{2}, \frac{1}{2}; +\right\rangle = \frac{1}{\sqrt{3}}\left[|{\uparrow\uparrow\downarrow}\rangle + \exp(2\pi i/3)\,|{\uparrow\downarrow\uparrow}\rangle + \exp(-2\pi i/3)\,|{\downarrow\uparrow\uparrow}\rangle\right]. \tag{17}$$

(v) Find an orthogonal state with $S = \frac{1}{2}$ and $M_S = \frac{1}{2}$ which has the negative eigenvalue $\left|\frac{1}{2}, \frac{1}{2}; -\right\rangle$.
How are the states $\left|\frac{1}{2}, -\frac{1}{2}; +\right\rangle$ related to these previous states?