1 Thermal Green's functions

The thermal Green's function is defined as

$$G_{ij}(\tau) = -\langle T_\tau \psi_i(\tau)\psi_j^\dagger(0)\rangle,$$

where T_τ denotes (imaginary) time ordering, $\psi_i(\tau) = e^{H\tau}\psi_i e^{-H\tau}$, the expectation value is defined as $\langle A\rangle = \frac{1}{Z}\operatorname{tr}(e^{-\beta H}A)$, and i, j denote single particle states. The path integral formulation of the Green's function of non-interacting particles (bosons or fermions) is:

$$G_{ij}(\tau) = -\frac{1}{Z}\int D(\bar\psi,\psi)\psi_i(\tau)\bar\psi_j(0)e^{-S(\bar\psi,\psi)}, \qquad S = \sum_i \int_0^\beta d\tau \bar\psi_i(\partial_\tau + \epsilon_i - \mu)\psi_i \tag{1}$$

1. Show that the path integral automatically takes care of the time ordering by repeating the construction of the path integral with $G_{ij}(\tau)$.

   Hint: Recall from the lecture the definition of time ordering for both bosons and fermions, and that in order to evaluate the path integral, operators have to be normal ordered, i.e., creation operators to the left of annihilation operators.

2. Evaluate the Gaussian integral and obtain $G_{ij}(\tau)$ as a Matsubara sum for bosons and for fermions.

3. The occupation number in a single particle state i is in general given by $\lim_{\tau\to 0^-} G_{ii}(\tau)$. Compute the occupation number for bosons and fermions by performing the Matsubara summation. (Note: This is of course not the easiest way to obtain this result for non-interacting particles, but it is easily generalizable to interacting systems.)

4. The generating functional for correlation functions is defined as

$$\mathcal{Z}[\bar J, J] = \int D(\bar\psi,\psi)\exp\left\{-S[\bar\psi,\psi] - \sum_i\int_0^\beta d\tau(\bar J_i\psi_i + \bar\psi_i J_i)\right\}, \tag{2}$$

   where the J_i are (virtual) sources (complex or Grassmannian) introduced for the evaluation of n-point correlation functions. These can be obtained as functional derivatives of $\mathcal{Z}[J]$, where the source fields are set to zero after the evaluation:

$$\langle T_\tau \psi_{i_1}(i\omega_1)\cdots\psi_{i_n}(i\omega_n)\bar\psi_{j_1}(i\omega_{n+1})\cdots\bar\psi_{j_n}(i\omega_{2n})\rangle \tag{3}$$

$$= \frac{\zeta^n}{\mathcal{Z}[0,0]}\frac{\delta^{2n}\mathcal{Z}[\bar J, J]}{\delta\bar J_{i_1}(i\omega_1)\cdots\delta\bar J_{i_n}(i\omega_n)\delta J_{j_1}(i\omega_{n+1})\cdots J_{j_m}(i\omega_{2n})}\bigg|_{J,\bar J=0} \tag{4}$$

   Formulate the generating functional $\mathcal{Z}[\bar J, J]$ explicitly for the action given in (1) and calculate the Green's function $G_{ij}(i\omega_n)$ in Matsubara space via functional differentiation. Compare to your calculation in question 2.

2 Nambu-Goldstone Modes in the Heisenberg Ferromagnet

We consider an isotropic Heisenberg ferromagnet with spin 1/2-particles fixed to the sites of a lattice:

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j \tag{5}$$

with $J_{ij} > 0 \ \forall i, j$. In the ground state, all spins are aligned in the same direction denoted by $\mathbf{n}$ with $|\mathbf{n}| = 1$. Let us label the ground states by their orientation in space: $|0_\mathbf{n}\rangle = \bigotimes_{i=1}^{N} |i, \mathbf{n}\rangle$ with the single site states satisfying $\mathbf{n} \cdot \hat{\mathbf{S}}_i |i, \mathbf{n}\rangle = -\frac{1}{2} |i, \mathbf{n}\rangle$.

1. Show that in the thermodynamic limit $N \to \infty$ (N being the number of sites), different ground states are orthogonal, i.e.

$$|\langle 0_{\mathbf{n_1}} | 0_{\mathbf{n_2}}\rangle| = \left[\cos\left(\frac{\theta_{\mathbf{n}_1\mathbf{n}_2}}{2}\right)\right]^N \overset{N\to\infty}{\to} 0 \quad \text{unless} \quad \mathbf{n}_1 = \mathbf{n}_2 \tag{6}$$

   where $\theta_{\mathbf{n}_1\mathbf{n}_2}$ is the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

   Hint: Find an expression for the (normalized) eigenvector of $\mathbf{n} \cdot \hat{\mathbf{S}}_i$ for a single site with eigenvalue -$\frac{1}{2}$. When computing the overlap $|\langle 0_{\mathbf{n_1}} | 0_{\mathbf{n_2}}\rangle|$, you can assume that one vector is parallel to the z-axis of the coordinate system.

2. Let us fix the orientation of the ground state $\mathbf{n} = \mathbf{e}_z$. We can describe the system now using the raising/lowering operators $\hat{S}_i^+$, $\hat{S}_i^-$ with $\hat{S}_i^- |i, \mathbf{e}_z\rangle = 0$. They satisfy the following relations:

$$[\hat{S}_i^+, \hat{S}_i^-] = 2\,\hat{S}_i^z \qquad\qquad \{\hat{S}_i^-, \hat{S}_i^+\} = 1$$
$$[\hat{S}_i^-, \hat{S}_j^+] = [\hat{S}_i^-, \hat{S}_j^-] = [\hat{S}_i^+, \hat{S}_j^+] = 0 \quad \text{if } i \neq j$$

   Show that using these operators, the Hamiltonian can be written as

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\left[-\frac{1}{2}\left(\hat{S}_i^+ - \hat{S}_j^+\right)\left(\hat{S}_i^- - \hat{S}_j^-\right) + \hat{S}_i^+\hat{S}_i^-\hat{S}_j^+\hat{S}_j^- + \frac{1}{4}\right] \tag{7}$$

3. Due to equation (6), the Hilbert space is given as the Fock space above the vacuum $|0_{\mathbf{e}_z}\rangle$. The Nambu-Goldstone boson is a spin wave, a travelling perturbation induced by flipping a single spin. Hence, we can restrict ourselves to the one-particle Hilbert space with states $|i\rangle = \hat{S}_i^+ |0_{\mathbf{e}_z}\rangle$. Neglecting constant terms, show that the one-particle Hamiltonian acts as

$$H_{1P} |i\rangle = \frac{1}{2}\sum_j J_{ij} \left(|i\rangle - |j\rangle\right) \tag{8}$$

   Assuming $J_{ij} = J(|\mathbf{x}_i - \mathbf{x}_j|)$, show that the plane wave state $|\mathbf{k}\rangle = \sum_j e^{i\mathbf{k}\cdot\mathbf{x}_j} |j\rangle$ is an eigenstate of H_{1P} with the energy

$$E_\mathbf{k} = \frac{1}{2}\left(J_\mathbf{0} - J_\mathbf{k}\right) . \tag{9}$$

Here, we defined $J_{\mathbf{k}} = \sum_j J(|\mathbf{x}_j|)e^{-i\mathbf{k}\cdot\mathbf{x}_j}$. Evaluate $E_{\mathbf{k}}$ for a constant nearest neighbour interaction on a square lattice.

Remark: The existence of a **gapless mode** is a general consequence of the **Nambu-Goldstone theorem** for spontaneous symmetry breaking in systems with a continuous symmetry group. In the lecture you have encountered this paradigm in the example of a Bose gas experiencing symmetry breaking of a $U(1)$ group selecting the phase of the complex bosonic field. The linear dispersing Goldstone mode is associated to phase fluctuations of the symmetry broken ground state. In the **Heisenberg Ferromagnet** we in contrast find a **quadratic dispersion** at low momenta for the gapless mode. The general expectation is that the symmetry breaking of the ground state from $SU(2) \to U(1) \times \mathbb{Z}_2$ would indicate the existence of 2 linear dispersion Goldstone modes, as two generators are broken. Following part 3. of this problem, however, we can excite only **one gapless spin wave** excitation, which shows an quadratic dispersion. This is a consequence of the fact that these generators are not independent when evaluating them with the FM ground state. By contrast, for the **Heisenberg Antiferromagnet** we can excite **two spin waves** (one located at momentum $k = 0$ and one at $k = \pi$) in the ordered phase and get the usual **linear dispersion** for both modes. For more details see the article Phys Rev D 84.125013 or the book by Anthony Zee *Group Theory in a Nutshell*.