Time and frequency domain analysis of an electronic low pass filter

Krish Shah
Lab Group 37
Girton College

This report details the theoretical and experimental time and frequency domain analysis of a first order Butterworth low pass filter. Using Ohms and Kirchoffs Laws, we derive equations for the gain and phase of the filter, as well as the charging response in DC current. Experimental data must be scrutinized due to the large uncertainties of electrical components. We use sophisticated error propogation tools to show that almost all experimental data lies well within the tolerances of the components used.

Introduction and Objectives

Low pass filters act to block the high frequency component of an arbitary signal. This property makes them very useful in acoustics and electronics, where they have applications ranging from subwoofer electronics to telecommunications equipment.

This report has the following aims and objectives:

- To compare the measurements obtained from the picoscope readings with expected theoretical calculations of the nominal value of the RC time constant of the components.
- To understand the impact of the resistor tolerance (5%) and the capacitator tolerance (20%) on the value of the RC time constant and the expected performance of the components.
- To investigate the attenuation vs frequency behaviour of a low pass filter in terms of dB/decade.

Experimental Method

Two experiments were completed for this report, following the guidance in the lab handouts provided. Both experiments involved the analysis of a $100\,\mathrm{k}\Omega$ resistor in series with a $1\mu\mathrm{F}$ capacitor. The circuit diagram below labels the input voltage, $v_{\mathrm{in}}(t)$ and the output voltage, $v_{\mathrm{out}}(t)$ as functions of time.

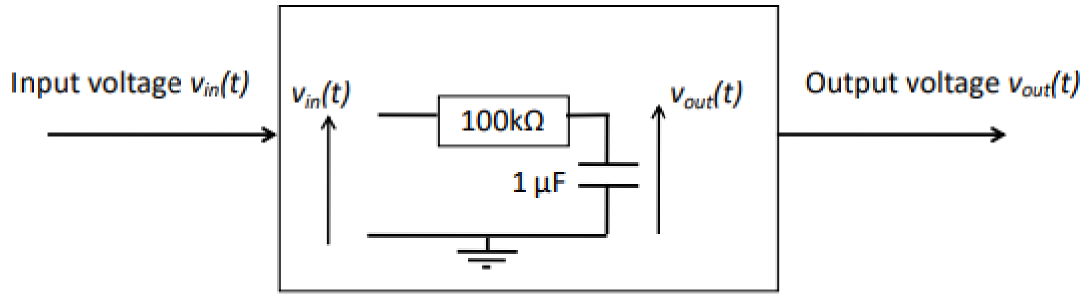

Figure 1: The circuit diagram used throughout this experiment

IEP Exercise B involved the time domain analysis of the circuit. This experiment measured the charging response of the capacitator, with $v_{\mathrm{in}}(t)$ equal to

$$v_{\mathrm{in}}(t) = \begin{cases} 0 & t < 0 \\ V_{max} & t \geq 0 \end{cases}$$

IEP Exercise C involved the frequency domain analysis of the circuit. In this experiment the frequency of the input voltage was varied and the output voltage was measured.

Theory

First, we consider the time domain analysis. The experiment begins with the voltage across all components set to zero, so $v_{\text{in}} = v_{\text{out}} = 0$ for $t < 0$.

At $t \geq 0$, $v_{\text{in}}(t)$ is equal to V_{max}. Applying Ohm's Law and the definition of capacitance, we can derive a differential equation for $v_{\text{out}}(t)$ for when $t \geq 0$.

$$I = \frac{V_{max} - v_{\text{out}}(t)}{R} = C\frac{d[v_{\text{out}}(t)]}{dt}$$

This can be readily solved to give an equation for $v_{\text{out}}(t)$. Putting everything together, we have:

$$v_{\text{out}}(t) = \begin{cases} 0 & t < 0 \\ V_{max}\left[1 - \exp\left(\frac{-t}{RC}\right)\right] & t \geq 0 \end{cases}$$

In practice, our charging response data is not perfectly zeroed. To counteract this, we shift the input times by a fixed amount, t_d, such that the data is closer the theoretical line of best fit.

For the second experiment, we begin by using Kirchoffs laws to get an expression for the ratio of v_{in} and v_{out}.

$$\frac{v_{\text{out}}}{v_{\text{in}}} = \frac{\frac{1}{R}}{\frac{1}{R} + j\omega C} = \frac{1}{\sqrt{1 + (\omega RC)^2}} \exp\left[-j\tan^{-1}(\omega RC)\right]$$

From this equation we can inferr that the phase shift between v_{in} and v_{out} is $\phi = -\tan^{-1}(\omega RC)$.

Next, we find the theoretical attenuation in decibels.

$$20\log_{10}\left(\frac{V_o}{V_i}\right) = 20\log_{10}\left[\frac{1}{\sqrt{1 + (\omega RC)^2}}\right] = -10\log_{10}\left[1 + (\omega RC)^2\right]$$

At very low frequencies, this equation is approximated by a straight line of zero slope. For large frequencies, $\log\left(1 + (\omega RC)^2\right) \approx 2\log(\omega RC)$, so the attenuation is approximated by a straight line of −20 dB per decade slope.

Results

Uncertainties are a decisive factor in our analysis. The resistor and capacitor have tolerances of 5% and 20% respectively, making the value of RC differ by up to 25%. We need error propagation tools in order to determine whether experiment differs significantly from theory.

In light of these issues, the analysis is conducted in the Julia programming language using the `Measurements.jl` module. The module manages uncertainties in a comprehensive manner and has functional correlation built in, so it understands that `sec²(x) - tan²(x) ≈ 1 ± 0`.

Entering values for R and C into the program, we can see that the uncertainty of $\frac{1}{RC}$ given by the program agrees with the answer when calculated by hand:

```
10.0 ± 2.0 s^-1
```

```
begin
    R    = 100e3 ± 1000    # Resistance values from lab handout
    C    = 1e-6  ± 2e-7    # Capacitance values from lab handout
    (1/(R * C))u"s^-1"     # Display the value of 1/RC (with units)
end
```

Using this software package, we can plot our analysis comparing theoretical quantities with measured ones. In each case the values of R and C used are the same as above. We represent the area of uncertainty on each graph below with a blue shaded region.

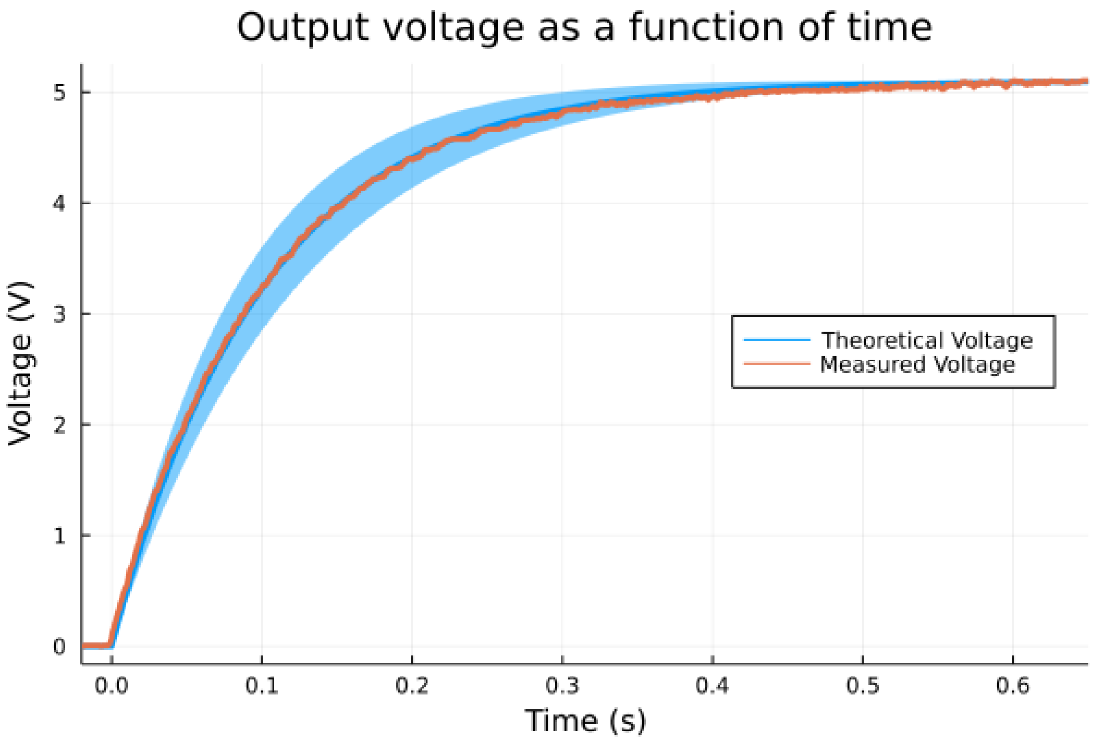

Figure 2: A graph showing the output voltage as a function of time. The theoretical voltage curve used $V_{max} = 5.102 \pm 0.0002\text{V}$, obtained from the picoscope measuring software. The measured voltage was shifted by $t_d = 0.018\text{s}$ in the time domain to zero the picoscope readings.

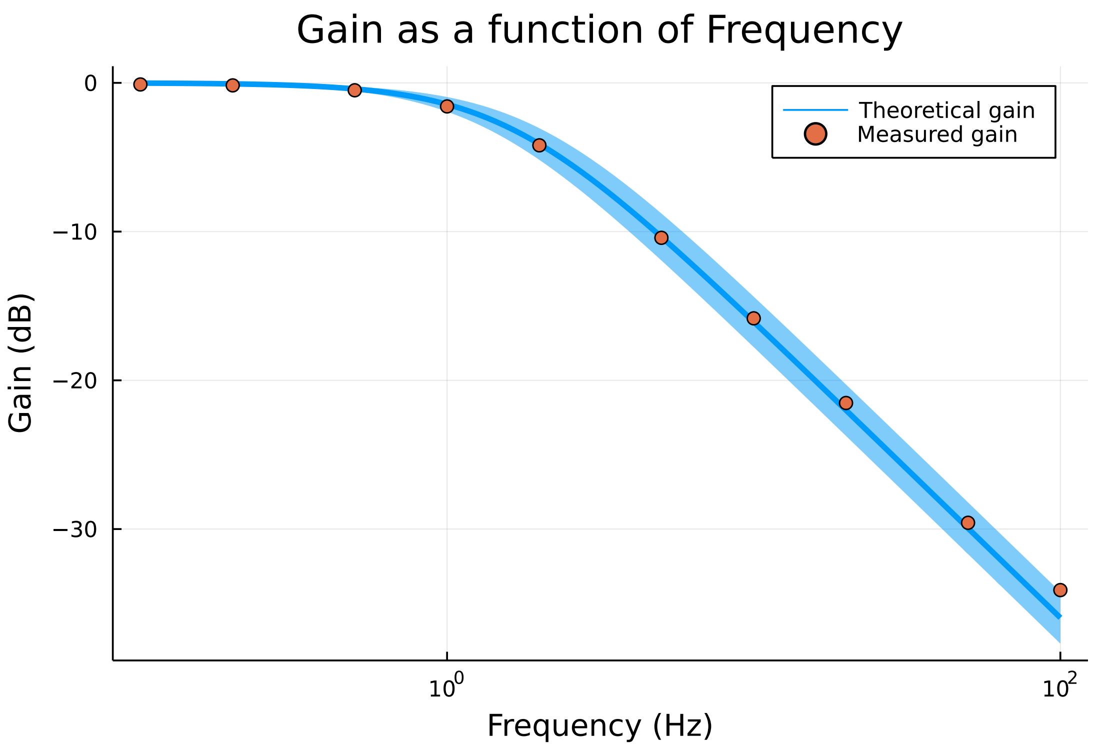

Figure 3: A Bode Magnitude plot showing gain as a function of frequency.

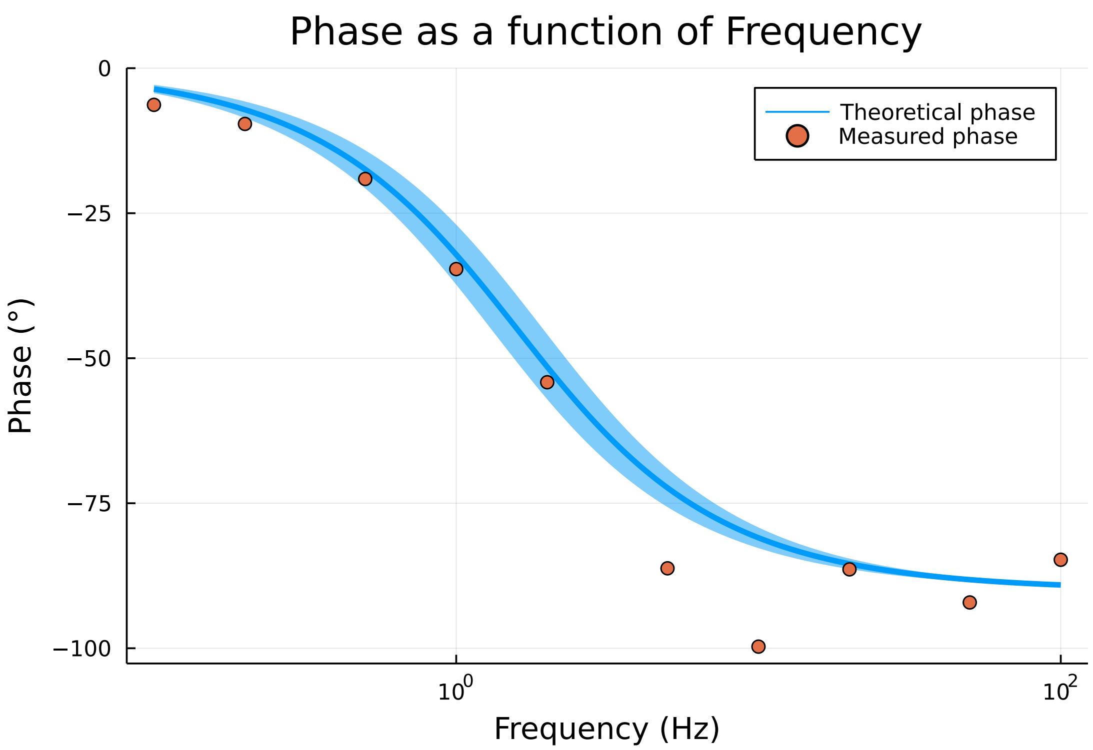

Figure 4: A Bode Phase plot showing phase as a function of frequency.

Discussion

Time domain analysis agrees almost perfectly with theoretical data.

- In figure 2, all points are within the shaded blue region, which shows that the measurement is consistent with the resistance and capacitance within the tolerances given.

Frequency domain analysis agrees strongly with theoretical data, but hints strongly towards human error.

- In figure 3, all points are within the shaded blue region, so the measurements are consistent with the resistance and capacitance within the tolerances given.
- Figure 4 required the phase to be recorded manually. This likely made the measurements far less accurate and more prone to human error. Indeed, the author particularly regrets their lack of discipline in measuring the last few data points.

Conclusions

1. Time and frequency domain analysis was performed on a low pass filter with a $1\mu\text{F}$ capacitor and a $1\text{k}\Omega$ resistor to obtain a value for the RC time constant and plot Bode plots for gain and phase as functions of frequency.
2. Time domain showed that the RC time constant measured was very accurate, virtually identical to the expected value
3. The measured data for the phase as a function of frequency was less accurate, almost certainly due to the manual process in which the phase was determined.
4. The Julia programming language presents a unique set of tools that could be useful in the Engineering Department

Appendix: Data

	V_{in}	V_{out}	Frequency	Delay	Gain	Phase
1	723.7 mV	714.8 mV	0.1 Hz	175.6 ms	-0.107481 dB	-6.3216°
2	703.4 mV	689.8 mV	0.2 Hz	133.5 ms	-0.169583 dB	-9.612°
3	680.9 mV	642.9 mV	0.5 Hz	106.1 ms	-0.498798 dB	-19.098°
4	687.7 mV	573.1 mV	1.0 Hz	96.2 ms	-1.58337 dB	-34.632°
5	686.2 mV	423.1 mV	2.0 Hz	75.2 ms	-4.20015 dB	-54.144°
6	682.8 mV	205.8 mV	5.0 Hz	47.9 ms	-10.417 dB	-86.22°
7	683.2 mV	110.4 mV	10.0 Hz	27.7 ms	-15.8316 dB	-99.72°
8	682.9 mV	57.3 mV	20.0 Hz	12.0 ms	-21.524 dB	-86.4°
9	682.6 mV	22.66 mV	50.0 Hz	5.117 ms	-29.5781 dB	-92.106°
10	682.7 mV	13.46 mV	100.0 Hz	2.354 ms	-34.1037 dB	-84.744°