EE 364
J. M. Mendel

A Survivor's Guide to the Mathematics Used in Probability

1. Set Theory

1. The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. Our textbook denotes the complement of A by A'. Many people denote it as $\overline{A}$ or A^c. Get used to different notations.

2. The **intersection** of two events A and B, denoted as $A \cap B$, is the event containing all elements that are common to A and B (i.e., that are in both A and B).

3. The **union** of two events A and B, denoted as $A \cup B$, is the event containing all the elements that belong to A or B or both.

4. Two events A and B are **mutually exclusive**, or **disjoint** if $A \cap B = \varnothing$, i.e. if A and B have no elements in common.

5. Set theoretic operations:

Idempotency: $A \cup A = A$ and $A \cap A = A$
Commutativity: $A \cup B = B \cup A$ and $A \cap B = B \cap A$
Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
Absorption: $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$
Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Involution: $\overline{\overline{A}} = A$
DeMorgan's Laws: $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$
Identity:

$$A \cup \varnothing = A,\ A \cap S = A$$

$$A \cup S = S,\ A \cap \varnothing = \varnothing$$

6. The inclusive versus the exclusive OR

a. inclusive OR: A OR B means A or B or both
e.g. You win a lottery if any number with a 6 or 7 shows up, means you win with a 6 or 7 or both, i.e. 6 OR 7

b. exclusive OR: A XOR B means A or B but not both
e.g. If you order a Coke or a 7-Up, you really mean a Coke or a 7-Up but not both, i.e. your Coke XOR 7-Up.

Unless otherwise stated we shall always mean the inclusive OR in our works.

7. The union $C + D$ of any two sets C, D can **always** be written as: $C + D = C + C'D = C * C'D$, where * represents a union with the understanding that the quantities it connects are *mutually exclusive*.

8. If $AB = \varnothing$, then $A + B = A * B$. Also, if $AB = \varnothing$, then $A + B = A * B = A * A'B$ since $A'B = B$.

2. Counting Techniques: Permutations and Combinations

1. A permutation is an arrangement of all or part of a set of objects.

2. The number of permutations of n distinct objects is $n!$

3. The number of permutations of n distinct objects taken r at a time is ${}_nP_r = \dfrac{n!}{(n-r)!}$

4. The number of permutations of n distinct objects arranged in a circle is $(n-1)!$

5. The number of distinct permutations of n things of which n_1 are of one kind, n_2 are of a second kind, ..., n_k of a kth kind is $n!/n_1!n_2!\cdots n_k!$.

6. The number of combinations of n distinct objects taken r at a time is $\dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$ (derivation included on next two pages).

7. The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second cell, and so forth, is
$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$$
where $n_1 + n_2 + \cdots + n_r = n$.

8. Some identities:

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$

$$\binom{n}{n} = \binom{n}{0} = 1$$

Combinations

Ref: Lady Luck: the theory of Probability, W. Weaver, Anchor Books, Doubleday & Co., Garden City, N.Y., 1963

Sometimes one is not concerned about the *order* of the objects (letters, cards, numbers, or whatever), but is only interested in the *constitution* of the sample in question. Thus suppose we are dealing with the five letters A, B, C, D, and E. How many essentially different groups or *combinations*, to use the technical term, of three letters can one make out of these five letters? ABC is one combination. ABD is another, for you call it a different combination if only one element differs. So ABE is another, BCD another, and so on. How many in all are there?

With only a few items one could get the answer just by writing them all down in some systematic way. But it is important to have a general formula which applies no matter how many items there are.

If you ask: How many combinations can I make of five things if I take them *all*? Clearly there is only *one* such choice. I just take all the five items available — and that is that. Indeed, the number of combinations of n things, taken all n at a time, is clearly *one*, no matter what n is. In the case of the *permutations* of n things, it is important to talk about the number taken n at a time, or taken r at a time, r being less than n. In the case of *combinations* of n things, it is trivial to take all n, so one proceeds at once to the general formula for the "combinations of n things taken r at a time."

This general formula can be derived by a very simple trick. Suppose we denote *the number of combinations of n things taken r at a time* by $\binom{n}{r}$. In older books you may see the notation ${}_nC_r$ which is naturally related to the notation for the permutations of n things taken r at a time, but nowadays this older notation for combinations is seldom used.

Now suppose we take each one of these *combinations*, which contains r elements, and rearrange (or "permute") these r elements so as to obtain all the permutations or arrangements of these particular r elements. We could, as the previous section proved, produce $r!$ arrangements of these r elements.

Now do this with another of the combinations, and with another, and so on—until you have made $r!$ different permutations out of each one of these $\binom{n}{r}$ number of combinations. Every single one of the r-fold permutations thus made will differ from every other one, either in *constitution* (because of having started out with a different combination) or in *order* (because of being different permutations of that combination).

The totality of all of these r-fold permutations will obviously be all the r-fold permutations one can make, starting out with n objects. But that total number is known, from the previous section, to be ${}_nP_r$ as given by formula (5) $[n!/(n-r)!]$

Thus we have $\binom{n}{r}$ combinations, each one of which makes $r!$ permutations, and the totality is ${}_nP_r$. In other words

$$r!\cdot\binom{n}{r} = {}_nP_r = \frac{n!}{(n-r)!}$$

or

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \tag{6}$$

3. INEQUALITIES AND THEIR GRAPHS

Ref: The Probability Tutoring Book, C. Ash, IEEE Press, IEEE, NY, 1993.

Here's one way to draw the graph of the inequality $x - y < 2$. First, draw the graph of the equation $x - y = 2$, a line. The line divides the plane into two regions (Fig. 1). One of the regions corresponds to $x - y < 2$ and the other to $x - y > 2$. To decide which region goes with which inequality, test a point. Point $(100, 0)$ for instance is in region 2, and it satisfies the inequality $x - y > 2$. So it is region 1 that must be the graph of $x - y < 2$.

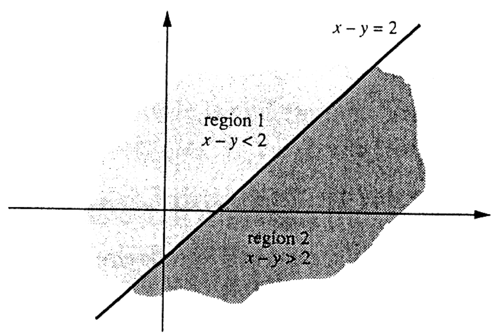


Figure 1

Another way to graph inequalities is to solve for y. The graph of

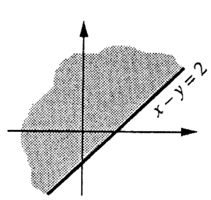
$$y < x - 2$$

is the region *below* line $y = x - 2$ and the graph of

$$y > x - 2$$

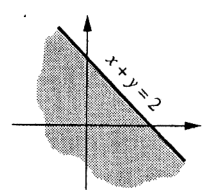
is the region *above* line $y = x - 2$.

Here are some more graphs of inequalities.



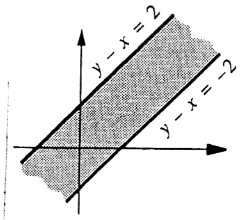
$x - y < 2$

Figure 2



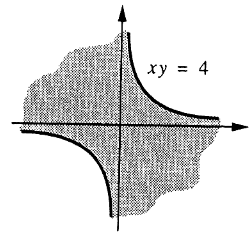
$x + y < 2$

Figure 3



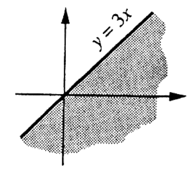
$|y - x| < 2$

Figure 4



$xy < 4$

Figure 5



$y < 3x$

Figure 6

Here are some trickier ones:

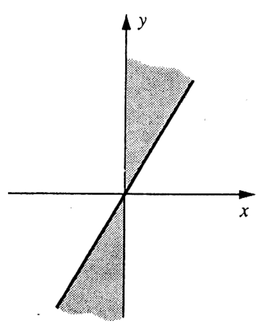
The inequality $y/x > 3$ is equivalent to

$$y > 3x \quad \text{if} \quad x > 0$$

$$y < 3x \quad \text{if} \quad x < 0$$

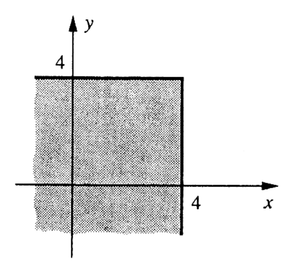
So the graph is *above* line $y = 3x$ on the right side and *below* $y = 3x$ on the left side (Fig. 1).

The inequality $\max(x, y) < 4$ means $x < 4$ *and* $y < 4$ so the graph is to the left of line $x = 4$ and below line $y = 4$ (Fig. 2).



$\frac{y}{x} > 3$

Figure 1

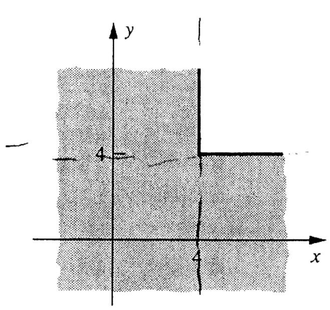


$\max(x, y) < 4$

Figure 2

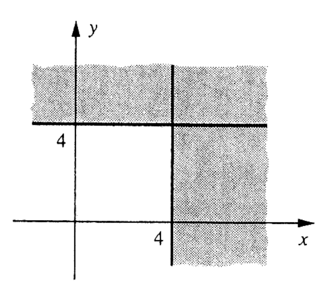
The inequality $\max(x, y) > 4$ means $x > 4$ *or* $y > 4$ so the graph includes all points that are either to the right of line $x = 4$ or above line $y = 4$ (Fig. 4).

The inequality $\min(x, y) < 4$ means $x < 4$ *or* $y < 4$ so the graph includes all points that are either to the left of line $x = 4$ or below line $y = 4$ (Fig. 3).



$\min(x, y) < 4$

Figure 3



$\max(x, y) > 4$

Figure 4

4. SERIES EXPANSIONS.

Note: All of the remaining items in this Guide are from the Ash book (see Section 3).

- Series for e^x

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x \text{ for all } x$$

- Series for e^{-x} (replace x by $-x$ in the e^x series)

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots = e^{-x} \text{ for all } x$$

- Limit for e^a $\qquad \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$

- Geometric Series

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \quad \text{for } -1 < x < 1$$

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r} \quad \text{for } -1 < r < 1$$

- Differentiated Geometric Series

$$1 + 2x + 3x^2 + 4x^3 + \cdots = D\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

for $-1 < x < 1$

- Finite Geometric Series

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r}$$

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

- Differentiated Finite Geometric Series

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = D\left(\frac{1-x^{n+1}}{1-x}\right)$$

$$= \frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}$$

5. INTEGRALS

Some Integrals for Reference

$$\int_0^1 x^n\,(1-x)^m\,dx = \frac{n!m!}{(n+m+1)}!$$

$$\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right)$$

$$\int x^2\,e^{ax}\,dx = e^{ax}\left(\frac{2}{a^3} - \frac{2x}{a^2} + \frac{x^2}{a}\right)$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$$

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \text{for } a > 0$$

$$\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$$

$$\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}} \qquad \text{for } a > 0$$

$$\int_0^\infty x^n\,e^{-x}\,dx = n! \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

$$\int_0^\infty x^n\,e^{-ax}\,dx = \frac{n!}{a^{n+1}} \qquad \text{for } a > 0; \quad n = 0, 1, 2, 3, \ldots$$

6. Differentiating & Integrating Functions That Change Formula

Suppose

$$f(x) = \begin{cases} x & \text{for } 2 \le x \le 3 \\ x^2 & \text{for } 3 \le x \le 5 \end{cases}$$

On line 2, change $3 \le x$ to $3 < x$. Observe that f(x) is discontinuous at $x = 3$.

Make the same change on line 2 of f'(x).

Then

$$f'(x) = \begin{cases} 1 & \text{for } 2 \le x \le 3 \\ 2x & \text{for } 3 \le x \le 5 \end{cases}$$

and

$$\int_2^5 f(x)\,dx = \int_2^3 x\,dx + \int_3^5 x^2\,dx = \frac{x^2}{2}\bigg|_2^3 + \frac{x^3}{3}\bigg|_3^5 = \frac{5}{2} + \frac{98}{3} = \frac{211}{6}$$

Similarly, if

$$g(x) = \begin{cases} x^2 & \text{if } 2 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

then

$$\int_{-\infty}^{\infty} g(x)\,dx = \int_2^5 x^2\,dx = \frac{x^3}{3}\bigg|_2^3 = \frac{19}{3}$$

7. Integrating With Absolute Values

$$\int_{-2}^{3} |x|\, dx = \int_{-2}^{0} -x\, dx + \int_{0}^{3} x\, dx \quad \text{(Fig. 1)}$$

$$= -\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{3} = \frac{13}{2}$$

$$\int_{-1}^{3} e^{|x|}\, dx = \int_{-1}^{0} e^{-x}\, dx + \int_{0}^{3} e^{x}\, dx \quad \text{(Fig. 2)}$$

$$= -e^{-x}\Big|_{-1}^{0} + e^{x}\Big|_{0}^{3} = e + e^3 - 2$$

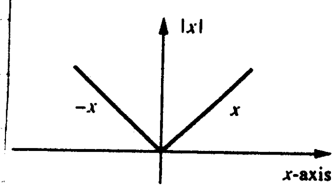


Figure 1

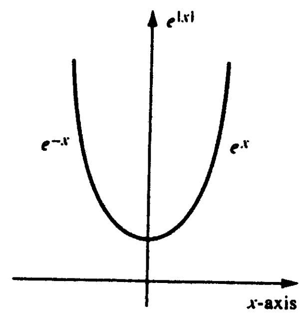


Figure 2

8. Double Integrals

There are two ways to set up $\int f(x,y)\,dA$ over a region R.

$$\int_R f(x,y)\,dA = \int_{\text{lowest } y \text{ in } R}^{\text{highest } y \text{ in } R} \int_{x \text{ on left bdry}}^{x \text{ on right bdry}} f(x,y)\,dx\,dy \text{ (Fig. 5)}$$

$$\int_R f(x,y)\,dA = \int_{\text{leftmost } x \text{ in } R}^{\text{rightmost } x \text{ in } R} \int_{y \text{ on lower bdry}}^{y \text{ on upper bdry}} f(x,y)\,dx\,dy \text{ (Fig. 6)}$$

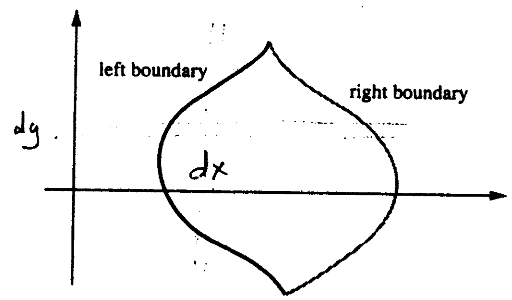


Figure 5

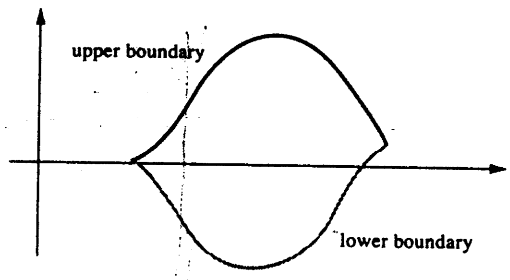


Figure 6

Example 1

I'll set up

$$\int_R f(x,y)\,dA$$

where R is the triangular region in Fig. 7.

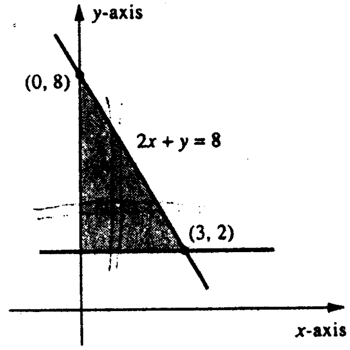


Figure 7

Method 1 (Fig. 8).

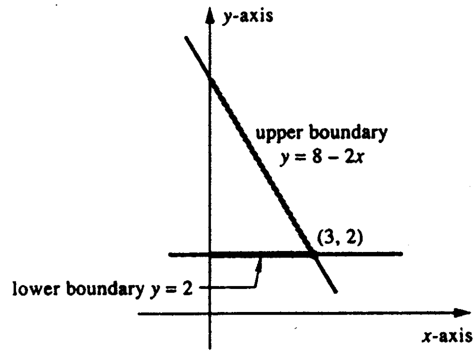


Figure 8

$$\int_R f(x,y)\,dA = \int_{x=0}^{3}\int_{y=2}^{y=8-2x} f(x,y)\,dy\,dx$$

Method 2 (Fig. 9).

$$\int_R f(x,y)\,dA = \int_{y=2}^{8}\int_{x=0}^{4-y/2} f(x,y)\,dx\,dy$$

Warning

Inner limits are *boundary* values; outer limits are *extreme* values.
The inner limits will *contain the other variable* unless the boundary in question is a vertical or horizontal line with an equation as simple as $x = 0$ or $y = 2$.
The limits on the outer integral are *always constants*.

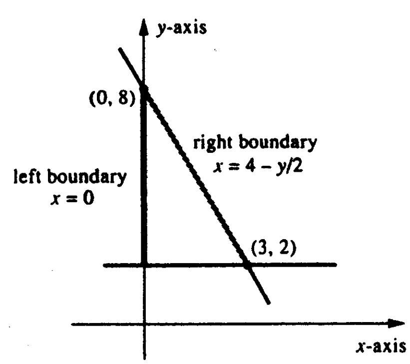


Figure 9

Integrating on a Region with a Two-Curve Boundary

Consider $\int_R f(x,y)\,dA$, where R is the region bounded by the line $x + y = 6$ and the parabola $x = y^2$ (Fig. 10).

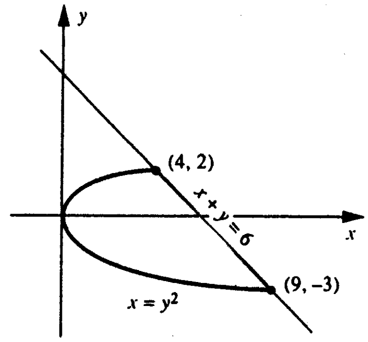


Figure 10

Method 1. (Fig. 11).

$$\int_R f(x,y)\,dA = \int_{y=-3}^{2}\int_{x=y^2}^{x=6-y} f(x,y)\,dx\,dy$$

Method 2. (Fig. 12).

The lower boundary is the parabola $x = y^2$; solve the equation for y to get $y = -\sqrt{x}$ (choose the negative square root because y is negative on the lower part of the parabola).

But *the upper boundary consists of two curves*, the parabola and the line. To continue with this order of integration, divide the region into the two indicated parts, I and II:

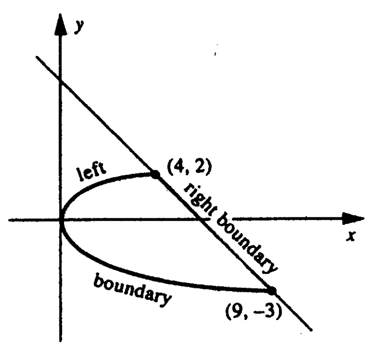


Figure 11

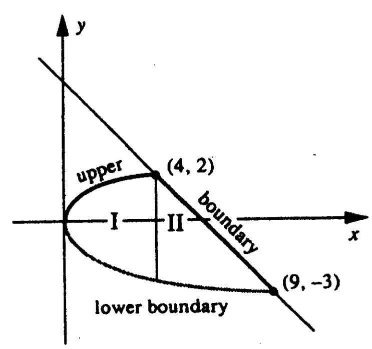


Figure 12

$$\int_R f(x,y)\,dA = \int_I f(x,y)\,dA + \int_{II} f(x,y)\,dA$$

$$= \int_{x=0}^{4}\int_{y=-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy\,dx + \int_{x=4}^{9}\int_{y=-\sqrt{x}}^{y=6-x} f(x,y)\,dy\,dx$$

Warning

Examine the boundaries carefully to catch the ones consisting of more than one curve.

Examples

Here are two regions of integration with limits put in both ways.

For the unbounded region in Fig. 13, the limits are

$$\int_{y=-\infty}^{0}\int_{x=1/y}^{\infty} + \int_{y=0}^{\infty}\int_{x=-\infty}^{1/y}$$

and also

$$\int_{x=-\infty}^{0}\int_{y=1/x}^{\infty} + \int_{x=0}^{\infty}\int_{y=-\infty}^{1/x}$$

For the region in Fig. 14, the limits are

$$\int_{y=0}^{2}\int_{x=y}^{x=y+3}$$

and also

$$\int_{x=0}^{2}\int_{y=0}^{x} + \int_{x=2}^{3}\int_{y=0}^{2} + \int_{x=3}^{5}\int_{y=x-3}^{2}$$

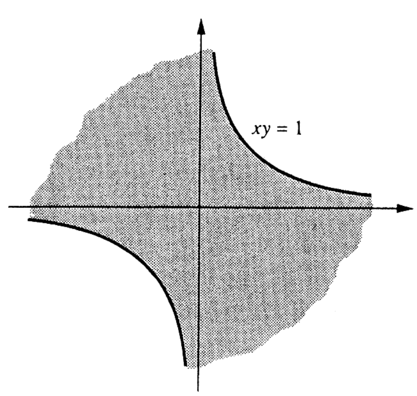


Figure 13

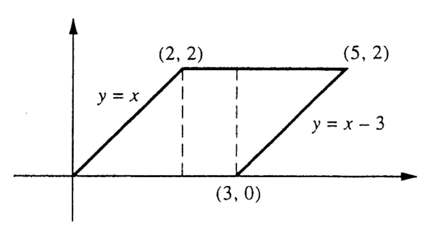


Figure 14

Double Integrating in Polar Coordinates

To find $\int_R f(X, y)\, dA$, let

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r^2 = x^2 + y^2$$
$$dA = r\ dr\ d\theta$$

and use these limits of integration (Fig. 15):

$$\int_{\text{smallest } \theta}^{\text{largest } \theta} \int_{r \text{ on inner bdry}}^{r \text{ on outer bdry}}$$

For example,

$$\int x^2 \, dA \quad \text{on the semicircular region in Fig. 16}$$

$$= \int_{\theta=\pi}^{2\pi} \int_{r=0}^{4} (r \cos\theta)^2 \, r \sin\theta \, r \, dr \, d\theta$$

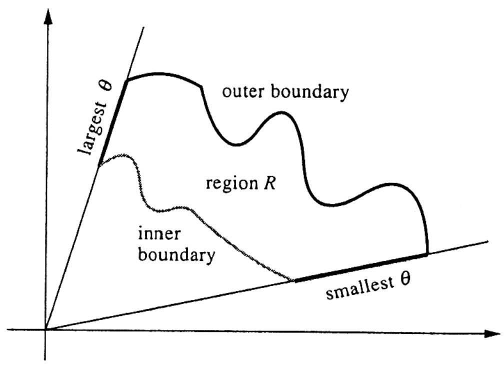


Figure 15

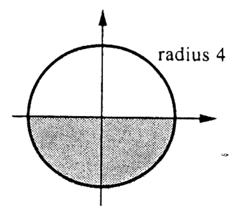


Figure 16

Warning

Don't forget that it's $dA = \boxed{r}\, dr \, d\theta$. Remember the extra r.