Fall 2005 EE 364 Midterm Exam 1 – Solution

1 Short Questions (40 Points)

1. **(10 points)** Since games are won or lost independently,

$$P(\text{Win Next 10}|\text{Win First 3}) = P(\text{Win Next 10}) = (0.9)^{10} = 0.349 \tag{1}$$

2. **(10 points)** There are $2^8 = 256$ different binary words. The number of words with exactly 3 ones is $\binom{8}{3}$. So, the desired probability is

$$P(\text{Exactly three 1's}) = \frac{\binom{8}{3}}{256} = \frac{56}{256} = 0.21875 \tag{2}$$

To find the probability that fewer than 4 ones occur in a randomly selected word, we note that this menas that either 0, 1, 2, or 3 ones occur. Thus, the desired probability is

$$P(\text{Fewer than four 1's}) = \frac{\binom{8}{3} + \binom{8}{2} + \binom{8}{1} + \binom{8}{0}}{256} = 0.3633 \tag{3}$$

3. **(10 points)** Starting with the definition,

$$P(A|C \cup D) = \frac{P(A \cap [C \cup D])}{P(C \cup D)} \tag{4}$$

$$= \frac{P([A \cap C] \cup [A \cap D])}{P(C) + P(D)} \tag{5}$$

$$= \frac{P(A \cap C) + P(A \cap D)}{P(C) + P(D)} \tag{6}$$

$$= \frac{P(A|C)P(C) + P(A|D)P(D)}{P(C) + P(D)} \tag{7}$$

Note the we have used the distributive property and the fact that $A \cap C$ and $A \cap D$ are disjoint when C and D are disjoint.

Plugging the values given into this expression yields $P(A|C \cup D) = 0.3$.

4. **(10 points)** This was from the homework and the solution is repeated from the HW solution:

- From the CDF plot, we have that $F_X(3) = 6a$. This gives us that

$$6a = 1 \Rightarrow a = \frac{1}{6}.$$

- $$Pr(X \leq 2) = F_X(2) = 5a = \frac{5}{6}.$$

- $$Pr(|X| \leq 2) = Pr(-2 < X < 2) = \lim_{\epsilon \to 2^-} F_X(\epsilon) - F_X(-2) = 4a = \frac{2}{3}$$

- $$Pr(X = 1) = \lim_{\epsilon \to 1^+} F_X(\epsilon) - \lim_{\epsilon \to 1^-} F_X(\epsilon) = 0.$$

- $$Pr(2.5 \leq X < 3) = \lim_{\epsilon \to 3^-} F_X(\epsilon) - \lim_{\epsilon \to 2.5^-} F_X(\epsilon) = 5a - 5a = 0.$$

- $$Pr(X = 3) = \lim_{\epsilon \to 3^+} F_X(\epsilon) - \lim_{\epsilon \to 3^-} F_X(\epsilon) = 6a - 5a = \frac{1}{6}.$$

2 Educational Values (40 points)

Denote events in the following manner: H, M, L are the event that an engineer makes a HIGH, MED, or LOW salary, respectively. Also, let USC, HMC, and Stan denote the events that an engineer is a USC, Mudd, or Stanford alumnus, respectively. Thus, the problem gives the following probabilities:

$$P(\text{USC}) = 0.5 \tag{8}$$
$$P(\text{HMC}) = 0.1 \tag{9}$$
$$P(H|\text{USC}) = 0.6 \tag{10}$$
$$P(M|\text{USC}) = 0.3 \tag{11}$$
$$P(H|\text{Stan}) = 0.5 \tag{12}$$
$$P(L|\text{Stan}) = 0.2 \tag{13}$$
$$P(H|\text{HMC}) = 0.9 \tag{14}$$

Also, it can be concluded from the problem description and basic probability relations that $P(L|\text{USC}) = 0.1$, $P(M|\text{Stan}) = 0.3$, and $P(M|\text{HMC}) = P(L|\text{HMC}) = 0.05$.

This is a Bayes' Law type of problem with the conceptual diagram shown in Fig. 1. Note that both the salary divisions and the schools form a partition of the sample space, but we associate the schools with the partition used in the Theorem of Total Probability because we are given conditional probabilities of the form $P(\text{income}|\text{school})$

(a) (10 points) This part just deals with the 'multiplicative rule' – *i.e.,* $P(A \cap B) = P(A|B)P(B)$.

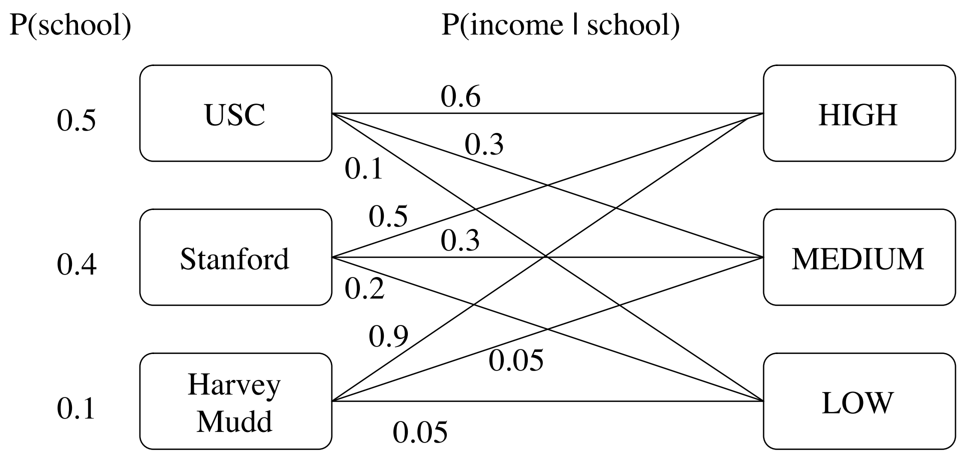


Figure 1: Problem diagram for the Educational Values problem.

The event that an engineer is an HMC graduate and makes a HIGH salary is that both HMC and H occur, so that

$$\begin{aligned} P(\text{HMC grad. and HIGH income}) &= P(\text{HMC} \cap H) && (15) \\ &= P(H|\text{HMC})P(\text{HMC}) = (0.9)(0.1) = 0.09 && (16) \end{aligned}$$

The event that an engineer is an HMC graduate and does not make a LOW salary is

$$\text{HMC} \cap L^c = \text{HMC} \cap (H \cup M) = [\text{HMC} \cap H] \cup [\text{HMC} \cap M] \tag{17}$$

The probability of this event can be evaluated now since the two events on the right (above) are mutually exclusive. Thus,

$$\begin{aligned} P(\text{HMC} \cap L^c) &= P(\text{HMC} \cap H) + P(\text{HMC} \cap M) && (18) \\ &= P(H|\text{HMC})P(\text{HMC}) + P(M|\text{HMC})P(\text{HMC}) = 0.095 && (19) \end{aligned}$$

Note that this is also $P(\text{HMC}) - P(\text{HMC} \cap L)$.

(b) (10 points) This part uses the Theorem of Total Probability. For example,

$$P(L) = P(L|\text{USC})P(\text{USC}) + P(L|\text{Stan})P(\text{Stan}) + P(L|\text{HMC})P(\text{HMC}) \tag{20}$$

The other results follow similarly and we obtain:

$$\begin{aligned} P(\text{LOW income}) &= 0.135 && (21) \\ P(\text{MEDIUM income}) &= 0.275 && (22) \\ P(\text{HIGH income}) &= 0.59 && (23) \end{aligned}$$

Note that these probabilities sum to one as they must because the income levels form a partition of the sample space in this problem.

(c) (10 points) This part uses Bayes' Law. Specifically, we have

$$P(\text{Stanford graduate, given LOW income}) = P(\text{Stan}|L) = \frac{P(L|\text{Stan})P(\text{Stan})}{P(L)} \quad (24)$$

which evalutes to 0.5926.

(d) (10 points) The most reasonable strategy would be to find the school that maximizes $P(\text{school}|H)$. This is the so-called MAP (maximum a-posteroiri probability) decision rule briefly discussed in lecture. These three probabilities can be computed using the same method used above to compute $P(\text{Stan}|L)$. Doing so yields

$$P(\text{USC}|H) = 0.584 \quad P(\text{Stan}|H) = 0.3389 \quad P(\text{HMC}|H) = 0.1525 \quad (25)$$

So, according to this rule, we would decide (guess) that a given HIGH salary engineer attended USC.

Note that another possible, though less reasonable, decision rule would be to select the school that maximizes $P(H|\text{school})$ which would yield HMC as the decision. The weakness of this rule is that it does not take into account the a-prioir probabilities – *i.e.,* does not account for the fact that only 10 percent of engineers are Mudders. This second rule is called the Maxium Likelihood (ML) rule.

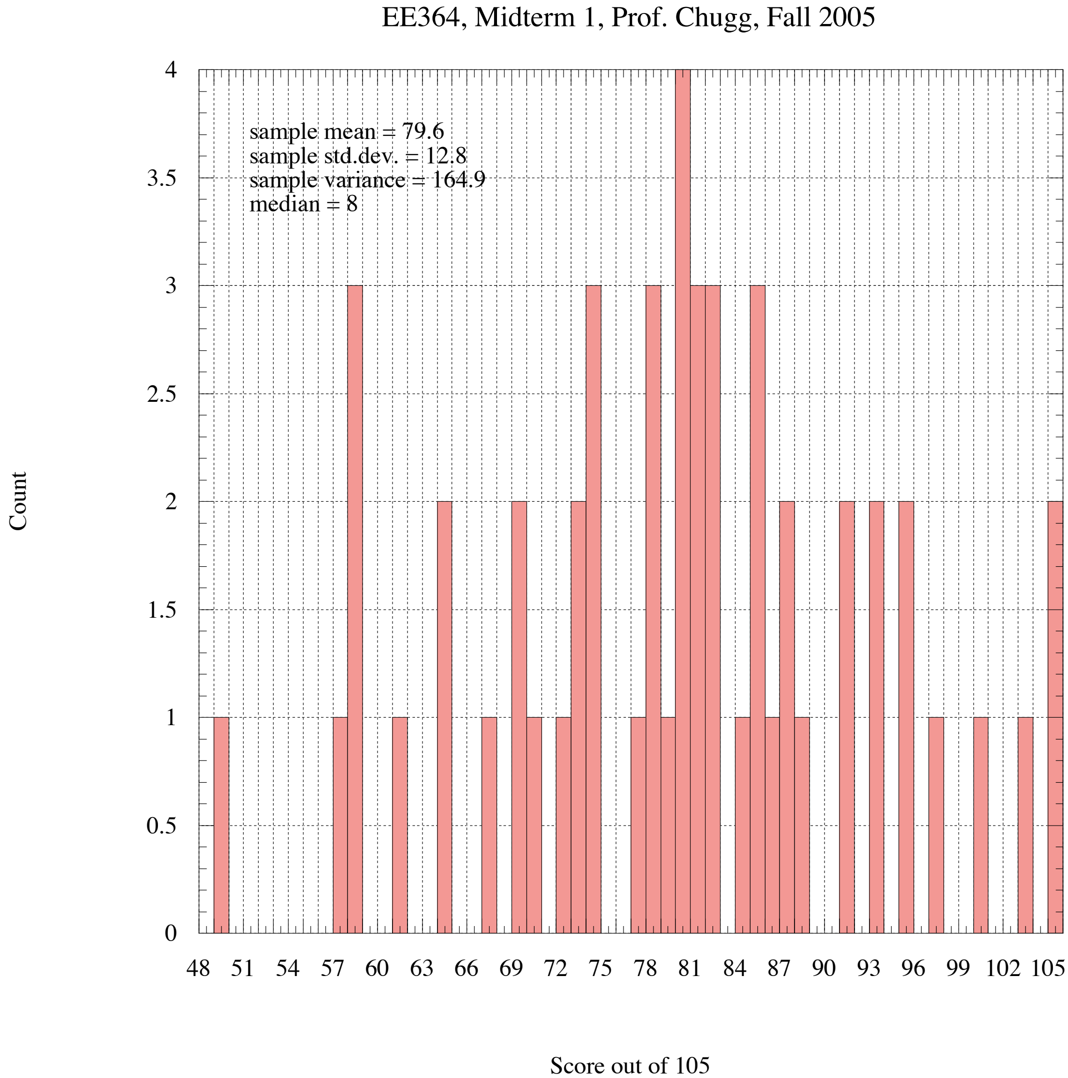
3 Making Change (25 points)

One way to solve this is to think of a table with column heading Q (quaters), D (dimes), N (nickels), and P (pennies) and think of filling in the table with all possible values. Then, for example, the Q column can take values 0, 1,2, 3, or four values. Similarly, the D, N, and P columns take on 3, 2, and 4 values, respectively. Thus, the total number of collection is $(4)(3)(2)(4) = 96$. One of these entries is all zeros, which the problem statement excludes. So, the number of different (non-empty) combinations of coins is 95.

For part (b), you need to consider how collections of coins have the same monetary value. With some thought, it can be concluded that the only such equivalence is two dimes and one nickel in place of a quarter. Thus, we should consider the following cases

Q	D	N	P
0	2	1	$\star$
1	2	1	$\star$
2	2	1	$\star$
3	2	1	$\star$

where $\star$ is a 'wildcard' and can take any value for the number of pennies (0,1,2,3). Note that the first row is equivalent to row of $(1, 0, 0, \star)$ – *i.e.,* one quarter and the same number of pennies. Similarly, the second and third rows are equivalent to $(2, 0, 0, \star)$ and $(3, 0, 0, \star)$, respectively. Note that the last row above is the same as 4 quarters, but that is not possible since there are only 3 quarters available. Therefore, we should eliminate from the table the rows of the form shown in the first three rows above and keep those of the form shown in the last. The number of rows of the form $(1, 2, 1, \star)$ is 4, one for each possible value of $\star$. Similarly, there are 4 row of the form given in rows 2 and 3 above. So, we should subtract $4 + 4 + 4 = 12$ rows from the total in part (a), yielding 83.



Regrade Policy

- All requests must be submitted in writing to me (Prof. Chugg) on or before Oct. 26, 2005.
- Requests are generally discouraged as the exams have been graded carefully with an effort to be consistent. *Exception:* Please submit any mistakes made in totaling points.
- Understand the solution and your answer before requesting any reconsideration. Demonstrate this in your written request.
- The entire exam will be regraded; not just the portion requested.