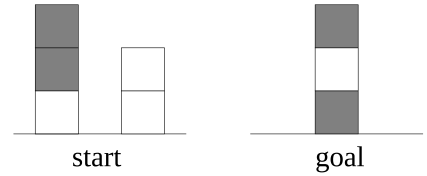
Artificial Intelligence
Markov Decision Problems

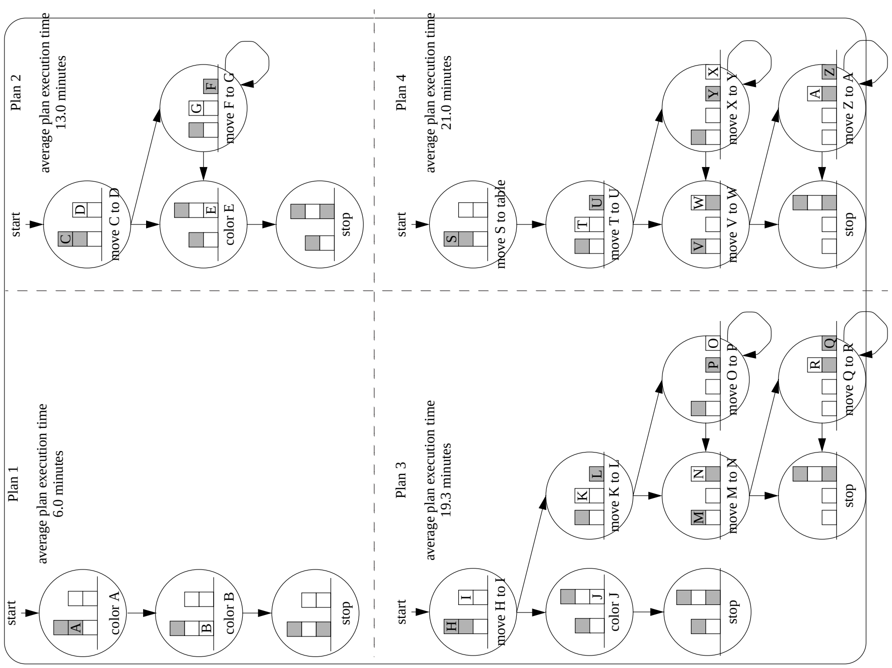
Nilsson - briefly mentioned in Chapter 10

Russell and Norvig - Chapter 17

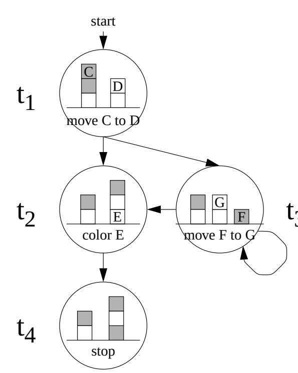
example: a probabilistic blocksworld



action	outcome	probability	time (= cost)
move	success	0.1	1 minute
	failure	0.9	1minute
paint	success	1.0	3 minutes



how to determine the average plan-execution time of a given plan

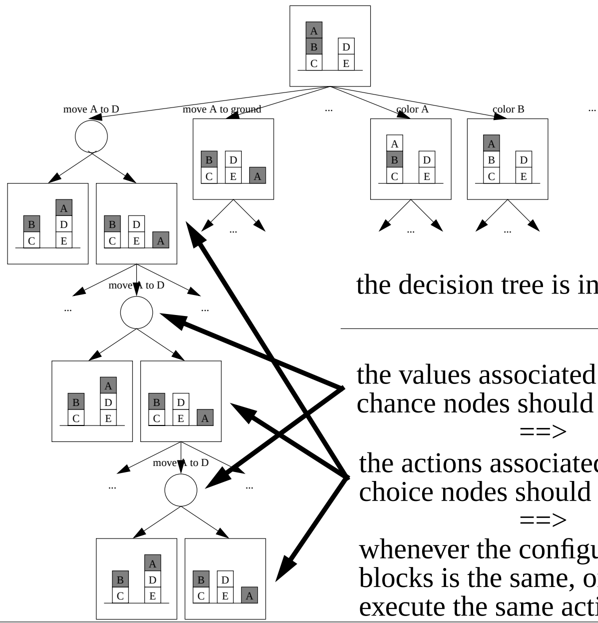


t_i = average plan-execution time until a goal state is reached if the agent starts in state i and follows the plan

$t_1 = 1 + 0.1\ t_2 + 0.9\ t_3$
$t_2 = 3 + 1.0\ t_4$
$t_3 = 1 + 0.1\ t_2 + 0.9\ t_3$
$t_4 = 0$

$t_1 = 13$ (= average plan-execution time)
$t_2 = 3$
$t_3 = 13$
$t_4 = 0$

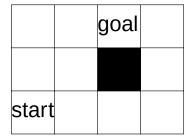
how to (almost) detemine a plan with minimal average plan-execution time



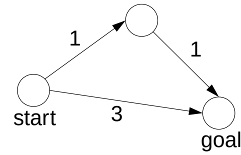
the decision tree is infinite!

the values associated with these chance nodes should be the same
==>
the actions associated with these choice nodes should be the same
==>
whenever the configuration of the blocks is the same, one wants to execute the same action (= policy)

deterministic planning and search



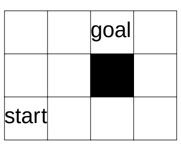
actions are deterministic



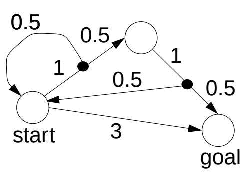
actions have deterministic effects
state and action determine uniquely the successor state
states are completely observable

a plan is a sequence of actions (= path)
minimize total cost
optimal plan is acyclic

probabilistic planning and search
=Markov Decision Problems (MDPs)



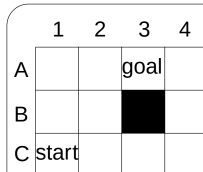
actions are probabilistic
the robot can “drift”



Markov property

actions have probabilistic effects
state and action uniquely determine prob distribution over successor states
states are completely observable

a plan is a mapping from states to actions (= policy)
minimize expected total cost
optimal plan can be cyclic



N N E E

deterministic planning and search

a plan is a sequence of actions (= path)
can be found using (forward or backward) search

Markov Decision Problems

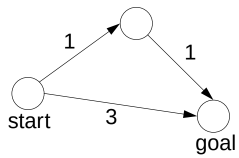
A1	E
A2	E
A3	-
A4	W
B1	N
B2	N
B4	N
C1	N
C2	N
C3	W
C4	N

a plan is a mapping from states to actions (= policy)
how to find it?

determine the expected goal distances of all states

greedily assign the action to each state
that decreases the expected goal distance the most

deterministic planning and search



s = state
a = action
A(s) = set of actions that can be executed in state s
succ(s,a) = the state that results from the execution of action a in state s
c(s,a) = the cost that results from the execution of action a in state s

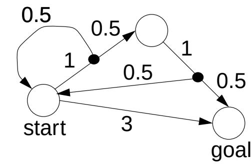
gd(s) = goal distance of state s

gd(s) = 0 if s is a goal state
$gd(s) = \min_{a \,\varepsilon\, A(s)} c(s,a) + gd(succ(s,a))$ if s is not a goal state

a(s) = the optimal action to execute in state s

$a(s) = \mathrm{argmin}_{a \,\varepsilon\, A(s)} c(s,a) + gd(succ(s,a))$ if s is not a goal state

Markov Decision Problems



Bellman equation

s = state
a = action
A(s) = set of actions that can be executed in state s
succ(s,a) = the set of states that can result from the execution of action a in state s
c(s,a) = the cost that results from the execution of action a in state s
p(s'|s,a) = the probability that state s' results from the execution of action a in state s

gd(s) = expected goal distance of state s

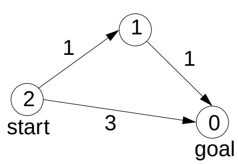
gd(s) = 0 if s is a goal state
$gd(s) = \min_{a \,\varepsilon\, A(s)} (c(s,a) + \Sigma_{s' \,\varepsilon\, succ(s,a)} p(s'|s,a)\, gd(s'))$ if s is not a goal state

a(s) = the optimal action to execute in state s

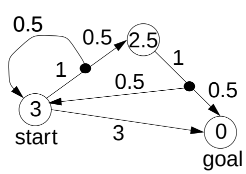
$a(s) = \mathrm{argmin}_{a \,\varepsilon\, A(s)} (c(s,a) + \Sigma_{s' \,\varepsilon\, succ(s,a)} p(s'|s,a)\, gd(s'))$ if s is not a goal state

examples

deterministic planning and search

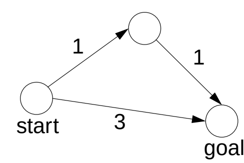


Markov Decision Problems



given the expected goal distances, we can use the definition to check them
but calculating them is a chicken-and-egg problem

deterministic planning and search



s = state
a = action
A(s) = set of actions that can be executed in state s
succ(s,a) = the state that results from the execution of action a in state s
c(s,a) = the cost that results from the execution of action a in state s

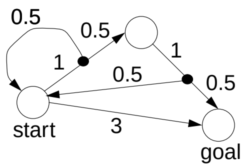
gd(s) = goal distance of state s (= minimal cost until a goal is reached if execution st
$gd_i(s)$ = minimal cost until a goal is reached or i actions have been executed
if execution starts in state s

for i larger than a constant: $gd(s) = gd_i(s)$ (= once $gd_i(s) = gd_{i-1}(s)$ for all states s)

$gd_0(s) = 0$

$gd_i(s) = 0$ if s is a goal state
$gd_i(s) = \min_{a \,\varepsilon\, A(s)} (c(s,a) + gd_{i-1}(succ(s,a)))$ if s is not a goal state

Markov Decision Problems



s = state
a = action
A(s) = set of actions that can be executed in state s
succ(s,a) = the set of states that can result from the execution of action a in state s
c(s,a) = the cost that results from the execution of action a in state s
p(s'|s,a) = the probability that state s' results from the execution of action a in state s

gd(s) = expected goal distance of state s
$gd_i(s)$ = minimal expected cost until a goal is reached or i actions have been executed if execution starts in state s

$gd(s) = \lim_{i \to \text{infinity}} gd_i(s)$ (not necessarily after a finite amount of time)

$gd_0(s) = 0$

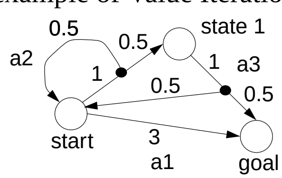
$gd_i(s) = 0$ if s is a goal state
$gd_i(s) = \min_{a \,\varepsilon\, A(s)} (c(s,a) + \Sigma_{s' \,\varepsilon\, succ(s,a)}\, p(s'|s,a)\, gd_{i-1}(s'))$ if s is not a goal state

Value Iteration

maintains approximations of the goal distances (= values)

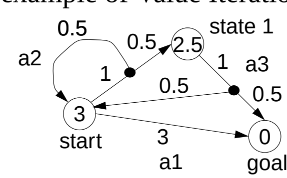
1. i := 0
2. Set (for all sεS) $gd_i(s) = 0$.
3. i := i+1
4. Set (for all sεS)
   $gd_i(s) = 0$ if s is a goal state
   $gd_i(s) = \min_{a \,\varepsilon\, A(s)} (c(s,a) + \Sigma_{s' \,\varepsilon\, succ(s,a)}\, p(s'|s,a)\, gd_{i-1}(s'))$ if s is not a goal state
5. If (for some sεS) $|gd_i(s) - gd_{i-1}(s)|$ > small constant, go to 3.
6. Set (for all sεS that are not goal states)
   $a(s) = \text{argmin}_{a \,\varepsilon\, A(s)} (c(s,a) + \Sigma_{s' \,\varepsilon\, succ(s,a)}\, p(s'|s,a)\, gd_i(s'))$

example of Value Iteration (1)



i		0	1		2		3		4		5		6	
start	a1	0	3	1	3	2	3	2.75	3	3	3	3	3	3
	a2		1		2		2.75		3.375		3.6875		3.75	
state 1	a3	0	1	1	1.5	1.5	2	2	2.375	2.375	2.5	2.5	2.5	2.5
goal		0	0	0	0	0	0	0	0	0	0	0	0	0

example of Value Iteration (2)



(no discounting)

which action to execute in the start state?

a1: 3 + 0 = 3
a2: 1 + 0.5 3 + 0.5 2.5 = 3.75

execute a1!

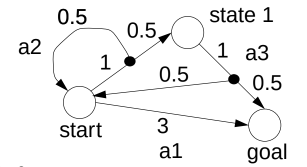
Policy Iteration

maintains a policy

1. $i := 0$
2. Set (for all $s \varepsilon S$ that are not goal states) $a_i(s)$ to an arbitrary action in $A(s)$.
3. Set $gd_i(s)$ to the average plan-execution time until a goal state is reached if the agent starts in state s and follows policy a_i
4. $i := i+1$
5. Set (for all $s \varepsilon S$ that are not goal states)
   $a_i(s) = \text{argmin}_{a \,\varepsilon\, A(s)} (c(s,a) + \Sigma_{s' \,\varepsilon\, succ(s,a)}\, p(s'|s,a)\, gd_i(s'))$
6. If (for some $s \varepsilon S$ that is not a goal state) $a_i(s)$ does not equal $a_{i-1}(s)$, go to 3
7. Set (for all $s \varepsilon S$ that are not goal states) $a(s) = a_i(s)$.

Note: The initial policy a_o has to guarantee that the agent reaches a goal state with probability one no matter which state it is started in.

example of Policy Iteration



(no discounting)

policy at i=0
a_0(start) = a2 (could also have been a1) and a_0(state 1) = a3

gd_0(start) = 1 + 0.5 gd_0(start) + 0.5 gd_0(state1) = 6
gd_0(state 1) = 1 + 0.5 gd_0(start) + 0.5 gd_0(goal) = 4
gd_0(goal) = 0

policy at i=1
a_1(start) = a1 and a_1(state 1) = a3

gd_1(start) = 3 + 1.0 gd_0(goal) = 3
gd_1(state 1) = 1 + 0.5 gd_1(start) + 0.5 gd_1(goal) = 2.5
gd_1(goal) = 0

policy at i=2
a_2(start) = a1 and a_2(state 1) = a3

execute action a1 in the start state!

extensions: no goal (1)

what if there is no goal?
“living in the world”

can no longer minimize expected cost until the goal is reached

here:
- can minimize expected cost per action execution
- can minimize expected total discounted cost

extensions: no goal (2)

cannot minimize expected total cost

1 2 3 4 4 4 4 ... expected total cost = infinite

1 1 1 1 1 1 1 ... expected total cost = infinite

extensions: no goal (3)

total discounted cost = $\xrightarrow{}$ discount factor

if the interest rate is $(1-\gamma)/\gamma$ (for $0 < \gamma < 1$),
how much money do I need to pay someone right now
so that there is no difference to paying the following yearly installments

1 2 3 4 4 4 4 ...

x dollars right now are worth $(1 + (1-\gamma)/\gamma)x = x/\gamma$ dollars in a year
so, y dollars in a year are worth γ y dollars right now

answer: $1 + \gamma\, 2 + \gamma^2\, 3 + \gamma^3\, 4 + \gamma^4\, 4 + \ldots$

extensions: no goal (4)

can minimize the expected total discounted cost - assume $\gamma = 0.9$

1 2 3 4 4 ... expected total discounted cost = 34.39

1 1 1 1 1 ... expected total discounted cost = 10.00

extensions: no goal (5)

- discounting makes the total cost finite

c c c c c ... expected total discounted cost = $c/(1-\gamma)$

- discounting smoothes out the horizon
- discounting can be interpreted as the probability of dying

discounting:
if the interest rate is $(1-\gamma)/\gamma$,
then y dollars in a year are worth γ y dollars right now

dying:
if I die later this year with probability $1-\gamma$,
then the expected value of y dollars in a year is γ y right now

Value-Iteration with or without discounting

γ	= discount factor ($0 < \gamma < 1$); if there is a goal, can set $\gamma = 1$ (no discounting)
s	= state
a	= action
A(s)	= set of actions that can be executed in state s
succ(s,a)	= the set of states that can result from the execution of action a in state s
c(s,a)	= the cost that results from the execution of action a in state s
p(s'|s,a)	= the probability that state s' results from the execution of action a in state s
gd(s)	= minimal expected discounted total cost if execution starts in state s

$gd(s) = 0$ if s is a goal state
$gd(s) = \min_{a \in A(s)} (c(s,a) + \gamma \Sigma_{s' \in succ(s,a)}\, p(s'|s,a)\, gd(s'))$ if s is not a goal state

$gd_i(s)$ = minimal expected discounted total cost until a goal is reached or i actions have been executed if execution starts in state s

$gd_0(s) = 0$ for all s
$gd_i(s) = 0$ if s is a goal state
$gd_i(s) = \min_{a \in A(s)} (c(s,a) + \gamma \Sigma_{s' \in succ(s,a)}\, p(s'|s,a)\, gd_{i-1}(s'))$ if s is not a goal state

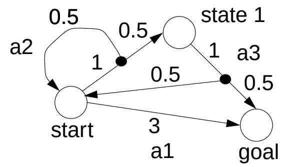
$gd(s) = \lim_{i \to infinity} gd_i(s)$

a(s) = the optimal action to execute in state s

$a(s) = \text{argmin}_{a \in A(s)} (c(s,a) + \gamma \Sigma_{s' \in succ(s,a)}\, p(s'|s,a)\, gd(s'))$ if s is not a goal state

gd(s) does not necessarily converge after a finite amount of time
a(s) converges after a finite amount of time if gd(s) is approximated with $gd_i(s)$ for all s

example of Value Iteration (1)

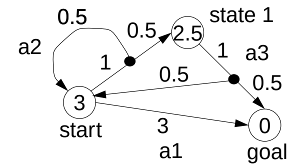


(discount factor = 0.9)

i		0	1		2		3		4	
start	a1	0	3	1	3	1.9	3	2.5075	3	2.9631
	a2		1		1.9		2.5075		2.9631	
state 1	a3	0	1	1	1.45	1.45	1.855	1.855	2.1284	2.1284
goal		0	0	0	0	0	0	0	0	0

5		6		7	
3	3	3	3	3	3
3.2912		3.4		3.4075	
2.3334	2.3334	2.35	2.35	2.35	2.35
0	0	0	0	0	0

example of Value Iteration (2)



(discount factor = 0.9)

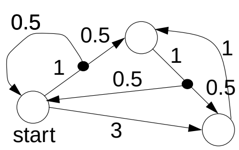
which action to execute in the start state?

a1: 3 + 0 = 3
a2: 1 + 0.9 0.5 3 + 0.9 0.5 2.35 = 3.4075

execute a1!

(In general, the optimal action depends on the discount factor!)

"learning" for optimization
"reinforcement learning" with Markov Decision Process Models



exam example

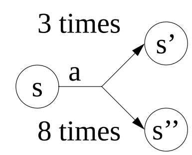
find a policy (behavior)
that maximizes the expected total discounted reward
even in the presence of delayed rewards

if you don't know the action outcomes (rewards and probabilities):
reinforcement learning

exploration/exploitation tradeoff

"learning" for optimization
"reinforcement learning" with Markov Decision Process Models
approach 1

estimate the probabilities and rewards



p(s'|s,a) = ?

use value-iteration

"learning" for optimization
"reinforcement learning" with Markov Decision Process Models
approach 2

use Q-learning

if you execute action a in state s and
you receive cost c and make a transition to state s'
then update

$$Q(s,a) = Q(s,a) + \alpha\,(c + \gamma\, V(s') - Q(s,a))$$

learning rate (α)

discount factor $0 < \gamma < 1$

$V(s') = \min_{a \,\varepsilon\, A(s')} Q(s',a)$

Q(s,a) = minimal expected discounted total cost until a goal is reached if execution starts in state s and the first action executed is a

V(s') = minimal expected discounted total cost until a goal is reached if execution starts in state s' (= "gd(s')")

"learning" for optimization
"reinforcement learning" with Markov Decision Process Models
approach 2

1. Initialize Q(s,a) = 0 for all states s and actions a.
2. s := the current state.
3. if s is a goal state then stop.
4. Choose an action a to execute in the current state s.
   (The action believed to be best is a := $\arg\min_{a \,\varepsilon\, A(s)} Q(s,a)$.)
5. Execute action a. Observe the cost c and successor state s'.
6. Update $Q(s,a) = Q(s,a) + \alpha\,(c + \gamma\, V(s') - Q(s,a))$.
7. Goto 2.