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Passage to the limit over small parameters in the viscous Cahn–Hilliard equation ☆

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ABSTRACT

We study singular passage to the limit over different small parameters for the viscous Cahn–Hilliard equation under weak growth assumptions on the nonlinearity φ. A rigorous proof of convergence to solutions of either the Cahn–Hilliard equation, or of the Allen–Cahn equation, or of the Sobolev equation, depending on the choice of the parameter, is provided. We also study the singular limit of the Cahn–Hilliard equation as the parameter in the fourth-order term goes to zero. In particular, we show that a *Radon measure-valued solution* of the limiting ill-posed problem can arise, depending on the behavior of the nonlinearity φ at infinity.

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1. Introduction

In this paper we investigate the initial–boundary value problem

$$\begin{cases} (1-\beta)u_t = \Delta\big[\varphi(u) - \alpha\Delta u + \beta u_t\big] & \text{in } \Omega\times(0,T) =: Q, \\ u = \Delta u = 0 & \text{on } \partial\Omega\times(0,T), \\ u = u_0 & \text{in } \Omega\times\{0\}, \end{cases} \tag{P}$$

where $\alpha\in(0,\infty)$, $\beta\in(0,1)$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ if $N\geq 2$, and $T>0$. Assumptions about the initial data function u_0 are given in Section 2. As for the function $\varphi:\mathbb{R}\to\mathbb{R}$, the following assumptions will be used:

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(H_0)
$$\varphi \in W^{1,\infty}_{loc}(\mathbb{R}), \qquad \varphi(u)u \geq 0 \quad \text{for any } u \in \mathbb{R};$$

(H_1) there exists $K > 0$ such that

$$\left|\varphi'(u)\right| \leq K\left(1 + |u|^{q-1}\right) \tag{1.1}$$

for some $q \in (1, \infty)$ if $N = 1, 2$, or $q \in (1, \frac{N+2}{N-2}]$ if $N \geq 3$. By assumption (H_0) the function φ is locally Lipschitz continuous, and there holds $\varphi(0) = 0$. Observe that no monotonicity of φ is assumed.

The first equation of problem (P) is a particular case of the *viscous Cahn–Hilliard equation*,

$$\nu u_t = \Delta\left[\varphi(u) - \alpha\Delta u + \beta u_t\right] \quad (\alpha, \beta, \nu > 0) \tag{1.2}$$

with $\nu = 1-\beta$ ($\beta \in (0,1)$). Eq. (1.2) has been derived by several authors on the strength of different physical considerations (in particular, see [11,12,14]). There exists a wide literature concerning the relationship between the viscous Cahn–Hilliard equation and *phase field models*, as well as generalized versions of the equation suggested in [11] (e.g., see [6,22] and references therein).

The main purpose of this paper is to investigate different *singular limits* of problem (P). When $\nu = 1$, $\beta = 0$ Eq. (1.2) *formally* gives the *Cahn–Hilliard equation*,

$$u_t = \Delta\left[\varphi(u) - \alpha\Delta u\right], \tag{1.3}$$

whereas for $\nu = 1$, $\alpha = 0$ it reduces *formally* to the *Sobolev equation*,

$$u_t = \Delta\left[\varphi(u) + \beta u_t\right]. \tag{1.4}$$

It is natural to ask whether the above formal remarks can be given in a sound analytical meaning (in fact, no rigorous proof of them is to our knowledge available, in spite of the wide literature concerning Eqs. (1.2)–(1.4)).

In this connection, observe that Cahn–Hilliard type equations arise in the context of the theory of phase transitions, in which case the function φ is cubic (i.e., $\varphi(u) = u^3 - u$). In this case, as always when φ is nonmonotonic, both equations (1.3) and (1.4) provide a *regularization* of the *forward–backward parabolic equation*

$$u_t = \Delta\left[\varphi(u)\right]. \tag{1.5}$$

It has been strongly emphasized [3] that regularizations should be specific for the given physical problem and justified at the level of mathematical modeling, since the limiting equation is *ill-posed*, thus different regularizations can lead to different dynamics of solutions. Therefore, it seems worth justifying rigorously both (1.3) and (1.4) as the limiting cases of a sound mathematical model. At the same time, it looks natural to investigate Eqs. (1.2)–(1.4) in a unified way by a *"vanishing viscosity"* approach.

In the light of the above remarks, the present investigation is deeply connected with that of forward–backward parabolic equations, which – beside phase transitions – arise in a variety of applications, such as edge detection in image processing [17], aggregation models in population dynamics [16], and stratified turbulent shear flow [2]. In these models a typical form of the nonlinear function φ is

$$\varphi(u) = \frac{u}{1+u^2},$$

thus φ is monotonically decreasing and vanishing as $u \to +\infty$ – a completely different behavior with respect to the case of phase transitions. In fact, the behavior of the function φ at infinity turns out to be a major feature of the problem (see the remarks following (1.7)).

In this general framework, it looks also natural to seek a proper definition of solution for the ill-posed equation (1.5) by a "vanishing viscosity" method, which makes use of either (1.3) or (1.4) as regularized equation. Although quite a few results have been obtained in this direction for the pseudoparabolic regularization (1.4), much less is known for the Cahn–Hilliard regularization (1.3) (see [19,18,20,23,25]).

Below we prove that solutions of problem (P) converge in a suitable sense to solutions of the problem for the Cahn–Hilliard equation:

$$\begin{cases} u_t = \Delta\big[\varphi(u) - \alpha\Delta u\big] & \text{in } Q, \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{CH}$$

as $\beta \to 0^+$, or respectively to solutions of the problem for the Sobolev equation:

$$\begin{cases} (1-\beta)u_t = \Delta\big[\varphi(u) + \beta u_t\big] & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{S}$$

as $\alpha \to 0^+$ (under additional assumptions on the function φ; see assumptions (H_2)–(H_3) and Theorem 2.5).

Also observe that Eq. (1.2) with $\nu = 0$, $\beta = 1$ formally gives the *Allen–Cahn equation*,

$$u_t = \alpha\Delta u - \varphi(u). \tag{1.6}$$

In this connection, it is proven below that solutions of (P) converge in a suitable sense to solutions of the problem:

$$\begin{cases} u_t = \alpha\Delta u - \varphi(u) & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{AC}$$

as $\beta \to 1^-$ (see Theorems 2.4–2.6). Similar results are proven for the companion Neumann problem (see Section 6, in particular Theorem 6.4).

Further, we study the limit of solutions of problem (CH) as $\alpha \to 0^+$. We prove the existence of a triple (u, v, μ) – where u, v are functions and μ is a *finite Radon measure* on Q – which satisfies a weak limiting equality (see Theorem 2.7, in particular equality (2.44)). We cannot maintain that this triple is in some sense a solution of the limiting problem

$$\begin{cases} u_t = \Delta\big[\varphi(u)\big] & \text{in } Q, \\ u = \Delta u = 0 & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{1.7}$$

since the relation between v and the function $\varphi(u)$, even in the sense of *Young measures*, is unclear. This point was addressed in [20] by taking advantage of the cubic-like growth of φ at infinity, which gives rise to better estimates of the family $\{u_\alpha\}$ of solutions of (CH); at the same time, this growth prevented the appearance of a Radon measure in the solution. Instead in the present case, if the antiderivative of φ grows linearly at infinity (see assumption (H_4)), we only have L^1-estimates of the family $\{u_\alpha\}$, which are compatible with the need of a Radon measure to describe solutions of the problem (in this connection, see [3, 21,23,25]). Similar and more enhanced phenomena can be expected, if φ either has a sublinear growth, or vanishes at infinity.

Our approach is based on a detailed analysis of solutions of problem (P), which relies on an *approximation method* already used in similar cases [3–5,21,23–25]. Beside lending in a natural way the estimates needed to study the singular limits, it allows to improve in several ways on the available existence results for the viscous Cahn–Hilliard equation – in particular, by weakening the assumptions usually made about the space dimension and/or the growth of the function φ at infinity ([7,9]; see Theorem 2.2).

2. Mathematical framework and results

2.1. Well-posedness and a priori estimates

Let us state the following definition.

Definition 2.1. Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. By a strict solution of problem (P) we mean any function $u \in C([0,T]; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ such that $\varphi(u) \in C([0,T]; L^2(\Omega))$, and

$$\begin{cases} u_t = \Delta v & \text{in } Q, \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{2.1}$$

in strong sense. Here $v \in C([0,T]; H^2(\Omega) \cap H^1_0(\Omega))$ and for every $t \in [0,T]$ the function $v(\cdot,t)$ is the unique solution of the elliptic problem

$$\begin{cases} -\beta\Delta v(\cdot,t) + (1-\beta)v(\cdot,t) = \varphi(u)(\cdot,t) - \alpha\Delta u(\cdot,t) & \text{in } \Omega, \\ v(\cdot,t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

As often in the pertaining literature (e.g., see [6]), the function v will be referred to as the *chemical potential*.

A first well-posedness result for problem (P), if assumptions (H_0)–(H_1) are replaced by the stronger condition

$$(H_2) \qquad \varphi \in Lip(\mathbb{R}), \qquad \varphi(u)u \geq 0 \quad \text{for any } u \in \mathbb{R},$$

is the content of the following theorem.

Theorem 2.1. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let φ satisfy assumption (H_2). Then for every $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ there exists a unique strict solution of problem (P).*

Let us now address well-posedness when φ satisfies assumptions (H_0)–(H_1), and the Cauchy data function u_0 belongs to $H^1_0(\Omega)$. In this case solutions of problem (P) are meant in the following sense.

Definition 2.2. Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let $u_0 \in H^1_0(\Omega)$. By a solution of problem (P) we mean any function $u \in C([0,T]; H^1_0(\Omega))$ such that:

(i) $u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$, $\Delta u \in L^2(Q)$;
(ii) problems (2.1) and (2.2) are satisfied in strong sense, with $v \in L^\infty(0,T; H^1_0(\Omega)) \cap L^2(0,T; H^2(\Omega))$.

Remark 2.1. The concept of solution given by Definition 2.1 is the natural one in the context of the semigroup approach to problem (P) (see [7,9]). On the other hand, that given by Definition 2.2 is naturally suggested by combining the semigroup approach with the approximation method described below (see Section 3).

In the limiting case $\beta \to 0^+$ it leads to a notion of solution of problem (CH) (see Definition 2.3), which is weaker than that given e.g. in [8]. Observe that the latter relies on the use of the Galerkin method in some suitable evolution triple (in this connection, see [1]). Similarly, the solutions of problem (S) given by Definition 2.4 (which are suggested by Definition 2.2) in the limiting case $\alpha \to 0^+$ are weaker than those considered in [15].

The following result will be proven.

Theorem 2.2. *Let $\alpha \in (0, \infty)$, $\beta \in (0, 1)$, and let φ satisfy assumptions (H_0)–(H_1). Then for every $u_0 \in H_0^1(\Omega)$ there exists a solution of problem (P). If φ satisfies assumption (H_2), the solution is unique.*

Moreover, for every $\bar{\alpha} > 0$ there exists $M > 0$ (which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$) such that for any $\alpha \in (0, \bar{\alpha})$ and $\beta \in (0, 1)$

$$\big\|\Phi(u)\big\|_{L^\infty(0,T;L^1(\Omega))} \le M, \tag{2.3}$$

where

$$\Phi(u) := \int_0^u \varphi(z)\,dz \quad (u \in \mathbb{R}); \tag{2.4}$$

$$\sqrt{\alpha}\|u\|_{L^\infty(0,T;H_0^1(\Omega))} \le M; \tag{2.5}$$

$$\sqrt{\beta}\|u_t\|_{L^2(Q)} \le M; \tag{2.6}$$

$$\sqrt{\alpha}\big\|\varphi(u)\big\|_{L^2(Q)} \le M; \tag{2.7}$$

$$\alpha^{\frac{3}{2}}\|\Delta u\|_{L^2(Q)} \le M; \tag{2.8}$$

$$\sqrt{1-\beta}\|v\|_{L^2(0,T;H_0^1(\Omega))} \le M; \tag{2.9}$$

$$\sqrt{\alpha}\beta\|v\|_{L^\infty(0,T;H_0^1(\Omega))} \le M; \tag{2.10}$$

$$\sqrt{\beta(1-\beta)}\|v\|_{L^2(0,T;H^2(\Omega))} \le M. \tag{2.11}$$

Let us mention for future reference the following result, which provides further estimates of the solution given by Theorem 2.2.

Theorem 2.3. *Let $\alpha \in (0, \infty)$, $\beta \in (0, 1)$ and $u_0 \in H_0^1(\Omega)$. Let φ satisfy either assumption (H_2), or assumptions (H_0), (H_1) and the following one:*

(H_3) *there exists $u_0 > 0$ such that $\varphi'(u) > 0$ if $|u| \ge u_0$.*

Let u be the solution of problem (P) given by Theorem 2.2. Then for every $\alpha \in (0, \infty)$ and $\beta \in (0, 1)$

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} \le \|u_0\|_{H_0^1(\Omega)}\sqrt{\frac{1+e^{\frac{2LT}{\beta}}}{1-\beta}}; \tag{2.12}$$

$$\|u\|_{L^\infty(0,T;H_0^1(\Omega))} \le \|u_0\|_{H_0^1(\Omega)}\sqrt{\frac{2(1+e^{\frac{2LT}{\beta}})}{\beta}}; \tag{2.13}$$

$$\|\Delta u\|_{L^2(Q)} \le \|u_0\|_{H_0^1(\Omega)}\sqrt{\frac{1+e^{\frac{2LT}{\beta}}}{\alpha}}. \tag{2.14}$$

Moreover, for every $\bar{\alpha} > 0$ and $\beta \in (0,1)$ there exists $\bar{M} > 0$ (which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$ and on β, and diverges as $\beta \to 0^+$, $\beta \to 1^-$) such that for any $\alpha \in (0,\bar{\alpha})$ and $n \in \mathbb{N}$

$$\big\|\varphi(u)\big\|_{L^2(Q)} \le \bar{M}. \tag{2.15}$$

Remark 2.2. In connection with the above theorem, observe that:

(i) if (H_2) is satisfied, there exists $L > 0$ such that $|\varphi_n'(u)| \le L$ for any $n \in \mathbb{N}$ and $u \in \mathbb{R}$;
(ii) if (H_3) is satisfied, there holds $\varphi_n'(u) > 0$ for any $n \in \mathbb{N}$, $n > u_0$ and $u \in \mathbb{R}$, $|u| \ge u_0$.

2.2. Asymptotical limits

2.2.1. The limit $\beta \to 0^+$ (for fixed $\alpha > 0$)

As $\beta \to 0^+$, inequalities (2.6) and (2.10)–(2.15) get lost. Accordingly, solutions of problem (CH) for the Cahn–Hilliard equation are meant in the following sense.

Definition 2.3. Let $\alpha \in (0,\infty)$ and $u_0 \in H_0^1(\Omega)$. By a solution of problem (CH) we mean any function $u \in L^\infty(0,T;H_0^1(\Omega))$ such that:

(i) $\varphi(u) \in L^2(Q)$, $\Delta u \in L^2(Q)$, and $v := \varphi(u) - \alpha\Delta u \in L^2(0,T;H_0^1(\Omega))$;
(ii) there holds

$$\iint_Q u\zeta_t \, dx\, dt + \iint_Q \big[\varphi(u) - \alpha\Delta u\big]\Delta\zeta \, dx\, dt = -\int_\Omega u_0(x)\zeta(x,0)\, dx \tag{2.16}$$

for every $\zeta \in C^1([0,T];C_c^2(\Omega))$ such that $\zeta(\cdot,T) = 0$ in Ω.

By studying the limiting points of the sequence $\{u_n\}$ we shall prove the following result.

Theorem 2.4. *Let $u_0 \in H_0^1(\Omega)$, and let φ satisfy assumptions (H_0)–(H_1). Let $u_{\alpha,\beta}$ be the solution of problem (P) given by Theorem 2.2 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\alpha \in (0,\infty)$ there exist $u_\alpha \in L^\infty(0,T;H_0^1(\Omega))$, $v_\alpha \in L^2(0,T;H_0^1(\Omega))$ and two subsequences $\{u_{\alpha,\beta_k}\} \subseteq \{u_{\alpha,\beta}\}$, $\{v_{\alpha,\beta_k}\} \subseteq \{v_{\alpha,\beta}\}$ such that*

(i) *$\varphi(u_\alpha) \in L^2(Q)$, and $\Delta u_\alpha \in L^2(Q)$. Moreover, the function u_α is a solution of problem (CH);*
(ii) *as $\beta_k \to 0^+$ there hold*

$$u_{\alpha,\beta_k} \overset{*}{\rightharpoonup} u_\alpha \quad \text{in } L^\infty\big(0,T;H_0^1(\Omega)\big), \tag{2.17}$$

$$u_{\alpha,\beta_k} \to u_\alpha \quad \text{almost everywhere in } Q, \tag{2.18}$$

$$\varphi(u_{\alpha,\beta_k}) \rightharpoonup \varphi(u_\alpha) \quad \text{in } L^2(Q), \tag{2.19}$$

$$\Delta u_{\alpha,\beta_k} \rightharpoonup \Delta u_\alpha \quad \text{in } L^2(Q); \tag{2.20}$$

$$v_{\alpha,\beta_k} \rightharpoonup v_\alpha \quad \text{in } L^2\big(0,T;H_0^1(\Omega)\big); \tag{2.21}$$

(iii) *the function u_α satisfies inequalities (2.3)–(2.5) and (2.7)–(2.8), whereas v_α satisfies the a priori estimate*

$$\|v_\alpha\|_{L^2(0,T;H_0^1(\Omega))} \le M \tag{2.22}$$

with some constant $M > 0$ which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$.

2.2.2. The limit $\alpha \to 0^+$ (for fixed $\beta \in (0,1)$)

In this case inequalities (2.5), (2.7)–(2.8), (2.10) and (2.14) are lost. Solutions of problem (S) are meant in the following sense.

Definition 2.4. Let $\beta \in (0,1)$, and let $u_0 \in H_0^1(\Omega)$. By a solution of problem (S) we mean any function $u \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ such that:

(i) $u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$;
(ii) problem (2.1) is satisfied in strong sense with $v := \frac{1}{1-\beta}[\varphi(u) + \beta u_t] \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega))$.

Theorem 2.5. *Let $u_0 \in H_0^1(\Omega)$, and let φ satisfy either assumption (H_2), or assumptions (H_0), (H_1) and (H_3). Let $u_{\alpha,\beta}$ be the solution of problem (P) given by Theorem 2.2 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\beta \in (0,1)$ there exist functions $u_\beta \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $v_\beta \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega))$ and a subsequence $\{u_{\alpha_k,\beta}\} \subseteq \{u_{\alpha,\beta}\}$ such that*

(i) *$u_{\beta t} \in L^2(Q)$, and $\varphi(u_\beta) \in L^2(Q)$. Moreover, the function u_β is a solution of problem (S) with $v_\beta = \frac{1}{1-\beta}[\varphi(u_\beta) + \beta u_{\beta t}]$;*
(ii) *as $\alpha_k \to 0^+$ there hold*

$$u_{\alpha_k,\beta} \overset{*}{\rightharpoonup} u_\beta \quad \text{in } L^\infty\big(0,T;H_0^1(\Omega)\big), \tag{2.23}$$

$$u_{\alpha_k,\beta} \to u_\beta \quad \text{almost everywhere in } Q, \tag{2.24}$$

$$(u_{\alpha_k,\beta})_t \rightharpoonup u_{\beta t} \quad \text{in } L^2(Q), \tag{2.25}$$

$$\varphi(u_{\alpha_k,\beta}) \rightharpoonup \varphi(u_\beta) \quad \text{in } L^2(Q), \tag{2.26}$$

$$v_{\alpha_k,\beta} \rightharpoonup v_\beta \quad \text{in } L^2\big(0,T;H_0^1(\Omega)\big), \tag{2.27}$$

$$\Delta v_{\alpha_k,\beta} \rightharpoonup \Delta v_\beta \quad \text{in } L^2(Q); \tag{2.28}$$

(iii) *the function u_β satisfies inequalities (2.3), (2.6), (2.11)–(2.13) and (2.15), whereas v_β satisfies inequality (2.9).*

2.2.3. The limit $\beta \to 1^-$ (for fixed $\alpha > 0$)

As $\beta \to 1^-$, inequalities (2.9), (2.11)–(2.12) and (2.15) are lost. Solutions of the initial–boundary value problem (AC) are defined as follows.

Definition 2.5. Let $\alpha \in (0,\infty)$, and let $u_0 \in H_0^1(\Omega)$. By a solution of problem (AC) we mean any function $u \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ such that:

(i) $u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$, $\Delta u \in L^2(Q)$;
(ii) problem (AC) is satisfied in strong sense.

Theorem 2.6. *Let $u_0 \in H_0^1(\Omega)$, and let φ satisfy assumptions (H_0)–(H_1). Let $u_{\alpha,\beta}$ be the solution of problem (P) given by Theorem 2.2 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\alpha \in (0,\infty)$ there exist a function $u_\alpha \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ and a subsequence $\{u_{\alpha,\beta_k}\} \subseteq \{u_{\alpha,\beta}\}$ such that*

(i) *$u_{\alpha t} \in L^2(Q)$, $\varphi(u_\alpha) \in L^2(Q)$, and $\Delta u_\alpha \in L^2(Q)$. Moreover, the function u_α is a solution of problem (AC);*

(ii) *as* $\beta_k \to 1^-$ *there hold*

$$u_{\alpha,\beta_k} \overset{*}{\rightharpoonup} u_\alpha \quad \text{in } L^\infty\big(0,T;H_0^1(\Omega)\big), \tag{2.29}$$

$$u_{\alpha,\beta_k} \to u_\alpha \quad \text{almost everywhere in } Q, \tag{2.30}$$

$$(u_{\alpha,\beta_k})_t \rightharpoonup u_{\alpha t} \quad \text{in } L^2(Q), \tag{2.31}$$

$$\varphi(u_{\alpha,\beta_k}) \rightharpoonup \varphi(u_\alpha) \quad \text{in } L^2(Q), \tag{2.32}$$

$$\Delta u_{\alpha,\beta_k} \rightharpoonup \Delta u_\alpha \quad \text{in } L^2(Q); \tag{2.33}$$

(iii) *the function* u_α *satisfies inequalities (2.3)–(2.5), (2.7)–(2.8), and the a priori estimate*

$$\|u_{\alpha t}\|_{L^2(Q)} \le M \tag{2.34}$$

with some constant $M > 0$ *which only depends on the norm* $\|u_0\|_{H_0^1(\Omega)}$.

Moreover, if φ *satisfies either assumption* (H_2), *or assumptions* (H_0), (H_1) *and* (H_3), *then* u_α *satisfies the a priori estimates*

$$\|u_\alpha\|_{L^\infty(0,T;H_0^1(\Omega))} \le \sqrt{2}\|u_0\|_{H_0^1(\Omega)}\big(1+e^{2LT}\big)^{\frac{1}{2}}; \tag{2.35}$$

$$\sqrt{\alpha}\|\Delta u_\alpha\|_{L^2(Q)} \le \|u_0\|_{H_0^1(\Omega)}\big(1+e^{2LT}\big)^{\frac{1}{2}} \quad \big(\alpha \in (0,\infty)\big). \tag{2.36}$$

2.3. Letting $\alpha \to 0^+$ in problem (CH)

Observe that the solution u_α of problem (CH) given by Theorem 2.4 satisfies inequalities (2.3)–(2.5) and (2.7)–(2.8), whereas $v_\alpha := \varphi(u_\alpha) - \alpha\Delta u_\alpha$ satisfies inequality (2.22) (see Theorem 2.4(iii)). As $\alpha \to 0^+$, inequalities (2.5), (2.7) and (2.8) get lost, thus the only a priori estimate of the family $\{u_\alpha\}$ (uniform with respect to α) is given by inequality (2.3). As a consequence, different situations will expectedly arise in the limit as $\alpha \to 0^+$, depending on the behavior at infinity of the function Φ. This motivates the following assumption:

(H_4) there exists $k > 0$ such that

$$k|u|^r \le \Phi(u) \tag{2.37}$$

for some $r \in [1,\infty)$ if $N = 1, 2$, or $r \in [1, \frac{2N}{N-2}]$ if $N \ge 3$.

Observe that the above conditions on the exponent r follow from assumption (H_1) and the compatibility condition $r \le q+1$ (see (H_1), (H_4)).

Let us introduce some definitions concerning Radon measures on the set Q. By $\mathcal{M}(\Omega)$ we denote the space of finite Radon measures on Ω, and by $\mathcal{M}^+(\Omega)$ the cone of positive (finite) Radon measures on Ω. We denote by $\langle\cdot,\cdot\rangle_\Omega$ the duality map between $\mathcal{M}(\Omega)$ and the space $C_c(\Omega)$ of continuous functions with compact support. For $\mu \in \mathcal{M}(\Omega)$ and $\rho \in L^1(\Omega,\mu)$ we set, by abuse of notation,

$$\langle\mu,\rho\rangle_\Omega := \int_\Omega \rho\, d\mu \quad \text{and} \quad \|\mu\|_{\mathcal{M}(\Omega)} := |\mu|(\overline{\Omega}). \tag{2.38}$$

Similar notations will be used for the space of finite Radon measures on Q.

We denote by $L^\infty(0,T;\mathcal{M}(\Omega))$ the set of finite Radon measures $u \in \mathcal{M}(Q)$ with the following properties: for almost every $t \in (0,T)$ there exists a measure $u(\cdot,t) \in \mathcal{M}(\Omega)$ such that

(i) for every $\zeta \in C(\overline{Q})$ the map $t \to \langle u(\cdot,t), \zeta(\cdot,t)\rangle_\Omega$ belongs to $L^1(0,T)$, and

$$\langle u, \zeta\rangle_Q = \int_0^T \big\langle u(\cdot,t), \zeta(\cdot,t)\big\rangle_\Omega \, dt; \tag{2.39}$$

(ii) the map $t \to \|u(\cdot,t)\|_{\mathcal{M}(\Omega)}$ belongs to $L^\infty(0,T)$.

The definition of the positive cone $L^\infty(0,T;\mathcal{M}^+(\Omega))$ is now obvious.

Let us also recall the following definition.

Definition 2.6. A subset $\mathcal{U} \subseteq L^1(Q)$ is said to be uniformly integrable if:

(i) there exists $M > 0$ such that

$$\|f\|_{L^1(Q)} := \iint_Q \big|f(x,t)\big| \, dx \, dt \le M \quad \text{for any } f \in \mathcal{U};$$

(ii) for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $f \in \mathcal{U}$ and any Lebesgue measurable set $E \subseteq Q$

$$|E| < \delta \quad \Rightarrow \quad \iint_E \big|f(x,t)\big| \, dx \, dt < \varepsilon.$$

Then we can state the following result.

Theorem 2.7. *Let $u_0 \in H_0^1(\Omega)$, and let φ satisfy assumptions (H_0), (H_1) and (H_4). Let u_α be the solution of problem (CH) given by Theorem 2.4 ($\alpha \in (0,\infty)$). Then there exist $u \in L^\infty(0,T;L^r(\Omega))$, $\mu \in L^\infty(0,T;\mathcal{M}(\Omega))$ and $v \in L^2(0,T;H_0^1(\Omega))$ with the following properties:*

(i) *there exist two subsequences $\{u_{\alpha_k}\} \subseteq \{u_\alpha\}$, $\{v_{\alpha_k}\} \subseteq \{v_\alpha\}$ and a decreasing sequence of measurable sets $E_k \subseteq Q$ of Lebesgue measure $|E_k| \to 0$, such that the sequence $\{u_{\alpha_k}\chi_{Q\setminus E_k}\}$ is uniformly integrable, and as $\alpha_k \to 0^+$ there hold*

$$u_{\alpha_k}\chi_{Q\setminus E_k} \rightharpoonup u \quad \text{in } L^r(Q), \tag{2.40}$$

$$u_{\alpha_k}\chi_{E_k} \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}(Q), \tag{2.41}$$

$$\varphi(u_{\alpha_k}) \rightharpoonup v \quad \text{in } \mathcal{D}(Q), \tag{2.42}$$

$$v_{\alpha_k} \rightharpoonup v \quad \text{in } L^2\big(0,T;H_0^1(\Omega)\big); \tag{2.43}$$

(ii) *there holds*

$$\iint_Q u\zeta_t \, dx \, dt + \int_0^T \big\langle \mu(\cdot,t), \zeta_t(\cdot,t)\big\rangle_\Omega \, dt = \iint_Q \nabla v \cdot \nabla\zeta \, dx \, dt - \int_\Omega u_0(x)\zeta(x,0) \, dx \tag{2.44}$$

for every $\zeta \in C^1([0,T];C_c^1(\Omega))$ such that $\zeta(\cdot,T) = 0$ in Ω.

Moreover, the measure μ is equal to 0 if assumption (H_4) is satisfied with $r > 1$. In this case $E_k = \emptyset$ for every $k \in \mathbb{N}$, the convergence in (2.40) reads

$$u_{\alpha_k} \rightharpoonup u \quad \text{in } L^r(Q), \tag{2.45}$$

and (2.41) is trivially satisfied.

3. Well-posedness: proofs

Theorem 2.1 is easily proven by standard methods of the theory of abstract evolution equations, if problem (P) is rephrased as a Cauchy problem in the Banach space $X = L^2(\Omega)$ – an approach already used in [7,9]. To this purpose, denote by $[I - \varepsilon\Delta]^{-1}$ $(\varepsilon > 0)$ the operator

$$[I - \varepsilon\Delta]^{-1} : L^2(\Omega) \to H^2(\Omega) \cap H_0^1(\Omega), \qquad [I - \varepsilon\Delta]^{-1} z := w \quad \bigl(z \in L^2(\Omega)\bigr),$$

where $w \in H^2(\Omega) \cap H_0^1(\Omega)$ is the unique solution of the elliptic problem

$$\begin{cases} -\epsilon\Delta w + w = z & \text{in } \Omega, \\ w = 0 & \text{in } \partial\Omega \end{cases} \tag{3.1}$$

for any $z \in L^2(\Omega)$. Observe that the operatorial identity

$$\Delta\bigl[(1-\beta)I - \beta\Delta\bigr]^{-1} = \frac{1}{\beta}\bigl\{(1-\beta)\bigl[(1-\beta)I - \beta\Delta\bigr]^{-1} - I\bigr\}, \tag{3.2}$$

where

$$\bigl[(1-\beta)I - \beta\Delta\bigr]^{-1} := \frac{1}{1-\beta}\left[I - \frac{\beta}{1-\beta}\Delta\right]^{-1},$$

holds in the strong sense in $L^2(\Omega)$. Then consider the operator $\mathcal{A} \equiv \mathcal{A}_{\alpha\beta} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ defined as follows:

$$\begin{cases} D(\mathcal{A}) := H^2(\Omega) \cap H_0^1(\Omega), \\ \mathcal{A}u := -\alpha\Delta\bigl[(1-\beta)I - \beta\Delta\bigr]^{-1}\Delta u \quad \bigl(u \in D(\mathcal{A})\bigr). \end{cases} \tag{3.3}$$

Also observe that, if φ satisfies assumption (H_2), there holds

$$\bigl\|\varphi(u)\bigr\|_{L^2(\Omega)} \le \bigl\|\varphi'\bigr\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\Omega)}$$

for every $u \in L^2(\Omega)$. Hence the nonlinear operator $\mathcal{F} \equiv \mathcal{F}_\beta : L^2(\Omega) \to L^2(\Omega)$,

$$\mathcal{F}(u) := \Delta\bigl[(1-\beta)I - \beta\Delta\bigr]^{-1}\varphi(u) \quad \bigl(u \in L^2(\Omega)\bigr) \tag{3.4}$$

is well defined.

From the first equation of problem (P) we plainly obtain

$$(1-\beta)u_t - \beta\Delta u_t = \frac{1}{\beta}\bigl\{\beta\Delta\bigl(\varphi(u) - \alpha\Delta u\bigr) - (1-\beta)\bigl(\varphi(u) - \alpha\Delta u\bigr)\bigr\} + \frac{(1-\beta)}{\beta}\bigl[\varphi(u) - \alpha\Delta u\bigr],$$

whence by (3.2)

$$
\begin{aligned}
u_t &= -\frac{1}{\beta}\big[\varphi(u) - \alpha\Delta u\big] + \frac{(1-\beta)}{\beta}\big[(1-\beta)I - \beta\Delta\big]^{-1}\big(\varphi(u) - \alpha\Delta u\big) \\
&= \Delta\big[(1-\beta)I - \beta\Delta\big]^{-1}\big(\varphi(u) - \alpha\Delta u\big).
\end{aligned} \tag{3.5}
$$

Then problem (P) reads

$$
\begin{cases} u_t = \mathcal{A}u + \mathcal{F}(u) & (t > 0), \\ u(0) = u_0 \in X. \end{cases} \tag{3.6}
$$

To prove existence and uniqueness of solutions to (3.6), we need some properties of the operators $\mathcal{A}$ and $\mathcal{F}$.

Proposition 3.1. *For every $\alpha \in (0, \infty)$, $\beta \in (0, 1)$ the linear operator $\mathcal{A} \equiv \mathcal{A}_{\alpha\beta}$ defined in (3.3) is self-adjoint.*

Proof. (i) Firstly, let us show that $\mathcal{A}$ is symmetric. For every $u \in H^2(\Omega) \cap H_0^1(\Omega)$ we have

$$
\mathcal{A}u = -\alpha\Delta g_u, \tag{3.7}
$$

where $g_u \in H^2(\Omega) \cap H_0^1(\Omega)$ is the unique solution of the elliptic problem

$$
\begin{cases} -\beta\Delta g_u + (1-\beta)g_u = \Delta u & \text{in } \Omega, \\ g_u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}
$$

For every $u, v \in D(\mathcal{A})$ there holds

$$
(\mathcal{A}u, v)_{L^2(\Omega)} - (u, \mathcal{A}v)_{L^2(\Omega)} = -\frac{\alpha(1-\beta)}{\beta}\int_\Omega (g_u v - g_v u)\,dx.
$$

Hence the claim will follow, if we show that

$$
\int_\Omega (g_u v - g_v u)\,dx = 0. \tag{3.9}
$$

To this purpose, let $h_u, h_v \in H^4(\Omega) \cap H_0^1(\Omega)$ be the unique solutions of the problems

$$
\begin{cases} -\beta\Delta h_u + (1-\beta)h_u = u & \text{in } \Omega, \\ h_u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}
$$

and

$$
\begin{cases} -\beta\Delta h_v + (1-\beta)h_v = v & \text{in } \Omega, \\ h_v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}
$$

respectively; here u and v belong to $H^2(\Omega) \cap H_0^1(\Omega)$. In particular, there hold $\Delta h_u, \Delta h_v \in H_0^1(\Omega)$, hence from (3.8)–(3.11) by uniqueness we have $\Delta h_u = g_u$ and $\Delta h_v = g_v$. Then we obtain

$$
\begin{aligned}
g_u v &= -\beta\Delta h_v g_u + (1-\beta)h_v g_u = -\beta g_v g_u + (1-\beta)h_v\Delta h_u, \\
g_v u &= -\beta\Delta h_u g_v + (1-\beta)h_u g_v = -\beta g_u g_v + (1-\beta)h_u\Delta h_v,
\end{aligned}
$$

whence

$$\int_\Omega (g_u v - g_v u)\,dx = (1-\beta)\int_\Omega [h_v \Delta h_u - h_u \Delta h_v]\,dx = 0.$$

Then the claim follows.

(ii) It is easily seen that the operator $\mathcal{A}$ is dissipative. In fact, for every $u \in \mathcal{D}(\mathcal{A})$, we have

$$(\mathcal{A}u, u)_{L^2(\Omega)} = -\alpha \int_\Omega u \Delta g_u\,dx, \tag{3.12}$$

with $g_u \in H^2(\Omega) \cap H_0^1(\Omega)$ as in (3.8). Since there holds

$$\begin{aligned}(\mathcal{A}u, u)_{L^2(\Omega)} &= -\alpha \int_\Omega u \Delta g_u\,dx = -\alpha \int_\Omega g_u \Delta u\,dx \\ &= \alpha\beta \int_\Omega g_u \Delta g_u\,dx - \alpha(1-\beta)\int_\Omega g_u^2\,dx \\ &= -\alpha\beta \int_\Omega |\nabla g_u|^2\,dx - \alpha(1-\beta)\int_\Omega g_u^2\,dx \le 0,\end{aligned}$$

the claim follows.

(iii) Next, let us prove that for every $f \in L^2(\Omega)$ there exists a unique $u \in \mathcal{D}(\mathcal{A})$ such that

$$u - \mathcal{A}u = f. \tag{3.13}$$

In this connection, observe that

$$\begin{aligned}u - \mathcal{A}u &= u + \alpha\Delta\big\{\big[(1-\beta)I - \beta\Delta\big]^{-1}\Delta u\big\} \\ &= u + \alpha\big\{\big[(1-\beta)I - \beta\Delta\big]^{-1}\Delta^2 u\big\}\end{aligned} \tag{3.14}$$

(the proof of the latter equality is postponed until the end of the proof). In view of (3.14), Eq. (3.13) reads

$$\alpha\Delta^2 u = (1-\beta)(f-u) - \beta\Delta(f-u),$$

namely

$$\alpha\Delta^2 u - \beta\Delta u + (1-\beta)u = -\beta\Delta f + (1-\beta)f. \tag{3.15}$$

By a solution of (3.15) we mean any $u \in \mathcal{D}(\mathcal{A})$ such that for every $\xi \in \mathcal{D}(\mathcal{A})$

$$\alpha \int_\Omega \Delta u \Delta\xi\,dx + \beta \int_\Omega \nabla u \nabla\xi\,dx + (1-\beta)\int_\Omega u\xi\,dx = -\beta \int_\Omega f\Delta\xi\,dx + (1-\beta)\int_\Omega f\xi\,dx.$$

To this purpose, consider the bilinear form $a : \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}) \to \mathbb{R}$ defined by setting

$$a(u,v) := \alpha \int_\Omega \Delta u \Delta v\,dx + \beta \int_\Omega \nabla u \cdot \nabla v\,dx + (1-\beta)\int_\Omega uv\,dx$$

for every $u, v \in \mathcal{D}(\mathcal{A})$. Then there exists $C > 0$ such that

$$\begin{aligned}\left|a(u,v)\right| &\le \alpha\|\Delta u\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)}+\beta\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}+(1-\beta)\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\\ &\le C\|u\|_{H^2(\Omega)}\|v\|_{H^2(\Omega)},\\ a(u,u) &= \alpha\int_\Omega |\Delta u|^2\,dx+\beta\int_\Omega |\nabla u|^2\,dx+(1-\beta)\int_\Omega u^2\,dx \ge C\|u\|^2_{H^2(\Omega)}\end{aligned}$$

for every $u,v\in\mathcal{D}(\mathcal{A})$. By the above inequalities and the Lax–Milgram Theorem well-posedness of (3.15) for every $f\in L^2(\Omega)$ easily follows.

Finally, let us prove the latter equality in (3.14). By $W:=[\beta\Delta+(1-\beta)I]^{-1}(\Delta^2 u)$ with $u\in H^2(\Omega)\cap H^1_0(\Omega)$ we mean any $W\in L^2(\Omega)$ such that

$$-\beta\int_\Omega W\Delta\zeta\,dx+(1-\beta)\int_\Omega W\zeta\,dx=\int_\Omega \Delta u\Delta\zeta\,dx \tag{3.16}$$

for every $\zeta\in H^2(\Omega)\cap H^1_0(\Omega)$. Since

$$\Delta\left[-\beta\Delta+(1-\beta)I\right]^{-1}(\Delta u)=\Delta g_u\in L^2(\Omega)$$

(with g_u defined in (3.8)) satisfies (3.16), the equality follows if we show that there is a unique solution of (3.16) in $L^2(\Omega)$. In fact, let $W_1,W_2\in L^2(\Omega)$ solve (3.16). Then the difference $W_0:=W_1-W_2\in L^2(\Omega)$ satisfies the equality

$$-\beta\int_\Omega W_0\Delta\zeta\,dx+(1-\beta)\int_\Omega W_0\zeta\,dx=0 \tag{3.17}$$

for every $\zeta\in H^2(\Omega)\cap H^1_0(\Omega)$. Choosing the test function $\zeta=\zeta_0\in H^2(\Omega)\cap H^1_0(\Omega)$, where ζ_0 satisfies the problem

$$\begin{cases}-\Delta\zeta_0=W_0 & \text{in } \Omega,\\ \zeta_0=0 & \text{on } \partial\Omega,\end{cases}$$

we obtain

$$\beta\int_\Omega W_0^2\,dx+(1-\beta)\int_\Omega |\nabla\zeta_0|^2\,dx=0, \tag{3.18}$$

whence $W_0=0$. This completes the proof. □

Proposition 3.2. *Let φ satisfy assumption (H_2). Then for every $\beta\in(0,1)$ the nonlinear operator $\mathcal{F}\equiv\mathcal{F}_\beta$ defined in (3.4) is Lipschitz continuous.*

Proof. By equality (3.4) and the very definition of the operator $\mathcal{F}$

$$\left\|\mathcal{F}(u)-\mathcal{F}(v)\right\|_{L^2(\Omega)}\le C\left\|\varphi(u)-\varphi(v)\right\|_{L^2(\Omega)}\le C\left\|\varphi'\right\|_{L^\infty(\mathbb{R})}\|u-v\|_{L^2(\Omega)} \tag{3.19}$$

for every $u,v\in L^2(\Omega)$. Hence the claim follows. □

Proof of Theorem 2.1. The result follows from Propositions 3.1–3.2 by standard results of semigroup theory (e.g., see [13, Proposition 6.1.2]). □

To prove Theorem 2.2 we shall make use of the following approximation procedure. Consider the family of approximating problems

$$\begin{cases} (1-\beta)u_{nt} = \Delta\big[\varphi_n(u_n) - \alpha\Delta u_n + \beta u_{nt}\big] & \text{in } Q, \\ u_n = \Delta u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n = u_{0n} & \text{in } \Omega \times \{0\}, \end{cases} \tag{P$_n$}$$

with u_{0n} and φ_n defined as follows. For every $u_0 \in H^1_0(\Omega)$, let $\{u_{0n}\} \subseteq C^\infty_c(\Omega)$ be any sequence such that

$$\|u_{0n}\|_{H^1_0(\Omega)} \le \|u_0\|_{H^1_0(\Omega)}, \tag{3.20}$$

$$u_{0n} \to u_0 \quad \text{in } H^1_0(\Omega). \tag{3.21}$$

For any $n \in \mathbb{N}$ set

$$\varphi_n(u) := \begin{cases} \varphi(u) & \text{if } |u| \le n, \\ \varphi(n) + K(u-n) & \text{if } u > n, \\ \varphi(-n) + K(u+n) & \text{if } u < -n, \end{cases} \tag{3.22}$$

where $K > 0$ is the constant in assumption (H_1). It is immediately seen that $\varphi_n(u)u \ge 0$ for any $u \in \mathbb{R}$, and

$$\varphi_n(u) \to \varphi(u) \quad \text{for any } u \in \mathbb{R} \tag{3.23}$$

as $n \to \infty$. Moreover, for every $n \in \mathbb{N}$ there holds $\varphi_n \in Lip(\mathbb{R})$, since

$$\big|\varphi_n'(u)\big| \le K\big\{\big(1+|u|^{q-1}\big)\chi_{\{|u|\le n\}}(u) + \chi_{\{|u|>n\}}(u)\big\}, \tag{3.24}$$

thus in particular

$$\big|\varphi_n'(u)\big| \le K\big(1+n^{q-1}\big) \quad \text{for every } n \in \mathbb{N}\ (u \in \mathbb{R}). \tag{3.25}$$

Observe that inequality (3.24) and the equality $\varphi_n(0) = 0$ also imply the estimate:

$$\big|\varphi_n'(u)\big| \le K\big(1+|u|^{q-1}\big) \quad \text{for every } u \in \mathbb{R}\ (n \in \mathbb{N}). \tag{3.26}$$

By the above remarks (in particular, see inequality (3.25)), every function φ_n satisfies assumption (H_2), whereas u_{0n} belongs to $H^2(\Omega) \cap H^1_0(\Omega)$. Then by Theorem 2.1 for every $n \in \mathbb{N}$ there exists a unique strict solution u_n of problem (P$_n$). By studying the limiting points of the sequence $\{u_n\}$ we shall prove Theorem 2.2.

Remark 3.1. By inequality (3.26) there exists $K_1 > 0$ such that

$$\big|\varphi_n(u)\big| \le K_1\big(1+|u|^q\big) \quad \text{for every } u \in \mathbb{R}\ (n \in \mathbb{N}), \tag{3.27}$$

$$0 \le \Phi_n(u) \le K_1\big(1+|u|^{q+1}\big) \quad \text{for every } u \in \mathbb{R}, \tag{3.28}$$

where

$$\Phi_n(u) := \int_0^u \varphi_n(z)\,dz \quad (u \in \mathbb{R},\ n \in \mathbb{N}) \tag{3.29}$$

(observe that $\Phi_n(u) \geq 0$ for any $u \in \mathbb{R}$, since $\varphi_n(u)u \geq 0$). Clearly, analogous inequalities hold for φ and for its antiderivative Φ (see (2.4)). Moreover, as $n \to \infty$ there holds

$$\Phi_n(u) \to \Phi(u) \quad \text{for any } u \in \mathbb{R}. \tag{3.30}$$

Similar considerations hold true for the following functions:

$$\psi(u) := \int_0^u |z|^{q-1}\,dz, \tag{3.31}$$

$$\Psi(u) := \int_0^u \psi(z)\,dz \quad (u \in \mathbb{R}). \tag{3.32}$$

Observe that by inequality (3.26) there exists $\tilde{K} > 0$ such that

$$\bigl|\varphi_n(u)\bigr| \leq \tilde{K}\bigl|\psi(u)\bigr| \quad \text{for every } u \in \mathbb{R}\ (n \in \mathbb{N}). \tag{3.33}$$

Remark 3.2. Observe that problem (P_n) can be recast in the form

$$\begin{cases} u_{nt} = \Delta v_n & \text{in } Q, \\ u_n = u_{0n} & \text{in } \Omega \times \{0\} \end{cases} \tag{3.34}$$

where for every $t \in [0,T]$ the function $v_n(\cdot,t)$ solves the elliptic problem

$$\begin{cases} -\beta\Delta v_n(\cdot,t) + (1-\beta)v_n(\cdot,t) = \varphi_n(u_n)(\cdot,t) - \alpha\Delta u_n(\cdot,t) & \text{in } \Omega, \\ v_n(\cdot,t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.35}$$

To prove Theorem 2.2 we need some a priori estimates of solutions of the approximating problems (P_n). To this purpose the following lemma is expedient.

Lemma 3.3. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, $u_0 \in H_0^1(\Omega)$, and let φ satisfy assumptions (H_0)–(H_1). Let $\{u_n\}$ be the sequence of solutions to problems (P_n) given by Theorem 2.1, with a sequence $\{u_{0n}\}$ of initial data which satisfy (3.20)–(3.21). Then for every $t \in (0,T]$ there holds*

$$\begin{aligned} &\int_\Omega \Phi_n(u_n)(x,t)\,dx + \frac{\alpha}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx + \beta\int_0^t\int_\Omega u_{nt}^2\,dx\,ds + (1-\beta)\int_0^t\int_\Omega |\nabla v_n|^2\,dx\,ds \\ &\quad = \int_\Omega \Phi_n(u_{0n})(x)\,dx + \frac{\alpha}{2}\int_\Omega |\nabla u_{0n}|^2\,dx, \end{aligned} \tag{3.36}$$

where Φ_n is the function defined in (3.29).

Proof. Let us multiply the first equation of (3.34) by

$$v_n = \frac{1}{1-\beta}\bigl[\varphi(u_n) - \alpha\Delta u_n + \beta u_{nt}\bigr]$$

and integrate over $\Omega \times (0,t)$. Then we obtain

$$\int_\Omega \Phi_n(u_n)(x,t)\,dx - \alpha\int_0^t\int_\Omega \Delta u_n u_{nt}\,dx\,ds + \beta\int_0^t\int_\Omega u_{nt}^2\,dx\,ds + (1-\beta)\int_0^t\int_\Omega |\nabla v_n|^2\,dx\,ds$$
$$= \int_\Omega \Phi_n(u_{0n})\,dx. \tag{3.37}$$

Since $u_n \in C([0,T];H^2(\Omega)\cap H_0^1(\Omega))$ and $u_{nt} \in C([0,T];L^2(\Omega))$, by standard approximation arguments there holds

$$\int_0^t\int_\Omega \Delta u_n u_{nt}\,dx\,ds = \frac{1}{2}\int_\Omega |\nabla u_{0n}|^2\,dx - \frac{1}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx. \tag{3.38}$$

From (3.37)–(3.38) equality (3.36) follows. □

Further a priori estimates of the sequence $\{u_n\}$ are given by the following lemma.

Lemma 3.4. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$ and $u_0 \in H_0^1(\Omega)$. Let φ satisfy either assumption (H_2), or assumptions (H_0), (H_1) and (H_3). Let $\{u_n\}$ be the sequence of solutions to problems (P_n) given by Theorem 2.1, with a sequence $\{u_{0n}\}$ of initial data which satisfy (3.20)–(3.21). Then there exists $L > 0$ such that for every $t \in (0,T]$*

$$(1-\beta)\int_\Omega u_n^2(x,t)\,dx + \frac{\beta}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\int_\Omega (\Delta u_n)^2\,dx\,ds \le \|u_0\|^2_{H_0^1(\Omega)}\big(1+e^{\frac{2LT}{\beta}}\big). \tag{3.39}$$

Proof. Let us multiply the first equation of (P_n) by u_n and integrate over $\Omega \times (0,t)$. By (3.38) we obtain

$$(1-\beta)\int_\Omega u_n^2(x,t)\,dx + \frac{\beta}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\int_\Omega (\Delta u_n)^2\,dx\,ds$$
$$= -\int_0^t\int_\Omega \varphi_n'(u_n)|\nabla u_n|^2\,dx\,ds + (1-\beta)\int_\Omega u_{0n}^2\,dx + \frac{\beta}{2}\int_\Omega |\nabla u_{0n}|^2\,dx. \tag{3.40}$$

If assumption (H_2) is satisfied, there exists $L > 0$ such that

$$\left|\int_0^t\int_\Omega \varphi_n'(u_n)|\nabla u_n|^2\,dx\,ds\right| \le L\int_0^t\int_\Omega |\nabla u_n|^2\,dx\,ds$$

for any $n \in \mathbb{N}$ (see Remark 2.2(i)). On the other hand, if assumption (H_3) is satisfied, for any $n \in \mathbb{N}$, $n > u_0$ there holds

$$-\int_0^t\int_\Omega \varphi_n'(u_n)|\nabla u_n|^2\,dx\,ds \le -\int_0^t\int_{\{|u_n|\le u_0\}} \varphi'(u_n)|\nabla u_n|^2\,dx\,ds$$
$$\le L\int_0^t\int_\Omega |\nabla u_n|^2\,dx\,ds$$

with some constant $L > 0$, since $\varphi \in W^{1,\infty}_{loc}(\mathbb{R})$ by assumption (H_0) (see Remark 2.2(ii)). In either case from equality (3.40) we get

$$(1-\beta)\int_\Omega u_n^2(x,t)\,dx + \frac{\beta}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx + \alpha\int_0^t\int_\Omega (\Delta u_n)^2\,dx\,ds$$
$$\le \|u_0\|^2_{H_0^1(\Omega)} + L\int_0^t\int_\Omega |\nabla u_n|^2\,dx\,ds; \tag{3.41}$$

here use of inequality (3.20) has been made. Then by the Gronwall Lemma from (3.41) we obtain

$$\frac{\beta}{2}\int_\Omega |\nabla u_n|^2(x,t)\,dx \le \|u_0\|^2_{H_0^1(\Omega)} e^{\frac{2Lt}{\beta}}$$

for every $t \in (0,T]$. By integrating the above inequality on $(0,T]$ we plainly obtain

$$L\int_0^t\int_\Omega |\nabla u_n|^2\,dx\,ds \le \|u_0\|^2_{H_0^1(\Omega)} e^{\frac{2LT}{\beta}},$$

which upon substitution in (3.41) gives inequality (3.39). Then the result follows. □

Proposition 3.5. *Let the assumptions of Lemma 3.3 be satisfied. Then for every $\bar\alpha > 0$ there exists $M > 0$ (which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$) such that for any $\alpha \in (0,\bar\alpha)$, $\beta \in (0,1)$ and $n \in \mathbb{N}$*

$$\big\|\Phi_n(u_n)\big\|_{L^\infty(0,T;L^1(\Omega))} \le M; \tag{3.42}$$
$$\sqrt{\alpha}\|u_n\|_{L^\infty(0,T;H_0^1(\Omega))} \le M; \tag{3.43}$$
$$\sqrt{\beta}\|u_{nt}\|_{L^2(Q)} \le M; \tag{3.44}$$
$$\sqrt{1-\beta}\|v_n\|_{L^2(0,T;H_0^1(\Omega))} \le M; \tag{3.45}$$
$$\sqrt{\beta(1-\beta)}\|v_n\|_{L^2(0,T;H^2(\Omega))} \le M. \tag{3.46}$$

Proof. By inequality (3.28) and assumption (H_1) we have

$$0 \le \int_\Omega \Phi_n(u_{0n})\,dx \le K_1\Big(2|\Omega| + \int_\Omega |u_{0n}|^{q+1}\,dx\Big)$$
$$\le \begin{cases} K_1\Big(2|\Omega| + \int_\Omega |u_{0n}|^{\frac{2N}{N-2}}\,dx\Big) & \text{if } N \ge 3,\\ K_1|\Omega|\big(1 + \|u_{0n}\|^{q+1}_{L^\infty(\Omega)}\big) & \text{if } N = 1,2.\end{cases}$$

By the above estimate, inequality (3.20) and the Sobolev embedding results there exists $C > 0$ (which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$) such that for every $n \in \mathbb{N}$

$$\int_\Omega \Phi_n(u_{0n})(x)\,dx + \frac{\alpha}{2}\int_\Omega |\nabla u_{0n}|^2\,dx \le C + \frac{\alpha}{2}\|u_0\|^2_{H_0^1(\Omega)}.$$

Then, by recalling that $\Phi_n(u_n) \geq 0$, from equality (3.36) we obtain estimates (3.42)–(3.45). Inequality (3.46) follows from (3.44)–(3.45), since $u_{nt} = \Delta v_n$ (see (3.34)). □

Proposition 3.6. *Let the assumptions of Lemma 3.3 be satisfied. Then for every $\bar{\alpha} > 0$ there exists $M > 0$ (which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$) such that for any $\alpha \in (0, \bar{\alpha})$, $\beta \in (0,1)$ and $n \in \mathbb{N}$*

$$\sqrt{\alpha}\big\|\varphi_n(u_n)\big\|_{L^2(Q)} \leq M; \tag{3.47}$$

$$\alpha^{\frac{3}{2}}\|\Delta u_n\|_{L^2(Q)} \leq M; \tag{3.48}$$

$$\sqrt{\alpha}\beta\|v_n\|_{L^\infty(0,T;H_0^1(\Omega))} \leq M. \tag{3.49}$$

Proof. Observe that by (3.34)–(3.35) there holds

$$\alpha\Delta u_n = \varphi_n(u_n) + \beta u_{nt} - (1-\beta)v_n \quad \text{in } Q, \tag{3.50}$$

whence by inequalities (3.44)–(3.45)

$$\begin{aligned}\alpha\|\Delta u_n\|_{L^2(Q)} &\leq \big\|\varphi_n(u_n)\big\|_{L^2(Q)} + \beta\|u_{nt}\|_{L^2(Q)} + (1-\beta)\|v_n\|_{L^2(Q)} \\ &\leq M + \big\|\varphi_n(u_n)\big\|_{L^2(Q)}.\end{aligned} \tag{3.51}$$

Therefore inequality (3.47), together with (3.44)–(3.45), implies (3.48). To prove (3.47), let us distinguish two cases:

(i) either $N = 1, 2$ and $q \in (1, \infty)$, or $N \geq 3$ and $q \in (1, \frac{N}{N-2}]$;
(ii) $N \geq 3$ and $q \in (\frac{N}{N-2}, \frac{N+2}{N-2}]$.

(i) By inequality (3.27), for every $t \in (0,T]$ we have

$$\begin{aligned}\big\|\varphi_n(u_n)(\cdot,t)\big\|_{L^2(\Omega)} &= \left(\int_\Omega \big|\varphi_n(u_n)\big|^2(x,t)\,dx\right)^{\frac{1}{2}} \leq 2K_1\left(2|\Omega| + \int_\Omega |u_n|^{2q}(x,t)\,dx\right) \\ &\leq \begin{cases} 2K_1(2|\Omega| + \int_\Omega |u_n|^{\frac{2N}{N-2}}(x,t)\,dx) & \text{if } N \geq 3 \text{ and } q \in (1, \frac{N}{N-2}], \\ 2K_1|\Omega|(1 + \|u_n(\cdot,t)\|^{2q}_{L^\infty(\Omega)}) & \text{if } N = 1,2 \text{ and } q \in (1,\infty). \end{cases}\end{aligned}$$

By the above estimate, inequality (3.43) and the Sobolev embedding results we obtain (3.47) in the present case.

(ii) In this case, by the Sobolev embedding results there hold $u_n(\cdot,t) \in L^{\frac{2N}{N-4}}(\Omega)$ and $|\nabla u_n(\cdot,t)| \in L^{\frac{2N}{N-2}}(\Omega)$ for every $t \in (0,T]$, since $u_n \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$. Then by Remark 3.1 (in particular, see (3.27) and (3.31)) there holds

$$\begin{aligned}\big\|\psi(u_n)(\cdot,t)\big\|_{L^2(\Omega)} &= \left(\int_\Omega \big|\psi(u_n)\big|^2(x,t)\,dx\right)^{\frac{1}{2}} \leq 2K_1\left(2|\Omega| + \int_\Omega |u_n|^{2q}(x,t)\,dx\right) \\ &\leq 2K_1\left(2|\Omega| + \int_\Omega |u_n|^{\frac{2(N+2)}{N-2}}(x,t)\,dx\right) < \infty,\end{aligned}$$

since $L^{\frac{2N}{N-4}}(\Omega) \hookrightarrow L^{\frac{2(N+2)}{N-2}}(\Omega)$. Similarly,

$$\begin{aligned}\big\|\big[\psi'(u_n)|\nabla u_n|^2\big](\cdot,t)\big\|_{L^1(\Omega)} &= \int\limits_\Omega \big[\psi'(u_n)|\nabla u_n|^2\big](x,t)\,dx \\ &\le \Big(\int\limits_\Omega \big|\psi'(u_n)\big|^{\frac{N}{2}}(x,t)\,dx\Big)^{\frac{2}{N}}\Big(\int\limits_\Omega |\nabla u_n|^{\frac{2N}{N-2}}(x,t)\,dx\Big)^{\frac{N-2}{N}} \\ &= \Big(\int\limits_\Omega |u_n|^{\frac{N}{2}(q-1)}(x,t)\,dx\Big)^{\frac{2}{N}}\Big(\int\limits_\Omega |\nabla u_n|^{\frac{2N}{N-2}}(x,t)\,dx\Big)^{\frac{N-2}{N}} \\ &\le C\Big(\int\limits_\Omega |u_n|^{\frac{2N}{N-2}}(x,t)\,dx\Big)^{\frac{2}{N}}\Big(\int\limits_\Omega |\nabla u_n|^{\frac{2N}{N-2}}(x,t)\,dx\Big)^{\frac{N-2}{N}} < \infty\end{aligned}$$

for some $C > 0$.

Now let us multiply equality (3.50) by $\psi(u_n)$ and integrate over Q. By the above remarks we obtain plainly

$$\begin{aligned}&\iint\limits_Q \big|\varphi_n(u_n)\psi(u_n)\big|\,dx\,dt + \alpha\iint\limits_Q \big[\psi'(u_n)|\nabla u_n|^2\big](x,t)\,dx \\ &\quad\le \beta\int\limits_\Omega \Psi(u_{0n})\,dx + (1-\beta)\iint\limits_Q v_n\psi(u_n)\,dx\,dt,\end{aligned} \tag{3.52}$$

where Ψ is the function defined in (3.32) and the inequality

$$\varphi_n(u)\psi(u) = \frac{[\varphi_n(u)u][\psi(u)u]}{u^2} \ge 0 \quad (u \ne 0)$$

has been used.

Concerning the right-hand side of (3.52), by the Hölder inequality we have

$$\iint\limits_Q \big|v_n\psi(u_n)\big|\,dx\,dt \le \|v_n\|_{L^{\frac{2N}{N-2}}(Q)}\big\|\psi(u_n)\big\|_{L^{\frac{2N}{N+2}}(Q)}. \tag{3.53}$$

Let us show that for some constant $M > 0$ there holds

$$\sqrt{\alpha}\big\|\psi(u_n)\big\|_{L^\infty(0,T;L^{\frac{2N}{N+2}}(\Omega))} \le M. \tag{3.54}$$

In fact, by Remark 3.1 there exists $K_2 > 0$ such that for every $t \in (0,T]$

$$\begin{aligned}\big\|\psi(u_n)(\cdot,t)\big\|_{L^{\frac{2N}{N+2}}(\Omega)} &\le K_2\Big(2|\Omega| + \int\limits_\Omega |u_n|^{\frac{2qN}{N+2}}(x,t)\,dx\Big)^{\frac{N+2}{2N}} \\ &\le K_2\Big(2|\Omega| + \int\limits_\Omega |u_n|^{\frac{2N}{N-2}}(x,t)\,dx\Big)^{\frac{N+2}{2N}},\end{aligned}$$

since by assumption $q \le \frac{N+2}{N-2}$. Then by the above estimate, inequality (3.43) and the Sobolev embedding we obtain (3.54). Further, from inequalities (3.45) and (3.53)–(3.54) by the Sobolev embedding results we obtain

$$\sqrt{\alpha(1-\beta)} \iint_Q \left|v_n\psi(u_n)\right| dx\,dt \le M \tag{3.55}$$

with some constant $M > 0$.

On the other hand, since $|\Psi(u_n)| \le K_1(1+|u_n|^{\frac{2N}{N-2}})$, by inequality (3.20) and the Sobolev embedding results there exists $M > 0$ (which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$) such that for every $n \in \mathbb{N}$

$$\int_\Omega \Psi(u_{0n})(x)\,dx \le M. \tag{3.56}$$

Then from (3.52), (3.55) and (3.56) we easily obtain

$$\alpha \iint_Q \left|\varphi_n(u_n)\psi(u_n)\right| dx\,dt \le M$$

for some constant $M > 0$, whence by inequality (3.33) the estimate (3.47) follows also in this case. This completes the proof of (3.47).

To prove inequality (3.49), for any $t \in (0,T]$ let us multiply the elliptic equation (3.35) by $v_n(\cdot,t)$ and integrate over Ω. Using the Hölder inequality as in (3.53), by the Sobolev embedding results we have

$$\begin{aligned}\beta \int_\Omega \left|\nabla v_n(x,t)\right|^2 dx + (1-\beta)\int_\Omega v_n^2(x,t)\,dx &= \int_\Omega \left[v_n\varphi_n(u_n)\right](x,t)\,dx + \alpha\int_\Omega [\nabla v_n \cdot \nabla u_n](x,t)\,dx \\ &\le \left(\int_\Omega \left|\varphi_n(u_n)(x,t)\right|^{\frac{2N}{N+2}} dx\right)^{\frac{N+2}{2N}} \left(\int_\Omega \left|v_n(x,t)\right|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}} \\ &\quad + \alpha\left(\int_\Omega \left|\nabla u_n(x,t)\right|^2 dx\right)^{\frac12}\left(\int_\Omega \left|\nabla v_n(x,t)\right|^2 dx\right)^{\frac12}.\end{aligned} \tag{3.57}$$

Using inequality (3.27) and arguing as in the proof of (3.54), it is proven that for some constant $M > 0$

$$\sqrt{\alpha}\left\|\varphi_n(u_n)\right\|_{L^\infty(0,T;L^{\frac{2N}{N+2}}(\Omega))} \le M. \tag{3.58}$$

Then by the Sobolev embedding results from inequalities (3.43) and (3.57)–(3.58) we obtain for every $t \in (0,T]$

$$\beta \int_\Omega \left|\nabla v_n(x,t)\right|^2 dx + (1-\beta)\int_\Omega v_n^2(x,t)\,dx \le \frac{M}{\sqrt{\alpha}}\left(\int_\Omega \left|\nabla v_n(x,t)\right|^2 dx\right)^{\frac12}$$

for some constant $M > 0$, whence inequality (3.49) immediately follows. This completes the proof. □

Since the assumptions of Lemma 3.4 imply those of Lemma 3.3, we have the following result.

Proposition 3.7. *Let the assumptions of Lemma 3.4 be satisfied. Then for every $\beta \in (0,1)$ and $n \in \mathbb{N}$*

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \le \|u_0\|_{H_0^1(\Omega)}\sqrt{\frac{1+e^{\frac{2LT}{\beta}}}{1-\beta}}; \tag{3.59}$$

$$\|u_n\|_{L^\infty(0,T;H_0^1(\Omega))} \le \|u_0\|_{H_0^1(\Omega)} \sqrt{\frac{2(1+e^{\frac{2LT}{\beta}})}{\beta}}; \tag{3.60}$$

$$\|\Delta u_n\|_{L^2(Q)} \le \|u_0\|_{H_0^1(\Omega)} \sqrt{\frac{1+e^{\frac{2LT}{\beta}}}{\alpha}} \quad \bigl(\alpha \in (0,\infty)\bigr). \tag{3.61}$$

Moreover, for every $\bar{\alpha} > 0$ and $\beta \in (0,1)$ there exists $\bar{M} > 0$ (which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$ and on β, and diverging as $\beta \to 0^+$, $\beta \to 1^-$) such that for any $\alpha \in (0,\bar{\alpha})$ and $n \in \mathbb{N}$

$$\bigl\|\varphi_n(u_n)\bigr\|_{L^2(Q)} \le \bar{M}. \tag{3.62}$$

Proof. Estimates (3.59), (3.60) and (3.61) follow directly from inequality (3.39). Concerning (3.62), from the first equality in (3.35) and inequalities (3.46), (3.61) we plainly obtain

$$\begin{aligned}\bigl\|\varphi_n(u_n)\bigr\|_{L^2(Q)} &\le \alpha\|\Delta u_n\|_{L^2(Q)} + \beta\|\Delta v_n\|_{L^2(Q)} + (1-\beta)\|v_n\|_{L^2(Q)} \\ &\le \|u_0\|_{H_0^1(\Omega)} \sqrt{\bar{\alpha}\bigl(1+e^{\frac{2LT}{\beta}}\bigr)} + M\left(\sqrt{\frac{\beta}{1-\beta}} + \sqrt{1-\beta}\right).\end{aligned}$$

Then the claim follows. □

Proposition 3.8. *Let the assumptions of Lemma 3.3 be satisfied. Then there exist $u \in L^\infty(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$, $v \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and subsequences $\{u_{n_k}\}$, $\{v_{n_k}\}$ such that*

(i) *$u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$, $\Delta u \in L^2(Q)$;*
(ii) *there hold*

$$u_{n_k} \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\bigl(0,T;H_0^1(\Omega)\bigr), \tag{3.63}$$
$$u_{n_k} \to u \quad \text{almost everywhere in } Q, \tag{3.64}$$
$$u_{n_k t} \rightharpoonup u_t \quad \text{in } L^2(Q), \tag{3.65}$$
$$\varphi_{n_k}(u_{n_k}) \to \varphi(u) \quad \text{almost everywhere in } Q, \tag{3.66}$$
$$\varphi_{n_k}(u_{n_k}) \rightharpoonup \varphi(u) \quad \text{in } L^2(Q), \tag{3.67}$$
$$\Phi_{n_k}(u_{n_k}) \to \Phi(u) \quad \text{almost everywhere in } Q, \tag{3.68}$$
$$\Delta u_{n_k} \rightharpoonup \Delta u \quad \text{in } L^2(Q), \tag{3.69}$$
$$v_{n_k} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty\bigl(0,T;H_0^1(\Omega)\bigr), \tag{3.70}$$
$$v_{n_k} \rightharpoonup v \quad \text{in } L^2\bigl(0,T;H^2(\Omega)\bigr). \tag{3.71}$$

Proof. The convergence claims in (3.63)–(3.64) and in (3.70)–(3.71) follow from the a priori estimates (3.43), respectively (3.49) and (3.46). Concerning (3.65), by estimate (3.44) there exists $w \in L^2(Q)$ such that (possibly by extracting a subsequence, denoted again $\{u_{n_k t}\}$ for simplicity)

$$u_{n_k t} \rightharpoonup w \quad \text{in } L^2(Q),$$

thus in particular

$$\iint_Q u_{n_k t}\zeta \, dx\, dt = -\iint_Q u_{n_k}\zeta_t \, dx\, dt \to \iint_Q w\zeta \, dx\, dt$$

for every $\zeta \in C_c^1(Q)$. Since

$$\iint_Q u_{n_k}\zeta_t\,dx\,dt \to \iint_Q u\zeta_t\,dx\,dt$$

by (3.63), there holds $w = u_t \in L^2(Q)$, thus the first claim in (i) and (3.65) follow. The third claim in (i) and (3.69) are similarly proven by using estimate (3.48).

It is easily seen that the convergence in (3.66) (respectively in (3.68)) follows from that in (3.64) by using (3.23) and (3.26) (respectively (3.30) and (3.27)). Let us now address (3.67). By inequality (3.47) there exists $z \in L^2(Q)$ such that (possibly by extracting a subsequence, denoted again $\{\varphi_{n_k}(u_{n_k})\}$ for simplicity)

$$\varphi_{n_k}(u_{n_k}) \rightharpoonup z \quad \text{in } L^2(Q). \tag{3.72}$$

Inequality (3.47) also implies that the sequence $\{\varphi_n(u_n)\}$ is uniformly integrable in $L^1(Q)$. In fact,

$$\big\|\varphi_n(u_n)\big\|_{L^1(Q)} \le \sqrt{|Q|}\big\|\varphi_n(u_n)\big\|_{L^2(Q)} \le M\sqrt{|Q|} \quad \text{for any } n \in \mathbb{N},$$

and for any measurable subset $E \subseteq Q$ with Lebesgue measure $|E| < \delta$

$$\iint_E \big|\varphi_n(u_n)\big|\,dx\,dt \le M\sqrt{|\delta|} \quad \text{for any } n \in \mathbb{N}.$$

Then by the Prokhorov Theorem and the convergence in (3.66) (e.g., see [26, Proposition 1]) we obtain that (possibly by extracting a subsequence, denoted again $\{\varphi_{n_k}(u_{n_k})\}$ for simplicity)

$$\varphi_{n_k}(u_{n_k}) \rightharpoonup \varphi(u) \quad \text{in } L^1(Q). \tag{3.73}$$

From (3.72) and (3.73) we obtain that $z = \varphi(u) \in L^2(Q)$, thus the second claim in (i) and (3.67) follow.

Finally, let us show that $u \in C([0,T]; L^2(\Omega))$. By estimate (3.44), for each $t_1, t_2 \in [0,T]$ and $n \in \mathbb{N}$ there holds

$$\begin{aligned}\int_\Omega \big|u_n(x,t_2) - u_n(x,t_1)\big|^2\,dx &= \int_\Omega \left|\int_{t_1}^{t_2} u_{nt}(x,t)\,dt\right|^2 dx \\ &\le |t_1 - t_2| \int_{t_1}^{t_2}\int_\Omega u_{nt}^2(x,t)\,dx\,dt \le \frac{M}{\beta}|t_1 - t_2|.\end{aligned}$$

By inequality (3.43) and the above estimate the sequence $\{u_n\}$ is equibounded and equicontinuous in $C([0,T]; L^2(\Omega))$. Then by the Ascoli–Arzelà Theorem there exists a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k} \to u \quad \text{in } C\big([0,T]; L^2(\Omega)\big), \tag{3.74}$$

whence the claim follows. This completes the proof. □

The following result follows from Proposition 3.8 (in particular, see (3.63), (3.66)) by a standard localization argument; we leave the proof to the reader.

Proposition 3.9. *Let the assumptions of Proposition 3.8 be satisfied. Let u, v be the functions and $\{u_{n_k}\}$, $\{v_{n_k}\}$ the subsequences given by Proposition 3.8. Then for almost every $t \in (0,T)$*

$$u_{n_k}(\cdot,t) \rightharpoonup u(\cdot,t) \quad \text{in } H_0^1(\Omega), \tag{3.75}$$

$$\varphi_{n_k}(u_{n_k})(\cdot,t) \rightharpoonup \varphi(u)(\cdot,t) \quad \text{in } L^{\frac{2N}{N+2}}(\Omega), \tag{3.76}$$

$$v_{n_k}(\cdot,t) \rightharpoonup v(\cdot,t) \quad \text{in } H_0^1(\Omega). \tag{3.77}$$

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. As already proven, the functions u, v considered in Proposition 3.8 have the regularity properties stated in Definition 2.2. Let $n_k \to \infty$ in the weak formulation of problems (3.34), (3.35) (written with $n = n_k$). Then the convergence results of Propositions 3.8–3.9 imply that problems (2.1) and (2.2) are satisfied (in $L^2(Q)$, respectively in $L^2(\Omega)$ for almost every $t \in (0,T)$).

Let us prove that $u \in C([0,T];H_0^1(\Omega))$. First recall that $u \in C([0,T];L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$ and $u_t, \varphi(u), \Delta u \in L^2(Q)$. Let us multiply the first equation in (2.1) by

$$v = \frac{1}{1-\beta}\left[\varphi(u) - \alpha\Delta u + \beta u_t\right]$$

and argue as in the proof of Lemma 3.3. Then we get

$$\begin{aligned}(1-\beta)\int_\Omega \Phi(u)(x,t_2)\,dx &+ \frac{\alpha(1-\beta)}{2}\int_\Omega |\nabla u|^2(x,t_2)\,dx + \beta(1-\beta)\int_{t_1}^{t_2}\int_\Omega u_t^2\,dx\,ds + \int_{t_1}^{t_2}\int_\Omega |\nabla v|^2\,dx\,ds \\ &= (1-\beta)\int_\Omega \Phi(u)(x,t_1)\,dx + \frac{\alpha(1-\beta)}{2}\int_\Omega |\nabla u|^2(x,t_1)\,dx \end{aligned} \tag{3.78}$$

for every $0 \le t_1 < t_2 \le T$ (since $u \in C([0,T];L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega))$ it is not restrictive to assume that $u(\cdot,t) \in H_0^1(\Omega)$ for every $t \in [0,T]$).

Next, let us choose in the above equality (3.78) $t_2 = t_n$, where $t_n \to t_1^+$ (the case $t_1 = t_n$, with $t_n \to t_2^-$ is analogous). There holds

$$\begin{aligned}\frac{\alpha(1-\beta)}{2}\lim_{n\to\infty}\int_\Omega |\nabla u_n|^2(x,t_n)\,dx = &-\lim_{n\to\infty}\left\{\beta(1-\beta)\int_{t_1}^{t_n}\int_\Omega \left(u_t^2 + |\nabla v|^2\right)dx\,dt\right\} \\ &+ (1-\beta)\lim_{n\to\infty}\left\{\int_\Omega \Phi(u)(x,t_1)\,dx - \int_\Omega \Phi(u)(x,t_n)\,dx\right\} \\ &+ \frac{\alpha(1-\beta)}{2}\int_\Omega |\nabla u|^2(x,t_1)\,dx.\end{aligned}$$

Here we have

$$u(\cdot,t_n) \rightharpoonup u(\cdot,t_1) \quad \text{in } H_0^1(\Omega), \tag{3.79}$$

$$\lim_{n\to\infty}\left\{\beta(1-\beta)\int_{t_1}^{t_n}\int_\Omega \left(u_t^2 + |\nabla v|^2\right)dx\,dt\right\} = 0. \tag{3.80}$$

$$\lim_{n\to\infty}\left|\int_\Omega \Phi(u)(x,t_1)\,dx - \int_\Omega \Phi(u)(x,t_n)\,dx\right| = \lim_{n\to\infty}\left|\int_{t_1}^{t_n}\int_\Omega \varphi(u)u_t\,dx\,dt\right| = 0 \tag{3.81}$$

since $\varphi(u)u_t \in L^1(Q)$. By (3.80)–(3.81) there holds

$$\lim_{n\to\infty}\int_\Omega |\nabla u|^2(x,t_n)\,dx = \int_\Omega |\nabla u|^2(x,t_1)\,dx,$$

whence $u(\cdot,t_n) \to u(\cdot,t_1)$ in $H_0^1(\Omega)$ (see also (3.79)).

Therefore, the function u is a solution of problem (P), and the a priori estimates (2.5)–(2.11) follow from the analogous inequalities (3.43)–(3.49) by the lower semicontinuity of the norm. Concerning (2.3), by estimate (3.42), the convergence in (3.68) and the Fatou Lemma there holds

$$0 \le \int_\Omega \Phi(u)(x,t)\,dx \le \liminf_{n_k\to\infty}\int_\Omega \Phi_{n_k}(u_{n_k})(x,t)\,dx \le M$$

for almost every $t \in (0,T)$. Then inequality (2.3) follows.

It remains to prove uniqueness. To this purpose, let (u_1,v_1) and (u_2,v_2) be two solutions of problem (P). Then the differences u_1-u_2, v_1-v_2 satisfy the problems

$$\begin{cases} (u_1-u_2)_t = \Delta(v_1-v_2) & \text{in } Q, \\ u_1-u_2 = 0 & \text{in } \Omega\times\{0\}, \end{cases} \tag{3.82}$$

$$\begin{cases} -\beta\big[\Delta(v_1-v_2)\big](\cdot,t) + (1-\beta)(v_1-v_2)(\cdot,t) & \\ \quad = \big(\varphi(u_1)-\varphi(u_2)\big)(\cdot,t) - \alpha\big[\Delta(u_1-u_2)\big](\cdot,t) & \text{in } \Omega, \\ (v_1-v_2)(\cdot,t) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.83}$$

Let us multiply the first equation of (3.82) by the function

$$\psi(x,t) := -\int_t^\tau (v_1-v_2)(x,s)\,ds,$$

for any fixed $\tau \in (0,T)$, and integrate over $Q_\tau := \Omega\times(0,\tau)$. Then we obtain

$$\begin{aligned}\iint_{Q_\tau}(u_1-u_2)(v_1-v_2)\,dx\,dt &= -\iint_{Q_\tau}\big[\nabla(v_2-v_2)\big](x,t)\left(\int_t^\tau \big[\nabla(v_2-v_2)\big](x,s)\,ds\right)dx\,dt \\ &= -\frac{1}{2}\int_\Omega\left|\int_0^\tau \big[\nabla(v_2-v_2)\big](x,s)\,ds\right|^2 dx \le 0,\end{aligned}$$

whence

$$\iint_{Q_\tau}(u_1-u_2)(v_1-v_2)\,dx\,dt \le 0. \tag{3.84}$$

On the other hand, let us multiply the first equation of (3.83) by the difference $(u_1-u_2)(\cdot,t)$ and integrate over Q_τ. This gives

$$(1-\beta)\iint_{Q_\tau}(u_1-u_2)(v_1-v_2)\,dx\,dt = \iint_{Q_\tau}\left[\varphi(u_1)-\varphi(u_2)\right][u_1-u_2]\,dx\,dt$$
$$+\alpha\iint_{Q_\tau}\left|\nabla(u_1-u_2)\right|^2 dx\,dt + \frac{\beta}{2}\int_\Omega (u_1-u_2)^2(x,\tau)\,dx,$$

whence by inequality (3.84) and assumption (H_2)

$$\frac{\beta}{2}\int_\Omega (u_1-u_2)^2(x,\tau)\,dx \le C\iint_{Q_\tau}(u_1-u_2)^2\,dx\,dt$$

with some constant $C>0$. By the above estimate and the Gronwall Lemma uniqueness follows. This completes the proof. □

Proof of Theorem 2.3. Estimates (2.12)–(2.15) follow from the analogous inequalities (3.59)–(3.62) by the lower semicontinuity of the norm, in view of the convergence in (3.63) and (3.67)–(3.69). □

4. Asymptotic limits: proofs

Let us first prove Theorem 2.4.

Proof of Theorem 2.4. The a priori estimates (2.5) and (2.9) ensure the existence of two limiting functions $u_\alpha \in L^\infty(0,T;H_0^1(\Omega))$, $v_\alpha \in L^2(0,T;H_0^1(\Omega))$ such that both the convergences in (2.17) and (2.21) hold. Moreover, by (2.5), (2.9) and the first equation in (2.1), it can be easily seen that for every $\rho \in H_0^1(\Omega)$ the sequence

$$F_k^\rho(t) = \int_\Omega u_{\alpha,\beta_k}(x,t)\rho(x)\,dx$$

is uniformly bounded – hence weakly relatively compact – in the Sobolev space $H^1(0,T)$. By such consideration and (2.17), it follows that for almost every $t\in(0,T)$ there holds

$$F_k^\rho(t) \to \int_\Omega u_\alpha(x,t)\rho(x)\,dx. \tag{4.1}$$

Thus, by (2.17) and (4.1) we obtain

$$u_{\alpha,\beta_k}(\cdot,t) \rightharpoonup u_\alpha(\cdot,t) \quad \text{in } H_0^1(\Omega) \tag{4.2}$$

for almost every $t\in(0,T)$. Finally, by (4.2), (2.17) and the Dominated Convergence Theorem, (2.18) follows.

The convergences in (2.19)–(2.21) and claim (i) follow from (2.7)–(2.9) by arguing as in the proof of Propositions 3.5–3.6.

To prove claim (iii) let us consider the weak formulation of problems (2.1)–(2.2) written with $u=u_{\alpha,\beta}$ and $v=v_{\alpha,\beta}$, namely

$$\iint_Q u_{\alpha,\beta}\zeta_t\,dx\,dt + \iint_Q v_{\alpha,\beta}\Delta\zeta\,dx\,dt = -\int_\Omega u_0(x)\zeta(x,0)\,dx, \tag{4.3}$$

$$\beta\iint_Q \nabla v_{\alpha,\beta}\cdot\nabla\zeta\,dx\,dt + (1-\beta)\iint_Q v_{\alpha,\beta}\zeta\,dx\,dt = \iint_Q\left[\varphi(u_{\alpha,\beta})-\alpha\Delta u_{\alpha,\beta}\right]\zeta\,dx\,dt \tag{4.4}$$

for every $\zeta \in C^1([0,T]; C_c^2(\Omega))$ such that $\zeta(.,T) = 0$ in Ω. Let $\beta_k \to 0^+$ in equalities (4.3)–(4.4) (written with $\beta = \beta_k$). Then by (2.19)–(2.21) we obtain

$$\iint_Q u_\alpha \zeta_t \, dx\, dt + \iint_Q v_\alpha \Delta\zeta \, dx\, dt = -\int_\Omega u_0(x)\zeta(x,0)\, dx,$$

$$\iint_Q v_\alpha \zeta \, dx\, dt = \iint_Q \big[\varphi(u_\alpha) - \alpha\Delta u_\alpha\big]\zeta \, dx\, dt$$

for every ζ as above, whence equality (2.16) and claim (iii) follow. Finally, the statements in claim (iii) concerning inequalities (2.5), (2.7)–(2.8) and (2.22) follow from the analogous estimates (2.5), (2.7)–(2.9) and (2.22) for $u = u_{\alpha,\beta}$ and $v = v_{\alpha,\beta}$, by the convergence in (2.17), (2.19)–(2.21) and the lower semicontinuity of the norm. On the other hand, the statement concerning inequality (2.3) follows from the same inequality for $u = u_{\alpha,\beta}$ by the convergence in (2.18) and the Fatou Lemma, as in the proof of Theorem 2.2. Then the conclusion follows. □

Proof of Theorem 2.6. We only prove claim (iii), the proof of the others following by the same arguments used in the proof of Theorem 2.4. Consider the weak formulation of problems (2.1)–(2.2) written with $u = u_{\alpha,\beta}$ and $v = v_{\alpha,\beta}$, namely

$$\iint_Q (u_{\alpha,\beta})_t \zeta \, dx\, dt = -\iint_Q \nabla v_{\alpha,\beta} \cdot \nabla\zeta \, dx\, dt, \tag{4.5}$$

$$\beta \iint_Q \nabla v_{\alpha,\beta} \cdot \nabla\zeta \, dx\, dt + (1-\beta) \iint_Q v_{\alpha,\beta}\zeta \, dx\, dt = \iint_Q \big[\varphi(u_{\alpha,\beta}) - \alpha\Delta u_{\alpha,\beta}\big]\zeta \, dx\, dt \tag{4.6}$$

for every $\zeta \in C([0,T]; C^2(\bar{\Omega}))$ such that $\zeta(.,t) = 0$ on $\partial\Omega$ for every $t \in [0,T]$. By inequality (2.10) there exist a function $v_\alpha \in L^\infty(0,T; H_0^1(\Omega))$ and a subsequence $\{v_{\alpha,\beta_k}\} \subseteq \{v_{\alpha,\beta}\}$ such that

$$v_{\alpha,\beta_k} \overset{*}{\rightharpoonup} v_\alpha \quad \text{in } L^\infty\big(0,T; H_0^1(\Omega)\big)$$

as $\beta_k \to 1^-$. Let $\beta_k \to 1^-$ in equalities (4.5)–(4.6) (written with $\beta = \beta_k$). Then we obtain

$$\begin{aligned} \iint_Q u_{\alpha t}\zeta \, dx\, dt &= -\iint_Q \nabla v_\alpha \cdot \nabla\zeta \, dx\, dt \\ &= -\iint_Q \big[\varphi(u_\alpha) - \alpha\Delta u_\alpha\big]\zeta \, dx\, dt \end{aligned}$$

for every ζ as above. By using inequality (2.34) and arguing as in the proof of Proposition 3.8, it is easily seen that $u_\alpha \in C([0,T]; L^2(\Omega))$. Then the result follows. □

Proof of Theorem 2.5. The convergence claims in (2.23)–(2.24) follow from the a priori estimates (2.6) and (2.13), those in (2.25)–(2.26) follow from (2.6) and (2.15), respectively, and those in (2.27)–(2.28) follow from (2.9) and (2.11).

To prove claim (iii), consider the following weak formulation of problems (2.1)–(2.2):

$$\iint_Q (u_{\alpha,\beta})_t \zeta \, dx\, dt = \iint_Q \Delta v_{\alpha,\beta}\zeta \, dx\, dt, \tag{4.7}$$

$$-\beta \iint_Q \Delta v_{\alpha,\beta}\zeta\,dx\,dt + (1-\beta)\iint_Q v_{\alpha,\beta}\zeta\,dx\,dt = \iint_Q \varphi(u_{\alpha,\beta})\zeta\,dx\,dt - \alpha\iint_Q u_{\alpha,\beta}\Delta\zeta\,dx\,dt \tag{4.8}$$

for every $\zeta \in C([0,T]; C_c^2(\Omega))$. Let $\alpha_k \to 0^+$ in (4.8) (written with $\alpha = \alpha_k$). Then by (4.7) and the arbitrariness of ζ we obtain the equalities

$$\iint_Q (u_\beta)_t\zeta\,dx\,dt = \iint_Q \Delta v_\beta\zeta\,dx\,dt, \tag{4.9}$$

$$-\beta\iint_Q \Delta v_\beta\zeta\,dx\,dt + (1-\beta)\iint_Q v_\beta\zeta\,dx\,dt = \iint_Q \varphi(u_\beta)\zeta\,dx\,dt. \tag{4.10}$$

By the arbitrariness of ζ, from (4.10) we obtain

$$v_\beta = \frac{1}{1-\beta}\big[\varphi(u_\beta) + \beta u_{\beta t}\big],$$

which upon substitution in (4.9) shows that problem (S) is satisfied in strong sense. The proof of claim (iii) is analogous to those given for Theorems 2.4–2.6, hence the result follows. □

5. Letting $\alpha \to 0^+$ in problem (CH): proofs

To prove Theorem 2.7 we need some definitions and results concerning Young measures on $Q \times \mathbb{R}$ (e.g., see [10,26] and references therein).

Definition 5.1. By a Young measure on $Q \times \mathbb{R}$ we mean any positive Radon measure τ such that

$$\tau(E \times \mathbb{R}) = |E| \tag{5.1}$$

for any Lebesgue measurable set $E \subseteq Q$. The set of Young measures on $Q \times \mathbb{R}$ will be denoted by $\mathcal{Y}(Q;\mathbb{R})$.

If $f : Q \to \mathbb{R}$ is Lebesgue measurable, the Young measure associated to f is the measure $\tau \in \mathcal{Y}(Q;\mathbb{R})$ such that

$$\tau(E \times F) = \big|E \cap f^{-1}(F)\big| \tag{5.2}$$

for any Lebesgue measurable set $E \subseteq Q$ and any Borel set $F \subseteq \mathbb{R}$.

Proposition 5.1. *Let $\tau \in \mathcal{Y}(Q;\mathbb{R})$. Then for almost every $(x,t) \in Q$ there exists a measure $\tau_{(x,t)} \in \mathcal{P}\infty(\mathbb{R})$, such that for any function $\psi : Q \times \mathbb{R} \to \mathbb{R}$ bounded and continuous:*

(i) *the map*

$$(x,t) \to \big\langle \tau_{(x,t)}, \psi(x,t,\cdot)\big\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \psi(x,t,\xi)\,d\tau_{(x,t)}(\xi)$$

is Lebesgue measurable;

(ii) *there holds*

$$\langle \tau, \psi\rangle_{Q\times\mathbb{R}} := \int_{Q\times\mathbb{R}} \psi\,d\tau = \iint_Q \big\langle \tau_{(x,t)}, \psi(x,t,\cdot)\big\rangle_{\mathbb{R}}\,dx\,dt$$
$$= \iint_Q dx\,dt \int_{\mathbb{R}} \psi(x,t,\xi)\,d\tau_{(x,t)}(\xi). \tag{5.3}$$

Therefore, every $\tau \in \mathcal{Y}(Q \times \mathbb{R})$ can be identified with the associated family $\{\tau_{(x,t)} \mid (x,t) \in Q\}$, which is called the disintegration of τ.

Definition 5.2. Let $\{\tau^n\} \subseteq \mathcal{Y}(Q;\mathbb{R})$, $\tau \in \mathcal{Y}(Q;\mathbb{R})$ $(n \in \mathbb{N})$. We say that $\tau^n \to \tau$ narrowly in $Q \times \mathbb{R}$, if

$$\int_{Q\times\mathbb{R}} \psi\, d\tau^n \to \int_{Q\times\mathbb{R}} \psi\, d\tau \tag{5.4}$$

for any function $\psi : Q \times \mathbb{R} \to \mathbb{R}$ bounded and measurable, such that $\psi(x,t,\cdot)$ is continuous for almost every $(x,t) \in Q$.

The following result, concerning bounded sequences of functions in $L^1(Q)$, will be used.

Theorem 5.2. *Let $\{f_n\}$ be a bounded sequence in $L^1(Q)$, and $\{\tau^n\}$ the sequence of associated Young measures. Then:*

(i) *there exist subsequences $\{f_k\} \equiv \{f_{n_k}\} \subseteq \{f_n\}$, $\{\tau^k\} \equiv \{\tau^{n_k}\} \subseteq \{\tau^n\}$ and a Young measure τ on $Q \times \mathbb{R}$ such that $\tau^k \to \tau$ narrowly in $Q \times \mathbb{R}$;*

(ii) *for any $\rho \in C(\mathbb{R})$ such that the sequence $\{\rho \circ f_n\} \subseteq L^1(Q)$ is uniformly integrable, there holds*

$$\rho \circ f_k \equiv \rho \circ f_{n_k} \rightharpoonup \rho^* \quad \text{in } L^1(Q), \tag{5.5}$$

where

$$\rho^*(x,t) := \langle \tau_{(x,t)}, \rho \rangle_{\mathbb{R}} = \int_{\mathbb{R}} \rho(\xi)\, d\tau_{(x,t)}(\xi) \quad \text{for a.e. } (x,t) \in Q \tag{5.6}$$

and $\{\tau_{(x,t)}\}$ is the disintegration of τ.

If a sequence $\{f_n\}$ is bounded in $L^1(Q)$ but not uniformly integrable, we can extract from it a uniformly integrable subsequence "by removing sets of small measure". This is the content of the following theorem (e.g., see [10]).

Theorem 5.3 *(Biting Lemma). Let $\{f_n\}$ be a bounded sequence in $L^1(Q)$. Then there exist a subsequence $\{f_k\} \equiv \{f_{n_k}\} \subseteq \{f_n\}$ and a decreasing sequence of measurable sets $E_k \subseteq Q$ of Lebesgue measure $|E_k| \to 0$, such that the sequence $\{f_k \chi_{Q\setminus E_k}\}$ is uniformly integrable.*

Some consequences of the Biting Lemma are discussed in the following.

Remark 5.1. Let $\{f_k\}$ be the subsequence considered in Theorem 5.3, and let $\{\tau^k\}$ be the associated sequence of Young measures. Let τ denote the narrow limit of the sequence $\{\tau^k\}$, which exists by Theorem 5.2(i) (possibly by extracting a subsequence, still denoted $\{\tau^k\}$ for simplicity), and let $\{E_k\}$ be the sequence of measurable sets considered in Theorem 5.3. Since the sequence $\{f_k \chi_{Q\setminus E_k}\}$ is uniformly integrable, by Theorem 5.2(ii) there holds

$$f_k \chi_{Q\setminus E_k} \rightharpoonup u := \int_{[0,\infty)} \xi\, d\tau(\xi) \quad \text{in } L^1(Q). \tag{5.7}$$

The function $u \in L^1(Q)$ in (2.40) is called the barycenter of the disintegration $\{\tau_{(x,t)}\}$ of τ. Besides, since the sequence $\{f_k \chi_{E_k}\}$ is bounded in $L^1(Q)$, there exists a measure $\mu \in \mathcal{M}(Q)$ such that

$$f_k \chi_{E_k} \stackrel{*}{\rightharpoonup} (1-\beta) \quad \text{in } \mathcal{M}(Q). \tag{5.8}$$

We can now proceed to prove Theorem 2.7. Let us first mention the following result.

Proposition 5.4. *Let $u_0 \in H_0^1(\Omega)$, and let φ satisfy assumptions (H_0), (H_1) and (H_4). Let u_α be the solution of problem (CH) given by Theorem 2.4 $(\alpha \in (0,\infty))$. Then there exists $M > 0$ (which only depends on the norm $\|u_0\|_{H_0^1(\Omega)}$) such that for any $\alpha \in (0,\bar{\alpha})$ and r as in (H_4)*

$$\|u_\alpha\|_{L^\infty(0,T;L^r(\Omega))} \le M. \tag{5.9}$$

Proof. Follows immediately from inequality (2.3) and assumption (H_4). □

We can now prove Theorem 2.7.

Proof of Theorem 2.7. The claims concerning the function $v \in L^2(0,T;H_0^1(\Omega))$, the subsequence $\{v_{\alpha_k}\} \subseteq \{v_\alpha\}$ and the convergence in (2.43) follow from inequality (2.34).

On the other hand, by inequality (5.9) the family $\{u_\alpha\}$ is bounded in $L^1(Q)$, thus the Biting Lemma and its consequences described in Remark 5.1 can be used. In particular, there exist a subsequence $\{u_{\alpha_k}\} \subseteq \{u_\alpha\}$, and a decreasing sequence of measurable sets $E_k \subseteq Q$ of Lebesgue measure $|E_k| \to 0$, such that the convergence in (5.7)–(5.8) holds with $f_k = u_{\alpha_k}$. Moreover, by arguing as in [25], it is easily seen that $u \in L^\infty(0,T;L^1(\Omega))$ and $\mu \in L^\infty(0,T;\mathcal{M}(\Omega))$.

To prove claim (ii) and the convergence in (2.42), recall that $v_\alpha := \varphi(u_\alpha) - \alpha\Delta u_\alpha$ belongs to $L^2(0,T;H_0^1(\Omega))$. Then by standard approximation arguments from equality (2.16) we get

$$\iint_Q u_\alpha \zeta_t \, dx\, dt = \iint_Q \nabla v_\alpha \cdot \nabla \zeta \, dx\, dt - \int_\Omega u_0(x)\zeta(x,0)\, dx \tag{5.10}$$

for every $\zeta \in C^1([0,T];C_c^1(\Omega))$ such that $\zeta(\cdot,T) = 0$ in Ω, and

$$\iint_Q v_\alpha \zeta \, dx\, dt = \iint_Q \varphi(u_\alpha)\zeta \, dx\, dt - \alpha \iint_Q u_\alpha \Delta\zeta \, dx\, dt \tag{5.11}$$

for every $\zeta \in C^1([0,T];C_c^2(\Omega))$ such that $\zeta(.,T) = 0$ in Ω. Observe that

$$\iint_Q u_{\alpha_k}\zeta_t \, dx\, dt = \iint_Q u_{\alpha_k}\chi_{E_k}\zeta_t \, dx\, dt + \iint_Q u_{\alpha_k}\chi_{Q\setminus E_k}\zeta_t \, dx\, dt.$$

Now let $\alpha_k \to 0^+$ in equality (5.10) (written with $\alpha = \alpha_k$). Then by the convergence in (5.7)–(5.8) we obtain equality (2.44). On the other hand, by the convergence in (2.43), letting $\alpha_k \to 0^+$ in equality (5.11) (written with $\alpha = \alpha_k$) gives the convergence in (2.42).

Finally, if assumption (H_4) holds with $r > 1$, by inequality (5.9) the family $\{u_\alpha\}$ is uniformly integrable in $L^1(Q)$. Then by Theorem 5.2(ii) the convergence in (2.45) and the remaining claims follow. This completes the proof. □

6. Neumann boundary conditions

Consider the companion problem of (P) with homogeneous Neumann boundary conditions:

$$\begin{cases} (1-\beta)u_t = \Delta\big[\varphi(u) - \alpha\Delta u + \beta u_t\big] & \text{in } Q, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{NP}$$

Set

$$H^2_E(\Omega) := \left\{ u \in H^2(\Omega) \;\middle|\; \frac{\partial u}{\partial n} = 0 \right\},$$

where the normal derivative $\frac{\partial u}{\partial n}$ is meant in the sense of traces on $\partial\Omega$. By abuse of notation, hereafter we denote by $[I - \varepsilon\Delta]^{-1}$ ($\varepsilon > 0$) the operator

$$[I - \varepsilon\Delta]^{-1} : L^2(\Omega) \to H^2_E(\Omega), \qquad [I - \varepsilon\Delta]^{-1} z := w \quad \big(z \in L^2(\Omega)\big),$$

where $w \in H^2_E(\Omega)$ is the unique solution of the elliptic problem

$$\begin{cases} -\epsilon\Delta w + w = z & \text{in } \Omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{in } \partial\Omega \end{cases} \tag{6.1}$$

for any $z \in L^2(\Omega)$. Accordingly, by $\mathcal{A} \equiv \mathcal{A}_{\alpha\beta} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ we denote the following operator:

$$\begin{cases} D(\mathcal{A}) := H^2_E(\Omega), \\ \mathcal{A}u := -\alpha\Delta\big[(1-\beta)I - \beta\Delta\big]^{-1}\Delta u \quad \big(u \in D(\mathcal{A})\big). \end{cases} \tag{6.2}$$

By the methods used in Section 3 we have a first existence result concerning solutions of problem (NP), which is the analogue of Theorem 2.1.

Theorem 6.1. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let φ satisfy assumption (H_2). Then for every $u_0 \in H^2_E(\Omega)$ there exists a unique function u such that:*

(i) *$u \in C([0,T]; H^2_E(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, and $\varphi(u) \in C([0,T]; L^2(\Omega))$;*
(ii) *u satisfies in strong sense problem (2.1), where $v \in C([0,T]; H^2_E(\Omega))$ and for every $t \in [0,T]$ the function $v(\cdot,t)$ is the unique solution of the elliptic problem*

$$\begin{cases} -\beta\Delta v(\cdot,t) + (1-\beta)v(\cdot,t) = \varphi(u)(\cdot,t) - \alpha\Delta u(\cdot,t) & \text{in } \Omega, \\ \dfrac{\partial v(\cdot,t)}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

A more general well-posedness result for problem (NP), analogous to Theorem 2.2, is obtained as before by considering the family of approximating problems

$$\begin{cases} (1-\beta)u_{nt} = \Delta\big[\varphi_n(u_n) - \alpha\Delta u_n + \beta u_{nt}\big] & \text{in } Q, \\ \dfrac{\partial u_n}{\partial n} = \dfrac{\partial \Delta u_n}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n = u_{0n} & \text{in } \Omega \times \{0\}. \end{cases} \tag{NP$_n$}$$

Here φ_n is defined as above (see (3.22)), and for every $u_0 \in H^1(\Omega)$ $\{u_{0n}\} \subseteq H^2_E(\Omega)$ is any sequence such that

$$\|u_{0n}\|_{H^1(\Omega)} \le \|u_0\|_{H^1(\Omega)}, \tag{6.4}$$

$$u_{0n} \to u_0 \quad \text{in } H^1(\Omega). \tag{6.5}$$

Observe that by mass conservation, for every $t \in (0,T]$ and $n \in \mathbb{N}$

$$\int_\Omega u_n(x,t)\,dx = \int_\Omega u_{0n}(x)\,dx. \tag{6.6}$$

Concerning solutions of the approximating problems (NP_n) we have the following estimates, which are the counterpart of those of Propositions 3.5–3.6.

Proposition 6.2. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, $u_0 \in H^1(\Omega)$, and let φ satisfy assumptions (H_0)–(H_1). Let $\{u_n\}$ be the sequence of solutions to problems (NP_n) given by Theorem 6.1, with a sequence $\{u_{0n}\}$ of initial data which satisfies (6.4)–(6.5). Then for every $\bar\alpha > 0$ there exists $M > 0$ (which only depends on the norm $\|u_0\|_{H^1(\Omega)}$) such that for any $\alpha \in (0,\bar\alpha)$, $\beta \in (0,1)$ and $n \in \mathbb{N}$*

$$\big\|\Phi_n(u_n)\big\|_{L^\infty(0,T;L^1(\Omega))} \le M; \tag{6.7}$$

$$\sqrt{\alpha}\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \le M; \tag{6.8}$$

$$\sqrt{\alpha}\big\||\nabla u_n|\big\|_{L^\infty(0,T;L^2(\Omega))} \le M; \tag{6.9}$$

$$\sqrt{\beta}\|u_{nt}\|_{L^2(Q)} \le M; \tag{6.10}$$

$$\sqrt{\alpha}\big\|\varphi_n(u_n)\big\|_{L^2(Q)} \le M; \tag{6.11}$$

$$\sqrt{\alpha^3\beta}\|\Delta u_n\|_{L^2(Q)} \le M; \tag{6.12}$$

$$\sqrt{\alpha\beta}(1-\beta)\|v_n\|_{L^2(Q)} \le M; \tag{6.13}$$

$$\sqrt{1-\beta}\big\||\nabla v_n|\big\|_{L^2(Q)} \le M; \tag{6.14}$$

$$\sqrt{\beta}\|\Delta v_n\|_{L^2(Q)} \le M. \tag{6.15}$$

Proof. It is easily checked that equality (3.36) holds under the present assumption, hence inequalities (6.7), (6.9)–(6.10) and (6.14) follow. The proof of inequality (6.11) is the same as that of (3.47), by using (6.9) and (6.14) instead of (3.43) and (3.45), respectively. Inequality (6.15) follows from (6.10) by the equality $u_{nt} = \Delta v_n$.

Concerning (6.8), observe that by the Poincaré inequality and mass conservation (see (6.6)) there exists $C > 0$ such that

$$\begin{aligned}\big\|u_n(\cdot,t)\big\|^2_{L^2(\Omega)} &\le C\left[\big\||\nabla u_n|(\cdot,t)\big\|^2_{L^2(\Omega)} + \left(\int_\Omega u_n(x,t)\,dx\right)^2\right] \\ &= C\left[\big\||\nabla u_n|(\cdot,t)\big\|^2_{L^2(\Omega)} + \left(\int_\Omega u_{0n}(x)\,dx\right)^2\right]\end{aligned}$$

for every $t \in (0,T)$. Then by inequalities (6.4) and (6.9) we have

$$\big\|u_n(\cdot,t)\big\|^2_{L^2(\Omega)} \le C\left(\frac{M}{\alpha} + |\Omega|\|u_0\|^2_{H^1(\Omega)}\right)$$

for every $t \in (0,T)$, whence (6.8) follows.

From the first equality in (3.57) (which follows from equality (2.2) as for Dirichlet boundary conditions) by the Sobolev embedding results and inequalities (6.9), (6.11) we have

$$\begin{aligned}
&\beta\big\||\nabla v_n|(\cdot,t)\big\|^2_{L^2(\Omega)} + (1-\beta)\big\|v_n(\cdot,t)\big\|^2_{L^2(\Omega)} \\
&\quad\le \big\|v_n(\cdot,t)\big\|_{L^2(\Omega)}\big\|\varphi_n(u_n)\big\|_{L^2(\Omega)} + \alpha\big\||\nabla v_n|(\cdot,t)\big\|_{L^2(\Omega)}\big\||\nabla u_n|(\cdot,t)\big\|_{L^2(\Omega)} \\
&\quad\le \frac{1-\beta}{2}\big\|v_n(\cdot,t)\big\|^2_{L^2(\Omega)} + \frac{1}{2(1-\beta)}\big\|\varphi_n(u_n)\big\|^2_{L^2(\Omega)} + \frac{\beta}{2}\big\||\nabla v_n|(\cdot,t)\big\|^2_{L^2(\Omega)} + \frac{\alpha^2}{2\beta}\big\||\nabla u_n|(\cdot,t)\big\|^2_{L^2(\Omega)},
\end{aligned}$$

namely

$$\beta\big\||\nabla v_n|(\cdot,t)\big\|^2_{L^2(\Omega)} + (1-\beta)\big\|v_n(\cdot,t)\big\|^2_{L^2(\Omega)} \le \frac{1}{1-\beta}\big\|\varphi_n(u_n)\big\|^2_{L^2(\Omega)} + \frac{\alpha^2}{\beta}\big\||\nabla u_n|(\cdot,t)\big\|^2_{L^2(\Omega)}.$$

Let us integrate the above inequality on $(0,T)$ and make use of inequalities (6.9), (6.11). Then we obtain

$$\beta\big\||\nabla v_n|\big\|^2_{L^2(Q)} + (1-\beta)\|v_n\|^2_{L^2(Q)} \le M^2\left(\frac{1}{\alpha(1-\beta)} + \frac{\alpha}{\beta}\right),$$

whence inequality (6.13) follows. Finally, from equality (3.50), by using (6.10), (6.11) and (6.13) we obtain (6.12). This completes the proof. □

By arguing as in Section 3, from the above estimates we obtain the following analogue of Theorem 2.2.

Theorem 6.3. *Let $\alpha \in (0,\infty)$, $\beta \in (0,1)$, and let φ satisfy assumptions (H_0)–(H_1). Then for every $u_0 \in H^1(\Omega)$ there exists a solution of problem (NP), which is meant in the following sense:*

(i) $u \in L^2(0,T;H^2_E(\Omega)) \cap C([0,T];H^1(\Omega))$, $u_t \in L^2(Q)$, $\varphi(u) \in L^2(Q)$;
(ii) *problems (2.1) and (6.3) are satisfied in strong sense, with $v \in L^2(0,T;H^2_E(\Omega))$.*

For every $t \in (0,T]$ there holds

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx, \tag{6.16}$$

and, if φ satisfies assumption (H_2), the solution is unique.

Moreover, for every $\bar\alpha > 0$ there exists $M > 0$ (which only depends on the norm $\|u_0\|_{H^1(\Omega)}$) such that for any $\alpha \in (0,\bar\alpha)$ and $\beta \in (0,1)$

$$\big\|\Phi(u)\big\|_{L^\infty(0,T;L^1(\Omega))} \le M, \tag{6.17}$$

where the function Φ is defined by (2.4);

$$\sqrt{\alpha}\|u\|_{L^\infty(0,T;H^1(\Omega))} \le M; \tag{6.18}$$

$$\sqrt{\beta}\|u_t\|_{L^2(Q)} \le M; \tag{6.19}$$

$$\sqrt{\alpha}\big\|\varphi(u)\big\|_{L^2(Q)} \le M; \tag{6.20}$$

$$\sqrt{\alpha^3\beta}\|\Delta u\|_{L^2(Q)} \le M; \tag{6.21}$$

$$\sqrt{\alpha\beta}(1-\beta)\|v\|_{L^2(Q)} \le M; \tag{6.22}$$

$$\sqrt{1-\beta}\big\||\nabla v|\big\|_{L^2(Q)} \le M; \tag{6.23}$$

$$\sqrt{\beta}\|\Delta v\|_{L^2(Q)} \le M. \tag{6.24}$$

Remark 6.1. If φ satisfies either assumption (H_2), or assumptions (H_0), (H_1) and (H_3), the counterparts of Lemma 3.4, Proposition 3.7 and Theorem 2.3 are easily proven; we leave their formulation to the reader. Similarly, we shall not discuss in the present case results analogous to Theorem 2.5, concerning the limit $\alpha \to 0^+$.

The following theorem shows that as $\beta \to 0^+$ the solution of problem (NP) obtained above gives a solution of the Neumann initial–boundary value problem for the Cahn–Hilliard equation:

$$\begin{cases} u_t = \Delta\big[\varphi(u) - \alpha\Delta u\big] & \text{in } Q, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \Delta u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{NCH}$$

The proof (which makes use of estimates (6.18), (6.20), (6.22) and (6.23)) is similar to that of Theorem 2.4, thus is omitted.

Theorem 6.4. *Let $u_0 \in H^1(\Omega)$, and let φ satisfy assumptions (H_0)–(H_1). Let $u_{\alpha,\beta}$ be the solution of problem (NP) given by Theorem 6.3 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\alpha \in (0,\infty)$ there exist $u_\alpha \in L^\infty(0,T;H^1(\Omega))$ with $\varphi(u_\alpha) \in L^2(Q)$, $v_\alpha \in L^2(0,T;H^1(\Omega))$ and two subsequences $\{u_{\alpha,\beta_k}\} \subseteq \{u_{\alpha,\beta}\}$, $\{v_{\alpha,\beta_k}\} \subseteq \{v_{\alpha,\beta}\}$ such that*

(i) *as $\beta_k \to 0^+$ there hold*

$$u_{\alpha,\beta_k} \overset{*}{\rightharpoonup} u_\alpha \quad \text{in } L^\infty\big(0,T;H^1(\Omega)\big), \tag{6.25}$$

$$u_{\alpha,\beta_k} \to u_\alpha \quad \text{almost everywhere in } Q, \tag{6.26}$$

$$\varphi(u_{\alpha,\beta_k}) \rightharpoonup \varphi(u_\alpha) \quad \text{in } L^2(Q), \tag{6.27}$$

$$v_{\alpha,\beta_k} \rightharpoonup v_\alpha \quad \text{in } L^2\big(0,T;H^1(\Omega)\big); \tag{6.28}$$

(ii) *the function u_α is a solution of problem (NCH), in the sense that*

$$\iint\limits_Q u_\alpha \zeta_t \, dx\, dt - \iint\limits_Q \nabla v_\alpha \cdot \nabla\zeta \, dx\, dt = -\int\limits_\Omega u_0(x)\zeta(x,0)\, dx,$$

$$\iint\limits_Q v_\alpha \zeta \, dx\, dt = \iint\limits_Q \varphi(u_\alpha)\zeta \, dx\, dt + \alpha \iint\limits_Q \nabla u_\alpha \cdot \nabla\zeta \, dx\, dt$$

for every $\zeta \in C^1(\bar{Q})$ such that $\zeta(.,T) = 0$ in Ω;

(iii) *the function u_α satisfies equality (6.16) for almost every $t \in (0,T)$ and inequalities (6.17), (6.18), (6.20), whereas v_α satisfies the a priori estimate*

$$\big\||\nabla v_\alpha|\big\|_{L^2(Q)} \le M \tag{6.29}$$

with some constant $M > 0$ which only depends on the norm $\|u_0\|_{H^1(\Omega)}$.

When $\beta \to 1^-$, in the present case we obtain solutions of the problem

$$\begin{cases} u_t = \alpha \Delta u - \varphi(u) + \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \varphi(u)\,dx & \text{in } Q_T, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{NRD}$$

(where the presence of the nonlocal term in the right-hand side of the first equation stems from Neumann boundary condition; see (AC)). This is the content of the following theorem.

Theorem 6.5. *Let $u_0 \in H^1(\Omega)$, and let φ satisfy assumptions (H_0)–(H_1). Let $u_{\alpha,\beta}$ be the solution of problem (NP) given by Theorem 6.3 ($\alpha \in (0,\infty)$, $\beta \in (0,1)$). Then for every $\alpha \in (0,\infty)$ there exist a function $u_\alpha \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ and a subsequence $\{u_{\alpha,\beta_k}\} \subseteq \{u_{\alpha,\beta}\}$ such that*

(i) *$u_{\alpha t} \in L^2(Q)$, $\varphi(u_\alpha) \in L^2(Q)$, $\Delta u_\alpha \in L^2(Q)$;*
(ii) *as $\beta_k \to 1^-$ there hold*

$$u_{\alpha,\beta_k} \overset{*}{\rightharpoonup} u_\alpha \quad \text{in } L^\infty\big(0,T;H^1(\Omega)\big), \tag{6.30}$$

$$u_{\alpha,\beta_k} \to u_\alpha \quad \text{almost everywhere in } Q, \tag{6.31}$$

$$(u_{\alpha,\beta_k})_t \rightharpoonup u_{\alpha t} \quad \text{in } L^2(Q), \tag{6.32}$$

$$\varphi(u_{\alpha,\beta_k}) \rightharpoonup \varphi(u_\alpha) \quad \text{in } L^2(Q), \tag{6.33}$$

$$\Delta u_{\alpha,\beta_k} \rightharpoonup \Delta u_\alpha \quad \text{in } L^2(Q), \tag{6.34}$$

(iii) *the function u_α is a solution of problem (NRD), in the sense that*

$$\int_Q \left(\int_\Omega \varphi(u_\alpha)(x',t)\,dx' \right) \zeta(x,t)\,dx\,dt + \int_\Omega u_0(x)\zeta(x,0)\,dx$$
$$= \iint_Q \varphi(u_\alpha)\zeta\,dx\,dt + \alpha \iint_Q \nabla u_\alpha \cdot \nabla\zeta\,dx\,dt + \iint_Q u_\alpha \zeta_t\,dx\,dt$$

for every $\zeta \in C^1(\bar{Q})$ such that $\zeta(.,T) = 0$ in Ω;
(iv) *the function u_α satisfies equality (6.16), inequalities (6.17), (6.18), (6.20) and the a priori estimates*

$$\|u_{\alpha t}\|_{L^2(Q)} \le M, \qquad \sqrt{\alpha^3}\|\Delta u\|_{L^2(Q)} \le M, \tag{6.35}$$

with some constant $M > 0$ which only depends on the norm $\|u_0\|_{H^1(\Omega)}$.

Proof. We only prove claim (iii). Setting $z_{\alpha,\beta} := (1-\beta)v_{\alpha,\beta}$, by inequalities (6.22)–(6.23) we have

$$\sqrt{\alpha\beta}\|z_{\alpha,\beta}\|_{L^2(Q)} \le M, \qquad \big\||\nabla z_{\alpha,\beta}|\big\|_{L^2(Q)} \le M\sqrt{1-\beta}.$$

Then for every $\alpha \in (0,\infty)$ there exist a function $z_\alpha \in L^2(0,T;H^1(\Omega))$ and a subsequence $\{z_{\alpha,\beta_k}\} \subseteq \{z_{\alpha,\beta}\}$ such that

$$z_{\alpha,\beta_k} \rightharpoonup z_\alpha \quad \text{in } L^2(Q), \qquad \nabla z_{\alpha,\beta_k} \to 0 \quad \text{almost everywhere in } Q. \tag{6.36}$$

Therefore z_α is a function of t alone, and $z_\alpha \in L^2(0,T)$.

Now observe that the first equation in (6.3) reads

$$z(t) = \varphi(u)(\cdot,t) - \alpha\Delta u(\cdot,t) + \beta u_t(\cdot,t) \quad \text{in } \Omega \ \big(t \in (0,T)\big),$$

whence plainly

$$\iint_Q z_{\alpha,\beta}\zeta\,dx\,dt = \iint_Q \varphi(u_{\alpha,\beta})\zeta\,dx\,dt + \alpha\iint_Q \nabla u_{\alpha,\beta}\cdot\nabla\zeta\,dx\,dt + \beta\iint_Q (u_{\alpha,\beta})_t\zeta\,dx\,dt$$

for every $\zeta \in C^1(\bar{Q})$. Let us write the above equality with $\beta = \beta_k$ and let $\beta_k \to 1^-$. Then by (6.30), (6.32), (6.33) and (6.36) we obtain

$$\int_0^T z_\alpha(t)\int_\Omega \zeta\,dx\,dt = \iint_Q \varphi(u_\alpha)\zeta\,dx\,dt + \alpha\iint_Q \nabla u_\alpha\cdot\nabla\zeta\,dx\,dt + \iint_Q u_{\alpha t}\zeta\,dx\,dt \tag{6.37}$$

for every $\zeta \in C^1(\bar{Q})$. Let us choose $\zeta = h \in C_c(0,T)$ in (6.37). By the conservation of mass (see (6.16)) and standard approximation arguments we get

$$|\Omega| z_\alpha(t) = \int_\Omega \varphi(u_\alpha)(x,t)\,dx \tag{6.38}$$

for every $t \in (0,T)$. From equalities (6.37)–(6.38) the conclusion follows. □

Finally, let us discuss the limit $\alpha \to 0^+$ of problem (NCH). Up to obvious changes, the analogue of Theorem 2.7 holds true; we leave its formulation to the reader. Remarkably, thanks to the conservation of mass (see equality (6.16) and Theorem 6.4(iii)), the same holds under the weaker assumption

(H_5) there exists $k > 0$ such that

$$ku^\pm \le \Phi(u) \quad \text{for any } u \in \mathbb{R}, \tag{6.39}$$

where $r^\pm := \max\{\pm r, 0\}$ $(r \in \mathbb{R})$, and either sign holds in the above inequality. In fact, the following holds.

Theorem 6.6. *Let $u_0 \in H^1(\Omega)$, and let φ satisfy assumptions (H_0), (H_1) and (H_5). Let u_α be the solution of problem (NCH) given by Theorem 6.4 $(\alpha \in (0,\infty))$. Then there exist $u \in L^\infty(0,T;L^1(\Omega))$, $\mu \in L^\infty(0,T;\mathcal{M}^+(\Omega))$ and $v \in L^2(0,T;H^1(\Omega))$ with the following properties:*

(i) *there exist two subsequences $\{u_{\alpha_k}\} \subseteq \{u_\alpha\}$, $\{v_{\alpha_k}\} \subseteq \{v_\alpha\}$ and a decreasing sequence of measurable sets $E_k \subseteq Q$ of Lebesgue measure $|E_k| \to 0$, such that the sequence $\{u_{\alpha_k}\chi_{Q\setminus E_k}\}$ is uniformly integrable, and as $\alpha_k \to 0^+$ the convergence in (2.40)–(2.42) holds true;*

(ii) *as $\alpha_k \to 0^+$ there holds*

$$v_{\alpha_k} \rightharpoonup v \quad \text{in } L^2\big(0,T;H^1(\Omega)\big); \tag{6.40}$$

(iii) *equality (2.44) is satisfied.*

Proof. By the considerations in Section 5, it suffices to prove that the family $\{u_\alpha\}$ is bounded in $L^1(Q)$. To this purpose, observe that for almost every $t \in (0,T)$

$$\begin{aligned}\int_\Omega \left|u_\alpha(x,t)\right| dx &= \pm \int_\Omega u_\alpha(x,t)\,dx + 2\int_\Omega u_\alpha^\mp(x,t)\,dx \\ &= \pm \int_\Omega u_0(x)\,dx + 2\int_\Omega u_\alpha^\mp(x,t)\,dx \\ &\le \sqrt{|\Omega|}\|u_0\|_{H^1(\Omega)} + 2\int_\Omega \Phi(u_\alpha)(x,t)\,dx \le M;\end{aligned}$$

here use of equality (6.16) and inequality (6.17) has been made. Then the conclusion follows. □

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