ON THE CAHN-HILLIARD REGULARIZATION OF A PERONA-MALIK TYPE EQUATION

BUI LE TRONG THANH AND SILVIA TOZZA

Abstract. We study existence, uniqueness and long-time behaviour of global solutions of the Cahn-Hilliard equation with nonlinearity growing at most linearly at infinity. Motivation for the study comes from the fourth-order regularization of a forward-backward parabolic equation with nonlinearity of Perona-Malik type.

1. Introduction

In this paper we study the problem

$$(P_T) \qquad \begin{cases} u_t = \Delta[\varphi(u) - \epsilon\Delta u] & \text{in } Q_T := \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $T > 0$, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $\frac{\partial}{\partial\nu}$ denotes the outer normal derivative at $\partial\Omega$. We are interested in nonlinearities φ of the following type:

$$\varphi(u) = \frac{u}{1+u^2}\,, \quad \varphi(u) = u\exp(-u)\,. \tag{1.1}$$

Precise assumptions concerning the function φ (and the initial data function u_0) are made below (see Section 2).

Our motivation comes from the Perona-Malik equation [PM] in one space dimension

$$z_t = [\varphi(z_x)]_x\,, \tag{1.2}$$

where φ is as in (1.1), which also appears in a mathematical model for the formation of layers of constant temperature or salinity in the ocean (see [BBDU]). The relationship with problem (P_T), with φ as in (1.1), is easily seen: differentiating

---

Date: December 1, 2014.

1991 *Mathematics Subject Classification.* Primary: 35D99, 35K55, 35R25; Secondary: 28A33, 28A50.

Key words and phrases. Forward-backward parabolic equations, Perona-Malik equation.

equation (1.2) formally with respect to x and setting $u := z_x$ gives the equation

$$u_t = [\varphi(u)]_{xx}\,. \tag{1.3}$$

The first equation in problem (P_T) is a specific regularisation, called of *Cahn-Hilliard type*, of equation (1.3). In fact, the Perona-Malik equation is a well-known example of *forward-backward* parabolic equation, which is ill-posed forward in time.

Beside in image processing [PM] and in modelling of stratified turbulent shear flow [BBDU], forward-backward equations arise in many applications, *e.g.*, in aggregation models of population dynamics [Pa]. To regularize these equations, first a higher order term depending on a small parameter $\epsilon > 0$ is added to the right-hand side (on the strength of different physical and biological considerations, *e.g.*, see [BFJ, BS, G]), then the "vanishing viscosity limit" as $\epsilon \to 0$ is investigated. In carrying out this program, mainly two classes of additional terms have been used in the literature:
(i) $\epsilon\Delta[\psi(u)]_t$, with $\psi' > 0$, leading to third order *pseudo-parabolic equations* [BBDU, BST1, BST2, BuST1, BuST2, EP, MTT, NP, Pl1, Pl2, Pl3, S, STe, ST1, ST2, ST3]. If $\psi(u) = u$, this regularization is called *Sobolev regularization*;
(ii) $-\epsilon\Delta^2 u$, leading to fourth-order *Cahn-Hilliard type equations* (see [BFG, Pl4, Sl] and references therein). This kind of regularization has been less addressed, possibly since studying its singular limit as $\epsilon \to 0$ appears to be more difficult than for the regularization mentioned in (i).

Cahn-Hilliard type equations have been widely investigated in the context of the theory of phase transitions (in particular, see [BS, CGS, EZ, Z]). In this case the non-linearity φ suggested by modelling is cubic, *i.e.*, $\varphi(u) = u^3 - u$. Existence of suitably defined global solutions was proven in [EZ], and their asymptotic behaviour for large time was studied in [Z], under the assumption $N \le 3$ on the space dimension. The singular limit as $\epsilon \to 0$ was studied in [Pl4], taking advantage of the fourth order growth at infinity of the associated free energy Φ. Unfortunately, the approach in [Pl4] does not apply when φ is as in (1.1), due to the very slow (only logarithmic) growth at infinity of the associated potential.

In the light of the above remarks, our motivation for the present study is investigating the regularization mentioned in (ii) for forward-backward equations, whose nonlinearity φ grows at most linearly at infinity (see assumption (A_1)). This is meant as a preliminary step before addressing the singular limit of the problem as $\epsilon \to 0$. Specifically, we prove existence and uniqueness of global solutions in a suitable function space under the assumption $N \le 5$ (see Theorem 2.3). We also study, using the same approach as in [Z], the asymptotic behaviour as $t \to \infty$ of solutions of the problem

$$(P_\infty) \qquad \begin{cases} u_t = \Delta[\varphi(u) - \epsilon\Delta u] & \text{in } Q_\infty := \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases}$$

(in particular, see Theorem 2.11). In doing so, we take advantage of conservation of mass: for any solution u of problem (P_∞)

$$\frac{1}{|\Omega|}\int_\Omega u(x,t)dx = M\,,$$

where

$$M := \frac{1}{|\Omega|}\int_\Omega u_0 dx \tag{1.4}$$

(see Proposition 2.4). Finally, in the case $N = 1$ we address existence and multiplicity of equilibrium solutions of (P_∞) when $\varphi(u) = \frac{u}{1+u^2}$. At variance from the cases of a polynomial φ (see [CGS, NPe, Z]), a complete analytical investigation reveals to be cumbersome, thus recourse to numerical methods has been expedient.

The paper is organized as follows. In Section 2 we describe the mathematical framework and state our main results. Proofs are to be found in Sections 3, 4. Equilibrium solutions of (P_∞) in one space dimension are studied in Section 5.

2. Mathematical framework

2.1. **Preliminaries.** The following function spaces will be used in the sequel:

$$H^2_E(\Omega) := \left\{u \in H^2(\Omega)\,\Big|\, \frac{\partial u}{\partial \nu} = 0\right\}\,,$$

$$H^2_{E^*}(\Omega) := \left\{u \in H^2_E(\Omega)\,\Big|\, \int_\Omega u dx = 0\right\}\,,$$

$$H^4_E(\Omega) := \left\{u \in H^4(\Omega)\,\Big|\, \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0\right\}\,,$$

$$H^4_{E^*}(\Omega) := \left\{u \in H^4_E(\Omega)\,\Big|\, \int_\Omega u dx = 0\right\}\,,$$

$$H^{4,1}(Q_T) := \left\{u \in L^2(0,T;H^4(\Omega))\,|\, u_t \in L^2(Q_T)\right\}\,.$$

We always suppose that $\varphi \in C^3(\mathbf{R})$, $\varphi(0) = 0$; moreover, the following assumptions concerning φ will be used:

$$(A_1) \qquad \varphi' \in L^\infty(\mathbf{R})\,;$$

$$(A_2) \qquad \varphi'' \in L^\infty(\mathbf{R})\,;$$

$$(A_3) \qquad \varphi''' \in L^\infty(\mathbf{R})\,;$$

$$(A_4) \qquad s\varphi(s) \geq 0 \quad \text{for any } \ s \in \mathbf{R}\,.$$

The following proposition (*e.g.*, see [Ze, Proposition 23.23]) will be used to prove existence results.

Proposition 2.1. *Let V be a separable reflexive Banach space with dual space V', and let H be a separable Hilbert space such that:*
(i) $V \subset H \subset V'$;
(ii) V is continuously embedded into H and dense in H.
Then for any $p \in (1,\infty)$ the space

$$Z := \{u \,|\, u \in L^p((0,T);V),\, u_t \in L^q((0,T);V')\}\ ,$$

where $\frac{1}{p}+\frac{1}{q}=1$, is continuously embedded into $C([0,T];H)$.

2.2. **Existence.** Let us state the following definition.

Definition 2.2. *Let $u_0 \in H^2_E(\Omega)$. By a solution of problem (P_T) we mean any function $u=u(x,t)$, $u \in C\left([0,T];H^2_E(\Omega)\right) \cap H^{4,1}(Q_T)$ such that $\varphi(u) \in C\left([0,T];H^2_E(\Omega)\right)$, $u(\cdot,0)=u_0$, $\frac{\partial \Delta u}{\partial \nu}=0$ in the sense of distribution on $\partial\Omega$ and*

$$\int\!\!\int_{Q_T} u_t \eta \, dxdt = -\int\!\!\int_{Q_T} \nabla\left[\varphi(u)-\epsilon\Delta u\right]\cdot\nabla\eta \, dxdt \tag{2.5}$$

for any $\eta \in L^2((0,T),H^1(\Omega))$ (the central dot "$\cdot$" denoting the scalar product in $\mathbb{R}^N$).

Theorem 2.3. *Let assumptions (A_1)-(A_2) be satisfied. Let $u_0 \in H^2_E(\Omega)$, and let $N \leq 5$. Then for every $T>0$ there exists a unique solution of problem (P_T).*

Choosing $\eta \equiv 1$ in equality (2.5) immediately gives the following result.

Proposition 2.4. *Let the assumptions of Theorem 2.3 be satisfied, and let u be the solution of problem (P_T) given by the same theorem. Then for every $t \in (0,T)$*

$$\int_\Omega u(x,t)dx = \int_\Omega u_0 dx\,. \tag{2.6}$$

By Theorem 2.3 and Proposition 2.4 we have the following result.

Corollary 2.5. *Let assumptions (A_1)-(A_2) be satisfied. Let $u_0 \in H^2_{E^*}(\Omega)$, and let $N \leq 5$. Then for every $T > 0$ there exists a unique solution of problem (P_T), which belongs to $C\left([0,T]; H^2_{E^*}(\Omega)\right)$.*

2.3. **Asymptotic behaviour.** Let us first state the following definitions.

Definition 2.6. *Let $u_0 \in H^2_E(\Omega)$. By a global solution of problem (P_∞) we mean any function $u \in C\left([0,\infty); H^2_E(\Omega)\right) \cap H^{4,1}(Q_\infty)$, with $\varphi(u) \in C\left([0,\infty); H^2_E(\Omega)\right)$, which is a solution of problem (P_T) for every $T > 0$.*

Definition 2.7. *Let $u_0 \in H^2_E(\Omega)$. Let u be the global solution of problem (P_∞) given by Theorem 2.3. By the ω−limit set of the solution u we mean the set*

$$\omega(u_0) := \{\bar{u} \mid \exists \{t_n\} \subseteq (0,\infty) \text{ such that } u(x,t_n) \to \bar{u} \text{ in } H^2_E(\Omega)\}. \tag{2.7}$$

Definition 2.8. *By an equilibrium solution of problem (P_∞) we mean any function $w \in H^4_E(\Omega)$, with $\varphi(w) \in H^2_E(\Omega)$, which satisfies in the strong sense the equality*

$$\Delta[\varphi(w) - \epsilon\Delta w] = 0 \quad \text{in } \Omega\,. \tag{2.8}$$

Remark 2.9. *It is immediately seen that there is one-to-one correspondence between equilibrium solutions of problem (P_∞) and functions $w \in H^2_E(\Omega)$, with $\varphi(w) \in H^2_E(\Omega)$, which satisfy in the strong sense the equality*

$$\epsilon\Delta w = \varphi(w) + \sigma \quad \text{in } \Omega \tag{2.9}$$

with some constant $\sigma \in \mathbf{R}$.

By Theorem 2.3 and a standard prolongation argument, for every $u_0 \in H^2_E(\Omega)$ there exists a unique global solution of problem (P_∞). Let us study the asymptotic behaviour of this solution as $t \to \infty$.

To this purpose, Proposition 2.6 suggests the change of unknown $z := u - M$, where M denotes the mass defined in (1.4). Then $u = z + M$ and $z(\cdot,t) \in H^2_{E^*}(\Omega)$ for every $t \in (0,\infty)$. Therefore, it is not restrictive to study problem (P_∞) with initial data $u_0 \in H^2_{E^*}(\Omega)$ (clearly, this implies $u(\cdot,t) \in H^2_{E^*}(\Omega)$ for every $t \in (0,\infty)$). In doing so, the advantage is that we can obtain uniform estimates of the solution on the whole half-line $(0,\infty)$. In fact, the following proposition will be proven.

Proposition 2.10. *Let $u_0 \in H^4_{E^*}(\Omega)$, let assumptions (A_1)-(A_4) be satisfied, and let $N \leq 5$. Let u be the unique global solution of problem (P_∞) given by Theorem*

2.3. Then for every $t \in (0, \infty)$

$$\|u(\cdot,t)\|_{H^3(\Omega)} \le C_2^* \,. \tag{2.10}$$

Then we have the following result.

Theorem 2.11. *Let $u_0 \in H^4_{E^*}(\Omega)$, let assumptions (A_1)-(A_4) be satisfied, and let $N \le 5$. Then:*
(i) the ω-limit set $\omega(u_0)$ is nonempty;
(ii) the function $t \to F(u)(t)$, where

$$F(u)(t) := \int_\Omega \left\{ \Phi(u)(x,t) + \frac{\epsilon}{2} |\nabla u(x,t)|^2 \right\} dx \,, \quad \Phi(u) := \int_0^u \varphi(s) ds \tag{2.11}$$

for any $u \in C\left([0,\infty); H^1(\Omega)\right)$, is nonicreasing;
(iii) every point of $\omega(u_0)$ is an equilibrium solution of problem (P_∞).

Claim (i) of the above theorem, whose proof is omitted, is an obvious consequence of Proposition 2.10, whereas claims (i)-(ii) follow by standard arguments ($e.g.$, see [H]). We leave the details to the reader.

Finally, observe that problem (P_∞) and its solution depends parametrically on ϵ. The following proposition shows that for ϵ sufficiently large the solution decays to zero as $t \to \infty$.

Proposition 2.12. *Let $u_0 \in H^4_{E^*}(\Omega)$, let assumptions (A_1)-(A_4) be satisfied, and let $N \le 5$. Let u be the unique global solution of problem (P_∞) given by Theorem 2.3. Then there exists $\epsilon_0 > 0$ such that for any $\epsilon > \epsilon_0$*

$$\|u(\cdot,t)\|_{H^1(\Omega)} \to 0 \qquad as \;\; t \to \infty \,.$$

In agreement with Proposition 2.12, it can be proven that every nontrivial equilibrium solution of problem (P_∞) is trivial if ϵ is large enough. On the other hand, by similar methods it can be proven that problem (P_∞) admits nontrivial equilibrium solutions, if ϵ is sufficiently small and the mass M sufficiently large. The proof of these statements is analogous to those of [Z, Lemma 3.2 and Theorem 3.3]. The latter statement is in agreement with the considerations of Section 4 (based on numerical evidence), if $N = 1$ and $\varphi(s) = \frac{s}{1+s^2}$.

3. Proof of existence results

This section is devoted to the proof of Theorem 2.3. Set

$$v := \varphi(u) - \epsilon\Delta u\,. \tag{3.12}$$

Then problem (P_T) can be rewritten in the equivalent form

$$\begin{cases} u_t = \Delta v & \text{in } Q_T \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,. \end{cases} \tag{3.13}$$

According to Definition 2.2, we seek a couple of functions $u \in C\left([0,T]; H^2_E(\Omega)\right) \cap H^{4,1}(Q_T)$, $v \in C\left([0,T]; H^2_E(\Omega)\right)$ satisfying problem (3.13), in the sense that $u(\cdot,0) = u_0$ and

$$\iint_{Q_T} u_t \eta \, dxdt = -\iint_{Q_T} \nabla v \cdot \nabla \eta \, dxdt \tag{3.14}$$

for any $\eta \in L^2((0,T), H^1(\Omega))$. This will be achieved using the Faedo-Galerkin method.

Let ψ_k $(k \in \mathbf{N})$ denote the eigenfunctions of the Laplace operator with Neumann boundary conditions

$$\begin{cases} -\Delta\psi_k = \lambda_k \psi_k & \text{in } \Omega \\ \dfrac{\partial \psi_k}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.15}$$

which combined with the constant function $\Phi_0 \equiv 1$ form an orthogonal basis of $H^2_E(\Omega)$. Since by assumption the boundary $\partial\Omega$ is smooth, the functions ψ_k are smooth and there holds

$$\frac{\partial \Delta\psi_k}{\partial \nu} = 0 \quad \text{on } \partial\Omega \qquad (k \in \mathbf{N} \cup \{0\})\,. \tag{3.16}$$

Thus they are a suitable choice for the Faedo-Galerkin method.

In view of the above remarks, we consider approximate solutions of (3.13) of the form

$$u_m := \sum_{j=0}^{m} w_{jm}\psi_j\,, \quad v_m := \varphi(u_m) - \epsilon\Delta u_m \qquad (m \in \mathbf{N} \cup \{0\})\,, \tag{3.17}$$

with coefficients $w_{jm} = w_{jm}(t)$ $(t \in (0,T)$ to be determined. Since

$$\frac{\partial u_m}{\partial \nu} = \sum_{j=0}^{m} w_{jm} \frac{\partial \psi_j}{\partial \nu}$$

and

$$\frac{\partial v_m}{\partial \nu} = \sum_{j=0}^{m} w_{jm} \left[\varphi'(u_m)\frac{\partial \psi_j}{\partial \nu} - \epsilon\,\frac{\partial \Delta\psi_j}{\partial \nu}\right],$$

by (3.15)-(3.16) there holds

$$\frac{\partial u_m(\cdot,t)}{\partial \nu} = \frac{\partial v_m(\cdot,t)}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{3.18}$$

for every $m \in \mathbf{N} \cup \{0\}$. Clearly,

$$u_{mt} = \sum_{j=0}^{m} w'_{jm}(t)\psi_j\,, \quad \Delta u_m = -\sum_{j=0}^{m} \lambda_j w_{jm}\psi_j\,, \quad \Delta^2 u_m = \sum_{j=0}^{m} \lambda_j^2 w_{jm}\psi_j\,. \tag{3.19}$$

Denoting by $(\cdot,\cdot)_{L^2(\Omega)}$ the scalar product in $L^2(\Omega)$,

$$(f,g)_{L^2(\Omega)} := \int_\Omega fg dx \qquad \text{for any } f,g \in L^2(\Omega)\,,$$

by (3.19) we have

$$(u_{mt},\psi_k)_{L^2(\Omega)} = w'_{km}(t)\,,$$

$$\begin{aligned}(\Delta v_m,\psi_k)_{L^2(\Omega)} &= (v_m,\Delta\psi_k)_{L^2(\Omega)} = -\lambda_k\,(v_m,\psi_k)_{L^2(\Omega)} \\ &= -\lambda_k \left\{(\varphi(u_m),\psi_k)_{L^2(\Omega)} - \epsilon\,(\Delta u_m,\psi_k)_{L^2(\Omega)}\right\} \\ &= -\epsilon\lambda_k^2 w_{km}(t) - \lambda_k\,(\varphi(u_m),\psi_k)_{L^2(\Omega)} \qquad (k = 0,1,...,m)\,.\end{aligned}$$

According to the Faedo-Galerkin method, we require that the equalities

$$(u_{mt},\psi_k)_{L^2(\Omega)} = (\Delta v_m,\psi_k)_{L^2(\Omega)}$$

be satisfied for each $m \in \mathbf{N} \cup \{0\}$ and $k = 0,1,...,m$. This gives the system of ordinary differential equations

$$\begin{cases} w'_{km} = -\epsilon\lambda_k^2 w_{km} - \lambda_k\,(\varphi(u_m),\psi_k)_{L^2(\Omega)} & \text{in } (0,T) \\ w_{km}(0) = \alpha_{km} \end{cases} \tag{3.20}$$

for the coefficients $w_{0m}, w_{1m}, ...w_{mm}$. Here

$$\alpha_{km} := (u_{0m},\psi_k)_{L^2(\Omega)}\;,$$

with

$$\begin{cases} u_{0m} := \sum_{j=0}^{m} \alpha_{jm}\psi_j, \quad u_{0m} \to u_0 \text{ in } H^2_E(\Omega)\,, \\ \|u_{0m}\|_{H^2(\Omega)} \le \|u_0\|_{H^2(\Omega)}\,. \end{cases} \tag{3.21}$$

For every $m \in \mathbf{N} \cup \{0\}$ and $k = 0$, since $\lambda_0 = 0$ the unique solution of system (3.21) is $w_{0m}(t) = w_{0m}(0) = \alpha_{0m}$ (observe that the nonlinear term $(\varphi(u_m),\psi_k)_{L^2(\Omega)}$ is locally Lipschitz continuous with respect to w_{km} by assumption (A_1)). On the other hand, for for every $m \in \mathbf{N}$ and $k = 1,...,m$ there exists $T_m > 0$ such that system (3.20) has a unique solution in the maximal interval $(0,T_m)$. In view of the a priori estimates below, this solution is global - namely, it exists in $(0,T)$ for every $m \in \mathbf{N}$.

Lemma 3.1. *Let $u_0 \in H^2_E(\Omega)$, and let assumption (A_1) be satisfied. Let u_m be defined by (3.17) with coefficients w_{km} satisfying system (3.20)-(3.21) in the maximal interval $(0,T_m)$. Then there exists $C_1 > 0$ (only depending on ϵ, T and the norm $\|u_0\|_{H^2(\Omega)}$) such that for every $t \in (0,T_m)$*

$$\|u_m(\cdot,t)\|^2_{L^2(\Omega)} + \int_0^t \|\Delta u_m(\cdot,s)\|^2_{L^2(\Omega)} ds \le C_1\,, \tag{3.22}$$

$$\||\nabla u_m(\cdot,t)|\|^2_{L^2(\Omega)} + \int_0^t \||\nabla \Delta u_m(\cdot,s)|\|^2_{L^2(\Omega)} ds \le C_1\,. \tag{3.23}$$

Proof. By (3.17) and (3.19) we have

$$\|u_m(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{j=0}^m |w_{jm}|^2\,, \quad \|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{j=0}^m \lambda_j^2 |w_{jm}|^2\,. \tag{3.24}$$

Multiplying the first equation of (3.20) by w_{km} and summing over $k = 0,...,m$ plainly we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_m(\cdot,t)\|^2_{L^2(\Omega)} &= -\epsilon\,\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} + (\varphi(u_m),\Delta u_m)_{L^2(\Omega)} \\ &= -\epsilon\,\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} - \int_\Omega \varphi'(u_m)|\nabla u_m(x,t)|^2 dx \\ &\le -\epsilon\,\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} + C\int_\Omega |\nabla u_m(x,t)|^2 dx \\ &= -\epsilon\,\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} - C\int_\Omega u_m(x,t)\Delta u_m(x,t) dx \\ &\le -\frac{\epsilon}{2}\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} + \frac{C^2}{2\epsilon}\|u_m(\cdot,t)\|^2_{L^2(\Omega)} \end{aligned} \tag{3.25}$$

with some constant $C > 0$; here use of assumption (A_1) and equality (3.18) has been made. By Gronwall's inequality and (3.21), from (3.25) we get

$$\|u_m(\cdot,t)\|^2_{L^2(\Omega)} \le \|u_0\|^2_{H^2(\Omega)} e^{C^2 T/\epsilon} \qquad (t \in (0,T_m))\,. \tag{3.26}$$

Integrating inequality (3.25) on $(0,T_m)$ and using (3.26) gives inequality (3.22).

To prove (3.23) observe preliminarily that

$$\||\nabla u_m(\cdot,t)|\|^2_{L^2(\Omega)} = -\,(\Delta u_m, u_m)_{L^2(\Omega)} = \sum_{j=0}^m \lambda_j |w_{jm}|^2,$$

$$\||\nabla \Delta u_m(\cdot,s)|\|^2_{L^2(\Omega)} = -\left(\Delta u_m, \Delta^2 u_m\right)_{L^2(\Omega)} = \sum_{j=0}^m \lambda_j^3 |w_{jm}|^2$$

(see (3.16), (3.18) and (3.19)). Then multiplying the first equation of (3.20) by $\lambda_k w_{km}$ and summing over $k = 0, ..., m$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u_m(\cdot,t)\|^2_{L^2(\Omega)} &= -\epsilon\,\|\nabla\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} - \big(\varphi(u_m),\Delta^2 u_m\big)_{L^2(\Omega)} \\ &= -\epsilon\,\|\nabla\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} + \int_\Omega \varphi'(u_m)\nabla u_m(x,t)\cdot\nabla\Delta u_m(x,t)dx \\ &\le -\epsilon\|\nabla\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} + C\int_\Omega |\nabla u_m(x,t)||\nabla\Delta u_m(x,t)|dx \\ &\le -\frac{\epsilon}{2}\|\nabla\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} + \frac{C^2}{2\epsilon}\int_\Omega |\nabla u_m(x,t)|^2 dx\end{aligned}$$

with some constant $C > 0$; here use of assumption (A_1) and equality (3.16) has been made. Using Gronwall's inequality and and arguing as for (3.25), from the above inequality we obtain (3.23). This completes the proof. □

Under the stronger assumptions of Theorem 2.3 we can improve on the estimates of the above lemma. To this purpose, following [EZ] we shall make use of the Nirenberg inequality:

$$\|D^j v\|_{L^p(\Omega)} \le K_1\|D^m v\|^a_{L^r(\Omega)}\|v\|^{1-a}_{L^q(\Omega)} + K_2\|v\|_{L^q(\Omega)} \tag{3.27}$$

where $D \equiv \frac{\partial}{\partial x_k}$ $(k = 1,\dots,N)$, $K_1, K_2 > 0$ and

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{m}{N}\right) + \frac{1-a}{q}\,, \tag{3.28}$$

with $j \in \mathbf{N}\cup\{0\}$, $m \in \mathbf{N}$, $j \le m$, $a \in [\frac{j}{m}, 1]$ and $p, q, r \in (1,\infty)$ (*e.g.*, see [A]).

Lemma 3.2. *Let $u_0 \in H^2_E(\Omega)$, let assumptions (A_1)-(A_2) be satisfied, and let $N \le 5$. Let u_m be defined by (3.17) with coefficients w_{km} satisfying system (3.20)-(3.21) in the maximal interval $(0, T_m)$. Then there exists $C_2 > 0$ (only depending on ϵ, T and the norm $\|u_0\|_{H^2(\Omega)}$) such that for every $t \in (0, T_m)$*

$$\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} + \epsilon\int_0^t \|\Delta^2 u_m(\cdot,s)\|^2_{L^2(\Omega)}ds \le C_2\,. \tag{3.29}$$

Proof. Multiplying the first equation of (3.20) by $\lambda_k^2 w_{km}$ and summing over $k = 0, ..., m$, we get

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_m(\cdot,t)\|^2_{L^2(\Omega)} = -\epsilon\,\|\Delta^2 u_m(\cdot,t)\|^2_{L^2(\Omega)} + \big(\Delta\varphi(u_m)(\cdot,t),\Delta^2 u_m(\cdot,t)\big)_{L^2(\Omega)} \tag{3.30}$$

(see (3.19)). Set $\Delta[\varphi(u_m)] \equiv \Delta[\varphi(u_m)(\cdot,t)]$, $\Delta^2 u_m \equiv \Delta^2 u_m(\cdot,t)$ for simplicity. Since

$$\Delta[\varphi(u_m)] = \varphi'(u_m)\Delta u_m + \varphi''(u_m)|\nabla u_m|^2\,,$$

using assumptions (A_1)-(A_2) plainly we have

$$\begin{aligned} \left|\left(\Delta[\varphi(u_m)], \Delta^2 u_m\right)_{L^2(\Omega)}\right| &\leq \|\Delta[\varphi(u_m)]\|_{L^2(\Omega)} \left\|\Delta^2 u_m\right\|_{L^2(\Omega)} \\ &\leq C \left\|\Delta^2 u_m\right\|_{L^2(\Omega)} \left\{\||\nabla u_m|\|^2_{L^4(\Omega)} + \|\Delta u_m\|_{L^2(\Omega)}\right\} . \end{aligned} \tag{3.31}$$

To estimate the term $\||\nabla u_m|\|^2_{L^4(\Omega)}$ in the right-hand side of (3.31), we use the Nirenberg inequality (3.27) with $v = |\nabla u_m|$, $j = 0, m = 3, a = N/12, p = 4, q = r = 2$. Then we obtain

$$\||\nabla u_m|\|_{L^4(\Omega)} \leq \tilde{K_1} \|\Delta^2 u_m\|^a_{L^2(\Omega)} \| |\nabla u_m| \|^{1-a}_{L^2(\Omega)} + K_2 \| |\nabla u_m| \|_{L^2(\Omega)} \tag{3.32}$$

for some $\tilde{K_1} > 0$. Since by assumption $2a = N/6 \leq 5/6$, by inequalities (3.23) and (3.32) there exist $M_1 > 0, M_2 > 0$ such that for every $t \in (0, T_m)$

$$\||\nabla u_m|(\cdot, t)\|^2_{L^4(\Omega)} \leq M_1 \|\Delta^2 u_m(\cdot, t)\|^{\frac{5}{6}}_{L^2(\Omega)} + M_2 \,. \tag{3.33}$$

Similarly, the term $\|\Delta u_m\|_{L^2(\Omega)}$ in the right-hand side of (3.31) can be estimated using the Nirenberg inequality with $v = |\nabla u_m|$, $j = 1, m = 3, a = 1/3, p = q = r = 2$. This gives

$$\|\Delta u_m\|_{L^2(\Omega)} \leq \bar{K_1} \|\Delta^2 u_m\|^{\frac{1}{3}}_{L^2(\Omega)} \| |\nabla u_m| \|^{\frac{2}{3}}_{L^2(\Omega)} + K_2 \| |\nabla u_m| \|_{L^2(\Omega)} \tag{3.34}$$

for some $\bar{K_1} > 0$. Hence by inequalities (3.23) and (3.33) there exist $N_1 > 0, N_2 > 0$ such that for every $t \in (0, T_m)$

$$\|\Delta u_m(\cdot, t)\|_{L^2(\Omega)} \leq N_1 \|\Delta^2 u_m(\cdot, t)\|^{\frac{1}{3}}_{L^2(\Omega)} + N_2 \,. \tag{3.35}$$

By equality (3.30) and inequalities (3.31), (3.33) and (3.35), it is easily seen that there exists $M > 0$ (depending on ϵ) such that

$$\frac{1}{2}\frac{d}{dt}\|\Delta u_m(\cdot, t)\|^2_{L^2(\Omega)} + \frac{\epsilon}{2} \|\Delta^2 u_m(\cdot, t)\|^2_{L^2(\Omega)} \leq M$$

for every $t \in (0, T_m)$, whence inequality (3.29) immediately follows. This completes the proof. □

Now we can prove Theorem 2.3.

Proof of Theorem 2.3. Let $\{u_m\}$, $\{v_m\}$ be the sequences defined in (3.17), with coefficients w_{jm} satisfying system (3.20). Observe preliminarily that by estimate (3.22) there holds $T_m = T$ for every $m \in \mathbf{N}$, thus estimates (3.22), (3.23) and (3.29) hold for every $t \in (0, T)$ and $m \in \mathbf{N}$. As a consequence, the sequence $\{u_m\}$ belongs to a bounded subset of $L^2((0,T); H^4(\Omega))$. Then there exists a subsequence $\{u_k\} \equiv \{u_{m_k}\} \subseteq \{u_m\}$ and a function $u \in L^2((0,T); H^4(\Omega))$ such that

$$u_k \rightharpoonup u \text{ in } L^2((0,T); H^4(\Omega)) \,. \tag{3.36}$$

Moreover, by estimates (3.22)-(3.23), it is not restrictive to assume that

$$u_k \to u \text{ almost everywhere in } Q_T\,. \tag{3.37}$$

By the assumed properties of φ, there holds $\varphi(u) \in L^2\big((0,T);H^2(\Omega)\big)$. Let us prove that u is a solution of problem (P_T).

For every $k \in \mathbf{N}$ and any $\eta \in L^2(0,T;H^1(\Omega))$ there holds

$$\begin{aligned}\iint_{Q_T} u_{kt}\eta\,dxdt &= -\iint_{Q_T} \nabla\left[\varphi(u_k)\right]\cdot\nabla\eta\,dxdt - \epsilon\iint_{Q_T}\Delta^2 u_k\eta\,dxdt \\ &= -\iint_{Q_T}\varphi'(u_k)\nabla u_k\cdot\nabla\eta\,dxdt - \epsilon\iint_{Q_T}\Delta^2 u_k\eta\,dxdt\,.\end{aligned} \tag{3.38}$$

As $k \to \infty$, using the convergence in (3.36)-(3.37) and assumption (A_1), by the Dominated Convergence Theorem we easily get

$$\iint_{Q_T}\varphi'(u_k)\nabla u_k\cdot\nabla\eta\,dxdt \quad\to\quad \iint_{Q_T}\varphi'(u)\nabla u\cdot\nabla\eta\,dxdt\,,$$

whereas by (3.36)

$$\iint_{Q_T}\Delta^2 u_k\eta\,dxdt \quad\to\quad \iint_{Q_T}\Delta^2 u\,\eta\,dxdt$$

for any η as above. By the above convergence results, letting $k \to \infty$ in equality (3.38) for any $\eta \in C_0^\infty(Q_T)$ we have

$$\begin{aligned}\iint_{Q_T} u_{kt}\eta\,dxdt &= -\iint_{Q_T} u_k\eta_t\,dxdt \quad\to\quad -\iint_{Q_T} u\eta_t\,dxdt \\ &= \iint_{Q_T}\left\{\Delta\left[\varphi(u)\right] - \epsilon\Delta^2 u\right\}\eta\,dxdt\,.\end{aligned}$$

By the regularity of u, it follows that the distributional derivative u_t belongs to $L^2(Q_T)$, thus $u \in H^{4,1}(Q_T)$. Moreover, equality (3.14) holds for any $\eta \in L^2(0,T;H^1(\Omega))$.

From (3.36) and $\frac{\partial\Delta u_k}{\partial\nu} = 0$ on $\partial\Omega$, it is easy to see that $\frac{\partial\Delta u}{\partial\nu} = 0$ in the sense of distribution on $\partial\Omega$. Further observe that, in view of estimate (3.23), it is not restrictive to assume that

$$u_k \to u \text{ in } L^2((0,T);H^2(\Omega))\,. \tag{3.39}$$

Since $\{u_k\} \subseteq L^2((0,T);H^2_E(\Omega))$, from (3.39) we obtain that $u \in L^2((0,T);H^2_E(\Omega))$. Observe that $H^4(\Omega)\cap H^2_E(\Omega)$ endowed with the norm $\|\cdot\|_{H^4(\Omega)}$ is a closed subspace of $H^4(\Omega)$ hence is a reflexive Banach space. Therefore, applying Proposition 2.1 with $p = 2$, $V = H^4(\Omega)\cap H^2_E(\Omega)$ and $H = H^2_E(\Omega)$ we obtain that $u \in C\big([0,T];H^2_E(\Omega)\big)$, thus $\varphi(u) \in C\big([0,T];H^2_E(\Omega)\big)$.

Finally, observe that by (3.38) and subsequent remarks there holds

$$u_{kt} \to u_t \text{ in } L^2(Q_T)\,, \tag{3.40}$$

thus by Sobolev embedding

$$u_k \to u \text{ in } C\left([0,T];L^2(\Omega)\right).$$

In particular,

$$u_k(\cdot,0) = u_{0k} \to u(\cdot,0) \text{ in } L^2(\Omega),$$

whence $u(\cdot,0) = u_0$ by (3.21).

It remains to prove uniqueness. By a standard argument, let u, v be solutions of problem (P_T). Then we have

$$(u-v)_t + \epsilon\,\Delta^2(u-v) = \Delta[\varphi(u) - \varphi(v)].$$

Multiplying by $u-v$ and integrating over Ω yields

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|(u-v)(\cdot,t)\|^2_{L^2(\Omega)} + \epsilon\,\|\Delta(u-v)(\cdot,t)\|^2_{L^2(\Omega)} \\ = \; & \int_\Omega \left[(\varphi(u)-\varphi(v))\,\Delta(u-v)\right](x,t)dx \\ \le \; & C\int_\Omega \left[|u-v|\,|\Delta(u-v)|\right](x,t)dx \\ \le \; & \frac{\epsilon}{2}\|\Delta(u-v)(\cdot,t)\|^2_{L^2(\Omega)} + \frac{C^2}{2\epsilon}\|(u-v)(\cdot,t)\|^2_{L^2(\Omega)} \end{aligned}$$

(here use of assumption (A_1) has been made).Then we have

$$\frac{d}{dt}\|(u-v)(\cdot,t)\|^2_{L^2(\Omega)} + \epsilon\|\Delta(u-v)(\cdot,t)\|^2_{L^2(\Omega)} \le \frac{C^2}{\epsilon}\|(u-v)(\cdot,t)\|^2_{L^2(\Omega)},$$

whence by Gronwall's inequality the equality $u = v$ immediately follows. This completes the proof. □

Remark 3.3. *Let $\langle\cdot,\cdot\rangle$ denote the duality map between the spaces $(H^1(\Omega))'$ and $H^1(\Omega)$. It is worth observing that, under the weaker assumptions of Lemma 3.1, for any $u_0 \in H^2_E(\Omega)$ a solution of problem (P_T) exists in the following weaker sense:*
(i) $u \in C\left([0,T];H^2_E(\Omega)\right) \cap L^2((0,T);H^3(\Omega))$, $u(\cdot,0) = u_0$, $\varphi(u) \in C\left([0,T];H^2_E(\Omega)\right)$ and $u_t \in L^2((0,T);(H^1(\Omega))')$;
(ii) there holds

$$\int_0^T \langle u_t,\eta\rangle dt = -\iint_{Q_T} \nabla\left[\varphi(u) - \epsilon\Delta u\right]\cdot\nabla\eta\,dxdt \tag{3.41}$$

for every $\eta \in L^2((0,T);H^1(\Omega))$

In fact, by estimates (3.22)-(3.23) *there exist a subsequence $\{u_k\} \equiv \{u_{m_k}\} \subseteq \{u_m\}$ and a function $u \in L^2((0,T);H^3(\Omega))$ such that the convergence in* (3.37) *and* (3.39) *holds, and moreover*

$$u_k \rightharpoonup u \text{ in } L^2\left((0,T);H^3(\Omega)\right). \tag{3.42}$$

Then applying Proposition 2.1 with $p = 2$, $V = H^3(\Omega) \cap H^2_E(\Omega)$ *and* $H = H^2_E(\Omega)$ *we obtain that* $u \in C\left([0,T]; H^2_E(\Omega)\right)$, *thus* $\varphi(u) \in C\left([0,T]; H^2_E(\Omega)\right)$.

Further observe that for any $\xi \in H^1(\Omega)$ *and* $t \in [0,T]$ *there holds*

$$\begin{aligned}\langle u_{mt}(\cdot,t),\xi\rangle &= (u_{mt}(\cdot,t),\xi)_{L^2(\Omega)}\\ &= (\Delta v_m,\xi)_{L^2(\Omega)} = -(\nabla v_m,\nabla\xi)_{L^2(\Omega)}\\ &= -([\varphi'(u_m)\nabla u_m - \epsilon\nabla\Delta u_m](\cdot,t),\nabla\xi)_{L^2(\Omega)}\,,\end{aligned}$$

whence

$$\|u_{mt}(\cdot,t)\|_{(H^1(\Omega))'} \le \|[\varphi'(u_m)\nabla u_m - \epsilon\nabla\Delta u_m](\cdot,t)\|_{L^2(\Omega)}$$

for any $t \in [0,T]$. *By estimate* (3.23) *and assumption* (A_1) *this plainly gives*

$$\int_0^T \|u_{mt}(\cdot,t)\|^2_{(H^1(\Omega))'} \le 2C_1\left(C^2 + \epsilon^2\right)$$

for every $m \in \mathbf{N}$, *proving that the sequence* $\{u_{mt}\}$ *belongs to a bounded subset of* $L^2((0,T);(H^1(\Omega))')$. *Then it is not restrictive to assume that*

$$u_{kt} \rightharpoonup u_t \text{ in } L^2((0,T);(H^1(\Omega))')\,. \tag{3.43}$$

By the above remarks, letting $k \to \infty$ *in the equality*

$$\iint_{Q_T} u_{kt}\eta\,dxdt = -\iint_{Q_T} \nabla\left[\varphi(u_k) - \epsilon\Delta u_k\right]\cdot\nabla\eta\,dxdt$$

(which holds for any $k \in \mathbf{N}$ *and* $\eta \in L^2((0,T);H^1(\Omega))$*) we obtain* (3.41). *Arguing as in the proof of Theorem 2.3, the equality* $u(x,0) = u_0(x)$ *is easily proven. Hence the claim follows.*

We conclude this section by proving for further reference the following analogue of Lemma 3.2.

Lemma 3.4. *Let* $u_0 \in H^4_E(\Omega)$, *let assumptions* (A_1)-(A_3) *be satisfied, and let* $N \le 5$. *Let* u_m *be defined by* (3.17) *with coefficients* w_{km} *satisfying system* (3.20) *in the maximal interval* $(0,T_m)$, *with initial data* α_{km} *such that*

$$\begin{cases} u_{0m} := \sum\limits_{j=0}^{m} \alpha_{jm}\psi_j, \quad u_{0m} \to u_0 \text{ in } H^4_E(\Omega)\,,\\ \|u_{0m}\|_{H^4(\Omega)} \le \|u_0\|_{H^4(\Omega)}\,. \end{cases} \tag{3.44}$$

Then there exists $C_3 > 0$ *(only depending on* ϵ, T *and the norm* $\|u_0\|_{H^4(\Omega)}$*) such that for every* $t \in (0,T_m)$

$$\|\,|\nabla\Delta u_m(\cdot,t)|\,\|^2_{L^2(\Omega)} + \epsilon\int_0^t \|\,|\nabla\Delta^2 u_m(\cdot,s)|\,\|^2_{L^2(\Omega)}ds \le C_3\,. \tag{3.45}$$

Proof. Multiplying the first equation of (3.20) by $\lambda_k^3 w_{km}$ and summing over $k = 0, ..., m$, we get

$$\begin{aligned}
(3.46)\qquad & \frac{1}{2}\frac{d}{dt}\| \, |\nabla \Delta u_m(\cdot,t)| \, \|^2_{L^2(\Omega)} \\
&= -\epsilon \| \, |\nabla \Delta^2 u_m(\cdot,t)| \, \|^2_{L^2(\Omega)} - \big(\nabla[\Delta\varphi(u_m)](\cdot,t), \nabla[\Delta^2 u_m](\cdot,t)\big)_{L^2(\Omega)} \\
&\leq -\frac{\epsilon}{2} \| \, |\nabla \Delta^2 u_m(\cdot,t)| \, \|^2_{L^2(\Omega)} + \frac{1}{2\epsilon} \| \, |\nabla[\Delta\varphi(u_m)](\cdot,t)| \, \|^2_{L^2(\Omega)} \, ;
\end{aligned}$$

here use of the equalities

$$\sum_{j=0}^{m} \lambda_k^3 \, |w_{km}|^2 = \| \, |\nabla \Delta u_m(\cdot,t)| \, \|^2_{L^2(\Omega)} \, ,$$

$$\sum_{k=0}^{m} \lambda_k^5 \, |w_{km}|^2 = \| \, |\nabla \Delta^2 u_m(\cdot,t)| \, \|^2_{L^2(\Omega)} \, ,$$

$$\sum_{k=0}^{m} \lambda_k^4 w_{km} \, (\varphi(u_m), \psi_k)_{L^2(\Omega)} = \big(\nabla[\Delta\varphi(u_m)](\cdot,t), \nabla[\Delta^2 u_m](\cdot,t)\big)_{L^2(\Omega)}$$

has been made (see (3.19)). Set $\nabla[\Delta\varphi(u_m)] \equiv \nabla[\Delta\varphi(u_m)](\cdot,t)$ for simplicity. Since

$$\begin{aligned}
\nabla[\Delta\varphi(u_m)] &= \nabla[\varphi'(u_m)\Delta u_m + \varphi''(u_m)|\nabla u_m|^2] \\
&= \varphi'(u_m)\nabla\Delta u_m + \varphi'''(u_m)\nabla u_m |\nabla u_m|^2 \\
&+ \varphi''(u_m)\left(\nabla u_m \Delta u_m + 2\sum_{j=1}^{N} \frac{\partial u_m}{\partial x_j}\frac{\partial \nabla u_m}{\partial x_j}\right),
\end{aligned}$$

using assumptions (A_1)-(A_3) and estimates (3.22), (3.23) and (3.29) plainly we have

(3.47)

$$\| \, |\nabla[\Delta\varphi(u_m)]| \, \|_{L^2(\Omega)} \leq \bar{C}_1 \left\{ \| \, |\nabla\Delta u_m| \, \|_{L^2(\Omega)} + \| \, |\nabla u_m| \, \|^3_{L^6(\Omega)} + \| \, |\nabla u_m| \, |\Delta u_m| \, \|_{L^2(\Omega)} \right\} + \bar{C}_2$$

with some $\bar{C}_1$, $\bar{C}_2 > 0$. To estimate the term $\| |\nabla u_m| \|^3_{L^6(\Omega)}$ in the right-hand side of (3.47), we use the Nirenberg inequality (3.27) with $v = u_m$, $j = 1, m = 5, p = 6, = 2$. As for q, by estimate (3.29) and Sobolev embedding we can choose any $q \in (1, \infty)$ if $N \leq 3$, respectively $q \in (1, \frac{2N}{N-4})$ if $N = 4, 5$. If $N \leq 3$ equality (3.28) is satisfied with

$$a = a(q, N) := \frac{1}{3}\frac{(6-N)q + 6N}{(10-N)q + 2N} < \frac{1}{3} \, ,$$

respectively with $a = \frac{N-3}{9} \leq \frac{2}{9}$ if $N = 4, 5$. Then we obtain

$$(3.48)\qquad \| |\nabla u_m| \|_{L^6(\Omega)} \leq \tilde{K_1} \| \, |\nabla \Delta^2 u_m| \, \|^a_{L^2(\Omega)} \| \, |\nabla u_m| \, \|^{1-a}_{L^q(\Omega)} + K_2 \| \, |\nabla u_m| \, \|_{L^q(\Omega)}$$

for some $\tilde{K_1} > 0$. Then by estimate (3.29) there exist $P_1 > 0, P_2 > 0$ such that for every $t \in (0, T_m)$

$$(3.49)\qquad \| |\nabla u_m|(\cdot,t) \|^3_{L^6(\Omega)} \leq P_1 \| \, |\nabla \Delta^2 u_m| \, \|^{3a}_{L^2(\Omega)} + P_2 \, ,$$

with $3a < 1$.

Similarly, the term $\| |\nabla \Delta u_m| \|_{L^2(\Omega)}$ in the right-hand side of (3.47) can be estimated using the Nirenberg inequality with $v = \Delta u_m$, $j = 1, m = 3, a = 1/3, p = q = r = 2$. This gives

$$\| |\nabla \Delta u_m| \|_{L^2(\Omega)} \le \tilde{K_1} \| |\nabla \Delta^2 u_m| \|_{L^2(\Omega)}^{\frac{1}{3}} \| \Delta u_m \|_{L^2(\Omega)}^{\frac{2}{3}} + K_2 \| \Delta u_m \|_{L^2(\Omega)} \tag{3.50}$$

for some $\tilde{K_1} > 0$. Hence by estimate (3.29) there exist $Q_1 > 0, Q_2 > 0$ such that for every $t \in (0, T_m)$

$$\| |\nabla \Delta u_m(\cdot, t)| \|_{L^2(\Omega)} \le Q_1 \| |\nabla \Delta^2 u_m(\cdot, t)| \|_{L^2(\Omega)}^{\frac{1}{3}} + Q_2 \,. \tag{3.51}$$

Furthermore, again using Nirenberg and Holder inequality,

$$\| |\nabla u_m| |\Delta u_m| \|_{L^2(\Omega)} \le \| |\nabla u_m| \|_{L^k(\Omega)} \| |\Delta u_m| \|_{L^l(\Omega)} \text{ with } \frac{1}{k} + \frac{1}{l} = \frac{1}{2}$$

and

$$\| |\nabla u_m| \|_{L^k(\Omega)} \le C_1 \| |\nabla \Delta^2 u_m| \|_{L^2(\Omega)}^{N/4l} + C_2,$$
$$\| |\Delta u_m| \|_{L^l(\Omega)} \le C_3 \| |\nabla \Delta^2 u_m| \|_{L^2(\Omega)}^{1/4+N/4k} + C_4,$$

yield

$$\| |\nabla u_m| |\Delta u_m| \|_{L^2(\Omega)} \le R_1 \| |\nabla \Delta^2 u_m| \|_{L^2(\Omega)}^{1/4+N/8} + R_2. \tag{3.52}$$

By the last inequality in (3.46) and inequalities (3.47), (3.49), (3.51) and (3.52), it is easily seen that there exists $\tilde{M} > 0$ (depending on ϵ) such that

$$\frac{1}{2} \frac{d}{dt} \| |\nabla \Delta u_m(\cdot, t)| \|_{L^2(\Omega)}^2 + \frac{\epsilon}{2} \| |\nabla \Delta^2 u_m(\cdot, t)| \|_{L^2(\Omega)}^2 \le \tilde{M}$$

for every $t \in (0, T_m)$, whence inequality (3.45) follows. This completes the proof. □

4. Asymptotic behaviour: Proofs

Let $u_0 \in H^2_{E^*}(\Omega)$. In this case the approximating sequence in (3.21) becomes

$$\begin{cases} u_{0m} := \sum_{j=1}^{m} \alpha_{jm} \psi_j, \quad u_{0m} \to u_0 \text{ in } H^2_E(\Omega)\,, \\ \|u_{0m}\|_{H^2(\Omega)} \le \|u_0\|_{H^2(\Omega)} \qquad (m \in \mathbf{N})\,, \end{cases} \tag{4.53}$$

thus

$$\int_\Omega u_{0m} \, dx = 0 \quad \text{for any } m \in \mathbf{N}\,.$$

Accordingly, in the proof of Theorem 2.3 now we have

$$u_m := \sum_{k=1}^{m} w_{km} \psi_k \tag{4.54}$$

with coefficients w_{km} $(k = 1, ..., m)$ determined by system (3.20). Hence there holds

$$\int_\Omega u_m(x,t) \, dx = \sum_{k=1}^{m} w_{km}(t) \int_\Omega \psi_k \, dx = 0 \tag{4.55}$$

for any $m \in \mathbf{N}$ and $t \in (0, T_m)$.

Since $u_m(\cdot, t) \in H^2_{E^*}(\Omega)$, it is easily seen that estimates analogous to those of Lemmata 3.1, 3.2 and 3.4 hold with constants independent of T. In fact, the following holds.

Lemma 4.1. *Let $u_0 \in H^2_{E^*}(\Omega)$, and let assumptions (A_1) and (A_4) be satisfied. Let u_m be defined by* (4.54) *with coefficients w_{km} satisfying system* (3.20)*,* (4.53) *in the maximal interval $(0, T_m)$. Then there exists $C_1^* > 0$ (only depending on ϵ and the norm $\|u_0\|_{H^2(\Omega)}$) such that for every $t \in (0, T_m)$*

$$\|u_m(\cdot, t)\|_{H^1(\Omega)} \le C_1^* \,. \tag{4.56}$$

Lemma 4.2. *Let $u_0 \in H^4_{E^*}(\Omega)$, let assumptions (A_1)-(A_4) be satisfied, and let $N \le 5$. Let u_m be defined by* (4.54) *with coefficients w_{km} satisfying system* (3.20)*,* (4.53) *in the maximal interval $(0, T_m)$. Then there exists $C_2^* > 0$ (only depending on ϵ and the norm $\|u_0\|_{H^4(\Omega)}$) such that for every $t \in (0, T_m)$*

$$\|u_m(\cdot, t)\|_{H^3(\Omega)} \le C_2^* \,. \tag{4.57}$$

The proof of Lemma 4.2 is similar to that of Lemmata 3.2 and 3.4, using estimate (4.56) instead of (3.22)-(3.23). We leave the details to the reader.

Proof of Lemma 4.1. Let F be the functional defined in (2.11). Using (3.20) plainly gives

$$\begin{aligned} \frac{d}{dt}\left[F(u_m)\right](t) &= \int_\Omega [\varphi(u_m)(x,t) - \epsilon\Delta u_m(x,t)]u_{mt}(x,t)\,dx \\ &= \int_\Omega v_m(x,t)u_{mt}(x,t)\,dx = -\sum_{k=1}^m \lambda_k |(v_m(\cdot,t),\psi_k)_{L^2(\Omega)}|^2 \le 0 \,. \end{aligned}$$

This yields $F(u_m)(t) \le F(u_m)(0)$, namely

$$\begin{aligned} \int_\Omega \left\{\Phi(u_m)(x,t) + \frac{\epsilon}{2}|\nabla u_m(x,t)|^2\right\} dx &\le \int_\Omega \left\{\Phi(u_{0m}) + \frac{\epsilon}{2}|\nabla u_{0m}|^2\right\} dx \\ &\le (M+\epsilon)\|u_0\|_{H^2(\Omega)} \end{aligned}$$

with some constant $M > 0$, for any $t \in (0, T_m)$; here use of assumption (A_1) has been made. Since by assumption $(A4)$ there holds $\Phi(u) \ge 0$ for any $u \in \mathbf{R}$, from the above equality we get

$$\|\,|\nabla u_m(\cdot,t)|\,\|_{L^2(\Omega)} \le \frac{2(M+\epsilon)}{\epsilon}\,\|u_0\|_{H^2(\Omega)} \,,$$

whence (4.56) follows by Poincaré's inequality. This proves the result. □

Now we can prove Proposition 2.10.

Proof of Proposition 2.10. Let $\{u_k\}$ be the subsequence considered in the proof of Theorem 2.3. By estimate (3.45) and a diagonal argument, there exists a subsequence $\{u_l\} \equiv \{u_{k_l}\} \subseteq \{u_k\}$ such that

$$u_l \to u \text{ in } L^2((0,T); H^3(\Omega)) .$$

Plainly, this gives

$$\begin{aligned}&\int_0^T \left| \|u_l(\cdot,t)\|_{H^3(\Omega)} - \|u(\cdot,t)\|_{H^3(\Omega)} \right| dt \\ &\le \int_0^T \|u_l(\cdot,t) - u(\cdot,t)\|_{H^3(\Omega)}\, dt \\ &\le \sqrt{T} \left(\int_0^T \|u_l(\cdot,t) - u(\cdot,t)\|^2_{H^3(\Omega)}\, dt \right)^{\frac{1}{2}} \to 0\end{aligned}$$

as $l \to \infty$. Hence there exists a subsequence, denoted again by $\{u_l\}$ for simplicity, such that

$$\|u_l(\cdot,t)\|_{H^3(\Omega)} \to \|u(\cdot,t)\|_{H^3(\Omega)} \quad \text{for almost every } \; t \in (0,T) .$$

By inequality (4.57), this yields inequality (2.10) for any $t \in (0,T)$. Then by the arbitrariness of $T \in (0,\infty)$ the conclusion follows. □

Proof of Proposition 2.12. Multiplying the first equation of (P_∞) by u and integrating over Ω we get

$$\begin{aligned}(4.58)\quad &\frac{1}{2}\frac{d}{dt}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} + \epsilon \|\nabla \Delta u(\cdot,t)\|^2_{L^2(\Omega)} = -\int_\Omega \varphi'(u) \nabla u(x,t) \nabla \Delta u(x,t) dx \\ \le\quad &\frac{\epsilon}{2}\|\nabla \Delta u(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\int_\Omega (\varphi'(u))^2 |\nabla u(x,t)|^2 dx ,\end{aligned}$$

whence by assumption (A_1)

$$\frac{d}{dt}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} + \epsilon \|\nabla \Delta u(\cdot,t)\|^2_{L^2(\Omega)} \le \frac{C^2}{\epsilon}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)}$$

for some $C > 0$. Since $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, by Poincaré's inequality there holds

$$\begin{aligned}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} &\le C_0 \|\Delta u(\cdot,t)\|^2_{L^2(\Omega)} \\ &= -C_0 \int_\Omega \nabla u(x,t) \cdot \nabla \Delta u(x,t) dx \\ &\le \frac{1}{2}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} + \frac{C_0^2}{2}\| \, |\nabla \Delta u(\cdot,t)| \, \|^2_{L^2(\Omega)} ,\end{aligned}$$

namely

$$(4.59)\qquad \| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} \le C_0^2 \| \, |\nabla \Delta u(\cdot,t)| \, \|^2_{L^2(\Omega)}$$

for some $C_0 > 0$. By inequalities (4.58)-(4.59) we have

$$\frac{d}{dt}\| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} + \left(\frac{\epsilon}{C_0^2} - \frac{C^2}{\epsilon} \right) \| \, |\nabla u(\cdot,t)| \, \|^2_{L^2(\Omega)} \le 0 ,$$

whence

$$\lim_{t\to\infty} \| \, |\nabla u(\cdot,t)| \, \|_{L^2(\Omega)} = 0$$

if $\epsilon > \epsilon_0 := C_0 C$. Since $u(\cdot,t) \in H^2_{E^*}(\Omega)$ for any $t \in (0,\infty)$, by Poincaré's inequality the above equality implies

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{L^2(\Omega)} = 0\,.$$

Then the conclusion follows. □

5. Stationary problem in one space dimension

In this section we address existence and multiplicity of solutions of the problem

$$\begin{cases} [\varphi(u) - \epsilon u'']'' = 0 \text{ in } (-L,L) \\ u'(-L) = u'(L) = u'''(-L) = u'''(L) = 0 \\ \dfrac{1}{2M}\displaystyle\int_\Omega u(x)dx = L\,, \end{cases} \tag{5.60}$$

where the primes denote differentiation and M is the mass defined by (1.4). Integrating twice the first equation of (5.60) we obtain (in agreement with Remark 2.9) the equivalent second order problem

$$\begin{cases} \epsilon u'' = \varphi(u) - \sigma \text{ in } (-L,L) \\ u'(-L) = u'(L) = 0 \\ \dfrac{1}{2M}\displaystyle\int_\Omega u(x)dx = L\,, \end{cases} \tag{E}$$

where $\sigma \in \mathbf{R}$ is a constant to be chosen. The above problem with $\varphi(u) = u^3 - u$ (and $\epsilon = 1$) was investigated in [Z, NPe]. As motivated in the Introduction, we are interested in obtaining similar results choosing

$$\varphi(u) = \frac{u}{1+u^2}\,. \tag{5.61}$$

Specifically, we address the existence of *simple* solutions, *i.e.*, solutions which are strictly monotone and bounded.

Let $u = u(x)$ be a solution of problem (E). Multiplying the first equation of problem (E) by u' and integrating we obtain

$$\frac{\epsilon}{2}\,[u'(x)]^2 = \mathcal{W}(u(x),\sigma) - b\,, \tag{5.62}$$

where

$$\mathcal{W}(u,\sigma) := \frac{\log(1+u^2)}{2} - \sigma u$$

and b is another constant of integration.

Due to the boundary conditions, we must have

$$\mathcal{W}(u(\pm L),\sigma) - b = 0\,. \tag{5.63}$$

If u is a simple solution, there holds $u(-L) \neq u(L)$. Therefore, the equation $\mathcal{W}(u,\sigma) - b = 0$ must have at least two roots $u_1 < u_2$ such that $\mathcal{W}(u,\sigma) - b > 0$ for $u_1 < u < u_2$ (see (5.62)). If φ is of the form (5.61) and $\sigma > 0$, it is easily seen that this implies

$$\mathcal{W}(\alpha,\sigma) < b < \mathcal{W}(\beta,\sigma)\,,$$

where α and β are local extremum points of $\mathcal{W}$ - namely, solve the equation

$$(1+u^2)\mathcal{W}_u(u,\sigma) = u - \sigma(1+u^2) = 0\,.$$

Clearly, the above situation gives the condition $\sigma < \frac{1}{2}$; if this is the case, there holds

$$\alpha \equiv \alpha(\sigma) = \frac{1-\sqrt{1-4\sigma^2}}{2\sigma}\,, \quad \beta \equiv \beta(\sigma) = \frac{1+\sqrt{1-4\sigma^2}}{2\sigma}\,.$$

Similar considerations hold for $\sigma < 0$, yielding the condition $\sigma > -\frac{1}{2}$. We conclude that the above strategy to determine simple solutions of problem (E) requires that the parameters (σ, b) belong to the following *admissible region* (see Figure 1)

$$\Sigma := \left\{(\sigma,b)\,\middle|\,\sigma\in\left(-\frac{1}{2},0\right)\cup\left(0,\frac{1}{2}\right),\ \mathcal{W}(\alpha,\sigma) < b < \mathcal{W}(\beta,\sigma)\right\}. \tag{5.64}$$

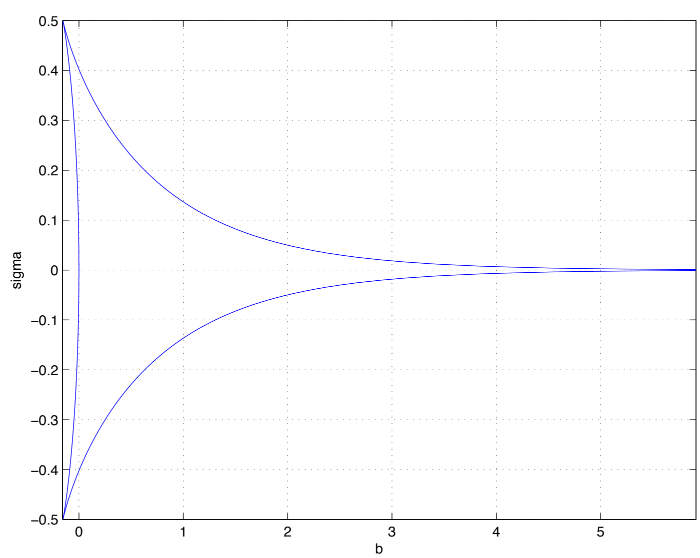

FIGURE 1. Admissible region.

Now let $(\sigma, b) \in \Sigma$, and let u be a simple solution of problem (E); suppose $u' > 0$ without loss of generality. Then equality (5.62) gives

$$u'(x) = \sqrt{\frac{2}{\epsilon}}\sqrt{\mathcal{W}(u(x),\sigma) - b}\,,$$

whence by integration

$$2L = \int_{-L}^{L} dx = \sqrt{\frac{\epsilon}{2}}\int_{u_1(\sigma,b)}^{u_2(\sigma,b)} \frac{ds}{\sqrt{\mathcal{W}(s,\sigma) - b}} =: 2\mathcal{L}(\sigma,b)\,. \tag{5.65}$$

(see (5.63)). By the same token, the third equation of problem (E) gives

$$2LM = \int_{-L}^{L} u(x)dx = \sqrt{\frac{\epsilon}{2}}\int_{u_1(\sigma,b)}^{u_2(\sigma,b)} \frac{sds}{\sqrt{\mathcal{W}(s,\sigma) - b}} =: 2\mathcal{M}(\sigma,b). \tag{5.66}$$

Therefore, proving the existence of a simple solution of problem (E) amounts to finding a pair $(\sigma, b) \in \Sigma$ such that

$$\begin{cases} \mathcal{L}(\sigma,b) = L \\ \mathcal{M}(\sigma,b) = LM\,. \end{cases} \tag{5.67}$$

Investigating existence and multiplicity of solutions of system (5.67) requires information about the monotonicity properties of the functions $\mathcal{L}$ and $\mathcal{M}$. Unfortunately, at variance from the cases of φ of polynomial type dealt with in [CGS, NPe, Z], for the present choice of φ a complete analytical investigation of this point could not be carried out (similar difficulties are encountered if $\varphi(u) = u\exp(-u)$; see (1.1)). Therefore, the investigation has been pursued by a numerical method, whose main steps are described as follows in the case $\sigma \in (0, \frac{1}{2})$ (similarly it is possible to obtain the case with $\sigma \in (-\frac{1}{2}, 0)$):

Step 1: We fix the step size $\Delta\sigma = 0.001$ and $\Delta b = 0.005$ and we construct the vector $\sigma \in (0, \frac{1}{2})$ with grid step $\Delta\sigma$. In this way we have 499 nodes for σ.

Step 2: For each σ we calculate $\mathcal{W}(\alpha,\sigma)$ and $\mathcal{W}(\beta,\sigma)$ in order to construct the matrix of the values for b.

Step 3: For each value of σ and each value of b inside the admissible region visible in Fig. 1, we calculate the two roots $u_1(\sigma,b)$ and $u_2(\sigma,b)$ by using the bisection method.

Step 4: By a recursive adaptive Simpson quadrature method, we numerically evaluate the integral defined between the two roots calculated in the previous step in order to calculate the functions $\mathcal{L}(\sigma,b)$ and $\mathcal{M}(\sigma,b)$ defined in (5.65) and (5.66), respectively.

Step 5: We calculate the partial derivative of the functions $\mathcal{L}(\sigma, b)$ and $\mathcal{M}(\sigma, b)$ with respect to σ and b by replacing the derivatives with their incremental ratios, that is by using the centered finite differences method.

We can arrive to our conclusion by using Figures 2, 3 and 4. In order to obtain a more clear vision, just some level curves are represented. The computations were done on a computer Mac OS X version 10.6.8 with processor 2.66 GHz Intel Core 2 Duo, RAM 4 GB.

On the numerical evidence, the following conclusions can be drawn.

Conjecture 1 (Monotonicity properties of $\mathcal{L}$ and $\mathcal{M}$). *For any $(\sigma, b) \in \Sigma$ there holds*

$$\frac{\partial \mathcal{L}}{\partial b}, \frac{\partial \mathcal{L}}{\partial \sigma}, \frac{\partial \mathcal{M}}{\partial b}, \frac{\partial \mathcal{M}}{\partial \sigma} \leq 0.$$

Consider the level curves

$$C_{\mathcal{L}} := \{(\sigma, b) \in \Sigma | \mathcal{L}(\sigma, b) = L\}, \quad D_{\mathcal{M}} := \{(\sigma, b) \in \Sigma | \mathcal{M}(\sigma, b) = LM\}.$$

Conjecture 2 (Monotonicity properties of the curves $C_{\mathcal{L}}$, $D_{\mathcal{M}}$). *For any $L > 0$, both curves $C_{\mathcal{L}}$ and $D_{\mathcal{M}}$ are decreasing in the admissible region Σ.*

Conjecture 3 (Intersections of the curves $C_{\mathcal{L}}$, $D_{\mathcal{M}}$) *For any $L > 0$ large enough, there exist some $M > 0$ such that problem (E) has a simple solution.*

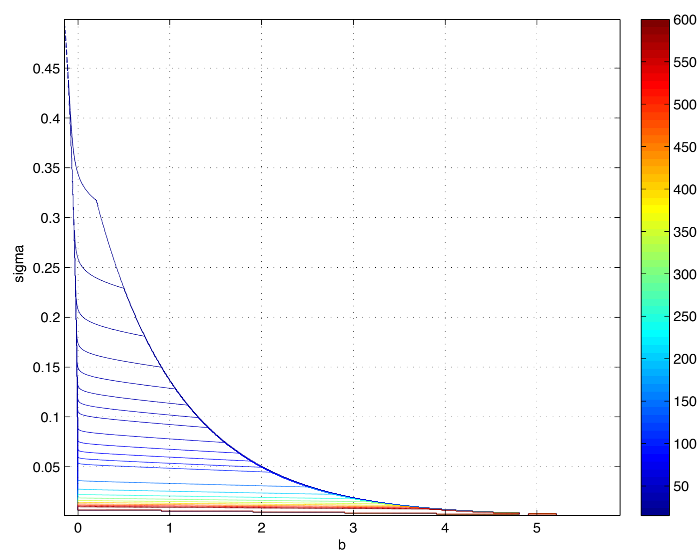

FIGURE 2. Level curves of $\mathcal{L}(\sigma, b)$.

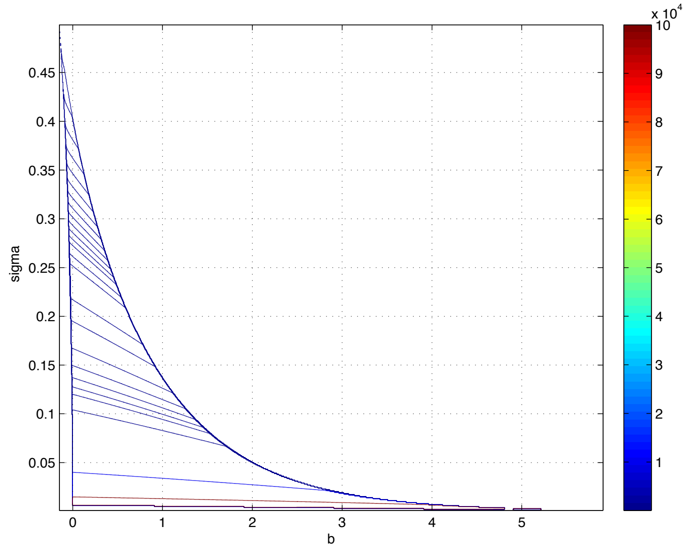

FIGURE 3. Level curves of $\mathcal{M}(\sigma, b)$.

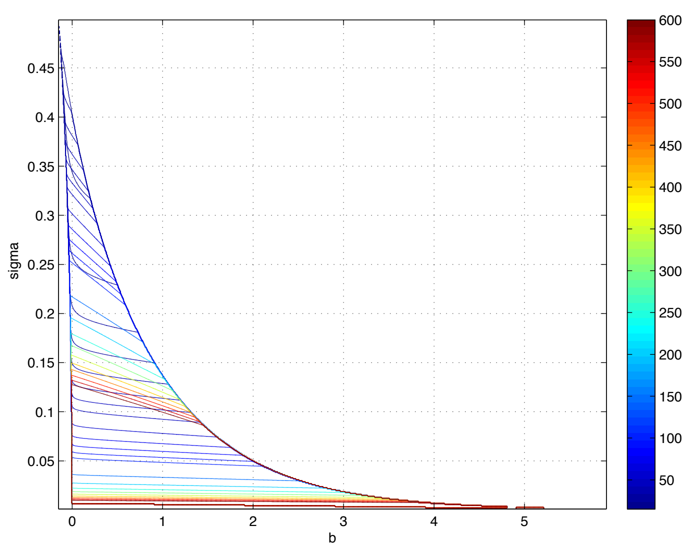

FIGURE 4. Level curves of $\mathcal{L}(\sigma, b)$ and $\mathcal{M}(\sigma, b)$.

Acknowledgments. The research leading to the present results has received funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement n. PITN-GA-2008-238702.

References

[A] R. A. Adams, *Sobolev Spaces* (Academic Press, 1975).

[BBDU] G. I. Barenblatt, M. Bertsch, R. Dal Passo & M. Ughi, *A degenerate pseudoparabolic regularization of a nonlinear forward-backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow*, SIAM J. Math. Anal. **24** (1993), 1414–1439.

[BFG] G. Bellettini, G. Fusco & N. Guglielmi, *A concept of solution for forward-backward equations of the form* $u_t = \frac{1}{2}(\zeta'(u_x))_x$ *and numerical experiments for the singular perturbation* $u_t = -\epsilon^2 u_{xxxx} + \frac{1}{2}(\zeta'(u_x))_x$, Discrete Cont. Dyn. Syst. **16** (2006), 259-274.

[BFJ] K. Binder, H. L. Frisch & J. Jäckle, *Kinetics of phase separation in the presence of slowly relaxing structural variables*, J. Chem. Phys. **85** (1986), 1505-1512.

[BS] M. Brokate & J. Sprekels, *Hysteresis and Phase Transitions*, Applied Mathematical Sciences **121** (Springer, 1996).

[BST1] M. Bertsch, F. Smarrazzo & A. Tesei, *Pseudo-parabolic regularization of forward-backward parabolic equations: Power-type nonlinearities*, (preprint, 2012).

[BST2] M. Bertsch, F. Smarrazzo & A. Tesei, *Pseudo-parabolic regularization of forward-backward parabolic equations: A logarithmic nonlinearity*, (preprint, 2012).

[BuST1] L.T.T. Bui, F. Smarrazzo & A. Tesei, *Sobolev regularization of a class of forward-backward parabolic equation*, (preprint, 2013).

[BuST2] L.T.T. Bui, F. Smarrazzo & A. Tesei, *Passage to the limit over small parameters in a viscous Cahn-Hilliard equation*, (preprint, 2013).

[CGS] J. Carr, M. E. Gurtin & M. Slemrod, *Structured phase transitions on a finite interval*, Arch. Rational Mech. Anal. **(**86) (1984), 317-351.

[EP] L. C. Evans & M. Portilheiro, *Irreversibility and hysteresis for a forward-backward diffusion equation*, Math. Mod. Meth. Appl. Sci. **14** (2004), 1599-1620.

[EZ] Charles M. Elliott & Zheng Songmu, *On the Cahn-Hilliard equation*, Arch. rational Mech. Anal. 96 **4** (1986), 339-357.

[G] M. E. Gurtin, *Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance*, Physica D **92** (1996), 178-192.

[H] D. Henry, *Geometric Theory of Semilinear Parabolic Equation* (Springer- Verlag, Berlin, 1981).

[MTT] C. Mascia, A. Terracina & A. Tesei, *Two-phase entropy solutions of a forward-backward parabolic equation*, Arch. Rational Mech. Anal. **194** (2009), 887-925.

[NP] A. Novick-Cohen & R. L. Pego, *Stable patterns in a viscous diffusion equation*, Trans. Amer. Math. Soc. **324** (1991), 331-351.

[NPe] Novick-Cohen, A.; Peletier, L. A, *Steady states of the one-dimensional Cahn-Hilliard equation*, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), no. 6, 1071?1098.

[Pa] V. Padrón, *Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations*, Comm. Partial Differential Equations **23** (1998), 457-486.

[PM] P. Perona & J. Malik, *Scale space and edge detection using anisotropic diffusion*, IEEE Trans. Pattern Anal. Mach. Intell. **12** (1990), 629–639.

[Pl1] P. I. Plotnikov, *Passing to the limit with respect to viscosity in an equation with variable parabolicity direction*, Differential Equations **30** (1994), 614-622.

[Pl2] P. I. Plotnikov, *Equations with alternating direction of parabolicity and the hysteresis effect*, Russian Acad. Sci. Dokl. Math. **47** (1993), 604-608.

[Pl3] P. I. Plotnikov, *Forward-backward parabolic equations and hysteresis*, J. Math. Sci. **93** (1999), 747-766.

[Pl4] P. I. Plotnikov, *Passage to the limit over a small parameter in the Cahn-Hilliard equations*, Siberian Math. J. **38** (1997), 550-566.

[S] F. Smarrazzo, *On a class of equations with variable parabolicity direction*, Discrete Contin. Dyn. Syst. **22** (2008), 729-758.

[Sl] M. Slemrod, *Dynamics of Young measure solutions to a backward-forward heat equation*, J. Dynam. Differential Equations **3** (1991), 1-28.

[ST1] F. Smarrazzo & A. Tesei, *Long-time behaviour of solutions to a class of forward-backward parabolic equations* , SIAM J. Math. Anal. **42** (2010), 1046-1093.

[ST2] F. Smarrazzo & A. Tesei, *Degenerate regularization of forward-backward parabolic equations: The regularized problem*, Arch. Rational Mech. Anal. **204** (2012), 85-139.

[ST3] F. Smarrazzo & A. Tesei, *Degenerate regularization of forward-backward parabolic equations: The vanishing viscosity limit*, Math. Ann. **355** (2013), 551-584.

[STe] F. Smarrazzo & A. Terracina, *Sobolev approximation for two-phase solutions of forward-backward parabolic problems*, Discrete Contin. Dyn. Syst. **33** (2013), 1657-1697.

[Z] S. Zheng, *Asymptotic behavior of solution to the Cahn-Hilliard equation*, Appl. Anal. **3** (1986), 165-184.

[Ze] E. Zeidler, *Nonlinear Functional Analysis and its applications. Part II: Linear monotone operators* (Springer- Verlag, Berlin, 1990).

Department of Mathematics and Computer Science, University of Science, Ho Chi Minh city, 227 Nguyen Van Cu, D. 5, HCM city, Vietnam

E-mail address: trongthanhptnk@yahoo.com

E-mail address: bltthanh@hcmus.edu.vn

Dipartimento di Matematica "G. Castelnuovo", Sapienza - Università di Roma, P.le A. Moro 5, I-00185 Roma, Italy

E-mail address: tozza@mat.uniroma1.it