Elementary Linear Algebra - MATH 2250 - Day 9

Name:

1. Mark each of the followings as True or False. If a set is dependent, provide a nonzero combination of the vectors which gives the zero vector.

   $\boxed{\text{T}}$ $\boxed{\text{F}}$ The vectors $(1, 0)$ and $(0, 0)$ are dependent.

   $0(1,0) + 1(0,0) = (0,0)$

   $\boxed{\text{T}}$ $\boxed{\text{F}}$ The vectors $(1, 0)$ and $(0, 1)$ are dependent.

   $\boxed{\text{T}}$ $\boxed{\text{F}}$ The vectors $(1, 0)$ and $(0, 1)$ and $(\pi, -\pi)$ are dependent.

   $\pi(1,0) - \pi(0,1) - 1(\pi, -\pi) = (0,0)$

2. The vectors $v_1, v_2, \ldots, v_n$ are independent if $c_1 v_1 + \cdots + c_n v_n = 0$ implies $c_1, \ldots, c_n = 0$.

3. Let A be an $m \times n$ matrix with rank ~~n~~ r whose columns are linearly dependent, then $r \boxed{<} n$. (Fill the box with the best choice from $<, =, >, \leq, \geq$.) Explain.

   $A = [A_1 | \cdots | A_n]$ if the columns are linearly dependent, then at least one of them depends on the others, hence rank is less than n.

4. Give two distinct bases for $\mathbb{R}^3$.

   $$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$$

5. Can the zero vector be in any basis? Could it be in any independent set?

   Nope! NO.

6. The vectors $\boldsymbol{v} = (1, 0)$, $\boldsymbol{w} = (0, 1)$ and $\boldsymbol{u} = (\pi, \pi)$ are linearly dependent. To show this build a matrix $A = \left[\boldsymbol{v} \middle| \boldsymbol{w} \middle| \boldsymbol{u} \right]$, and show that its null space is nonzero.

   $$A = \begin{bmatrix} 1 & 0 & \pi \\ 0 & 1 & \pi \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & \pi \\ 0 & 1 & \pi \end{bmatrix} \longrightarrow N = \begin{bmatrix} -\pi \\ -\pi \\ 1 \end{bmatrix}$$

   $$N(A) = \left\{ c \begin{bmatrix} -\pi \\ -\pi \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\} \neq Z$$

7. The vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ are linearly independent. To show this, we need to show that the only linear combination of them that is equal to the zero vector is the zero combination. That is, if $c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\0\end{bmatrix} + c_3 \begin{bmatrix}0\\0\\1\end{bmatrix} = \mathbf{0}$, then $c_1, c_2, \underset{c_3}{\cancel{c+3}} = 0$. Write a system of linear equations with variables c_1, c_2, c_3 corresponding to the above equation. Then find **all** the solutions to that system.

$$\begin{aligned} c_1 + 0 + 0 &= 0 \\ 0 + c_2 + 0 &= 0 \\ 0 + 0 + c_3 &= 0 \end{aligned} \longrightarrow \text{all solutions: } \begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

8. Show that the vectors $(1,-1,0)$, $(0,1,-1)$, and $(0,0,1)$ are linearly independent.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{all solutions to } A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0 \text{ are } \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}.$$

That is, if $c_1(1,-1,0) + c_2(0,1,-1) + c_3(0,0,1) = 0$, then $c_1 = c_2 = c_3 = 0$; hence the 3 given vectors are lin. ind. by definition.

9. Are there 4 vectors in $\mathbb{R}^3$ that are linearly independent? Why?

Nope, $A = [v_1 | v_2 | v_3 | v_4]$ can have at most 3 pivots, hence nonzero null space! that is a nonzero lin. comb. of v_1, v_2, v_3, v_4 is the zero vector.

10. Does 1 vector span $\mathbb{R}^3$? What the **maximum** dimension of a space that it could span?

Nope! 1

11. Do 2 vectors span $\mathbb{R}^3$? What the **maximum** dimension of a space that they could span?

NO! 2

12. Do 3 vectors span $\mathbb{R}^3$? Could they span a plane? A line? → Yes, if they are multiples of one vector.

→ Maybe! if they are lin. ind.

→ Yes, if two of them are lin. ind. and third one is in the plane spanned by them.

13. Do 4 vectors span $\mathbb{R}^3$? Could they span a 3-D space? A plane? A line?

Maybe, if 3 of them are lin. ind. Yes. Yes. Yes. (see ↑)

14. Are vectors $(1,1,1)$, $(1,1,1)$, and $(1,2,3)$ linearly independent? Why?

Nope: $1(1,1,1) - 1(1,1,1) + 0(1,2,3) = 0$

15. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2 pivots

Do the columns of A span the columns space of A?

Yes, that's how we constructed $C(A)$.

Are the columns of A linearly independent?

Yes, because at most 3 of them could be lin. ind. (why?)

Find three different bases for the column space of A.

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}$.

Note: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is not a basis for $C(A)$, why?.

What is the rank of A? 2

What is the dimension of the column space of A?

2

Find two **more** bases for the column space of A.

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}$.

I can choose any two lin. ind. vectors in $C(A)$ and that would form a basis for $C(A)$.

Find a basis for the ~~column~~ null space of A.

$N = \begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $N(A)$

Find a nonzero linear combination of columns of A which gives the zero vector.

Any thing in the null space would work. e.g.

$1\begin{bmatrix} A_1 \end{bmatrix} - 2\begin{bmatrix} A_2 \end{bmatrix} + 1\begin{bmatrix} A_3 \end{bmatrix} + 0\begin{bmatrix} A_4 \end{bmatrix}$, or $2\begin{bmatrix} A_1 \end{bmatrix} - 3\begin{bmatrix} A_2 \end{bmatrix} + 0\begin{bmatrix} A_3 \end{bmatrix} + 1\begin{bmatrix} A_4 \end{bmatrix}$, or $3\begin{bmatrix} A_1 \end{bmatrix} - 5\begin{bmatrix} A_2 \end{bmatrix} + 1\begin{bmatrix} A_3 \end{bmatrix} + 1\begin{bmatrix} A_4 \end{bmatrix}$

or ...

Find a nonzero linear combination of columns of A that is not a 'multiple' of the previous linear combination, but still gives the zero vector.

see above!

16. Assume that the rank of a matrix $A = \left[\begin{array}{c|c|c|c|c} A_1 & A_2 & A_3 & A_4 & A_5 \end{array}\right]$ is 3, and the vectors

$$A\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, A\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}, \text{ and } A\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

are linearly independent. Find a basis for the column space of A. What is the dimension of $C(A)$?

A_1, A_2, A_5 3

Find a basis for the null space of A. What is the dimension of $N(A)$?

Let $\text{rref}(A) = \left[\begin{array}{cc|cc|c} 1 & 0 & a & b & 0 \\ 0 & 1 & c & d & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, then $N = \begin{bmatrix} a & b \\ c & d \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} a \\ c \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ d \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $N(A)$, and $\dim N(A) = 2$.

F

why?

17. **(Important)** Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 4 \\ 0 & -1 & -1 & -5 \end{bmatrix}$. Is $(0, \overset{-1}{\cancel{0}}, 1)$ in the column space of A? How do you know?

Yes, $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & b_1 \\ -1 & 1 & 0 & 4 & b_2 \\ 0 & -1 & -1 & -5 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & b_1 \\ 0 & 1 & 1 & 5 & b_2+b_1 \\ 0 & 0 & 0 & 0 & b_3+b_2+b_1 \end{array}\right] \to$ solvability condition:

$b_1 + b_2 + b_3 = 0$

and $1 + (-1) + (1) = 0$. ✓

Find $R = \text{rref(A)}$. Is $(0, \overset{-1}{\cancel{0}}, 1)$ in the column space of R?

Nope, $R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, no lin. comb. of col's of R produce a 1 in the last row.

How are the column spaces of A and R related to each other?

$C(A) \neq C(R)$