Elementary Linear Algebra - MATH 2250 - Day 3

Name:

1. Let the subscripts denote the number of rows and columns of each matrix, respectively. Mark all the matrix products that are defined. In each case that the product is defined write down the number of rows and columns of the product.
   - ☐ $A_{2\times 3}B_{3\times 2}$
   - ☐ $A_{3\times 3}B_{2\times 2}$
   - ☐ $A_{3\times 3}B_{3\times 3}$
   - ☐ $A_{2\times 1}B_{2\times 1}$
   - ☐ $A_{1\times 2}B_{2\times 1}$

2. Find the inverse of A using Gauss-Jordan method. Show steps AND check your solution!

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

3. Find the inverse of A using Gauss-Jordan method. Show steps AND check your solution!

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$$

4. This afternoon, read the worked examples 2.4A and 2.4B in the book (page 72). ← Don't forget this!

5. For the following matrices, by giving conditions of the entries p, q, r, z explain when does $AB = BA$? When does $BC = CB$? When does A times BC equal AB times C?

$$A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$$

6. True or False. Give a specific example when false:

(a) If columns of 1 and 3 of B are the same, so are columns 1 and 3 of AB.

T: A operates on rows of B, so whatever happens to col 1 of B happens to col 3 of it, too. [also, see below]

(b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of BA.

T: see above. [also: (row 1 of BA) = (row 1 of B)$\cdot A$ = (row 3 of B)$\cdot A$ = (row 3 of BA)]

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC.

T: see (b) and that $ABC = A\cdot(BC)$

(d) $(AB)^2 = A^2B^2$.

F: $A=\begin{bmatrix}1&0\\0&2\end{bmatrix}, B=\begin{bmatrix}1&1\\1&1\end{bmatrix} \Rightarrow A^2=\begin{bmatrix}1&0\\0&4\end{bmatrix}, B^2=\begin{bmatrix}2&2\\2&2\end{bmatrix}, A^2B^2=\begin{bmatrix}2&2\\8&8\end{bmatrix}, AB=\begin{bmatrix}1&1\\2&2\end{bmatrix}, (AB)^2=\begin{bmatrix}3&3\\6&6\end{bmatrix}$

$\neq$

(e) $(A+B)^2 = A^2 + 2AB + B^2$.

F: $A=\begin{bmatrix}1&0\\0&2\end{bmatrix}, B=\begin{bmatrix}1&1\\1&1\end{bmatrix} \Rightarrow (A+B)^2=\begin{bmatrix}2&1\\1&3\end{bmatrix}^2=\begin{bmatrix}5&5\\5&10\end{bmatrix}, A^2+2AB+B^2=\begin{bmatrix}5&4\\4&10\end{bmatrix}$

$\neq$

(f) If $AB = B$ then $A = I$.

F: $A: \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}, B=\begin{bmatrix}2&4\\6&8\\0&0\end{bmatrix}, AB=\begin{bmatrix}2&4\\6&8\\0&0\end{bmatrix}$

(g) If A^2 is defined then A is necessarily square.

T: $A_{m\times n} \rightarrow A_{m\times n}\cdot A_{m\times n}$ $\quad m=n$.

(h) If AB and BA are defined then A and B are square.

F: $A: [\ \ ]_{2\times 3}$ $B: [\ \ ]_{3\times 2} \rightarrow AB = [\ \ ]_{2\times 2}, BA = [\ \ ]_{3\times 3}$

(i) If AB and BA are defined then AB and BA are square.

T: $A_{m\times n}, B_{n\times m} \rightarrow AB_{m\times m}, BA_{n\times n}$

(j) $(AB)^{-1} = A^{-1}B^{-1}$

F: $A=\begin{bmatrix}1&0\\0&2\end{bmatrix}, B=\begin{bmatrix}1&0\\1&1\end{bmatrix} \rightarrow AB=\begin{bmatrix}1&0\\2&2\end{bmatrix}, (AB)^{-1}=\begin{bmatrix}1&0\\-1&1/2\end{bmatrix}, A^{-1}=\begin{bmatrix}1&0\\0&1/2\end{bmatrix}, B^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$

$A^{-1}B^{-1}=\begin{bmatrix}1&0\\-\frac{1}{2}&\frac{1}{2}\end{bmatrix}$

$\neq$

7. What is the determinant of the matrix of coefficients for the following system of linear equations?

$$\begin{cases} x + 2y = 5 \\ -2x + 3y = 0 \end{cases}$$

What does the determinant tell you about the existence of a solution to the above system?

det(A)=0 ⇝ there is a zero in a pivot position ⇝ System either has infinitely many solutions, or no solutions.

8. Recall that if A is invertible, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for any right hand side $\boldsymbol{b}$. What is that solution $\boldsymbol{x}$ in terms of A and $\boldsymbol{b}$?

$Ax=b \rightsquigarrow \underbrace{A^{-1}A}_{=I}x = A^{-1}b \rightsquigarrow x = A^{-1}b$ is "the" solution.

In problem 2 you found the inverse of a matrix. Use that inverse to solve each of the following systems of equations:

$$\begin{cases} x + 2y = 1 \\ 2x + 5y = 0 \end{cases} \qquad \begin{cases} x + 2y = 0 \\ 2x + 5y = 1 \end{cases} \qquad \begin{cases} x + 2y = 1 \\ 2x + 5y = 1 \end{cases}$$

9. Recall that when you multiplied the following equations by $1, 1, -1$ and added them you got $0 = 1$.

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 4 \end{aligned}$$

Let A be the matrix of coefficients for the above system. Does there exist a nonzero vector $\boldsymbol{v} = (x, y, z)$ such that $A\boldsymbol{v} = \mathbf{0}$? What is it?

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} x+y+z=0 \\ y \quad =0 \\ 0 \quad = 0 \end{cases} \to$ any vector of the form $(a, 0, -a)$ is a solution to $Av = 0$, for example $v = (1, 0, -1)$

Is A invertible? Why?

No, because if A was invertible, then from $Av = 0 \rightsquigarrow A^{-1}Av = A^{-1}0 \rightsquigarrow v = 0$ would be the only solution to $Av = 0$, but we know there are other solutions, such as $(1, 0, -1)$.

10. Let A be a square matrix (of any size) with a zero column. Is A invertible? Why?

No, there will be a zero in a pivot position of the zero column, hence, it will have a nonzero solution to $Ax = 0$, and thus A wouldn't be invertible, as in problem 9.

11. Let A be a square matrix (of any size) with a repeated ~~column~~ row. Is A invertible? Why?

See above!

(Q: What about if A has a repeated column?)

12. Find the inverse of this lower triangular matrix, using Gauss-Jordan method: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

Do you expect the inverse of any lower triangular matrix to be lower triangular? Why?

Yes, because the elementary matrices to eliminate the entries below the diagonal will be all lower triangular, and the product of lower triangular matrices is lower triangular. [The scalar matrices are diagonal, hence lower triangular, too.]