Elementary Linear Algebra - MATH 2250 - Day 10

Name:

1. **Define** the four fundamental subspaces.

   In the book.

2. The set of all the linear combinations of the rows of a matrix A is the <u>row space</u> of A.

3. How are the column space of A^T and the row space of A related?

   colums of A^T span the row space of A. (they might not be a basis for it. why?)

4. Find the left null space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\left[\begin{array}{cc|ccc} 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|ccc} 1 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & 1 & 0 \\ 0 & 0 & -\frac{8}{5} & \frac{1}{5} & 1 \end{array}\right]$$

$\Rightarrow \begin{bmatrix} -8 & 1 & 5 \end{bmatrix}$ is a basis for $LN(A)$.

5. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$.

   - The column space of A is a subspace of $\mathbb{R}^{\boxed{4}}$.
   - The row space of A is a subspace of $\mathbb{R}^{\boxed{3}}$.
   - The null space of A is a subspace of $\mathbb{R}^{\boxed{3}}$.
   - The left null space of A is a subspace of $\mathbb{R}^{\boxed{4}}$.

6. Let A be an $m \times n$ matrix.

   - The column space of A is a subspace of $\mathbb{R}^{\boxed{m}}$.
   - The row space of A is a subspace of $\mathbb{R}^{\boxed{n}}$.

- The null space of A is a subspace of $\mathbb{R}^{\boxed{n}}$.
- The left null space of A is a subspace of $\mathbb{R}^{\boxed{m}}$.

7. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. $\longrightarrow R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\leftarrow$ 2 pivots in the first two columns.

- Find a basis for the column space of A. What is its dimension?

$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$. 2.

Note: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is not a basis for $C(A)$. why?

- Find a basis for the row space of A. What is its dimension?

$\{ [1 \quad 0 \quad 1], [0 \quad 1 \quad 1] \}$. 2.

- Find a basis for the null space of A. What is its dimension?

$N = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \longrightarrow \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $N(A)$. 1.

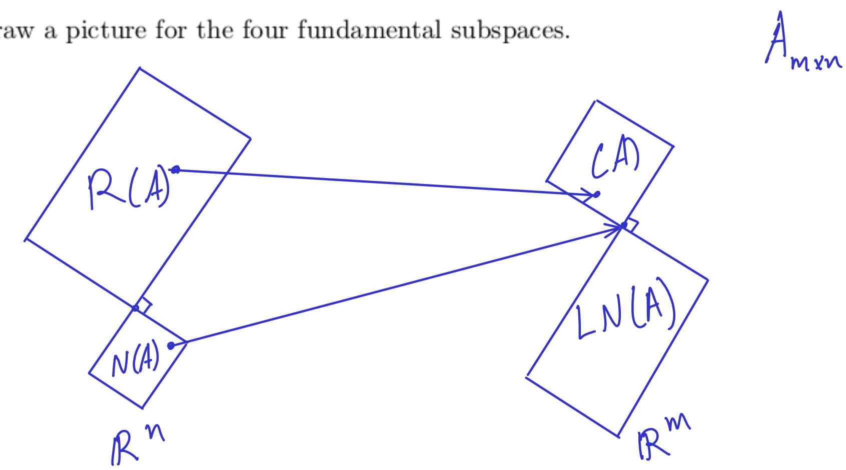
- Find a basis for the left null space of A. What is its dimension?

sol 1: Find E such that $EA = R$, the last <u>two</u> rows of it form a basis for $LN(A)$.

sol 2: Find a basis for $N(A^T)$.

8. Mark each of the followings as True or False.
   - [✓T] [F] The dimension of the row space of a matrix is equal to the dimension of the column space of it.
   - [T] [✓F] Let $R = \text{rref}(A)$. Then the column space of R is the same as the column space of A.
   - [✓T] [F] Let $R = \text{rref}(A)$. Then the row space of R is the same as the row space of A.
   - [✓T] [F] Let $R = \text{rref}(A)$. Then the null space of R is the same as the null space of A.
   - [T] [✓F] Let $R = \text{rref}(A)$. Then the left null space of R is the same as the left null space of A.

9. Draw a picture for the four fundamental subspaces.



10. Go back to the video (or (the cover of) the book) and look at the picture of the four fundamental subspaces. What is that point of intersection of the row space and the null space? What about the point of intersection of the columns space and the left null space?

The zero vector. Also the zero vector.

11. Let U be the set of all the 3×3 upper-triangular matrices.

(a) Show that if $A, B \in U$, then $A + B \in U$.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} + \begin{bmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{bmatrix} = \begin{bmatrix} a+a' & b+b' & c+c' \\ 0 & d+d' & e+e' \\ 0 & 0 & f+f' \end{bmatrix} \in U$$

(b) Show that if $A \in U$, then $cA \in U$ for any real number c.

(c) Is U a vector space?

Yes, because (a) and (b)

(d) What is the dimension of U?

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(e) Find a basis for U. (You need to provide a set of "vectors" that are linearly independent and span the whole vector space, that is every upper-triangular matrix is a linear combination of them.)

$$\left\{ E_{11}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, E_{12}=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, E_{13}=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, E_{22}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, E_{23}=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}, E_{33}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} \right\}$$

Note that if $c_1E_{11}+c_2E_{12}+\cdots+c_6E_{33}=0$, then $\begin{bmatrix}c_1&c_2&c_3\\&c_4&c_4\\&&c_5\end{bmatrix}=0$, then $c_1, c_2, \dots, c_6=0$

and any $\underset{\underset{U}{\uparrow}}{A}=\begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}=aE_{11}+bE_{12}+\cdots+fE_{33}$.

span.

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12. Show that S, the set of all 3×3 symmetric matrices is a vector space and find its dimension by finding a basis for it.

$A, B\in S \Rightarrow \begin{cases}A^T=A\\B^T=B\end{cases}$, $(A+B)^T=A^T+B^T=A+B \Rightarrow A+B\in S$

$(cA)^T=cA^T=cA \Rightarrow cA\in S$