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Existence of a not necessarily symmetric matrix with given distinct eigenvalues and graph

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ABSTRACT

For given distinct numbers $\lambda_1 \pm \mu_1 \mathrm{i}, \lambda_2 \pm \mu_2 \mathrm{i}, \ldots, \lambda_k \pm \mu_k \mathrm{i} \in \mathbb{C} \setminus \mathbb{R}$ and $\gamma_1, \gamma_2, \ldots, \gamma_l \in \mathbb{R}$, and a given graph G with a matching of size at least k, we will show that there is a real matrix whose eigenvalues are the given numbers and its graph is G. In particular, this implies that any real matrix with distinct eigenvalues is similar to a real, irreducible, tridiagonal matrix.

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1. Introduction

A *directed graph* $G = (V, E)$ is a pair of sets V and E where V is the set of vertices of G, and E, the set of edges of G, is a subset of $V \times V$. That is, each element of E is an ordered pair (u, v), with $u, v \in V$. We say a graph G is *loopless* when for each $(u, v) \in E$, we have $u \neq v$. In this paper we only consider loopless graphs. If $(u, v) \in E$ then we say u is adjacent to v and denote it by $u \to v$. Note that such a graph might have both edges (u, v) and (v, u), but since E is a set, there are no multiple edges from u to v. A directed loopless graph $G = (V, E)$ is said to have a *matching* of size k if E contains k vertex-disjoint edges $(u_1, v_1), \ldots, (u_k, v_k)$ and their reverses $(v_1, u_1), \ldots, (v_k, u_k)$.

A graph G is called undirected if for each $u \neq v$ the edge $(u, v) \in E$ if and only if $(v, u) \in E$. Hence we can ignore the directions of edges and consider E as a set of 2-subsets of V. That is, $E \subset \big\{\{u, v\} \,\big|\, u, v \in V\big\}$. In this case we call G an *undirected graph*. An undirected loopless graph $G = (V, E)$ is said to have a *matching* of size k if E contains k vertex-disjoint edges $\{u_1, v_1\}, \ldots, \{u_k, v_k\}$

Let $A \in \mathbb{R}^{n \times n}$. We say a (directed or undirected) loopless graph G is the *graph of the matrix* A when for each $i \neq j$ we have $A_{i,j} \neq 0$ if and only if $i \to j$. Note that the diagonal entries of A can be zero or nonzero.

It is of interest to study the existence of matrices with given spectral properties and graph, see [1, Chapter 4]. For the problems when the solution matrix is not necessarily symmetric see [2] for a survey on the structured inverse eigenvalue problems with an extensive bibliography, specially SIEP6b, and see [3] for the related minimum rank problems. In this paper we examine a the problem of existence of a solution matrix $A \in \mathbb{R}^{n \times n}$ where G, the graph of A, and Λ, its spectrum, are prescribed. An obvious necessary condition for the existence of a solution is that Λ to be closed under complex conjugation. Assume that Λ consists of k distinct complex conjugate pairs in $\mathbb{C} \setminus \mathbb{R}$. We will show that a sufficient condition for the existence of a solution is that

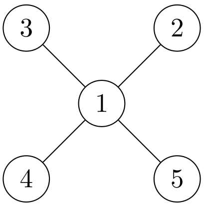
C1. G has a matching of size at least k, and
C2. all the eigenvalues are distinct.

Example 1.1 shows that condition C1 is not necessary, and Example 1.2 shows that condition C2 is not necessary.

Example 1.1. Let G be a star on 5 vertices with 1 as the center vertex, and let

$$A = \begin{bmatrix} 3 & -2 & -6 & 1 & 6 \\ 5 & -2 & 0 & 0 & 0 \\ -5 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 3 & 0 \\ -6 & 0 & 0 & 0 & 5 \end{bmatrix},$$

G:



It is straightforward to check that G is the graph of A, and it does not have a matching of size 2, but its eigenvalues approximately are 2.957, $-1.170 \pm 1.836\mathrm{i}$, and $5.192 \pm 4.222\mathrm{i}$, which contains two conjugate pairs of non-real numbers.

Example 1.2. Let G be the empty graph on 5 vertices, and let A be the identity matrix of order 5. It is clear that G is the graph of A and the eigenvalues of A are not distinct.

In Section 2 we provide some machinery in order to prove the main theorem in Section 3 about the existence of a solution for the inverse eigenvalue problem for a graph when the solution matrix is not necessarily symmetric.

2. Preliminaries

In this section we first introduce the notion of transversality, and use it to show that simple real roots of a real polynomial remain real under small perturbations. Then we give an example of a real matrix whose spectrum is a given set of real numbers and pairs of complex conjugate numbers. For the given matrix we define a neighborhood of its spectrum and assign an order to it. We finally study how small perturbations of the matrix change its simple eigenvalues.

2.1. Transversal intersections

In this section we formally prove that if a real polynomial has only simple roots, then a small real perturbation of it does not change the number of its simple roots. Furthermore, the real roots remain real and the complex roots remain complex. We first define two families of manifolds and show that they intersect transversally at some points. Then we will use this result to show that small perturbation of a real polynomial does not change the number of its simple real roots.

Let $\mathbb{R}[x]$ be the set of polynomials in x with coefficients in $\mathbb{R}$ and

$$p_t(x) = x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t) \in \mathbb{R}[x], \tag{1}$$

where for each $i = 0, 1, \ldots, n-1$ the coefficient $a_i(t)$ is a continuous function of t from $(-1, 1)$ to $\mathbb{R}$. For each $t \in (-1, 1)$ define

$$P(t) = \{(x, p_t(x)) \in \mathbb{R}^2 \,\big|\, x \in \mathbb{R}\}, \tag{2}$$

and

$$S = \{(x, 0) \in \mathbb{R}^2 \,\big|\, x \in \mathbb{R}\}. \tag{3}$$

Note that S and $P(t)$ for each $t \in (-1, 1)$ are smooth manifolds of $\mathbb{R}^2$. The tangent space to S at any point $(x_0, 0) \in \mathbb{R}^2$, $\mathcal{T}_{S.(x_0,0)}$, is S itself, and the tangent space to $P(t)$

at any point $(x_0, p_t(x_0))$, $\mathcal{T}_{P(t).(x_0,p_t(x_0))}$ is the tangent line to the graph of $y = p_t(x)$ at the point $(x_0, p_t(x_0))$. The latter tangent space is the set

$$\mathcal{T}_{P(t).(x_0,p_t(x_0))} = \{(x_0, p_t'(x_0)(x - x_0) + p_t(x_0) \,|\, x \in \mathbb{R}\},$$

where $p_t'(x_0)$ denotes the derivative of $p_t(x)$ evaluated at x_0. It is evident that $\mathcal{T}_{P(t).(x_0,p_t(x_0))}$ is a line not parallel to S when x_0 is not a root of $p_t'(x)$. In particular, when x_0 is a root of $p_t(x)$, then $P(t)$ and S intersect transversally at x_0 if and only if x_0 is a simple root of $p_t(x)$. We shall need the following special case of [4, Lemma 2.1].

Lemma 2.1. *Let $P(t)$ and $S(t)$ be smooth families of manifolds in $\mathbb{R}^N$, for some positive integer N, and assume that $P(0)$ and $S(0)$ intersect transversally at x. Then there exists a neighborhood $W \subseteq \mathbb{R}^2$ of the origin, such that for each $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2) \in W$, the manifolds $P(\varepsilon_1)$ and $S(\varepsilon_2)$ intersect transversally at a point $x(\varepsilon)$, so that $x(0) = x$ and $x(\varepsilon)$ depends continuously on ε.*

The following lemma shows that if $p(x)$ is a polynomial in $\mathbb{R}[x]$ with k simple real roots, then any sufficiently small perturbation of $p(x)$ also has k simple real roots.

Lemma 2.2. *Let $p_t(x)$ be defined as in Equation (1). If $p_0(x)$ has a simple real root, then there is an $\varepsilon > 0$ such that for each $-\varepsilon < t < \varepsilon$ the polynomial $p_t(x)$ has a simple real root.*

Proof. Let x_0 be a simple root of $p_0(x)$, and let S and $P(t)$ be defined be Equations (2) and (3). Then S and $P(0)$ intersect transversally at x_0, and thus by Lemma 2.1 there is an $\varepsilon > 0$ such that for any $-\varepsilon < t < \varepsilon$ the manifolds $P(t)$ and S intersect transversally at $x_0(t)$ where $x_0(0) = x_0$ and $x_0(t)$ depends continuously on t. In particular, $x_0(t)$ is a simple root of $p_t(x)$. □

Corollary 2.3. *If $p(x)$ is a polynomial of degree n in $\mathbb{R}[x]$ with k simple real roots and $n - k$ distinct non-real roots, then any sufficiently small real perturbation of $p(x)$ also has k simple real roots and $n - k$ distinct non-real roots.*

2.2. A matrix with a given spectrum

Now, we show that for given l distinct real numbers, and $2k$ distinct non-real numbers which are conjugate pairs, there is a real matrix whose eigenvalues are the given numbers. Then we define some (ordered) neighborhood of its spectrum.

Example 2.4. There is a real matrix A whose eigenvalues are given numbers $\lambda_j \pm \mu_j \mathrm{i} \in \mathbb{C} \setminus \mathbb{R}$ for $j = 1, 2, \ldots, k$, and $\gamma_j \in \mathbb{R}$ for $j = 1, 2, \ldots, l$, namely

$$A = \left(\bigoplus_{j=1}^{k} \begin{bmatrix} \lambda_j & \mu_j \\ -\mu_j & \lambda_j \end{bmatrix} \right) \oplus \left(\bigoplus_{j=1}^{l} \begin{bmatrix} \gamma_j \end{bmatrix} \right). \tag{4}$$

Note that this is the real Jordan form for a semisimple matrix, and that a unit eigenvector corresponding to the eigenvalue $\lambda_j \pm \mu_j \mathrm{i}$ is

$$\boldsymbol{v}_j = \frac{1}{\sqrt{2}} \begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \pm\mathrm{i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow 2j-1 \\ \leftarrow 2j \\ \\ \\ \\ \end{array} \end{array}.$$

Furthermore, note that the corresponding eigenvector of the same eigenvalue for $A^\top$ is $\boldsymbol{w}_j = \overline{\boldsymbol{v}}_j$.

Now, we define a matrix of variables for a graph G and we will consider the rate of change of its eigenvalues as the variables change. Let G be a graph on $n = 2k + l$ vertices and $k + m$ edges. Assume that G has a matching $\mathcal{M} = \big\{\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}\big\}$, and let the rest of the edges of G be denoted by $\{i_r, j_r\}$, with $i_r < j_r$ for $r = 1, 2, \ldots, m$. Also, let $\boldsymbol{x} = (x_1, x_2, \ldots, x_k), \boldsymbol{y} = (y_1, y_2, \ldots, y_k) \in \mathbb{R}^k$, $\boldsymbol{z} = (z_1, z_2, \ldots, z_l) \in \mathbb{R}^l$, and $\boldsymbol{u} = (u_1, u_2, \ldots, u_m), \boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m) \in \mathbb{R}^m$.

Define $M = M(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{u}, \boldsymbol{\omega})$ with x_j on the $(2j-1, 2j-1)$ and $(2j, 2j)$ positions and y_j on the $(2j-1, 2j)$ position and $-y_j$ on the $(2j, 2j-1)$ position, for $j = 1, 2, \ldots, k$; and z_j on the $(2k+j, 2k+j)$ position, for $j = 1, 2, \ldots, l$; and u_r on (i_r, j_r) position, and ω_r on (j_r, i_r) position, for $r = 1, 2, \ldots, m$. The matrix M has the following form

$$M = \left[\begin{array}{cc|c|cc|ccc}
x_1 & y_1 & & & & & & \\
-y_1 & x_1 & & & & & & \\ \hline
 & & \ddots & & & & & \\ \hline
 & & & x_k & y_k & & & \\
 & & & -y_k & x_k & & & \\ \hline
 & & & & & z_1 & & \\
 & & & & & & \ddots & \\
 & & & & & & & z_l
\end{array}\right], \tag{5}$$

where the entries not shown are either 0, some u_r, or some ω_r. Note that

$$M(\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k, \gamma_1, \dots, \gamma_l, 0, \dots, 0) = A,$$

where A is the Jordan matrix (4).

Now, and for the rest of this paper, assume that

$$\Lambda = \{\lambda_j \pm \mu_j \mathrm{i} \in \mathbb{C} \setminus \mathbb{R} \mid j = 1, 2, \dots, k\} \cup \{\gamma_j \in \mathbb{R} \mid j = 1, 2, \dots, l\}$$

is a fixed set of $n = 2k + l$ distinct numbers. Define an ε-neighborhood of a set $S \subseteq \mathbb{C}$ to be the set of points that are of distance at most ε from a point of S, that is,

$$N_\varepsilon(S) = \{z \in \mathbb{C} \mid |z - s| < \varepsilon \text{ for some } s \in S\}.$$

Since Λ consists of n distinct points in the complex plain, there is an ε such that $N_\varepsilon(\Lambda)$ consists of n disjoint disks $\mathbb{D}_1^+, \mathbb{D}_2^+, \dots, \mathbb{D}_k^+$, $\mathbb{D}_1^-, \mathbb{D}_2^-, \dots, \mathbb{D}_k^-$, and $\mathbb{D}_1, \mathbb{D}_2, \dots, \mathbb{D}_l$ where $\mathbb{D}_j^+$ contains $\lambda_j + \mu_j\mathrm{i}$, $\mathbb{D}_j^-$ contains $\lambda_j - \mu_j\mathrm{i}$, and $\mathbb{D}_j$ contains γ_j. Also, let $\mathbb{D}'_j$ denote $\mathbb{D}_j \cap \mathbb{R}$, and

$$\mathbb{D} = \left(\bigcup_{j=1}^{k} \mathbb{D}_j^+\right) \cup \left(\bigcup_{j=1}^{k} \mathbb{D}_j^-\right) \cup \left(\bigcup_{j=1}^{k} \mathbb{D}'_j\right).$$

Proposition 2.5. *Let $A \in \mathbb{R}^{n\times n}$ have n distinct eigenvalues. Then for an eigenvalue λ and corresponding left eigenvector $\boldsymbol{u}^\top$ and right eigenvector $\boldsymbol{v}$, we have $\boldsymbol{u}^\top \boldsymbol{v} \neq 0$.*

Proof. Let J be the Jordan canonical form of A. Since all the eigenvalues of A are simple, J is diagonal. Let $A = SJS^{-1}$ for some invertible matrix S. Observe that for an eigenvalue λ there is an $1 \leq i \leq n$ such that the i-th column of S, $\boldsymbol{s}^i$, is a right eigenvector of A for the eigenvalue λ, and the i-th row of S^{-1}, ${\boldsymbol{s}_i}^\top$, is a left eigenvector of A for the eigenvalue λ. Since $S^{-1}S = I$ we have ${\boldsymbol{s}_i}^\top \boldsymbol{s}^i = 1$. This implies $\boldsymbol{u}^\top \boldsymbol{v} \neq 0$. $\square$

2.3. Small perturbations of a matrix and its eigenvalues

In this part we study the effect of small perturbations of a matrix on its eigenvalues and define a function that maps a matrix to its eigenvalues. Then we will show that the Jacobian matrix of this function evaluated at a certain point has full rank.

If the matrix M is in a small neighborhood of A, then its eigenvalues lie in $N_\varepsilon(\Lambda)$. Moreover, Lemma 2.3 implies that the real eigenvalues of M lie in $\mathbb{D}'_j$'s. Let $\lambda_j(M)$ denote the real part of the eigenvalue of M that lies in $\mathbb{D}_j^+$, $\mu_j(M)$ denote the imaginary part of the eigenvalue of M that lies in $\mathbb{D}_j^+$, and $\gamma_j(M)$ denote the real eigenvalue of M that lies in $\mathbb{D}'_j$.

Now define a function f in a small neighborhood of A as follows:

$$f\colon \mathbb{R}^{(2k+l)+2m} \to \mathbb{R}^{2k+l} \tag{6}$$

$$M \mapsto \big(\lambda_1(M), \dots, \lambda_k(M)\,,\, \mu_1(M), \dots, \mu_k(m)\,,\, \gamma_1(M), \dots, \gamma_l(M)\big). \tag{7}$$

Thus, f maps a small neighborhood of A to $\mathbb{D}$. It is important to note that f is differentiable in a this small neighborhood [5, see Chapter Two, Section 5.8], and the goal is to show that the Jacobian of this function evaluated at A has full row rank. The following two lemmas calculate the derivative of each of components of f. Note that the derivative of the real eigenvalues are also included in the following lemma, as they happen precisely when $\mu_r = 0$. For more detailed discussion on this topic see [6, Section 11.6].

Lemma 2.6. *Let A and B be real matrices where A has distinct simple eigenvalues $\lambda_r \pm \mu_r \mathrm{i} \in \mathbb{C}$, for $r = 1, 2, \dots, k+l$, where $\mu_r = 0$ if and only if $r > k$. Let $\boldsymbol{v}_r$ be a right unit eigenvectors of A corresponding to $\lambda_r + \mu_r \mathrm{i}$, and $\boldsymbol{w}_r^\top$ be a left unit eigenvectors of A corresponding to the same eigenvalue. Also, let $A(t) = A + tB$, for $t \in (-1, 1)$, and let $\lambda_r(t) + \mu_r(t)\mathrm{i}$ denote an eigenvalue of $A(t)$ that approaches $\lambda + \mu \mathrm{i}$ as t approaches 0. Then the following hold:*

$$\frac{\mathrm{d}}{\mathrm{dt}}\lambda_j(0) = \mathrm{Re}(\zeta) \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{dt}}\mu_j(0) = \mathrm{Im}(\zeta),$$

where $\zeta = \dfrac{\boldsymbol{w}_r^\top B \boldsymbol{v}_r}{\boldsymbol{w}_r^\top \boldsymbol{v}_r}$.

Proof. Let $\boldsymbol{v}_r(t)$ be a unit eigenvector of $A(t)$ corresponding to $\lambda_r(t) + \mu_r(t)\mathrm{i}$. Note that

$$A(t) \to A, \quad \boldsymbol{v}_r(t) \to \boldsymbol{v}_r, \quad \lambda_r(t) \to \lambda_r, \quad \text{and } \mu_r(t) \to \mu_r,$$

as $t \to 0$. Note that

$$A(t)\boldsymbol{v}_r(t) = (\lambda_r(t) + \mu_r(t)\mathrm{i})\boldsymbol{v}_r(t).$$

Differentiating with respect to t we have

$$\dot{A}(t)\boldsymbol{v}_r(t) + A(t)\dot{\boldsymbol{v}}_r(t) = (\dot{\lambda}_r(t) + \dot{\mu}_r(t)\mathrm{i})\boldsymbol{v}_r(t) + (\lambda_r(t) + \mu_r(t)\mathrm{i})\dot{\boldsymbol{v}}_r(t).$$

Letting $t = 0$ we have

$$B\boldsymbol{v}_r + A\dot{\boldsymbol{v}}_r(0) = (\dot{\lambda}_r(0) + \dot{\mu}_r(0)\mathrm{i})\boldsymbol{v}_r + (\lambda_r + \mu_r\mathrm{i})\dot{\boldsymbol{v}}_r(0).$$

Multiply both sides by $\boldsymbol{w}_r^\top$ from the left

$$\boldsymbol{w}_r^\top B\boldsymbol{v}_r + \boldsymbol{w}_r^\top A\dot{\boldsymbol{v}}_r(0) = (\dot{\lambda}_r(0) + \dot{\mu}_r(0)\mathrm{i})\boldsymbol{w}_r^\top\boldsymbol{v}_r + (\lambda_r + \mu_r\mathrm{i})\boldsymbol{w}_r^\top\dot{\boldsymbol{v}}_r(0),$$

since $\boldsymbol{w}_r^\top A = (\lambda_r + \mu_r \mathrm{i})\boldsymbol{w}_r^\top$ we get

$$\boldsymbol{w}_r^\top B\boldsymbol{v}_r + (\lambda_r + \mu_r\mathrm{i})\boldsymbol{w}_r^\top \dot{\boldsymbol{v}}_r(0) = (\dot{\lambda}_r(0) + \dot{\mu}_r(0)\mathrm{i})\boldsymbol{w}_r^\top \boldsymbol{v}_r + (\lambda_r + \mu_r\mathrm{i})\boldsymbol{w}_r^\top \dot{\boldsymbol{v}}_r(0).$$

The second terms in the left hand side and right hand side of the equation are equal. Thus

$$\boldsymbol{w}_r^\top B\boldsymbol{v}_r = (\dot{\lambda}_r(0) + \dot{\mu}_r(0)\mathrm{i})\boldsymbol{w}_r^\top \boldsymbol{v}_r.$$

By Proposition 2.5 we have $\boldsymbol{w}_r^\top \boldsymbol{v}_r \neq 0$. Hence

$$\dot{\lambda}_r(0) + \dot{\mu}_r(0)\mathrm{i} = \frac{\boldsymbol{w}_r^\top B\boldsymbol{v}_r}{\boldsymbol{w}_r^\top \boldsymbol{v}_r}. \tag{8}$$

Conjugating both sides we get

$$\dot{\lambda}_r(0) - \dot{\mu}_r(0)\mathrm{i} = \frac{\overline{\boldsymbol{w}}_r^\top B\overline{\boldsymbol{v}}_r}{\overline{\boldsymbol{w}}_r^\top \overline{\boldsymbol{v}}_r}. \tag{9}$$

Now once add equations (8) and (9) and once subtract them to get

$$\dot{\lambda}_r(0) = \mathrm{Re}\left(\frac{\boldsymbol{w}_r^\top B\boldsymbol{v}_r}{\boldsymbol{w}_r^\top \boldsymbol{v}_r}\right), \tag{10}$$

$$\dot{\mu}_r(0) = \mathrm{Im}\left(\frac{\boldsymbol{w}_r^\top B\boldsymbol{v}_r}{\boldsymbol{w}_r^\top \boldsymbol{v}_r}\right). \quad \square \tag{11}$$

The following lemma summarizes the above machinery as the final calculation of derivative of the r-th eigenvalue of A with respect to the variables on the 2×2 or 1×1 block diagonals, that is with respect to x_j's, y_j's, and z_j's. Then, this will be used in computing the Jacobian matrix of the function f in the following corollary.

Lemma 2.7. *Let A be the matrix (4), and let E_{ij} denote the matrix of appropriate size with a 1 in its (i, j)-entry and zeros elsewhere. Also, let B in Lemma 2.6 be one of the following:*

1. $B = E_{2j-1,2j-1} + E_{2j,2j}$, *for some* $j = 1, 2, \dots, k$,
2. $B = E_{2j-1,2j} - E_{2j,2j-1}$, *for some* $j = 1, 2, \dots, k$, *or*
3. $B = E_{2k+j,2k+j}$, *for some* $j = 1, 2, \dots, l$.

Then

$$\dot{\lambda}_r(0) + \dot{\mu}_r(0)\mathrm{i} = \begin{cases} 1; \text{ if and only if } r = j & \text{(in case 1)}, \\ \mathrm{i}; \text{ if and only if } r = j & \text{(in case 2)}, \\ 1; \text{ if and only if } r = 2k + j & \text{(in case 3)}. \end{cases}$$

Proof. Note that for matrix A in (4) we have $w_r = \overline{v}_r$. Thus $w_r^\top v_r = 1$ and

$$w_r^\top B v_r = \begin{cases} \boldsymbol{w}_{r_{2j-1}} \boldsymbol{v}_{r_{2j-1}} + \boldsymbol{w}_{r_{2j}} \boldsymbol{v}_{r_{2j}} & \text{(in case 1)}, \\ \boldsymbol{w}_{r_{2j-1}} \boldsymbol{v}_{r_{2j}} - \boldsymbol{w}_{r_{2j}} \boldsymbol{v}_{r_{2j-1}} & \text{(in case 2)}, \\ \boldsymbol{w}_{r_{2k+j}} \boldsymbol{v}_{r_{2k+j}} & \text{(in case 3)}. \end{cases}$$

Thus

$$w_r^\top B v_r = \begin{cases} 1; \text{ if and only if } r = j & \text{(in case 1)}, \\ \mathrm{i}; \text{ if and only if } r = j & \text{(in case 2)}, \\ 1; \text{ if and only if } r = 2k+j & \text{(in case 3)}. \end{cases} \qquad \square$$

Now we are ready to evaluate the Jacobian of the function f.

Corollary 2.8. *Let A be the matrix (4), and the function f be defined by Equation (6). Also, let $\mathrm{Jac}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}$ denote the matrix obtained from the Jacobian matrix of f by keeping only the columns corresponding to the derivatives with respect to x_j's, y_j's, and z_j's. Then*

$$\mathrm{Jac}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}(f)\Big|_A = I,$$

where I denotes the identity matrix of size $2k + l$, and thus it is nonsingular.

3. Main result

In this section we prove that for given l distinct real numbers, $2k$ distinct non-real numbers which are conjugate pairs, and a graph on n vertices with a matching of size at least k, there is a real matrix whose eigenvalues are the given numbers and its graph is the given graph.

Theorem 3.1. *For given distinct numbers $\lambda_1 \pm \mu_1\mathrm{i}, \lambda_2 \pm \mu_2\mathrm{i}, \ldots, \lambda_k \pm \mu_k\mathrm{i} \in \mathbb{C} \setminus \mathbb{R}$ and $\gamma_1, \gamma_2, \ldots, \gamma_l \in \mathbb{R}$, and a given graph G on $2k + l$ vertices with a matching of size at least k there is a real matrix whose eigenvalues are the given numbers and its graph is G.*

Proof. Let A be the matrix (4), matrix M be defined as in (5), and function f be defined by Equation (6). Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$, and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_l)$. Also, let $\mathbf{0}$ denote a zero vector of appropriate size. Note that

$$M(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\gamma}, \mathbf{0}, \mathbf{0}) = A.$$

Also note that

$$f(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\gamma}, \mathbf{0}, \mathbf{0}) = (\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\gamma}).$$

Furthermore, by Corollary 2.8 we have

$$\mathrm{Jac}_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}}(f)\Big|_{A} = I,$$

and hence it is nonsingular. Then by the Implicit Function Theorem for any small $\boldsymbol{\varepsilon}, \boldsymbol{\delta} \in \mathbb{R}^m$, there are $\overline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\mu}}, \overline{\boldsymbol{\gamma}}$ close to $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\gamma}$ such that $f(\overline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\mu}}, \overline{\boldsymbol{\gamma}}, \boldsymbol{\varepsilon}, \boldsymbol{\delta}) = (\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\gamma})$. Choose $\overline{\boldsymbol{\lambda}}$, $\overline{\boldsymbol{\mu}}$, and $\overline{\boldsymbol{\gamma}}$ such that they have no zero entries, and let $\tilde{A} = M(\overline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\mu}}, \overline{\boldsymbol{\gamma}}, \boldsymbol{\varepsilon}, \boldsymbol{\delta})$. Then eigenvalues of $\tilde{A}$ are $\lambda_j \pm \mu_j \mathrm{i}$ for $j = 1, 2, \ldots, k$ and γ_j for $j = 1, 2, \ldots l$, and the graph of $\tilde{A}$ is G. □

Remark 3.2. If all the prescribed eigenvalues are real, one can always choose $\boldsymbol{\varepsilon} = \boldsymbol{\delta}$ to find a symmetric matrix $\tilde{A}$. Also, if all the prescribed eigenvalues are purely imaginary, one can always choose $\boldsymbol{\varepsilon} = -\boldsymbol{\delta}$ to make $\tilde{A}$ the sum of a skew-symmetric matrix and a diagonal matrix. The case with all real eigenvalues was previously proven in [7] and the case with all purely imaginary eigenvalues was shown in [8], and the constructed matrix is shown to have a zero diagonal, that is, it is a skew-symmetric matrix.

Corollary 3.3. *Let G be a graph with a matching of size k. Then any real matrix with distinct eigenvalues of which at most $2k$ are non-real, is similar to a real matrix whose graph is G.*

Note that this implies any real matrix with distinct eigenvalues is similar to a tridiagonal real matrix with nonzero superdiagonal and subdiagonal entries. On the other hand any real tridiagonal matrix with nonzero superdiagonal and subdiagonal entries has distinct eigenvalues. Thus we have the following corollary.

Corollary 3.4. *A real matrix has distinct eigenvalues if and only if it is similar to a real irreducible tridiagonal matrix.*

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