Elementary Linear Algebra - MATH 2250 - Day 25

Name:

1. The eigenvalues of a real symmetric matrix are ____________ numbers. For example the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are ______ and ______. But the eigenvalues of $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ are ____________ and ____________.

2. Let $x = a + \text{ib}$ be a complex number. Find $\bar{x}x$. What do you know about this quantity?

3. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Find $\bar{x}^T x$. What do you know about this quantity?

4. Let's see why are the eigenvalues of a real symmetric matrix are real. Recall that a number a is real if and only if its complex conjugate $\bar{a}$ is equal to a. Assume that λ is an eigenvalue of A with corresponding eigenvector x. Then

$$Ax = \_\_\_\_\_\_\_\_. \qquad (1)$$

Multiply both sides by $\bar{x}^T$ from left:

$$\bar{x}^T Ax = \_\_\_\_\_\_\_\_. \qquad (2)$$

Take the complex conjugate of both sides of (1):

$$\overline{Ax} = \_\_\_\_\_\_\_\_. \qquad (3)$$

Since A is a real matrix, and $\overline{ab} = \bar{a}\bar{b}$:

$$A\bar{x} = \_\_\_\_\_\_\_\_. \qquad (4)$$

Take transpose of both sides:

$$(A\bar{x})^T = \_\_\_\_\_\_\_\_. \qquad (5)$$

Simplify:

$$\bar{x}^T A^T = \_\_\_\_\_\_\_\_. \qquad (6)$$

But A is symmetric, that is $A^T =$ ____________, hence

$$\bar{x}^T A = \_\_\_\_\_\_\_\_. \qquad (7)$$

Multiply both sides by x from right:

$$\bar{x}^T Ax = \_\_\_\_\_\_\_\_\_\_\_\_. \qquad (8)$$

Compare (2) and (8):

$$\bar{x}^T \bar{\lambda} x = \_\_\_\_\_\_\_\_\_\_\_\_. \qquad (9)$$

But x is a nonzero vector (why?), so $\bar{x}^T x$ is a ____________ number. Divide both sides by $\bar{x}^T x$:

$$\bar{\lambda} = \_\_\_\_\_\_. \qquad (10)$$

So λ is ____________.

5. The eigenvectors of a real symmetric matrix can be chosen ________________________.

6. For a (real or complex) matrix A if $\bar{A}^T = A$, then A is called to be a Hermitian matrix. Use problem 4 to show that the eigenvalues of A are ____________.

7. What can you tell about the eigenvalues of real skew-symmetric matrices? (A is skew-symmetric if $A^T = -A$.)

8. Let A be symmetric with eigenvalues $\lambda_1, \dots, \lambda_n$, with corresponding eigenvectors $q_1, \dots, q_n$, such that $Q = \left[\begin{array}{c|c|c} q_1 & \cdots & q_n \end{array}\right]$ is orthonormal. Then $A = Q\Lambda Q^T$, where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Then

$$A = \lambda_1\_\_\_\_\_\_\_\_\_\_\_\_ + \lambda_2\_\_\_\_\_\_\_\_\_\_\_\_ + \cdots + \lambda_n\_\_\_\_\_\_\_\_\_\_\_\_.$$

This is called the spectral decomposition of A.

9. Recall that the eigenvalues of a matrix are not the same as the pivots of it. But the ____________ of the eigenvalues of a matrix are the same as the ____________ of the pivots of it, and the product of the eigenvalues of a matrix is equal to the product of the ________________________.

10. What is the determinant and the signs of the eigenvalues of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$? Is it positive definite?