Elementary Linear Algebra - MATH 2250 - Day 1

Name:

Consider the following system.

$$\begin{cases} 2x + y = 5 \\ -x + 3y = 0 \end{cases}$$

1. Mark this as True or False: (Explain why)

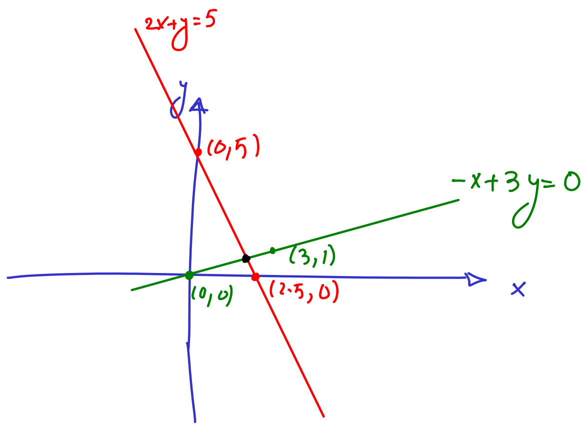
   If the two lines $2x + y = 5$ and $-x + 3y = 0$ meet at a point (a, b), then (a, b) is a solution to the system.

   True, because (a,b) is a point on the first line, so it solves the first equation; and it is a point on the second line, so it solves the second equation. That is it solves the system.

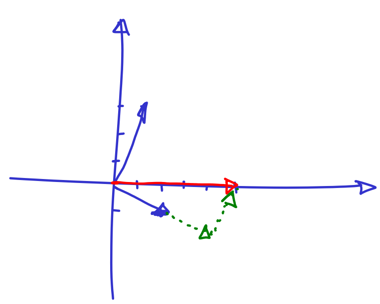
2. Write the matrix form for the system.

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

3. Draw the row picture of the system.



4. Draw the column picture of the system.



5. Give an example of a linear combination of two vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$2\begin{bmatrix} 2 \\ -1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 3 \end{bmatrix} \left(= \begin{bmatrix} 0 \\ -14 \end{bmatrix} \right)$$

6. The linear combinations of $\boldsymbol{v} = (2, -1, 0)$ and $\boldsymbol{w} = (1, 3, 0)$ fill a plane. Describe that plane.

The xy-plane. (The plane $\{(a, b, 0) \mid a, b \in \mathbb{R}\}$)

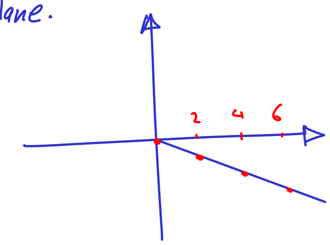
Find a vector that is not a combination of $\boldsymbol{v}$ and w.

Any vector with nonzero z-coordinate works, an example is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

7. For $\boldsymbol{v} = (2, -1)$ describe all points $c\boldsymbol{v}$ with (1) whole numbers c (2) nonnegative $c \geq 0$.

(1) The points on the line $y = -2x$ with their first coordinate from $\{0, 2, 4, 6, \ldots\}$

(2) The half line $y = -2x$ on the right half plane.



8. Find two equations for the unknowns c and d so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$cv + dw = b \rightarrow c\begin{bmatrix} -2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2c \\ c \end{bmatrix} + \begin{bmatrix} 0 \\ -d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2c \\ c-d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

equations: $\begin{cases} -2c = 1 \\ c - d = 0 \end{cases}$

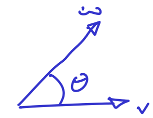
9. For $\boldsymbol{v} = (1, 2)$ and $\boldsymbol{w} = (2, 1)$ test the Schwarz inequality on $\boldsymbol{v} \cdot \boldsymbol{w}$, and the triangle inequality of $||\boldsymbol{v} + \boldsymbol{w}||$.

Schwarz ineq: $|v \cdot w| \leq \|v\| \|w\|$. $\quad \|v\| = \sqrt{5}$, $\|w\| = \sqrt{5}$, $|v \cdot w| = 4$ $\rightsquigarrow$ $4 \leq \sqrt{5}\sqrt{5}$ ✓

Triangle ineq: $\|v + w\| \leq \|v\| + \|w\|$. $\quad \|v + w\| = \sqrt{18}$ $\rightsquigarrow$ $\underset{\underset{4.2}{\text{IS}}}{\sqrt{18}} \leq \underset{\underset{4.5}{\text{IS}}}{\sqrt{5} + \sqrt{5}}$ ✓

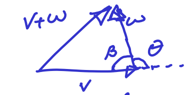
Find $\cos\theta$ for the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$.

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|} = \frac{4}{\sqrt{5}\sqrt{5}} = \frac{4}{5}$$



When will we have equality $|\boldsymbol{v}\cdot\boldsymbol{w}| = ||\boldsymbol{v}||\,||\boldsymbol{w}||$, and $||\boldsymbol{v}+\boldsymbol{w}|| = ||\boldsymbol{v}|| + ||\boldsymbol{w}||$?

- We know: $|v\cdot w| = \|v\|\,\|w\|\,|\cos\theta|$, hence $|v\cdot w| = \|v\|\,\|w\|$ if and only if $|\cos\theta| = 1$, that is, when $\theta = 0^\circ, 180^\circ$. In other words when $v \parallel w$.



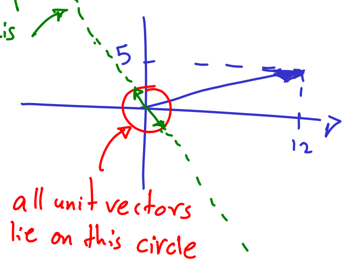
- Laws of cosine assert that $\|v+w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\beta$, so we have $\|v+w\| = \|v\| + \|w\| \iff \|v+w\|^2 = (\|v\|+\|w\|)^2 = \|v\|^2 + \|w\|^2 + 2\|v\|\|w\|$, hence we have equality if and only if $\cos\beta = -1 \iff \beta = 180^\circ \iff \theta = 0^\circ$, that is when v and w are in the same direction.

10. Find a unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v} = (12, 5)$.

$|v| = \sqrt{12^2+5^2} = 13$

$\Rightarrow u_v = \frac{1}{13}(12,5) = \left(\frac{12}{3}, \frac{5}{13}\right)$

all vectors perp to v lie on this line



all unit vectors lie on this circle

11. Find **all** the unit vectors which are perpendicular to $\boldsymbol{v} = (12, 5)$.

Let $u = (a,b)$ be a unit vector perp. to $v = (12,5)$

$$\Rightarrow \begin{cases} a^2 + b^2 = 1 \\ 12a + 5b = 0 \end{cases}$$

$$\Rightarrow a = \frac{-5}{12}b \rightsquigarrow \left(\frac{-5}{12}b\right)^2 + b^2 = 1 \Rightarrow \left(\frac{25}{12}+1\right)b^2 = 1 \Rightarrow b^2 = \frac{12}{37} \Rightarrow b = \pm\sqrt{\frac{12}{37}},\ a = \mp\frac{5}{12}\sqrt{\frac{12}{13}}$$

$u_1 = \left(\sqrt{\frac{12}{37}}, \frac{-5}{12}\sqrt{\frac{12}{37}}\right)$ and $u_2 = \left(-\sqrt{\frac{12}{37}}, \frac{5}{12}\sqrt{\frac{12}{37}}\right)$

12. Find a vector $x = (c, d)$ that has dot product $\boldsymbol{x}\cdot\boldsymbol{r} = 1$, and $\boldsymbol{x}\cdot\boldsymbol{s} = 0$ with the given vectors $\boldsymbol{r} = (1, 3)$ and $\boldsymbol{s} = (-3, 1)$.

$1 = x\cdot r = (c,d)\cdot(1,3) = c + 3d$

$0 = x\cdot s = (c,d)\cdot(-3,1) = -3c + d$

$$\rightsquigarrow \begin{cases} c + 3d = 1 \\ -3c + d = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 1 \\ -3 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 10 & 3 \end{array}\right]$$

$$\begin{cases} c + 3d = 1 \longrightarrow c = 0.1 \\ 10d = 3 \longrightarrow d = 0.3 \end{cases}$$

$x = (0.1, 0.3)$