**Elementary Linear Algebra - MATH 2250 - Exam 1**

**Name:** ______________________________

1. Describe geometrically (line, plane, or all of $\mathbb{R}^3$) all linear combinations of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\6\\9\end{bmatrix}$.

$\begin{bmatrix}3\\6\\9\end{bmatrix} = 3\begin{bmatrix}1\\2\\3\end{bmatrix} \Rightarrow$ all linear combinations of them is a line.

2. How long is the vector $\boldsymbol{v} = (1,1,1,1,1,1,1,1,1)$ in 9 dimensions?

$\|v\| = \sqrt{v \cdot v} = \sqrt{1+1+1+1+1+1+1+1+1} = \sqrt{9} = 3$

3. Write down three 'independent' vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^3$.

$u = \begin{bmatrix}1\\1\\1\end{bmatrix}, v = \begin{bmatrix}0\\1\\1\end{bmatrix}, w = \begin{bmatrix}0\\0\\1\end{bmatrix}$. (v is not a multiple of u, and no lin. comb. of u, v gives w.)

Using the three vectors above as columns, form a matrix $A = \left[\begin{array}{c|c|c} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{array}\right]$, and find its $LU$-decomposition.

We could have chosen $u = \begin{bmatrix}1\\0\\0\end{bmatrix}, v = \begin{bmatrix}0\\1\\0\end{bmatrix}$ for simplicity, but then the LU-Decomposition would've been even more boring than this one!

$A = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$

$\rightarrow U_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}, \quad L_1 = \begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}$

$\rightarrow U_2 = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \quad L_2 = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$

$\rightarrow L = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}, \quad U = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

check: $LU = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} = A$

4. Solve the system of linear equations

$$\begin{cases} x + 2y + 4z = 18 \\ -2x + 5y + z = 5 \\ -4x + y + 2z = 0 \end{cases}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 18 \\ -2 & 5 & 1 & 5 \\ -4 & 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{elimination}} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 18 \\ 0 & 9 & 9 & 41 \\ 0 & 9 & 18 & 72 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 18 \\ 0 & 9 & 9 & 41 \\ 0 & 0 & 9 & 31 \end{array}\right]$$

$$x + 2y + 4z = 18 \longrightarrow x + \frac{20}{9} + \frac{124}{9} = 18 \longrightarrow x = 2$$

$$9y + 9z = 41 \longrightarrow 9y = 10 \longrightarrow y = 10/9$$

$$9z = 31 \longrightarrow z = 31/9$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 10/9 \\ 31/9 \end{bmatrix}$$

5. Find the inverse of $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 3 & -1 & -1 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ -1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 3 & -1 & -1 & 1 & 0 & 0 \\ 0 & 3-\frac{1}{3} & -1-\frac{1}{3} & \frac{1}{3} & 1 & 0 \\ 0 & -1-\frac{1}{3} & 3-\frac{1}{3} & \frac{1}{3} & 0 & 1 \end{array}\right] \left( = \left[\begin{array}{ccc|ccc} 3 & -1 & -1 & 1 & 0 & 0 \\ 0 & \frac{8}{3} & \frac{-4}{3} & \frac{1}{3} & 1 & 0 \\ 0 & -\frac{4}{3} & \frac{8}{3} & \frac{1}{3} & 0 & 1 \end{array}\right] \right)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 3 & -1 & -1 & 1 & 0 & 0 \\ 0 & 8/3 & -\frac{4}{3} & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 2 & \frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & \frac{-1}{3} & \frac{-1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{8} & \frac{3}{8} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{3} & 0 & \frac{5}{12} & \frac{1}{12} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 & 1/4 & 1/4 & 1/2 \end{array}\right] \Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad \text{check: } AA^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

6. Describe the column space and the null space of $A$ **and** the complete solution to $A\boldsymbol{x} = \boldsymbol{b}$ for

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}.$$

$$C(A) = \left\{ c_1 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix} + c_4 \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} : c_1, c_2, c_3, c_4 \in \mathbb{R} \right\}$$

find: rref$(A)$:

$$\left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 2 & 4 & 6 & 4 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{c} \begin{array}{cccc} I & & F & \end{array} \\ \left[\begin{array}{cc|cc|c} 1 & 0 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} = R|b'$$

$$N = \begin{bmatrix} -1 & 2 \\ -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to N(A) = \left\{ c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

$$x_1 + \cancel{x_3} - 2\cancel{x_4} = 4$$
$$x_2 + \cancel{x_3} + 2\cancel{x_4} = -1$$
$$\qquad \quad 0 \qquad 0$$

$$\Rightarrow \begin{cases} x_1 = 4 \\ x_2 = -1 \end{cases}$$

$$\Rightarrow x_p = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_c = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}; \; c_1, c_2 \in \mathbb{R}$$

7. Answer ONLY ONE of the following questions:

- I) A man is three times as old as his son was at the time when the father was twice as old as his son will be two years from now. Find the present age of each person if the sum of their ages is 55.
- II) A movie star, unwilling to give his age, posed the following riddle to a gossip columnist. "9 years ago, I was 15 times as old as my daughter. Now I am 6 times as old as she is." How old are the star and his daughter?

I) M: man's age now
S: son's age now
n: years ago

$$\begin{cases} M = 3(S-n) \\ M-n = 2(S+2) \\ M+S = 55 \end{cases} \rightarrow \begin{array}{l} M-3S+3n=0 \\ M-2S-n=4 \\ M+S=55 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 1 & -2 & -1 & 4 \\ 1 & 1 & 0 & 55 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 4 & -3 & 55 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 13 & 39 \end{array}\right]$$

$$\rightarrow \begin{cases} M - 3S + 3n = 0 & \rightarrow M = 39 \\ S - 4n = 4 & \rightarrow S = 16 \\ 13n = 39 & \rightarrow n = 3 \end{cases}$$

Man is 39 years old and his son is 16 years old.

II) M: movie star's age
D: his daughter's age

$$\begin{array}{l} M-9 = 15(D-9) \\ M = 6D \end{array} \rightarrow \begin{array}{l} M-15D = -14\cdot 9 \\ M-6D = 0 \end{array}$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -15 & -14\cdot 9 \\ 1 & -6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -15 & -14\cdot 9 \\ 0 & 9 & +14\cdot 9 \end{array}\right] \rightarrow \begin{cases} M-15D = -14\cdot 9 \\ 9D = 14\cdot 9 \rightarrow D = 14 \end{cases}$$

$$M = 14\cdot 6 = 84$$

The moive star is 84 years old and his daughter is 14 years old.