Elementary Linear Algebra - MATH 2250 - Exam 3

Please read and sign (papers without printed name and signature will not be graded):
"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Print name: ______________________________ Sign: ______________________________

1. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

   (a) What are the eigenvalues of A? Explain.

   (b) What is the rank of A? Explain.

   (c) Compute in simplest form e^{tA}.

a) $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 \longrightarrow -\lambda^3 = 0 \Rightarrow \lambda = 0$ (multiplicity 3)

all the e-values of A are: 0, 0, 0.

b) $\mathrm{rref}(A) = \begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow$ 2 pivots $\Rightarrow r = 2$

c) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; for $k \geq 3$

$$\Rightarrow e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \frac{t^0A^0}{0!} + \frac{t^1A^1}{1!} + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + 0 + \cdots$$

$$= I + \begin{bmatrix} 0 & t & t \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t+t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

2. Consider the matrix $A = \begin{bmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ with parameter x in the $(1,1)$ position.

(a) Specify all numbers x, if any, for which A is positive definite. Explain.

(b) Specify all numbers x, for which e^A is positive definite. Explain.

a) $\begin{bmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow x > 0$

$\Rightarrow x \cdot 1 - 1 \cdot 1 = x - 1 > 0 \rightarrow x > 1$

$\Rightarrow x \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 1 - x \cdot 1 \cdot 1 - 1 \cdot 1 \cdot 1 - 1 \cdot 1 \cdot 1 = 0 > 0 \leftarrow$ impossible!

For no x, A is positive definite!

b) e^A is always positive definit, refer to Thursday's lecture.

$\Rightarrow$ All $x \in \mathbb{R}$ works

3. If A is symmetric, which of these four matrices are necessary positive definite. Explain.

$$A^3, (A^2+I)^{-1}, A+I, e^A.$$

P.D. P.D

Let λ be an e-val of A, ① then λ^3 is an e-val of A^3 and if $\lambda \leq 0$, then $\lambda^3 \leq 0$.

② then $\lambda^2+1 > 0$ is an e-val of A^2+I and $\frac{1}{\lambda^2+1} > 0$ is an e-val of $(A^2+I)^{-1}$ $\Rightarrow (A^2+I)^{-1}$ is P.D.

[all e-val's of $(A^2+I)^{-1}$ are $\frac{1}{\lambda^2+1} > 0$, where λ is an e-val of A.]

③ $\lambda+1$ is an e-val of $A+I$, and if $\lambda \leq -1 \Rightarrow \lambda+1 \leq 0$

④ $e^\lambda > 0$ is an e-val of e^A.

[all e-vals of e^A are e^λ, where λ is an eval of A]

4. P is a 3×3 permutation matrix. List all the possible eigenvalues of P.

$PP^T = I \Rightarrow$ evals of P are λ's s.t. $\lambda^2 = 1$

$\Rightarrow \boxed{\lambda = \pm 1}$

5. We are told that A is 2×2, symmetric, and Markov, and one of the real eigenvalues is y with $-1 < y < 1$.

(a) What is the matrix A in terms of y?

(b) Compute the eigenvectors of A.

(c) What is A^{2014} in simplest form?

a) Any markov matrix has an e-val 1, so the two e-vals of A are 1 and y. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, but A is symm.

So, $b = c \rightsquigarrow A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$. Also, A is Markov $\rightarrow \begin{cases} a+b=1 \\ b+d=1 \end{cases}$

$$\Longrightarrow A = \begin{bmatrix} a & 1-a \\ 1-a & a \end{bmatrix}$$

$$\text{trace}(A) = 2a = 1+y \Longrightarrow a = \frac{1+y}{2}, \quad 1-a = 1 - \frac{1+y}{2} = \frac{1-y}{2}$$

$$\Longrightarrow A = \begin{bmatrix} \frac{1+y}{2} & \frac{1-y}{2} \\ \frac{1-y}{2} & \frac{1+y}{2} \end{bmatrix}$$

b) The e-vector corresponding to 1: $A - I = \begin{bmatrix} \frac{1+y}{2} - 1 & \frac{1-y}{2} \\ \frac{1-y}{2} & \frac{1+y}{2} - 1 \end{bmatrix} = \begin{bmatrix} -\frac{1-y}{2} & \frac{1-y}{2} \\ \frac{1-y}{2} & -\frac{1-y}{2} \end{bmatrix}$

$$\text{rref}(A-I) \rightsquigarrow \begin{bmatrix} -\frac{1-y}{2} & \frac{1-y}{2} \\ 0 & 0 \end{bmatrix} \overset{\text{if } y \neq 1}{\rightsquigarrow} \left[\begin{array}{c|c} 1 & -1 \\ \hline 0 & 0 \end{array}\right] \rightsquigarrow N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(we knew this w/o calculating!)

e-vec corresponding to y: $A - yI = \begin{bmatrix} \frac{1+y}{2} - y & \frac{1-y}{2} \\ \frac{1-y}{2} & \frac{1+y}{2} - y \end{bmatrix} = \begin{bmatrix} \frac{1-y}{2} & \frac{1-y}{2} \\ \frac{1-y}{2} & \frac{1-y}{2} \end{bmatrix}$

$$\text{rref}(A-yI) \rightsquigarrow \begin{bmatrix} \frac{1-y}{2} & \frac{1-y}{2} \\ 0 & 0 \end{bmatrix} \xrightarrow{\text{if } y\neq 1} \left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & 0 \end{array}\right] \rightsquigarrow N = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The e-vectors are: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for y,

and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for 1.

[Note that the same e-vectors work for when $y=1$, i.e. when $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.]

c) $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix}$

$\Rightarrow A = S\Lambda S^{-1}$

$$\Rightarrow A^{2014} = S\Lambda^{2014}S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & y^{2014} \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & y^{2014} \\ 1 & -y^{2014} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+y^{2014} & 1-y^{2014} \\ 1-y^{2014} & 1+y^{2014} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+y^{2014}}{2} & \frac{1-y^{2014}}{2} \\ \frac{1-y^{2014}}{2} & \frac{1+y^{2014}}{2} \end{bmatrix}$$

(Did we expect it to be symmetric?)

6. (*Optional: extra credit*) Suppose C is $n \times n$ and positive definite. If A is $n \times m$ and $M = A^T CA$ is not positive definite, fine the smallest eigenvalues of M. Explain.

Solution. The smallest eigenvalue of M is 0.

The problem only asks for brief explanations, but to help students understand the material better, I will give lengthy ones.

First of all, note that $M^T = A^T C^T A = A^T CA = M$, so M is symmetric. That implies that all the eigenvalues of M are real. (Otherwise, the question wouldn't even make sense; what would the "smallest" of a set of complex numbers mean?)

Since we are assuming that M is *not* positive definite, at least one of its eigenvalues must be nonpositive. So, to solve the problem, we just have to explain why M cannot have any negative eigenvalues. The explanation is that M is **positive semidefinite**. That's the buzzword we were looking for.

Why is M positive semidefinite? Well, note that, since C is positive definite, we know that for every vector y in $\mathbb{R}^n$

$$y^T Cy \geqslant 0,$$

with equality if and only if y is the zero vector. Then, for any vector x in $\mathbb{R}^m$, we may set $y = Ax$, and see that

$$x^T Mx = x^T A^T CAx = (Ax)^T C(Ax) \geqslant 0. \tag{*}$$

Since M is symmetric, the fact that $x^T Mx$ is always non-negative means that M is positive semidefinite. Such a matrix never has negative eigenvalues. Why? Well, if M did have a negative eigenvalue, say $\lambda < 0$, with a corresponding eigenvector $v \neq 0$, then

$$v^T Mv = v^T(\lambda v) = \lambda v^T v = \lambda \|v\|^2 < 0,$$

which would contradict $(*)$ above, which is supposed to hold for *every* x in $\mathbb{R}^m$. $\square$