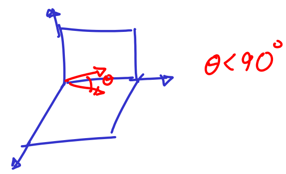
Elementary Linear Algebra - MATH 2250 - Day 13

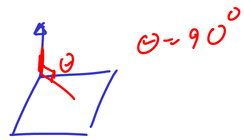
Name:

1. Mark each of the followings as True or False. In each case draw some pictures to clarify your answer.

   T F The xy-plane and the yz-plane are perpendicular to each other as vector spaces.



   T F The xy-plane and the z-axis are perpendicular to each other as vector spaces.



   T F The zero vector is perpendicular to any vector.

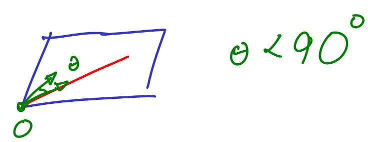
   T F Two planes through the origin are perpendicular to each other.

   As planes yes, as vector spaces in $\mathbb{R}^3$ no.
   as vector spaces in $\mathbb{R}^k$ could be, but not necessarily! $(k \geq 4)$

   T F Two planes through the origin could be perpendicular to each other.

   As planes yes, as vector spaces in $\mathbb{R}^3$ no,
   as vector spaces in $\mathbb{R}^k$ yes! $(k \geq 4)$

   T F A lines in the plane through the origin is perpendicular to that plane.



2. Let A be a matrix. Find the intersection of the null space of A and the row space of A.

   $N(A) \cap R(A) = \{0\}$ because if $v \in N(A) \cap R(A)$,

   then $\begin{bmatrix} \; \\ \boxed{\quad v \quad} \end{bmatrix} \begin{bmatrix} v \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, in particular $v \cdot v = 0 \Rightarrow v = 0$.

   [note that I'm taking v as a row of A, and not just "something" in the row space. Why is this enough?]

3. Let A be a matrix. Find the intersection of the left null space of A and the column space of A.

Same as above: conside A^T. [Or, similarly argue that if $\begin{bmatrix} & v & \end{bmatrix}\left[\ \boxed{v}\ \right]=0 \Rightarrow v=0$]

4. Give an example of two matrices A and B such that $AB = I$ but $BA \neq I$.

$A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$ $\qquad AB=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_{2\times2}$

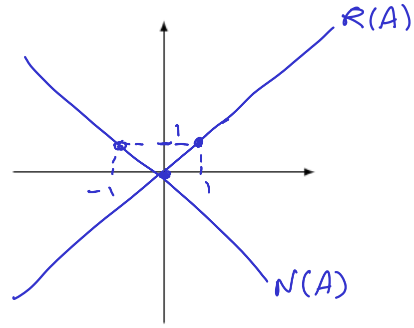
$B=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$ $\qquad BA=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\neq I_{3\times3}$

[You can delete any number of rows or col's of any invertible matrix and the corresponding col's or rows of its inverse, and it would work]

important Q: Are there any other examples? other than these?

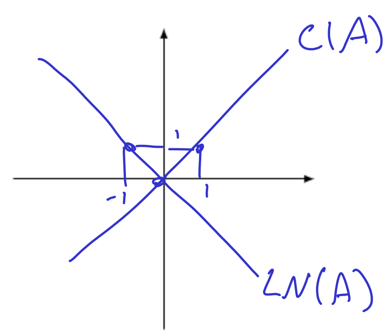
5. Consider the matrix $A = \begin{bmatrix}1&1\\1&1\end{bmatrix}$. "Draw" the row space and the ~~column~~ null space of A in one system of coordinates, $\mathbb{R}^2$.

Redo this problem for $\begin{bmatrix}1&1\\2&2\end{bmatrix}$.



What is the angle between the two spaces? 90°

"Draw" the column space and the left null space of A in one system of coordinates, $\mathbb{R}^2$.



What is the angle between the two spaces? 90°

6. Find a vector orthogonal to $v = (2, 2, -1)$.

$w=(a,b,c)$ $\qquad v\cdot w=0$

$(2,2,-1)\cdot(a,b,c)=2a+2b-c=0 \to$ one solution is $\begin{bmatrix}0\\0\\0\end{bmatrix}$, another is $\underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{w_1}, \underbrace{\begin{bmatrix}1/2\\0\\1\end{bmatrix}}_{w_2}$

7. Build a matrix whose row space has the basis $v = (2, 2, -1)$, and call it A.

$A=\begin{bmatrix}2&2&-1\end{bmatrix} \to \text{rref}(A)=\begin{bmatrix}1&1&-\frac{1}{2}\end{bmatrix}$

$N=\begin{bmatrix}-1&1/2\\1&0\\0&1\end{bmatrix}$

Find the null space of A Find the dot product of any vector in null space of A with v.

$$N(A)=\left\{c_1\begin{bmatrix}-1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1/2\\0\\1\end{bmatrix} : c_1,c_2\in\mathbb{R}\right\}$$

$w\in N(A) \Rightarrow w=c_1\underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{w_1}+c_2\underbrace{\begin{bmatrix}1/2\\0\\1\end{bmatrix}}_{w_2} \Rightarrow v\cdot w = v\cdot(c_1w_1+c_2w_2)=c_1(v\cdot w_1)+c_2(v\cdot w_2)=0+0=0$

Find another matrix whose row space has the basis $v = (2, 2, -1)$, and call it B. Then find the null space of B.

$B = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix} \rightsquigarrow rref(B) = \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \to N = \begin{bmatrix} -1 & 1/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$N(A) = \left\{ c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$

Find the dot product of any vector in null space of ~~A~~ B with v.

Same as A.

Find **all** the vectors that are orthogonal to v.

any $w \in N(A) = N(B)$.

8. Is it true that if x is orthogonal to v and to w, then x is perpendicular to $v + w$? Why?

if $\overline{x \cdot v = 0}$ $\overline{x \cdot w = 0}$ $\Rightarrow$ $\overline{x \cdot (v+w)} = x \cdot v + x \cdot w = 0 + 0 = 0$ ✓

Yes.

Is it true that if x is orthogonal to v, then x is perpendicular to cv, for any real number c? Why?

$\overline{\text{if } x \cdot v = 0}$ $\overline{x \cdot (cv)} = c(x \cdot v) = c \times 0 = 0$

Is the set of all vectors perpendicular to v a subspace?

Yes, because

9. Is there a matrix with 5 columns whose row space and null space have the same dimension? If yes, give an example, if no, why?

Nope, let $\dim(N(A)) = N$ and $\dim(R(A)) = r$ $\Rightarrow N + r = 5 \to$ not even!

But if $N = r \Rightarrow N + r$ is even!

10. Is there a matrix with 4 columns whose row space and null space have the same dimension? If yes, give an example, if no, why?

Yes

$$A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\underbrace{\quad}_{2 \text{ pivots}}$ $\underbrace{\quad}_{2 \text{ free var's}}$

$\dim R(A) = 2$

$\dim N(A) = 2$

$2+2=4$

11. Recall that each column of AB is a linear combination of the columns of A. Then, $\dim C(AB) \square \dim C(A)$. That is, column rank AB $\boxed{\leq}$ column rank A.

12. Recall that each row of AB is a linear combination of the rows of B. Then, $\dim R(AB) \square \dim R(A)$. That is, row rank AB $\boxed{\leq}$ row rank B.

13. Recall that for any matrix X, $\dim R(X) = \dim C(X)$ (why?). Then, row rank of X $\boxed{=}$ column rank of X. $=\text{rank}(X)$

#pivots $= r$

14. Using the results from problems 11–13,

$$\text{rank(AB)} \boxed{\leq} \text{rank(A)},$$

and

$$\text{rank(AB)} \boxed{\leq} \text{rank(B)}.$$

$$\Rightarrow \quad \text{rank}(AB) \leq \min\left(\text{rank}(A), \text{rank}(B)\right)$$

15. Give an example of two matrices A and B such that rank(AB) = rank(A) = rank(B).

$A = B = I$

16. Give an example of two matrices A and B such that rank(AB) < rank(A) **AND** rank(AB) < rank(B).

$$A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = \text{rank}(B) = 1$

but $\text{rank}(AB) = 0$