Elementary Linear Algebra - MATH 2250 - Day 2

Name:

Consider the following system and answer the following questions.

$$\begin{cases} x + 2y = 5 \\ -2x + 3y = 0 \end{cases}$$

1. T $\boxed{F}$ ✓ A pivot can be any number. What couldn't it be? Can't be zero.

   T $\boxed{F}$ ✓ for two matrices A and B always $AB = BA$. Give an example.

$$\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}, \quad \begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}$$

$$\begin{bmatrix}1&1\\2&2\end{bmatrix} \neq \begin{bmatrix}1&2\\1&2\end{bmatrix}$$

(Q: Let $A = \begin{bmatrix}1&0\\0&2\end{bmatrix}$, what are all the matrices B such that $AB = BA$?)

2. What are the two pivots of the above system after elimination? Show steps.

$$\begin{bmatrix}1&2\\-2&3\end{bmatrix} \longrightarrow \begin{bmatrix}1&2\\0&7\end{bmatrix} \rightsquigarrow \text{pivots are } 1, 7.$$

3. Does the elimination process for the system above fail or succeed? Why?

   It succeeds because there is no zeros in a pivot position.

4. Write down the augmented matrix for the above system and solve the system, using forward elimination and back substitution.

$$\left[\begin{array}{cc|c}1&2&5\\-2&3&0\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1&2&5\\0&7&10\end{array}\right] \longrightarrow \begin{cases} x+2y=5 \\ 7y=10 \end{cases} \Rightarrow y = 10/7 \qquad x + \frac{20}{7} = 5 \Rightarrow x = \frac{15}{7}$$

5. Find the elementary matrix $E_{3,1}$ that satisfies the following matrix multiplication:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-2&5&3&9\\0&6&-1&2\\2&3&0&1\\1&0&-6&7\end{bmatrix} \overset{\text{same}}{=} \begin{bmatrix}-2&5&3&9\\0&6&-1&2\\0&8&3&10\\1&0&-6&7\end{bmatrix}$$

6. What is the inverse of the matrix $E_{3,1}$ you found in the previous problem?

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7. What is the $(3,2)$-entry of the matrix M?

$$M = \begin{bmatrix} -2 & 5 & 3 & 9 \\ 0 & -6 & -1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & -6 & 7 \end{bmatrix} \begin{bmatrix} -2 & 5 & 3 & 9 \\ 0 & -6 & -1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & -6 & 7 \end{bmatrix}$$

$$2\cdot 5 + 3(-6) + 0\cdot 1 + 1\cdot 0 = -8$$

8. Do the following multiplications:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & 5 & 3 & 9 \\ 0 & -6 & -1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & -6 & 7 \end{bmatrix} =$$

$$\begin{bmatrix} -2 & 5 & 3 & 9 \\ 0 & -6 & -1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & -6 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} =$$

9. If the the columns of a matrix A lie in a plane, then they can be combined into $A\boldsymbol{x} = \mathbf{0}$, and then each row has $\boldsymbol{r} \cdot \boldsymbol{x} = \mathbf{0}$.

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and by rows:} \quad \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{x} \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} \\ \boldsymbol{r}_3 \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The three rows also lie in a plane. Why is that plane perpendicular to $\boldsymbol{x}$?

Any vector in the plane of r_1, r_2, r_3 is a linear combination of r_1, r_2, r_3. Assume w is in that plane. Then there are real numbers c, d, e such that $w = cr_1 + dr_2 + er_3$. Then $w \cdot x = (cr_1 + dr_2 + er_3)\cdot x = c\underbrace{r_1 \cdot x}_{=0} + d\underbrace{r_2 \cdot x}_{=0} + e\underbrace{r_3 \cdot x}_{=0} = 0 \Rightarrow x$ is perp to any vector in that plane, hence x is perp to the plane.

10. This system has no solution. The planes in the row picture don't meet at a point.

$$\begin{aligned} x + y + z &= 2 \\ x + 2y + z &= 3 \\ 2x + 3y + 2z &= 4 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = b$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 \end{array}$$

(a) Multiply the equations by $1, 1, -1$ and add to get $0 = 1$. No solution. Are any two of the planes parallel? (What are the equations of planes parallel to $x + y + z = 2$?) $\Rightarrow$ $x+y+z = c$, where c is any real number.

(b) Take the dot product of each column of A (and also $\boldsymbol{b}$) with $y = (1, 1, -1)$. How do those dot products show that the system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution?

If $Ax=b$ has a solution, then b is a linear combination of columns of A, that is, (why?)

$b = ca_1 + da_2 + ea_3$, for some real numbers c, d, e. Then

$$(1,1,-1)\cdot b = (1,1,-1)\cdot(ca_1+da_2+ea_3)$$

$$\underset{\displaystyle 1}{\shortparallel} \quad \neq \quad \underset{\displaystyle 0}{\shortparallel}$$

$\leftarrow$ contradiction.

(c) Find three right side vectors $\boldsymbol{b}^*$ and $\boldsymbol{b}^{**}$ and $\boldsymbol{b}^{***}$ tht do allow solutions.

Any linear combination of the columns

Let $b^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $b^{**} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $b^{***} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(Can you find more? How many?)

11. Find the matrix P that multiplies (x, y, z) to give (y, z, x).

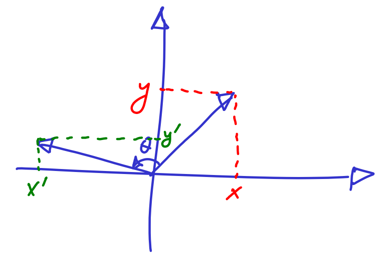
12. Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z).

13. What 2×2 matrix R rotates every vector by $90°$? (R times $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} y \\ -x \end{bmatrix}$.)

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(What can you tell about the matrix that rotates every vector θ degrees?)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R_\theta \begin{bmatrix} x \\ y \end{bmatrix}$$



14. Draw the row and columns pictures for the equations $x - 2y = 0$, $y + x = 6$.