Elementary Linear Algebra - MATH 2250 - Day 26

Name:

1. What is the length of a complex vector $v = (v_1, \ldots, v_n)$?

$$\sqrt{\bar{v}^T v} = \sqrt{\begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}} = \sqrt{\bar{v}_1 v_1 + \bar{v}_2 v_2 + \cdots + \bar{v}_n v_n}$$

$$= \sqrt{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}$$

$|z|$ = magnitude of the complex number $z = a + bi = \sqrt{a^2+b^2}$

2. What is a unitary matrix?

3. The n-th roots of unity are all the (complex) numbers z that $z^n = 1$. That is, all the solutions to the equation $z^n - 1 = 0$. First, recall that if $z^n = 1$, then $|z| = 1$, that is z lives on the unit circle. Also, note that any point on the unit circle can be written as $\cos(t) + \mathrm{i}\sin(\mathrm{t})$, for some $0 \le t < 2\pi$. Recall that $\cos(t) + \mathrm{i}\sin(\mathrm{t}) = \mathrm{e}^{\mathrm{it}}$. That is $z = e^{\mathrm{it}}$, and $z^n = 1$ if $(e^{\mathrm{it}})^n = 1$. But

$$1 = e^{2\pi i}$$

That is

$$z^n = (e^{\mathrm{it}})^n = 1 = e^{2\pi i}$$

Hence

$$z = e^{\mathrm{it}} = \underline{\qquad}.$$

$2\pi i/n$

The number $z = \underline{e}$ is called the *primitive* n-th root of unity, and us usually denoted by ω (read: omega). All the n-th roots of unity are powers of ω: $1, \omega, \omega^2, \ldots, \omega^{n-1}$.

Find all the 4-th roots of unity: $\omega^0, \omega, \omega^2, \omega^3$, and draw them in a complex plane.

$-1, 1, -i, i$

Check that $(\omega^i)^4 = 1$, for each $i = 1, \ldots, 3$.

4. Write F_4, the 4×4 Fourier matrix.

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

5. Find F_4^{-1}.

$$F_4^{-1} = \frac{1}{4} \overline{F_4}^T = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

6. Find F_2 and F_2^{-1}.

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad F_2^{-1} = \frac{1}{2} \overline{F_2}^T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

7. Let ω be the primitive 4-th root of unity, and $D = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$. Evaluate the matrix

$$\left[\begin{array}{c|c} I_2 & D \\ \hline I_2 & -D \end{array}\right] \left[\begin{array}{c|c} F_2 & O \\ \hline O & F_2 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Compare this with F_4.

They're equal