Elementary Linear Algebra - MATH 2250 - Day 25

Name:

1. The eigenvalues of a real symmetric matrix are real numbers. For example the eigenvalues of $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are 3 and 1. But the eigenvalues of $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ are $2+i$ and $2-i$.

2. Let $x = a + \mathrm{i}b$ be a complex number. Find $\bar{x}x$. What do you know about this quantity?

$\bar{x} = a - ib \rightarrow \bar{x}x = (a-ib)(a+ib) = a^2 + b^2$, is a non-negative real number.

3. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Find $\bar{x}^T x$. What do you know about this quantity?

$\begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \cdots + \bar{x}_n x_n$, is a sum of non-negative real numbers, hence it is a non-negative real number.

4. Let's see why are the eigenvalues of a real symmetric matrix are real. Recall that a number a is real if and only if its complex conjugate $\bar{a}$ is equal to a. Assume that λ is an eigenvalue of A with corresponding eigenvector x. Then

$$Ax = \lambda x. \quad (1)$$

Multiply both sides by $\bar{x}^T$ from left:

$$\bar{x}^T Ax = \lambda \bar{x} x. \quad (2)$$

Take the complex conjugate of both sides of (1):

$$\overline{Ax} = \overline{\lambda x}. \quad (3)$$

Since A is a real matrix, and $\overline{ab} = \bar{a}\bar{b}$:

$$A\bar{x} = \bar{\lambda}\bar{x}. \quad (4)$$

Take transpose of both sides:

$$(A\bar{x})^T = (\bar{\lambda}\bar{x})^T. \quad (5)$$

Simplify:

$$\bar{x}^T A^T = \bar{\lambda}\bar{x}^T. \quad (6)$$

But A is symmetric, that is $A^T = A$, hence

$$\bar{x}^T A = \bar{\lambda}\bar{x}^T. \quad (7)$$

Multiply both sides by x from right:

$$\bar{x}^T Ax = \bar{\lambda}\bar{x}^T x. \quad (8)$$

Compare (2) and (8):

$$\bar{x}^T \bar{\lambda} x = \lambda \bar{x}^T x. \quad (9)$$

But x is a nonzero vector (why?), so $\bar{x}^T x$ is a nonzero number. Divide both sides by $\bar{x}^T x$:

x is an e-vector

$$\bar{\lambda} = \lambda. \qquad (10)$$

So λ is real.

5. The eigenvectors of a real symmetric matrix can be chosen perpendicular.

6. For a (real or complex) matrix A if $\bar{A}^T = A$, then A is called to be a Hermitian matrix. Use problem 4 to show that the eigenvalues of A are real.

Similar proof: $Ax = \lambda x$

$\Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x \ (*)$

also: $\bar{x}^T \bar{A}^T = \bar{\lambda} \bar{x}^T$

$(\bar{A}^T = A)$ $\Rightarrow \bar{x}^T A = \bar{\lambda} \bar{x}^T$

$\Rightarrow \bar{x}^T A x = \bar{\lambda} \bar{x}^T x \ (**)$

$(*), (**) \Rightarrow \lambda \bar{x}^T x = \bar{\lambda} \bar{x}^T x$

$\Rightarrow \lambda = \bar{\lambda}$

$\Rightarrow \lambda \in \mathbb{R}$.

7. What can you tell about the eigenvalues of real skew-symmetric matrices? (A is skew-symmetric if $A^T = -A$.)

They are purely imaginary. Similar proof:

$Ax = \lambda x$

$\Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x \ (*)$

also: $\bar{x}^T \bar{A}^T = \bar{\lambda} \bar{x}^T$

$(A^T = -A$ and $\bar{A} = A)$ $\Rightarrow -\bar{x}^T A = \bar{\lambda} \bar{x}^T$

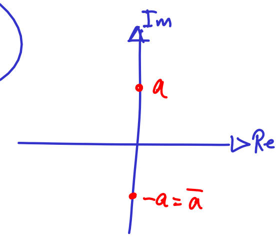
$\Rightarrow -\bar{x}^T A x = \bar{\lambda} \bar{x}^T x$

$\Rightarrow \bar{x}^T A x = -\bar{\lambda} \bar{x}^T x \ (**)$

$(*), (**) \Rightarrow \lambda \bar{x}^T x = -\bar{\lambda} \bar{x}^T x$

$\Rightarrow \lambda = -\bar{\lambda}$

happens only for purely imaginary numbers.



8. Let A be symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n$, with corresponding eigenvectors $q_1, \ldots, q_n$, such that $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ is orthonormal. Then $A = Q\Lambda Q^T$, where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T.$$

This is called the spectral decomposition of A.

9. Recall that the eigenvalues of a matrix are not the same as the pivots of it. But the signs of the eigenvalues of a matrix are the same as the signs of the pivots of it, and the product of the eigenvalues of a matrix is equal to the product of the pivots.

10. What is the determinant and the signs of the eigenvalues of $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$? Is it positive definite?

row reduce: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\rightarrow$ all pivots are positive

$\Rightarrow$ e-val's are positive $\Rightarrow$ matrix is PD.

product of pivots is 1 $\Rightarrow$ determinant is 1.