Elementary Linear Algebra - MATH 2250 - Exam 2

Please read and sign (papers without printed name and signature will not be graded):
"On my honor, I have neither given nor received unauthorized aid in doing this assignment."

Print name: ______________________ Sign: ______________________

1. Which of the following (if any) are subspaces. For any that are **not** subspaces give an example of how they violate a property of subspaces.

   (a) Given a 3×5 matrix with full row rank, the set of all solutions to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (b) All 3×5 matrices with $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ in their column space.

   (c) All 5×3 matrices with $(2, 1, 3)$ in their column space.

   (d) All vectors $\boldsymbol{x}$ with $||\boldsymbol{x} - \boldsymbol{y}|| = ||\boldsymbol{y}||$, for some given fixed vector $\boldsymbol{y} \neq \mathbf{0}$.

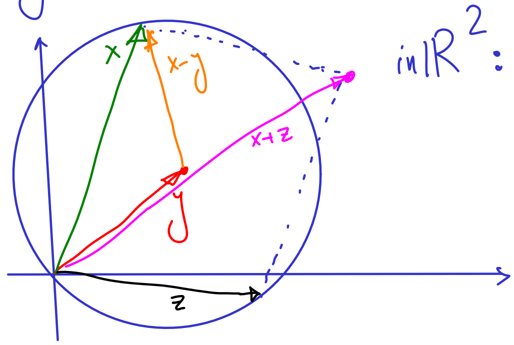
(a) Is not a V.S. because it doesn't include zero: $A0 \neq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) Is not a V.S. because it doesn't include $0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

doesn't have $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ in C.S.

(c) There is not such a matrix. So this space is empty. But a V.S. cannot be empty, since it has to include zero.

(d)



in $\mathbb{R}^2$: The set of all such vectors x is a circle centered at y, passing through origin. (it includes zero.)
But x+z is not on the circle, for some x, z on the circle, hence it's not a vector space.

2. (a) Find the matrix P that projects every vector $\boldsymbol{b}$ in $\mathbb{R}^3$ onto the line in the direction of $(1, 2, 3)$.

(b) Describe the Four fundamental subspaces of P by providing a basis for each of them.

(a) $$P = \frac{aa^T}{a^Ta} = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \rightsquigarrow \operatorname{rref}(P) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) $C(P) = \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\rangle$

$R(P) = \langle [1 \ \ 2 \ \ 3] \rangle$

$P^T = P$

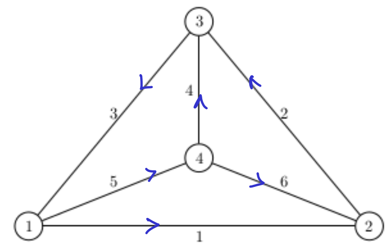
$N(P) = \left\langle \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\rangle$

$LN(P) = \langle [-2 \ \ 1 \ \ 0], [-3 \ \ 0 \ \ 1] \rangle$

OR: $$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \rightarrow EA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(A)$$

$\Rightarrow LN(P) = \langle [-2 \ \ 1 \ \ 0], [-3 \ \ 0 \ \ 1] \rangle$

3. Write down the 6×4 incidence matrix A of this graph. What is the dimension of the column space $C(A)$? Describe the null space $N(A)$.



$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \end{array}$$

rows 1,2,4 of A are lin. ind. $\leftarrow 3$

and $[1\ 1\ 1\ 1]^T$ is in the null space. $\leftarrow 1$

$3+1=4$

$\Rightarrow \begin{cases} \text{rank}(A) = \dim(\text{row space of } A) = 3 = \dim(\text{col space of } A). \\ \text{nullity of } A = 1 \end{cases}$

$N(A) = \left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle.$

4. (a) Consider the following data:

Year	US Population (million)
1900	70
1920	100
1940	130
1980	230

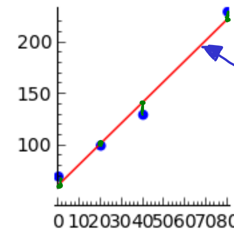
Suppose the population growth is linear, and you want to fit the best line $y = Cx + D$ to these values, where $x = 0$ represents the year 1900. What is the matrix A in the system $A\begin{bmatrix} C \\ D \end{bmatrix} = \boldsymbol{b}$? Find the best $\hat{C}, \hat{D}$, and the heights p_1, p_2, p_3, p_4 of that line $y = \hat{C}x + \hat{D}$ at years 1900, 1920, 1940, and 1980. What is the error vector $\boldsymbol{e}$? Show by numbers that e is perpendicular to $C(A)$.

(b) What is your estimate for the population in year 1960? 2000? 2020? 3000?

(a)

$$70 = D$$
$$100 = 20C + D$$
$$130 = 40C + D$$
$$230 = 80C + D$$

$$A = \begin{bmatrix} 0 & 1 \\ 20 & 1 \\ 40 & 1 \\ 80 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 70 \\ 100 \\ 130 \\ 230 \end{bmatrix}$$



$\boxed{A^TA\hat{x} = A^Tb}$ ← $\hat{x}$ is the best solution. Let's find it:

$$A^TA = \begin{bmatrix} 0 & 20 & 40 & 80 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 20 & 1 \\ 40 & 1 \\ 80 & 1 \end{bmatrix} = \begin{bmatrix} 8400 & 140 \\ 140 & 4 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 0 & 20 & 40 & 80 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 70 \\ 100 \\ 130 \\ 230 \end{bmatrix} = \begin{bmatrix} 25600 \\ 530 \end{bmatrix} \longrightarrow \begin{array}{l} \hat{C} = 141/70 \\ \hat{D} = 62 \end{array} \rightsquigarrow \boxed{y = \frac{141}{70}x + 62}$$

$$\rightsquigarrow p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 62 \\ 102.29 \\ 142.57 \\ 223.14 \end{bmatrix} \qquad e = b - p = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \approx \begin{bmatrix} 8 \\ -2.29 \\ -12.57 \\ 6.86 \end{bmatrix}$$

$$C(A) = \left\langle \underbrace{\begin{pmatrix} 0 \\ 20 \\ 40 \\ 80 \end{pmatrix}}_{=a_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{=a_2} \right\rangle, \text{ and}$$

$$a_1 \cdot e = 0 \cdot 8 + 20(-2.29) + 40(-12.57) + 80(6.86)$$
$$= 0.2 \approx 0$$

$$a_2 \cdot e = 8 - 2.29 - 12.57 + 6.86 = 0$$

(b)

x	60 (1960)	100 (2000)	120 (2020)	1100 (3000)
$y = \frac{141}{70}x + 62$ (million)	182.86	263.43	303.71	2277.71

5. Start with the two vectors (columns of A):

$$\boldsymbol{a}_1 = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix} \text{ and } \boldsymbol{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) With $\boldsymbol{q}_1 = \boldsymbol{a}_1$ find an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ for the space spanned by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (column space of A).

(b) What shape is the matrix R in $A = QR$ and why is $R = Q^T A$ (Here Q has columns $\boldsymbol{q}_1, \boldsymbol{q}_2$)? Compute R.

(c) Find the projection matrices P_A and P_Q onto the column spaces of A and Q.

(a) $a = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$, $b = a_2 - \mathrm{Proj}_{a_1} a_2 = a_2 - \frac{a_2^T a_1}{a_1^T a_1} a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{\sin\theta}{\underbrace{\sin^2\theta + \cos^2\theta}_{=1}} \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix} = \begin{bmatrix} 1-\sin^2\theta \\ 0 \\ -\sin\theta\cos\theta \end{bmatrix}$

$\rightarrow b = \begin{bmatrix} \cos^2\theta \\ 0 \\ -\sin\theta\cos\theta \end{bmatrix}$

$\|a\| = \sqrt{\sin^2\theta + \cos^2\theta} = 1 \rightsquigarrow q_1 = \frac{a}{\|a\|} = \begin{bmatrix} \sin\theta \\ 0 \\ \cos\theta \end{bmatrix}$

$\|b\| = \sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta} = \sqrt{\cos^2\theta(\sin^2\theta + \cos^2\theta)} = |\cos\theta| \rightsquigarrow q_2 = \frac{b}{\|b\|} = \begin{bmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{bmatrix}$

$= \cos\theta$

(assuming $0° < \theta < 90°$)

(b) R is alway upper triangular.

$R = Q^T A$, $Q = \begin{bmatrix} \sin\theta & \cos\theta \\ 0 & 0 \\ \cos\theta & -\sin\theta \end{bmatrix}$, $A = \begin{bmatrix} \sin\theta & 1 \\ 0 & 0 \\ \cos\theta & 0 \end{bmatrix}$

$A = QR$
$Q^T Q = I$
$Q^T A = Q^T Q R$
$\boxed{Q^T A = R}$

$\rightsquigarrow R = Q^T A = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} \sin\theta & 1 \\ 0 & 0 \\ \cos\theta & 0 \end{bmatrix} = \begin{bmatrix} 1 & \sin\theta \\ 0 & \cos\theta \end{bmatrix}$

(c) $P_A = A(A^TA)^{-1}A^T$, and $A^TA = \begin{pmatrix} 1 & \sin\theta \\ \sin\theta & 1 \end{pmatrix}$ and $(A^TA)^{-1} = \begin{pmatrix} \sec^2\theta & -\sec\theta\tan\theta \\ -\sec\theta\tan\theta & \sec^2\theta \end{pmatrix}$

$\Rightarrow P_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Did we expect them to be equal? What can you tell about the column space of A?

$P_Q = Q\underbrace{(Q^TQ)^{-1}}_{=I}Q^T = QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$