Elementary Linear Algebra - MATH 2250 - Day 21

Name:

1. T ~~F~~ If λ is an eigenvalue of A and μ is an eigenvalue of B then $\lambda + \mu$ is an eigenvalue of $A + B$. Explain.

2. ~~T~~ F If λ is an eigenvalue of A and μ is an eigenvalue of B then with the same eigenvector x, then $\lambda + \mu$ is an eigenvalue of $A + B$. Explain.

$Ax = \lambda x,\ Bx = \mu x$

$(A+B)x = Ax + Bx = \lambda x + \mu x = (\lambda + \mu)x$

3. Let $\boldsymbol{x} = (2, 3, 1)$ be an eigenvector of A corresponding to the eigenvalue 3. Evaluate Ax. $Ax = 3(2,3,1) = (6,9,3)$

4. The Fundamental Theorem of Algebra asserts that any polynomial of degree n has exactly n (complex) roots. How many eigenvalues does an $n \times n$ matrix have? Why?

n e-val's. Because the characteristic polynomial of an $n \times n$ matrix is a polynomial of degree n, and its roots are the e-val's of the matrix.

5. If A is singular then one of its eigenvalues is $\underline{0}$.

6. If P is a nonzero projection matrix in $\mathbb{R}^3$, then two of its eigenvalues are $\underline{1}$, and $\underline{0}$.

7. If λ is an eigenvalue of A, then $A - \lambda I$ is a(n) <u>singular</u> matrix.

8. What is the sum of the eigenvalues of the $n \times n$ identity matrix? n

9. What is the sum of the eigenvalues of $A = \text{diag}(d_1, \ldots, d_n)$? $d_1 + d_2 + \cdots + d_n$

10. Let's find (guess?) all the eigenvalues of $A = \text{diag}(d_1, \ldots, d_n)$. Let $\boldsymbol{e}_i$ be the vector with a 1 in its i-th position and 0's elsewhere, e.g. $\boldsymbol{e}_1 = (1, 0, 0, \ldots, 0)$ etc. What is $A\boldsymbol{e}_i$, for each i?

$Ae_i = d_i e_i \longrightarrow \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-th} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{-th}$

11. What is the trace (the sum of the eigenvalues) of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$? $1+2+6=9$

12. If λ is an eigenvalue of A, then $A - \lambda I$ is a(n) <u>singular</u> matrix.

13. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What is the characteristic equation of A?

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \longrightarrow (a-\lambda)(d-\lambda) - bc = 0 \longrightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\left(\Rightarrow \lambda^2 - tr(A)\,\lambda + \det(A) = 0 \right)$$

14. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$. What is the characteristic equation of A? What are all the eigenvalues of A? $\lambda^2 - 6\lambda + 4 = 0$

roots of

15. Let $\boldsymbol{x}$ be an eigenvector of A for an eigenvalue λ. is $2\boldsymbol{x}$ an eigenvector of A? For what eigenvalue?

Yes, $\underline{A(2x)} = 2(Ax) = 2(\lambda x) = \underline{\lambda(2x)}$, for the <u>same</u> e-value λ.

What are all the eigenvectors of A for the eigenvalue λ? (*Agreement: we do not consider the zero vector, and eigenvalue for any eigenvalue, not even for the zero eigenvalue!*)

cx, for $c \in \mathbb{R} \setminus \{0\}$

16. Find a matrix with eigenvalues $1, 2, 3$, and 4.

$$\begin{bmatrix} 1 & & & \\ & 2 & & 0 \\ & & 3 & \\ 0 & & & 4 \end{bmatrix}$$

17. Let A be a matrix with an eigenvalue λ and the corresponding eigenvector $\boldsymbol{x}$. Let $B = 2A$, and evaluate $B\boldsymbol{x}$.

$$Bx = 2(Ax) = 2(\lambda x) = (2\lambda)\, x$$

What can you tell about the eigenvalue of B in terms of the eigenvalues of A?

e-val's of B are 2 times e-val's of A.

18. Find an eigenvector for each of the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$.

19. Let A be as in problem 14, and let $B = A + 2I$. What are the eigenvalues of B?

... (Do calculations) ... $\Rightarrow$ (e-values of B) $=$ (e'values of A) $+ 2$

Find an eigenvector for each of the eigenvalues of B.

$Ax = \lambda x \Rightarrow Bx = (A + 2I)x = Ax + 2Ix = \lambda x + 2x = (\lambda + 2)x$

Same as e-vectors of A.

What relations hold between the eigenvalues and eigenvectors of A and B?

20. Find all the eigenvalues and their corresponding eigenvectors of $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. (refer to problem 10)

e-vec: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

e-val: $1 \quad 1$

21. Find all the eigenvalues and their corresponding eigenvectors of $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. Do eigenvectors of the matrix from problem 20 work?

$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \rightarrow$ not an e-vector

e-vals of $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ are roots of $\lambda^2 - 1 \rightarrow \pm 1$

e-vectors: $\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} \rightsquigarrow \begin{bmatrix}+1 & -1\\ 0 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \rightsquigarrow \begin{bmatrix}-1\\1\end{bmatrix}$

22. We are not going to prove this, but it is good to remember that

(a) the eigenvalues of any symmetric matrix are <u>real</u> numbers, and

(b) the eigenvalues of any skew-symmetric matrix are <u>purely imaginary</u> numbers.