Elementary Linear Algebra - MATH 2250 - Day 18

Name:

Use the 'big formula' to answer the following questions:

1. $\begin{vmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{vmatrix} =$ abc + (a bunch of zeros) = abc

2. $\begin{vmatrix} 0 & a & d \\ 0 & 0 & b \\ c & 0 & 0 \end{vmatrix}$ abc + (a bunch of zeros) = abc

3. $\begin{vmatrix} 0 & a & d \\ 0 & e & b \\ c & 0 & 0 \end{vmatrix}$ $abc + (-1)\,dec = abc - dec = (ab-de)c$

4. $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{vmatrix} =$ abcd

5. $\begin{vmatrix} 0 & 0 & e & a \\ 0 & f & b & 0 \\ g & c & 0 & 0 \\ d & 0 & 0 & h \end{vmatrix} = (+1)abcd + (-1)efgh = abcd - efgh$

6. How many 4×4 permutation matrices are there? What are their determinants?

   $4!$ ± 1

7. How many terms are in the 'big formula' for the determinant of a 4×4 matrix? What are their 'signs'? 24, ± 1

8. Let's go back to the pivot formula for determinant. Recall that if elimination turns A into U with $PA = LU$, where P is a permutation matrix, L is a lower triangular matrix and U is an upper triangular matrix with $d_1, d_2, \ldots, d_n$ in pivot positions, then $\det(L) =$ 1, $\det(P) =$ ± 1, and $\det(U) =$ $d_1 d_2 \cdots d_n$. So,

$$\det(A) = \pm d_1 d_2 \cdots d_n.$$

9. Using the big formula find $\begin{vmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{vmatrix} =$ aehj

10. Using the big formula find $\begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{vmatrix} =$ $1 \cdot 1 \cdot 1 \cdot c = c$

11. $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 4 \end{matrix} =$ $(-1)(-1)(1 \cdot 1 \cdot 1 \cdot 1) = 1$

12. $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} =$ $(-1)(-1)(1 \cdot 1 \cdot 1 \cdot 1) = 1$

13. Evaluate the followings using the big formula:

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\begin{vmatrix} e & f \\ g & h \end{vmatrix} = ef - gh$

$\begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{vmatrix} =$ $adeh - adfg - bceh + bcfg = (ad-bc)(ef-gh)$

14. Using the big formula evaluate $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} =$ $2 \cdot 2 - (-1)(-1) = 4 - 1 = 3$

15. Using the big formula evaluate $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} =$ $8 - 2(-1)(-1) - (-1)(-1)2 =$

$8 - 2 - 2 = 4$

16. Using the cofactor formula evaluate the determinant of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $\rightarrow 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$

$= 2(4-1) + (-2)$

$= 4$

Find A^{-1}.

Recall that the cofactor $C_{ij} = (-1)^{i+j} \det M_{ij}$. Find all the cofactors of the matrix A and put them in a matrix C.

Find AC^T.

Compare C^T with A^{-1}.

17. Using the cofactor formula evaluate $\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} =$

18. What formula would you use to evaluate the determinant $\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$?

Evaluate it.

19. Recall the big formula for the determinant of an $n \times n$ matrix:

$$\det(A) = \sum_{\text{all } n! \text{ permutations}} (\det P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}.$$

Using this formula, explain if you multiply each a_{ij} by the fraction $\frac{i}{j}$, why is $\det(A)$ unchanged?

We can factor an i from row i and a $1/j$ from row j.

$\det\left[\frac{i}{j} a_{ij}\right] = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n} \det[a_{ij}] = \det[a_{ij}]$.

20. Use cofactor formula to evaluate $\begin{vmatrix} a & b & c & d \\ e & 0 & 0 & 0 \\ f & 0 & 0 & 0 \\ g & 0 & 0 & 0 \end{vmatrix} = a\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} - b\begin{vmatrix} e & 0 & 0 \\ f & 0 & 0 \\ g & 0 & 0 \end{vmatrix} + c\begin{vmatrix} e & 0 & 0 \\ f & 0 & 0 \\ g & 0 & 0 \end{vmatrix} - d\begin{vmatrix} e & 0 & 0 \\ f & 0 & 0 \\ g & 0 & 0 \end{vmatrix}$

$= 0 - 0 + 0 - 0 = 0$

21. What is the rank of the matrix in problem 20? 0, 1, or 2.

22. Let the 4×4 Vandermonde matrix be $V_4 = \begin{bmatrix} \boxed{1} & a & a^2 & a^3 \\ 1 & \boxed{b} & b^2 & b^3 \\ 1 & c & \boxed{c^2} & c^3 \\ 1 & x & x^2 & \boxed{x^3} \end{bmatrix}$. Explain why the determinant of V_4 contains x^3, but not x^4 or x^5.

in the big formula there is 1, or x, or x^2 or x^3 in any term, but not a product of those, since they are all in the same row.

The determinant is zero at $x =$ a, b, and c. The cofactor of x^3 is $|V_3| = (b-a)(c-a)(c-b)$. Then $|V_4| =$ $(b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$

because if $x = a, b,$ or c, then there is a repeated row.