Elementary Linear Algebra - MATH 2250 - Quiz 14

Name:

1. $\boxed{\text{T}}$ $\boxed{\text{F}}$ The projection of $2\boldsymbol{a}$ onto $\boldsymbol{b}$ is equal to 2 times the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$. Give a detailed example.

Let $a=(1,0) \rightarrow 2a=(2,0)$
$b=(1,0)$

$v=\mathrm{Proj}_b\, a=(1,0)$, $w=\mathrm{Proj}_b\, 2a=(2,0)$, $w=2v$ ✓

2. $\boxed{\text{T}}$ $\boxed{\text{F}}$ The projection of $\boldsymbol{a}$ onto $2\boldsymbol{b}$ is equal to 2 times the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$. Give a detailed example.

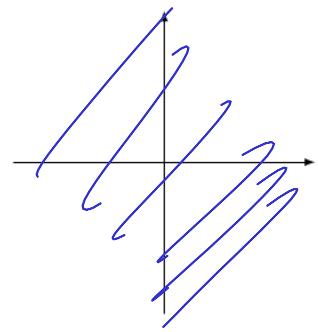
a, b: as above.

$v=\mathrm{Proj}_b\, a=(1,0)$, $w=\mathrm{Proj}_{2b}\, a=(1,0)$, $w\neq 2v$

$\boxed{\Rightarrow v=w.}$

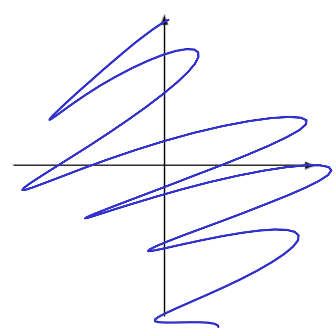
3. Let $\boldsymbol{a}=(1,1,1,1)$ and $\boldsymbol{b}=(1,-1,1,-1)$.
Find $\boldsymbol{p}$, the projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$, ~~and draw all three vectors.~~

$p=\mathrm{Proj}_b\, a=0$, because $a\cdot b=0$, $a\perp b$.



4. Let $\boldsymbol{a}=(1,1,1,1)$ and $\boldsymbol{b}=(2,2,2,2)$. Find $\boldsymbol{q}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$, ~~and draw all three vectors.~~

$q=\mathrm{Proj}_a\, b=\frac{8}{4}(1,1,1,1)=(2,2,2,2)=b$



5. If $\boldsymbol{b}=c\boldsymbol{a}$, for a real number c, then projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is ____b____.

6. If $\boldsymbol{b}=c\boldsymbol{a}$, for a real number c, then projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is ____a____.

7. Let $\boldsymbol{a} = (1, 2, 0, 2)$.

(a) Find $\boldsymbol{a}^T\boldsymbol{a}$. Is it nonzero?

Yes.

$$a^Ta = 1+4+0+4 = 9 \neq 0$$

(b) Find $\boldsymbol{a}\boldsymbol{a}^T$. What is its rank?

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 4 \end{bmatrix}, \quad \text{rank}(A) = 1.$$

(c) Find the projection matrix P that projects every vector onto $\boldsymbol{a}$. What is its rank?

$$P = \frac{aa^T}{a^Ta} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 4 \end{bmatrix}.$$

(d) Find the column space of P.

$$C(P) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} : c_1 \in \mathbb{R} \right\}$$

(e) Is P symmetric?

Yes, $P^T = P$

(f) If you find the vector $P\boldsymbol{b}$ for some vector $\boldsymbol{b}$, where do you expect it to live? Be as precise as possible.

Pb is the projecton of b onto a, so it will be a multiple of a. Hence I expect it to live on the line along a.

(g) Let $\boldsymbol{b} = (2, 0, 3, 6)$. Find $\boldsymbol{a}^T\boldsymbol{b}$, and $\cancel{\frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}\boldsymbol{a}^T}}$. $\frac{a^Tb}{a^Ta}$

$$a^Tb = \begin{bmatrix} 1 & 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2+0+0+12 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$$

$$a^Tb / a^Ta = \begin{bmatrix} 14/9 \end{bmatrix}$$

(h) Find $P\boldsymbol{b}$.

$$Pb = \frac{1}{9}\begin{bmatrix} 1 & 2 & 0 & 2 \\ 2 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 3 \\ 6 \end{bmatrix} = \frac{14}{9}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

(i) $\underbrace{\text{Note that } P\boldsymbol{b} \text{ is a multiple of } \boldsymbol{a}}_{\text{duh! didn't we expect that?!}}$. What multiple is it?

$\frac{14}{9}$.

Wait a minute, if all we need to find Pb is the $\frac{14}{9} = \frac{a^Tb}{a^Ta}$, why do we even bother finding P?

(j) Find $P^2\boldsymbol{b}$.

$P^2 = P \Rightarrow P^2 b = Pb = \frac{14}{9} a.$

do the calculations

(k) What do you expect about $P^3\boldsymbol{b}$, $P^4\boldsymbol{b}$ etc?

Same! $P = P^2 = P^3 = P^4 \Rightarrow Pb = P^2 b = P^3 b = P^4 b = \cdots$

(l) Is $P^2 = P$?

Yes!

(m) What about P^3, P^4 and P^5?

Yes!

if $P^2 = P$, multiply both sides with P again and you'll get $P^3 = P^2$, but $P^2 = P \Rightarrow P^3 = P^2 = P$ ...

8. Consider the equation $A\boldsymbol{x} = \boldsymbol{b}$, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. Does it have a solution?

Nope!

$$\left[\begin{array}{cc|c} 1 & 0 & 6 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & -6 \\ \hline 0 & 0 & 6 \end{array}\right]$$ no solution.

Find $B = A^T A$.

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

Find $\cancel{\boldsymbol{p}}\ \hat{b} = A^T\boldsymbol{b}$.

$$\hat{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Solve the system $B\hat{\boldsymbol{x}} = \cancel{\boldsymbol{p}}\ \hat{b}$.

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & 3 & 6 \\ 0 & 2 & -6 \end{array}\right] \rightarrow \hat{x}_2 = -3 \rightarrow \hat{x}_1 = 5 \Rightarrow \hat{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Is $(A^TA)^{-1}$ invertible? Why? Find its inverse.

Yes, because A has ind. col's.

$$(A^TA)^{-1} = B^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}$$

Find $(A^TA)^{-1}A^T\boldsymbol{b}$, and compare it with $\hat{\boldsymbol{x}}$.

$$(A^TA)^{-1}A^T b = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \hat{x}$$

Find $A(A^TA)^{-1}A^T\boldsymbol{b}$, ~~and compare it with $\boldsymbol{p}$~~. $=p$

$$p = A(A^TA)^{-1}A^T b = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

Find $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$.

$$e = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Find $\boldsymbol{b} - A\hat{\boldsymbol{x}}$, and compare it with $\boldsymbol{e}$.

$$\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = e$$

Find $A^T\boldsymbol{e}$. Is $\boldsymbol{e}$ orthogonal to $C(A)$? Why?

$$A^T e = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0.$$

Yes, it is, because e is the part of b which is "orthogonal" to the column space of A.

9. What is the projection matrix for projection onto column space of A? Is it symmetric? What is P^2?

$$P = A(A^TA)^{-1}A^T, \text{ Yes: } P^T = \left(A(A^TA)^{-1}A^T\right)^T = A^{TT}\left(\underbrace{(A^TA)^{-1}}_{}\right)^T A^T = A(A^TA)^{-1}A^T = P$$

$$((A^TA)^T)^{-1} = (A^TA)^{-1}$$

$$P^2 = \left(A\cancel{(A^TA)^{-1}A^T}\right)\left(\cancel{A}(A^TA)^{-1}A^T\right) = A(A^TA)^{-1}A^T = P.$$

10. Draw a picture similar to the one in page 221 of the book for Problem 8.