Elementary Linear Algebra - MATH 2250 - Day 24

Name:

1. What is a Markov matrix?

2. If A is a Markov matrix, then the largest eigenvalue of A is unique and it is 1.

3. Find all the eigenvalues of $A = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}$.

   Find all the eigenvectors of A.

4. Find all the eigenvalues of $A = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0.7 & 0.3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   Same as $\begin{bmatrix} .3 & .7 \\ .7 & .3 \end{bmatrix}$ and 1.

   Find all the eigenvectors of A.

5. Is $A - I$ for the matrix A in problem 3 singular? Yes, because 1 is an e-val of A.

6. In order to find the eigenvalues of the transpose of a matrix A we start with the polynomial equation

$$\det(A^T - \lambda I) = 0.$$

Recall that $\det(A^T) = \det(\underline{\ A\ })$. Then $\det(A^T - \lambda I) = \underline{\det(A - \lambda I)} = 0$. Thus the eigenvalues of A^T are the same as the eigenvalues of $\underline{\ A^T\ }$.

7. $\boxed{T}$ $\boxed{F}$ If the rows of a square matrix are linearly dependent, then the matrix is singular. Why?

We get a zero pivot while row reducing the matrix.

8. Is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ a Markov matrix? How many of its eigenvalues have absolute value equal to 1? 2 of them!

9. ("Everybody moves") Start with three groups of people, and at each time step, half of group 1 goes to group 2 and the other half goes to group 3. The other groups also split in half and move. Write down the matrix A that represents one step move, that is, $A \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ represents the population after one time step.

$$A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ \frac{1}{2} & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Find A^2.

Find the eigenvalues of A and A^2.

Find an eigenvector x_1 (the steady state) of A.

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Start with population $u_0 = (8, 16, 32)$, evaluate the states u_1, u_2, and u_3.

$u_1 = A u_0 = \cdots$

$u_2 = A u_1 = \cdots$

$u_3 = A u_2 = \cdots$

What is the sum of each vector u_i? How do you explain this in terms of the population?

56, the total population doesn't change!

What is the population of each group (approximately) after 10000 time steps? Why?

It will be close to the steady state. So, it is a multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$, but $a + a + a = 56 \Rightarrow a = 56/3$

$\Rightarrow u_{10000} \simeq \begin{bmatrix} 56/3 \\ 56/3 \\ 56/3 \end{bmatrix}$

10. (Perron-Frobenius Theorem) Let A be a matrix with all positive entries, and λ be the maximum eigenvalue of A with corresponding eigenvector x. Then $\lambda \boxed{>} 0$, and all numbers in x are positive.

11. For two function f and g then the inner product of f and g is

$$(f, g) = \int_a^b f \cdot g \qquad dx$$

Then the length squared of f is

$$||f||^2 = \int ||f||^2 \, dx$$