Elementary Linear Algebra - MATH 2250 - Day 8

Name:

1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & -3 & -4 & -5 \end{bmatrix}$.

   Find all the right hand sides $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ has a solution (the solvability condition).

reduce : $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 1 & 2 & 3 & 4 & b_2 \\ -2 & -3 & -4 & -5 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & 3 & b_2-b_1 \\ 0 & 0 & 0 & 0 & b_3+b_2+b_1 \end{array}\right] \rightarrow$ condition: $b_3+b_2+b_1=0$

All right hand sides: $\left\{ \begin{bmatrix} b_1 \\ b_2 \\ -b_2-b_1 \end{bmatrix}, \ b_1, b_2 \in \mathbb{R} \right\}$

2. Fill in the blank: For $A\boldsymbol{x} = \boldsymbol{b}$ to have a solution, If a combination of rows of A gives the zero row, then <u>the same combination</u> of entries of $\boldsymbol{b}$ must be <u>zero</u>.

3. what is rank of A? 2 $\rightarrow$ there are 2 pivots

4. Find all the solutions to $A\boldsymbol{x} = \boldsymbol{0}$, for A given in problem 1 (that is, find the null space of A).

$R = \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow N = \begin{bmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow N(A) = \left\{ c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$

5. Using the results of problem 1, does the equation $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ have a solution?

$1+0+(-1)=0$ ✓ yes, it does.

6. Find a particular solution to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ by letting the free variables equal to zero (that is, a $x_{\text{particular}}$).

reduce $\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$

$x_1 + 0 - \cancel{x_3} - \cancel{x_4} = 2$

$0 + x_2 + 2\cancel{x_3} + 3\cancel{x_4} = -1$

$\underbrace{x_3 = 0,\ x_4 = 0}_{\text{free}}$

$\rightarrow x_1 = 2$, $x_2 = -1$

$\rightarrow x_p = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

7. Find all the solutions to $A\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ (that is, the x_{complete}).

$$X_c = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

8. Find all the solutions to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

① $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to N = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \to N(A) = \left\{ c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$

② $x_p = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

③ $x_c = x_p + x_n$; $x_n \in N(A) \Rightarrow X_c = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $c_1, c_2 \in \mathbb{R}$

9. If an $m \times n$ matrix A has full column rank, then how many free variables are there? why?

none, each col has a pivot.

What is $N(A)$, the null space of A? Explain.

It's $\left\{ \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1} \right\} = Z$, because there is no free col's.

How many solution are there for $A\boldsymbol{x} = \boldsymbol{b}$?

At most 1; $\begin{cases} \text{if } b \in C(A), \text{ exactly one.} \\ \text{if } b \notin C(A), \text{ none.} \end{cases}$

10. Give an example of a 3×2 matrix that has full column rank, and call it A.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

What is the rref of A?

$\text{rref}(A) = A$

Find all the right hand sides $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ has a solution. Then pick one of them and call it $\boldsymbol{c}$.

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 \end{array}\right] \longrightarrow b_3 = 0 \Rightarrow b = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}; \; b_1, b_2 \in \mathbb{R}; \quad c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find all the solutions to $A\boldsymbol{x} = \boldsymbol{c}$

$$x_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad N(A) = Z \Rightarrow x_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

11. Give an example of a 2×3 matrix that has full column rank. Explain your thoughts.

Not possible, to have full col rank it needs 3 pivots, but there's only two rows.

12. Give an example of a 2×3 matrix that has full row rank, and call it A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Find all the right hand sides $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \end{array}\right] \rightarrow \text{no zero rows,}$$

Any $b \in \mathbb{R}^2$ works.

How many free variables are there?

1

How many solutions are there for $A\boldsymbol{x} = \boldsymbol{0}$? *Describe* all of them.

∞'ly many. $\quad N(A) = \left\{ c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : c \in \mathbb{R} \right\}$

all sol'ns: $\quad x_n = c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \; c \in \mathbb{R}$

Describe all the solutions to $A\boldsymbol{x} = \boldsymbol{b}$ for your favorite nonzero $\boldsymbol{b}$.

$$b = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \longrightarrow x_c = \underbrace{\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}}_{x_p} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \; c \in \mathbb{R}$$

13. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$. Find the reduced row echelon form for A.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the rank of A. 4

Is A invertible? Explain. Yes, there are 4 pivots.

or $C(A) = \mathbb{R}^4$

or $N(A) = Z$

$\}$ ← for square matrices either one is enough to be invertible.

How many free variables are there for A?

none.

What is the null space of A?

$$Z = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Find all the right hand sides $\boldsymbol{b}$ such that $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

all of $\mathbb{R}^4$, i.e. $\{b : b \in \mathbb{R}^4\}$

How many solutions are there for $A\boldsymbol{x} = \boldsymbol{b}$ for any $\boldsymbol{b}$?

One, since $N(A) = Z$

14. Before watching the next video, go back and watch the last 5 minutes of the previous video (Lecture 8) and make sure that you understand it well. Write what's on the last board below.

Do this!

15. ‘Here lies Diophantus,’
the wonder behold.

Through art algebraic,
the stone tells how old:

‘God gave him his boyhood
one-sixth of his life,

One twelfth more as youth
while whiskers grew rife;

And then yet one-seventh
ere marriage begun;

In five years there came
a bouncing new son.

Alas, the dear child of master
and sage After attaining half the measure of his father’s life
chill fate took him.

After consoling his fate
by the science of numbers for four years,
he ended his life.’

Stated in prose, the poem says that Diophantus’s youth lasts 1/6 of his life. He grew a beard after 1/12 more of his life. After 1/7 more of his life, Diophantus married. Five years later, he had a son. The son lived exactly half as long as his father, and Diophantus died just four years after his son’s death. All of this totals the years Diophantus lived. [1]

Let D: # years that Diophantus lived.
S: # years that his son lived.

How many year Diophantus and his son lived, each?

$$\begin{cases} \frac{1}{6}D + \frac{1}{12}D + \frac{1}{7}D + 5 + S + 4 = D \\ \frac{1}{2}D = S \end{cases}$$

$$\rightarrow \begin{cases} \frac{51}{84}D - S = 9 \\ \frac{1}{2}D - S = 0 \end{cases} \rightarrow \text{solve for } D, S \rightarrow \ldots$$

$$\rightarrow \begin{cases} D = 84 \\ S = 42 \end{cases}$$

[1] Adapted from: Weisstein, Eric W. “Diophantus’s Riddle.” From MathWorld–A Wolfram Web Resource. `http://mathworld.wolfram.com/DiophantussRiddle.html`.