Elementary Linear Algebra - MATH 2250 - Day 22

Name:

1. ~~Find the eigenvalues and the eigenvectors of a triangular matrix: Once with distinct eigenvalues and once with equal eigenvalues.~~

2. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find the eigenvalues of A and their corresponding eigenvectors. Call these eigenvalues λ_1 and λ_2, and the corresponding eigenvectors v_1 and v_2.

$$\lambda_1 = 2 \longrightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1$$
$$\lambda_2 = 0 \longrightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v_2$$

Let $S = \left[\begin{array}{c|c} v_1 & v_2 \end{array} \right]$. Evaluate AS.

$$AS = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$$

Let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$. Evaluate $S\Lambda$.

$$S\Lambda = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$$

Evaluate S^{-1}.

$$S^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Evaluate $S^{-1}AS$.

$$S^{-1}AS = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \Lambda$$

Evaluate $S\Lambda S^{-1}$.

$$S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = A$$

Find the echelon form of A, and multiply its pivots.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to 1 \cdot 0 = 0$$

Multiply the eigenvalues of A, and compare it with the product of the pivots.

$$2 \cdot 2 = 0 = 1 \cdot 0$$

Is it true that the pivots of A are the same as eigenvalues of A?

Nope!

3. Using the same matrix A as in problem ~~3~~ 2, recall that $A = S\Lambda S^{-1}$. Evaluate A^2, A^3, and A^4. What are the eigenvalues of A^2, A^3, A^4?

$$A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1}) = S(\Lambda^2)S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \cdots$$

$$A^3 = S\Lambda^3 S^{-1} = \cdots$$

$$A^4 = S\Lambda^4 S^{-1} = \cdots$$

e-val's of A^n are $\underline{0}$ and $\underline{2^n}$.

4. What are the eigenvalues of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Is the matrix diagonalizable? What is the diagonlized A?

1, 1, 1. yes, it is already diagonal. I

5. What are the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. 1, 1, 1.

What are the algebraic multiplicities of the eigenvalues of A?

1 has alg. mult. 3.

$x^3 - 1 = 0 \rightarrow x = 1$ (3 times!)

What are the geometric multiplicities of the eigenvalues of A? 1

$$A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow N(A-I)$$

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow N = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Is the matrix diagonalizable? Why?

No! because the e-vectors are not lin. ind.

6. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. Find λ and x such that $Ax = \lambda x$.

$$\lambda_1 = 1, \quad \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow N = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2, \quad \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow N = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 3, \quad \begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow N = \begin{bmatrix} 3/2 \\ 2 \\ 1 \end{bmatrix} \rightarrow x_3 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

Switch two rows of A to get B. Is it true that $Bx = \lambda x$?

$$B = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad B \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad \text{NO!}$$

What are all the eigenvalues of B?

$\ldots \rightarrow -1, 2, 3$

7. If the all the eigenvalues of $A_{n\times n}$ are distinct, then the eigenvectors of A are linearly independent, and hence they form a basis for $\mathbb{R}^n$. That is, every vector $v \in \mathbb{R}^n$ can be written as a linear combination of the eigenvectors of A.

8. Is it true that if A is diagonalizable, then the eigenvalues of A are distinct? Check it for the identity matrix.

Nope! e.g. I has all e-val's $= 1$

but it is diag'ble.

9. If A diagonalizable, is it true that the diagonalized A is the same as rref(A)? Give an example.

Nope! $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

10. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.