Elementary Linear Algebra - MATH 2250 - Day 17

Name:

1. T F $\det(A+B) = \det(A) + \det(B)$, for any two matrices A and B.
   T F $\det(cA) = c\det(A)$, for any real number c and any matrix A.
   T F $\det(A) = 0$, if and only if $A = O$, the zero matrix.
   T F $\det(P) = 1$, for any permutation matrix P.
   T F $\det(A) = \det(R)$, where R is the reduced row echelon form of A, with no row-exchanges.
   T F $\det(AB) = \det(A)\det(B)$, for any two matrices A and B.
   T F $\det(A^{-1}) = \det(A)$.
   T F $\det(I + A) = 1 + \det(A)$.
   T F $\det(ABC) = \det(A)\det(B)\det(C)$.
   T F $\det(2A) = 2\det(A)$.
   T F $\det(AB) = \det(BA)$.
   T F $\det(A^n) = \det(A)^n$
   T F $\det(A) = \det(A^T)$.
   T F If A is not invertible, then AB is not invertible

   $$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}}_{B}=\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{AB}$$
   (it's true for square matrices, though!)

   T F The determinant of A is always the product of its pivots.
   T F $\det(A-B) = \det(A) - \det(B)$.
   T F $\det(cA) = c^n\det(A)$, for any $n \times n$ matrix A. Why?

2. What are the main three properties of the determinant? Mention briefly.

1) $\det(I) = 1$.
2) If B is obtained from A by 1 row exchange, then $\det(B) = -\det(A)$.
3) (multilinearity): $\det\begin{bmatrix} cA_1 + dA_1' \\ \hline A_2 \\ \hline \vdots \\ \hline A_n \end{bmatrix} = c\cdot\det\begin{bmatrix} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_n \end{bmatrix} + d\cdot\det\begin{bmatrix} A_1' \\ \hline A_2 \\ \hline \vdots \\ \hline A_n \end{bmatrix}$

3. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1$ then $\begin{vmatrix} a & b & c \\ -d & -e & -f \\ g & h & i \end{vmatrix} = -1$

4. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1$, and $\begin{vmatrix} a & b & c \\ d & e & f \\ g' & h' & i' \end{vmatrix} = 2$, then $\begin{vmatrix} a & b & c \\ d & e & f \\ g+g' & h+h' & i+i' \end{vmatrix} = 1+2=3$

5. $\begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ g & h & i \end{vmatrix} = 0$

~~6.~~ $\cancel{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1,}$

7. If U is the matrix obtained from A by reducing it, $\det(A) = \pm\det(U)$. Explain when it is $+$ and when it is $-$.

it's "$-$"; If there are odd number of row exchanges to get from A to U

"$+$" otherwise

8. $\begin{vmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & i & j \\ 0 & 0 & 0 & k \end{vmatrix} = aeik$

9. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -1$

10. If $\det(A_{3\times 3}) = 2$, then $\det(2A) = 2^3 \det(A) = 2^3 \cdot 2 = 16$

11. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

12. $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$

13. $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -\left(-\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}\right) = -(-1) = 1$

14. Let $A = LU$ be the LU-decomposition of A. Then $\det(A) = \overbrace{\det(L)}^{=1}\det(U) = \det(U)$

not that this means no row exchanges are there, since otherwise $PA = LU$.

15. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} =$

$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$

16. $\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{vmatrix} = \cancel{2}\cdot\cancel{\frac{3}{2}}\cdot\cancel{\frac{4}{3}}\cdot\frac{5}{\cancel{4}} = 5.$

17. We want to show that for ant orthogonal matrix Q $\det(Q) = \pm 1$. In order to show this, recall that

$$Q^T Q = \underline{\ I\ }.$$

Also, recall that

$$\det(AB) = \underline{\det A \ \det B}.$$

Then

$$\det(Q^T Q) = \det(\underline{Q^T})\det(\underline{Q}).$$

Furthermore, recall that

$$\det(A^T) = \underline{\det A},$$

and

$$\det(I) = \underline{\ 1\ }.$$

Hence

$$\det(Q^T Q) = \underline{(\det Q)(\det Q)} = (\det Q)^2 = 1 \Rightarrow \det Q = \pm 1.$$

18. Let $PAP^{-1} = \Lambda = \text{diag}(1, 2, -2, 3)$ be a diagonal matrix with $1, 2, -2$ and 3 on the main diagonal. Find $\det(A)$.

$PAP^{-1} = \Lambda \Rightarrow A = P^{-1}\Lambda P \Rightarrow \det(A) = \det(P^{-1}\Lambda P) = \det P^{-1} \det\Lambda \det P$

But $\det(P^{-1}) = \frac{1}{\det P} \Rightarrow \det(A) = \frac{1}{\cancel{\det P}} \det\Lambda \cancel{\det P} = \det\Lambda = 1\cdot 2\cdot(-2)\cdot 3 = -12.$

Find $\det(A^2)$.

$\det(A^2) = (\det(A))^2 = (-12)^2 = 144.$

Find $\det(A^n)$, for $n \geq 1$.

$\det(A^n) = (\det(A))^n = (-12)^n$

~~19. The commutator of A and B is $[A, B] = AB - BA$. Find $\det([A, B])$.~~

20. From $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$, find $A - xI$, where x is a real number. Which two numbers x lead to $\det(A - xI) = 0$?

$A - xI = \begin{bmatrix} 4-x & 1 \\ 2 & 3-x \end{bmatrix}$, $\det(A - xI) = (4-x)(3-x) - 1\cdot 2 = x^2 - 7x + 10 = 0$

$\Rightarrow x = 5, 2$