Elementary Linear Algebra - MATH 2250 - Day 4

Name:

1. Let A, B, and C be invertible, and A^{-1}, B^{-1}, and C^{-1} be their inverses, respectively. What is the inverse of ABC, in terms of A^{-1}, B^{-1}, and C^{-1}?

   $(ABC)^{-1} =$ $C^{-1}B^{-1}A^{-1}$

   Check your solution.

   $$(ABC)(C^{-1}B^{-1}A^{-1}) = A(B(\underbrace{CC^{-1}}_{=I})B^{-1})A^{-1} = A(\underbrace{BIB^{-1}}_{=I})A^{-1} = AIA^{-1} = I \checkmark$$

2. Let

   $$M = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}.$$

   What is the inverse of M? (Hint: note that M is the product of three elementary matrices.) How can you

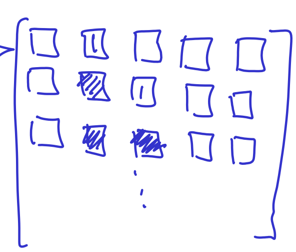
   $$M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \cdots & \end{bmatrix}$$

   check your solution without finding M? Check your solution. - - -

3. How many 4×4 permutation matrices are there? $4! = 24$

4. How many 5×5 permutation matrices are there? Explain

   $5! = 120$

   ← 5 choices for 1.

   ← 4 choices for 1.

   ⋮

5. If $A = LU$ is the LU-decomposition of A, for a lower triangular matrix L and an upper triangular matrix U, then to solve $A\boldsymbol{x} = \boldsymbol{b}$, one can solve $LU\boldsymbol{x} = \boldsymbol{b}$, by solving $L\boldsymbol{y} = b$ first, and then $U\boldsymbol{x} = \boldsymbol{y}$. Solve the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$, using LU-decomposition of A when

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\vdots$

$L = \begin{bmatrix} \quad \end{bmatrix}, \quad U = \begin{bmatrix} \quad \end{bmatrix},$

$Ax = b$

$(LU)x = b$

$L(Ux) = b$

call $Ux = y$

$\Rightarrow Ly = b \rightarrow$ solve for y $\Delta[\;] = b$

$\Rightarrow Ux = y \rightarrow$ solve for x. $\nabla[\;] = y$

6. Find the LU-decomposition of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What do you expect about the LU-decomposition of a lower-triangular matrix?

Lower triangular entries of L equal those of A, U is diagonal and its diag. entries equal those of A

What about the LU-decomposition of an upper-triangular matrix?

$L = I, \quad U = A.$

7. Find the LU-decomposition of $A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$.

$\vdots$

8. Find the inverse of

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

make these = zero

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to$$

make these = zero

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to$$

make this = zero

$$\to \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}}_{A^{-1}} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solve $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \to x = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$

9. Find A^2, A^3, A^4, and A^5 for $A = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

$$A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = A^5 = 0_{4\times 4}$$

10. What rows or columns or matrices do you multiply to find

(a) the third column of AB? A times the 3rd col of B.

(b) the first row of AB? first row of A times B.

(c) the entry in row 3, column 4 of AB row 3 of A times col 4 of B.

(d) the entry in row 1, column 1 of CDE? (first row of C times D) times first col of E.

or first row of C times (D times first col of E)

11. Compute:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a+b+c & a+b+c & a+b+c \\ d+e+f & d+e+f & d+e+f \\ g+h+i & g+h+i & g+h+i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a+d+g & b+e+h & c+f+i \\ a+d+g & b+e+h & c+f+i \\ a+d+g & b+e+h & c+f+i \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}_{1\times 3} \begin{bmatrix} a+b+c \\ d+e+f \\ g+h+i \end{bmatrix}_{3\times 1} = \begin{bmatrix} a+b+c+d+e+f+g+h+i \end{bmatrix}_{1\times 1}$$