INTERMEDIATE MICROECONOMICS

NINTH EDITION

HAL R. VARIAN

Intermediate Microeconomics

A Modern Approach

Ninth Edition

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A Modern Approach

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Hal R. Varian

University of California at Berkeley

W. W. Norton & Company • New York • London

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NINTH EDITION

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ISBN 978-0-393-12396-8

W. W. Norton & Company, Inc., 500 Fifth Avenue, New York, N.Y. 10110
W. W. Norton & Company, Ltd., Castle House, 75/76 Wells Street, London W1T 3QT
www.wwnorton.com

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To Carol

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PREFACE

The success of the first eight editions of *Intermediate Microeconomics* has pleased me very much. It has confirmed my belief that the market would welcome an analytic approach to microeconomics at the undergraduate level.

My aim in writing the original text was to present a treatment of the methods of microeconomics that would allow students to apply these tools on their own and not just passively absorb the predigested cases described in the text. I have found that the best way to do this is to emphasize the fundamental conceptual foundations of microeconomics and to provide concrete examples of their application rather than to attempt to provide an encyclopedia of terminology and anecdote.

A challenge in pursuing this approach arises from the lack of mathematical prerequisites for economics courses at many colleges and universities. The lack of calculus and problem-solving experience in general makes it difficult to present some of the analytical methods of economics. However, it is not impossible. One can go a long way with a few simple facts about linear demand and supply functions, and some elementary algebra. It is perfectly possible to be analytical without being excessively mathematical.

The distinction is worth emphasizing. An analytical approach to economics is one that uses rigorous, logical reasoning. This does not necessarily require the use of advanced mathematical methods. The language of mathematics certainly helps to ensure a rigorous analysis and using it is undoubtedly the best way to proceed when possible, but it may not be appropriate for all students.

Many undergraduate majors in economics are students who *should* know calculus, but don't—at least, not very well. For this reason I have kept calculus out of the main body of the text. However, I have provided complete calculus appendices to many of the chapters. This means that the calculus methods are there for the students who can handle them, but they do not pose a barrier to understanding for the others.

I think that this approach manages to convey the idea that calculus is not just a footnote to the argument of the text, but is instead a deeper way to examine the same issues that one can also explore verbally and graphically. Many arguments are much simpler with a little mathematics, and all economics students should learn that. In many cases I've found that with a little motivation, and a few nice economic examples, students become quite enthusiastic about looking at things from an analytic perspective.

For students who are comfortable with calculus, I also offer a version of the text that incorporates the material in the chapter appendices into the body of chapters.

There are several other innovations in this text. First, the chapters are generally very short. I've tried to make most of them roughly "lecture size," so that they can be read in one sitting. I have followed the standard order of discussing first consumer theory and then producer theory, but I've spent a bit more time on consumer theory than is normally the case. This is not because I think that consumer theory is necessarily the most important part of microeconomics; rather, I have found that this is the material that students find the most mysterious, so I wanted to provide a more detailed treatment of it.

Second, I've tried to put in a lot of examples of how to use the theories described here. In most books, students look at a lot of diagrams of shifting curves, but they don't see much algebra, or much calculation of any sort for that matter. But it is the algebra that is used to solve problems in practice. Graphs can provide insight, but the real power of economic analysis comes in calculating quantitative answers to economic problems. Every economics student should be able to translate an economic story into an equation or a numerical example, but all too often the development of this skill is neglected. For this reason I have also provided a workbook that I feel is an integral accompaniment to this book. The workbook was written with my colleague Theodore Bergstrom, and we have put a lot of effort into generating interesting and instructive problems. We think that it provides an important aid to the student of microeconomics.

Third, I believe that the treatment of the topics in this book is more accurate than is usually the case in intermediate micro texts. It is true that I've sometimes chosen special cases to analyze when the general case is too difficult, but I've tried to be honest about that when I did it. In general, I've tried to spell out every step of each argument in detail. I believe that the discussion I've provided is not only more complete and more

accurate than usual, but this attention to detail also makes the arguments easier to understand than the loose discussion presented in many other books.

There Are Many Paths to Economic Enlightenment

There is more material in this book than can comfortably be taught in one semester, so it is worthwhile picking and choosing carefully the material that you want to study. If you start on page 1 and proceed through the chapters in order, you will run out of time long before you reach the end of the book. The modular structure of the book allows the instructor a great deal of freedom in choosing how to present the material, and I hope that more people will take advantage of this freedom. The following chart illustrates the chapter dependencies.

The darker colored chapters are "core" chapters—they should probably be covered in every intermediate microeconomics course. The lighter-colored chapters are "optional" chapters: I cover some but not all of these every semester. The gray chapters are chapters I usually don't cover in my course, but they could easily be covered in other courses. A solid line going from Chapter A to Chapter B means that Chapter A should be read before

chapter B. A broken line means that Chapter B requires knowing some material in Chapter A, but doesn't depend on it in a significant way.

I generally cover consumer theory and markets and then proceed directly to producer theory. Another popular path is to do exchange right after consumer theory; many instructors prefer this route and I have gone to some trouble to make sure that this path is possible.

Some people like to do producer theory before consumer theory. This is possible with this text, but if you choose this path, you will need to supplement the textbook treatment. The material on isoquants, for example, assumes that the students have already seen indifference curves.

Much of the material on public goods, externalities, law, and information can be introduced earlier in the course. I've arranged the material so that it is quite easy to put it pretty much wherever you desire.

Similarly, the material on public goods can be introduced as an illustration of Edgeworth box analysis. Externalities can be introduced right after the discussion of cost curves, and topics from the information chapter can be introduced almost anywhere after students are familiar with the approach of economic analysis.

Changes for the Ninth Edition

I have added a new chapter on measurement which describes some of the issues involved in estimating economic relationships. The idea is to introduce the student to some basic concepts from econometrics and try to bridge the theoretical treatment in the book with the practical problems encountered in practice.

I have offered some new examples drawn from Silicon Valley firms such as Apple, eBay, Google, Yahoo, and others. I discuss topics such as the complementarity between the iPod and iTunes, the positive feedback associated with companies such as Facebook, and the ad auction models used by Google, Microsoft, and Yahoo. I believe that these are fresh and interesting examples of economics in action.

I've also added an extended discussion of mechanism design issues, including two-sided matching markets and the Vickrey-Clarke-Groves mechanisms. This field, which was once primarily theoretical in nature, has now taken on considerable practical importance.

The Test Bank and Workbook

The workbook, *Workouts in Intermediate Microeconomics,* is an integral part of the course. It contains hundreds of fill-in-the-blank exercises that lead the students through the steps of actually applying the tools they have learned in the textbook. In addition to the exercises, *Workouts* contains a collection of short multiple-choice quizzes based on the workbook problems in each chapter. Answers to the quizzes are also included in *Workouts.*

These quizzes give a quick way for the student to review the material he or she has learned by working the problems in the workbook.

But there is more . . . instructors who have adopted *Workouts* for their course can make use of the *Test Bank* offered with the textbook. The *Test Bank* contains several alternative versions of each *Workouts* quiz. The questions in these quizzes use different numerical values but the same internal logic. They can be used to provide additional problems for students to work on, or to give quizzes to be taken in class. Grading is quick and reliable because the quizzes are multiple choice and can be graded electronically.

In our course, we tell the students to work through all the quiz questions for each chapter, either by themselves or with a study group. Then during the term we have a short in-class quiz every other week or so, using the alternative versions from the *Test Bank*. These are essentially the *Workouts* quizzes with different numbers. Hence, students who have done their homework find it easy to do well on the quizzes.

We firmly believe that you can't learn economics without working some problems. The quizzes provided in *Workouts* and in the *Test Bank* make the learning process much easier for both the student and the teacher.

A hard copy of the *Test Bank* is available from the publisher, as is the textbook's *Instructor's Manual*, which includes my teaching suggestions and lecture notes for each chapter of the textbook, and solutions to the exercises in *Workouts*.

A number of other useful ancillaries are also available with this textbook. These include a comprehensive set of PowerPoint slides, as well as the Norton Economic News Service, which alerts students to economic news related to specific material in the textbook. For information on these and other ancillaries, please visit the homepage for the book at `http://www.wwnorton.com/varian`.

The Production of the Book

The entire book was typeset by the author using TEX, the wonderful typesetting system designed by Donald Knuth. I worked on a Linux system and using GNU `emacs` for editing, `rcs` for version control and the TEX Live system for processing. I used `makeindex` for the index, and Trevor Darrell's `psfig` software for inserting the diagrams.

The book design was by Nancy Dale Muldoon, with some modifications by Roy Tedoff and the author. Jack Repchek coordinated the whole effort in his capacity as editor.

Acknowledgments

Several people contributed to this project. First, I must thank my editorial assistants for the first edition, John Miller and Debra Holt. John provided

many comments, suggestions, and exercises based on early drafts of this text and made a significant contribution to the coherence of the final product. Debra did a careful proofreading and consistency check during the final stages and helped in preparing the index.

The following individuals provided me with many useful suggestions and comments during the preparation of the first edition: Ken Binmore (University of Michigan), Mark Bagnoli (Indiana University), Larry Chenault (Miami University), Jonathan Hoag (Bowling Green State University), Allen Jacobs (M.I.T.), John McMillan (University of California at San Diego), Hal White (University of California at San Diego), and Gary Yohe (Wesleyan University). In particular, I would like to thank Dr. Reiner Buchegger, who prepared the German translation, for his close reading of the first edition and for providing me with a detailed list of corrections. Other individuals to whom I owe thanks for suggestions prior to the first edition are Theodore Bergstrom, Jan Gerson, Oliver Landmann, Alasdair Smith, Barry Smith, and David Winch.

My editorial assistants for the second edition were Sharon Parrott and Angela Bills. They provided much useful assistance with the writing and editing. Robert M. Costrell (University of Massachusetts at Amherst), Ashley Lyman (University of Idaho), Daniel Schwallie (Case-Western Reserve), A. D. Slivinskie (Western Ontario), and Charles Plourde (York University) provided me with detailed comments and suggestions about how to improve the second edition.

In preparing the third edition I received useful comments from the following individuals: Doris Cheng (San Jose), Imre Csekó (Budapest), Gregory Hildebrandt (UCLA), Jamie Brown Kruse (Colorado), Richard Manning (Brigham Young), Janet Mitchell (Cornell), Charles Plourde (York University), Yeung-Nan Shieh (San Jose), and John Winder (Toronto). I especially want to thank Roger F. Miller (University of Wisconsin), and David Wildasin (Indiana) for their detailed comments, suggestions, and corrections.

The fifth edition benefited from the comments by Kealoah Widdows (Wabash College), William Sims (Concordia University), Jennifer R. Reinganum (Vanderbilt University), and Paul D. Thistle (Western Michigan University).

I received comments that helped in preparation of the sixth edition from James S. Jordon (Pennsylvania State University), Brad Kamp (University of South Florida), Sten Nyberg (Stockholm University), Matthew R. Roelofs (Western Washington University), Maarten-Pieter Schinkel (University of Maastricht), and Arthur Walker (University of Northumbria).

The seventh edition received reviews by Irina Khindanova (Colorado School of Mines), Istvan Konya (Boston College), Shomu Banerjee (Georgia Tech), Andrew Helms (University of Georgia), Marc Melitz (Harvard University), Andrew Chatterjea (Cornell University), and Cheng-Zhong Qin (UC Santa Barbara).

Finally, I received helpful comments on the eighth edition from Kevin Balsam (Hunter College), Clive Belfield (Queens College, CUNY), Reiner Buchegger (Johannes Kepler University), Lars Metzger (Technische Universitaet Dortmund), Jeffrey Miron (Harvard University), Babu Nahata (University of Louisville), and Scott J. Savage (University of Colorado).

Berkeley, California
December 2013

CHAPTER 1

THE MARKET

The conventional first chapter of a microeconomics book is a discussion of the "scope and methods" of economics. Although this material can be very interesting, it hardly seems appropriate to *begin* your study of economics with such material. It is hard to appreciate such a discussion until you have seen some examples of economic analysis in action.

So instead, we will begin this book with an *example* of economic analysis. In this chapter we will examine a model of a particular market, the market for apartments. Along the way we will introduce several new ideas and tools of economics. Don't worry if it all goes by rather quickly. This chapter is meant only to provide a quick overview of how these ideas can be used. Later on we will study them in substantially more detail.

1.1 Constructing a Model

Economics proceeds by developing **models** of social phenomena. By a model we mean a simplified representation of reality. The emphasis here is on the word "simple." Think about how useless a map on a one-to-one

scale would be. The same is true of an economic model that attempts to describe every aspect of reality. A model's power stems from the elimination of irrelevant detail, which allows the economist to focus on the essential features of the economic reality he or she is attempting to understand.

Here we are interested in what determines the price of apartments, so we want to have a simplified description of the apartment market. There is a certain art to choosing the right simplifications in building a model. In general we want to adopt the simplest model that is capable of describing the economic situation we are examining. We can then add complications one at a time, allowing the model to become more complex and, we hope, more realistic.

The particular example we want to consider is the market for apartments in a medium-size midwestern college town. In this town there are two sorts of apartments. There are some that are adjacent to the university, and others that are farther away. The adjacent apartments are generally considered to be more desirable by students, since they allow easier access to the university. The apartments that are farther away necessitate taking a bus, or a long, cold bicycle ride, so most students would prefer a nearby apartment . . . if they can afford one.

We will think of the apartments as being located in two large rings surrounding the university. The adjacent apartments are in the inner ring, while the rest are located in the outer ring. We will focus exclusively on the market for apartments in the inner ring. The outer ring should be interpreted as where people can go who don't find one of the closer apartments. We'll suppose that there are many apartments available in the outer ring, and their price is fixed at some known level. We'll be concerned solely with the determination of the price of the inner-ring apartments and who gets to live there.

An economist would describe the distinction between the prices of the two kinds of apartments in this model by saying that the price of the outer-ring apartments is an **exogenous variable**, while the price of the inner-ring apartments is an **endogenous variable**. This means that the price of the outer-ring apartments is taken as determined by factors not discussed in this particular model, while the price of the inner-ring apartments is determined by forces described in the model.

The first simplification that we'll make in our model is that all apartments are identical in every respect except for location. Thus it will make sense to speak of "the price" of apartments, without worrying about whether the apartments have one bedroom, or two bedrooms, or whatever.

But what determines this price? What determines who will live in the inner-ring apartments and who will live farther out? What can be said about the desirability of different economic mechanisms for allocating apartments? What concepts can we use to judge the merit of different assignments of apartments to individuals? These are all questions that we want our model to address.

1.2 Optimization and Equilibrium

Whenever we try to explain the behavior of human beings we need to have a framework on which our analysis can be based. In much of economics we use a framework built on the following two simple principles.

The optimization principle: People try to choose the best patterns of consumption that they can afford.

The equilibrium principle: Prices adjust until the amount that people demand of something is equal to the amount that is supplied.

Let us consider these two principles. The first is *almost* tautological. If people are free to choose their actions, it is reasonable to assume that they try to choose things they want rather than things they don't want. Of course there are exceptions to this general principle, but they typically lie outside the domain of economic behavior.

The second notion is a bit more problematic. It is at least conceivable that at any given time peoples' demands and supplies are not compatible, and hence something must be changing. These changes may take a long time to work themselves out, and, even worse, they may induce other changes that might "destabilize" the whole system.

This kind of thing can happen ... but it usually doesn't. In the case of apartments, we typically see a fairly stable rental price from month to month. It is this *equilibrium* price that we are interested in, not in how the market gets to this equilibrium or how it might change over long periods of time.

It is worth observing that the definition used for equilibrium may be different in different models. In the case of the simple market we will examine in this chapter, the demand and supply equilibrium idea will be adequate for our needs. But in more general models we will need more general definitions of equilibrium. Typically, equilibrium will require that the economic agents' actions must be consistent with each other.

How do we use these two principles to determine the answers to the questions we raised above? It is time to introduce some economic concepts.

1.3 The Demand Curve

Suppose that we consider all of the possible renters of the apartments and ask each of them the maximum amount that he or she would be willing to pay to rent one of the apartments.

Let's start at the top. There must be someone who is willing to pay the highest price. Perhaps this person has a lot of money, perhaps he is

very lazy and doesn't want to walk far . . . or whatever. Suppose that this person is willing to pay $500 a month for an apartment.

If there is only one person who is willing to pay $500 a month to rent an apartment, then if the price for apartments were $500 a month, exactly one apartment would be rented—to the one person who was willing to pay that price.

Suppose that the next highest price that anyone is willing to pay is $490. Then if the market price were $499, there would still be only one apartment rented: the person who was *willing* to pay $500 would rent an apartment, but the person who was willing to pay $490 wouldn't. And so it goes. Only one apartment would be rented if the price were $498, $497, $496, and so on . . . until we reach a price of $490. At that price, exactly two apartments would be rented: one to the $500 person and one to the $490 person.

Similarly, two apartments would be rented until we reach the maximum price that the person with the *third* highest price would be willing to pay, and so on.

Economists call a person's maximum willingness to pay for something that person's **reservation price**. The reservation price is the highest price that a given person will accept and still purchase the good. In other words, a person's reservation price is the price at which he or she is just indifferent between purchasing or not purchasing the good. In our example, if a person has a reservation price p it means that he or she would be just indifferent between living in the inner ring and paying a price p and living in the outer ring.

Thus the number of apartments that will be rented at a given price p^* will just be the number of people who have a reservation price greater than or equal to p^*. For if the market price is p^*, then everyone who is willing to pay at least p^* for an apartment will want an apartment in the inner ring, and everyone who is not willing to pay p^* will choose to live in the outer ring.

We can plot these reservation prices in a diagram as in Figure 1.1. Here the price is depicted on the vertical axis and the number of people who are willing to pay that price or more is depicted on the horizontal axis.

Another way to view Figure 1.1 is to think of it as measuring how many people would want to rent apartments at any particular price. Such a curve is an example of a **demand curve**—a curve that relates the quantity demanded to price. When the market price is above $500, zero apartments will be rented. When the price is between $500 and $490, one apartment will be rented. When it is between $490 and the third highest reservation price, two apartments will be rented, and so on. The demand curve describes the quantity demanded at each of the possible prices.

The demand curve for apartments slopes down: as the price of apartments decreases more people will be willing to rent apartments. If there are many people and their reservation prices differ only slightly from person to

The demand curve for apartments. The vertical axis measures the market price and the horizontal axis measures how many apartments will be rented at each price.

Figure 1.1

person, it is reasonable to think of the demand curve as sloping smoothly downward, as in Figure 1.2. The curve in Figure 1.2 is what the demand curve in Figure 1.1 would look like if there were many people who want to rent the apartments. The "jumps" shown in Figure 1.1 are now so small relative to the size of the market that we can safely ignore them in drawing the market demand curve.

1.4 The Supply Curve

We now have a nice graphical representation of demand behavior, so let us turn to supply behavior. Here we have to think about the nature of the market we are examining. The situation we will consider is where there are many independent landlords who are each out to rent their apartments for the highest price the market will bear. We will refer to this as the case of a **competitive market**. Other sorts of market arrangements are certainly possible, and we will examine a few later.

For now, let's consider the case where there are many landlords who all operate independently. It is clear that if all landlords are trying to do the best they can and if the renters are fully informed about the prices the landlords charge, then the equilibrium price of all apartments in the inner ring must be the same. The argument is not difficult. Suppose instead that there is some high price, p_h, and some low price, p_l, being charged

Figure 1.2 **Demand curve for apartments with many demanders**. Because of the large number of demanders, the jumps between prices will be small, and the demand curve will have the conventional smooth shape.

for apartments. The people who are renting their apartments for a high price could go to a landlord renting for a low price and offer to pay a rent somewhere between p_h and p_l. A transaction at such a price would make both the renter and the landlord better off. To the extent that all parties are seeking to further their own interests and are aware of the alternative prices being charged, a situation with different prices being charged for the same good cannot persist in equilibrium.

But what will this single equilibrium price be? Let us try the method that we used in our construction of the demand curve: we will pick a price and ask how many apartments will be supplied at that price.

The answer depends to some degree on the time frame in which we are examining the market. If we are considering a time frame of several years, so that new construction can take place, the number of apartments will certainly respond to the price that is charged. But in the "short run"—within a given year, say—the number of apartments is more or less fixed. If we consider only this short-run case, the supply of apartments will be constant at some predetermined level.

The **supply curve** in this market is depicted in Figure 1.3 as a vertical line. Whatever price is being charged, the same number of apartments will be rented, namely, all the apartments that are available at that time.

Short-run supply curve. The supply of apartments is fixed in the short run.

Figure 1.3

1.5 Market Equilibrium

We now have a way of representing the demand and the supply side of the apartment market. Let us put them together and ask what the equilibrium behavior of the market is. We do this by drawing both the demand and the supply curve on the same graph in Figure 1.4.

In this graph we have used p^* to denote the price where the quantity of apartments demanded equals the quantity supplied. This is the **equilibrium price** of apartments. At this price, each consumer who is willing to pay at least p^* is able to find an apartment to rent, and each landlord will be able to rent apartments at the going market price. Neither the consumers nor the landlords have any reason to change their behavior. This is why we refer to this as an *equilibrium:* no change in behavior will be observed.

To better understand this point, let us consider what would happen at a price other than p^*. For example, consider some price $p < p^*$ where demand is greater than supply. Can this price persist? At this price at least some of the landlords will have more renters than they can handle. There will be lines of people hoping to get an apartment at that price; there are more people who are willing to pay the price p than there are apartments. Certainly some of the landlords would find it in their interest to raise the price of the apartments they are offering.

Similarly, suppose that the price of apartments is some p greater than p^*.

Figure 1.4 **Equilibrium in the apartment market**. The equilibrium price, p^*, is determined by the intersection of the supply and demand curves.

Then some of the apartments will be vacant: there are fewer people who are willing to pay p than there are apartments. Some of the landlords are now in danger of getting no rent at all for their apartments. Thus they will have an incentive to lower their price in order to attract more renters.

If the price is above p^* there are too few renters; if it is below p^* there are too many renters. Only at the price of p^* is the number of people who are willing to rent at that price equal to the number of apartments available for rent. Only at that price does demand equal supply.

At the price p^* the landlords' and the renters' behaviors are compatible in the sense that the number of apartments demanded by the renters at p^* is equal to the number of apartments supplied by the landlords. This is the equilibrium price in the market for apartments.

Once we've determined the market price for the inner-ring apartments, we can ask who ends up getting these apartments and who is exiled to the farther-away apartments. In our model there is a very simple answer to this question: in the market equilibrium everyone who is willing to pay p^* or more gets an apartment in the inner ring, and everyone who is willing to pay less than p^* gets one in the outer ring. The person who has a reservation price of p^* is just indifferent between taking an apartment in the inner ring and taking one in the outer ring. The other people in the inner ring are getting their apartments at less than the maximum they would be willing to pay for them. Thus the assignment of apartments to renters is determined by how much they are willing to pay.

1.6 Comparative Statics

Now that we have an economic model of the apartment market, we can begin to use it to analyze the behavior of the equilibrium price. For example, we can ask how the price of apartments changes when various aspects of the market change. This kind of an exercise is known as **comparative statics**, because it involves comparing two "static" equilibria without worrying about how the market moves from one equilibrium to another.

The movement from one equilibrium to another can take a substantial amount of time, and questions about how such movement takes place can be very interesting and important. But we must walk before we can run, so we will ignore such dynamic questions for now. Comparative statics analysis is only concerned with comparing equilibria, and there will be enough questions to answer in this framework for the present.

Let's start with a simple case. Suppose that the supply of apartments is increased, as in Figure 1.5.

Increasing the supply of apartments. As the supply of apartments increases, the equilibrium price decreases.

Figure 1.5

It is easy to see in this diagram that the equilibrium price of apartments will fall. Similarly, if the supply of apartments were reduced the equilibrium price would rise.

Let's try a more complicated—and more interesting—example. Suppose that a developer decides to turn several of the apartments into condominiums. What will happen to the price of the remaining apartments?

Your first guess is probably that the price of apartments will go up, since the supply has been reduced. But this isn't necessarily right. It is true that the supply of apartments to rent has been reduced. But the *demand for apartments* has been reduced as well, since some of the people who were renting apartments may decide to purchase the new condominiums.

It is natural to assume that the condominium purchasers come from those who already live in the inner-ring apartments—those people who are willing to pay more than p^* for an apartment. Suppose, for example, that the demanders with the 10 highest reservation prices decide to buy condos rather than rent apartments. Then the new demand curve is just the old demand curve with 10 fewer demanders at each price. Since there are also 10 fewer apartments to rent, the new equilibrium price is just what it was before, and exactly the same people end up living in the inner-ring apartments. This situation is depicted in Figure 1.6. Both the demand curve and the supply curve shift left by 10 apartments, and the equilibrium price remains unchanged.

Figure 1.6 **Effect of creating condominiums**. If demand and supply both shift left by the same amount the equilibrium price is unchanged.

Most people find this result surprising. They tend to see just the reduction in the supply of apartments and don't think about the reduction in demand. The case we've considered is an extreme one: *all* of the condo purchasers were former apartment dwellers. But the other case—where none of the condo purchasers were apartment dwellers—is even more extreme.

The model, simple though it is, has led us to an important insight. If we want to determine how conversion to condominiums will affect the apartment market, we have to consider not only the effect on the supply of apartments but also the effect on the demand for apartments.

Let's consider another example of a surprising comparative statics analysis: the effect of an apartment tax. Suppose that the city council decides that there should be a tax on apartments of $50 a year. Thus each landlord will have to pay $50 a year to the city for each apartment that he owns. What will this do to the price of apartments?

Most people would think that at least some of the tax would get passed along to apartment renters. But, rather surprisingly, that is not the case. In fact, the equilibrium price of apartments will remain unchanged!

In order to verify this, we have to ask what happens to the demand curve and the supply curve. The supply curve doesn't change—there are just as many apartments after the tax as before the tax. And the demand curve doesn't change either, since the number of apartments that will be rented at each different price will be the same as well. If neither the demand curve nor the supply curve shifts, the price can't change as a result of the tax.

Here is a way to think about the effect of this tax. Before the tax is imposed, each landlord is charging the highest price that he can get that will keep his apartments occupied. The equilibrium price p^* is the highest price that can be charged that is compatible with all of the apartments being rented. After the tax is imposed can the landlords raise their prices to compensate for the tax? The answer is no: if they could raise the price and keep their apartments occupied, they would have already done so. If they were charging the maximum price that the market could bear, the landlords couldn't raise their prices any more: none of the tax can get passed along to the renters. The landlords have to pay the entire amount of the tax.

This analysis depends on the assumption that the supply of apartments remains fixed. If the number of apartments can vary as the tax changes, then the price paid by the renters will typically change. We'll examine this kind of behavior later on, after we've built up some more powerful tools for analyzing such problems.

1.7 Other Ways to Allocate Apartments

In the previous section we described the equilibrium for apartments in a competitive market. But this is only one of many ways to allocate a

resource; in this section we describe a few other ways. Some of these may sound rather strange, but each will illustrate an important economic point.

The Discriminating Monopolist

First, let us consider a situation where there is one dominant landlord who owns all of the apartments. Or, alternatively, we could think of a number of individual landlords getting together and coordinating their actions to act as one. A situation where a market is dominated by a single seller of a product is known as a **monopoly**.

In renting the apartments the landlord could decide to auction them off one by one to the highest bidders. Since this means that different people would end up paying different prices for apartments, we will call this the case of the **discriminating monopolist**. Let us suppose for simplicity that the discriminating monopolist knows each person's reservation price for apartments. (This is not terribly realistic, but it will serve to illustrate an important point.)

This means that he would rent the first apartment to the fellow who would pay the most for it, in this case $500. The next apartment would go for $490 and so on as we moved down the demand curve. Each apartment would be rented to the person who was willing to pay the most for it.

Here is the interesting feature of the discriminating monopolist: *exactly the same people will get the apartments as in the case of the market solution,* namely, everyone who valued an apartment at more than p^*. The last person to rent an apartment pays the price p^*—the same as the equilibrium price in a competitive market. The discriminating monopolist's attempt to maximize his own profits leads to the same allocation of apartments as the supply and demand mechanism of the competitive market. The amount the people *pay* is different, but who gets the apartments is the same. It turns out that this is no accident, but we'll have to wait until later to explain the reason.

The Ordinary Monopolist

We assumed that the discriminating monopolist was able to rent each apartment at a different price. But what if he were forced to rent all apartments at the same price? In this case the monopolist faces a tradeoff: if he chooses a low price he will rent more apartments, but he may end up making less money than if he sets a higher price.

Let us use $D(p)$ to represent the demand function—the number of apartments demanded at price p. Then if the monopolist sets a price p, he will rent $D(p)$ apartments and thus receive a revenue of $pD(p)$. The revenue that the monopolist receives can be thought of as the area of a box: the

height of the box is the price p and the width of the box is the number of apartments $D(p)$. The product of the height and the width—the area of the box—is the revenue the monopolist receives. This is the box depicted in Figure 1.7.

Revenue box. The revenue received by the monopolist is just the price times the quantity, which can be interpreted as the area of the box illustrated.

Figure 1.7

If the monopolist has no costs associated with renting an apartment, he would want to choose a price that has the largest associated revenue box. The largest revenue box in Figure 1.7 occurs at the price $\hat{p}$. In this case the monopolist will find it in his interest *not* to rent all of the apartments. In fact this will generally be the case for a monopolist. The monopolist will want to restrict the output available in order to maximize his profit. This means that the monopolist will generally want to charge a price that is higher than the equilibrium price in a competitive market, p^*. In the case of the ordinary monopolist, fewer apartments will be rented, and each apartment will be rented at a higher price than in the competitive market.

Rent Control

A third and final case that we will discuss will be the case of rent control. Suppose that the city decides to impose a maximum rent that can be

charged for apartments, say p_{max}. We suppose that the price p_{max} is less than the equilibrium price in the competitive market, p^*. If this is so we would have a situation of **excess demand**: there are more people who are willing to rent apartments at p_{max} than there are apartments available. Who will end up with the apartments?

The theory that we have described up until now doesn't have an answer to this question. We can describe what will happen when supply equals demand, but we don't have enough detail in the model to describe what will happen if supply doesn't equal demand. The answer to who gets the apartments under rent control depends on who has the most time to spend looking around, who knows the current tenants, and so on. All of these things are outside the scope of the simple model we've developed. It may be that exactly the same people get the apartments under rent control as under the competitive market. But that is an extremely unlikely outcome. It is much more likely that some of the formerly outer-ring people will end up in some of the inner-ring apartments and thus displace the people who would have been living there under the market system. So under rent control the same number of apartments will be rented at the rent-controlled price as were rented under the competitive price: they'll just be rented to different people.

1.8 Which Way Is Best?

We've now described four possible ways of allocating apartments to people:

- The competitive market.
- A discriminating monopolist.
- An ordinary monopolist.
- Rent control.

These are four different economic institutions for allocating apartments. Each method will result in different people getting apartments or in different prices being charged for apartments. We might well ask which economic institution is best. But first we have to define "best." What criteria might we use to compare these ways of allocating apartments?

One thing we can do is to look at the economic positions of the people involved. It is pretty obvious that the owners of the apartments end up with the most money if they can act as discriminating monopolists: this would generate the most revenues for the apartment owner(s). Similarly the rent-control solution is probably the worst situation for the apartment owners.

What about the renters? They are probably worse off on average in the case of a discriminating monopolist—most of them would be paying a higher price than they would under the other ways of allocating apartments.

Are the consumers better off in the case of rent control? Some of them are: the consumers *who end up getting the apartments* are better off than they would be under the market solution. But the ones who didn't get the apartments are *worse off* than they would be under the market solution.

What we need here is a way to look at the economic position of all the parties involved—all the renters *and* all the landlords. How can we examine the desirability of different ways to allocate apartments, taking everybody into account? What can be used as a criterion for a "good" way to allocate apartments taking into account *all* of the parties involved?

1.9 Pareto Efficiency

One useful criterion for comparing the outcomes of different economic institutions is a concept known as Pareto efficiency or economic efficiency.[1] We start with the following definition: if we can find a way to make some people better off without making anybody else worse off, we have a **Pareto improvement**. If an allocation allows for a Pareto improvement, it is called **Pareto inefficient**; if an allocation is such that no Pareto improvements are possible, it is called **Pareto efficient**.

A Pareto inefficient allocation has the undesirable feature that there is some way to make somebody better off without hurting anyone else. There may be other positive things about the allocation, but the fact that it is Pareto inefficient is certainly one strike against it. If there is a way to make someone better off without hurting anyone else, why not do it?

The idea of Pareto efficiency is an important one in economics and we will examine it in some detail later on. It has many subtle implications that we will have to investigate more slowly, but we can get an inkling of what is involved even now.

Here is a useful way to think about the idea of Pareto efficiency. Suppose that we assigned the renters to the inner- and outer-ring apartments randomly, but then allowed them to sublet their apartments to each other. Some people who really wanted to live close in might, through bad luck, end up with an outer-ring apartment. But then they could sublet an inner-ring apartment from someone who was assigned to such an apartment but who didn't value it as highly as the other person. If individuals were assigned randomly to apartments, there would generally be some who would want to trade apartments, if they were sufficiently compensated for doing so.

For example, suppose that person A is assigned an apartment in the inner ring that he feels is worth $200, and that there is some person B in the outer ring who would be willing to pay $300 for A's apartment. Then there is a

[1] Pareto efficiency is named after the nineteenth-century economist and sociologist Vilfredo Pareto (1848–1923) who was one of the first to examine the implications of this idea.

"gain from trade" if these two agents swap apartments and arrange a side payment from B to A of some amount of money between \$200 and \$300. The exact amount of the transaction isn't important. What is important is that the people who are willing to pay the most for the apartments get them—otherwise, there would be an incentive for someone who attached a low value to an inner-ring apartment to make a trade with someone who placed a high value on an inner-ring apartment.

Suppose that we think of all voluntary trades as being carried out so that all gains from trade are exhausted. The resulting allocation must be Pareto efficient. If not, there would be some trade that would make two people better off without hurting anyone else—but this would contradict the assumption that all voluntary trades had been carried out. An allocation in which all voluntary trades have been carried out is a Pareto efficient allocation.

1.10 Comparing Ways to Allocate Apartments

The trading process we've described above is so general that you wouldn't think that anything much could be said about its outcome. But there is one very interesting point that can be made. Let us ask who will end up with apartments in an allocation where all of the gains from trade have been exhausted.

To see the answer, just note that anyone who has an apartment in the inner ring must have a higher reservation price than anyone who has an apartment in the outer ring—otherwise, they could make a trade and make both people better off. Thus if there are S apartments to be rented, then the S people with the highest reservation prices end up getting apartments in the inner ring. This allocation is Pareto efficient—anything else is not, since any other assignment of apartments to people would allow for some trade that would make at least two of the people better off without hurting anyone else.

Let us try to apply this criterion of Pareto efficiency to the outcomes of the various resource allocation devices mentioned above. Let's start with the market mechanism. It is easy to see that the market mechanism assigns the people with the S highest reservation prices to the inner ring—namely, those people who are willing to pay more than the equilibrium price, p^*, for their apartments. Thus there are no further gains from trade to be had once the apartments have been rented in a competitive market. The outcome of the competitive market is Pareto efficient.

What about the discriminating monopolist? Is that arrangement Pareto efficient? To answer this question, simply observe that the discriminating monopolist assigns apartments to exactly the same people who receive apartments in the competitive market. Under each system everyone who is willing to pay more than p^* for an apartment gets an apartment. Thus the discriminating monopolist generates a Pareto efficient outcome as well.

Although both the competitive market and the discriminating monopolist generate Pareto efficient outcomes in the sense that there will be no further trades desired, they can result in quite different distributions of income. Certainly the consumers are much worse off under the discriminating monopolist than under the competitive market, and the landlord(s) are much better off. In general, Pareto efficiency doesn't have much to say about distribution of the gains from trade. It is only concerned with the *efficiency* of the trade: whether all of the possible trades have been made.

What about the ordinary monopolist who is constrained to charge just one price? It turns out that this situation is not Pareto efficient. All we have to do to verify this is to note that, since all the apartments will not in general be rented by the monopolist, he can increase his profits by renting an apartment to someone who doesn't have one at *any* positive price. There is some price at which both the monopolist and the renter must be better off. As long as the monopolist doesn't change the price that anybody else pays, the other renters are just as well off as they were before. Thus we have found a **Pareto improvement**—a way to make two parties better off without making anyone else worse off.

The final case is that of rent control. This also turns out not to be Pareto efficient. The argument here rests on the fact that an arbitrary assignment of renters to apartments will generally involve someone living in the inner ring (say Mr. In) who is willing to pay less for an apartment than someone living in the outer ring (say Ms. Out). Suppose that Mr. In's reservation price is $300 and Ms. Out's reservation price is $500.

We need to find a Pareto improvement—a way to make Mr. In and Ms. Out better off without hurting anyone else. But there is an easy way to do this: just let Mr. In sublet his apartment to Ms. Out. It is worth $500 to Ms. Out to live close to the university, but it is only worth $300 to Mr. In. If Ms. Out pays Mr. In $400, say, and trades apartments, they will both be better off: Ms. Out will get an apartment that she values at more than $400, and Mr. In will get $400 that he values more than an inner-ring apartment.

This example shows that the rent-controlled market will generally not result in a Pareto efficient allocation, since there will still be some trades that could be carried out after the market has operated. As long as some people get inner-ring apartments who value them less highly than people who don't get them, there will be gains to be had from trade.

1.11 Equilibrium in the Long Run

We have analyzed the equilibrium pricing of apartments in the **short run**—when there is a fixed supply of apartments. But in the **long run** the supply of apartments can change. Just as the demand curve measures the number of apartments that will be demanded at different prices, the supply curve measures the number of apartments that will be supplied at different prices.

The final determination of the market price for apartments will depend on the interaction of supply and demand.

And what is it that determines the supply behavior? In general, the number of new apartments that will be supplied by the private market will depend on how profitable it is to provide apartments, which depends, in part, on the price that landlords can charge for apartments. In order to analyze the behavior of the apartment market in the long run, we have to examine the behavior of suppliers as well as demanders, a task we will eventually undertake.

When supply is variable, we can ask questions not only about who gets the apartments, but about how many will be provided by various types of market institutions. Will a monopolist supply more or fewer apartments than a competitive market? Will rent control increase or decrease the equilibrium number of apartments? Which institutions will provide a Pareto efficient number of apartments? In order to answer these and similar questions we must develop more systematic and powerful tools for economic analysis.

Summary

1. Economics proceeds by making models of social phenomena, which are simplified representations of reality.

2. In this task, economists are guided by the optimization principle, which states that people typically try to choose what's best for them, and by the equilibrium principle, which says that prices will adjust until demand and supply are equal.

3. The demand curve measures how much people wish to demand at each price, and the supply curve measures how much people wish to supply at each price. An equilibrium price is one where the amount demanded equals the amount supplied.

4. The study of how the equilibrium price and quantity change when the underlying conditions change is known as comparative statics.

5. An economic situation is Pareto efficient if there is no way to make some group of people better off without making some other group of people worse off. The concept of Pareto efficiency can be used to evaluate different ways of allocating resources.

REVIEW QUESTIONS

1. Suppose that there were 25 people who had a reservation price of \$500, and the 26th person had a reservation price of \$200. What would the demand curve look like?

2. In the above example, what would the equilibrium price be if there were 24 apartments to rent? What if there were 26 apartments to rent? What if there were 25 apartments to rent?

3. If people have different reservation prices, why does the market demand curve slope down?

4. In the text we assumed that the condominium purchasers came from the inner-ring people—people who were already renting apartments. What would happen to the price of inner-ring apartments if all of the condominium purchasers were outer-ring people—the people who were not currently renting apartments in the inner ring?

5. Suppose now that the condominium purchasers were all inner-ring people, but that each condominium was constructed from two apartments. What would happen to the price of apartments?

6. What do you suppose the effect of a tax would be on the number of apartments that would be built in the long run?

7. Suppose the demand curve is $D(p) = 100 - 2p$. What price would the monopolist set if he had 60 apartments? How many would he rent? What price would he set if he had 40 apartments? How many would he rent?

8. If our model of rent control allowed for unrestricted subletting, who would end up getting apartments in the inner circle? Would the outcome be Pareto efficient?

CHAPTER 2

BUDGET CONSTRAINT

The economic theory of the consumer is very simple: economists assume that consumers choose the best bundle of goods they can afford. To give content to this theory, we have to describe more precisely what we mean by "best" and what we mean by "can afford." In this chapter we will examine how to describe what a consumer can afford; the next chapter will focus on the concept of how the consumer determines what is best. We will then be able to undertake a detailed study of the implications of this simple model of consumer behavior.

2.1 The Budget Constraint

We begin by examining the concept of the **budget constraint**. Suppose that there is some set of goods from which the consumer can choose. In real life there are many goods to consume, but for our purposes it is convenient to consider only the case of two goods, since we can then depict the consumer's choice behavior graphically.

We will indicate the consumer's **consumption bundle** by (x_1, x_2). This is simply a list of two numbers that tells us how much the consumer is choosing to consume of good 1, x_1, and how much the consumer is choosing to

consume of good 2, x_2. Sometimes it is convenient to denote the consumer's bundle by a single symbol like X, where X is simply an abbreviation for the list of two numbers (x_1, x_2).

We suppose that we can observe the prices of the two goods, (p_1, p_2), and the amount of money the consumer has to spend, m. Then the budget constraint of the consumer can be written as

$$p_1 x_1 + p_2 x_2 \leq m. \tag{2.1}$$

Here $p_1 x_1$ is the amount of money the consumer is spending on good 1, and $p_2 x_2$ is the amount of money the consumer is spending on good 2. The budget constraint of the consumer requires that the amount of money spent on the two goods be no more than the total amount the consumer has to spend. The consumer's *affordable* consumption bundles are those that don't cost any more than m. We call this set of affordable consumption bundles at prices (p_1, p_2) and income m the **budget set** of the consumer.

2.2 Two Goods Are Often Enough

The two-good assumption is more general than you might think at first, since we can often interpret one of the goods as representing everything else the consumer might want to consume.

For example, if we are interested in studying a consumer's demand for milk, we might let x_1 measure his or her consumption of milk in quarts per month. We can then let x_2 stand for everything else the consumer might want to consume.

When we adopt this interpretation, it is convenient to think of good 2 as being the dollars that the consumer can use to spend on other goods. Under this interpretation the price of good 2 will automatically be 1, since the price of one dollar is one dollar. Thus the budget constraint will take the form

$$p_1 x_1 + x_2 \leq m. \tag{2.2}$$

This expression simply says that the amount of money spent on good 1, $p_1 x_1$, plus the amount of money spent on all other goods, x_2, must be no more than the total amount of money the consumer has to spend, m.

We say that good 2 represents a **composite good** that stands for everything else that the consumer might want to consume other than good 1. Such a composite good is invariably measured in dollars to be spent on goods other than good 1. As far as the algebraic form of the budget constraint is concerned, equation (2.2) is just a special case of the formula given in equation (2.1), with $p_2 = 1$, so everything that we have to say about the budget constraint in general will hold under the composite-good interpretation.

2.3 Properties of the Budget Set

The **budget line** is the set of bundles that cost exactly m:

$$p_1x_1 + p_2x_2 = m. \tag{2.3}$$

These are the bundles of goods that just exhaust the consumer's income.

The budget set is depicted in Figure 2.1. The heavy line is the budget line—the bundles that cost exactly m—and the bundles below this line are those that cost strictly less than m.

Figure 2.1 **The budget set**. The budget set consists of all bundles that are affordable at the given prices and income.

We can rearrange the budget line in equation (2.3) to give us the formula

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1. \tag{2.4}$$

This is the formula for a straight line with a vertical intercept of m/p_2 and a slope of $-p_1/p_2$. The formula tells us how many units of good 2 the consumer needs to consume in order to just satisfy the budget constraint if she is consuming x_1 units of good 1.

Here is an easy way to draw a budget line given prices (p_1, p_2) and income m. Just ask yourself how much of good 2 the consumer could buy if she spent all of her money on good 2. The answer is, of course, m/p_2. Then ask how much of good 1 the consumer could buy if she spent all of her money on good 1. The answer is m/p_1. Thus the horizontal and vertical intercepts measure how much the consumer could get if she spent all of her money on goods 1 and 2, respectively. In order to depict the budget line just plot these two points on the appropriate axes of the graph and connect them with a straight line.

The slope of the budget line has a nice economic interpretation. It measures the rate at which the market is willing to "substitute" good 1 for good 2. Suppose for example that the consumer is going to increase her consumption of good 1 by Δx_1.[1] How much will her consumption of good 2 have to change in order to satisfy her budget constraint? Let us use Δx_2 to indicate her change in the consumption of good 2.

Now note that if she satisfies her budget constraint before and after making the change she must satisfy

$$p_1 x_1 + p_2 x_2 = m$$

and

$$p_1(x_1 + \Delta x_1) + p_2(x_2 + \Delta x_2) = m.$$

Subtracting the first equation from the second gives

$$p_1 \Delta x_1 + p_2 \Delta x_2 = 0.$$

This says that the total value of the change in her consumption must be zero. Solving for $\Delta x_2/\Delta x_1$, the rate at which good 2 can be substituted for good 1 while still satisfying the budget constraint, gives

$$\frac{\Delta x_2}{\Delta x_1} = -\frac{p_1}{p_2}.$$

This is just the slope of the budget line. The negative sign is there since Δx_1 and Δx_2 must always have opposite signs. If you consume more of good 1, you have to consume less of good 2 and vice versa if you continue to satisfy the budget constraint.

Economists sometimes say that the slope of the budget line measures the **opportunity cost** of consuming good 1. In order to consume more of good 1 you have to give up some consumption of good 2. Giving up the opportunity to consume good 2 is the true economic cost of more good 1 consumption; and that cost is measured by the slope of the budget line.

[1] The Greek letter Δ, delta, is pronounced "del-ta." The notation Δx_1 denotes the change in good 1. For more on changes and rates of changes, see the Mathematical Appendix.

2.4 How the Budget Line Changes

When prices and incomes change, the set of goods that a consumer can afford changes as well. How do these changes affect the budget set?

Let us first consider changes in income. It is easy to see from equation (2.4) that an increase in income will increase the vertical intercept and not affect the slope of the line. Thus an increase in income will result in a *parallel shift outward* of the budget line as in Figure 2.2. Similarly, a decrease in income will cause a parallel shift inward.

Figure 2.2 **Increasing income**. An increase in income causes a parallel shift outward of the budget line.

What about changes in prices? Let us first consider increasing price 1 while holding price 2 and income fixed. According to equation (2.4), increasing p_1 will not change the vertical intercept, but it will make the budget line steeper since p_1/p_2 will become larger.

Another way to see how the budget line changes is to use the trick described earlier for drawing the budget line. If you are spending all of your money on good 2, then increasing the price of good 1 doesn't change the maximum amount of good 2 you could buy—thus the vertical intercept of the budget line doesn't change. But if you are spending all of your money on good 1, and good 1 becomes more expensive, then your

consumption of good 1 must decrease. Thus the horizontal intercept of the budget line must shift inward, resulting in the tilt depicted in Figure 2.3.

Increasing price. If good 1 becomes more expensive, the budget line becomes steeper.

Figure 2.3

What happens to the budget line when we change the prices of good 1 and good 2 at the same time? Suppose for example that we double the prices of both goods 1 and 2. In this case both the horizontal and vertical intercepts shift inward by a factor of one-half, and therefore the budget line shifts inward by one-half as well. Multiplying both prices by two is just like dividing income by 2.

We can also see this algebraically. Suppose our original budget line is

$$p_1x_1 + p_2x_2 = m.$$

Now suppose that both prices become t times as large. Multiplying both prices by t yields

$$tp_1x_1 + tp_2x_2 = m.$$

But this equation is the same as

$$p_1x_1 + p_2x_2 = \frac{m}{t}.$$

Thus multiplying both prices by a constant amount t is just like dividing income by the same constant t. It follows that if we multiply both prices

by t *and* we multiply income by t, then the budget line won't change at all.

We can also consider price and income changes together. What happens if both prices go up and income goes down? Think about what happens to the horizontal and vertical intercepts. If m decreases and p_1 and p_2 both increase, then the intercepts m/p_1 and m/p_2 must both decrease. This means that the budget line will shift inward. What about the slope of the budget line? If price 2 increases more than price 1, so that $-p_1/p_2$ decreases (in absolute value), then the budget line will be flatter; if price 2 increases less than price 1, the budget line will be steeper.

2.5 The Numeraire

The budget line is defined by two prices and one income, but one of these variables is redundant. We could peg one of the prices, or the income, to some fixed value, and adjust the other variables so as to describe exactly the same budget set. Thus the budget line

$$p_1x_1 + p_2x_2 = m$$

is exactly the same budget line as

$$\frac{p_1}{p_2}x_1 + x_2 = \frac{m}{p_2}$$

or

$$\frac{p_1}{m}x_1 + \frac{p_2}{m}x_2 = 1,$$

since the first budget line results from dividing everything by p_2, and the second budget line results from dividing everything by m. In the first case, we have pegged $p_2 = 1$, and in the second case, we have pegged $m = 1$. Pegging the price of one of the goods or income to 1 and adjusting the other price and income appropriately doesn't change the budget set at all.

When we set one of the prices to 1, as we did above, we often refer to that price as the **numeraire** price. The numeraire price is the price relative to which we are measuring the other price and income. It will occasionally be convenient to think of one of the goods as being a numeraire good, since there will then be one less price to worry about.

2.6 Taxes, Subsidies, and Rationing

Economic policy often uses tools that affect a consumer's budget constraint, such as taxes. For example, if the government imposes a **quantity tax**, this means that the consumer has to pay a certain amount to the government

for each unit of the good he purchases. In the U.S., for example, we pay about 15 cents a gallon as a federal gasoline tax.

How does a quantity tax affect the budget line of a consumer? From the viewpoint of the consumer the tax is just like a higher price. Thus a quantity tax of t dollars per unit of good 1 simply changes the price of good 1 from p_1 to $p_1 + t$. As we've seen above, this implies that the budget line must get steeper.

Another kind of tax is a **value** tax. As the name implies this is a tax on the value—the price—of a good, rather than the quantity purchased of a good. A value tax is usually expressed in percentage terms. Most states in the U.S. have sales taxes. If the sales tax is 6 percent, then a good that is priced at \$1 will actually sell for \$1.06. (Value taxes are also known as **ad valorem** taxes.)

If good 1 has a price of p_1 but is subject to a sales tax at rate τ, then the actual price facing the consumer is $(1 + \tau)p_1$.[2] The consumer has to pay p_1 to the supplier and τp_1 to the government for each unit of the good so the total cost of the good to the consumer is $(1 + \tau)p_1$.

A **subsidy** is the opposite of a tax. In the case of a **quantity subsidy**, the government *gives* an amount to the consumer that depends on the amount of the good purchased. If, for example, the consumption of milk were subsidized, the government would pay some amount of money to each consumer of milk depending on the amount that consumer purchased. If the subsidy is s dollars per unit of consumption of good 1, then from the viewpoint of the consumer, the price of good 1 would be $p_1 - s$. This would therefore make the budget line flatter.

Similarly an ad valorem subsidy is a subsidy based on the price of the good being subsidized. If the government gives you back \$1 for every \$2 you donate to charity, then your donations to charity are being subsidized at a rate of 50 percent. In general, if the price of good 1 is p_1 and good 1 is subject to an ad valorem subsidy at rate σ, then the actual price of good 1 facing the consumer is $(1 - \sigma)p_1$.[3]

You can see that taxes and subsidies affect prices in exactly the same way except for the algebraic sign: a tax increases the price to the consumer, and a subsidy decreases it.

Another kind of tax or subsidy that the government might use is a **lump-sum** tax or subsidy. In the case of a tax, this means that the government takes away some fixed amount of money, regardless of the individual's behavior. Thus a lump-sum tax means that the budget line of a consumer will shift inward because his money income has been reduced. Similarly, a lump-sum subsidy means that the budget line will shift outward. Quantity

[2] The Greek letter τ, tau, rhymes with "wow" in mathematical discourse, though modern Greeks pronounce it "taf."

[3] The Greek letter σ is pronounced "sig-ma."

taxes and value taxes tilt the budget line one way or the other depending on which good is being taxed, but a lump-sum tax shifts the budget line inward.

Governments also sometimes impose *rationing* constraints. This means that the level of consumption of some good is fixed to be no larger than some amount. For example, during World War II the U.S. government rationed certain foods like butter and meat.

Suppose, for example, that good 1 were rationed so that no more than $\overline{x}_1$ could be consumed by a given consumer. Then the budget set of the consumer would look like that depicted in Figure 2.4: it would be the old budget set with a piece lopped off. The lopped-off piece consists of all the consumption bundles that are affordable but have $x_1 > \overline{x}_1$.

Figure 2.4 **Budget set with rationing**. If good 1 is rationed, the section of the budget set beyond the rationed quantity will be lopped off.

Sometimes taxes, subsidies, and rationing are combined. For example, we could consider a situation where a consumer could consume good 1 at a price of p_1 up to some level $\overline{x}_1$, and then had to pay a tax t on all consumption in excess of $\overline{x}_1$. The budget set for this consumer is depicted in Figure 2.5. Here the budget line has a slope of $-p_1/p_2$ to the left of $\overline{x}_1$, and a slope of $-(p_1 + t)/p_2$ to the right of $\overline{x}_1$.

Taxing consumption greater than $\bar{x}_1$. In this budget set the consumer must pay a tax only on the consumption of good 1 that is in excess of $\bar{x}_1$, so the budget line becomes steeper to the right of $\bar{x}_1$.

Figure 2.5

EXAMPLE: The Food Stamp Program

Since the Food Stamp Act of 1964 the U.S. federal government has provided a subsidy on food for poor people. The details of this program have been adjusted several times. Here we will describe the economic effects of one of these adjustments.

Before 1979, households who met certain eligibility requirements were allowed to purchase food stamps, which could then be used to purchase food at retail outlets. In January 1975, for example, a family of four could receive a maximum monthly allotment of $153 in food coupons by participating in the program.

The price of these coupons to the household depended on the household income. A family of four with an adjusted monthly income of $300 paid $83 for the full monthly allotment of food stamps. If a family of four had a monthly income of $100, the cost for the full monthly allotment would have been $25.[4]

The pre-1979 Food Stamp program was an ad valorem subsidy on food. The rate at which food was subsidized depended on the household income.

[4] These figures are taken from Kenneth Clarkson, *Food Stamps and Nutrition*, American Enterprise Institute, 1975.

The family of four that was charged \$83 for their allotment paid \$1 to receive \$1.84 worth of food (1.84 equals 153 divided by 83). Similarly, the household that paid \$25 was paying \$1 to receive \$6.12 worth of food (6.12 equals 153 divided by 25).

The way that the Food Stamp program affected the budget set of a household is depicted in Figure 2.6A. Here we have measured the amount of money spent on food on the horizontal axis and expenditures on all other goods on the vertical axis. Since we are measuring each good in terms of the money spent on it, the "price" of each good is automatically 1, and the budget line will therefore have a slope of -1.

If the household is allowed to buy \$153 of food stamps for \$25, then this represents roughly an 84 percent ($= 1 - 25/153$) subsidy of food purchases, so the budget line will have a slope of roughly $-.16$ ($= 25/153$) until the household has spent \$153 on food. Each dollar that the household spends on food up to \$153 would reduce its consumption of other goods by about 16 cents. After the household spends \$153 on food, the budget line facing it would again have a slope of -1.

Figure 2.6

Food stamps. How the budget line is affected by the Food Stamp program. Part A shows the pre-1979 program and part B the post-1979 program.

These effects lead to the kind of "kink" depicted in Figure 2.6. Households with higher incomes had to pay more for their allotment of food stamps. Thus the slope of the budget line would become steeper as household income increased.

In 1979 the Food Stamp program was modified. Instead of requiring that

households purchase food stamps, they are now simply given to qualified households. Figure 2.6B shows how this affects the budget set.

Suppose that a household now receives a grant of \$200 of food stamps a month. Then this means that the household can consume \$200 more food per month, regardless of how much it is spending on other goods, which implies that the budget line will shift to the right by \$200. The slope will not change: \$1 less spent on food would mean \$1 more to spend on other things. But since the household cannot legally sell food stamps, the maximum amount that it can spend on other goods does not change. The Food Stamp program is effectively a lump-sum subsidy, except for the fact that the food stamps can't be sold.

2.7 Budget Line Changes

In the next chapter we will analyze how the consumer chooses an optimal consumption bundle from his or her budget set. But we can already state some observations here that follow from what we have learned about the movements of the budget line.

First, we can observe that since the budget set doesn't change when we multiply all prices and income by a positive number, the optimal choice of the consumer from the budget set can't change either. Without even analyzing the choice process itself, we have derived an important conclusion: a perfectly balanced inflation—one in which all prices and all incomes rise at the same rate—doesn't change anybody's budget set, and thus cannot change anybody's optimal choice.

Second, we can make some statements about how well-off the consumer can be at different prices and incomes. Suppose that the consumer's income increases and all prices remain the same. We know that this represents a parallel shift outward of the budget line. Thus every bundle the consumer was consuming at the lower income is also a possible choice at the higher income. But then the consumer must be at least as well-off at the higher income as at the lower income—since he or she has the same choices available as before plus some more. Similarly, if one price declines and all others stay the same, the consumer must be at least as well-off. This simple observation will be of considerable use later on.

Summary

1. The budget set consists of all bundles of goods that the consumer can afford at given prices and income. We will typically assume that there are only two goods, but this assumption is more general than it seems.

2. The budget line is written as $p_1x_1 + p_2x_2 = m$. It has a slope of $-p_1/p_2$, a vertical intercept of m/p_2, and a horizontal intercept of m/p_1.

3. Increasing income shifts the budget line outward. Increasing the price of good 1 makes the budget line steeper. Increasing the price of good 2 makes the budget line flatter.

4. Taxes, subsidies, and rationing change the slope and position of the budget line by changing the prices paid by the consumer.

REVIEW QUESTIONS

1. Originally the consumer faces the budget line $p_1x_1 + p_2x_2 = m$. Then the price of good 1 doubles, the price of good 2 becomes 8 times larger, and income becomes 4 times larger. Write down an equation for the new budget line in terms of the original prices and income.

2. What happens to the budget line if the price of good 2 increases, but the price of good 1 and income remain constant?

3. If the price of good 1 doubles and the price of good 2 triples, does the budget line become flatter or steeper?

4. What is the definition of a numeraire good?

5. Suppose that the government puts a tax of 15 cents a gallon on gasoline and then later decides to put a subsidy on gasoline at a rate of 7 cents a gallon. What net tax is this combination equivalent to?

6. Suppose that a budget equation is given by $p_1x_1 + p_2x_2 = m$. The government decides to impose a lump-sum tax of u, a quantity tax on good 1 of t, and a quantity subsidy on good 2 of s. What is the formula for the new budget line?

7. If the income of the consumer increases and one of the prices decreases at the same time, will the consumer necessarily be at least as well-off?

CHAPTER 3

PREFERENCES

We saw in Chapter 2 that the economic model of consumer behavior is very simple: people choose the best things they can afford. The last chapter was devoted to clarifying the meaning of "can afford," and this chapter will be devoted to clarifying the economic concept of "best things."

We call the objects of consumer choice **consumption bundles**. This is a complete list of the goods and services that are involved in the choice problem that we are investigating. The word "complete" deserves emphasis: when you analyze a consumer's choice problem, make sure that you include all of the appropriate goods in the definition of the consumption bundle.

If we are analyzing consumer choice at the broadest level, we would want not only a complete list of the goods that a consumer might consume, but also a description of when, where, and under what circumstances they would become available. After all, people care about how much food they will have tomorrow as well as how much food they have today. A raft in the middle of the Atlantic Ocean is very different from a raft in the middle of the Sahara Desert. And an umbrella when it is raining is quite a different good from an umbrella on a sunny day. It is often useful to think of the

"same" good available in different locations or circumstances as a different good, since the consumer may value the good differently in those situations.

However, when we limit our attention to a simple choice problem, the relevant goods are usually pretty obvious. We'll often adopt the idea described earlier of using just two goods and calling one of them "all other goods" so that we can focus on the tradeoff between one good and everything else. In this way we can consider consumption choices involving many goods and still use two-dimensional diagrams.

So let us take our consumption bundle to consist of two goods, and let x_1 denote the amount of one good and x_2 the amount of the other. The complete consumption bundle is therefore denoted by (x_1, x_2). As noted before, we will occasionally abbreviate this consumption bundle by X.

3.1 Consumer Preferences

We will suppose that given any two consumption bundles, (x_1, x_2) and (y_1, y_2), the consumer can rank them as to their desirability. That is, the consumer can determine that one of the consumption bundles is strictly better than the other, or decide that she is indifferent between the two bundles.

We will use the symbol $\succ$ to mean that one bundle is **strictly preferred** to another, so that $(x_1, x_2) \succ (y_1, y_2)$ should be interpreted as saying that the consumer **strictly prefers** (x_1, x_2) to (y_1, y_2), in the sense that she definitely wants the x-bundle rather than the y-bundle. This preference relation is meant to be an operational notion. If the consumer prefers one bundle to another, it means that he or she would choose one over the other, given the opportunity. Thus the idea of preference is based on the consumer's *behavior.* In order to tell whether one bundle is preferred to another, we see how the consumer behaves in choice situations involving the two bundles. If she always chooses (x_1, x_2) when (y_1, y_2) is available, then it is natural to say that this consumer prefers (x_1, x_2) to (y_1, y_2).

If the consumer is **indifferent** between two bundles of goods, we use the symbol $\sim$ and write $(x_1, x_2) \sim (y_1, y_2)$. Indifference means that the consumer would be just as satisfied, according to her own preferences, consuming the bundle (x_1, x_2) as she would be consuming the other bundle, (y_1, y_2).

If the consumer prefers or is indifferent between the two bundles we say that she **weakly prefers** (x_1, x_2) to (y_1, y_2) and write $(x_1, x_2) \succeq (y_1, y_2)$.

These relations of strict preference, weak preference, and indifference are not independent concepts; the relations are themselves related! For example, if $(x_1, x_2) \succeq (y_1, y_2)$ and $(y_1, y_2) \succeq (x_1, x_2)$ we can conclude that $(x_1, x_2) \sim (y_1, y_2)$. That is, if the consumer thinks that (x_1, x_2) is at least as good as (y_1, y_2) *and* that (y_1, y_2) is at least as good as (x_1, x_2), then the consumer must be indifferent between the two bundles of goods.

Similarly, if $(x_1, x_2) \succeq (y_1, y_2)$ but we know that it is *not* the case that $(x_1, x_2) \sim (y_1, y_2)$, we can conclude that we must have $(x_1, x_2) \succ (y_1, y_2)$. This just says that if the consumer thinks that (x_1, x_2) is at least as good as (y_1, y_2), and she is not indifferent between the two bundles, then it must be that she thinks that (x_1, x_2) is strictly better than (y_1, y_2).

3.2 Assumptions about Preferences

Economists usually make some assumptions about the "consistency" of consumers' preferences. For example, it seems unreasonable—not to say contradictory—to have a situation where $(x_1, x_2) \succ (y_1, y_2)$ and, at the same time, $(y_1, y_2) \succ (x_1, x_2)$. For this would mean that the consumer strictly prefers the x-bundle to the y-bundle ... and vice versa.

So we usually make some assumptions about how the preference relations work. Some of the assumptions about preferences are so fundamental that we can refer to them as "axioms" of consumer theory. Here are three such axioms about consumer preference.

Complete. We assume that any two different bundles can be compared. That is, given any x-bundle and any y-bundle, we assume that $(x_1, x_2) \succeq (y_1, y_2)$, or $(y_1, y_2) \succeq (x_1, x_2)$, or both, in which case the consumer is indifferent between the two bundles.

Reflexive. We assume that any bundle is at least as good as itself: $(x_1, x_2) \succeq (x_1, x_2)$.

Transitive. If $(x_1, x_2) \succeq (y_1, y_2)$ and $(y_1, y_2) \succeq (z_1, z_2)$, then we assume that $(x_1, x_2) \succeq (z_1, z_2)$. In other words, if the consumer thinks that X is at least as good as Y and that Y is at least as good as Z, then the consumer thinks that X is at least as good as Z.

The first axiom, completeness, is hardly objectionable, at least for the kinds of choices economists generally examine. To say that any two bundles can be compared is simply to say that the consumer is able to make a choice between any two given bundles. One might imagine extreme situations involving life or death choices where ranking the alternatives might be difficult, or even impossible, but these choices are, for the most part, outside the domain of economic analysis.

The second axiom, reflexivity, is trivial. Any bundle is certainly at least as good as an identical bundle. Parents of small children may occasionally observe behavior that violates this assumption, but it seems plausible for most adult behavior.

The third axiom, transitivity, is more problematic. It isn't clear that transitivity of preferences is *necessarily* a property that preferences would have to have. The assumption that preferences are transitive doesn't seem

compelling on grounds of pure logic alone. In fact it's not. Transitivity is a hypothesis about people's choice behavior, not a statement of pure logic. Whether it is a basic fact of logic or not isn't the point: it is whether or not it is a reasonably accurate description of how people behave that matters.

What would you think about a person who said that he preferred a bundle X to Y, and preferred Y to Z, but then also said that he preferred Z to X? This would certainly be taken as evidence of peculiar behavior.

More importantly, how would this consumer behave if faced with choices among the three bundles X, Y, and Z? If we asked him to choose his most preferred bundle, he would have quite a problem, for whatever bundle he chose, there would always be one that was preferred to it. If we are to have a theory where people are making "best" choices, preferences must satisfy the transitivity axiom or something very much like it. If preferences were not transitive there could well be a set of bundles for which there is no best choice.

3.3 Indifference Curves

It turns out that the whole theory of consumer choice can be formulated in terms of preferences that satisfy the three axioms described above, plus a few more technical assumptions. However, we will find it convenient to describe preferences graphically by using a construction known as **indifference curves**.

Consider Figure 3.1 where we have illustrated two axes representing a consumer's consumption of goods 1 and 2. Let us pick a certain consumption bundle (x_1, x_2) and shade in all of the consumption bundles that are weakly preferred to (x_1, x_2). This is called the **weakly preferred set**. The bundles on the boundary of this set—the bundles for which the consumer is just indifferent to (x_1, x_2)—form the **indifference curve**.

We can draw an indifference curve through any consumption bundle we want. The indifference curve through a consumption bundle consists of all bundles of goods that leave the consumer indifferent to the given bundle.

One problem with using indifference curves to describe preferences is that they only show you the bundles that the consumer perceives as being indifferent to each other—they don't show you which bundles are better and which bundles are worse. It is sometimes useful to draw small arrows on the indifference curves to indicate the direction of the preferred bundles. We won't do this in every case, but we will do it in a few of the examples where confusion might arise.

If we make no further assumptions about preferences, indifference curves can take very peculiar shapes indeed. But even at this level of generality, we can state an important principle about indifference curves: *indifference curves representing distinct levels of preference cannot cross.* That is, the situation depicted in Figure 3.2 cannot occur.

Weakly preferred set. The shaded area consists of all bundles that are at least as good as the bundle (x_1, x_2).

Figure 3.1

In order to prove this, let us choose three bundles of goods, X, Y, and Z, such that X lies only on one indifference curve, Y lies only on the other indifference curve, and Z lies at the intersection of the indifference curves. By assumption the indifference curves represent distinct levels of preference, so one of the bundles, say X, is strictly preferred to the other bundle, Y. We know that $X \sim Z$ and $Z \sim Y$, and the axiom of transitivity therefore implies that $X \sim Y$. But this contradicts the assumption that $X \succ Y$. This contradiction establishes the result—indifference curves representing distinct levels of preference cannot cross.

What other properties do indifference curves have? In the abstract, the answer is: not many. Indifference curves are a way to describe preferences. Nearly any "reasonable" preferences that you can think of can be depicted by indifference curves. The trick is to learn what kinds of preferences give rise to what shapes of indifference curves.

3.4 Examples of Preferences

Let us try to relate preferences to indifference curves through some examples. We'll describe some preferences and then see what the indifference curves that represent them look like.

Figure 3.2

Indifference curves cannot cross. If they did, X, Y, and Z would all have to be indifferent to each other and thus could not lie on distinct indifference curves.

There is a general procedure for constructing indifference curves given a "verbal" description of the preferences. First plop your pencil down on the graph at some consumption bundle (x_1, x_2). Now think about giving a little more of good 1, Δx_1, to the consumer, moving him to $(x_1 + \Delta x_1, x_2)$. Now ask yourself how would you have to *change* the consumption of x_2 to make the consumer indifferent to the original consumption point? Call this change Δx_2. Ask yourself the question "For a given change in good 1, how does good 2 have to change to make the consumer just indifferent between $(x_1 + \Delta x_1, x_2 + \Delta x_2)$ and (x_1, x_2)?" Once you have determined this movement at one consumption bundle you have drawn a piece of the indifference curve. Now try it at another bundle, and so on, until you develop a clear picture of the overall shape of the indifference curves.

Perfect Substitutes

Two goods are **perfect substitutes** if the consumer is willing to substitute one good for the other at a *constant* rate. The simplest case of perfect substitutes occurs when the consumer is willing to substitute the goods on a one-to-one basis.

Suppose, for example, that we are considering a choice between red pencils and blue pencils, and the consumer involved likes pencils, but doesn't care about color at all. Pick a consumption bundle, say $(10, 10)$. Then for this consumer, any other consumption bundle that has 20 pencils in it is

just as good as $(10, 10)$. Mathematically speaking, any consumption bundle (x_1, x_2) such that $x_1 + x_2 = 20$ will be on this consumer's indifference curve through $(10, 10)$. Thus the indifference curves for this consumer are all parallel straight lines with a slope of -1, as depicted in Figure 3.3. Bundles with more total pencils are preferred to bundles with fewer total pencils, so the direction of increasing preference is up and to the right, as illustrated in Figure 3.3.

How does this work in terms of general procedure for drawing indifference curves? If we are at $(10, 10)$, and we increase the amount of the first good by one unit to 11, how much do we have to change the second good to get back to the original indifference curve? The answer is clearly that we have to decrease the second good by 1 unit. Thus the indifference curve through $(10, 10)$ has a slope of -1. The same procedure can be carried out at any bundle of goods with the same results—in this case all the indifference curves have a constant slope of -1.

Perfect substitutes. The consumer only cares about the total number of pencils, not about their colors. Thus the indifference curves are straight lines with a slope of -1.

Figure 3.3

The important fact about perfect substitutes is that the indifference curves have a *constant* slope. Suppose, for example, that we graphed blue pencils on the vertical axis and *pairs* of red pencils on the horizontal axis. The indifference curves for these two goods would have a slope of -2, since the consumer would be willing to give up two blue pencils to get one more *pair* of red pencils.

In the textbook we'll primarily consider the case where goods are perfect substitutes on a one-for-one basis, and leave the treatment of the general case for the workbook.

Perfect Complements

Perfect complements are goods that are always consumed together in fixed proportions. In some sense the goods "complement" each other. A nice example is that of right shoes and left shoes. The consumer likes shoes, but always wears right and left shoes together. Having only one out of a pair of shoes doesn't do the consumer a bit of good.

Let us draw the indifference curves for perfect complements. Suppose we pick the consumption bundle $(10, 10)$. Now add 1 more right shoe, so we have $(11, 10)$. By assumption this leaves the consumer indifferent to the original position: the extra shoe doesn't do him any good. The same thing happens if we add one more left shoe: the consumer is also indifferent between $(10, 11)$ and $(10, 10)$.

Thus the indifference curves are L-shaped, with the vertex of the L occurring where the number of left shoes equals the number of right shoes as in Figure 3.4.

Figure 3.4 **Perfect complements**. The consumer always wants to consume the goods in fixed proportions to each other. Thus the indifference curves are L-shaped.

Increasing both the number of left shoes and the number of right shoes at the same time will move the consumer to a more preferred position, so the direction of increasing preference is again up and to the right, as illustrated in the diagram.

The important thing about perfect complements is that the consumer prefers to consume the goods in fixed proportions, not necessarily that the proportion is one-to-one. If a consumer always uses two teaspoons of sugar in her cup of tea, and doesn't use sugar for anything else, then the indifference curves will still be L-shaped. In this case the corners of the L will occur at (2 teaspoons sugar, 1 cup tea), (4 teaspoons sugar, 2 cups tea) and so on, rather than at (1 right shoe, 1 left shoe), (2 right shoes, 2 left shoes), and so on.

In the textbook we'll primarily consider the case where the goods are consumed in proportions of one-for-one and leave the treatment of the general case for the workbook.

Bads

A **bad** is a commodity that the consumer doesn't like. For example, suppose that the commodities in question are now pepperoni and anchovies—and the consumer loves pepperoni but dislikes anchovies. But let us suppose there is some possible tradeoff between pepperoni and anchovies. That is, there would be some amount of pepperoni on a pizza that would compensate the consumer for having to consume a given amount of anchovies. How could we represent these preferences using indifference curves?

Pick a bundle (x_1, x_2) consisting of some pepperoni and some anchovies. If we give the consumer more anchovies, what do we have to do with the pepperoni to keep him on the same indifference curve? Clearly, we have to give him some extra pepperoni to compensate him for having to put up with the anchovies. Thus this consumer must have indifference curves that slope up and to the right as depicted in Figure 3.5.

The direction of increasing preference is down and to the right—that is, toward the direction of decreased anchovy consumption and increased pepperoni consumption, just as the arrows in the diagram illustrate.

Neutrals

A good is a **neutral good** if the consumer doesn't care about it one way or the other. What if a consumer is just neutral about anchovies?[1] In this case his indifference curves will be vertical lines as depicted in Figure 3.6.

[1] Is anybody neutral about anchovies?

Figure 3.5

Bads. Here anchovies are a "bad," and pepperoni is a "good" for this consumer. Thus the indifference curves have a positive slope.

Figure 3.6

A neutral good. The consumer likes pepperoni but is neutral about anchovies, so the indifference curves are vertical lines.

He only cares about the amount of pepperoni he has and doesn't care at all about how many anchovies he has. The more pepperoni the better, but adding more anchovies doesn't affect him one way or the other.

Satiation

We sometimes want to consider a situation involving **satiation**, where there is some overall best bundle for the consumer, and the "closer" he is to that best bundle, the better off he is in terms of his own preferences. For example, suppose that the consumer has some most preferred bundle of goods $(\overline{x}_1, \overline{x}_2)$, and the farther away he is from that bundle, the worse off he is. In this case we say that $(\overline{x}_1, \overline{x}_2)$ is a **satiation** point, or a **bliss** point. The indifference curves for the consumer look like those depicted in Figure 3.7. The best point is $(\overline{x}_1, \overline{x}_2)$ and points farther away from this bliss point lie on "lower" indifference curves.

Satiated preferences. The bundle $(\overline{x}_1, \overline{x}_2)$ is the satiation point or bliss point, and the indifference curves surround this point.

Figure 3.7

In this case the indifference curves have a negative slope when the consumer has "too little" or "too much" of both goods, and a positive slope when he has "too much" of one of the goods. When he has too much of one of the goods, it becomes a bad—reducing the consumption of the bad good moves him closer to his "bliss point." If he has too much of both goods, they both are bads, so reducing the consumption of each moves him closer to the bliss point.

Suppose, for example, that the two goods are chocolate cake and ice cream. There might well be some optimal amount of chocolate cake and

ice cream that you would want to eat per week. Any less than that amount would make you worse off, but any more than that amount would also make you worse off.

If you think about it, most goods are like chocolate cake and ice cream in this respect—you can have too much of nearly anything. But people would generally not voluntarily *choose* to have too much of the goods they consume. Why would you choose to have more than you want of something? Thus the interesting region from the viewpoint of economic choice is where you have *less* than you want of most goods. The choices that people actually care about are choices of this sort, and these are the choices with which we will be concerned.

Discrete Goods

Usually we think of measuring goods in units where fractional amounts make sense—you might on average consume 12.43 gallons of milk a month even though you buy it a quart at a time. But sometimes we want to examine preferences over goods that naturally come in discrete units.

For example, consider a consumer's demand for automobiles. We could define the demand for automobiles in terms of the time spent using an automobile, so that we would have a continuous variable, but for many purposes it is the actual number of cars demanded that is of interest.

There is no difficulty in using preferences to describe choice behavior for this kind of discrete good. Suppose that x_2 is money to be spent on other goods and x_1 is a **discrete good** that is only available in integer amounts. We have illustrated the appearance of indifference "curves" and a weakly preferred set for this kind of good in Figure 3.8. In this case the bundles indifferent to a given bundle will be a set of discrete points. The set of bundles at least as good as a particular bundle will be a set of line segments.

The choice of whether to emphasize the discrete nature of a good or not will depend on our application. If the consumer chooses only one or two units of the good during the time period of our analysis, recognizing the discrete nature of the choice may be important. But if the consumer is choosing 30 or 40 units of the good, then it will probably be convenient to think of this as a continuous good.

3.5 Well-Behaved Preferences

We've now seen some examples of indifference curves. As we've seen, many kinds of preferences, reasonable or unreasonable, can be described by these simple diagrams. But if we want to describe preferences in general, it will be convenient to focus on a few general shapes of indifference curves. In

A discrete good. Here good 1 is only available in integer amounts. In panel A the dashed lines connect together the bundles that are indifferent, and in panel B the vertical lines represent bundles that are at least as good as the indicated bundle.

Figure 3.8

this section we will describe some more general assumptions that we will typically make about preferences and the implications of these assumptions for the shapes of the associated indifference curves. These assumptions are not the only possible ones; in some situations you might want to use different assumptions. But we will take them as the defining features for **well-behaved indifference curves**.

First we will typically assume that more is better, that is, that we are talking about *goods*, not bads. More precisely, if (x_1, x_2) is a bundle of goods and (y_1, y_2) is a bundle of goods with at least as much of both goods and more of one, then $(y_1, y_2) \succ (x_1, x_2)$. This assumption is sometimes called **monotonicity** of preferences. As we suggested in our discussion of satiation, more is better would probably only hold up to a point. Thus the assumption of monotonicity is saying only that we are going to examine situations *before* that point is reached—before any satiation sets in—while more *still is* better. Economics would not be a very interesting subject in a world where everyone was satiated in their consumption of every good.

What does monotonicity imply about the shape of indifference curves? It implies that they have a *negative* slope. Consider Figure 3.9. If we start at a bundle (x_1, x_2) and move anywhere up and to the right, we must be moving to a preferred position. If we move down and to the left we must be moving to a worse position. So if we are moving to an *indifferent* position, we must be moving either left and up or right and down: the indifference curve must have a negative slope.

Second, we are going to assume that averages are preferred to extremes. That is, if we take two bundles of goods (x_1, x_2) and (y_1, y_2) on the same indifference curve and take a weighted average of the two bundles such as

$$\left(\frac{1}{2}x_1 + \frac{1}{2}y_1, \frac{1}{2}x_2 + \frac{1}{2}y_2\right),$$

then the average bundle will be at least as good as or strictly preferred to each of the two extreme bundles. This weighted-average bundle has the average amount of good 1 and the average amount of good 2 that is present in the two bundles. It therefore lies halfway along the straight line connecting the x–bundle and the y–bundle.

Figure 3.9 **Monotonic preferences**. More of both goods is a better bundle for this consumer; less of both goods represents a worse bundle.

Actually, we're going to assume this for any weight t between 0 and 1, not just 1/2. Thus we are assuming that if $(x_1, x_2) \sim (y_1, y_2)$, then

$$(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2) \succeq (x_1, x_2)$$

for any t such that $0 \leq t \leq 1$. This weighted average of the two bundles gives a weight of t to the x-bundle and a weight of $1 - t$ to the y-bundle. Therefore, the distance from the x-bundle to the average bundle is just a fraction t of the distance from the x-bundle to the y-bundle, along the straight line connecting the two bundles.

What does this assumption about preferences mean geometrically? It means that the set of bundles weakly preferred to (x_1, x_2) is a **convex set**. For suppose that (y_1, y_2) and (x_1, x_2) are indifferent bundles. Then, if averages are preferred to extremes, all of the weighted averages of (x_1, x_2) and (y_1, y_2) are weakly preferred to (x_1, x_2) and (y_1, y_2). A convex set has the property that if you take *any* two points in the set and draw the line segment connecting those two points, that line segment lies entirely in the set.

Figure 3.10A depicts an example of convex preferences, while Figures 3.10B and 3.10C show two examples of nonconvex preferences. Figure 3.10C presents preferences that are so nonconvex that we might want to call them "concave preferences."

Various kinds of preferences. Panel A depicts convex preferences, panel B depicts nonconvex preferences, and panel C depicts "concave" preferences.

Figure 3.10

Can you think of preferences that are not convex? One possibility might be something like my preferences for ice cream and olives. I like ice cream and I like olives . . . but I don't like to have them together! In considering my consumption in the next hour, I might be indifferent between consuming 8 ounces of ice cream and 2 ounces of olives, or 8 ounces of olives and 2 ounces of ice cream. But either one of these bundles would be better than consuming 5 ounces of each! These are the kind of preferences depicted in Figure 3.10C.

Why do we want to assume that well-behaved preferences are convex? Because, for the most part, goods are consumed together. The kinds of preferences depicted in Figures 3.10B and 3.10C imply that the con-

sumer would prefer to specialize, at least to some degree, and to consume only one of the goods. However, the normal case is where the consumer would want to trade some of one good for the other and end up consuming some of each, rather than specializing in consuming only one of the two goods.

In fact, if we look at my preferences for *monthly* consumption of ice cream and olives, rather than at my immediate consumption, they would tend to look much more like Figure 3.10A than Figure 3.10C. Each month I would prefer having some ice cream and some olives—albeit at different times—to specializing in consuming either one for the entire month.

Finally, one extension of the assumption of convexity is the assumption of **strict convexity**. This means that the weighted average of two indifferent bundles is *strictly* preferred to the two extreme bundles. Convex preferences may have flat spots, while *strictly* convex preferences must have indifferences curves that are "rounded." The preferences for two goods that are perfect substitutes are convex, but not strictly convex.

3.6 The Marginal Rate of Substitution

We will often find it useful to refer to the slope of an indifference curve at a particular point. This idea is so useful that it even has a name: the slope of an indifference curve is known as the **marginal rate of substitution (MRS)**. The name comes from the fact that the MRS measures the rate at which the consumer is just willing to substitute one good for the other.

Suppose that we take a little of good 1, Δx_1, away from the consumer. Then we give him Δx_2, an amount that is just sufficient to put him back on his indifference curve, so that he is just as well off after this substitution of x_2 for x_1 as he was before. We think of the ratio $\Delta x_2/\Delta x_1$ as being the *rate* at which the consumer is willing to substitute good 2 for good 1.

Now think of Δx_1 as being a very small change—a marginal change. Then the rate $\Delta x_2/\Delta x_1$ measures the *marginal* rate of substitution of good 2 for good 1. As Δx_1 gets smaller, $\Delta x_2/\Delta x_1$ approaches the slope of the indifference curve, as can be seen in Figure 3.11.

When we write the ratio $\Delta x_2/\Delta x_1$, we will always think of both the numerator and the denominator as being small numbers—as describing *marginal* changes from the original consumption bundle. Thus the ratio defining the MRS will always describe the slope of the indifference curve: the rate at which the consumer is just willing to substitute a little more consumption of good 2 for a little less consumption of good 1.

One slightly confusing thing about the MRS is that it is typically a *negative* number. We've already seen that monotonic preferences imply that indifference curves must have a negative slope. Since the MRS is the numerical measure of the slope of an indifference curve, it will naturally be a negative number.

The marginal rate of substitution (MRS). The marginal rate of substitution measures the slope of the indifference curve.

Figure 3.11

The marginal rate of substitution measures an interesting aspect of the consumer's behavior. Suppose that the consumer has well-behaved preferences, that is, preferences that are monotonic and convex, and that he is currently consuming some bundle (x_1, x_2). We now will offer him a trade: he can exchange good 1 for 2, or good 2 for 1, in any amount at a "rate of exchange" of E.

That is, if the consumer gives up Δx_1 units of good 1, he can get $E\Delta x_1$ units of good 2 in exchange. Or, conversely, if he gives up Δx_2 units of good 2, he can get $\Delta x_2/E$ units of good 1. Geometrically, we are offering the consumer an opportunity to move to any point along a line with slope $-E$ that passes through (x_1, x_2), as depicted in Figure 3.12. Moving up and to the left from (x_1, x_2) involves exchanging good 1 for good 2, and moving down and to the right involves exchanging good 2 for good 1. In either movement, the exchange rate is E. Since exchange always involves giving up one good in exchange for another, the exchange *rate* E corresponds to a *slope* of $-E$.

We can now ask what would the rate of exchange have to be in order for the consumer to want to stay put at (x_1, x_2)? To answer this question, we simply note that any time the exchange line *crosses* the indifference curve, there will be some points on that line that are preferred to (x_1, x_2)—that lie above the indifference curve. Thus, if there is to be no movement from

(x_1, x_2), the exchange line must be tangent to the indifference curve. That is, the slope of the exchange line, $-E$, must be the slope of the indifference curve at (x_1, x_2). At any other rate of exchange, the exchange line would cut the indifference curve and thus allow the consumer to move to a more preferred point.

Figure 3.12 **Trading at an exchange rate**. Here we are allowing the consumer to trade the goods at an exchange rate E, which implies that the consumer can move along a line with slope $-E$.

Thus the slope of the indifference curve, the marginal rate of substitution, measures the rate at which the consumer is just on the margin of trading or not trading. At any rate of exchange other than the MRS, the consumer would want to trade one good for the other. But if the rate of exchange equals the MRS, the consumer wants to stay put.

3.7 Other Interpretations of the MRS

We have said that the MRS measures the rate at which the consumer is just on the margin of being willing to substitute good 1 for good 2. We could also say that the consumer is just on the margin of being willing to "pay" some of good 1 in order to buy some more of good 2. So sometimes

you hear people say that the slope of the indifference curve measures the **marginal willingness to pay**.

If good 2 represents the consumption of "all other goods," and it is measured in dollars that you can spend on other goods, then the marginal-willingness-to-pay interpretation is very natural. The marginal rate of substitution of good 2 for good 1 is how many dollars you would just be willing to give up spending on other goods in order to consume a little bit more of good 1. Thus the MRS measures the marginal willingness to give up dollars in order to consume a small amount more of good 1. But giving up those dollars is just like paying dollars in order to consume a little more of good 1.

If you use the marginal-willingness-to-pay interpretation of the MRS, you should be careful to emphasize both the "marginal" and the "willingness" aspects. The MRS measures the amount of good 2 that one is *willing* to pay for a *marginal* amount of extra consumption of good 1. How much you actually *have* to pay for some given amount of extra consumption may be different than the amount you are willing to pay. How much you have to pay will depend on the price of the good in question. How much you are willing to pay doesn't depend on the price—it is determined by your preferences.

Similarly, how much you may be willing to pay for a large change in consumption may be different from how much you are willing to pay for a marginal change. How much you actually end up buying of a good will depend on your preferences for that good and the prices that you face. How much you would be willing to pay for a small amount extra of the good is a feature only of your preferences.

3.8 Behavior of the MRS

It is sometimes useful to describe the shapes of indifference curves by describing the behavior of the marginal rate of substitution. For example, the "perfect substitutes" indifference curves are characterized by the fact that the MRS is constant at -1. The "neutrals" case is characterized by the fact that the MRS is everywhere infinite. The preferences for "perfect complements" are characterized by the fact that the MRS is either zero or infinity, and nothing in between.

We've already pointed out that the assumption of monotonicity implies that indifference curves must have a negative slope, so the MRS always involves reducing the consumption of one good in order to get more of another for monotonic preferences.

The case of convex indifference curves exhibits yet another kind of behavior for the MRS. For strictly convex indifference curves, the MRS—the slope of the indifference curve—decreases (in absolute value) as we increase x_1. Thus the indifference curves exhibit a **diminishing marginal rate of**

substitution. This means that the amount of good 1 that the person is willing to give up for an additional amount of good 2 increases the amount of good 1 increases. Stated in this way, convexity of indifference curves seems very natural: it says that the more you have of one good, the more willing you are to give some of it up in exchange for the other good. (But remember the ice cream and olives example—for some pairs of goods this assumption might not hold!)

Summary

1. Economists assume that a consumer can rank various consumption possibilities. The way in which the consumer ranks the consumption bundles describes the consumer's preferences.

2. Indifference curves can be used to depict different kinds of preferences.

3. Well-behaved preferences are monotonic (meaning more is better) and convex (meaning averages are preferred to extremes).

4. The marginal rate of substitution (MRS) measures the slope of the indifference curve. This can be interpreted as how much the consumer is willing to give up of good 2 to acquire more of good 1.

REVIEW QUESTIONS

1. If we observe a consumer choosing (x_1, x_2) when (y_1, y_2) is available one time, are we justified in concluding that $(x_1, x_2) \succ (y_1, y_2)$?

2. Consider a group of people A, B, C and the relation "at least as tall as," as in "A is at least as tall as B." Is this relation transitive? Is it complete?

3. Take the same group of people and consider the relation "strictly taller than." Is this relation transitive? Is it reflexive? Is it complete?

4. A college football coach says that given any two linemen A and B, he always prefers the one who is bigger and faster. Is this preference relation transitive? Is it complete?

5. Can an indifference curve cross itself? For example, could Figure 3.2 depict a single indifference curve?

6. Could Figure 3.2 be a single indifference curve if preferences are monotonic?

7. If both pepperoni and anchovies are bads, will the indifference curve have a positive or a negative slope?

8. Explain why convex preferences means that "averages are preferred to extremes."

9. What is your marginal rate of substitution of $1 bills for $5 bills?

10. If good 1 is a "neutral," what is its marginal rate of substitution for good 2?

11. Think of some other goods for which your preferences might be concave.

CHAPTER 4

UTILITY

In Victorian days, philosophers and economists talked blithely of "utility" as an indicator of a person's overall well-being. Utility was thought of as a numeric measure of a person's happiness. Given this idea, it was natural to think of consumers making choices so as to maximize their utility, that is, to make themselves as happy as possible.

The trouble is that these classical economists never really described how we were to measure utility. How are we supposed to quantify the "amount" of utility associated with different choices? Is one person's utility the same as another's? What would it mean to say that an extra candy bar would give me twice as much utility as an extra carrot? Does the concept of utility have any independent meaning other than its being what people maximize?

Because of these conceptual problems, economists have abandoned the old-fashioned view of utility as being a measure of happiness. Instead, the theory of consumer behavior has been reformulated entirely in terms of **consumer preferences**, and utility is seen only as a *way to describe preferences.*

Economists gradually came to recognize that all that mattered about utility as far as choice behavior was concerned was whether one bundle had a higher utility than another—how much higher didn't really matter.

Originally, preferences were defined in terms of utility: to say a bundle (x_1, x_2) was preferred to a bundle (y_1, y_2) meant that the x-bundle had a higher utility than the y-bundle. But now we tend to think of things the other way around. The *preferences* of the consumer are the fundamental description useful for analyzing choice, and utility is simply a way of describing preferences.

A **utility function** is a way of assigning a number to every possible consumption bundle such that more-preferred bundles get assigned larger numbers than less-preferred bundles. That is, a bundle (x_1, x_2) is preferred to a bundle (y_1, y_2) if and only if the utility of (x_1, x_2) is larger than the utility of (y_1, y_2): in symbols, $(x_1, x_2) \succ (y_1, y_2)$ if and only if $u(x_1, x_2) > u(y_1, y_2)$.

The only property of a utility assignment that is important is how it *orders* the bundles of goods. The magnitude of the utility function is only important insofar as it *ranks* the different consumption bundles; the size of the utility difference between any two consumption bundles doesn't matter. Because of this emphasis on ordering bundles of goods, this kind of utility is referred to as **ordinal utility**.

Consider for example Table 4.1, where we have illustrated several different ways of assigning utilities to three bundles of goods, all of which order the bundles in the same way. In this example, the consumer prefers A to B and B to C. All of the ways indicated are valid utility functions that describe the same preferences because they all have the property that A is assigned a higher number than B, which in turn is assigned a higher number than C.

Different ways to assign utilities. **Table 4.1**

Bundle	U_1	U_2	U_3
A	3	17	-1
B	2	10	-2
C	1	.002	-3

Since only the ranking of the bundles matters, there can be no unique way to assign utilities to bundles of goods. If we can find one way to assign utility numbers to bundles of goods, we can find an infinite number of ways to do it. If $u(x_1, x_2)$ represents a way to assign utility numbers to the bundles (x_1, x_2), then multiplying $u(x_1, x_2)$ by 2 (or any other positive number) is just as good a way to assign utilities.

Multiplication by 2 is an example of a **monotonic transformation**. A

monotonic transformation is a way of transforming one set of numbers into another set of numbers in a way that preserves the order of the numbers.

We typically represent a monotonic transformation by a function $f(u)$ that transforms each number u into some other number $f(u)$, in a way that preserves the order of the numbers in the sense that $u_1 > u_2$ implies $f(u_1) > f(u_2)$. A monotonic transformation and a monotonic function are essentially the same thing.

Examples of monotonic transformations are multiplication by a positive number (e.g., $f(u) = 3u$), adding any number (e.g., $f(u) = u + 17$), raising u to an odd power (e.g., $f(u) = u^3$), and so on.[1]

The rate of change of $f(u)$ as u changes can be measured by looking at the change in f between two values of u, divided by the change in u:

$$\frac{\Delta f}{\Delta u} = \frac{f(u_2) - f(u_1)}{u_2 - u_1}.$$

For a monotonic transformation, $f(u_2) - f(u_1)$ always has the same sign as $u_2 - u_1$. Thus a monotonic function always has a positive rate of change. This means that the graph of a monotonic function will always have a positive slope, as depicted in Figure 4.1A.

Figure 4.1 **A positive monotonic transformation**. Panel A illustrates a monotonic function—one that is always increasing. Panel B illustrates a function that is *not* monotonic, since it sometimes increases and sometimes decreases.

[1] What we are calling a "monotonic transformation" is, strictly speaking, called a "positive monotonic transformation," in order to distinguish it from a "negative monotonic transformation," which is one that *reverses* the order of the numbers. Monotonic transformations are sometimes called "monotonous transformations," which seems unfair, since they can actually be quite interesting.

If $f(u)$ is *any* monotonic transformation of a utility function that represents some particular preferences, then $f(u(x_1, x_2))$ is also a utility function that represents those same preferences.

Why? The argument is given in the following three statements:

1. To say that $u(x_1, x_2)$ represents some particular preferences means that $u(x_1, x_2) > u(y_1, y_2)$ if and only if $(x_1, x_2) \succ (y_1, y_2)$.
2. But if $f(u)$ is a monotonic transformation, then $u(x_1, x_2) > u(y_1, y_2)$ if and only if $f(u(x_1, x_2)) > f(u(y_1, y_2))$.
3. Therefore, $f(u(x_1, x_2)) > f(u(y_1, y_2))$ if and only if $(x_1, x_2) \succ (y_1, y_2)$, so the function $f(u)$ represents the preferences in the same way as the original utility function $u(x_1, x_2)$.

We summarize this discussion by stating the following principle: *a monotonic transformation of a utility function is a utility function that represents the same preferences as the original utility function.*

Geometrically, a utility function is a way to label indifference curves. Since every bundle on an indifference curve must have the same utility, a utility function is a way of assigning numbers to the different indifference curves in a way that higher indifference curves get assigned larger numbers. Seen from this point of view a monotonic transformation is just a relabeling of indifference curves. As long as indifference curves containing more-preferred bundles get a larger label than indifference curves containing less-preferred bundles, the labeling will represent the same preferences.

4.1 Cardinal Utility

There are some theories of utility that attach a significance to the magnitude of utility. These are known as **cardinal utility** theories. In a theory of cardinal utility, the size of the utility difference between two bundles of goods is supposed to have some sort of significance.

We know how to tell whether a given person prefers one bundle of goods to another: we simply offer him or her a choice between the two bundles and see which one is chosen. Thus we know how to assign an ordinal utility to the two bundles of goods: we just assign a higher utility to the chosen bundle than to the rejected bundle. Any assignment that does this will be a utility function. Thus we have an operational criterion for determining whether one bundle has a higher utility than another bundle for some individual.

But how do we tell if a person likes one bundle twice as much as another? How could you even tell if *you* like one bundle twice as much as another?

One could propose various definitions for this kind of assignment: I like one bundle twice as much as another if I am willing to pay twice as much for it. Or, I like one bundle twice as much as another if I am willing to run

twice as far to get it, or to wait twice as long, or to gamble for it at twice the odds.

There is nothing wrong with any of these definitions; each one would give rise to a way of assigning utility levels in which the magnitude of the numbers assigned had some operational significance. But there isn't much right about them either. Although each of them is a possible interpretation of what it means to want one thing twice as much as another, none of them appears to be an especially compelling interpretation of that statement.

Even if we did find a way of assigning utility magnitudes that seemed to be especially compelling, what good would it do us in describing choice behavior? To tell whether one bundle or another will be chosen, we only have to know which is preferred—which has the larger utility. Knowing how much larger doesn't add anything to our description of choice. Since cardinal utility isn't needed to describe choice behavior and there is no compelling way to assign cardinal utilities anyway, we will stick with a purely ordinal utility framework.

4.2 Constructing a Utility Function

But are we assured that there is any way to assign ordinal utilities? Given a preference ordering can we always find a utility function that will order bundles of goods in the same way as those preferences? Is there a utility function that describes any reasonable preference ordering?

Not all kinds of preferences can be represented by a utility function. For example, suppose that someone had intransitive preferences so that $A \succ B \succ C \succ A$. Then a utility function for these preferences would have to consist of numbers $u(A)$, $u(B)$, and $u(C)$ such that $u(A) > u(B) > u(C) > u(A)$. But this is impossible.

However, if we rule out perverse cases like intransitive preferences, it turns out that we will typically be able to find a utility function to represent preferences. We will illustrate one construction here, and another one in Chapter 14.

Suppose that we are given an indifference map as in Figure 4.2. We know that a utility function is a way to label the indifference curves such that higher indifference curves get larger numbers. How can we do this?

One easy way is to draw the diagonal line illustrated and label each indifference curve with its distance from the origin measured along the line.

How do we know that this is a utility function? It is not hard to see that if preferences are monotonic then the line through the origin must intersect every indifference curve exactly once. Thus every bundle is getting a label, and those bundles on higher indifference curves are getting larger labels—and that's all it takes to be a utility function.

Constructing a utility function from indifference curves. Draw a diagonal line and label each indifference curve with how far it is from the origin measured along the line.

Figure 4.2

This gives us one way to find a labeling of indifference curves, at least as long as preferences are monotonic. This won't always be the most natural way in any given case, but at least it shows that the idea of an ordinal utility function is pretty general: nearly any kind of "reasonable" preferences can be represented by a utility function.

4.3 Some Examples of Utility Functions

In Chapter 3 we described some examples of preferences and the indifference curves that represented them. We can also represent these preferences by utility functions. If you are given a utility function, $u(x_1, x_2)$, it is relatively easy to draw the indifference curves: you just plot all the points (x_1, x_2) such that $u(x_1, x_2)$ equals a constant. In mathematics, the set of all (x_1, x_2) such that $u(x_1, x_2)$ equals a constant is called a **level set**. For each different value of the constant, you get a different indifference curve.

EXAMPLE: Indifference Curves from Utility

Suppose that the utility function is given by: $u(x_1, x_2) = x_1 x_2$. What do the indifference curves look like?

We know that a typical indifference curve is just the set of all x_1 and x_2 such that $k = x_1 x_2$ for some constant k. Solving for x_2 as a function of x_1, we see that a typical indifference curve has the formula:

$$x_2 = \frac{k}{x_1}.$$

This curve is depicted in Figure 4.3 for $k = 1, 2, 3 \cdots$.

Figure 4.3 **Indifference curves**. The indifference curves $k = x_1 x_2$ for different values of k.

Let's consider another example. Suppose that we were given a utility function $v(x_1, x_2) = x_1^2 x_2^2$. What do its indifference curves look like? By the standard rules of algebra we know that:

$$v(x_1, x_2) = x_1^2 x_2^2 = (x_1 x_2)^2 = u(x_1, x_2)^2.$$

Thus the utility function $v(x_1, x_2)$ is just the square of the utility function $u(x_1, x_2)$. Since $u(x_1, x_2)$ cannot be negative, it follows that $v(x_1, x_2)$ is a monotonic transformation of the previous utility function, $u(x_1, x_2)$. This means that the utility function $v(x_1, x_2) = x_1^2 x_2^2$ has to have exactly the same shaped indifference curves as those depicted in Figure 4.3. The labeling of the indifference curves will be different—the labels that were $1, 2, 3, \cdots$ will now be $1, 4, 9, \cdots$—but the set of bundles that has $v(x_1, x_2) =$

9 is exactly the same as the set of bundles that has $u(x_1, x_2) = 3$. Thus $v(x_1, x_2)$ describes exactly the same preferences as $u(x_1, x_2)$ since it *orders* all of the bundles in the same way.

Going the other direction—finding a utility function that represents some indifference curves—is somewhat more difficult. There are two ways to proceed. The first way is mathematical. Given the indifference curves, we want to find a function that is constant along each indifference curve and that assigns higher values to higher indifference curves.

The second way is a bit more intuitive. Given a description of the preferences, we try to think about what the consumer is trying to maximize—what combination of the goods describes the choice behavior of the consumer. This may seem a little vague at the moment, but it will be more meaningful after we discuss a few examples.

Perfect Substitutes

Remember the red pencil and blue pencil example? All that mattered to the consumer was the total number of pencils. Thus it is natural to measure utility by the total number of pencils. Therefore we provisionally pick the utility function $u(x_1, x_2) = x_1 + x_2$. Does this work? Just ask two things: is this utility function constant along the indifference curves? Does it assign a higher label to more-preferred bundles? The answer to both questions is yes, so we have a utility function.

Of course, this isn't the only utility function that we could use. We could also use the *square* of the number of pencils. Thus the utility function $v(x_1, x_2) = (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$ will also represent the perfect-substitutes preferences, as would any other monotonic transformation of $u(x_1, x_2)$.

What if the consumer is willing to substitute good 1 for good 2 at a rate that is different from one-to-one? Suppose, for example, that the consumer would require *two* units of good 2 to compensate him for giving up one unit of good 1. This means that good 1 is *twice* as valuable to the consumer as good 2. The utility function therefore takes the form $u(x_1, x_2) = 2x_1 + x_2$. Note that this utility yields indifference curves with a slope of -2.

In general, preferences for perfect substitutes can be represented by a utility function of the form

$$u(x_1, x_2) = ax_1 + bx_2.$$

Here a and b are some positive numbers that measure the "value" of goods 1 and 2 to the consumer. Note that the slope of a typical indifference curve is given by $-a/b$.

Perfect Complements

This is the left shoe–right shoe case. In these preferences the consumer only cares about the number of *pairs* of shoes he has, so it is natural to choose the number of pairs of shoes as the utility function. The number of complete pairs of shoes that you have is the *minimum* of the number of right shoes you have, x_1, and the number of left shoes you have, x_2. Thus the utility function for perfect complements takes the form $u(x_1, x_2) = \min\{x_1, x_2\}$.

To verify that this utility function actually works, pick a bundle of goods such as $(10, 10)$. If we add one more unit of good 1 we get $(11, 10)$, which should leave us on the same indifference curve. Does it? Yes, since $\min\{10, 10\} = \min\{11, 10\} = 10$.

So $u(x_1, x_2) = \min\{x_1, x_2\}$ is a possible utility function to describe perfect complements. As usual, any monotonic transformation would be suitable as well.

What about the case where the consumer wants to consume the goods in some proportion other than one-to-one? For example, what about the consumer who always uses 2 teaspoons of sugar with each cup of tea? If x_1 is the number of cups of tea available and x_2 is the number of teaspoons of sugar available, then the number of correctly sweetened cups of tea will be $\min\{x_1, \frac{1}{2}x_2\}$.

This is a little tricky so we should stop to think about it. If the number of cups of tea is greater than half the number of teaspoons of sugar, then we know that we won't be able to put 2 teaspoons of sugar in each cup. In this case, we will only end up with $\frac{1}{2}x_2$ correctly sweetened cups of tea. (Substitute some numbers in for x_1 and x_2 to convince yourself.)

Of course, any monotonic transformation of this utility function will describe the same preferences. For example, we might want to multiply by 2 to get rid of the fraction. This gives us the utility function $u(x_1, x_2) = \min\{2x_1, x_2\}$.

In general, a utility function that describes perfect-complement preferences is given by

$$u(x_1, x_2) = \min\{ax_1, bx_2\},$$

where a and b are positive numbers that indicate the proportions in which the goods are consumed.

Quasilinear Preferences

Here's a shape of indifference curves that we haven't seen before. Suppose that a consumer has indifference curves that are vertical translates of one another, as in Figure 4.4. This means that all of the indifference curves are just vertically "shifted" versions of one indifference curve. It follows that

the equation for an indifference curve takes the form $x_2 = k - v(x_1)$, where k is a different constant for each indifference curve. This equation says that the height of each indifference curve is some function of x_1, $-v(x_1)$, plus a constant k. Higher values of k give higher indifference curves. (The minus sign is only a convention; we'll see why it is convenient below.)

Quasilinear preferences. Each indifference curve is a vertically shifted version of a single indifference curve.

Figure 4.4

The natural way to label indifference curves here is with k—roughly speaking, the height of the indifference curve along the vertical axis. Solving for k and setting it equal to utility, we have

$$u(x_1, x_2) = k = v(x_1) + x_2.$$

In this case the utility function is linear in good 2, but (possibly) nonlinear in good 1; hence the name **quasilinear utility**, meaning "partly linear" utility. Specific examples of quasilinear utility would be $u(x_1, x_2) = \sqrt{x_1} + x_2$, or $u(x_1, x_2) = \ln x_1 + x_2$. Quasilinear utility functions are not particularly realistic, but they are very easy to work with, as we'll see in several examples later on in the book.

Cobb-Douglas Preferences

Another commonly used utility function is the **Cobb-Douglas** utility function

$$u(x_1, x_2) = x_1^c x_2^d,$$

where c and d are positive numbers that describe the preferences of the consumer.[2]

The Cobb-Douglas utility function will be useful in several examples. The preferences represented by the Cobb-Douglas utility function have the general shape depicted in Figure 4.5. In Figure 4.5A, we have illustrated the indifference curves for $c = 1/2$, $d = 1/2$. In Figure 4.5B, we have illustrated the indifference curves for $c = 1/5$, $d = 4/5$. Note how different values of the parameters c and d lead to different shapes of the indifference curves.

Figure 4.5 **Cobb-Douglas indifference curves**. Panel A shows the case where $c = 1/2$, $d = 1/2$ and panel B shows the case where $c = 1/5$, $d = 4/5$.

Cobb-Douglas indifference curves look just like the nice convex monotonic indifference curves that we referred to as "well-behaved indifference curves" in Chapter 3. Cobb-Douglas preferences are the standard example of indifference curves that look well-behaved, and in fact the formula describing them is about the simplest algebraic expression that generates well-behaved preferences. We'll find Cobb-Douglas preferences quite useful to present algebraic examples of the economic ideas we'll study later.

Of course a monotonic transformation of the Cobb-Douglas utility function will represent exactly the same preferences, and it is useful to see a couple of examples of these transformations.

[2] Paul Douglas was a twentieth-century economist at the University of Chicago who later became a U.S. senator. Charles Cobb was a mathematician at Amherst College. The Cobb-Douglas functional form was originally used to study production behavior.

First, if we take the natural log of utility, the product of the terms will become a sum so that we have

$$v(x_1, x_2) = \ln(x_1^c x_2^d) = c \ln x_1 + d \ln x_2.$$

The indifference curves for this utility function will look just like the ones for the first Cobb-Douglas function, since the logarithm is a monotonic transformation. (For a brief review of natural logarithms, see the Mathematical Appendix at the end of the book.)

For the second example, suppose that we start with the Cobb-Douglas form

$$v(x_1, x_2) = x_1^c x_2^d.$$

Then raising utility to the $1/(c+d)$ power, we have

$$x_1^{\frac{c}{c+d}} x_2^{\frac{d}{c+d}}.$$

Now define a new number

$$a = \frac{c}{c+d}.$$

We can now write our utility function as

$$v(x_1, x_2) = x_1^a x_2^{1-a}.$$

This means that we can always take a monotonic transformation of the Cobb-Douglas utility function that make the exponents sum to 1. This will turn out to have a useful interpretation later on.

The Cobb-Douglas utility function can be expressed in a variety of ways; you should learn to recognize them, as this family of preferences is very useful for examples.

4.4 Marginal Utility

Consider a consumer who is consuming some bundle of goods, (x_1, x_2). How does this consumer's utility change as we give him or her a little more of good 1? This rate of change is called the **marginal utility** with respect to good 1. We write it as MU_1 and think of it as being a ratio,

$$MU_1 = \frac{\Delta U}{\Delta x_1} = \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1},$$

that measures the rate of change in utility (ΔU) associated with a small change in the amount of good 1 (Δx_1). Note that the amount of good 2 is held fixed in this calculation.[3]

[3] See the appendix to this chapter for a calculus treatment of marginal utility.

This definition implies that to calculate the change in utility associated with a small change in consumption of good 1, we can just multiply the change in consumption by the marginal utility of the good:

$$\Delta U = MU_1 \Delta x_1.$$

The marginal utility with respect to good 2 is defined in a similar manner:

$$MU_2 = \frac{\Delta U}{\Delta x_2} = \frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{\Delta x_2}.$$

Note that when we compute the marginal utility with respect to good 2 we keep the amount of good 1 constant. We can calculate the change in utility associated with a change in the consumption of good 2 by the formula

$$\Delta U = MU_2 \Delta x_2.$$

It is important to realize that the magnitude of marginal utility depends on the magnitude of utility. Thus it depends on the particular way that we choose to measure utility. If we multiplied utility by 2, then marginal utility would also be multiplied by 2. We would still have a perfectly valid utility function in that it would represent the same preferences, but it would just be scaled differently.

This means that marginal utility itself has no behavioral content. How can we calculate marginal utility from a consumer's choice behavior? We can't. Choice behavior only reveals information about the way a consumer *ranks* different bundles of goods. Marginal utility depends on the particular utility function that we use to reflect the preference ordering and its magnitude has no particular significance. However, it turns out that marginal utility can be used to calculate something that does have behavioral content, as we will see in the next section.

4.5 Marginal Utility and MRS

A utility function $u(x_1, x_2)$ can be used to measure the marginal rate of substitution (MRS) defined in Chapter 3. Recall that the MRS measures the slope of the indifference curve at a given bundle of goods; it can be interpreted as the rate at which a consumer is just willing to substitute a small amount of good 2 for good 1.

This interpretation gives us a simple way to calculate the MRS. Consider a change in the consumption of each good, $(\Delta x_1, \Delta x_2)$, that keeps utility constant—that is, a change in consumption that moves us along the indifference curve. Then we must have

$$MU_1 \Delta x_1 + MU_2 \Delta x_2 = \Delta U = 0.$$

Solving for the slope of the indifference curve we have

$$\text{MRS} = \frac{\Delta x_2}{\Delta x_1} = -\frac{MU_1}{MU_2}. \tag{4.1}$$

(Note that we have 2 over 1 on the left-hand side of the equation and 1 over 2 on the right-hand side. Don't get confused!)

The algebraic sign of the MRS is negative: if you get more of good 1 you have to get *less* of good 2 in order to keep the same level of utility. However, it gets very tedious to keep track of that pesky minus sign, so economists often refer to the MRS by its absolute value—that is, as a positive number. We'll follow this convention as long as no confusion will result.

Now here is the interesting thing about the MRS calculation: the MRS can be measured by observing a person's actual behavior—we find that rate of exchange where he or she is just willing to stay put, as described in Chapter 3.

The utility function, and therefore the marginal utility function, is not uniquely determined. Any monotonic transformation of a utility function leaves you with another equally valid utility function. Thus, if we multiply utility by 2, for example, the marginal utility is multiplied by 2. Thus the magnitude of the marginal utility function depends on the choice of utility function, which is arbitrary. It doesn't depend on behavior alone; instead it depends on the utility function that we use to describe behavior.

But the *ratio* of marginal utilities gives us an observable magnitude—namely the marginal rate of substitution. The ratio of marginal utilities is independent of the particular transformation of the utility function you choose to use. Look at what happens if you multiply utility by 2. The MRS becomes

$$\text{MRS} = -\frac{2MU_1}{2MU_2}.$$

The 2s just cancel out, so the MRS remains the same.

The same sort of thing occurs when we take any monotonic transformation of a utility function. Taking a monotonic transformation is just relabeling the indifference curves, and the calculation for the MRS described above is concerned with moving along a given indifference curve. Even though the marginal utilities are changed by monotonic transformations, the *ratio* of marginal utilities is independent of the particular way chosen to represent the preferences.

4.6 Utility for Commuting

Utility functions are basically ways of describing choice behavior: if a bundle of goods X is chosen when a bundle of goods Y is available, then X must have a higher utility than Y. By examining choices consumers make we can estimate a utility function to describe their behavior.

This idea has been widely applied in the field of transportation economics to study consumers' commuting behavior. In most large cities commuters have a choice between taking public transit or driving to work. Each of these alternatives can be thought of as representing a bundle of different characteristics: travel time, waiting time, out-of-pocket costs, comfort, convenience, and so on. We could let x_1 be the amount of travel time involved in each kind of transportation, x_2 the amount of waiting time for each kind, and so on.

If $(x_1, x_2, \ldots, x_n)$ represents the values of n different characteristics of driving, say, and $(y_1, y_2, \ldots, y_n)$ represents the values of taking the bus, we can consider a model where the consumer decides to drive or take the bus depending on whether he prefers one bundle of characteristics to the other.

More specifically, let us suppose that the average consumer's preferences for characteristics can be represented by a utility function of the form

$$U(x_1, x_2, \ldots, x_n) = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n,$$

where the coefficients β_1, β_2, and so on are unknown parameters. Any monotonic transformation of this utility function would describe the choice behavior equally well, of course, but the linear form is especially easy to work with from a statistical point of view.

Suppose now that we observe a number of similar consumers making choices between driving and taking the bus based on the particular pattern of commute times, costs, and so on that they face. There are statistical techniques that can be used to find the values of the coefficients β_i for $i = 1, \ldots, n$ that best fit the observed pattern of choices by a set of consumers. These statistical techniques give a way to estimate the utility function for different transportation modes.

One study reports a utility function that had the form[4]

$$U(TW, TT, C) = -0.147TW - 0.0411TT - 2.24C, \tag{4.2}$$

where

TW = total walking time to and from bus or car
TT = total time of trip in minutes
C = total cost of trip in dollars

The estimated utility function in the Domenich-McFadden book correctly described the choice between auto and bus transport for 93 percent of the households in their sample.

[4] See Thomas Domenich and Daniel McFadden, *Urban Travel Demand* (North-Holland Publishing Company, 1975). The estimation procedure in this book also incorporated various demographic characteristics of the households in addition to the purely economic variables described here. Daniel McFadden was awarded the Nobel Prize in economics in 2000 for his work in developing techniques to estimate models of this sort.

The coefficients on the variables in Equation (4.2) describe the weight that an average household places on the various characteristics of their commuting trips; that is, the marginal utility of each characteristic. The *ratio* of one coefficient to another measures the marginal rate of substitution between one characteristic and another. For example, the ratio of the marginal utility of walking time to the marginal utility of total time indicates that walking time is viewed as being roughly 3 times as onerous as travel time by the average consumer. In other words, the consumer would be willing to substitute 3 minutes of additional travel time to save 1 minute of walking time.

Similarly, the ratio of cost to travel time indicates the average consumer's tradeoff between these two variables. In this study, the average commuter valued a minute of commute time at $0.0411/2.24 = 0.0183$ dollars per minute, which is \$1.10 per hour. For comparison, the hourly wage for the average commuter in 1967, the year of the study, was about \$2.85 an hour.

Such estimated utility functions can be very valuable for determining whether or not it is worthwhile to make some change in the public transportation system. For example, in the above utility function one of the significant factors explaining mode choice is the time involved in taking the trip. The city transit authority can, at some cost, add more buses to reduce this travel time. But will the number of extra riders warrant the increased expense?

Given a utility function and a sample of consumers we can forecast which consumers will drive and which consumers will choose to take the bus. This will give us some idea as to whether the revenue will be sufficient to cover the extra cost.

Furthermore, we can use the marginal rate of substitution to estimate the *value* that each consumer places on the reduced travel time. We saw above that in the Domenich-McFadden study the average commuter in 1967 valued commute time at a rate of \$1.10 per hour. Thus the commuter should be willing to pay about \$0.37 to cut 20 minutes from his or her trip. This number gives us a measure of the dollar benefit of providing more timely bus service. This benefit must be compared to the cost of providing more timely bus service in order to determine if such provision is worthwhile. Having a quantitative measure of benefit will certainly be helpful in making a rational decision about transport policy.

Summary

1. A utility function is simply a way to represent or summarize a preference ordering. The numerical magnitudes of utility levels have no intrinsic meaning.

2. Thus, given any one utility function, any monotonic transformation of it will represent the same preferences.

3. The marginal rate of substitution, MRS, can be calculated from the utility function via the formula $\text{MRS} = \Delta x_2/\Delta x_1 = -MU_1/MU_2$.

REVIEW QUESTIONS

1. The text said that raising a number to an odd power was a monotonic transformation. What about raising a number to an even power? Is this a monotonic transformation? (Hint: consider the case $f(u) = u^2$.)

2. Which of the following are monotonic transformations? (1) $u = 2v - 13$; (2) $u = -1/v^2$; (3) $u = 1/v^2$; (4) $u = \ln v$; (5) $u = -e^{-v}$; (6) $u = v^2$; (7) $u = v^2$ for $v > 0$; (8) $u = v^2$ for $v < 0$.

3. We claimed in the text that if preferences were monotonic, then a diagonal line through the origin would intersect each indifference curve exactly once. Can you prove this rigorously? (Hint: what would happen if it intersected some indifference curve twice?)

4. What kind of preferences are represented by a utility function of the form $u(x_1, x_2) = \sqrt{x_1 + x_2}$? What about the utility function $v(x_1, x_2) = 13x_1 + 13x_2$?

5. What kind of preferences are represented by a utility function of the form $u(x_1, x_2) = x_1 + \sqrt{x_2}$? Is the utility function $v(x_1, x_2) = x_1^2 + 2x_1\sqrt{x_2} + x_2$ a monotonic transformation of $u(x_1, x_2)$?

6. Consider the utility function $u(x_1, x_2) = \sqrt{x_1 x_2}$. What kind of preferences does it represent? Is the function $v(x_1, x_2) = x_1^2 x_2$ a monotonic transformation of $u(x_1, x_2)$? Is the function $w(x_1, x_2) = x_1^2 x_2^2$ a monotonic transformation of $u(x_1, x_2)$?

7. Can you explain why taking a monotonic transformation of a utility function doesn't change the marginal rate of substitution?

APPENDIX

First, let us clarify what is meant by "marginal utility." As elsewhere in economics, "marginal" just means a derivative. So the marginal utility of good 1 is just

$$MU_1 = \lim_{\Delta x_1 \to 0} \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1}.$$

Note that we have used the *partial* derivative here, since the marginal utility of good 1 is computed holding good 2 fixed.

Now we can rephrase the derivation of the MRS given in the text using calculus. We'll do it two ways: first by using differentials, and second by using implicit functions.

For the first method, we consider making a change (dx_1, dx_2) that keeps utility constant. So we want

$$du = \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 = 0.$$

The first term measures the increase in utility from the small change dx_1, and the second term measures the increase in utility from the small change dx_2. We want to pick these changes so that the total change in utility, du, is zero. Solving for dx_2/dx_1 gives us

$$\frac{dx_2}{dx_1} = -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2},$$

which is just the calculus analog of equation (4.1) in the text.

As for the second method, we now think of the indifference curve as being described by a function $x_2(x_1)$. That is, for each value of x_1, the function $x_2(x_1)$ tells us how much x_2 we need to get on that specific indifference curve. Thus the function $x_2(x_1)$ has to satisfy the identity

$$u(x_1, x_2(x_1)) \equiv k,$$

where k is the utility label of the indifference curve in question.

We can differentiate both sides of this identity with respect to x_1 to get

$$\frac{\partial u(x_1, x_2)}{\partial x_1} + \frac{\partial u(x_1, x_2)}{\partial x_2} \frac{\partial x_2(x_1)}{\partial x_1} = 0.$$

Notice that x_1 occurs in two places in this identity, so changing x_1 will change the function in two ways, and we have to take the derivative at each place that x_1 appears.

We then solve this equation for $\partial x_2(x_1)/\partial x_1$ to find

$$\frac{\partial x_2(x_1)}{\partial x_1} = -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2},$$

just as we had before.

The implicit function method is a little more rigorous, but the differential method is more direct, as long as you don't do something silly.

Suppose that we take a monotonic transformation of a utility function, say, $v(x_1, x_2) = f(u(x_1, x_2))$. Let's calculate the MRS for this utility function. Using the chain rule

$$\begin{aligned} \text{MRS} = -\frac{\partial v/\partial x_1}{\partial v/\partial x_2} &= -\frac{\partial f/\partial u}{\partial f/\partial u} \frac{\partial u/\partial x_1}{\partial u/\partial x_2} \\ &= -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} \end{aligned}$$

since the $\partial f/\partial u$ term cancels out from both the numerator and denominator. This shows that the MRS is independent of the utility representation.

This gives a useful way to recognize preferences that are represented by different utility functions: given two utility functions, just compute the marginal rates of substitution and see if they are the same. If they are, then the two utility functions have the same indifference curves. If the direction of increasing preference is the same for each utility function, then the underlying preferences must be the same.

EXAMPLE: Cobb-Douglas Preferences

The MRS for Cobb-Douglas preferences is easy to calculate by using the formula derived above.

If we choose the log representation where

$$u(x_1, x_2) = c \ln x_1 + d \ln x_2,$$

then we have

$$\begin{aligned} \text{MRS} &= -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} \\ &= -\frac{c/x_1}{d/x_2} \\ &= -\frac{c}{d}\frac{x_2}{x_1}. \end{aligned}$$

Note that the MRS only depends on the ratio of the two parameters and the quantity of the two goods in this case.

What if we choose the exponent representation where

$$u(x_1, x_2) = x_1^c x_2^d?$$

Then we have

$$\begin{aligned} \text{MRS} &= -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} \\ &= -\frac{c x_1^{c-1} x_2^d}{d x_1^c x_2^{d-1}} \\ &= -\frac{c x_2}{d x_1}, \end{aligned}$$

which is the same as we had before. Of course you knew all along that a monotonic transformation couldn't change the marginal rate of substitution!

CHAPTER 5

CHOICE

In this chapter we will put together the budget set and the theory of preferences in order to examine the optimal choice of consumers. We said earlier that the economic model of consumer choice is that people choose the best bundle they can afford. We can now rephrase this in terms that sound more professional by saying that "consumers choose the most preferred bundle from their budget sets."

5.1 Optimal Choice

A typical case is illustrated in Figure 5.1. Here we have drawn the budget set and several of the consumer's indifference curves on the same diagram. We want to find the bundle in the budget set that is on the highest indifference curve. Since preferences are well-behaved, so that more is preferred to less, we can restrict our attention to bundles of goods that lie *on* the budget line and not worry about those *beneath* the budget line.

Now simply start at the right-hand corner of the budget line and move to the left. As we move along the budget line we note that we are moving to higher and higher indifference curves. We stop when we get to the highest

indifference curve that just touches the budget line. In the diagram, the bundle of goods that is associated with the highest indifference curve that just touches the budget line is labeled (x_1^*, x_2^*).

The choice (x_1^*, x_2^*) is an **optimal choice** for the consumer. The set of bundles that she prefers to (x_1^*, x_2^*)—the set of bundles *above* her indifference curve—doesn't intersect the bundles she can afford—the bundles *beneath* her budget line. Thus the bundle (x_1^*, x_2^*) is the best bundle that the consumer can afford.

Figure 5.1 **Optimal choice**. The optimal consumption position is where the indifference curve is tangent to the budget line.

Note an important feature of this optimal bundle: at this choice, the indifference curve is *tangent* to the budget line. If you think about it a moment you'll see that this has to be the case: if the indifference curve weren't tangent, it would cross the budget line, and if it crossed the budget line, there would be some nearby point on the budget line that lies above the indifference curve—which means that we couldn't have started at an optimal bundle.

Does this tangency condition really *have* to hold at an optimal choice? Well, it doesn't hold in *all* cases, but it does hold for most interesting cases. What is always true is that at the optimal point the indifference curve can't cross the budget line. So when does "not crossing" imply tangent? Let's look at the exceptions first.

First, the indifference curve might not have a tangent line, as in Figure 5.2. Here the indifference curve has a kink at the optimal choice, and a tangent just isn't defined, since the mathematical definition of a tangent requires that there be a unique tangent line at each point. This case doesn't have much economic significance—it is more of a nuisance than anything else.

Kinky tastes. Here is an optimal consumption bundle where the indifference curve doesn't have a tangent.

Figure 5.2

The second exception is more interesting. Suppose that the optimal point occurs where the consumption of some good is zero as in Figure 5.3. Then the slope of the indifference curve and the slope of the budget line are different, but the indifference curve still doesn't *cross* the budget line.

We say that Figure 5.3 represents a **boundary optimum**, while a case like Figure 5.1 represents an **interior optimum**.

If we are willing to rule out "kinky tastes" we can forget about the example given in Figure 5.2.[1] And if we are willing to restrict ourselves only to *interior* optima, we can rule out the other example. If we have an interior optimum with smooth indifference curves, the slope of the indifference curve and the slope of the budget line must be the same . . . because if they were different the indifference curve would cross the budget line, and we couldn't be at the optimal point.

Figure 5.3

Boundary optimum. The optimal consumption involves consuming zero units of good 2. The indifference curve is not tangent to the budget line.

We've found a necessary condition that the optimal choice must satisfy. If the optimal choice involves consuming some of both goods—so that it is an interior optimum—then necessarily the indifference curve will be tangent to the budget line. But is the tangency condition a *sufficient* condition for a bundle to be optimal? If we find a bundle where the indifference curve is tangent to the budget line, can we be sure we have an optimal choice?

Look at Figure 5.4. Here we have three bundles where the tangency condition is satisfied, all of them interior, but only two of them are optimal.

[1] Otherwise, this book might get an R rating.

So in general, the tangency condition is only a necessary condition for optimality, not a sufficient condition.

More than one tangency. Here there are three tangencies, but only two optimal points, so the tangency condition is necessary but not sufficient.

Figure 5.4

However, there is one important case where it is sufficient: the case of convex preferences. In the case of convex preferences, any point that satisfies the tangency condition must be an optimal point. This is clear geometrically: since convex indifference curves must curve away from the budget line, they can't bend back to touch it again.

Figure 5.4 also shows us that in general there may be more than one optimal bundle that satisfies the tangency condition. However, again convexity implies a restriction. If the indifference curves are *strictly* convex—they don't have any flat spots—then there will be only one optimal choice on each budget line. Although this can be shown mathematically, it is also quite plausible from looking at the figure.

The condition that the MRS must equal the slope of the budget line at an interior optimum is obvious graphically, but what does it mean economically? Recall that one of our interpretations of the MRS is that it is that rate of exchange at which the consumer is just willing to stay put. Well, the market is offering a rate of exchange to the consumer of $-p_1/p_2$—if

you give up one unit of good 1, you can buy p_1/p_2 units of good 2. If the consumer is at a consumption bundle where he or she is willing to stay put, it must be one where the MRS is equal to this rate of exchange:

$$\text{MRS} = -\frac{p_1}{p_2}.$$

Another way to think about this is to imagine what would happen if the MRS were different from the price ratio. Suppose, for example, that the MRS is $\Delta x_2/\Delta x_1 = -1/2$ and the price ratio is $1/1$. Then this means the consumer is just willing to give up 2 units of good 1 in order to get 1 unit of good 2—but the market is willing to exchange them on a one-to-one basis. Thus the consumer would certainly be willing to give up some of good 1 in order to purchase a little more of good 2. Whenever the MRS is different from the price ratio, the consumer cannot be at his or her optimal choice.

5.2 Consumer Demand

The optimal choice of goods 1 and 2 at some set of prices and income is called the consumer's **demanded bundle**. In general when prices and income change, the consumer's optimal choice will change. The **demand function** is the function that relates the optimal choice—the quantities demanded—to the different values of prices and incomes.

We will write the demand functions as depending on both prices and income: $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$. For each different set of prices and income, there will be a different combination of goods that is the optimal choice of the consumer. Different preferences will lead to different demand functions; we'll see some examples shortly. Our major goal in the next few chapters is to study the behavior of these demand functions—how the optimal choices change as prices and income change.

5.3 Some Examples

Let us apply the model of consumer choice we have developed to the examples of preferences described in Chapter 3. The basic procedure will be the same for each example: plot the indifference curves and budget line and find the point where the highest indifference curve touches the budget line.

Perfect Substitutes

The case of perfect substitutes is illustrated in Figure 5.5. We have three possible cases. If $p_2 > p_1$, then the slope of the budget line is flatter than the slope of the indifference curves. In this case, the optimal bundle is

where the consumer spends all of his or her money on good 1. If $p_1 > p_2$, then the consumer purchases only good 2. Finally, if $p_1 = p_2$, there is a whole range of optimal choices—any amount of goods 1 and 2 that satisfies the budget constraint is optimal in this case. Thus the demand function for good 1 will be

$$x_1 = \begin{cases} m/p_1 & \text{when } p_1 < p_2; \\ \text{any number between 0 and } m/p_1 & \text{when } p_1 = p_2; \\ 0 & \text{when } p_1 > p_2. \end{cases}$$

Are these results consistent with common sense? All they say is that if two goods are perfect substitutes, then a consumer will purchase the cheaper one. If both goods have the same price, then the consumer doesn't care which one he or she purchases.

Optimal choice with perfect substitutes. If the goods are perfect substitutes, the optimal choice will usually be on the boundary.

Figure 5.5

Perfect Complements

The case of perfect complements is illustrated in Figure 5.6. Note that the optimal choice must always lie on the diagonal, where the consumer is purchasing equal amounts of both goods, no matter what the prices are.

In terms of our example, this says that people with two feet buy shoes in pairs.[2]

Let us solve for the optimal choice algebraically. We know that this consumer is purchasing the same amount of good 1 and good 2, no matter what the prices. Let this amount be denoted by x. Then we have to satisfy the budget constraint

$$p_1 x + p_2 x = m.$$

Solving for x gives us the optimal choices of goods 1 and 2:

$$x_1 = x_2 = x = \frac{m}{p_1 + p_2}.$$

The demand function for the optimal choice here is quite intuitive. Since the two goods are always consumed together, it is just as if the consumer were spending all of her money on a single good that had a price of $p_1 + p_2$.

Figure 5.6 **Optimal choice with perfect complements**. If the goods are perfect complements, the quantities demanded will always lie on the diagonal since the optimal choice occurs where x_1 equals x_2.

[2] Don't worry, we'll get some more exciting results later on.

Discrete goods. In panel A the demand for good 1 is zero, while in panel B one unit will be demanded.

Figure 5.7

Neutrals and Bads

In the case of a neutral good the consumer spends all of her money on the good she likes and doesn't purchase any of the neutral good. The same thing happens if one commodity is a bad. Thus, if commodity 1 is a good and commodity 2 is a bad, then the demand functions will be

$$x_1 = \frac{m}{p_1}$$
$$x_2 = 0.$$

Discrete Goods

Suppose that good 1 is a discrete good that is available only in integer units, while good 2 is money to be spent on everything else. If the consumer chooses $1, 2, 3, \cdots$ units of good 1, she will implicitly choose the consumption bundles $(1, m - p_1)$, $(2, m - 2p_1)$, $(3, m - 3p_1)$, and so on. We can simply compare the utility of each of these bundles to see which has the highest utility.

Alternatively, we can use the indifference-curve analysis in Figure 5.7. As usual, the optimal bundle is the one on the highest indifference "curve." If the price of good 1 is very high, then the consumer will choose zero units of consumption; as the price decreases the consumer will find it optimal to consume 1 unit of the good. Typically, as the price decreases further the consumer will choose to consume more units of good 1.

Concave Preferences

Consider the situation illustrated in Figure 5.8. Is X the optimal choice? No! The optimal choice for these preferences is always going to be a boundary choice, like bundle Z. Think of what nonconvex preferences mean. If you have money to purchase ice cream and olives, and you don't like to consume them together, you'll spend all of your money on one or the other.

Figure 5.8 **Optimal choice with concave preferences**. The optimal choice is the boundary point, Z, not the interior tangency point, X, because Z lies on a higher indifference curve.

Cobb-Douglas Preferences

Suppose that the utility function is of the Cobb-Douglas form, $u(x_1, x_2) = x_1^c x_2^d$. In the Appendix to this chapter we use calculus to derive the optimal

choices for this utility function. They turn out to be

$$x_1 = \frac{c}{c+d}\frac{m}{p_1}$$
$$x_2 = \frac{d}{c+d}\frac{m}{p_2}.$$

These demand functions are often useful in algebraic examples, so you should probably memorize them.

The Cobb-Douglas preferences have a convenient property. Consider the fraction of his income that a Cobb-Douglas consumer spends on good 1. If he consumes x_1 units of good 1, this costs him p_1x_1, so this represents a fraction p_1x_1/m of total income. Substituting the demand function for x_1 we have

$$\frac{p_1x_1}{m} = \frac{p_1}{m}\frac{c}{c+d}\frac{m}{p_1} = \frac{c}{c+d}.$$

Similarly the fraction of his income that the consumer spends on good 2 is $d/(c+d)$.

Thus the Cobb-Douglas consumer always spends a fixed fraction of his income on each good. The size of the fraction is determined by the exponent in the Cobb-Douglas function.

This is why it is often convenient to choose a representation of the Cobb-Douglas utility function in which the exponents sum to 1. If $u(x_1, x_2) = x_1^a x_2^{1-a}$, then we can immediately interpret a as the fraction of income spent on good 1. For this reason we will usually write Cobb-Douglas preferences in this form.

5.4 Estimating Utility Functions

We've now seen several different forms for preferences and utility functions and have examined the kinds of demand behavior generated by these preferences. But in real life we usually have to work the other way around: we observe demand behavior, but our problem is to determine what kind of preferences generated the observed behavior.

For example, suppose that we observe a consumer's choices at several different prices and income levels. An example is depicted in Table 5.1. This is a table of the demand for two goods at the different levels of prices and incomes that prevailed in different years. We have also computed the share of income spent on each good in each year using the formulas $s_1 = p_1x_1/m$ and $s_2 = p_2x_2/m$.

For these data, the expenditure shares are relatively constant. There are small variations from observation to observation, but they probably aren't large enough to worry about. The average expenditure share for good 1 is about 1/4, and the average income share for good 2 is about 3/4. It appears

Table 5.1 **Some data describing consumption behavior.**

Year	p_1	p_2	m	x_1	x_2	s_1	s_2	Utility
1	1	1	100	25	75	.25	.75	57.0
2	1	2	100	24	38	.24	.76	33.9
3	2	1	100	13	74	.26	.74	47.9
4	1	2	200	48	76	.24	.76	67.8
5	2	1	200	25	150	.25	.75	95.8
6	1	4	400	100	75	.25	.75	80.6
7	4	1	400	24	304	.24	.76	161.1

that a utility function of the form $u(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{3}{4}}$ seems to fit these data pretty well. That is, a utility function of this form would generate choice behavior that is pretty close to the observed choice behavior. For convenience we have calculated the utility associated with each observation using this estimated Cobb-Douglas utility function.

As far as we can tell from the observed behavior it appears as though the consumer is maximizing the function $u(x_1, x_2) = x_1^{\frac{1}{4}} x_2^{\frac{3}{4}}$. It may well be that further observations on the consumer's behavior would lead us to reject this hypothesis. But based on the data we have, the fit to the optimizing model is pretty good.

This has very important implications, since we can now use this "fitted" utility function to evaluate the impact of proposed policy changes. Suppose, for example, that the government was contemplating imposing a system of taxes that would result in this consumer facing prices $(2, 3)$ and having an income of 200. According to our estimates, the demanded bundle at these prices would be

$$x_1 = \frac{1}{4}\frac{200}{2} = 25$$
$$x_2 = \frac{3}{4}\frac{200}{3} = 50.$$

The estimated utility of this bundle is

$$u(x_1, x_2) = 25^{\frac{1}{4}} 50^{\frac{3}{4}} \approx 42.$$

This means that the new tax policy would make the consumer better off than he was in year 2, but worse off than he was in year 3. Thus we can use the observed choice behavior to value the implications of proposed policy changes on this consumer.

Since this is such an important idea in economics, let us review the logic one more time. Given some observations on choice behavior, we try to determine what, if anything, is being maximized. Once we have an estimate of what it is that is being maximized, we can use this both to

predict choice behavior in new situations and to evaluate proposed changes in the economic environment.

Of course we have described a very simple situation. In reality, we normally don't have detailed data on individual consumption choices. But we often have data on groups of individuals—teenagers, middle-class households, elderly people, and so on. These groups may have different preferences for different goods that are reflected in their patterns of consumption expenditure. We can estimate a utility function that describes their consumption patterns and then use this estimated utility function to forecast demand and evaluate policy proposals.

In the simple example described above, it was apparent that income shares were relatively constant so that the Cobb-Douglas utility function would give us a pretty good fit. In other cases, a more complicated form for the utility function would be appropriate. The calculations may then become messier, and we may need to use a computer for the estimation, but the essential idea of the procedure is the same.

5.5 Implications of the MRS Condition

In the last section we examined the important idea that observation of demand behavior tells us important things about the underlying preferences of the consumers that generated that behavior. Given sufficient observations on consumer choices it will often be possible to estimate the utility function that generated those choices.

But even observing *one* consumer choice at one set of prices will allow us to make some kinds of useful inferences about how consumer utility will change when consumption changes. Let us see how this works.

In well-organized markets, it is typical that everyone faces roughly the same prices for goods. Take, for example, two goods like butter and milk. If everyone faces the same prices for butter and milk, and everyone is optimizing, and everyone is at an interior solution . . . then everyone must have the same marginal rate of substitution for butter and milk.

This follows directly from the analysis given above. The market is offering everyone the same rate of exchange for butter and milk, and everyone is adjusting their consumption of the goods until their own "internal" marginal valuation of the two goods equals the market's "external" valuation of the two goods.

Now the interesting thing about this statement is that it is independent of income and tastes. People may value their *total* consumption of the two goods very differently. Some people may be consuming a lot of butter and a little milk, and some may be doing the reverse. Some wealthy people may be consuming a lot of milk and a lot of butter while other people may be consuming just a little of each good. But everyone who is consuming the two goods must have the same marginal rate of substitution. Everyone

who is consuming the goods must agree on how much one is worth in terms of the other: how much of one they would be willing to sacrifice to get some more of the other.

The fact that price ratios measure marginal rates of substitution is very important, for it means that we have a way to value possible changes in consumption bundles. Suppose, for example, that the price of milk is $1 a quart and the price of butter is $2 a pound. Then the marginal rate of substitution for all people who consume milk and butter must be 2: they have to have 2 quarts of milk to compensate them for giving up 1 pound of butter. Or conversely, they have to have 1 pound of butter to make it worth their while to give up 2 quarts of milk. Hence everyone who is consuming both goods will value a marginal change in consumption in the same way.

Now suppose that an inventor discovers a new way of turning milk into butter: for every 3 quarts of milk poured into this machine, you get out 1 pound of butter, and no other useful byproducts. Question: is there a market for this device? Answer: the venture capitalists won't beat a path to his door, that's for sure. For everyone is already operating at a point where they are just willing to trade 2 quarts of milk for 1 pound of butter; why would they be willing to substitute 3 quarts of milk for 1 pound of butter? The answer is they wouldn't; this invention isn't worth anything.

But what would happen if he got it to run in reverse so he could dump in a pound of butter get out 3 quarts of milk? Is there a market for this device? Answer: yes! The market prices of milk and butter tell us that people are just barely willing to trade one pound of butter for 2 quarts of milk. So getting 3 quarts of milk for a pound of butter is a better deal than is currently being offered in the marketplace. Sign me up for a thousand shares! (And several pounds of butter.)

The market prices show that the first machine is unprofitable: it produces $2 of butter by using $3 of milk. The fact that it is unprofitable is just another way of saying that people value the inputs more than the outputs. The second machine produces $3 worth of milk by using only $2 worth of butter. This machine is profitable because people value the outputs more than the inputs.

The point is that, since prices measure the rate at which people are just willing to substitute one good for another, they can be used to value policy proposals that involve making changes in consumption. The fact that prices are not arbitrary numbers but reflect how people value things on the margin is one of the most fundamental and important ideas in economics.

If we observe one choice at one set of prices we get the MRS at one consumption point. If the prices change and we observe another choice we get another MRS. As we observe more and more choices we learn more and more about the shape of the underlying preferences that may have generated the observed choice behavior.

5.6 Choosing Taxes

Even the small bit of consumer theory we have discussed so far can be used to derive interesting and important conclusions. Here is a nice example describing a choice between two types of taxes. We saw that a **quantity tax** is a tax on the amount consumed of a good, like a gasoline tax of 15 cents per gallon. An **income tax** is just a tax on income. If the government wants to raise a certain amount of revenue, is it better to raise it via a quantity tax or an income tax? Let's apply what we've learned to answer this question.

First we analyze the imposition of a quantity tax. Suppose that the original budget constraint is

$$p_1x_1 + p_2x_2 = m.$$

What is the budget constraint if we tax the consumption of good 1 at a rate of t? The answer is simple. From the viewpoint of the consumer it is just as if the price of good 1 has increased by an amount t. Thus the new budget constraint is

$$(p_1 + t)x_1 + p_2x_2 = m. \tag{5.1}$$

Therefore a quantity tax on a good increases the price perceived by the consumer. Figure 5.9 gives an example of how that price change might affect demand. At this stage, we don't know for certain whether this tax will increase or decrease the consumption of good 1, although the presumption is that it will decrease it. Whichever is the case, we do know that the optimal choice, (x_1^*, x_2^*), must satisfy the budget constraint

$$(p_1 + t)x_1^* + p_2x_2^* = m. \tag{5.2}$$

The revenue raised by this tax is $R^* = tx_1^*$.

Let's now consider an income tax that raises the same amount of revenue. The form of this budget constraint would be

$$p_1x_1 + p_2x_2 = m - R^*$$

or, substituting for R^*,

$$p_1x_1 + p_2x_2 = m - tx_1^*.$$

Where does this budget line go in Figure 5.9?

It is easy to see that it has the same slope as the original budget line, $-p_1/p_2$, but the problem is to determine its location. As it turns out, the budget line with the income tax must pass through the point (x_1^*, x_2^*). The way to check this is to plug (x_1^*, x_2^*) into the income-tax budget constraint and see if it is satisfied.

Figure 5.9

Income tax versus a quantity tax. Here we consider a quantity tax that raises revenue R^* and an income tax that raises the same revenue. The consumer will be better off under the income tax, since he can choose a point on a higher indifference curve.

Is it true that

$$p_1 x_1^* + p_2 x_2^* = m - t x_1^*?$$

Yes it is, since this is just a rearrangement of equation (5.2), which we know to be true.

This establishes that (x_1^*, x_2^*) lies on the income tax budget line: it is an *affordable* choice for the consumer. But is it an optimal choice? It is easy to see that the answer is no. At (x_1^*, x_2^*) the MRS is $-(p_1 + t)/p_2$. But the income tax allows us to trade at a rate of exchange of $-p_1/p_2$. Thus the budget line cuts the indifference curve at (x_1^*, x_2^*), which implies that there will be some point on the budget line that will be preferred to (x_1^*, x_2^*).

Therefore the income tax is definitely superior to the quantity tax in the sense that you can raise the same amount of revenue from a consumer and still leave him or her better off under the income tax than under the quantity tax.

This is a nice result, and worth remembering, but it is also worthwhile

understanding its limitations. First, it only applies to one consumer. The argument shows that for any given consumer there is an income tax that will raise as much money from that consumer as a quantity tax and leave him or her better off. But the amount of that income tax will typically differ from person to person. So a *uniform* income tax for all consumers is not necessarily better than a *uniform* quantity tax for all consumers. (Think about a case where some consumer doesn't consume any of good 1—this person would certainly prefer the quantity tax to a uniform income tax.)

Second, we have assumed that when we impose the tax on income the consumer's income doesn't change. We have assumed that the income tax is basically a lump sum tax—one that just changes the amount of money a consumer has to spend but doesn't affect any choices he has to make. This is an unlikely assumption. If income is earned by the consumer, we might expect that taxing it will discourage earning income, so that after-tax income might fall by even more than the amount taken by the tax.

Third, we have totally left out the supply response to the tax. We've shown how demand responds to the tax change, but supply will respond too, and a complete analysis would take those changes into account as well.

Summary

1. The optimal choice of the consumer is that bundle in the consumer's budget set that lies on the highest indifference curve.

2. Typically the optimal bundle will be characterized by the condition that the slope of the indifference curve (the MRS) will equal the slope of the budget line.

3. If we observe several consumption choices it may be possible to estimate a utility function that would generate that sort of choice behavior. Such a utility function can be used to predict future choices and to estimate the utility to consumers of new economic policies.

4. If everyone faces the same prices for the two goods, then everyone will have the same marginal rate of substitution, and will thus be willing to trade off the two goods in the same way.

REVIEW QUESTIONS

1. If two goods are perfect substitutes, what is the demand function for good 2?

2. Suppose that indifference curves are described by straight lines with a slope of $-b$. Given arbitrary prices and money income p_1, p_2, and m, what will the consumer's optimal choices look like?

3. Suppose that a consumer always consumes 2 spoons of sugar with each cup of coffee. If the price of sugar is p_1 per spoonful and the price of coffee is p_2 per cup and the consumer has m dollars to spend on coffee and sugar, how much will he or she want to purchase?

4. Suppose that you have highly nonconvex preferences for ice cream and olives, like those given in the text, and that you face prices p_1, p_2 and have m dollars to spend. List the choices for the optimal consumption bundles.

5. If a consumer has a utility function $u(x_1, x_2) = x_1 x_2^4$, what fraction of her income will she spend on good 2?

6. For what kind of preferences will the consumer be just as well-off facing a quantity tax as an income tax?

APPENDIX

It is very useful to be able to solve the preference-maximization problem and get algebraic examples of actual demand functions. We did this in the body of the text for easy cases like perfect substitutes and perfect complements, and in this Appendix we'll see how to do it in more general cases.

First, we will generally want to represent the consumer's preferences by a utility function, $u(x_1, x_2)$. We've seen in Chapter 4 that this is not a very restrictive assumption; most well-behaved preferences can be described by a utility function.

The first thing to observe is that we already *know* how to solve the optimal-choice problem. We just have to put together the facts that we learned in the last three chapters. We know from this chapter that an optimal choice (x_1, x_2) must satisfy the condition

$$\mathrm{MRS}(x_1, x_2) = -\frac{p_1}{p_2}, \tag{5.3}$$

and we saw in the Appendix to Chapter 4 that the MRS can be expressed as the negative of the ratio of derivatives of the utility function. Making this substitution and cancelling the minus signs, we have

$$\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2} = \frac{p_1}{p_2}. \tag{5.4}$$

From Chapter 2 we know that the optimal choice must also satisfy the budget constraint

$$p_1 x_1 + p_2 x_2 = m. \tag{5.5}$$

This gives us two equations—the MRS condition and the budget constraint—and two unknowns, x_1 and x_2. All we have to do is to solve these two equations

to find the optimal choices of x_1 and x_2 as a function of the prices and income. There are a number of ways to solve two equations in two unknowns. One way that always works, although it might not always be the simplest, is to solve the budget constraint for one of the choices, and then substitute that into the MRS condition.

Rewriting the budget constraint, we have

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1 \tag{5.6}$$

and substituting this into equation (5.4) we get

$$\frac{\partial u(x_1, m/p_2 - (p_1/p_2)x_1)/\partial x_1}{\partial u(x_1, m/p_2 - (p_1/p_2)x_1)/\partial x_2} = \frac{p_1}{p_2}.$$

This rather formidable looking expression has only one unknown variable, x_1, and it can typically be solved for x_1 in terms of (p_1, p_2, m). Then the budget constraint yields the solution for x_2 as a function of prices and income.

We can also derive the solution to the utility maximization problem in a more systematic way, using calculus conditions for maximization. To do this, we first pose the utility maximization problem as a constrained maximization problem:

$$\max_{x_1, x_2} u(x_1, x_2)$$

$$\text{such that } p_1x_1 + p_2x_2 = m.$$

This problem asks that we choose values of x_1 and x_2 that do two things: first, they have to satisfy the constraint, and second, they give a larger value for $u(x_1, x_2)$ than any other values of x_1 and x_2 that satisfy the constraint.

There are two useful ways to solve this kind of problem. The first way is simply to solve the constraint for one of the variables in terms of the other and then substitute it into the objective function.

For example, for any given value of x_1, the amount of x_2 that we need to satisfy the budget constraint is given by the linear function

$$x_2(x_1) = \frac{m}{p_2} - \frac{p_1}{p_2}x_1. \tag{5.7}$$

Now substitute $x_2(x_1)$ for x_2 in the utility function to get the *unconstrained* maximization problem

$$\max_{x_1} u(x_1, m/p_2 - (p_1/p_2)x_1).$$

This is an unconstrained maximization problem in x_1 alone, since we have used the function $x_2(x_1)$ to ensure that the value of x_2 will always satisfy the budget constraint, whatever the value of x_1 is.

We can solve this kind of problem just by differentiating with respect to x_1 and setting the result equal to zero in the usual way. This procedure will give us a first-order condition of the form

$$\frac{\partial u(x_1, x_2(x_1))}{\partial x_1} + \frac{\partial u(x_1, x_2(x_1))}{\partial x_2}\frac{dx_2}{dx_1} = 0. \tag{5.8}$$

Here the first term is the direct effect of how increasing x_1 increases utility. The second term consists of two parts: the rate of increase of utility as x_2 increases, $\partial u/\partial x_2$, times dx_2/dx_1, the rate of increase of x_2 as x_1 increases in order to continue to satisfy the budget equation. We can differentiate (5.7) to calculate this latter derivative

$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}.$$

Substituting this into (5.8) gives us

$$\frac{\partial u(x_1^*, x_2^*)/\partial x_1}{\partial u(x_1^*, x_2^*)/\partial x_2} = \frac{p_1}{p_2},$$

which just says that the marginal rate of substitution between x_1 and x_2 must equal the price ratio at the optimal choice (x_1^*, x_2^*). This is exactly the condition we derived above: the slope of the indifference curve must equal the slope of the budget line. Of course the optimal choice must also satisfy the budget constraint $p_1x_1^* + p_2x_2^* = m$, which again gives us two equations in two unknowns.

The second way that these problems can be solved is through the use of **Lagrange multipliers**. This method starts by defining an auxiliary function known as the *Lagrangian*:

$$L = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - m).$$

The new variable λ is called a **Lagrange multiplier** since it is multiplied by the constraint.[3] Then Lagrange's theorem says that an optimal choice (x_1^*, x_2^*) must satisfy the three first-order conditions

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= \frac{\partial u(x_1^*, x_2^*)}{\partial x_1} - \lambda p_1 = 0 \\
\frac{\partial L}{\partial x_2} &= \frac{\partial u(x_1^*, x_2^*)}{\partial x_2} - \lambda p_2 = 0 \\
\frac{\partial L}{\partial \lambda} &= p_1x_1^* + p_2x_2^* - m = 0.
\end{aligned}$$

There are several interesting things about these three equations. First, note that they are simply the derivatives of the Lagrangian with respect to x_1, x_2, and λ, each set equal to zero. The last derivative, with respect to λ, is just the budget constraint. Second, we now have three equations for the three unknowns, x_1, x_2, and λ. We have a hope of solving for x_1 and x_2 in terms of p_1, p_2, and m.

Lagrange's theorem is proved in any advanced calculus book. It is used quite extensively in advanced economics courses, but for our purposes we only need to know the statement of the theorem and how to use it.

In our particular case, it is worthwhile noting that if we divide the first condition by the second one, we get

$$\frac{\partial u(x_1^*, x_2^*)/\partial x_1}{\partial u(x_1^*, x_2^*)/\partial x_2} = \frac{p_1}{p_2},$$

which simply says the MRS must equal the price ratio, just as before. The budget constraint gives us the other equation, so we are back to two equations in two unknowns.

[3] The Greek letter λ is pronounced "lamb-da."

EXAMPLE: Cobb-Douglas Demand Functions

In Chapter 4 we introduced the **Cobb-Douglas utility function**

$$u(x_1, x_2) = x_1^c x_2^d.$$

Since utility functions are only defined up to a monotonic transformation, it is convenient to take logs of this expression and work with

$$\ln u(x_1, x_2) = c \ln x_1 + d \ln x_2.$$

Let's find the demand functions for x_1 and x_2 for the Cobb-Douglas utility function. The problem we want to solve is

$$\max_{x_1, x_2} c \ln x_1 + d \ln x_2$$

$$\text{such that } p_1 x_1 + p_2 x_2 = m.$$

There are at least three ways to solve this problem. One way is just to write down the MRS condition and the budget constraint. Using the expression for the MRS derived in Chapter 4, we have

$$\frac{c x_2}{d x_1} = \frac{p_1}{p_2}$$
$$p_1 x_1 + p_2 x_2 = m.$$

These are two equations in two unknowns that can be solved for the optimal choice of x_1 and x_2. One way to solve them is to substitute the second into the first to get

$$\frac{c(m/p_2 - x_1 p_1/p_2)}{d x_1} = \frac{p_1}{p_2}.$$

Cross multiplying gives

$$c(m - x_1 p_1) = d p_1 x_1.$$

Rearranging this equation gives

$$cm = (c + d) p_1 x_1$$

or

$$x_1 = \frac{c}{c + d} \frac{m}{p_1}.$$

This is the demand function for x_1. To find the demand function for x_2, substitute into the budget constraint to get

$$\begin{aligned} x_2 &= \frac{m}{p_2} - \frac{p_1}{p_2} \frac{c}{c + d} \frac{m}{p_1} \\ &= \frac{d}{c + d} \frac{m}{p_2}. \end{aligned}$$

The second way is to substitute the budget constraint into the maximization problem at the beginning. If we do this, our problem becomes

$$\max_{x_1} c \ln x_1 + d \ln(m/p_2 - x_1 p_1/p_2).$$

The first-order condition for this problem is

$$\frac{c}{x_1} - d\frac{p_2}{m - p_1 x_1}\frac{p_1}{p_2} = 0.$$

A little algebra—which you should do!—gives us the solution

$$x_1 = \frac{c}{c+d}\frac{m}{p_1}.$$

Substitute this back into the budget constraint $x_2 = m/p_2 - x_1 p_1/p_2$ to get

$$x_2 = \frac{d}{c+d}\frac{m}{p_2}.$$

These are the demand functions for the two goods, which, happily, are the same as those derived earlier by the other method.

Now for Lagrange's method. Set up the Lagrangian

$$L = c \ln x_1 + d \ln x_2 - \lambda(p_1 x_1 + p_2 x_2 - m)$$

and differentiate to get the three first-order conditions

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= \frac{c}{x_1} - \lambda p_1 = 0 \\
\frac{\partial L}{\partial x_2} &= \frac{d}{x_2} - \lambda p_2 = 0 \\
\frac{\partial L}{\partial \lambda} &= p_1 x_1 + p_2 x_2 - m = 0.
\end{aligned}$$

Now the trick is to solve them! The best way to proceed is to first solve for λ and then for x_1 and x_2. So we rearrange and cross multiply the first two equations to get

$$\begin{aligned}
c &= \lambda p_1 x_1 \\
d &= \lambda p_2 x_2.
\end{aligned}$$

These equations are just asking to be added together:

$$c + d = \lambda(p_1 x_1 + p_2 x_2) = \lambda m,$$

which gives us

$$\lambda = \frac{c+d}{m}.$$

Substitute this back into the first two equations and solve for x_1 and x_2 to get

$$\begin{aligned}
x_1 &= \frac{c}{c+d}\frac{m}{p_1} \\
x_2 &= \frac{d}{c+d}\frac{m}{p_2},
\end{aligned}$$

just as before.

CHAPTER 6

DEMAND

In the last chapter we presented the basic model of consumer choice: how maximizing utility subject to a budget constraint yields optimal choices. We saw that the optimal choices of the consumer depend on the consumer's income and the prices of the goods, and we worked a few examples to see what the optimal choices are for some simple kinds of preferences.

The consumer's **demand functions** give the optimal amounts of each of the goods as a function of the prices and income faced by the consumer. We write the demand functions as

$$x_1 = x_1(p_1, p_2, m)$$
$$x_2 = x_2(p_1, p_2, m).$$

The left-hand side of each equation stands for the quantity demanded. The right-hand side of each equation is the function that relates the prices and income to that quantity.

In this chapter we will examine how the demand for a good changes as prices and income change. Studying how a choice responds to changes in the economic environment is known as **comparative statics**, which we first described in Chapter 1. "Comparative" means that we want to compare

two situations: before and after the change in the economic environment. "Statics" means that we are not concerned with any adjustment process that may be involved in moving from one choice to another; rather we will only examine the equilibrium choice.

In the case of the consumer, there are only two things in our model that affect the optimal choice: prices and income. The comparative statics questions in consumer theory therefore involve investigating how demand changes when prices and income change.

6.1 Normal and Inferior Goods

We start by considering how a consumer's demand for a good changes as his income changes. We want to know how the optimal choice at one income compares to the optimal choice at another level of income. During this exercise, we will hold the prices fixed and examine only the change in demand due to the income change.

We know how an increase in money income affects the budget line when prices are fixed—it shifts it outward in a parallel fashion. So how does this affect demand?

We would normally think that the demand for each good would increase when income increases, as shown in Figure 6.1. Economists, with a singular lack of imagination, call such goods **normal** goods. If good 1 is a normal good, then the demand for it increases when income increases, and decreases when income decreases. For a normal good the quantity demanded always changes in the same way as income changes:

$$\frac{\Delta x_1}{\Delta m} > 0.$$

If something is called normal, you can be sure that there must be a *possibility* of being abnormal. And indeed there is. Figure 6.2 presents an example of nice, well-behaved indifference curves where an increase of income results in a *reduction* in the consumption of one of the goods. Such a good is called an **inferior** good. This may be "abnormal," but when you think about it, inferior goods aren't all that unusual. There are many goods for which demand decreases as income increases; examples might include gruel, bologna, shacks, or nearly any kind of low-quality good.

Whether a good is inferior or not depends on the income level that we are examining. It might very well be that very poor people consume more bologna as their income increases. But after a point, the consumption of bologna would probably decline as income continued to increase. Since in real life the consumption of goods can increase or decrease when income increases, it is comforting to know that economic theory allows for both possibilities.

Normal goods. The demand for both goods increases when income increases, so both goods are normal goods.

Figure 6.1

6.2 Income Offer Curves and Engel Curves

We have seen that an increase in income corresponds to shifting the budget line outward in a parallel manner. We can connect together the demanded bundles that we get as we shift the budget line outward to construct the **income offer curve**. This curve illustrates the bundles of goods that are demanded at the different levels of income, as depicted in Figure 6.3A. The income offer curve is also known as the **income expansion path**. If both goods are normal goods, then the income expansion path will have a positive slope, as depicted in Figure 6.3A.

For each level of income, m, there will be some optimal choice for each of the goods. Let us focus on good 1 and consider the optimal choice at each set of prices and income, $x_1(p_1, p_2, m)$. This is simply the demand function for good 1. If we hold the prices of goods 1 and 2 fixed and look at how demand changes as we change income, we generate a curve known as the **Engel curve**. The Engel curve is a graph of the demand for one of the goods as a function of income, with all prices being held constant. For an example of an Engel curve, see Figure 6.3B.

Figure 6.2 **An inferior good**. Good 1 is an inferior good, which means that the demand for it decreases when income increases.

Figure 6.3 **How demand changes as income changes**. The income offer curve (or income expansion path) shown in panel A depicts the optimal choice at different levels of income and constant prices. When we plot the optimal choice of good 1 against income, m, we get the Engel curve, depicted in panel B.

6.3 Some Examples

Let's consider some of the preferences that we examined in Chapter 5 and see what their income offer curves and Engel curves look like.

Perfect Substitutes

The case of perfect substitutes is depicted in Figure 6.4. If $p_1 < p_2$, so that the consumer is specializing in consuming good 1, then if his income increases he will increase his consumption of good 1. Thus the income offer curve is the horizontal axis, as shown in Figure 6.4A.

Perfect substitutes. The income offer curve (A) and an Engel curve (B) in the case of perfect substitutes.

Figure 6.4

Since the demand for good 1 is $x_1 = m/p_1$ in this case, the Engel curve will be a straight line with a slope of p_1, as depicted in Figure 6.4B. (Since m is on the vertical axis, and x_1 on the horizontal axis, we can write $m = p_1 x_1$, which makes it clear that the slope is p_1.)

Perfect Complements

The demand behavior for perfect complements is shown in Figure 6.5. Since the consumer will always consume the same amount of each good, no matter

what, the income offer curve is the diagonal line through the origin as depicted in Figure 6.5A. We have seen that the demand for good 1 is $x_1 = m/(p_1 + p_2)$, so the Engel curve is a straight line with a slope of $p_1 + p_2$ as shown in Figure 6.5B.

Figure 6.5 **Perfect complements**. The income offer curve (A) and an Engel curve (B) in the case of perfect complements.

Cobb-Douglas Preferences

For the case of Cobb-Douglas preferences it is easier to look at the algebraic form of the demand functions to see what the graphs will look like. If $u(x_1, x_2) = x_1^a x_2^{1-a}$, the Cobb-Douglas demand for good 1 has the form $x_1 = am/p_1$. For a fixed value of p_1, this is a *linear* function of m. Thus doubling m will double demand, tripling m will triple demand, and so on. In fact, multiplying m by any positive number t will just multiply demand by the same amount.

The demand for good 2 is $x_2 = (1-a)m/p_2$, and this is also clearly linear. The fact that the demand functions for both goods are linear functions of income means that the income expansion paths will be straight lines through the origin, as depicted in Figure 6.6A. The Engel curve for good 1 will be a straight line with a slope of p_1/a, as depicted in Figure 6.6B.

Cobb-Douglas. An income offer curve (A) and an Engel curve (B) for Cobb-Douglas utility.

Figure 6.6

Homothetic Preferences

All of the income offer curves and Engel curves that we have seen up to now have been straightforward—in fact they've been straight lines! This has happened because our examples have been so simple. Real Engel curves do not have to be straight lines. In general, when income goes up, the demand for a good could increase more or less rapidly than income increases. If the demand for a good goes up by a greater proportion than income, we say that it is a **luxury good**, and if it goes up by a lesser proportion than income we say that it is a **necessary good**.

The dividing line is the case where the demand for a good goes up by the same proportion as income. This is what happened in the three cases we examined above. What aspect of the consumer's preferences leads to this behavior?

Suppose that the consumer's preferences only depend on the *ratio* of good 1 to good 2. This means that if the consumer prefers (x_1, x_2) to (y_1, y_2), then she automatically prefers $(2x_1, 2x_2)$ to $(2y_1, 2y_2)$, $(3x_1, 3x_2)$ to $(3y_1, 3y_2)$, and so on, since the ratio of good 1 to good 2 is the same for all of these bundles. In fact, the consumer prefers (tx_1, tx_2) to (ty_1, ty_2) for any positive value of t. Preferences that have this property are known as **homothetic preferences**. It is not hard to show that the three examples of preferences given above—perfect substitutes, perfect complements, and Cobb-Douglas—are all homothetic preferences.

If the consumer has homothetic preferences, then the income offer curves are all straight lines through the origin, as shown in Figure 6.7. More specifically, if preferences are homothetic, it means that when income is scaled up or down by any amount $t > 0$, the demanded bundle scales up or down by the same amount. This can be established rigorously, but it is fairly clear from looking at the picture. If the indifference curve is tangent to the budget line at (x_1^*, x_2^*), then the indifference curve through (tx_1^*, tx_2^*) is tangent to the budget line that has t times as much income and the same prices. This implies that the Engel curves are straight lines as well. If you double income, you just double the demand for each good.

Figure 6.7

Homothetic preferences. An income offer curve (A) and an Engel curve (B) in the case of homothetic preferences.

Homothetic preferences are very convenient since the income effects are so simple. Unfortunately, homothetic preferences aren't very realistic for the same reason! But they will often be of use in our examples.

Quasilinear Preferences

Another kind of preferences that generates a special form of income offer curves and Engel curves is the case of quasilinear preferences. Recall the definition of quasilinear preferences given in Chapter 4. This is the case where all indifference curves are "shifted" versions of one indifference curve

as in Figure 6.8. Equivalently, the utility function for these preferences takes the form $u(x_1, x_2) = v(x_1) + x_2$. What happens if we shift the budget line outward? In this case, if an indifference curve is tangent to the budget line at a bundle (x_1^*, x_2^*), then another indifference curve must also be tangent at $(x_1^*, x_2^* + k)$ for any constant k. Increasing income doesn't change the demand for good 1 at all, and all the extra income goes entirely to the consumption of good 2. If preferences are quasilinear, we sometimes say that there is a "zero income effect" for good 1. Thus the Engel curve for good 1 is a vertical line—as you change income, the demand for good 1 remains constant. (See the appendix for a small qualification.)

Quasilinear preferences. An income offer curve (A) and an Engel curve (B) with quasilinear preferences.

Figure 6.8

What would be a real-life situation where this kind of thing might occur? Suppose good 1 is pencils and good 2 is money to spend on other goods. Initially I may spend my income only on pencils, but when my income gets large enough, I stop buying additional pencils—all of my extra income is spent on other goods. Other examples of this sort might be salt or toothpaste. When we are examining a choice between all other goods and some single good that isn't a very large part of the consumer's budget, the quasilinear assumption may well be plausible, at least when the consumer's income is sufficiently large.

6.4 Ordinary Goods and Giffen Goods

Let us now consider price changes. Suppose that we decrease the price of good 1 and hold the price of good 2 and money income fixed. Then what can happen to the quantity demanded of good 1? Intuition tells us that the quantity demanded of good 1 should increase when its price decreases. Indeed this is the ordinary case, as depicted in Figure 6.9.

Figure 6.9 **An ordinary good**. Ordinarily, the demand for a good increases when its price decreases, as is the case here.

When the price of good 1 decreases, the budget line becomes flatter. Or said another way, the vertical intercept is fixed and the horizontal intercept moves to the right. In Figure 6.9, the optimal choice of good 1 moves to the right as well: the quantity demanded of good 1 has increased. But we might wonder whether this always happens this way. Is it always the case that, no matter what kind of preferences the consumer has, the demand for a good must increase when its price goes down?

As it turns out, the answer is no. It is logically possible to find well-behaved preferences for which a decrease in the price of good 1 leads to a reduction in the demand for good 1. Such a good is called a **Giffen good**,

A Giffen good. Good 1 is a Giffen good, since the demand for it decreases when its price decreases.

Figure 6.10

after the nineteenth-century economist who first noted the possibility. An example is illustrated in Figure 6.10.

What is going on here in economic terms? What kind of preferences might give rise to the peculiar behavior depicted in Figure 6.10? Suppose that the two goods that you are consuming are gruel and milk and that you are currently consuming 7 bowls of gruel and 7 cups of milk a week. Now the price of gruel declines. If you consume the same 7 bowls of gruel a week, you will have money left over with which you can purchase more milk. In fact, with the extra money you have saved because of the lower price of gruel, you may decide to consume even more milk and reduce your consumption of gruel. The reduction in the price of gruel has freed up some extra money to be spent on other things—but one thing you might want to do with it is reduce your consumption of gruel! Thus the price change is to some extent *like* an income change. Even though *money* income remains constant, a change in the price of a good will change purchasing power, and thereby change demand.

So the Giffen good is not implausible purely on logical grounds, although Giffen goods are unlikely to be encountered in real-world behavior. Most goods are ordinary goods—when their price increases, the demand for them declines. We'll see why this is the ordinary situation a little later.

Incidentally, it is no accident that we used gruel as an example of both an inferior good and a Giffen good. It turns out that there is an intimate relationship between the two which we will explore in a later chapter.

But for now our exploration of consumer theory may leave you with the impression that nearly anything can happen: if income increases the demand for a good can go up or down, and if price increases the demand can go up or down. Is consumer theory compatible with *any* kind of behavior? Or are there some kinds of behavior that the economic model of consumer behavior rules out? It turns out that there *are* restrictions on behavior imposed by the maximizing model. But we'll have to wait until the next chapter to see what they are.

6.5 The Price Offer Curve and the Demand Curve

Suppose that we let the price of good 1 change while we hold p_2 and income fixed. Geometrically this involves pivoting the budget line. We can think of connecting together the optimal points to construct the **price offer curve** as illustrated in Figure 6.11A. This curve represents the bundles that would be demanded at different prices for good 1.

A Price offer curve **B** Demand curve

Figure 6.11

The price offer curve and demand curve. Panel A contains a price offer curve, which depicts the optimal choices as the price of good 1 changes. Panel B contains the associated demand curve, which depicts a plot of the optimal choice of good 1 as a function of its price.

We can depict this same information in a different way. Again, hold the price of good 2 and money income fixed, and for each different value of p_1 plot the optimal level of consumption of good 1. The result is the **demand curve** depicted in Figure 6.11B. The demand curve is a plot of the demand function, $x_1(p_1, p_2, m)$, holding p_2 and m fixed at some predetermined values.

Ordinarily, when the price of a good increases, the demand for that good will decrease. Thus the price and quantity of a good will move in *opposite* directions, which means that the demand curve will typically have a negative slope. In terms of rates of change, we would normally have

$$\frac{\Delta x_1}{\Delta p_1} < 0,$$

which simply says that demand curves usually have a negative slope.

However, we have also seen that in the case of Giffen goods, the demand for a good may decrease when its price decreases. Thus it is possible, but not likely, to have a demand curve with a positive slope.

6.6 Some Examples

Let's look at a few examples of demand curves, using the preferences that we discussed in Chapter 3.

Perfect Substitutes

The offer curve and demand curve for perfect substitutes—the red and blue pencils example—are illustrated in Figure 6.12. As we saw in Chapter 5, the demand for good 1 is zero when $p_1 > p_2$, any amount on the budget line when $p_1 = p_2$, and m/p_1 when $p_1 < p_2$. The offer curve traces out these possibilities.

In order to find the demand curve, we fix the price of good 2 at some price p_2^* and graph the demand for good 1 versus the price of good 1 to get the shape depicted in Figure 6.12B.

Perfect Complements

The case of perfect complements—the right and left shoes example—is depicted in Figure 6.13. We know that whatever the prices are, a consumer will demand the same amount of goods 1 and 2. Thus his offer curve will be a diagonal line as depicted in Figure 6.13A.

We saw in Chapter 5 that the demand for good 1 is given by

$$x_1 = \frac{m}{p_1 + p_2}.$$

If we fix m and p_2 and plot the relationship between x_1 and p_1, we get the curve depicted in Figure 6.13B.

A Price offer curve **B** Demand curve

Figure 6.12 **Perfect substitutes**. Price offer curve (A) and demand curve (B) in the case of perfect substitutes.

A Price offer curve **B** Demand curve

Figure 6.13 **Perfect complements**. Price offer curve (A) and demand curve (B) in the case of perfect complements.

A Discrete Good

Suppose that good 1 is a discrete good. If p_1 is very high then the consumer will strictly prefer to consume zero units; if p_1 is low enough the consumer will strictly prefer to consume one unit. At some price r_1, the consumer will be indifferent between consuming good 1 or not consuming it. The price

at which the consumer is just indifferent to consuming or not consuming the good is called the **reservation price**.[1] The indifference curves and demand curve are depicted in Figure 6.14.

Figure 6.14

A discrete good. As the price of good 1 decreases there will be some price, the reservation price, at which the consumer is just indifferent between consuming good 1 or not consuming it. As the price decreases further, more units of the discrete good will be demanded.

It is clear from the diagram that the demand behavior can be described by a sequence of reservation prices at which the consumer is just willing to purchase another unit of the good. At a price of r_1 the consumer is willing to buy 1 unit of the good; if the price falls to r_2, he is willing to buy another unit, and so on.

These prices can be described in terms of the original utility function. For example, r_1 is the price where the consumer is just indifferent between consuming 0 or 1 unit of good 1, so it must satisfy the equation

$$u(0, m) = u(1, m - r_1). \tag{6.1}$$

Similarly r_2 satisfies the equation

$$u(1, m - r_2) = u(2, m - 2r_2). \tag{6.2}$$

[1] The term reservation price comes from auction markets. When someone wanted to sell something in an auction he would typically state a minimum price at which he was willing to sell the good. If the best price offered was below this stated price, the seller reserved the right to purchase the item himself. This price became known as the seller's reservation price and eventually came to be used to describe the price at which someone was just willing to buy or sell some item.

The left-hand side of this equation is the utility from consuming one unit of the good at a price of r_2. The right-hand side is the utility from consuming two units of the good, each of which sells for r_2.

If the utility function is quasilinear, then the formulas describing the reservation prices become somewhat simpler. If $u(x_1, x_2) = v(x_1) + x_2$, and $v(0) = 0$, then we can write equation (6.1) as

$$v(0) + m = m = v(1) + m - r_1.$$

Since $v(0) = 0$, we can solve for r_1 to find

$$r_1 = v(1). \tag{6.3}$$

Similarly, we can write equation (6.2) as

$$v(1) + m - r_2 = v(2) + m - 2r_2.$$

Canceling terms and rearranging, this expression becomes

$$r_2 = v(2) - v(1).$$

Proceeding in this manner, the reservation price for the third unit of consumption is given by

$$r_3 = v(3) - v(2)$$

and so on.

In each case, the reservation price measures the increment in utility necessary to induce the consumer to choose an additional unit of the good. Loosely speaking, the reservation prices measure the marginal utilities associated with different levels of consumption of good 1. Our assumption of convex preferences implies that the sequence of reservation prices must decrease: $r_1 > r_2 > r_3 \cdots$.

Because of the special structure of the quasilinear utility function, the reservation prices do not depend on the amount of good 2 that the consumer has. This is certainly a special case, but it makes it very easy to describe demand behavior. Given any price p, we just find where it falls in the list of reservation prices. Suppose that p falls between r_6 and r_7, for example. The fact that $r_6 > p$ means that the consumer is willing to give up p dollars per unit bought to get 6 units of good 1, and the fact that $p > r_7$ means that the consumer is not willing to give up p dollars per unit to get the seventh unit of good 1.

This argument is quite intuitive, but let's look at the math just to make sure that it is clear. Suppose that the consumer demands 6 units of good 1. We want to show that we must have $r_6 \geq p \geq r_7$.

If the consumer is maximizing utility, then we must have

$$v(6) + m - 6p \geq v(x_1) + m - px_1$$

for all possible choices of x_1. In particular, we must have that

$$v(6) + m - 6p \geq v(5) + m - 5p.$$

Rearranging this equation we have

$$r_6 = v(6) - v(5) \geq p,$$

which is half of what we wanted to show.

By the same logic,

$$v(6) + m - 6p \geq v(7) + m - 7p.$$

Rearranging this gives us

$$p \geq v(7) - v(6) = r_7,$$

which is the other half of the inequality we wanted to establish.

6.7 Substitutes and Complements

We have already used the terms substitutes and complements, but it is now appropriate to give a formal definition. Since we have seen *perfect* substitutes and *perfect* complements several times already, it seems reasonable to look at the imperfect case.

Let's think about substitutes first. We said that red pencils and blue pencils might be thought of as perfect substitutes, at least for someone who didn't care about color. But what about pencils and pens? This is a case of "imperfect" substitutes. That is, pens and pencils are, to some degree, a substitute for each other, although they aren't as perfect a substitute for each other as red pencils and blue pencils.

Similarly, we said that right shoes and left shoes were perfect complements. But what about a pair of shoes and a pair of socks? Right shoes and left shoes are nearly always consumed together, and shoes and socks are *usually* consumed together. Complementary goods are those like shoes and socks that tend to be consumed together, albeit not always.

Now that we've discussed the basic idea of complements and substitutes, we can give a precise economic definition. Recall that the demand function for good 1, say, will typically be a function of the price of both good 1 and good 2, so we write $x_1(p_1, p_2, m)$. We can ask how the demand for good 1 changes as the price of good 2 changes: does it go up or down?

If the demand for good 1 goes up when the price of good 2 goes up, then we say that good 1 is a **substitute** for good 2. In terms of rates of change, good 1 is a substitute for good 2 if

$$\frac{\Delta x_1}{\Delta p_2} > 0.$$

The idea is that when good 2 gets more expensive the consumer switches to consuming good 1: the consumer *substitutes* away from the more expensive good to the less expensive good.

On the other hand, if the demand for good 1 goes down when the price of good 2 goes up, we say that good 1 is a **complement** to good 2. This means that

$$\frac{\Delta x_1}{\Delta p_2} < 0.$$

Complements are goods that are consumed together, like coffee and sugar, so when the price of one good rises, the consumption of both goods will tend to decrease.

The cases of perfect substitutes and perfect complements illustrate these points nicely. Note that $\Delta x_1/\Delta p_2$ is positive (or zero) in the case of perfect substitutes, and that $\Delta x_1/\Delta p_2$ is negative in the case of perfect complements.

A couple of warnings are in order about these concepts. First, the two-good case is rather special when it comes to complements and substitutes. Since income is being held fixed, if you spend more money on good 1, you'll have to spend less on good 2. This puts some restrictions on the kinds of behavior that are possible. When there are more than two goods, these restrictions are not so much of a problem.

Second, although the definition of substitutes and complements in terms of consumer demand behavior seems sensible, there are some difficulties with the definitions in more general environments. For example, if we use the above definitions in a situation involving more than two goods, it is perfectly possible that good 1 may be a substitute for good 3, but good 3 may be a complement for good 1. Because of this peculiar feature, more advanced treatments typically use a somewhat different definition of substitutes and complements. The definitions given above describe concepts known as **gross substitutes** and **gross complements**; they will be sufficient for our needs.

6.8 The Inverse Demand Function

If we hold p_2 and m fixed and plot p_1 against x_1 we get the **demand curve**. As suggested above, we typically think that the demand curve slopes downwards, so that higher prices lead to less demand, although the Giffen example shows that it could be otherwise.

As long as we do have a downward-sloping demand curve, as is usual, it is meaningful to speak of the **inverse demand function**. The inverse demand function is the demand function viewing price as a function of quantity. That is, for each level of demand for good 1, the inverse demand function measures what the price of good 1 would have to be in order for the consumer to choose that level of consumption. So the inverse demand

function measures the same relationship as the direct demand function, but just from another point of view. Figure 6.15 depicts the inverse demand function—or the direct demand function, depending on your point of view.

Inverse demand curve. If you view the demand curve as measuring price as a function of quantity, you have an inverse demand function.

Figure 6.15

Recall, for example, the Cobb-Douglas demand for good 1, $x_1 = am/p_1$. We could just as well write the relationship between price and quantity as $p_1 = am/x_1$. The first representation is the direct demand function; the second is the inverse demand function.

The inverse demand function has a useful economic interpretation. Recall that as long as both goods are being consumed in positive amounts, the optimal choice must satisfy the condition that the absolute value of the MRS equals the price ratio:

$$|\text{MRS}| = \frac{p_1}{p_2}.$$

This says that at the optimal level of demand for good 1, for example, we must have

$$p_1 = p_2|\text{MRS}|. \tag{6.4}$$

Thus, at the optimal level of demand for good 1, the price of good 1 is proportional to the absolute value of the MRS between good 1 and good 2.

Suppose for simplicity that the price of good 2 is one. Then equation (6.4) tells us that at the optimal level of demand, the price of good 1 measures how much the consumer is willing to give up of good 2 in order to get a little more of good 1. In this case the inverse demand function is simply measuring the absolute value of the MRS. For any optimal level of x_1 the inverse demand function tells how much of good 2 the consumer would want to have to compensate him for a small reduction in the amount of good 1. Or, turning this around, the inverse demand function measures how much the consumer would be willing to sacrifice of good 2 to make him just indifferent to having a little more of good 1.

If we think of good 2 as being money to spend on other goods, then we can think of the MRS as being how many dollars the individual would be willing to give up to have a little more of good 1. We suggested earlier that in this case, we can think of the MRS as measuring the marginal willingness to pay. Since the price of good 1 is just the MRS in this case, this means that the price of good 1 itself is measuring the marginal willingness to pay.

At each quantity x_1, the inverse demand function measures how many dollars the consumer is willing to give up for a little more of good 1; or, said another way, how many dollars the consumer was willing to give up for the last unit purchased of good 1. For a small enough amount of good 1, they come down to the same thing.

Looked at in this way, the downward-sloping demand curve has a new meaning. When x_1 is very small, the consumer is willing to give up a lot of money—that is, a lot of other goods, to acquire a little bit more of good 1. As x_1 is larger, the consumer is willing to give up less money, on the margin, to acquire a little more of good 1. Thus the marginal willingness to pay, in the sense of the marginal willingness to sacrifice good 2 for good 1, is decreasing as we increase the consumption of good 1.

Summary

1. The consumer's demand function for a good will in general depend on the prices of all goods and income.

2. A normal good is one for which the demand increases when income increases. An inferior good is one for which the demand decreases when income increases.

3. An ordinary good is one for which the demand decreases when its price increases. A Giffen good is one for which the demand increases when its price increases.

4. If the demand for good 1 increases when the price of good 2 increases, then good 1 is a substitute for good 2. If the demand for good 1 decreases in this situation, then it is a complement for good 2.

5. The inverse demand function measures the price at which a given quantity will be demanded. The height of the demand curve at a given level of consumption measures the marginal willingness to pay for an additional unit of the good at that consumption level.

REVIEW QUESTIONS

1. If the consumer is consuming exactly two goods, and she is always spending all of her money, can both of them be inferior goods?

2. Show that perfect substitutes are an example of homothetic preferences.

3. Show that Cobb-Douglas preferences are homothetic preferences.

4. The income offer curve is to the Engel curve as the price offer curve is to . . .?

5. If the preferences are concave will the consumer ever consume both of the goods together?

6. Are hamburgers and buns complements or substitutes?

7. What is the form of the inverse demand function for good 1 in the case of perfect complements?

8. True or false? If the demand function is $x_1 = -p_1$, then the inverse demand function is $x = -1/p_1$.

APPENDIX

If preferences take a special form, this will mean that the demand functions that come from those preferences will take a special form. In Chapter 4 we described quasilinear preferences. These preferences involve indifference curves that are all parallel to one another and can be represented by a utility function of the form

$$u(x_1, x_2) = v(x_1) + x_2.$$

The maximization problem for a utility function like this is

$$\max_{x_1, x_2} v(x_1) + x_2$$

$$\text{s.t. } p_1x_1 + p_2x_2 = m.$$

Solving the budget constraint for x_2 as a function of x_1 and substituting into the objective function, we have

$$\max_{x_1} v(x_1) + m/p_2 - p_1x_1/p_2.$$

Differentiating gives us the first-order condition

$$v'(x_1^*) = \frac{p_1}{p_2}.$$

This demand function has the interesting feature that the demand for good 1 must be independent of income—just as we saw by using indifference curves. The inverse demand curve is given by

$$p_1(x_1) = v'(x_1)p_2.$$

That is, the inverse demand function for good 1 is the derivative of the utility function times p_2. Once we have the demand function for good 1, the demand function for good 2 comes from the budget constraint.

For example, let us calculate the demand functions for the utility function

$$u(x_1, x_2) = \ln x_1 + x_2.$$

Applying the first-order condition gives

$$\frac{1}{x_1} = \frac{p_1}{p_2},$$

so the direct demand function for good 1 is

$$x_1 = \frac{p_2}{p_1},$$

and the inverse demand function is

$$p_1(x_1) = \frac{p_2}{x_1}.$$

The direct demand function for good 2 comes from substituting $x_1 = p_2/p_1$ into the budget constraint:

$$x_2 = \frac{m}{p_2} - 1.$$

A warning is in order concerning these demand functions. Note that the demand for good 1 is independent of income in this example. This is a general feature of a quasilinear utility function—the demand for good 1 remains constant as income changes. However, this can only be true for some values of income. A demand function can't literally be independent of income for *all* values of income; after all, when income is zero, all demands are zero. It turns

out that the quasilinear demand function derived above is only relevant when a positive amount of each good is being consumed.

In this example, when $m < p_2$, the optimal consumption of good 2 will be zero. As income increases the marginal utility of consumption of good 1 decreases. When $m = p_2$, the marginal utility from spending additional income on good 1 just equals the marginal utility from spending additional income on good 2. After that point, the consumer spends all additional income on good 2.

So a better way to write the demand for good 2 is:

$$x_2 = \begin{cases} 0 & \text{when } m \leq p_2 \\ m/p_2 - 1 & \text{when } m > p_2 \end{cases} .$$

For more on the properties of quasilinear demand functions see Hal R. Varian, *Microeconomic Analysis,* 3rd ed. (New York: Norton, 1992).

CHAPTER 7

REVEALED PREFERENCE

In Chapter 6 we saw how we can use information about the consumer's preferences and budget constraint to determine his or her demand. In this chapter we reverse this process and show how we can use information about the consumer's demand to discover information about his or her preferences. Up until now, we were thinking about what preferences could tell us about people's behavior. But in real life, preferences are not directly observable: we have to discover people's preferences from observing their behavior. In this chapter we'll develop some tools to do this.

When we talk of determining people's preferences from observing their behavior, we have to assume that the preferences will remain unchanged while we observe the behavior. Over very long time spans, this is not very reasonable. But for the monthly or quarterly time spans that economists usually deal with, it seems unlikely that a particular consumer's tastes would change radically. Thus we will adopt a maintained hypothesis that the consumer's preferences are stable over the time period for which we observe his or her choice behavior.

7.1 The Idea of Revealed Preference

Before we begin this investigation, let's adopt the convention that in this chapter, the underlying preferences—whatever they may be—are known to be strictly convex. Thus there will be a *unique* demanded bundle at each budget. This assumption is not necessary for the theory of revealed preference, but the exposition will be simpler with it.

Consider Figure 7.1, where we have depicted a consumer's demanded bundle, (x_1, x_2), and another arbitrary bundle, (y_1, y_2), that is beneath the consumer's budget line. Suppose that we are willing to postulate that this consumer is an optimizing consumer of the sort we have been studying. What can we say about the consumer's preferences between these two bundles of goods?

Revealed preference. The bundle (x_1, x_2) that the consumer chooses is revealed preferred to the bundle (y_1, y_2), a bundle that he could have chosen.

Figure 7.1

Well, the bundle (y_1, y_2) is certainly an affordable purchase at the given budget—the consumer could have bought it if he or she wanted to, and would even have had money left over. Since (x_1, x_2) is the *optimal* bundle, it must be better than anything else that the consumer could afford. Hence, in particular it must be better than (y_1, y_2).

The same argument holds for any bundle on or underneath the budget line other than the demanded bundle. Since it *could* have been bought at

the given budget but wasn't, then what *was* bought must be better. Here is where we use the assumption that there is a *unique* demanded bundle for each budget. If preferences are not strictly convex, so that indifference curves have flat spots, it may be that some bundles that are *on* the budget line might be just as good as the demanded bundle. This complication can be handled without too much difficulty, but it is easier to just assume it away.

In Figure 7.1 all of the bundles in the shaded area underneath the budget line are revealed worse than the demanded bundle (x_1, x_2). This is because they could have been chosen, but were rejected in favor of (x_1, x_2). We will now translate this geometric discussion of revealed preference into algebra.

Let (x_1, x_2) be the bundle purchased at prices (p_1, p_2) when the consumer has income m. What does it mean to say that (y_1, y_2) is affordable at those prices and income? It simply means that (y_1, y_2) satisfies the budget constraint

$$p_1 y_1 + p_2 y_2 \leq m.$$

Since (x_1, x_2) is actually bought at the given budget, it must satisfy the budget constraint with equality

$$p_1 x_1 + p_2 x_2 = m.$$

Putting these two equations together, the fact that (y_1, y_2) is affordable at the budget (p_1, p_2, m) means that

$$p_1 x_1 + p_2 x_2 \geq p_1 y_1 + p_2 y_2.$$

If the above inequality is satisfied and (y_1, y_2) is actually a different bundle from (x_1, x_2), we say that (x_1, x_2) is **directly revealed preferred** to (y_1, y_2).

Note that the left-hand side of this inequality is the expenditure on the bundle that is *actually chosen* at prices (p_1, p_2). Thus revealed preference is a relation that holds between the bundle that is actually demanded at some budget and the bundles that *could have been* demanded at that budget.

The term "revealed preference" is actually a bit misleading. It does not inherently have anything to do with preferences, although we've seen above that if the consumer is making optimal choices, the two ideas are closely related. Instead of saying "X is revealed preferred to Y," it would be better to say "X is chosen over Y." When we say that X is revealed preferred to Y, all we are claiming is that X is chosen when Y could have been chosen; that is, that $p_1 x_1 + p_2 x_2 \geq p_1 y_1 + p_2 y_2$.

7.2 From Revealed Preference to Preference

We can summarize the above section very simply. It follows from our model of consumer behavior—that people are choosing the best things they can

afford—that the choices they make are preferred to the choices that they could have made. Or, in the terminology of the last section, if (x_1, x_2) is *directly revealed preferred* to (y_1, y_2), then (x_1, x_2) is in fact *preferred* to (y_1, y_2). Let us state this principle more formally:

The Principle of Revealed Preference. *Let (x_1, x_2) be the chosen bundle when prices are (p_1, p_2), and let (y_1, y_2) be some other bundle such that $p_1x_1 + p_2x_2 \geq p_1y_1 + p_2y_2$. Then if the consumer is choosing the most preferred bundle she can afford, we must have $(x_1, x_2) \succ (y_1, y_2)$.*

When you first encounter this principle, it may seem circular. If X is revealed preferred to Y, doesn't that automatically mean that X is preferred to Y? The answer is no. "Revealed preferred" just means that X was chosen when Y was affordable; "preference" means that the consumer ranks X ahead of Y. If the consumer chooses the best bundles she can afford, then "revealed preference" implies "preference," but that is a consequence of the model of behavior, not the definitions of the terms.

This is why it would be better to say that one bundle is "chosen over" another, as suggested above. Then we would state the principle of revealed preference by saying: "If a bundle X is chosen over a bundle Y, then X must be preferred to Y." In this statement it is clear how the model of behavior allows us to use observed choices to infer something about the underlying preferences.

Whatever terminology you use, the essential point is clear: if we observe that one bundle is chosen when another one is affordable, then we have learned something about the preferences between the two bundles: namely, that the first is preferred to the second.

Now suppose that we happen to know that (y_1, y_2) is a demanded bundle at prices (q_1, q_2) and that (y_1, y_2) is itself revealed preferred to some other bundle (z_1, z_2). That is,

$$q_1y_1 + q_2y_2 \geq q_1z_1 + q_2z_2.$$

Then we know that $(x_1, x_2) \succ (y_1, y_2)$ and that $(y_1, y_2) \succ (z_1, z_2)$. From the transitivity assumption we can conclude that $(x_1, x_2) \succ (z_1, z_2)$.

This argument is illustrated in Figure 7.2. Revealed preference and transitivity tell us that (x_1, x_2) must be better than (z_1, z_2) for the consumer who made the illustrated choices.

It is natural to say that in this case (x_1, x_2) is **indirectly revealed preferred** to (z_1, z_2). Of course the "chain" of observed choices may be longer than just three: if bundle A is directly revealed preferred to B, and B to C, and C to D, ... all the way to M, say, then bundle A is still indirectly revealed preferred to M. The chain of direct comparisons can be of any length.

If a bundle is either directly or indirectly revealed preferred to another bundle, we will say that the first bundle is **revealed preferred** to the

Figure 7.2 **Indirect revealed preference**. The bundle (x_1, x_2) is indirectly revealed preferred to the bundle (z_1, z_2).

second. The idea of revealed preference is simple, but it is surprisingly powerful. Just looking at a consumer's choices can give us a lot of information about the underlying preferences. Consider, for example, Figure 7.2. Here we have several observations on demanded bundles at different budgets. We can conclude from these observations that since (x_1, x_2) is revealed preferred, either directly or indirectly, to all of the bundles in the shaded area, (x_1, x_2) is in fact *preferred* to those bundles by the consumer who made these choices. Another way to say this is to note that the true indifference curve through (x_1, x_2), whatever it is, must lie above the shaded region.

7.3 Recovering Preferences

By observing choices made by the consumer, we can learn about his or her preferences. As we observe more and more choices, we can get a better and better estimate of what the consumer's preferences are like.

Such information about preferences can be very important in making policy decisions. Most economic policy involves trading off some goods for others: if we put a tax on shoes and subsidize clothing, we'll probably end up having more clothes and fewer shoes. In order to evaluate the desirability of such a policy, it is important to have some idea of what consumer preferences between clothes and shoes look like. By examining consumer choices, we can extract such information through the use of revealed preference and related techniques.

If we are willing to add more assumptions about consumer preferences, we can get more precise estimates about the shape of indifference curves. For example, suppose we observe two bundles Y and Z that are revealed preferred to X, as in Figure 7.3, and that we are willing to postulate preferences are convex. Then we know that all of the weighted averages of Y and Z are preferred to X as well. If we are willing to assume that preferences are monotonic, then all the bundles that have more of both goods than X, Y, and Z—or any of their weighted averages—are also preferred to X.

Trapping the indifference curve. The upper shaded area consists of bundles preferred to X, and the lower shaded area consists of bundles revealed worse than X. The indifference curve through X must lie somewhere in the region between the two shaded areas.

Figure 7.3

The region labeled "Worse bundles" in Figure 7.3 consists of all the bundles to which X is revealed preferred. That is, this region consists of all the bundles that cost less than X, along with all the bundles that cost less than bundles that cost less than X, and so on.

Thus, in Figure 7.3, we can conclude that all of the bundles in the upper shaded area are better than X, and that all of the bundles in the lower shaded area are worse than X, according to the preferences of the consumer who made the choices. The true indifference curve through X must lie somewhere between the two shaded sets. We've managed to trap the indifference curve quite tightly simply by an intelligent application of the idea of revealed preference and a few simple assumptions about preferences.

7.4 The Weak Axiom of Revealed Preference

All of the above relies on the assumption that the consumer *has* preferences and that she is always choosing the best bundle of goods she can afford. If the consumer is not behaving this way, the "estimates" of the indifference curves that we constructed above have no meaning. The question naturally arises: how can we tell if the consumer is following the maximizing model? Or, to turn it around: what kind of observation would lead us to conclude that the consumer was *not* maximizing?

Consider the situation illustrated in Figure 7.4. Could both of these choices be generated by a maximizing consumer? According to the logic of revealed preference, Figure 7.4 allows us to conclude two things: (1) (x_1, x_2) is preferred to (y_1, y_2); and (2) (y_1, y_2) is preferred to (x_1, x_2). This is clearly absurd. In Figure 7.4 the consumer has apparently chosen (x_1, x_2) when she could have chosen (y_1, y_2), indicating that (x_1, x_2) was preferred to (y_1, y_2), but then she chose (y_1, y_2) when she could have chosen (x_1, x_2)—indicating the opposite!

Clearly, this consumer cannot be a maximizing consumer. Either the consumer is not choosing the best bundle she can afford, or there is some other aspect of the choice problem that has changed that we have not observed. Perhaps the consumer's tastes or some other aspect of her economic environment have changed. In any event, a violation of this sort is not consistent with the model of consumer choice in an unchanged environment.

The theory of consumer choice implies that such observations will not occur. If the consumers are choosing the best things they can afford, then things that are affordable, but not chosen, must be worse than what is chosen. Economists have formulated this simple point in the following basic axiom of consumer theory

Weak Axiom of Revealed Preference (WARP). *If (x_1, x_2) is directly revealed preferred to (y_1, y_2), and the two bundles are not the same, then it cannot happen that (y_1, y_2) is directly revealed preferred to (x_1, x_2).*

In other words, if a bundle (x_1, x_2) is purchased at prices (p_1, p_2) and a different bundle (y_1, y_2) is purchased at prices (q_1, q_2), then if

$$p_1 x_1 + p_2 x_2 \geq p_1 y_1 + p_2 y_2,$$

Violation of the Weak Axiom of Revealed Preference. A consumer who chooses both (x_1, x_2) and (y_1, y_2) violates the Weak Axiom of Revealed Preference.

Figure 7.4

it must *not* be the case that

$$q_1 y_1 + q_2 y_2 \geq q_1 x_1 + q_2 x_2.$$

In English: if the y-bundle is affordable when the x-bundle is purchased, then when the y-bundle is purchased, the x-bundle must not be affordable.

The consumer in Figure 7.4 has *violated* WARP. Thus we know that this consumer's behavior could not have been maximizing behavior.[1]

There is no set of indifference curves that could be drawn in Figure 7.4 that could make both bundles maximizing bundles. On the other hand, the consumer in Figure 7.5 satisfies WARP. Here it is possible to find indifference curves for which his behavior is optimal behavior. One possible choice of indifference curves is illustrated.

7.5 Checking WARP

Optional

It is important to understand that WARP is a condition that must be satisfied by a consumer who is always choosing the best things he or she can afford. The Weak Axiom of Revealed Preference is a logical implication

[1] Could we say his behavior is WARPed? Well, we could, but not in polite company.

Figure 7.5 **Satisfying WARP**. Consumer choices that satisfy the Weak Axiom of Revealed Preference and some possible indifference curves.

of that model and can therefore be used to check whether or not a particular consumer, or an economic entity that we might want to model as a consumer, is consistent with our economic model.

Let's consider how we would go about systematically testing WARP in practice. Suppose that we observe several choices of bundles of goods at different prices. Let us use (p_1^t, p_2^t) to denote the tth observation of prices and (x_1^t, x_2^t) to denote the tth observation of choices. To use a specific example, let's take the data in Table 7.1.

Table 7.1 **Some consumption data.**

Observation	p_1	p_2	x_1	x_2
1	1	2	1	2
2	2	1	2	1
3	1	1	2	2

Given these data, we can compute how much it would cost the consumer to purchase each bundle of goods at each different set of prices, as we've

done in Table 7.2. For example, the entry in row 3, column 1, measures how much money the consumer would have to spend at the third set of prices to purchase the first bundle of goods.

Cost of each bundle at each set of prices. **Table 7.2**

		Bundles		
		1	2	3
Prices	1	5	4*	6
	2	4*	5	6
	3	3*	3*	4

The diagonal terms in Table 7.2 measure how much money the consumer is spending at each choice. The other entries in each row measure how much she would have spent if she had purchased a different bundle. Thus we can see whether bundle 3, say, is revealed preferred to bundle 1, by seeing if the entry in row 3, column 1 (how much the consumer would have to spend at the third set of prices to purchase the first bundle) is less than the entry in row 3, column 3 (how much the consumer actually spent at the third set of prices to purchase the third bundle). In this particular case, bundle 1 was affordable when bundle 3 was purchased, which means that bundle 3 is revealed preferred to bundle 1. Thus we put a star in row 3, column 1, of the table.

From a mathematical point of view, we simply put a star in the entry in row s, column t, if the number in that entry is less than the number in row s, column s.

We can use this table to check for violations of WARP. In this framework, a violation of WARP consists of two observations t and s such that row t, column s, contains a star *and* row s, column t, contains a star. For this would mean that the bundle purchased at s is revealed preferred to the bundle purchased at t and vice versa.

We can use a computer (or a research assistant) to check and see whether there are any pairs of observations like these in the observed choices. If there are, the choices are inconsistent with the economic theory of the consumer. Either the theory is wrong for this particular consumer, or something else has changed in the consumer's environment that we have not controlled for. Thus the Weak Axiom of Revealed Preference gives us an easily checkable condition for whether some observed choices are consistent with the economic theory of the consumer.

In Table 7.2, we observe that row 1, column 2, contains a star and row 2, column 1, contains a star. This means that observation 2 could have been

chosen when the consumer actually chose observation 1 and vice versa. This is a violation of the Weak Axiom of Revealed Preference. We can conclude that the data depicted in Tables 7.1 and 7.2 could not be generated by a consumer with stable preferences who was always choosing the best things he or she could afford.

7.6 The Strong Axiom of Revealed Preference

The Weak Axiom of Revealed Preference described in the last section gives us an observable condition that must be satisfied by all optimizing consumers. But there is a stronger condition that is sometimes useful.

We have already noted that if a bundle of goods X is revealed preferred to a bundle Y, and Y is in turn revealed preferred to a bundle Z, then X must in fact be preferred to Z. If the consumer has consistent preferences, then we should never observe a sequence of choices that would reveal that Z was preferred to X.

The Weak Axiom of Revealed Preference requires that if X is *directly* revealed preferred to Y, then we should never observe Y being *directly* revealed preferred to X. The **Strong Axiom of Revealed Preference (SARP)** requires that the same sort of condition hold for *indirect* revealed preference. More formally, we have the following.

Strong Axiom of Revealed Preference (SARP). *If (x_1, x_2) is revealed preferred to (y_1, y_2) (either directly or indirectly) and (y_1, y_2) is different from (x_1, x_2), then (y_1, y_2) cannot be directly or indirectly revealed preferred to (x_1, x_2).*

It is clear that if the observed behavior is optimizing behavior then it must satisfy the SARP. For if the consumer is optimizing and (x_1, x_2) is revealed preferred to (y_1, y_2), either directly or indirectly, then we must have $(x_1, x_2) \succ (y_1, y_2)$. So having (x_1, x_2) revealed preferred to (y_1, y_2) *and* (y_1, y_2) revealed preferred to (x_1, x_2) would imply that $(x_1, x_2) \succ (y_1, y_2)$ *and* $(y_1, y_2) \succ (x_1, x_2)$, which is a contradiction. We can conclude that either the consumer must not be optimizing, or some other aspect of the consumer's environment—such as tastes, other prices, and so on—must have changed.

Roughly speaking, since the underlying preferences of the consumer must be transitive, it follows that the *revealed* preferences of the consumer must be transitive. Thus SARP is a *necessary* implication of optimizing behavior: if a consumer is always choosing the best things that he can afford, then his observed behavior must satisfy SARP. What is more surprising is that any behavior satisfying the Strong Axiom can be thought of as being generated by optimizing behavior in the following sense: if the observed choices satisfy SARP, we can always find nice, well-behaved preferences

that *could have* generated the observed choices. In this sense SARP is a *sufficient* condition for optimizing behavior: if the observed choices satisfy SARP, then it is always possible to find preferences for which the observed behavior is optimizing behavior. The proof of this claim is unfortunately beyond the scope of this book, but appreciation of its importance is not.

What it means is that SARP gives us *all* of the restrictions on behavior imposed by the model of the optimizing consumer. For if the observed choices satisfy SARP, we can "construct" preferences that could have generated these choices. Thus SARP is both a necessary and a sufficient condition for observed choices to be compatible with the economic model of consumer choice.

Does this prove that the constructed preferences actually generated the observed choices? Of course not. As with any scientific statement, we can only show that observed behavior is not inconsistent with the statement. We can't prove that the economic model is correct; we can just determine the implications of that model and see if observed choices are consistent with those implications.

7.7 How to Check SARP

Optional

Let us suppose that we have a table like Table 7.2 that has a star in row t and column s if observation t is directly revealed preferred to observation s. How can we use this table to check SARP?

The easiest way is first to transform the table. An example is given in Table 7.3. This is a table just like Table 7.2, but it uses a different set of numbers. Here the stars indicate direct revealed preference. The star in parentheses will be explained below.

How to check SARP. Table 7.3

		Bundles 1	2	3
Prices	1	20	10^*	$22^{(*)}$
	2	21	20	15^*
	3	12	15	10

Now we systematically look through the entries of the table and see if there are any *chains* of observations that make some bundle indirectly revealed preferred to that one. For example, bundle 1 is directly revealed preferred to bundle 2 since there is a star in row 1, column 2. And bundle

2 is directly revealed preferred to bundle 3, since there is a star in row 2, column 3. Therefore bundle 1 is *indirectly* revealed preferred to bundle 3, and we indicate this by putting a star (in parentheses) in row 1, column 3.

In general, if we have many observations, we will have to look for chains of arbitrary length to see if one observation is indirectly revealed preferred to another. Although it may not be exactly obvious how to do this, it turns out that there are simple computer programs that can calculate the indirect revealed preference relation from the table describing the direct revealed preference relation. The computer can put a star in location st of the table if observation s is revealed preferred to observation t by any chain of other observations.

Once we have done this calculation, we can easily test for SARP. We just see if there is a situation where there is a star in row t, column s, *and* also a star in row s, column t. If so, we have found a situation where observation t is revealed preferred to observation s, either directly or indirectly, and, at the same time, observation s is revealed preferred to observation t. This is a violation of the Strong Axiom of Revealed Preference.

On the other hand, if we do not find such violations, then we know that the observations we have are consistent with the economic theory of the consumer. These observations could have been made by an optimizing consumer with well-behaved preferences. Thus we have a completely operational test for whether or not a particular consumer is acting in a way consistent with economic theory.

This is important, since we can model several kinds of economic units as behaving like consumers. Think, for example, of a household consisting of several people. Will its consumption choices maximize "household utility"? If we have some data on household consumption choices, we can use the Strong Axiom of Revealed Preference to see. Another economic unit that we might think of as acting like a consumer is a nonprofit organization like a hospital or a university. Do universities maximize a utility function in making their economic choices? If we have a list of the economic choices that a university makes when faced with different prices, we can, in principle, answer this kind of question.

7.8 Index Numbers

Suppose we examine the consumption bundles of a consumer at two different times and we want to compare how consumption has changed from one time to the other. Let b stand for the base period, and let t be some other time. How does "average" consumption in year t compare to consumption in the base period?

Suppose that at time t prices are (p_1^t, p_2^t) and that the consumer chooses (x_1^t, x_2^t). In the base period b, the prices are (p_1^b, p_2^b), and the consumer's

choice is (x_1^b, x_2^b). We want to ask how the "average" consumption of the consumer has changed.

If we let w_1 and w_2 be some "weights" that go into making an average, then we can look at the following kind of quantity index:

$$I_q = \frac{w_1 x_1^t + w_2 x_2^t}{w_1 x_1^b + w_2 x_2^b}.$$

If I_q is greater than 1, we can say that the "average" consumption has gone up in the movement from b to t; if I_q is less than 1, we can say that the "average" consumption has gone down.

The question is, what do we use for the weights? A natural choice is to use the prices of the goods in question, since they measure in some sense the relative importance of the two goods. But there are two sets of prices here: which should we use?

If we use the base period prices for the weights, we have something called a **Laspeyres** index, and if we use the t period prices, we have something called a **Paasche** index. Both of these indices answer the question of what has happened to "average" consumption, but they just use different weights in the averaging process.

Substituting the t period prices for the weights, we see that the **Paasche quantity index** is given by

$$P_q = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^t x_1^b + p_2^t x_2^b},$$

and substituting the b period prices shows that the **Laspeyres quantity index** is given by

$$L_q = \frac{p_1^b x_1^t + p_2^b x_2^t}{p_1^b x_1^b + p_2^b x_2^b}.$$

It turns out that the magnitude of the Laspeyres and Paasche indices can tell us something quite interesting about the consumer's welfare. Suppose that we have a situation where the Paasche quantity index is greater than 1:

$$P_q = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^t x_1^b + p_2^t x_2^b} > 1.$$

What can we conclude about how well-off the consumer is at time t as compared to his situation at time b?

The answer is provided by revealed preference. Just cross multiply this inequality to give

$$p_1^t x_1^t + p_2^t x_2^t > p_1^t x_1^b + p_2^t x_2^b,$$

which immediately shows that the consumer must be better off at t than at b, since he could have consumed the b consumption bundle in the t situation but chose not to do so.

What if the Paasche index is *less* than 1? Then we would have

$$p_1^t x_1^t + p_2^t x_2^t < p_1^t x_1^b + p_2^t x_2^b,$$

which says that when the consumer chose bundle (x_1^t, x_2^t), bundle (x_1^b, x_2^b) was not affordable. But that doesn't say anything about the consumer's ranking of the bundles. Just because something costs more than you can afford doesn't mean that you prefer it to what you're consuming now.

What about the Laspeyres index? It works in a similar way. Suppose that the Laspeyres index is *less* than 1:

$$L_q = \frac{p_1^b x_1^t + p_2^b x_2^t}{p_1^b x_1^b + p_2^b x_2^b} < 1.$$

Cross multiplying yields

$$p_1^b x_1^b + p_2^b x_2^b > p_1^b x_1^t + p_2^b x_2^t,$$

which says that (x_1^b, x_2^b) is revealed preferred to (x_1^t, x_2^t). Thus the consumer is better off at time b than at time t.

7.9 Price Indices

Price indices work in much the same way. In general, a price index will be a weighted average of prices:

$$I_p = \frac{p_1^t w_1 + p_2^t w_2}{p_1^b w_1 + p_2^b w_2}.$$

In this case it is natural to choose the quantities as the weights for computing the averages. We get two different indices, depending on our choice of weights. If we choose the t period quantities for weights, we get the **Paasche price index**:

$$P_p = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^b x_1^t + p_2^b x_2^t},$$

and if we choose the base period quantities we get the **Laspeyres price index:**

$$L_p = \frac{p_1^t x_1^b + p_2^t x_2^b}{p_1^b x_1^b + p_2^b x_2^b}.$$

Suppose that the Paasche price index is less than 1; what does revealed preference have to say about the welfare situation of the consumer in periods t and b?

Revealed preference doesn't say anything at all. The problem is that there are now different prices in the numerator and in the denominator of the fractions defining the indices, so the revealed preference comparison can't be made.

Let's define a new index of the change in total expenditure by

$$M = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^b x_1^b + p_2^b x_2^b}.$$

This is the ratio of total expenditure in period t to the total expenditure in period b.

Now suppose that you are told that the Paasche price index was greater than M. This means that

$$P_p = \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^b x_1^t + p_2^b x_2^t} > \frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^b x_1^b + p_2^b x_2^b}.$$

Canceling the numerators from each side of this expression and cross multiplying, we have

$$p_1^b x_1^b + p_2^b x_2^b > p_1^b x_1^t + p_2^b x_2^t.$$

This statement says that the bundle chosen at year b is revealed preferred to the bundle chosen at year t. This analysis implies that if the Paasche price index is greater than the expenditure index, then the consumer must be better off in year b than in year t.

This is quite intuitive. After all, if prices rise by more than income rises in the movement from b to t, we would expect that would tend to make the consumer worse off. The revealed preference analysis given above confirms this intuition.

A similar statement can be made for the Laspeyres price index. If the Laspeyres price index is less than M, then the consumer must be better off in year t than in year b. Again, this simply confirms the intuitive idea that if prices rise less than income, the consumer would become better off. In the case of price indices, what matters is not whether the index is greater or less than 1, but whether it is greater or less than the expenditure index.

EXAMPLE: Indexing Social Security Payments

Many elderly people have Social Security payments as their sole source of income. Because of this, there have been attempts to adjust Social Security payments in a way that will keep purchasing power constant even when prices change. Since the amount of payments will then depend on the movement of some price index or cost-of-living index, this kind of scheme is referred to as **indexing**.

One indexing proposal goes as follows. In some base year b, economists measure the average consumption bundle of senior citizens. In each subsequent year the Social Security system adjusts payments so that the "purchasing power" of the average senior citizen remains constant in the sense that the average Social Security recipient is just able to afford the consumption bundle available in year b, as depicted in Figure 7.6.

Figure 7.6 **Social Security**. Changing prices will typically make the consumer better off than in the base year.

One curious result of this indexing scheme is that the average senior citizen will almost always be better off than he or she was in the base year b. Suppose that year b is chosen as the base year for the price index. Then the bundle (x_1^b, x_2^b) is the optimal bundle at the prices (p_1^b, p_2^b). This means that the budget line at prices (p_1^b, p_2^b) must be tangent to the indifference curve through (x_1^b, x_2^b).

Now suppose that prices change. To be specific, suppose that prices increase so that the budget line, in the absence of Social Security, would shift inward and tilt. The inward shift is due to the increase in prices; the tilt is due to the change in relative prices. The indexing program would then increase the Social Security payment so as to make the original bundle (x_1^b, x_2^b) affordable at the new prices. But this means that the budget line would cut the indifference curve, and there would be some other bundle

on the budget line that would be strictly preferred to (x_1^b, x_2^b). Thus the consumer would typically be able to choose a better bundle than he or she chose in the base year.

Summary

1. If one bundle is chosen when another could have been chosen, we say that the first bundle is revealed preferred to the second.

2. If the consumer is always choosing the most preferred bundles he or she can afford, this means that the chosen bundles must be preferred to the bundles that were affordable but weren't chosen.

3. Observing the choices of consumers can allow us to "recover" or estimate the preferences that lie behind those choices. The more choices we observe, the more precisely we can estimate the underlying preferences that generated those choices.

4. The Weak Axiom of Revealed Preference (WARP) and the Strong Axiom of Revealed Preference (SARP) are necessary conditions that consumer choices have to obey if they are to be consistent with the economic model of optimizing choice.

REVIEW QUESTIONS

1. When prices are $(p_1, p_2) = (1, 2)$ a consumer demands $(x_1, x_2) = (1, 2)$, and when prices are $(q_1, q_2) = (2, 1)$ the consumer demands $(y_1, y_2) = (2, 1)$. Is this behavior consistent with the model of maximizing behavior?

2. When prices are $(p_1, p_2) = (2, 1)$ a consumer demands $(x_1, x_2) = (1, 2)$, and when prices are $(q_1, q_2) = (1, 2)$ the consumer demands $(y_1, y_2) = (2, 1)$. Is this behavior consistent with the model of maximizing behavior?

3. In the preceding exercise, which bundle is preferred by the consumer, the x-bundle or the y-bundle?

4. We saw that the Social Security adjustment for changing prices would typically make recipients at least as well-off as they were at the base year. What kind of price changes would leave them just as well-off, no matter what kind of preferences they had?

5. In the same framework as the above question, what kind of preferences would leave the consumer just as well-off as he was in the base year, for *all* price changes?

CHAPTER 8

SLUTSKY EQUATION

Economists often are concerned with how a consumer's behavior changes in response to changes in the economic environment. The case we want to consider in this chapter is how a consumer's choice of a good responds to changes in its price. It is natural to think that when the price of a good rises the demand for it will fall. However, as we saw in Chapter 6 it is possible to construct examples where the optimal demand for a good *decreases* when its price falls. A good that has this property is called a **Giffen good**.

Giffen goods are pretty peculiar and are primarily a theoretical curiosity, but there are other situations where changes in prices might have "perverse" effects that, on reflection, turn out not to be so unreasonable. For example, we normally think that if people get a higher wage they will work more. But what if your wage went from $10 an hour to $1000 an hour? Would you really work more? Might you not decide to work fewer hours and use some of the money you've earned to do other things? What if your wage were $1,000,000 an hour? Wouldn't you work less?

For another example, think of what happens to your demand for apples when the price goes up. You would probably consume fewer apples. But

how about a family who grew apples to sell? If the price of apples went up, their income might go up so much that they would feel that they could now afford to consume more of their own apples. For the consumers in this family, an increase in the price of apples might well lead to an increase in the consumption of apples.

What is going on here? How is it that changes in price can have these ambiguous effects on demand? In this chapter and the next we'll try to sort out these effects.

8.1 The Substitution Effect

When the price of a good changes, there are two sorts of effects: the rate at which you can exchange one good for another changes, and the total purchasing power of your income is altered. If, for example, good 1 becomes cheaper, it means that you have to give up less of good 2 to purchase good 1. The change in the price of good 1 has changed the rate at which the market allows you to "substitute" good 2 for good 1. The trade-off between the two goods that the market presents the consumer has changed.

At the same time, if good 1 becomes cheaper it means that your money income will buy more of good 1. The purchasing power of your money has gone up; although the number of dollars you have is the same, the amount that they will buy has increased.

The first part—the change in demand due to the change in the rate of exchange between the two goods—is called the **substitution effect**. The second effect—the change in demand due to having more purchasing power—is called the **income effect**. These are only rough definitions of the two effects. In order to give a more precise definition we have to consider the two effects in greater detail.

The way that we will do this is to break the price movement into two steps: first we will let the *relative* prices change and adjust money income so as to hold purchasing power constant, then we will let purchasing power adjust while holding the relative prices constant.

This is best explained by referring to Figure 8.1. Here we have a situation where the price of good 1 has declined. This means that the budget line rotates around the vertical intercept m/p_2 and becomes flatter. We can break this movement of the budget line up into two steps: first *pivot* the budget line around the *original* demanded bundle and then *shift* the pivoted line out to the *new* demanded bundle.

This "pivot-shift" operation gives us a convenient way to decompose the change in demand into two pieces. The first step—the pivot—is a movement where the slope of the budget line changes while its purchasing power stays constant, while the second step is a movement where the slope stays constant and the purchasing power changes. This decomposition is only a hypothetical construction—the consumer simply observes a change

Figure 8.1

Pivot and shift. When the price of good 1 changes and income stays fixed, the budget line pivots around the vertical axis. We will view this adjustment as occurring in two stages: first pivot the budget line around the *original* choice, and then shift this line outward to the new demanded bundle.

in price and chooses a new bundle of goods in response. But in analyzing how the consumer's choice changes, it is useful to think of the budget line changing in two stages—first the pivot, then the shift.

What are the economic meanings of the pivoted and the shifted budget lines? Let us first consider the pivoted line. Here we have a budget line with the same slope and thus the same relative prices as the final budget line. However, the money income associated with this budget line is different, since the vertical intercept is different. Since the original consumption bundle (x_1, x_2) lies on the pivoted budget line, that consumption bundle is just affordable. The purchasing power of the consumer has remained constant in the sense that the original bundle of goods is just affordable at the new pivoted line.

Let us calculate how much we have to adjust money income in order to keep the old bundle just affordable. Let m' be the amount of money income that will just make the original consumption bundle affordable; this will be the amount of money income associated with the pivoted budget line. Since (x_1, x_2) is affordable at both (p_1, p_2, m) and (p_1', p_2, m'), we have

$$m' = p_1' x_1 + p_2 x_2$$
$$m = p_1 x_1 + p_2 x_2.$$

Subtracting the second equation from the first gives

$$m' - m = x_1[p_1' - p_1].$$

This equation says that the change in money income necessary to make the old bundle affordable at the new prices is just the original amount of consumption of good 1 times the change in prices.

Letting $\Delta p_1 = p_1' - p_1$ represent the change in price 1, and $\Delta m = m' - m$ represent the change in income necessary to make the old bundle just affordable, we have

$$\Delta m = x_1 \Delta p_1. \tag{8.1}$$

Note that the change in income and the change in price will always move in the same direction: if the price goes up, then we have to raise income to keep the same bundle affordable.

Let's use some actual numbers. Suppose that the consumer is originally consuming 20 candy bars a week, and that candy bars cost 50 cents a piece. If the price of candy bars goes up by 10 cents—so that $\Delta p_1 = .60 - .50 = .10$—how much would income have to change to make the old consumption bundle affordable?

We can apply the formula given above. If the consumer had \$2.00 more income, he would just be able to consume the same number of candy bars, namely, 20. In terms of the formula:

$$\Delta m = \Delta p_1 \times x_1 = .10 \times 20 = \$2.00.$$

Now we have a formula for the pivoted budget line: it is just the budget line at the new price with income changed by Δm. Note that if the price of good 1 goes down, then the adjustment in income will be negative. When a price goes down, a consumer's purchasing power goes up, so we will have to decrease the consumer's income in order to keep purchasing power fixed. Similarly, when a price goes up, purchasing power goes down, so the change in income necessary to keep purchasing power constant must be positive.

Although (x_1, x_2) is still affordable, it is not generally the optimal purchase at the pivoted budget line. In Figure 8.2 we have denoted the optimal purchase on the pivoted budget line by Y. This bundle of goods is the optimal bundle of goods when we change the price and then adjust dollar income so as to keep the old bundle of goods just affordable. The movement from X to Y is known as the **substitution effect**. It indicates how the consumer "substitutes" one good for the other when a price changes but purchasing power remains constant.

More precisely, the substitution effect, Δx_1^s, is the change in the demand for good 1 when the price of good 1 changes to p_1' and, at the same time, money income changes to m':

$$\Delta x_1^s = x_1(p_1', m') - x_1(p_1, m).$$

In order to determine the substitution effect, we must use the consumer's demand function to calculate the optimal choices at (p_1', m') and (p_1, m). The change in the demand for good 1 may be large or small, depending

Figure 8.2 **Substitution effect and income effect**. The pivot gives the substitution effect, and the shift gives the income effect.

on the shape of the consumer's indifference curves. But given the demand function, it is easy to just plug in the numbers to calculate the substitution effect. (Of course the demand for good 1 may well depend on the price of good 2; but the price of good 2 is being held constant during this exercise, so we've left it out of the demand function so as not to clutter the notation.)

The substitution effect is sometimes called the change in **compensated demand**. The idea is that the consumer is being compensated for a price rise by having enough income given back to him to purchase his old bundle. Of course if the price goes down he is "compensated" by having money taken away from him. We'll generally stick with the "substitution" terminology, for consistency, but the "compensation" terminology is also widely used.

EXAMPLE: Calculating the Substitution Effect

Suppose that the consumer has a demand function for milk of the form

$$x_1 = 10 + \frac{m}{10p_1}.$$

Originally his income is \$120 per week and the price of milk is \$3 per quart. Thus his demand for milk will be $10 + 120/(10 \times 3) = 14$ quarts per week.

Now suppose that the price of milk falls to \$2 per quart. Then his demand at this new price will be $10 + 120/(10 \times 2) = 16$ quarts of milk per week. The *total* change in demand is +2 quarts a week.

In order to calculate the substitution effect, we must first calculate how much income would have to change in order to make the original consumption of milk just affordable when the price of milk is \$2 a quart. We apply the formula (8.1):

$$\Delta m = x_1 \Delta p_1 = 14 \times (2 - 3) = -\$14.$$

Thus the level of income necessary to keep purchasing power constant is $m' = m + \Delta m = 120 - 14 = 106$. What is the consumer's demand for milk at the new price, \$2 per quart, and this level of income? Just plug the numbers into the demand function to find

$$x_1(p_1', m') = x_1(2, 106) = 10 + \frac{106}{10 \times 2} = 15.3.$$

Thus the substitution effect is

$$\Delta x_1^s = x_1(2, 106) - x_1(3, 120) = 15.3 - 14 = 1.3.$$

8.2 The Income Effect

We turn now to the second stage of the price adjustment—the shift movement. This is also easy to interpret economically. We know that a parallel shift of the budget line is the movement that occurs when income changes while relative prices remain constant. Thus the second stage of the price adjustment is called the **income effect**. We simply change the consumer's income from m' to m, keeping the prices constant at (p_1', p_2). In Figure 8.2 this change moves us from the point (y_1, y_2) to (z_1, z_2). It is natural to call this last movement the income effect since all we are doing is changing income while keeping the prices fixed at the new prices.

More precisely, the income effect, Δx_1^n, is the change in the demand for good 1 when we change income from m' to m, holding the price of good 1 fixed at p_1':

$$\Delta x_1^n = x_1(p_1', m) - x_1(p_1', m').$$

We have already considered the income effect earlier in section 6.1. There we saw that the income effect can operate either way: it will tend to increase or decrease the demand for good 1 depending on whether we have a normal good or an inferior good.

When the price of a good decreases, we need to decrease income in order to keep purchasing power constant. If the good is a normal good, then this decrease in income will lead to a decrease in demand. If the good is an inferior good, then the decrease in income will lead to an increase in demand.

EXAMPLE: Calculating the Income Effect

In the example given earlier in this chapter we saw that

$$x_1(p_1', m) = x_1(2, 120) = 16$$
$$x_1(p_1', m') = x_1(2, 106) = 15.3.$$

Thus the income effect for this problem is

$$\Delta x_1^n = x_1(2, 120) - x_1(2, 106) = 16 - 15.3 = 0.7.$$

Since milk is a normal good for this consumer, the demand for milk increases when income increases.

8.3 Sign of the Substitution Effect

We have seen above that the income effect can be positive or negative, depending on whether the good is a normal good or an inferior good. What about the substitution effect? If the price of a good goes down, as in Figure 8.2, then the change in the demand for the good due to the substitution effect *must* be nonnegative. That is, if $p_1 > p_1'$, then we *must* have $x_1(p_1', m') \geq x_1(p_1, m)$, so that $\Delta x_1^s \geq 0$.

The proof of this goes as follows. Consider the points on the pivoted budget line in Figure 8.2 where the amount of good 1 consumed is less than at the bundle X. These bundles were all affordable at the old prices (p_1, p_2) but they weren't purchased. Instead the bundle X was purchased. If the consumer is always choosing the best bundle he can afford, then X must be preferred to all of the bundles on the part of the pivoted line that lies inside the original budget set.

This means that the optimal choice on the pivoted budget line must not be one of the bundles that lies underneath the original budget line. The optimal choice on the pivoted line would have to be either X or some point to the right of X. But this means that the new optimal choice must involve consuming at least as much of good 1 as originally, just as we wanted to show. In the case illustrated in Figure 8.2, the optimal choice at the pivoted budget line is the bundle Y, which certainly involves consuming more of good 1 than at the original consumption point, X.

The substitution effect always moves opposite to the price movement. We say that *the substitution effect is negative*, since the change in demand due to the substitution effect is opposite to the change in price: if the price increases, the demand for the good due to the substitution effect decreases.

8.4 The Total Change in Demand

The total change in demand, Δx_1, is the change in demand due to the change in price, holding income constant:

$$\Delta x_1 = x_1(p_1', m) - x_1(p_1, m).$$

We have seen above how this change can be broken up into two changes: the substitution effect and the income effect. In terms of the symbols defined above,

$$\begin{aligned} \Delta x_1 &= \Delta x_1^s + \Delta x_1^n \\ x_1(p_1', m) - x_1(p_1, m) &= [x_1(p_1', m') - x_1(p_1, m)] \\ &\quad + [x_1(p_1', m) - x_1(p_1', m')]. \end{aligned}$$

In words this equation says that the total change in demand equals the substitution effect plus the income effect. This equation is called the **Slutsky identity**.[1] Note that it is an identity: it is true for all values of p_1, p_1', m, and m'. The first and fourth terms on the right-hand side cancel out, so the right-hand side is *identically* equal to the left-hand side.

The content of the Slutsky identity is not just the algebraic identity—that is a mathematical triviality. The content comes in the interpretation of the two terms on the right-hand side: the substitution effect and the income effect. In particular, we can use what we know about the signs of the income and substitution effects to determine the sign of the total effect.

While the substitution effect must always be negative—opposite the change in the price—the income effect can go either way. Thus the total effect may be positive or negative. However, if we have a normal good, then the substitution effect and the income effect work in the same direction. An increase in price means that demand will go down due to the substitution effect. If the price goes up, it is like a decrease in income, which, for a normal good, means a decrease in demand. Both effects reinforce each other. In terms of our notation, the change in demand due to a price increase for a normal good means that

$$\begin{array}{ccccc} \Delta x_1 & = & \Delta x_1^s & + & \Delta x_1^n. \\ (-) & & (-) & & (-) \end{array}$$

(The minus signs beneath each term indicate that each term in this expression is negative.)

[1] Named for Eugen Slutsky (1880–1948), a Russian economist who investigated demand theory.

Note carefully the sign on the income effect. Since we are considering a situation where the price rises, this implies a decrease in purchasing power—for a normal good this will imply a decrease in demand.

On the other hand, if we have an inferior good, it might happen that the income effect outweighs the substitution effect, so that the total change in demand associated with a price increase is actually positive. This would be a case where

$$\begin{array}{ccc} \Delta x_1 = & \Delta x_1^s + & \Delta x_1^n. \\ (?) & (-) & (+) \end{array}$$

If the second term on the right-hand side—the income effect—is large enough, the total change in demand could be positive. This would mean that an increase in price could result in an *increase* in demand. This is the perverse Giffen case described earlier: the increase in price has reduced the consumer's purchasing power so much that he has increased his consumption of the inferior good.

But the Slutsky identity shows that this kind of perverse effect can only occur for inferior goods: if a good is a normal good, then the income and substitution effects reinforce each other, so that the total change in demand is always in the "right" direction.

Thus a Giffen good must be an inferior good. But an inferior good is not necessarily a Giffen good: the income effect not only has to be of the "wrong" sign, it also has to be large enough to outweigh the "right" sign of the substitution effect. This is why Giffen goods are so rarely observed in real life: they would not only have to be inferior goods, but they would have to be *very* inferior.

This is illustrated graphically in Figure 8.3. Here we illustrate the usual pivot-shift operation to find the substitution effect and the income effect. In both cases, good 1 is an inferior good, and the income effect is therefore negative. In Figure 8.3A, the income effect is large enough to outweigh the substitution effect and produce a Giffen good. In Figure 8.3B, the income effect is smaller, and thus good 1 responds in the ordinary way to the change in its price.

8.5 Rates of Change

We have seen that the income and substitution effects can be described graphically as a combination of pivots and shifts, or they can be described algebraically in the Slutsky identity

$$\Delta x_1 = \Delta x_1^s + \Delta x_1^n,$$

which simply says that the total change in demand is the substitution effect plus the income effect. The Slutsky identity here is stated in terms

Inferior goods. Panel A shows a good that is inferior enough to cause the Giffen case. Panel B shows a good that is inferior, but the effect is not strong enough to create a Giffen good.

Figure 8.3

of absolute changes, but it is more common to express it in terms of *rates* of change.

When we express the Slutsky identity in terms of rates of change it turns out to be convenient to define Δx_1^m to be the *negative* of the income effect:

$$\Delta x_1^m = x_1(p_1', m') - x_1(p_1', m) = -\Delta x_1^n.$$

Given this definition, the Slutsky identity becomes

$$\Delta x_1 = \Delta x_1^s - \Delta x_1^m.$$

If we divide each side of the identity by Δp_1, we have

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1^m}{\Delta p_1}. \tag{8.2}$$

The first term on the right-hand side is the rate of change of demand when price changes and income is adjusted so as to keep the old bundle affordable—the substitution effect. Let's work on the second term. Since we have an income change in the numerator, it would be nice to get an income change in the denominator.

Remember that the income change, Δm, and the price change, Δp_1, are related by the formula

$$\Delta m = x_1 \Delta p_1.$$

Solving for Δp_1 we find

$$\Delta p_1 = \frac{\Delta m}{x_1}.$$

Now substitute this expression into the last term in (8.2) to get our final formula:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1^m}{\Delta m} x_1.$$

This is the Slutsky identity in terms of rates of change. We can interpret each term as follows:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{x_1(p_1', m) - x_1(p_1, m)}{\Delta p_1}$$

is the rate of change in demand as price changes, holding income fixed;

$$\frac{\Delta x_1^s}{\Delta p_1} = \frac{x_1(p_1', m') - x_1(p_1, m)}{\Delta p_1}$$

is the rate of change in demand as the price changes, adjusting income so as to keep the old bundle just affordable, that is, the substitution effect; and

$$\frac{\Delta x_1^m}{\Delta m} x_1 = \frac{x_1(p_1', m') - x_1(p_1', m)}{m' - m} x_1 \tag{8.3}$$

is the rate of change of demand holding prices fixed and adjusting income, that is, the income effect.

The income effect is itself composed of two pieces: how demand changes as income changes, times the original level of demand. When the price changes by Δp_1, the change in demand due to the income effect is

$$\Delta x_1^m = \frac{x_1(p_1', m') - x_1(p_1', m)}{\Delta m} x_1 \Delta p_1.$$

But this last term, $x_1 \Delta p_1$, is just the change in income necessary to keep the old bundle feasible. That is, $x_1 \Delta p_1 = \Delta m$, so the change in demand due to the income effect reduces to

$$\Delta x_1^m = \frac{x_1(p_1', m') - x_1(p_1', m)}{\Delta m} \Delta m,$$

just as we had before.

8.6 The Law of Demand

In Chapter 5 we voiced some concerns over the fact that consumer theory seemed to have no particular content: demand could go up or down when a price increased, and demand could go up or down when income increased. If a theory doesn't restrict observed behavior in *some* fashion it isn't much of a theory. A model that is consistent with all behavior has no real content.

However, we know that consumer theory does have some content—we've seen that choices generated by an optimizing consumer must satisfy the Strong Axiom of Revealed Preference. Furthermore, we've seen that any price change can be decomposed into two changes: a substitution effect that is sure to be negative—opposite the direction of the price change—and an income effect whose sign depends on whether the good is a normal good or an inferior good.

Although consumer theory doesn't restrict how demand changes when price changes or how demand changes when income changes, it does restrict how these two kinds of changes interact. In particular, we have the following.

The Law of Demand. *If the demand for a good increases when income increases, then the demand for that good must decrease when its price increases.*

This follows directly from the Slutsky equation: if the demand increases when income increases, we have a normal good. And if we have a normal good, then the substitution effect and the income effect reinforce each other, and an increase in price will unambiguously reduce demand.

8.7 Examples of Income and Substitution Effects

Let's now consider some examples of price changes for particular kinds of preferences and decompose the demand changes into the income and the substitution effects.

We start with the case of perfect complements. The Slutsky decomposition is illustrated in Figure 8.4. When we pivot the budget line around the chosen point, the optimal choice at the new budget line is the same as at the old one—this means that the substitution effect is zero. The change in demand is due entirely to the income effect in this case.

What about the case of perfect substitutes, illustrated in Figure 8.5? Here when we tilt the budget line, the demand bundle jumps from the vertical axis to the horizontal axis. There is no shifting left to do! The entire change in demand is due to the substitution effect in this case.

Figure 8.4 **Perfect complements**. Slutsky decomposition with perfect complements.

As a third example, let us consider the case of quasilinear preferences. This situation is somewhat peculiar. We have already seen that a shift in income causes no change in demand for good 1 when preferences are quasilinear. This means that the entire change in demand for good 1 is due to the substitution effect, and that the income effect is zero, as illustrated in Figure 8.6.

EXAMPLE: Rebating a Tax

In 1974 the Organization of Petroleum Exporting Countries (OPEC) instituted an oil embargo against the United States. OPEC was able to stop oil shipments to U.S. ports for several weeks. The vulnerability of the United States to such disruptions was very disturbing to Congress and the president, and there were many plans proposed to reduce the United States's dependence on foreign oil.

One such plan involved increasing the gasoline tax. Increasing the cost of gasoline to the consumers would make them reduce their consumption of gasoline, and the reduced demand for gasoline would in turn reduce the demand for foreign oil.

But a straight increase in the tax on gasoline would hit consumers where it hurts—in the pocketbook—and by itself such a plan would be politically

Perfect substitutes. Slutsky decomposition with perfect substitutes.

Figure 8.5

infeasible. So it was suggested that the revenues raised from consumers by this tax would be returned to the consumers in the form of direct money payments, or via the reduction of some other tax.

Critics of this proposal argued that paying the revenue raised by the tax back to the consumers would have no effect on demand since they could just use the rebated money to purchase more gasoline. What does economic analysis say about this plan?

Let us suppose, for simplicity, that the tax on gasoline would end up being passed along entirely to the consumers of gasoline so that the price of gasoline will go up by exactly the amount of the tax. (In general, only part of the tax would be passed along, but we will ignore that complication here.) Suppose that the tax would raise the price of gasoline from p to $p' = p + t$, and that the average consumer would respond by reducing his demand from x to x'. The average consumer is paying t dollars more for gasoline, and he is consuming x' gallons of gasoline after the tax is imposed, so the amount of revenue raised by the tax from the average consumer would be

$$R = tx' = (p' - p)x'.$$

Note that the revenue raised by the tax will depend on how much gasoline the consumer *ends up* consuming, x', not how much he was initially

Figure 8.6 **Quasilinear preferences**. In the case of quasilinear preferences, the entire change in demand is due to the substitution effect.

consuming, x.

If we let y be the expenditure on all other goods and set its price to be 1, then the original budget constraint is

$$px + y = m, \tag{8.4}$$

and the budget constraint in the presence of the tax-rebate plan is

$$(p + t)x' + y' = m + tx'. \tag{8.5}$$

In budget constraint (8.5) the average consumer is choosing the left-hand side variables—the consumption of each good—but the right-hand side—his income and the rebate from the government—are taken as fixed. The rebate depends on what all consumers do, not what the average consumer does. In this case, the rebate turns out to be the taxes collected from the average consumer—but that's because he is average, not because of any causal connection.

If we cancel tx' from each side of equation (8.5), we have

$$px' + y' = m.$$

Thus (x', y') is a bundle that was affordable under the original budget constraint and rejected in favor of (x, y). Thus it must be that (x, y)

is preferred to (x', y'): the consumers are made worse off by this plan. Perhaps that is why it was never put into effect!

The equilibrium with a rebated tax is depicted in Figure 8.7. The tax makes good 1 more expensive, and the rebate increases money income. The original bundle is no longer affordable, and the consumer is definitely made worse off. The consumer's choice under the tax-rebate plan involves consuming less gasoline and more of "all other goods."

Rebating a tax. Taxing a consumer and rebating the tax revenues makes the consumer worse off.

Figure 8.7

What can we say about the amount of consumption of gasoline? The average consumer could afford his old consumption of gasoline, but because of the tax, gasoline is now more expensive. In general, the consumer would choose to consume less of it.

EXAMPLE: Voluntary Real Time Pricing

Electricity production suffers from an extreme capacity problem: it is relatively cheap to produce up to capacity, at which point it is, by definition, impossible to produce more. Building capacity is extremely expensive, so

finding ways to reduce the use of electricity during periods of peak demand is very attractive from an economic point of view.

In states with warm climates, such as Georgia, roughly 30 percent of usage during periods of peak demand is due to air conditioning. Furthermore, it is relatively easy to forecast temperature one day ahead so that potential users will have time to adjust their demand by setting their air conditioning to a higher temperature, wearing light clothes, and so on. The challenge is to set up a pricing system so that those users who are able to cut back on their electricity use will have an incentive to reduce their consumption.

One way to accomplish this is through the use of Real Time Pricing (RTP). In a Real Time Pricing program, large industrial users are equipped with special meters that allow the price of electricity to vary from minute to minute, depending on signals sent from the electricity generating company. As the demand for electricity approaches capacity, the generating company increases the price so as to encourage users to cut back on their usage. The price schedule is determined as a function of the total demand for electricity.

Georgia Power Company claims that it runs the largest real time pricing program in the world. In 1999 it was able to reduce demand by 750 megawatts on high-price days by inducing some large customers to cut their demand by as much as 60 percent.

Georgia Power has devised several interesting variations on the basic real time pricing model. In one pricing plan, customers are assigned a baseline quantity, which represents their normal usage. When electricity is in short supply and the real time price increases, these users face a higher price for electricity use in excess of their baseline quantity. But they also receive a *rebate* if they can manage to cut their electricity use below their baseline amount.

Figure 8.8 shows how this affects the budget line of the users. The vertical axis is "money to spend on things other than electricity" and the horizontal axis is "electricity use." In normal times, users choose their electricity consumption to maximize utility subject to a budget constraint which is determined by the baseline price of electricity. The resulting choice is their baseline consumption.

When the temperature rises, the real time price increases, making electricity more expensive. But this increase in price is a good thing for users who can cut back their consumption, since they receive a rebate based on the high real time price for every kilowatt of reduced usage. If usage stays at the baseline amount, then the user's bill will not change.

It is not hard to see that this pricing plan is a Slutsky pivot around the baseline consumption. Thus we can be confident that electricity usage will decline, and that users will be at least as well off at the real time price as at the baseline price. Indeed, the program has been quite popular, with over 1,600 voluntary participants.

Figure 8.8

Voluntary real time pricing. Users pay higher rates for additional electricity when the real time price rises, but they also get rebates at the same price if they cut back their use. This results in a pivot around the baseline use and tends to make the customers better off.

8.8 Another Substitution Effect

The substitution effect is the name that economists give to the change in demand when prices change but a consumer's purchasing power is held constant, so that the original bundle remains affordable. At least this is *one* definition of the substitution effect. There is another definition that is also useful.

The definition we have studied above is called the **Slutsky substitution effect**. The definition we will describe in this section is called the **Hicks substitution effect**.[2]

Suppose that instead of pivoting the budget line around the original consumption bundle, we now *roll* the budget line around the indifference curve through the original consumption bundle, as depicted in Figure 8.9. In this way we present the consumer with a new budget line that has the same relative prices as the final budget line but has a different income. The purchasing power he has under this budget line will no longer be sufficient to

[2] The concept is named for Sir John Hicks, an English recipient of the Nobel Prize in Economics.

purchase his original bundle of goods—but it will be sufficient to purchase a bundle that is just *indifferent* to his original bundle.

Figure 8.9 **The Hicks substitution effect**. Here we pivot the budget line around the indifference curve rather than around the original choice.

Thus the Hicks substitution effect keeps *utility* constant rather than keeping purchasing power constant. The Slutsky substitution effect gives the consumer just enough money to get back to his old level of consumption, while the Hicks substitution effect gives the consumer just enough money to get back to his old indifference curve. Despite this difference in definition, it turns out that the Hicks substitution effect must be negative—in the sense that it is in a direction opposite that of the price change—just like the Slutsky substitution effect.

The proof is again by revealed preference. Let (x_1, x_2) be a demanded bundle at some prices (p_1, p_2), and let (y_1, y_2) be a demanded bundle at some other prices (q_1, q_2). Suppose that income is such that the consumer is indifferent between (x_1, x_2) and (y_1, y_2). Since the consumer is indifferent between (x_1, x_2) and (y_1, y_2), neither bundle can be revealed preferred to the other.

Using the definition of revealed preference, this means that the following

two inequalities are *not* true:

$$p_1x_1 + p_2x_2 > p_1y_1 + p_2y_2$$

$$q_1y_1 + q_2y_2 > q_1x_1 + q_2x_2.$$

It follows that these inequalities *are* true:

$$p_1x_1 + p_2x_2 \leq p_1y_1 + p_2y_2$$

$$q_1y_1 + q_2y_2 \leq q_1x_1 + q_2x_2.$$

Adding these inequalities together and rearranging them we have

$$(q_1 - p_1)(y_1 - x_1) + (q_2 - p_2)(y_2 - x_2) \leq 0.$$

This is a general statement about how demands change when prices change if income is adjusted so as to keep the consumer on the same indifference curve. In the particular case we are concerned with, we are only changing the first price. Therefore $q_2 = p_2$, and we are left with

$$(q_1 - p_1)(y_1 - x_1) \leq 0.$$

This equation says that the change in the quantity demanded must have the opposite sign from that of the price change, which is what we wanted to show.

The total change in demand is still equal to the substitution effect plus the income effect—but now it is the Hicks substitution effect. Since the Hicks substitution effect is also negative, the Slutsky equation takes exactly the same form as we had earlier and has exactly the same interpretation. Both the Slutsky and Hicks definitions of the substitution effect have their place, and which is more useful depends on the problem at hand. It can be shown that for small changes in price, the two substitution effects are virtually identical.

8.9 Compensated Demand Curves

We have seen how the quantity demanded changes as a price changes in three different contexts: holding income fixed (the standard case), holding purchasing power fixed (the Slutsky substitution effect), and holding utility fixed (the Hicks substitution effect). We can draw the relationship between price and quantity demanded holding any of these three variables fixed. This gives rise to three different demand curves: the standard demand curve, the Slutsky demand curve, and the Hicks demand curve.

The analysis of this chapter shows that the Slutsky and Hicks demand curves are always downward sloping curves. Furthermore the ordinary

demand curve is a downward sloping curve for normal goods. However, the Giffen analysis shows that it is theoretically possible that the ordinary demand curve may slope upwards for an inferior good.

The Hicksian demand curve—the one with utility held constant—is sometimes called the **compensated demand curve.** This terminology arises naturally if you think of constructing the Hicksian demand curve by adjusting income as the price changes so as to keep the consumer's utility constant. Hence the consumer is "compensated" for the price changes, and his utility is the same at every point on the Hicksian demand curve. This is in contrast to the situation with an ordinary demand curve. In this case the consumer is worse off facing higher prices than lower prices since his income is constant.

The compensated demand curve turns out to be very useful in advanced courses, especially in treatments of benefit-cost analysis. In this sort of analysis it is natural to ask what size payments are necessary to compensate consumers for some policy change. The magnitude of such payments gives a useful estimate of the cost of the policy change. However, actual calculation of compensated demand curves requires more mathematical machinery than we have developed in this text.

Summary

1. When the price of a good decreases, there will be two effects on consumption. The change in relative prices makes the consumer want to consume more of the cheaper good. The increase in purchasing power due to the lower price may increase or decrease consumption, depending on whether the good is a normal good or an inferior good.

2. The change in demand due to the change in relative prices is called the substitution effect; the change due to the change in purchasing power is called the income effect.

3. The substitution effect is how demand changes when prices change and purchasing power is held constant, in the sense that the original bundle remains affordable. To hold real purchasing power constant, money income will have to change. The necessary change in money income is given by $\Delta m = x_1 \Delta p_1$.

4. The Slutsky equation says that the total change in demand is the sum of the substitution effect and the income effect.

5. The Law of Demand says that normal goods must have downward-sloping demand curves.

REVIEW QUESTIONS

1. Suppose a consumer has preferences between two goods that are perfect substitutes. Can you change prices in such a way that the entire demand response is due to the income effect?

2. Suppose that preferences are concave. Is it still the case that the substitution effect is negative?

3. In the case of the gasoline tax, what would happen if the rebate to the consumers were based on their original consumption of gasoline, x, rather than on their final consumption of gasoline, x'?

4. In the case described in the preceding question, would the government be paying out more or less than it received in tax revenues?

5. In this case would the consumers be better off or worse off if the tax with rebate based on original consumption were in effect?

APPENDIX

Let us derive the Slutsky equation using calculus. Consider the Slutsky definition of the substitution effect, in which the income is adjusted so as to give the consumer just enough to buy the original consumption bundle, which we will now denote by $(\overline{x}_1, \overline{x}_2)$. If the prices are (p_1, p_2), then the consumer's actual choice with this adjustment will depend on (p_1, p_2) and $(\overline{x}_1, \overline{x}_2)$. Let's call this relationship the **Slutsky demand function** for good 1, and write it as $x_1^s(p_1, p_2, \overline{x}_1, \overline{x}_2)$.

Suppose the original demanded bundle is $(\overline{x}_1, \overline{x}_2)$ at prices $(\overline{p}_1, \overline{p}_2)$ and income $\overline{m}$. The Slutsky demand function tells us what the consumer would demand facing some different prices (p_1, p_2) and having income $p_1\overline{x}_1 + p_2\overline{x}_2$. Thus the Slutsky demand function at $(p_1, p_2, \overline{x}_1, \overline{x}_2)$ is the ordinary demand at (p_1, p_2) and income $p_1\overline{x}_1 + p_2\overline{x}_2$. That is,

$$x_1^s(p_1, p_2, \overline{x}_1, \overline{x}_2) \equiv x_1(p_1, p_2, p_1\overline{x}_1 + p_2\overline{x}_2).$$

This equation says that the Slutsky demand at prices (p_1, p_2) is that amount which the consumer would demand if he had enough income to purchase his original bundle of goods $(\overline{x}_1, \overline{x}_2)$. This is just the definition of the Slutsky demand function.

Differentiating this identity with respect to p_1, we have

$$\frac{\partial x_1^s(p_1, p_2, \overline{x}_1, \overline{x}_2)}{\partial p_1} = \frac{\partial x_1(p_1, p_2, \overline{m})}{\partial p_1} + \frac{\partial x_1(p_1, p_2, \overline{m})}{\partial m}\overline{x}_1.$$

Rearranging we have

$$\frac{\partial x_1(p_1, p_2, \overline{m})}{\partial p_1} = \frac{\partial x_1^s(p_1, p_2, \overline{x}_1, \overline{x}_2)}{\partial p_1} - \frac{\partial x_1(p_1, p_2, \overline{m})}{\partial m}\overline{x}_1.$$

Note the use of the chain rule in this calculation.

This is a derivative form of the Slutsky equation. It says that the total effect of a price change is composed of a substitution effect (where income is adjusted to keep the bundle $(\overline{x}_1, \overline{x}_2)$ feasible) and an income effect. We know from the text that the substitution effect is negative and that the sign of the income effect depends on whether the good in question is inferior or not. As you can see, this is just the form of the Slutsky equation considered in the text, except that we have replaced the Δ's with derivative signs.

What about the Hicks substitution effect? It is also possible to define a Slutsky equation for it. We let $x_1^h(p_1, p_2, \overline{u})$ be the Hicksian demand function, which measures how much the consumer demands of good 1 at prices (p_1, p_2) if income is adjusted to keep the level of *utility* constant at the original level $\overline{u}$. It turns out that in this case the Slutsky equation takes the form

$$\frac{\partial x_1(p_1, p_2, m)}{\partial p_1} = \frac{\partial x_1^h(p_1, p_2, \overline{u})}{\partial p_1} - \frac{\partial x_1(p_1, p_2, m)}{\partial m}\overline{x}_1.$$

The proof of this equation hinges on the fact that

$$\frac{\partial x_1^h(p_1, p_2, \overline{u})}{\partial p_1} = \frac{\partial x_1^s(p_1, p_2, \overline{x}_1, \overline{x}_2)}{\partial p_1}$$

for infinitesimal changes in price. That is, for derivative size changes in price, the Slutsky substitution and the Hicks substitution effect are the same. The proof of this is not terribly difficult, but it involves some concepts that are beyond the scope of this book. A relatively simple proof is given in Hal R. Varian, *Microeconomic Analysis,* 3rd ed. (New York: Norton, 1992).

EXAMPLE: Rebating a Small Tax

We can use the calculus version of the Slutsky equation to see how consumption choices would react to a small change in a tax when the tax revenues are rebated to the consumers.

Assume, as before, that the tax causes the price to rise by the full amount of the tax. Let x be the amount of gasoline, p its original price, and t the amount of the tax. Then the change in consumption will be given by

$$dx = \frac{\partial x}{\partial p}t + \frac{\partial x}{\partial m}tx.$$

The first term measures how demand responds to the price change times the amount of the price change—which gives us the price effect of the tax. The second terms tells us how demand responds to a change in income times the amount that income has changed—income has gone up by the amount of the tax revenues rebated to the consumer.

Now use Slutsky's equation to expand the first term on the right-hand side to get the substitution and income effects of the price change itself:

$$dx = \frac{\partial x^s}{\partial p}t - \frac{\partial x}{\partial m}tx + \frac{\partial x}{\partial m}tx = \frac{\partial x^s}{\partial p}t.$$

The income effect cancels out, and all that is left is the pure substitution effect. Imposing a small tax and rebating the revenues of the tax is just like imposing a price change and adjusting income so that the old consumption bundle is feasible—as long as the tax is small enough so that the derivative approximation is valid.

CHAPTER 9

BUYING AND SELLING

In the simple model of the consumer that we considered in the preceding chapters, the income of the consumer was given. In reality people earn their income by selling things that they own: items that they have produced, assets that they have accumulated, or, most commonly, their own labor. In this chapter we will examine how the earlier model must be modified so as to describe this kind of behavior.

9.1 Net and Gross Demands

As before, we will limit ourselves to the two-good model. We now suppose that the consumer starts off with an **endowment** of the two goods, which we will denote by (ω_1, ω_2).[1] This is how much of the two goods the consumer has *before* he enters the market. Think of a farmer who goes to market with ω_1 units of carrots and ω_2 units of potatoes. The farmer inspects the prices available at the market and decides how much he wants to buy and sell of the two goods.

[1] The Greek letter ω, omega, is pronounced "o–may–gah."

Let us make a distinction here between the consumer's **gross demands** and his **net demands**. The gross demand for a good is the amount of the good that the consumer actually ends up consuming: how much of each of the goods he or she takes home from the market. The net demand for a good is the *difference* between what the consumer ends up with (the gross demand) and the initial endowment of goods. The net demand for a good is simply the amount that is bought or sold of the good.

If we let (x_1, x_2) be the gross demands, then $(x_1 - \omega_1, x_2 - \omega_2)$ are the net demands. Note that while the gross demands are typically positive numbers, the net demands may be positive or negative. If the net demand for good 1 is negative, it means that the consumer wants to consume less of good 1 than she has; that is, she wants to *supply* good 1 to the market. A negative net demand is simply an amount supplied.

For purposes of economic analysis, the gross demands are the more important, since that is what the consumer is ultimately concerned with. But the net demands are what are actually exhibited in the market and thus are closer to what the layman means by demand or supply.

9.2 The Budget Constraint

The first thing we should do is to consider the form of the budget constraint. What constrains the consumer's final consumption? It must be that the value of the bundle of goods that she goes home with must be equal to the value of the bundle of goods that she came with. Or, algebraically:

$$p_1 x_1 + p_2 x_2 = p_1 \omega_1 + p_2 \omega_2.$$

We could just as well express this budget line in terms of net demands as

$$p_1(x_1 - \omega_1) + p_2(x_2 - \omega_2) = 0.$$

If $(x_1 - \omega_1)$ is positive we say that the consumer is a **net buyer** or **net demander** of good 1; if it is negative we say that she is a **net seller** or **net supplier**. Then the above equation says that the value of what the consumer buys must equal the value of what she sells, which seems sensible enough.

We could also express the budget line when the endowment is present in a form similar to the way we described it before. Now it takes two equations:

$$\begin{aligned} p_1 x_1 + p_2 x_2 &= m \\ m &= p_1 \omega_1 + p_2 \omega_2. \end{aligned}$$

Once the prices are fixed, the value of the endowment, and hence the consumer's money income, is fixed.

What does the budget line look like graphically? When we fix the prices, money income is fixed, and we have a budget equation just like we had before. Thus the slope must be given by $-p_1/p_2$, just as before, so the only problem is to determine the location of the line.

The location of the line can be determined by the following simple observation: the endowment bundle is always on the budget line. That is, one value of (x_1, x_2) that satisfies the budget line is $x_1 = \omega_1$ and $x_2 = \omega_2$. The endowment is always just affordable, since the amount you have to spend is precisely the value of the endowment.

Putting these facts together shows that the budget line has a slope of $-p_1/p_2$ and passes through the endowment point. This is depicted in Figure 9.1.

Figure 9.1 **The budget line**. The budget line passes through the endowment and has a slope of $-p_1/p_2$.

Given this budget constraint, the consumer can choose the optimal consumption bundle just as before. In Figure 9.1 we have shown an example of an optimal consumption bundle (x_1^*, x_2^*). Just as before, it will satisfy the optimality condition that the marginal rate of substitution is equal to the price ratio.

In this particular case, $x_1^* > \omega_1$ and $x_2^* < \omega_2$, so the consumer is a net buyer of good 1 and a net seller of good 2. The net demands are simply the net amounts that the consumer buys or sells of the two goods. In general the consumer may decide to be either a buyer or a seller depending on the relative prices of the two goods.

9.3 Changing the Endowment

In our previous analysis of choice we examined how the optimal consumption changed as the money income changed while the prices remained fixed. We can do a similar analysis here by asking how the optimal consumption changes as the *endowment* changes while the prices remain fixed.

For example, suppose that the endowment changes from (ω_1, ω_2) to some other value (ω_1', ω_2') such that

$$p_1\omega_1 + p_2\omega_2 > p_1\omega_1' + p_2\omega_2'.$$

This inequality means that the new endowment (ω_1', ω_2') is worth less than the old endowment—the money income that the consumer could achieve by selling her endowment is less.

This is depicted graphically in Figure 9.2A: the budget line shifts inward. Since this is exactly the same as a reduction in money income, we can conclude the same two things that we concluded in our examination of that case. First, the consumer is definitely worse off with the endowment (ω_1', ω_2') than she was with the old endowment, since her consumption possibilities have been reduced. Second, her demand for each good will change according to whether that good is a normal good or an inferior good.

For example, if good 1 is a normal good and the consumer's endowment changes in a way that reduces its value, we can conclude that the consumer's demand for good 1 will decrease.

The case where the value of the endowment increases is depicted in Figure 9.2B. Following the above argument we conclude that if the budget line shifts outward in a parallel way, the consumer must be made better off. Algebraically, if the endowment changes from (ω_1, ω_2) to (ω_1', ω_2') and $p_1\omega_1 + p_2\omega_2 < p_1\omega_1' + p_2\omega_2'$, then the consumer's new budget set must contain her old budget set. This in turn implies that the optimal choice of the consumer with the new budget set must be preferred to the optimal choice given the old endowment.

It is worthwhile pondering this point a moment. In Chapter 7 we argued that just because a consumption bundle had a higher cost than another didn't mean that it would be preferred to the other bundle. But that only holds for a bundle that must be *consumed*. If a consumer can sell a bundle of goods on a free market at constant prices, then she will always prefer a higher-valued bundle to a lower-valued bundle, simply because a

Figure 9.2 **Changes in the value of the endowment.** In case A the value of the endowment decreases, and in case B it increases.

higher-valued bundle gives her more income, and thus more consumption possibilities. Therefore, an *endowment* that has a higher value will always be preferred to an endowment with a lower value. This simple observation will turn out to have some important implications later on.

There's one more case to consider: what happens if $p_1\omega_1 + p_2\omega_2 = p_1\omega_1' + p_2\omega_2'$? Then the budget set doesn't change at all: the consumer is just as well-off with (ω_1, ω_2) as with (ω_1', ω_2'), and her optimal choice should be exactly the same. The endowment has just shifted along the original budget line.

9.4 Price Changes

Earlier, when we examined how demand changed when price changed, we conducted our investigation under the hypothesis that money income remained constant. Now, when money income is determined by the value of the endowment, such a hypothesis is unreasonable: if the value of a good you are selling changes, your money income will certainly change. Thus in the case where the consumer has an endowment, changing prices automatically implies changing income.

Let us first think about this geometrically. If the price of good 1 decreases, we know that the budget line becomes flatter. Since the endowment bundle is always affordable, this means that the budget line must pivot around the endowment, as depicted in Figure 9.3.

Decreasing the price of good 1. Lowering the price of good 1 makes the budget line pivot around the endowment. If the consumer remains a supplier she must be worse off.

Figure 9.3

In this case, the consumer is initially a seller of good 1 and remains a seller of good 1 even after the price has *declined*. What can we say about this consumer's welfare? In the case depicted, the consumer is on a lower indifference curve after the price change than before, but will this be true in general? The answer comes from applying the principle of revealed preference.

If the consumer remains a supplier, then her new consumption bundle must be on the colored part of the new budget line. But this part of the new budget line is inside the original budget set: all of these choices were open to the consumer before the price changed. Therefore, by revealed preference, all of these choices are worse than the original consumption bundle. We can therefore conclude that if the price of a good that a consumer is selling goes down, and the consumer decides to remain a seller, then the consumer's welfare must have declined.

What if the price of a good that the consumer is selling decreases and the consumer decides to switch to being a buyer of that good? In this case, the consumer may be better off or she may be worse off—there is no way to tell.

Let us now turn to the situation where the consumer is a net buyer of a good. In this case everything neatly turns around: if the consumer is a net

buyer of a good, its price *increases*, and the consumer optimally decides to remain a buyer, then she must definitely be worse off. But if the price increase leads her to become a seller, it could go either way—she may be better off, or she may be worse off. These observations follow from a simple application of revealed preference just like the cases described above, but it is good practice for you to draw a graph just to make sure you understand how this works.

Revealed preference also allows us to make some interesting points about the decision of whether to remain a buyer or to become a seller when prices change. Suppose, as in Figure 9.4, that the consumer is a net buyer of good 1, and consider what happens if the price of good 1 *decreases*. Then the budget line becomes flatter as in Figure 9.4.

Figure 9.4 **Decreasing the price of good 1**. If a person is a buyer and the price of what she is buying decreases, she remains a buyer.

As usual we don't know for certain whether the consumer will buy more or less of good 1—it depends on her tastes. However, we can say something for sure: *the consumer will continue to be a net buyer of good 1—she will not switch to being a seller.*

How do we know this? Well, consider what would happen if the consumer did switch. Then she would be consuming somewhere on the colored part of the new budget line in Figure 9.4. But those consumption bundles were feasible for her when she faced the original budget line, and she rejected

them in favor of (x_1^*, x_2^*). So (x_1^*, x_2^*) must be better than any of those points. And under the *new* budget line, (x_1^*, x_2^*) is a feasible consumption bundle. So whatever she consumes under the new budget line, it must be better than (x_1^*, x_2^*)—and thus better than any points on the colored part of the new budget line. This implies that her consumption of x_1 must be to the right of her endowment point—that is, she must remain a net demander of good 1.

Again, this kind of observation applies equally well to a person who is a net seller of a good: if the price of what she is selling goes *up*, she will not switch to being a net buyer. We can't tell for sure if the consumer will consume more or less of the good she is selling—but we know that she will keep selling it if the price goes up.

9.5 Offer Curves and Demand Curves

Recall from Chapter 6 that price offer curves depict those combinations of both goods that may be demanded by a consumer and that demand curves depict the relationship between the price and the quantity demanded of some good. Exactly the same constructions work when the consumer has an endowment of both goods.

Consider, for example, Figure 9.5, which illustrates the price offer curve and the demand curve for a consumer. The offer curve will always pass through the endowment, because at some price the endowment will be a demanded bundle; that is, at some prices the consumer will optimally choose not to trade.

As we've seen, the consumer may decide to be a buyer of good 1 for some prices and a seller of good 1 for other prices. Thus the offer curve will generally pass to the left and to the right of the endowment point.

The demand curve illustrated in Figure 9.5B is the gross demand curve—it measures the total amount the consumer chooses to consume of good 1. We have illustrated the net demand curve in Figure 9.6.

Note that the net demand for good 1 will typically be negative for some prices. This will be when the price of good 1 becomes so high that the consumer chooses to become a seller of good 1. At some price the consumer switches between being a net demander to being a net supplier of good 1.

It is conventional to plot the supply curve in the positive orthant, although it actually makes more sense to think of supply as just a negative demand. We'll bow to tradition here and plot the net supply curve in the normal way—as a positive amount, as in Figure 9.6.

Algebraically the net demand for good 1, $d_1(p_1, p_2)$, is the difference between the gross demand $x_1(p_1, p_2)$ and the endowment of good 1, when this difference is positive; that is, when the consumer wants more of the good than he or she has:

$$d_1(p_1, p_2) = \begin{cases} x_1(p_1, p_2) - \omega_1 & \text{if this is positive;} \\ 0 & \text{otherwise.} \end{cases}$$

Figure 9.5 **The offer curve and the demand curve**. These are two ways of depicting the relationship between the demanded bundle and the prices when an endowment is present.

The net supply curve is the difference between how much the consumer has of good 1 and how much he or she wants when *this* difference is positive:

$$s_1(p_1, p_2) = \begin{cases} \omega_1 - x_1(p_1, p_2) & \text{if this is positive;} \\ 0 & \text{otherwise.} \end{cases}$$

Everything that we've established about the properties of demand behavior applies directly to the supply behavior of a consumer—because supply is just negative demand. If the *gross* demand curve is always downward sloping, then the net demand curve will be downward sloping and the supply curve will be upward sloping. Think about it: if an increase in the price makes the net demand more negative, then the net supply will be more positive.

9.6 The Slutsky Equation Revisited

The above applications of revealed preference are handy, but they don't really answer the main question: how does the demand for a good react to a change in its price? We saw in Chapter 8 that if money income was held constant, and the good was a normal good, then a reduction in its price must lead to an increase in demand.

The catch is the phrase "money income was held constant." The case we are examining here necessarily involves a change in money income, since the value of the endowment will necessarily change when a price changes.

Figure 9.6

Gross demand, net demand, and net supply. Using the gross demand and net demand to depict the demand and supply behavior.

In Chapter 8 we described the Slutsky equation that decomposed the change in demand due to a price change into a substitution effect and an income effect. The income effect was due to the change in purchasing power when prices change. But now, purchasing power has two reasons to change when a price changes. The first is the one involved in the definition of the Slutsky equation: when a price falls, for example, you can buy just as much of a good as you were consuming before and have some extra money left over. Let us refer to this as the **ordinary income effect**. But the second effect is new. When the price of a good changes, it changes the value of your endowment and thus changes your money income. For example, if you are a net supplier of a good, then a fall in its price will reduce your money income directly since you won't be able to sell your endowment for as much money as you could before. We will have the same effects that we had before, plus an extra income effect from the influence of the prices on the value of the endowment bundle. We'll call this the **endowment income effect**.

In the earlier form of the Slutsky equation, the amount of money income you had was fixed. Now we have to worry about how your money income changes as the value of your endowment changes. Thus, when we calculate the effect of a change in price on demand, the Slutsky equation will take the form:

total change in demand = change due to substitution effect + change in demand due to ordinary income effect + change in demand due to endowment income effect.

The first two effects are familiar. As before, let us use Δx_1 to stand for the total change in demand, Δx_1^s to stand for the change in demand due to the substitution effect, and Δx_1^m to stand for the change in demand due to the ordinary income effect. Then we can substitute these terms into the above "verbal equation" to get the Slutsky equation in terms of rates of change:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - x_1 \frac{\Delta x_1^m}{\Delta m} + \text{endowment income effect.} \tag{9.1}$$

What will the last term look like? We'll derive an explicit expression below, but let us first think about what is involved. When the price of the endowment changes, money income will change, and this change in money income will induce a change in demand. Thus the endowment income effect will consist of two terms:

$$\text{endowment income effect} = \text{change in demand when income changes} \times \text{the change in income when price changes.} \tag{9.2}$$

Let's look at the second effect first. Since income is defined to be

$$m = p_1\omega_1 + p_2\omega_2,$$

we have

$$\frac{\Delta m}{\Delta p_1} = \omega_1.$$

This tells us how money income changes when the price of good 1 changes: if you have 10 units of good 1 to sell, and its price goes up by \$1, your money income will go up by \$10.

The first term in equation (9.2) is just how demand changes when income changes. We already have an expression for this: it is $\Delta x_1^m/\Delta m$: the change in demand divided by the change in income. Thus the endowment income effect is given by

$$\text{endowment income effect} = \frac{\Delta x_1^m}{\Delta m}\frac{\Delta m}{\Delta p_1} = \frac{\Delta x_1^m}{\Delta m}\omega_1. \tag{9.3}$$

Inserting equation (9.3) into equation (9.1) we get the final form of the Slutsky equation:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + (\omega_1 - x_1)\frac{\Delta x_1^m}{\Delta m}.$$

This equation can be used to answer the question posed above. We know that the sign of the substitution effect is always negative—opposite the direction of the change in price. Let us suppose that the good is a normal

good, so that $\Delta x_1^m/\Delta m > 0$. Then the sign of the combined income effect depends on whether the person is a net demander or a net supplier of the good in question. If the person is a net demander of a normal good, and its price increases, then the consumer will necessarily buy less of it. If the consumer is a net supplier of a normal good, then the sign of the total effect is ambiguous: it depends on the magnitude of the (positive) combined income effect as compared to the magnitude of the (negative) substitution effect.

As before, each of these changes can be depicted graphically, although the graph gets rather messy. Refer to Figure 9.7, which depicts the Slutsky decomposition of a price change. The total change in the demand for good 1 is indicated by the movement from A to C. This is the sum of three separate movements: the substitution effect, which is the movement from A to B, and two income effects. The ordinary income effect, which is the movement from B to D, is the change in demand *holding money income fixed*—that is, the same income effect that we examined in Chapter 8. But since the value of the endowment changes when prices change, there is now an extra income effect: because of the change in the value of the endowment, money income changes. This change in money income shifts the budget line back inward so that it passes through the endowment bundle. The change in demand from D to C measures this endowment income effect.

The Slutsky equation revisited. Breaking up the effect of the price change into the substitution effect (A to B), the ordinary income effect (B to D), and the endowment income effect (D to C).

Figure 9.7

9.7 Use of the Slutsky Equation

Suppose that we have a consumer who sells apples and oranges that he grows on a few trees in his backyard, like the consumer we described at the beginning of Chapter 8. We said there that if the price of apples increased, then this consumer might actually consume more apples. Using the Slutsky equation derived in this chapter, it is not hard to see why. If we let x_a stand for the consumer's demand for apples, and let p_a be the price of apples, then we know that

$$\frac{\Delta x_a}{\Delta p_a} = \underset{(-)}{\frac{\Delta x_a^s}{\Delta p_a}} + \underset{(+)}{(\omega_a - x_a)} \underset{(+)}{\frac{\Delta x_a^m}{\Delta m}}.$$

This says that the total change in the demand for apples when the price of apples changes is the substitution effect plus the income effect. The substitution effect works in the right direction—increasing the price decreases the demand for apples. But if apples are a normal good for this consumer, the income effect works in the wrong direction. Since the consumer is a net supplier of apples, the increase in the price of apples increases his money income so much that he wants to consume more apples due to the income effect. If the latter term is strong enough to outweigh the substitution effect, we can easily get the "perverse" result.

EXAMPLE: Calculating the Endowment Income Effect

Let's try a little numerical example. Suppose that a dairy farmer produces 40 quarts of milk a week. Initially the price of milk is \$3 a quart. His demand function for milk, for his own consumption, is

$$x_1 = 10 + \frac{m}{10p_1}.$$

Since he is producing 40 quarts at \$3 a quart, his income is \$120 a week. His initial demand for milk is therefore $x_1 = 14$. Now suppose that the price of milk changes to \$2 a quart. His money income will then change to $m' = 2 \times 40 = \$80$, and his demand will be $x_1' = 10 + 80/20 = 14$.

If his money income had remained fixed at $m = \$120$, he would have purchased $x_1 = 10 + 120/10 \times 2 = 16$ quarts of milk at this price. Thus the endowment income effect—the change in his demand due to the change in the value of his endowment—is -2. The substitution effect and the ordinary income effect for this problem were calculated in Chapter 8.

9.8 Labor Supply

Let us apply the idea of an endowment to analyzing a consumer's labor supply decision. The consumer can choose to work a lot and have relatively high consumption, or can choose to work a little and have a small consumption. The amount of consumption and labor will be determined by the interaction of the consumer's preferences and the budget constraint.

The Budget Constraint

Let us suppose that the consumer initially has some money income M that she receives whether she works or not. This might be income from investments or from relatives, for example. We call this amount the consumer's **nonlabor income**. (The consumer could have zero nonlabor income, but we want to allow for the possibility that it is positive.)

Let us use C to indicate the amount of consumption the consumer has, and use p to denote the price of consumption. Then letting w be the wage rate, and L the amount of labor supplied, we have the budget constraint:

$$pC = M + wL.$$

This says that the value of what the consumer consumes must be equal to her nonlabor income plus her labor income.

Let us try to compare the above formulation to the previous examples of budget constraints. The major difference is that we have something that the consumer is choosing—labor supply—on the right-hand side of the equation. We can easily transpose it to the left-hand side to get

$$pC - wL = M.$$

This is better, but we have a minus sign where we normally have a plus sign. How can we remedy this? Let us suppose that there is some maximum amount of labor supply possible—24 hours a day, 7 days a week, or whatever is compatible with the units of measurement we are using. Let $\overline{L}$ denote this amount of labor time. Then adding $w\overline{L}$ to each side and rearranging we have

$$pC + w(\overline{L} - L) = M + w\overline{L}.$$

Let us define $\overline{C} = M/p$, the amount of consumption that the consumer would have if she didn't work at all. That is, $\overline{C}$ is her endowment of consumption, so we write

$$pC + w(\overline{L} - L) = p\overline{C} + w\overline{L}.$$

Now we have an equation very much like those we've seen before. We have two choice variables on the left-hand side and two endowment variables on the right-hand side. The variable $\overline{L}-L$ can be interpreted as the amount of "leisure"—that is, time that isn't labor time. Let us use the variable R (for relaxation!) to denote leisure, so that $R = \overline{L} - L$. Then the total amount of time you have available for leisure is $\overline{R} = \overline{L}$ and the budget constraint becomes

$$pC + wR = p\overline{C} + w\overline{R}.$$

The above equation is formally identical to the very first budget constraint that we wrote in this chapter. However, it has a much more interesting interpretation. It says that the value of a consumer's consumption plus her leisure has to equal the value of her endowment of consumption and her endowment of time, where her endowment of time is valued at her wage rate. The wage rate is not only the price of labor, it is also the price of *leisure*.

After all, if your wage rate is \$10 an hour and you decide to consume an extra hour's leisure, how much does it cost you? The answer is that it costs you \$10 in forgone income—that's the price of that extra hour's consumption of leisure. Economists sometimes say that the wage rate is the **opportunity cost** of leisure.

The right-hand side of this budget constraint is sometimes called the consumer's **full income** or **implicit income**. It measures the value of what the consumer owns—her endowment of consumption goods, if any, and her endowment of her own time. This is to be distinguished from the consumer's **measured income**, which is simply the income she receives from selling off some of her time.

The nice thing about this budget constraint is that it is just like the ones we've seen before. It passes through the endowment point $(\overline{L}, \overline{C})$ and has a slope of $-w/p$. The endowment would be what the consumer would get if she did not engage in market trade at all, and the slope of the budget line tells us the rate at which the market will exchange one good for another.

The optimal choice occurs where the marginal rate of substitution—the tradeoff between consumption and leisure—equals w/p, the **real wage**, as depicted in Figure 9.8. The value of the extra consumption to the consumer from working a little more has to be just equal to the value of the lost leisure that it takes to generate that consumption. The real wage is the amount of consumption that the consumer can purchase if she gives up an hour of leisure.

9.9 Comparative Statics of Labor Supply

First let us consider how a consumer's labor supply changes as money income changes with the price and wage held fixed. If you won the state

Labor supply. The optimal choice describes the demand for leisure measured from the origin to the right, and the supply of labor measured from the endowment to the left.

Figure 9.8

lottery and got a big increase in nonlabor income, what would happen to your supply of labor? What would happen to your demand for leisure?

For most people, the supply of labor would drop when their money income increased. In other words, leisure is probably a normal good for most people: when their money income rises, people choose to consume more leisure. There seems to be a fair amount of evidence for this observation, so we will adopt it as a maintained hypothesis: we will assume that leisure is a normal good.

What does this imply about the response of the consumer's labor supply to changes in the wage rate? When the wage rate increases there are two effects: the return to working more increase and the cost of consuming leisure increases. By using the ideas of income and substitution effects and the Slutsky equation we can isolate these individual effects and analyze them.

When the wage rate increases, leisure becomes more expensive, which by itself leads people to want less of it (the substitution effect). Since leisure is a normal good, we would then predict that an increase in the wage rate would necessarily lead to a decrease in the demand for leisure—that is, an increase in the supply of labor. This follows from the Slutsky equation given in Chapter 8. A normal good must have a negatively sloped demand curve. If leisure is a normal good, then the supply curve of labor must be positively sloped.

But there is a problem with this analysis. First, at an intuitive level, it does not seem reasonable that increasing the wage would *always* result in an increased supply of labor. If my wage becomes very high, I might well "spend" the extra income in consuming leisure. How can we reconcile this apparently plausible behavior with the economic theory given above?

If the theory gives the wrong answer, it is probably because we've misapplied the theory. And indeed in this case we have. The Slutsky example described earlier gave the change in demand *holding money income constant.* But if the wage rate changes, then money income must change as well. The change in demand resulting from a change in money income is an extra income effect—the endowment income effect. It occurs on top of the ordinary income effect.

If we apply the *appropriate* version of the Slutsky equation given earlier in this chapter, we get the following expression:

$$\frac{\Delta R}{\Delta w} = \underset{(-)}{\text{substitution effect}} + \underset{(+)}{(\overline{R} - R)}\ \underset{(+)}{\frac{\Delta R}{\Delta m}}. \tag{9.4}$$

In this expression the substitution effect is definitely negative, as it always is, and $\Delta R/\Delta m$ is positive since we are assuming that leisure is a normal good. But $(\overline{R} - R)$ is positive as well, so the sign of the whole expression is ambiguous. Unlike the usual case of consumer demand, the demand for leisure will have an ambiguous sign, even if leisure is a normal good. As the wage rate increases, people may work more or less.

Why does this ambiguity arise? When the wage rate increases, the substitution effect says work more in order to substitute consumption for leisure. But when the wage rate increases, the value of the endowment goes up as well. This is just like extra income, which may very well be consumed in taking extra leisure. Which is the larger effect is an empirical matter and cannot be decided by theory alone. We have to look at people's actual labor supply decisions to determine which effect dominates.

The case where an increase in the wage rate results in a decrease in the supply of labor is represented by a **backward-bending labor supply curve**. The Slutsky equation tells us that this effect is more likely to occur the larger is $(\overline{R} - R)$, that is, the larger is the supply of labor. When $\overline{R} = R$, the consumer is consuming only leisure, so an increase in the wage will result in a pure substitution effect and thus an increase in the supply of labor. But as the labor supply increases, each increase in the wage gives the consumer additional income for all the hours he is working, so that after some point he may well decide to use this extra income to "purchase" additional leisure—that is, to *reduce* his supply of labor.

A backward-bending labor supply curve is depicted in Figure 9.9. When the wage rate is small, the substitution effect is larger than the income effect, and an increase in the wage will decrease the demand for leisure and hence increase the supply of labor. But for larger wage rates the income

effect may outweigh the substitution effect, and an increase in the wage will *reduce* the supply of labor.

Backward-bending labor supply. As the wage rate increases, the supply of labor increases from L_1 to L_2. But a further increase in the wage rate reduces the supply of labor back to L_1.

Figure 9.9

EXAMPLE: Overtime and the Supply of Labor

Consider a worker who has chosen to supply a certain amount of labor $L^* = \overline{R} - R^*$ when faced with the wage rate w as depicted in Figure 9.10. Now suppose that the firm offers him a higher wage, $w' > w$, for extra time that he chooses to work. Such a payment is known as an **overtime wage**.

In terms of Figure 9.10, this means that the slope of the budget line will be steeper for labor supplied in excess of L^*. But then we know that the worker will optimally choose to supply more labor, by the usual sort of revealed preference argument: the choices involving working less than L^* were available before the overtime was offered and were rejected.

Note that we get an unambiguous increase in labor supply with an overtime wage, whereas just offering a higher wage for all hours worked has an ambiguous effect—as discussed above, labor supply may increase or it may decrease. The reason is that the response to an overtime wage is essentially a pure substitution effect—the change in the optimal choice resulting from

Figure 9.10

Overtime versus an ordinary wage increase. An increase in the overtime wage definitely increases the supply of labor, while an increase in the straight wage could decrease the supply of labor.

pivoting the budget line around the chosen point. Overtime gives a higher payment for the *extra* hours worked, whereas a straight increase in the wage gives a higher payment for *all* hours worked. Thus a straight-wage increase involves both a substitution and an income effect while an overtime-wage increase results in a pure substitution effect. An example of this is shown in Figure 9.10. There an increase in the straight wage results in a *decrease* in labor supply, while an increase in the overtime wage results in an increase in labor supply.

Summary

1. Consumers earn income by selling their endowment of goods.

2. The gross demand for a good is the amount that the consumer ends up consuming. The net demand for a good is the amount the consumer buys. Thus the net demand is the difference between the gross demand and the endowment.

3. The budget constraint has a slope of $-p_1/p_2$ and passes through the endowment bundle.

4. When a price changes, the value of what the consumer has to sell will change and thereby generate an additional income effect in the Slutsky equation.

5. Labor supply is an interesting example of the interaction of income and substitution effects. Due to the interaction of these two effects, the response of labor supply to a change in the wage rate is ambiguous.

REVIEW QUESTIONS

1. If a consumer's net demands are $(5, -3)$ and her endowment is $(4, 4)$, what are her gross demands?

2. The prices are $(p_1, p_2) = (2, 3)$, and the consumer is currently consuming $(x_1, x_2) = (4, 4)$. There is a perfect market for the two goods in which they can be bought and sold costlessly. Will the consumer necessarily prefer consuming the bundle $(y_1, y_2) = (3, 5)$? Will she necessarily prefer having the bundle (y_1, y_2)?

3. The prices are $(p_1, p_2) = (2, 3)$, and the consumer is currently consuming $(x_1, x_2) = (4, 4)$. Now the prices change to $(q_1, q_2) = (2, 4)$. Could the consumer be better off under these new prices?

4. The U.S. currently imports about half of the petroleum that it uses. The rest of its needs are met by domestic production. Could the price of oil rise so much that the U.S. would be made better off?

5. Suppose that by some miracle the number of hours in the day increased from 24 to 30 hours (with luck this would happen shortly before exam week). How would this affect the budget constraint?

6. If leisure is an inferior good, what can you say about the slope of the labor supply curve?

APPENDIX

The derivation of the Slutsky equation in the text contained one bit of hand waving. When we considered how changing the monetary value of the endowment affects demand, we said that it was equal to $\Delta x_1^m/\Delta m$. In our old version of the Slutsky equation this was the rate of change in demand when income changed so as to keep the original consumption bundle affordable. But that will not

necessarily be equal to the rate of change of demand when the value of the endowment changes. Let's examine this point in a little more detail.

Let the price of good 1 change from p_1 to p_1', and use m'' to denote the new money income at the price p_1' due to the change in the value of the endowment. Suppose that the price of good 2 remains fixed so we can omit it as an argument of the demand function.

By definition of m'', we know that

$$m'' - m = \Delta p_1 \omega_1.$$

Note that it is identically true that

$$\begin{aligned}
&\frac{x_1(p_1', m'') - x_1(p_1, m)}{\Delta p_1} = \\
&+ \frac{x_1(p_1', m') - x_1(p_1, m)}{\Delta p_1} \quad \text{(substitution effect)} \\
&- \frac{x_1(p_1', m') - x_1(p_1', m)}{\Delta p_1} \quad \text{(ordinary income effect)} \\
&+ \frac{x_1(p_1', m'') - x_1(p_1', m)}{\Delta p_1} \quad \text{(endowment income effect).}
\end{aligned}$$

(Just cancel out identical terms with opposite signs on the right-hand side.)

By definition of the ordinary income effect,

$$\Delta p_1 = \frac{m' - m}{x_1}$$

and by definition of the endowment income effect,

$$\Delta p_1 = \frac{m'' - m}{\omega_1}.$$

Making these replacements gives us a Slutsky equation of the form

$$\begin{aligned}
&\frac{x_1(p_1', m'') - x_1(p_1, m)}{\Delta p_1} = \\
&+ \frac{x_1(p_1', m') - x_1(p_1, m)}{\Delta p_1} \quad \text{(substitution effect)} \\
&- \frac{x_1(p_1', m') - x_1(p_1', m)}{m' - m} x_1 \quad \text{(ordinary income effect)} \\
&+ \frac{x_1(p_1', m'') - x_1(p_1', m)}{m'' - m} \omega_1 \quad \text{(endowment income effect).}
\end{aligned}$$

Writing this in terms of Δs, we have

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1^m}{\Delta m} x_1 + \frac{\Delta x_1^w}{\Delta m} \omega_1.$$

The only new term here is the last one. It tells how the demand for good 1 changes as income changes, times the *endowment* of good 1. This is precisely the endowment income effect.

Suppose that we are considering a very small price change, and thus a small associated income change. Then the fractions in the two income effects will be virtually the same, since the *rate* of change of good 1 when income changes from m to m' should be about the same as when income changes from m to m''. For such small changes we can collect terms and write the last two terms—the income effects—as

$$\frac{\Delta x_1^m}{\Delta m}(\omega_1 - x_1),$$

which yields a Slutsky equation of the same form as that derived earlier:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + (\omega_1 - x_1)\frac{\Delta x_1^m}{\Delta m}.$$

If we want to express the Slutsky equation in calculus terms, we can just take limits in this expression. Or, if you prefer, we can calculate the correct equation directly, just by taking partial derivatives. Let $x_1(p_1, m(p_1))$ be the demand function for good 1 where we hold price 2 fixed and recognize that money income depends on the price of good 1 via the relationship $m(p_1) = p_1\omega_1 + p_2\omega_2$. Then we can write

$$\frac{dx_1(p_1, m(p_1))}{dp_1} = \frac{\partial x_1(p_1, m)}{\partial p_1} + \frac{\partial x_1(p_1, m)}{\partial m}\frac{dm(p_1)}{dp_1}. \tag{9.5}$$

By the definition of $m(p_1)$ we know how income changes when price changes:

$$\frac{\partial m(p_1)}{\partial p_1} = \omega_1, \tag{9.6}$$

and by the Slutsky equation we know how demand changes when price changes, holding money income fixed:

$$\frac{\partial x_1(p_1, m)}{\partial p_1} = \frac{\partial x_1^s(p_1)}{\partial p_1} - \frac{\partial x(p_1, m)}{\partial m}x_1. \tag{9.7}$$

Inserting equations (9.6) and (9.7) into equation (9.5) we have

$$\frac{dx_1(p_1, m(p_1))}{dp_1} = \frac{\partial x_1^s(p_1)}{\partial p_1} + \frac{\partial x(p_1, m)}{\partial m}(\omega_1 - x_1),$$

which is the form of the Slutsky equation that we want.

CHAPTER 10

INTERTEMPORAL CHOICE

In this chapter we continue our examination of consumer behavior by considering the choices involved in saving and consuming over time. Choices of consumption over time are known as **intertemporal choices**.

10.1 The Budget Constraint

Let us imagine a consumer who chooses how much of some good to consume in each of two time periods. We will usually want to think of this good as being a composite good, as described in Chapter 2, but you can think of it as being a specific commodity if you wish. We denote the amount of consumption in each period by (c_1, c_2) and suppose that the prices of consumption in each period are constant at 1. The amount of money the consumer will have in each period is denoted by (m_1, m_2).

Suppose initially that the only way the consumer has of transferring money from period 1 to period 2 is by saving it without earning interest. Furthermore let us assume for the moment that he has no possibility of

Budget constraint. This is the budget constraint when the rate of interest is zero and no borrowing is allowed. The less the individual consumes in period 1, the more he can consume in period 2.

Figure 10.1

borrowing money, so that the most he can spend in period 1 is m_1. His budget constraint will then look like the one depicted in Figure 10.1.

We see that there will be two possible kinds of choices. The consumer could choose to consume at (m_1, m_2), which means that he just consumes his income each period, or he can choose to consume less than his income during the first period. In this latter case, the consumer is saving some of his first-period consumption for a later date.

Now, let us allow the consumer to borrow and lend money at some interest rate r. Keeping the prices of consumption in each period at 1 for convenience, let us derive the budget constraint. Suppose first that the consumer decides to be a saver so his first period consumption, c_1, is less than his first-period income, m_1. In this case he will earn interest on the amount he saves, $m_1 - c_1$, at the interest rate r. The amount that he can consume next period is given by

$$\begin{aligned} c_2 &= m_2 + (m_1 - c_1) + r(m_1 - c_1) \\ &= m_2 + (1 + r)(m_1 - c_1). \end{aligned} \tag{10.1}$$

This says that the amount that the consumer can consume in period 2 is his income plus the amount he saved from period 1, plus the interest that he earned on his savings.

Now suppose that the consumer is a borrower so that his first-period consumption is greater than his first-period income. The consumer is a

borrower if $c_1 > m_1$, and the interest he has to *pay* in the second period will be $r(c_1 - m_1)$. Of course, he also has to pay back the amount that he borrowed, $c_1 - m_1$. This means his budget constraint is given by

$$\begin{aligned} c_2 &= m_2 - r(c_1 - m_1) - (c_1 - m_1) \\ &= m_2 + (1 + r)(m_1 - c_1), \end{aligned}$$

which is just what we had before. If $m_1 - c_1$ is positive, then the consumer earns interest on this savings; if $m_1 - c_1$ is negative, then the consumer pays interest on his borrowings.

If $c_1 = m_1$, then necessarily $c_2 = m_2$, and the consumer is neither a borrower nor a lender. We might say that this consumption position is the "Polonius point."[1]

We can rearrange the budget constraint for the consumer to get two alternative forms that are useful:

$$(1 + r)c_1 + c_2 = (1 + r)m_1 + m_2 \tag{10.2}$$

and

$$c_1 + \frac{c_2}{1 + r} = m_1 + \frac{m_2}{1 + r}. \tag{10.3}$$

Note that both equations have the form

$$p_1 x_1 + p_2 x_2 = p_1 m_1 + p_2 m_2.$$

In equation (10.2), $p_1 = 1 + r$ and $p_2 = 1$. In equation (10.3), $p_1 = 1$ and $p_2 = 1/(1 + r)$.

We say that equation (10.2) expresses the budget constraint in terms of **future value** and that equation (10.3) expresses the budget constraint in terms of **present value**. The reason for this terminology is that the first budget constraint makes the price of future consumption equal to 1, while the second budget constraint makes the price of present consumption equal to 1. The first budget constraint measures the period-1 price *relative* to the period-2 price, while the second equation does the reverse.

The geometric interpretation of present value and future value is given in Figure 10.2. The present value of an endowment of money in two periods is the amount of money in period 1 that would generate the same budget set as the endowment. This is just the horizontal intercept of the budget line, which gives the maximum amount of first-period consumption possible.

[1] "Neither a borrower, nor a lender be; For loan oft loses both itself and friend, And borrowing dulls the edge of husbandry." *Hamlet*, Act I, scene *iii*; Polonius giving advice to his son.

Present and future values. The vertical intercept of the budget line measures future value, and the horizontal intercept measures the present value.

Figure 10.2

Examining the budget constraint, this amount is $\bar{c}_1 = m_1 + m_2/(1 + r)$, which is the present value of the endowment.

Similarly, the vertical intercept is the maximum amount of second-period consumption, which occurs when $c_1 = 0$. Again, from the budget constraint, we can solve for this amount $\bar{c}_2 = (1+r)m_1 + m_2$, the future value of the endowment.

The present-value form is the more important way to express the intertemporal budget constraint since it measures the future relative to the present, which is the way we naturally look at it.

It is easy from any of these equations to see the form of this budget constraint. The budget line passes through (m_1, m_2), since that is always an *affordable* consumption pattern, and the budget line has a slope of $-(1 + r)$.

10.2 Preferences for Consumption

Let us now consider the consumer's preferences, as represented by his indifference curves. The shape of the indifference curves indicates the consumer's tastes for consumption at different times. If we drew indifference curves with a constant slope of -1, for example, they would represent tastes of a consumer who didn't care whether he consumed today or tomorrow. His marginal rate of substitution between today and tomorrow is -1.

If we drew indifference curves for perfect complements, this would indicate that the consumer wanted to consume equal amounts today and tomorrow. Such a consumer would be unwilling to substitute consumption from one time period to the other, no matter what it might be worth to him to do so.

As usual, the intermediate case of well-behaved preferences is the more reasonable situation. The consumer is willing to substitute some amount of consumption today for consumption tomorrow, and how much he is willing to substitute depends on the particular pattern of consumption that he has.

Convexity of preferences is very natural in this context, since it says that the consumer would rather have an "average" amount of consumption each period rather than have a lot today and nothing tomorrow or vice versa.

10.3 Comparative Statics

Given a consumer's budget constraint and his preferences for consumption in each of the two periods, we can examine the optimal choice of consumption (c_1, c_2). If the consumer chooses a point where $c_1 < m_1$, we will say that she is a **lender**, and if $c_1 > m_1$, we say that she is a **borrower**. In Figure 10.3A we have depicted a case where the consumer is a borrower, and in Figure 10.3B we have depicted a lender.

Figure 10.3 **Borrower and lender**. Panel A depicts a borrower, since $c_1 > m_1$, and panel B depicts a lender, since $c_1 < m_1$.

Let us now consider how the consumer would react to a change in the

interest rate. From equation (10.1) we see that increasing the rate of interest must tilt the budget line to a steeper position: for a given reduction in c_1 you will get more consumption in the second period if the interest rate is higher. Of course the endowment always remains affordable, so the tilt is really a pivot around the endowment.

We can also say something about how the choice of being a borrower or a lender changes as the interest rate changes. There are two cases, depending on whether the consumer is initially a borrower or initially a lender. Suppose first that he is a lender. Then it turns out that if the interest rate increases, the consumer must remain a lender.

This argument is illustrated in Figure 10.4. If the consumer is initially a lender, then his consumption bundle is to the left of the endowment point. Now let the interest rate increase. Is it possible that the consumer shifts to a new consumption point to the *right* of the endowment?

No, because that would violate the principle of revealed preference: choices to the right of the endowment point were available to the consumer when he faced the original budget set and were rejected in favor of the chosen point. Since the original optimal bundle is still available at the new budget line, the new optimal bundle must be a point *outside* the old budget set—which means it must be to the left of the endowment. The consumer must remain a lender when the interest rate increases.

There is a similar effect for borrowers: if the consumer is initially a borrower, and the interest rate declines, he or she will remain a borrower. (You might sketch a diagram similar to Figure 10.4 and see if you can spell out the argument.)

Thus if a person is a lender and the interest rate increases, he will remain a lender. If a person is a borrower and the interest rate decreases, he will remain a borrower. On the other hand, if a person is a lender and the interest rate decreases, he may well decide to switch to being a borrower; similarly, an increase in the interest rate may induce a borrower to become a lender. Revealed preference tells us nothing about these last two cases.

Revealed preference can also be used to make judgments about how the consumer's welfare changes as the interest rate changes. If the consumer is initially a borrower, and the interest rate rises, but he decides to remain a borrower, then he must be worse off at the new interest rate. This argument is illustrated in Figure 10.5; if the consumer remains a borrower, he must be operating at a point that was affordable under the old budget set but was rejected, which implies that he must be worse off.

10.4 The Slutsky Equation and Intertemporal Choice

The Slutsky equation can be used to decompose the change in demand due to an interest rate change into income effects and substitution effects, just

Figure 10.4

If a person is a lender and the interest rate rises, he or she will remain a lender. Increasing the interest rate pivots the budget line around the endowment to a steeper position; revealed preference implies that the new consumption bundle must lie to the left of the endowment.

as in Chapter 9. Suppose that the interest rate rises. What will be the effect on consumption in each period?

This is a case that is easier to analyze by using the future-value budget constraint, rather than the present-value constraint. In terms of the future-value budget constraint, raising the interest rate is just like raising the price of consumption today as compared to consumption tomorrow. Writing out the Slutsky equation we have

$$\underset{(?)}{\frac{\Delta c_1^t}{\Delta p_1}} = \underset{(-)}{\frac{\Delta c_1^s}{\Delta p_1}} + \underset{(?)}{(m_1 - c_1)} \underset{(+)}{\frac{\Delta c_1^m}{\Delta m}}.$$

The substitution effect, as always, works opposite the direction of price. In this case the price of period-1 consumption goes up, so the substitution effect says the consumer should consume less first period. This is the meaning of the minus sign under the substitution effect. Let's assume that consumption this period is a normal good, so that the very last term—how consumption changes as income changes—will be positive. So we put a plus sign under the last term. Now the sign of the whole expression will depend on the sign of $(m_1 - c_1)$. If the person is a borrower, this term will be negative and the whole expression will therefore unambiguously be

A borrower is made worse off by an increase in the interest rate. When the interest rate facing a borrower increases and the consumer chooses to remain a borrower, he or she is certainly worse off.

Figure 10.5

negative—for a borrower, an increase in the interest rate must lower today's consumption.

Why does this happen? When the interest rate rises, there is always a substitution effect towards consuming less today. For a borrower, an increase in the interest rate means that he will have to pay more interest tomorrow. This effect induces him to borrow less, and thus consume less, in the first period.

For a lender the effect is ambiguous. The total effect is the sum of a negative substitution effect and a positive income effect. From the viewpoint of a lender an increase in the interest rate may give him so much extra income that he will want to consume even more first period.

The effects of changing interest rates are not terribly mysterious. There is an income effect and a substitution effect as in any other price change. But without a tool like the Slutsky equation to separate out the various effects, the changes may be hard to disentangle. With such a tool, the sorting out of the effects is quite straightforward.

10.5 Inflation

The above analysis has all been conducted in terms of a general "consump-

tion" good. Giving up Δc units of consumption today buys you $(1+r)\Delta c$ units of consumption tomorrow. Implicit in this analysis is the assumption that the "price" of consumption doesn't change—there is no inflation or deflation.

However, the analysis is not hard to modify to deal with the case of inflation. Let us suppose that the consumption good now has a different price in each period. It is convenient to choose today's price of consumption as 1 and to let p_2 be the price of consumption tomorrow. It is also convenient to think of the endowment as being measured in units of the consumption goods as well, so that the monetary value of the endowment in period 2 is $p_2 m_2$. Then the amount of money the consumer can spend in the second period is given by

$$p_2 c_2 = p_2 m_2 + (1+r)(m_1 - c_1),$$

and the amount of consumption available second period is

$$c_2 = m_2 + \frac{1+r}{p_2}(m_1 - c_1).$$

Note that this equation is very similar to the equation given earlier—we just use $(1+r)/p_2$ rather than $1+r$.

Let us express this budget constraint in terms of the rate of inflation. The inflation rate, π, is just the rate at which prices grow. Recalling that $p_1 = 1$, we have

$$p_2 = 1 + \pi,$$

which gives us

$$c_2 = m_2 + \frac{1+r}{1+\pi}(m_1 - c_1).$$

Let's create a new variable ρ, the **real interest rate**, and define it by[2]

$$1 + \rho = \frac{1+r}{1+\pi}$$

so that the budget constraint becomes

$$c_2 = m_2 + (1+\rho)(m_1 - c_1).$$

One plus the real interest rate measures how much extra *consumption* you can get in period 2 if you give up some *consumption* in period 1. That is why it is called the *real* rate of interest: it tells you how much extra consumption you can get, not how many extra dollars you can get.

[2] The Greek letter ρ, rho, is pronounced "row."

The interest rate on dollars is called the **nominal** rate of interest. As we've seen above, the relationship between the two is given by

$$1 + \rho = \frac{1 + r}{1 + \pi}.$$

In order to get an explicit expression for ρ, we write this equation as

$$\begin{aligned}\rho &= \frac{1 + r}{1 + \pi} - 1 = \frac{1 + r}{1 + \pi} - \frac{1 + \pi}{1 + \pi} \\ &= \frac{r - \pi}{1 + \pi}.\end{aligned}$$

This is an exact expression for the real interest rate, but it is common to use an approximation. If the inflation rate isn't too large, the denominator of the fraction will be only slightly larger than 1. Thus the real rate of interest will be approximately given by

$$\rho \approx r - \pi,$$

which says that the real rate of interest is just the nominal rate minus the rate of inflation. (The symbol $\approx$ means "approximately equal to.") This makes perfectly good sense: if the interest rate is 18 percent, but prices are rising at 10 percent, then the real interest rate—the extra consumption you can buy next period if you give up some consumption now—will be roughly 8 percent.

Of course, we are always looking into the future when making consumption plans. Typically, we know the nominal rate of interest for the next period, but the rate of inflation for next period is unknown. The real interest rate is usually taken to be the current interest rate minus the *expected* rate of inflation. To the extent that people have different estimates about what the next year's rate of inflation will be, they will have different estimates of the real interest rate. If inflation can be reasonably well forecast, these differences may not be too large.

10.6 Present Value: A Closer Look

Let us return now to the two forms of the budget constraint described earlier in section 10.1 in equations (10.2) and (10.3):

$$(1 + r)c_1 + c_2 = (1 + r)m_1 + m_2$$

and

$$c_1 + \frac{c_2}{1 + r} = m_1 + \frac{m_2}{1 + r}.$$

Consider just the right-hand sides of these two equations. We said that the first one expresses the value of the endowment in terms of future value and that the second one expresses it in terms of present value.

Let us examine the concept of future value first. If we can borrow and lend at an interest rate of r, what is the future equivalent of \$1 today? The answer is $(1 + r)$ dollars. That is, \$1 today can be turned into $(1 + r)$ dollars next period simply by lending it to the bank at an interest rate r. In other words, $(1 + r)$ dollars next period is equivalent to \$1 today since that is how much you would have to pay next period to purchase—that is, borrow—\$1 today. The value $(1 + r)$ is just the price of \$1 today, relative to \$1 next period. This can be easily seen from the first budget constraint: it is expressed in terms of future dollars—the second-period dollars have a price of 1, and first-period dollars are measured relative to them.

What about present value? This is just the reverse: everything is measured in terms of today's dollars. How much is a dollar next period worth in terms of a dollar today? The answer is $1/(1+r)$ dollars. This is because $1/(1 + r)$ dollars can be turned into a dollar next period simply by saving it at the rate of interest r. The *present value* of a dollar to be delivered next period is $1/(1 + r)$.

The concept of present value gives us another way to express the budget for a two-period consumption problem: a consumption plan is affordable if *the present value of consumption equals the present value of income.*

The idea of present value has an important implication that is closely related to a point made in Chapter 9: if the consumer can freely buy and sell goods at constant prices, then the consumer would always prefer a higher-valued endowment to a lower-valued one. In the case of intertemporal decisions, this principle implies that *if a consumer can freely borrow and lend at a constant interest rate, then the consumer would always prefer a pattern of income with a higher present value to a pattern with a lower present value.*

This is true for the same reason that the statement in Chapter 9 was true: an endowment with a higher value gives rise to a budget line that is farther out. The new budget set contains the old budget set, which means that the consumer would have all the consumption opportunities she had with the old budget set plus some more. Economists sometimes say that an endowment with a higher present value **dominates** one with a lower present value in the sense that the consumer can have larger consumption in *every* period by selling the endowment with the higher present value that she could get by selling the endowment with the lower present value.

Of course, if the present value of one endowment is higher than another, then the future value will be higher as well. However, it turns out that the present value is a more convenient way to measure the purchasing power of an endowment of money over time, and it is the measure to which we will devote the most attention.

10.7 Analyzing Present Value for Several Periods

Let us consider a three-period model. We suppose that we can borrow or lend money at an interest rate r each period and that this interest rate will remain constant over the three periods. Thus the price of consumption in period 2 in terms of period-1 consumption will be $1/(1+r)$, just as before.

What will the price of period-3 consumption be? Well, if I invest \$1 today, it will grow into $(1+r)$ dollars next period; and if I leave this money invested, it will grow into $(1+r)^2$ dollars by the third period. Thus if I start with $1/(1+r)^2$ dollars today, I can turn this into \$1 in period 3. The price of period-3 consumption relative to period-1 consumption is therefore $1/(1+r)^2$. Each extra dollar's worth of consumption in period 3 costs me $1/(1+r)^2$ dollars today. This implies that the budget constraint will have the form

$$c_1 + \frac{c_2}{1+r} + \frac{c_3}{(1+r)^2} = m_1 + \frac{m_2}{1+r} + \frac{m_3}{(1+r)^2}.$$

This is just like the budget constraints we've seen before, where the price of period-t consumption in terms of today's consumption is given by

$$p_t = \frac{1}{(1+r)^{t-1}}.$$

As before, moving to an endowment that has a higher present value at these prices will be preferred by any consumer, since such a change will necessarily shift the budget set farther out.

We have derived this budget constraint under the assumption of constant interest rates, but it is easy to generalize to the case of changing interest rates. Suppose, for example, that the interest earned on savings from period 1 to 2 is r_1, while savings from period 2 to 3 earn r_2. Then \$1 in period 1 will grow to $(1+r_1)(1+r_2)$ dollars in period 3. The present value of \$1 in period 3 is therefore $1/(1+r_1)(1+r_2)$. This implies that the correct form of the budget constraint is

$$c_1 + \frac{c_2}{1+r_1} + \frac{c_3}{(1+r_1)(1+r_2)} = m_1 + \frac{m_2}{1+r_1} + \frac{m_3}{(1+r_1)(1+r_2)}.$$

This expression is not so hard to deal with, but we will typically be content to examine the case of constant interest rates.

Table 10.1 contains some examples of the present value of \$1 T years in the future at different interest rates. The notable fact about this table is how quickly the present value goes down for "reasonable" interest rates. For example, at an interest rate of 10 percent, the value of \$1 20 years from now is only 15 cents.

Table 10.1 **The present value of \$1 t years in the future.**

Rate	1	2	5	10	15	20	25	30
.05	.95	.91	.78	.61	.48	.37	.30	.23
.10	.91	.83	.62	.39	.24	.15	.09	.06
.15	.87	.76	.50	.25	.12	.06	.03	.02
.20	.83	.69	.40	.16	.06	.03	.01	.00

10.8 Use of Present Value

Let us start by stating an important general principle: *present value is the only correct way to convert a stream of payments into today's dollars.* This principle follows directly from the definition of present value: the present value measures the value of a consumer's endowment of money. As long as the consumer can borrow and lend freely at a constant interest rate, an endowment with higher present value can always generate *more* consumption in every period than an endowment with lower present value. Regardless of your own tastes for consumption in different periods, you should always prefer a stream of money that has a higher present value to one with lower present value—since that always gives you more consumption possibilities in every period.

This argument is illustrated in Figure 10.6. In this figure, (m_1', m_2') is a worse consumption bundle than the consumer's original endowment, (m_1, m_2), since it lies beneath the indifference curve through her endowment. Nevertheless, the consumer would prefer (m_1', m_2') to (m_1, m_2) if she is able to borrow and lend at the interest rate r. This is true because with the endowment (m_1', m_2') she can afford to consume a bundle such as (c_1, c_2), which is unambiguously better than her current consumption bundle.

One very useful application of present value is in valuing the income streams offered by different kinds of investments. If you want to compare two different investments that yield different streams of payments to see which is better, you simply compute the two present values and choose the larger one. The investment with the larger present value always gives you more consumption possibilities.

Sometimes it is necessary to purchase an income stream by making a stream of payments over time. For example, one could purchase an apartment building by borrowing money from a bank and making mortgage payments over a number of years. Suppose that the income stream (M_1, M_2) can be purchased by making a stream of payments (P_1, P_2).

In this case we can evaluate the investment by comparing the present

Higher present value. An endowment with higher present value gives the consumer more consumption possibilities in each period if she can borrow and lend at the market interest rates.

Figure 10.6

value of the income stream to the present value of the payment stream. If

$$M_1 + \frac{M_2}{1+r} > P_1 + \frac{P_2}{1+r}, \tag{10.4}$$

the present value of the income stream exceeds the present value of its cost, so this is a good investment—it will increase the present value of our endowment.

An equivalent way to value the investment is to use the idea of **net present value**. In order to calculate this number we calculate at the *net* cash flow in each period and then discount this stream back to the present. In this example, the net cash flow is $(M_1 - P_1, M_2 - P_2)$, and the net present value is

$$NPV = M_1 - P_1 + \frac{M_2 - P_2}{1+r}.$$

Comparing this to equation (10.4) we see that the investment should be purchased if and only if the net present value is positive.

The net present value calculation is very convenient since it allows us to add all of the positive and negative cash flows together in each period and then discount the resulting stream of cash flows.

EXAMPLE: Valuing a Stream of Payments

Suppose that we are considering two investments, A and B. Investment A

pays \$100 now and will also pay \$200 next year. Investment B pays \$0 now, and will generate \$310 next year. Which is the better investment?

The answer depends on the interest rate. If the interest rate is zero, the answer is clear—just add up the payments. For if the interest rate is zero, then the present-value calculation boils down to summing up the payments.

If the interest rate is zero, the present value of investment A is

$$PV_A = 100 + 200 = 300,$$

and the present value of investment B is

$$PV_B = 0 + 310 = 310,$$

so B is the preferred investment.

But we get the opposite answer if the interest rate is high enough. Suppose, for example, that the interest rate is 20 percent. Then the present-value calculation becomes

$$PV_A = 100 + \frac{200}{1.20} = 266.67$$
$$PV_B = 0 + \frac{310}{1.20} = 258.33.$$

Now A is the better investment. The fact that A pays back more money earlier means that it will have a higher present value when the interest rate is large enough.

EXAMPLE: The True Cost of a Credit Card

Borrowing money on a credit card is expensive: many companies quote yearly interest charges of 15 to 21 percent. However, because of the way these finance charges are computed, the true interest rate on credit card debt is much higher than this.

Suppose that a credit card owner charges a \$2000 purchase on the first day of the month and that the finance charge is 1.5 percent a month. If the consumer pays the entire balance by the end of the month, he does not have to pay the finance charge. If the consumer pays none of the \$2,000, he has to pay a finance charge of \$2000 × .015 = \$30 at the beginning of the next month.

What happens if the consumer pays \$1,800 towards the \$2000 balance on the last day of the month? In this case, the consumer has borrowed only \$200, so the finance charge should be \$3. However, many credit card companies charge the consumers much more than this. The reason is that many companies base their charges on the "average monthly balance," even if part of that balance is paid by the end of the month. In this example,

the average monthly balance would be about $2000 (30 days of the $2000 balance and 1 day of the $200 balance). The finance charge would therefore be slightly less than $30, even though the consumer has only borrowed $200. Based on the actual amount of money borrowed, this is an interest rate of 15 percent a month!

EXAMPLE: Extending Copyright

Article I, Section 8 of the U.S. Constitution enables Congress to grant patents and copyrights using this language: "To promote the Progress of Science and useful Arts, by securing for limited Times to Authors and Inventors the exclusive Right to their respective Writings and Discoveries."

But what does "limited Times" mean? The lifetime of a patent in the United States is fixed at 20 years; the lifetime for copyright is quite different.

The first copyright act, passed by Congress in 1790, offered a 14-year term along with a 14-year renewal. Subsequently, the copyright term was lengthened to 28 years in 1831, with a 28-year renewal option added in 1909. In 1962 the term became 47 years, and 67 years in 1978. In 1967 the term was defined as the life of the author plus 50 years, or 75 years for "works for hire." The 1998 Sonny Bono Copyright Term Extension Act lengthened this term to the life of the author plus 70 years for individuals and 75–95 years for works for hire.

It is questionable whether "the life of the author plus 70 years" should be considered a limited time. One might ask what *additional* incentive the 1998 extension creates for authors to create works?

Let us look at a simple example. Suppose that the interest rate is 7%. Then the increase in present value of extending the copyright term from 80 to 100 years is about 0.33% of the present value of the first 80 years. Those extra 20 years have almost no impact on the present value of the copyright at time of creation since they come so far in the future. Hence they likely provide miniscule incremental incentive to create the works in the first place.

Given this tiny increase in value from extending the copyright term why would it pay anybody to lobby for such a change? The answer is that the 1998 act extended the copyright term *retroactively* so that works that were near expiration were given a new lease on life.

For example, it has been widely claimed that Disney lobbied heavily for the copyright term extension, since the original Mickey Mouse film, *Steamboat Willie,* was about to go out of copyright.

Retroactive copyright extensions of this sort make no economic sense, since what matters for the authors are the incentives present at the time the work is created. If there were no such retroactive extension, it is unlikely

that anyone would have bothered to ask for copyright extensions given the low economic value of the additional years of protection.

10.9 Bonds

Securities are financial instruments that promise certain patterns of payment schedules. There are many kinds of financial instruments because there are many kinds of payment schedules that people want. Financial markets give people the opportunity to trade different patterns of cash flows over time. These cash flows are typically used to finance consumption at some time or other.

The particular kind of security that we will examine here is a **bond**. Bonds are issued by governments and corporations. They are basically a way to borrow money. The borrower—the agent who issues the bond—promises to pay a fixed number of dollars x (the **coupon**) each period until a certain date T (the **maturity date**), at which point the borrower will pay an amount F (the **face value**) to the holder of the bond.

Thus the payment stream of a bond looks like $(x, x, x, \ldots, F)$. If the interest rate is constant, the present discounted value of such a bond is easy to compute. It is given by

$$PV = \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \cdots + \frac{F}{(1+r)^T}.$$

Note that the present value of a bond will decline if the interest rate increases. Why is this? When the interest rate goes up the price now for \$1 delivered in the future goes down. So the future payments of the bond will be worth less now.

There is a large and developed market for bonds. The market value of outstanding bonds will fluctuate as the interest rate fluctuates since the present value of the stream of payments represented by the bond will change.

An interesting special kind of a bond is a bond that makes payments forever. These are called **consols** or **perpetuities**. Suppose that we consider a consol that promises to pay x a year forever. To compute the value of this consol we have to compute the infinite sum:

$$PV = \frac{x}{1+r} + \frac{x}{(1+r)^2} + \cdots.$$

The trick to computing this is to factor out $1/(1+r)$ to get

$$PV = \frac{1}{1+r}\left[x + \frac{x}{(1+r)} + \frac{x}{(1+r)^2} + \cdots\right].$$

But the term in the brackets is just x plus the present value! Substituting and solving for PV:

$$\begin{aligned} PV &= \frac{1}{(1+r)}\,[x + PV] \\ &= \frac{x}{r}. \end{aligned}$$

This wasn't hard to do, but there is an easy way to get the answer right off. How much money, V, would you need at an interest rate r to get x dollars forever? Just write down the equation

$$Vr = x,$$

which says that the interest on V must equal x. But then the value of such an investment is given by

$$V = \frac{x}{r}.$$

Thus it must be that the present value of a consol that promises to pay x dollars forever must be given by x/r.

For a consol it is easy to see directly how increasing the interest rate reduces the value of a bond. Suppose, for example, that a consol is issued when the interest rate is 10 percent. Then if it promises to pay \$10 a year forever, it will be worth \$100 now—since \$100 would generate \$10 a year in interest income.

Now suppose that the interest rate goes up to 20 percent. The value of the consol must fall to \$50, since it only takes \$50 to earn \$10 a year at a 20 percent interest rate.

The formula for the consol can be used to calculate an approximate value of a long-term bond. If the interest rate is 10 percent, for example, the value of \$1 30 years from now is only 6 cents. For the size of interest rates we usually encounter, 30 years might as well be infinity.

EXAMPLE: Installment Loans

Suppose that you borrow \$1000 that you promise to pay back in 12 monthly installments of \$100 each. What rate of interest are you paying?

At first glance it seems that your interest rate is 20 percent: you have borrowed \$1000, and you are paying back \$1200. But this analysis is incorrect. For you haven't really borrowed \$1000 for an entire year. You have borrowed \$1000 for a month, and then you pay back \$100. Then you only have borrowed \$900, and you owe only a month's interest on the \$900. You borrow that for a month and then pay back another \$100. And so on.

The stream of payments that we want to value is

$$(1000, -100, -100, \ldots, -100).$$

We can find the interest rate that makes the present value of this stream equal to zero by using a calculator or a computer. The actual interest rate that you are paying on the installment loan is about 35 percent!

10.10 Taxes

In the United States, interest payments are taxed as ordinary income. This means that you pay the same tax on interest income as on labor income. Suppose that your marginal tax bracket is t, so that each *extra* dollar of income, Δm, increases your tax liability by $t\Delta m$. Then if you invest X dollars in an asset, you'll receive an interest payment of rX. But you'll also have to pay taxes of trX on this income, which will leave you with only $(1-t)rX$ dollars of after-tax income. We call the rate $(1-t)r$ the **after-tax interest rate**.

What if you decide to borrow X dollars, rather than lend them? Then you'll have to make an interest payment of rX. In the United States, some interest payments are tax deductible and some are not. For example, the interest payments for a mortgage are tax deductable, but interest payments on ordinary consumer loans are not. On the other hand, businesses can deduct most kinds of the interest payments that they make.

If a particular interest payment is tax deductible, you can subtract your interest payment from your other income and only pay taxes on what's left. Thus the rX dollars you pay in interest will reduce your tax payments by trX. The total cost of the X dollars you borrowed will be $rX - trX = (1-t)rX$.

Thus the after-tax interest rate is the same whether you are borrowing or lending, for people in the same tax bracket. The tax on saving will reduce the amount of money that people want to save, but the subsidy on borrowing will increase the amount of money that people want to borrow.

EXAMPLE: Scholarships and Savings

Many students in the United States receive some form of financial aid to help defray college costs. The amount of financial aid a student receives depends on many factors, but one important factor is the family's ability to pay for college expenses. Most U.S. colleges and universities use a standard measure of ability to pay calculated by the College Entrance Examination Board (CEEB).

If a student wishes to apply for financial aid, his or her family must fill out a questionnaire describing their financial circumstances. The CEEB uses the information on the income and assets of the parents to construct a measure of "adjusted available income." The fraction of their adjusted available income that parents are expected to contribute varies between 22 and 47 percent, depending on income. In 1985, parents with a total before-tax income of around $35,000 dollars were expected to contribute about $7000 toward college expenses.

Each additional dollar of assets that the parents accumulate increases their expected contribution and decreases the amount of financial aid that their child can hope to receive. The formula used by the CEEB effectively imposes a tax on parents who save for their children's college education. Martin Feldstein, President of the National Bureau of Economic Research (NBER) and Professor of Economics at Harvard University, calculated the magnitude of this tax.[3]

Consider the situation of some parents contemplating saving an additional dollar just as their daughter enters college. At a 6 percent rate of interest, the future value of a dollar 4 years from now is $1.26. Since federal and state taxes must be paid on interest income, the dollar yields $1.19 in after-tax income in 4 years. However, since this additional dollar of savings increases the total assets of the parents, the amount of aid received by the daughter goes down during *each* of her four college years. The effect of this "education tax" is to reduce the future value of the dollar to only 87 cents after 4 years. This is equivalent to an income tax of 150 percent!

Feldstein also examined the savings behavior of a sample of middle-class households with pre-college children. He estimates that a household with income of $40,000 a year and two college-age children saves about 50 percent less than they would otherwise due to the combination of federal, state, and "education" taxes that they face.

10.11 Choice of the Interest Rate

In the above discussion, we've talked about "the interest rate." In real life there are many interest rates: there are nominal rates, real rates, before-tax rates, after-tax rates, short-term rates, long-term rates, and so on. Which is the "right" rate to use in doing present-value analysis?

The way to answer this question is to think about the fundamentals. The idea of present discounted value arose because we wanted to be able to convert money at one point in time to an equivalent amount at another point in time. "The interest rate" is the return on an investment that allows us to transfer funds in this way.

If we want to apply this analysis when there are a variety of interest rates available, we need to ask which one has the properties most like the stream of payments we are trying to value. If the stream of payments is not taxed, we should use an after-tax interest rate. If the stream of payments will continue for 30 years, we should use a long-term interest rate. If the stream of payments is risky, we should use the interest rate on an investment with similar risk characteristics. (We'll have more to say later about what this last statement actually means.)

[3] Martin Feldstein, "College Scholarship Rules and Private Savings," *American Economic Review*, 85, 3 (June 1995).

The interest rate measures the **opportunity cost** of funds—the value of alternative uses of your money. So every stream of payments should be compared to your best alternative that has similar characteristics in terms of tax treatment, risk, and liquidity.

Summary

1. The budget constraint for intertemporal consumption can be expressed in terms of present value or future value.

2. The comparative statics results derived earlier for general choice problems can be applied to intertemporal consumption as well.

3. The real rate of interest measures the extra consumption that you can get in the future by giving up some consumption today.

4. A consumer who can borrow and lend at a constant interest rate should always prefer an endowment with a higher present value to one with a lower present value.

REVIEW QUESTIONS

1. How much is $1 million to be delivered 20 years in the future worth today if the interest rate is 20 percent?

2. As the interest rate rises, does the intertemporal budget constraint become steeper or flatter?

3. Would the assumption that goods are perfect substitutes be valid in a study of intertemporal food purchases?

4. A consumer, who is initially a lender, remains a lender even after a decline in interest rates. Is this consumer better off or worse off after the change in interest rates? If the consumer becomes a borrower after the change is he better off or worse off?

5. What is the present value of $100 one year from now if the interest rate is 10%? What is the present value if the interest rate is 5%?

CHAPTER 11

ASSET MARKETS

Assets are goods that provide a flow of services over time. Assets can provide a flow of consumption services, like housing services, or can provide a flow of money that can be used to purchase consumption. Assets that provide a monetary flow are called **financial assets**.

The bonds that we discussed in the last chapter are examples of financial assets. The flow of services they provide is the flow of interest payments. Other sorts of financial assets such as corporate stock provide different patterns of cash flows. In this chapter we will examine the functioning of asset markets under conditions of complete certainty about the future flow of services provided by the asset.

11.1 Rates of Return

Under this admittedly extreme hypothesis, we have a simple principle relating asset rates of return: if there is no uncertainty about the cash flow provided by assets, then all assets have to have the same rate of return. The reason is obvious: if one asset had a higher rate of return than another, and both assets were otherwise identical, then no one would want to buy

the asset with the lower rate of return. So in equilibrium, all assets that are actually held must pay the same rate of return.

Let us consider the process by which these rates of return adjust. Consider an asset A that has current price p_0 and is expected to have a price of p_1 tomorrow. Everyone is certain about what today's price of the asset is, and everyone is certain about what tomorrow's price will be. We suppose for simplicity that there are no dividends or other cash payments between periods 0 and 1. Suppose furthermore that there is another investment, B, that one can hold between periods 0 and 1 that will pay an interest rate of r. Now consider two possible investment plans: either invest one dollar in asset A and cash it in next period, or invest one dollar in asset B and earn interest of r dollars over the period.

What are the values of these two investment plans at the end of the first period? We first ask how many units of the asset we must purchase to make a one dollar investment in it. Letting x be this amount we have the equation

$$p_0 x = 1$$

or

$$x = \frac{1}{p_0}.$$

It follows that the future value of one dollar's worth of this asset next period will be

$$FV = p_1 x = \frac{p_1}{p_0}.$$

On the other hand, if we invest one dollar in asset B, we will have $1 + r$ dollars next period. If assets A and B are both held in equilibrium, then a dollar invested in either one of them must be worth the same amount second period. Thus we have an equilibrium condition:

$$1 + r = \frac{p_1}{p_0}.$$

What happens if this equality is not satisfied? Then there is a sure way to make money. For example, if

$$1 + r > \frac{p_1}{p_0},$$

people who own asset A can sell one unit for p_0 dollars in the first period and invest the money in asset B. Next period their investment in asset B will be worth $p_0(1 + r)$, which is greater than p_1 by the above equation. This will guarantee that second period they will have enough money to repurchase asset A, and be back where they started from, but now with extra money.

This kind of operation—buying some of one asset and selling some of another to realize a sure return—is known as **riskless arbitrage**, or **arbitrage** for short. As long as there are people around looking for "sure things" we would expect that well-functioning markets should quickly eliminate any opportunities for arbitrage. Therefore, another way to state our equilibrium condition is to say that in equilibrium *there should be no opportunities for arbitrage.* We'll refer to this as the **no arbitrage condition**.

But how does arbitrage actually work to eliminate the inequality? In the example given above, we argued that if $1 + r > p_1/p_0$, then anyone who held asset A would want to sell it first period, since they were guaranteed enough money to repurchase it second period. But who would they sell it to? Who would want to buy it? There would be plenty of people willing to supply asset A at p_0, but there wouldn't be anyone foolish enough to demand it at that price.

This means that supply would exceed demand and therefore the price will fall. How far will it fall? Just enough to satisfy the arbitrage condition: until $1 + r = p_1/p_0$.

11.2 Arbitrage and Present Value

We can rewrite the arbitrage condition in a useful way by cross multiplying to get

$$p_0 = \frac{p_1}{1 + r}.$$

This says that the current price of an asset must be its present value. Essentially we have converted the future-value comparison in the arbitrage condition to a present-value comparison. So if the no arbitrage condition is satisfied, then we are assured that assets must sell for their present values. Any deviation from present-value pricing leaves a sure way to make money.

11.3 Adjustments for Differences among Assets

The no arbitrage rule assumes that the asset services provided by the two assets are identical, except for the purely monetary difference. If the services provided by the assets have different characteristics, then we would want to adjust for those differences before we blandly assert that the two assets must have the same equilibrium rate of return.

For example, one asset might be easier to sell than the other. We sometimes express this by saying that one asset is more **liquid** than another. In this case, we might want to adjust the rate of return to take account of the difficulty involved in finding a buyer for the asset. Thus a house that is worth \$100,000 is probably a less liquid asset than \$100,000 in Treasury bills.

Similarly, one asset might be riskier than another. The rate of return on one asset may be guaranteed, while the rate of return on another asset may be highly risky. We'll examine ways to adjust for risk differences in Chapter 13.

Here we want to consider two other types of adjustment we might make. One is adjustment for assets that have some return in consumption value, and the other is for assets that have different tax characteristics.

11.4 Assets with Consumption Returns

Many assets pay off only in money. But there are other assets that pay off in terms of consumption as well. The prime example of this is housing. If you own a house that you live in, then you don't have to rent living quarters; thus part of the "return" to owning the house is the fact that you get to live in the house without paying rent. Or, put another way, you get to pay the rent for your house to yourself. This latter way of putting it sounds peculiar, but it contains an important insight.

It is true that you don't make an *explicit* rental payment to yourself for the privilege of living in your house, but it turns out to be fruitful to think of a homeowner as *implicitly* making such a payment. The **implicit rental rate** on your house is the rate at which you could rent a similar house. Or, equivalently, it is the rate at which you could rent your house to someone else on the open market. By choosing to "rent your house to yourself" you are forgoing the opportunity of earning rental payments from someone else, and thus incurring an opportunity cost.

Suppose that the implicit rental payment on your house would work out to T dollars per year. Then part of the return to owning your house is the fact that it generates for you an implicit income of T dollars per year—the money that you would otherwise have to pay to live in the same circumstances as you do now.

But that is not the entire return on your house. As real estate agents never tire of telling us, a house is also an *investment*. When you buy a house you pay a significant amount of money for it, and you might reasonably expect to earn a monetary return on this investment as well, through an increase in the value of your house. This increase in the value of an asset is known as **appreciation**.

Let us use A to represent the expected appreciation in the dollar value of your house over a year. The total return to owning your house is the sum of the rental return, T, and the investment return, A. If your house initially cost P, then the *total* rate of return on your initial investment in housing is

$$h = \frac{T + A}{P}.$$

This total rate of return is composed of the consumption rate of return, T/P, and the investment rate of return, A/P.

Let us use r to represent the rate of return on other financial assets. Then the total rate of return on housing should, in equilibrium, be equal to r:

$$r = \frac{T + A}{P}.$$

Think about it this way. At the beginning of the year, you can invest P in a bank and earn rP dollars, or you can invest P dollars in a house and save T dollars of rent and earn A dollars by the end of the year. The total return from these two investments has to be the same. If $T + A < rP$ you would be better off investing your money in the bank and paying T dollars in rent. You would then have $rP - T > A$ dollars at the end of the year. If $T + A > rP$, then housing would be the better choice. (Of course, this is ignoring the real estate agent's commission and other transactions costs associated with the purchase and sale.)

Since the total return should rise at the rate of interest, the financial rate of return A/P will generally be less than the rate of interest. Thus in general, assets that pay off in consumption will in equilibrium have a lower financial rate of return than purely financial assets. This means that buying consumption goods such as houses, or paintings, or jewelry *solely* as a financial investment is probably not a good idea since the rate of return on these assets will probably be lower than the rate of return on purely financial assets, because part of the price of the asset reflects the consumption return that people receive from owning such assets. On the other hand, if you place a sufficiently high value on the consumption return on such assets, or you can generate rental income from the assets, it may well make sense to buy them. The *total* return on such assets may well make this a sensible choice.

11.5 Taxation of Asset Returns

The Internal Revenue Service distinguishes two kinds of asset returns for purposes of taxation. The first kind is the **dividend** or **interest** return. These are returns that are paid periodically—each year or each month—over the life of the asset. You pay taxes on interest and dividend income at your ordinary tax rate, the same rate that you pay on your labor income.

The second kind of returns are called **capital gains**. Capital gains occur when you sell an asset at a price higher than the price at which you bought it. Capital gains are taxed only when you actually sell the asset. Under the current tax law, capital gains are taxed at the same rate as ordinary income, but there are some proposals to tax them at a more favorable rate.

It is sometimes argued that taxing capital gains at the same rate as ordinary income is a "neutral" policy. However, this claim can be disputed for at least two reasons. The first reason is that the capital gains taxes are only paid when the asset is sold, while taxes on dividends or interest are

paid every year. The fact that the capital gains taxes are deferred until time of sale makes the effective tax rate on capital gains *lower* than the tax rate on ordinary income.

A second reason that equal taxation of capital gains and ordinary income is not neutral is that the capital gains tax is based on the increase in the *dollar* value of an asset. If asset values are increasing just because of inflation, then a consumer may owe taxes on an asset whose *real* value hasn't changed. For example, suppose that a person buys an asset for $100 and 10 years later it is worth $200. Suppose that the general price level also doubles in this same ten-year period. Then the person would owe taxes on a $100 capital gain even though the purchasing power of his asset hadn't changed at all. This tends to make the tax on capital gains *higher* than that on ordinary income. Which of the two effects dominates is a controversial question.

In addition to the differential taxation of dividends and capital gains there are many other aspects of the tax law that treat asset returns differently. For example, in the United States, **municipal bonds**, bonds issued by cities or states, are not taxed by the Federal government. As we indicated earlier, the consumption return from owner-occupied housing is not taxed. Furthermore, in the United States even part of the capital gains from owner-occupied housing is not taxed.

The fact that different assets are taxed differently means that the arbitrage rule must adjust for the tax differences in comparing rates of return. Suppose that one asset pays a before-tax interest rate, r_b, and another asset pays a return that is tax exempt, r_e. Then if both assets are held by individuals who pay taxes on income at rate t, we must have

$$(1 - t)r_b = r_e.$$

That is, the after-tax return on each asset must be the same. Otherwise, individuals would not want to hold both assets—it would always pay them to switch exclusively to holding the asset that gave them the higher after-tax return. Of course, this discussion ignores other differences in the assets such as liquidity, risk, and so on.

11.6 Market Bubbles

Suppose you are contemplating buying a house that is absolutely certain to be worth $220,000 a year from now and that the current interest rate (reflecting your alternative investment opportunities) is 10%. A fair price for the house would be the present value, $200,000.

Now suppose that things aren't quite so certain: many people believe that the house will be worth $220,000 in a year, but there are no guarantees. We would expect that the house would sell for somewhat less than $200,000 due to the additional risk associated with purchase.

Suppose the year goes by and the house is worth $240,000, far more than anticipated. The house value went up by 20%, even though the prevailing interest rate was 10%. It may be that this experience will lead people to revise their view about how much the house will be worth in the future—who knows, maybe it will go up by 20% or even more next year.

If many people hold such beliefs, they can bid up the price of housing now—which may encourage others to make even more optimistic forecasts about the housing market. As in our discussion of price adjustment, assets that people expect to have a higher return than the rate of interest get pushed up in price. The higher price will tend to reduce current demand but it also may encourage people to expect an even higher return in the future.

The first effect—high prices reducing demand—tends to stablize prices. The second effect—high prices leading to an expectation of even higher prices in the future—tends to destabilize prices.

This is an example of an **asset bubble**. In a bubble, the price of an asset increases, for one reason or another, and this leads people to expect the price to go up even more in the future. But if they expect the asset price to rise significantly in the future, they will try to buy more today, pushing prices up even more rapidly.

Financial markets may be subject to such bubbles, particularly when the participants are inexperienced. For example, in 2000–01 we saw a dramatic run-up in the prices of technology stocks and in 2005–06 we saw a bubble in house prices in much of the United States and many other countries.

All bubbles eventually burst. Prices fall and some people are left holding assets that are worth much less than they paid for them.

The key to avoiding bubbles is to look at economic fundamentals. In the midst of the housing bubble in the United States, the ratio between the price of a house and the yearly rental rate on an identical house became far larger than historical norms. This gap presumably reflected buyers' expectations of future price increases.

Similarly, the ratio of median house prices to median income reached historical highs. Both of these were warning signs that the high prices were unsustainable.

"This time it's different" can be a very hazardous belief to hold, particularly when it comes to financial markets.

11.7 Applications

The fact that all riskless assets must earn the same return is obvious, but very important. It has surprisingly powerful implications for the functioning of asset markets.

Depletable Resources

Let us study the market equilibrium for a depletable resource like oil. Consider a competitive oil market, with many suppliers, and suppose for simplicity that there are zero costs to extract oil from the ground. Then how will the price of oil change over time?

It turns out that the price of oil must rise at the rate of interest. To see this, simply note that oil in the ground is an asset like any other asset. If it is worthwhile for a producer to hold it from one period to the next, it must provide a return to him equivalent to the financial return he could get elsewhere. If we let p_{t+1} and p_t be the prices at times $t+1$ and t, then we have

$$p_{t+1} = (1+r)p_t$$

as our no arbitrage condition in the oil market.

The argument boils down to this simple idea: oil in the ground is like money in the bank. If money in the bank earns a rate of return of r, then oil in the ground must earn the same rate of return. If oil in the ground earned a higher return than money in the bank, then no one would take oil out of the ground, preferring to wait till later to extract it, thus pushing the current price of oil up. If oil in the ground earned a lower return than money in the bank, then the owners of oil wells would try to pump their oil out immediately in order to put the money in the bank, thereby depressing the current price of oil.

This argument tells us how the price of oil changes. But what determines the price level itself? The price level turns out to be determined by the demand for oil. Let us consider a very simple model of the demand side of the market.

Suppose that the demand for oil is constant at D barrels a year and that there is a total world supply of S barrels. Thus we have a total of $T = S/D$ years of oil left. When the oil has been depleted we will have to use an alternative technology, say liquefied coal, which can be produced at a constant cost of C dollars per barrel. We suppose that liquefied coal is a perfect substitute for oil in all applications.

Now, T years from now, when the oil is just being exhausted, how much must it sell for? Clearly it must sell for C dollars a barrel, the price of its perfect substitute, liquefied coal. This means that the price today of a barrel of oil, p_0, must grow at the rate of interest r over the next T years to be equal to C. This gives us the equation

$$p_0(1+r)^T = C$$

or

$$p_0 = \frac{C}{(1+r)^T}.$$

This expression gives us the current price of oil as a function of the other variables in the problem. We can now ask interesting comparative statics questions. For example, what happens if there is an unforeseen new discovery of oil? This means that T, the number of years remaining of oil, will increase, and thus $(1+r)^T$ will increase, thereby decreasing p_0. So an increase in the supply of oil will, not surprisingly, decrease its current price.

What if there is a technological breakthrough that decreases the value of C? Then the above equation shows that p_0 must decrease. The price of oil has to be equal to the price of its perfect substitute, liquefied coal, when liquefied coal is the only alternative.

When to Cut a Forest

Suppose that the size of a forest—measured in terms of the lumber that you can get from it—is some function of time, $F(t)$. Suppose further that the price of lumber is constant and that the rate of growth of the tree starts high and gradually declines. If there is a competitive market for lumber, when should the forest be cut for timber?

Answer: when the rate of growth of the forest equals the interest rate. Before that, the forest is earning a higher rate of return than money in the bank, and after that point it is earning less than money in the bank. The optimal time to cut a forest is when its growth rate just equals the interest rate.

We can express this more formally by looking at the present value of cutting the forest at time T. This will be

$$PV = \frac{F(T)}{(1+r)^T}.$$

We want to find the choice of T that maximizes the present value—that is, that makes the value of the forest as large as possible. If we choose a very small value of T, the rate of growth of the forest will exceed the interest rate, which means that the PV would be increasing so it would pay to wait a little longer. On the other hand, if we consider a very large value of T, the forest would be growing more slowly than the interest rate, so the PV would be decreasing. The choice of T that maximizes present value occurs when the rate of growth of the forest just equals the interest rate.

This argument is illustrated in Figure 11.1. In Figure 11.1A we have plotted the *rate* of growth of the forest and the *rate* of growth of a dollar invested in a bank. If we want to have the largest amount of money at some unspecified point in the future, we should always invest our money in the asset with the highest return available at each point in time. When

Figure 11.1 **Harvesting a forest**. The optimal time to cut a forest is when the rate of growth of the forest equals the interest rate.

the forest is young, it is the asset with the highest return. As it matures, its rate of growth declines, and eventually the bank offers a higher return.

The effect on total wealth is illustrated in Figure 11.1B. Before T wealth grows most rapidly when invested in the forest. After T it grows most rapidly when invested in the bank. Therefore, the optimal strategy is to invest in the forest up until time T, then harvest the forest, and invest the proceeds in the bank.

EXAMPLE: Gasoline Prices during the Gulf War

In the Summer of 1990 Iraq invaded Kuwait. As a response to this, the United Nations imposed a blockade on oil imports from Iraq. Immediately after the blockade was announced the price of oil jumped up on world markets. At the same time price of gasoline at U.S. pumps increased significantly. This in turn led to cries of "war profiteering" and several segments about the oil industry on the evening news broadcasts.

Those who felt the price increase was unjustified argued that it would take at least 6 weeks for the new, higher-priced oil to wend its way across to the Atlantic and to be refined into gasoline. The oil companies, they argued, were making "excessive" profits by raising the price of gasoline that had already been produced using cheap oil.

Let's think about this argument as economists. Suppose that you own an asset—say gasoline in a storage tank—that is currently worth $1 a gallon. Six weeks from now, you know that it will be worth $1.50 a gallon. What

price will you sell it for now? Certainly you would be foolish to sell it for much less than $1.50 a gallon—at any price much lower than that you would be better off letting the gasoline sit in the storage tank for 6 weeks. The same intertemporal arbitrage reasoning about extracting oil from the ground applies to gasoline in a storage tank. The (appropriate discounted) price of gasoline tomorrow has to equal the price of gasoline today if you want firms to supply gasoline today.

This makes perfect sense from a welfare point of view as well: if gasoline is going to be more expensive in the near future, doesn't it make sense to consume less of it today? The increased price of gasoline encourages immediate conservation measures and reflects the true scarcity price of gasoline.

Ironically, the same phenomenon occured two years later in Russia. During the transition to a market economy, Russian oil sold for about $3 a barrel at a time when the world price was about $19 a barrel. The oil producers anticipated that the price of oil would soon be allowed to rise—so they tried to hold back as much oil as possible from current production. As one Russian producer put it, "Have you seen anyone in New York selling one dollar for 10 cents?" The result was long lines in front of the gasoline pumps for Russian consumers.[1]

11.8 Financial Institutions

Asset markets allow people to change their pattern of consumption over time. Consider, for example, two people A and B who have different endowments of wealth. A might have $100 today and nothing tomorrow, while B might have $100 tomorrow and nothing today. It might well happen that each would rather have $50 today and $50 tomorrow. But they can reach this pattern of consumption simply by trading: A gives B $50 today, and B gives A $50 tomorrow.

In this particular case, the interest rate is zero: A lends B $50 and only gets $50 in return the next day. If people have convex preferences over consumption today and tomorrow, they would like to smooth their consumption over time, rather than consume everything in one period, even if the interest rate were zero.

We can repeat the same kind of story for other patterns of asset endowments. One individual might have an endowment that provides a steady stream of payments and prefer to have a lump sum, while another might have a lump sum and prefer a steady stream. For example, a twenty-year-old individual might want to have a lump sum of money now to buy a house, while a sixty-year-old might want to have a steady stream of money

[1] See Louis Uchitelle, "Russians Line Up for Gas as Refineries Sit on Cheap Oil," *New York Times,* July 12, 1992, page 4.

to finance his retirement. It is clear that both of these individuals could gain by trading their endowments with each other.

In a modern economy financial institutions exist to facilitate these trades. In the case described above, the sixty-year-old can put his lump sum of money in the bank, and the bank can then lend it to the twenty-year-old. The twenty-year-old then makes mortgage payments to the bank, which are, in turn, transferred to the sixty-year-old as interest payments. Of course, the bank takes its cut for arranging the trade, but if the banking industry is sufficiently competitive, this cut should end up pretty close to the actual costs of doing business.

Banks aren't the only kind of financial institution that allow one to reallocate consumption over time. Another important example is the stock market. Suppose that an entrepreneur starts a company that becomes successful. In order to start the company, the entrepreneur probably had some financial backers who put up money to help him get started—to pay the bills until the revenues started rolling in. Once the company has been established, the owners of the company have a claim to the profits that the company will generate in the future: they have a claim to a stream of payments.

But it may well be that they prefer a lump-sum reward for their efforts now. In this case, the owners can decide to sell the firm to other people via the stock market. They issue shares in the company that entitle the shareholders to a cut of the future profits of the firm in exchange for a lump-sum payment now. People who want to purchase part of the stream of profits of the firm pay the original owners for these shares. In this way, both sides of the market can reallocate their wealth over time.

There are a variety of other institutions and markets that help facilitate intertemporal trade. But what happens when the buyers and sellers aren't evenly matched? What happens if more people want to sell consumption tomorrow than want to buy it? Just as in any market, if the supply of something exceeds the demand, the price will fall. In this case, the price of consumption tomorrow will fall. We saw earlier that the price of consumption tomorrow was given by

$$p = \frac{1}{1+r},$$

so this means that the interest rate must rise. The increase in the interest rate induces people to save more and to demand less consumption now, and thus tends to equate demand and supply.

Summary

1. In equilibrium, all assets with certain payoffs must earn the same rate of return. Otherwise there would be a riskless arbitrage opportunity.

2. The fact that all assets must earn the same return implies that all assets will sell for their present value.

3. If assets are taxed differently, or have different risk characteristics, then we must compare their after-tax rates of return or their risk-adjusted rates of return.

REVIEW QUESTIONS

1. Suppose asset A can be sold for \$11 next period. If assets similar to A are paying a rate of return of 10%, what must be asset A's current price?

2. A house, which you could rent for \$10,000 a year and sell for \$110,000 a year from now, can be purchased for \$100,000. What is the rate of return on this house?

3. The payments of certain types of bonds (e.g., municipal bonds) are not taxable. If similar taxable bonds are paying 10% and everyone faces a marginal tax rate of 40%, what rate of return must the nontaxable bonds pay?

4. Suppose that a scarce resource, facing a constant demand, will be exhausted in 10 years. If an alternative resource will be available at a price of \$40 and if the interest rate is 10%, what must the price of the scarce resource be today?

APPENDIX

Suppose that you invest \$1 in an asset yielding an interest rate r where the interest is paid once a year. Then after T years you will have $(1+r)^T$ dollars. Suppose now that the interest is paid monthly. This means that the monthly interest rate will be $r/12$, and there will be $12T$ payments, so that after T years you will have $(1+r/12)^{12T}$ dollars. If the interest rate is paid daily, you will have $(1+r/365)^{365T}$ and so on.

In general, if the interest is paid n times a year, you will have $(1+r/n)^{nT}$ dollars after T years. It is natural to ask how much money you will have if the interest is paid *continuously*. That is, we ask what is the limit of this expression as n goes to infinity. It turns out that this is given by the following expression:

$$e^{rT} = \lim_{n\to\infty} (1+r/n)^{nT},$$

where e is $2.7183\ldots$, the base of natural logarithms.

This expression for continuous compounding is very convenient for calculations. For example, let us verify the claim in the text that the optimal time to harvest

the forest is when the rate of growth of the forest equals the interest rate. Since the forest will be worth $F(T)$ at time T, the present value of the forest harvested at time T is

$$V(T) = \frac{F(T)}{e^{rT}} = e^{-rT}F(T).$$

In order to maximize the present value, we differentiate this with respect to T and set the resulting expression equal to zero. This yields

$$V'(T) = e^{-rT}F'(T) - re^{-rT}F(T) = 0$$

or

$$F'(T) - rF(T) = 0.$$

This can be rearranged to establish the result:

$$r = \frac{F'(T)}{F(T)}.$$

This equation says that the optimal value of T satisfies the condition that the rate of interest equals the rate of growth of the value of the forest.

CHAPTER 12

UNCERTAINTY

Uncertainty is a fact of life. People face risks every time they take a shower, walk across the street, or make an investment. But there are financial institutions such as insurance markets and the stock market that can mitigate at least some of these risks. We will study the functioning of these markets in the next chapter, but first we must study individual behavior with respect to choices involving uncertainty.

12.1 Contingent Consumption

Since we now know all about the standard theory of consumer choice, let's try to use what we know to understand choice under uncertainty. The first question to ask is what is the basic "thing" that is being chosen?

The consumer is presumably concerned with the **probability distribution** of getting different consumption bundles of goods. A probability distribution consists of a list of different outcomes—in this case, consumption bundles—and the probability associated with each outcome. When a consumer decides how much automobile insurance to buy or how much to

invest in the stock market, he is in effect deciding on a pattern of probability distribution across different amounts of consumption.

For example, suppose that you have \$100 now and that you are contemplating buying lottery ticket number 13. If number 13 is drawn in the lottery, the holder will be paid \$200. This ticket costs, say, \$5. The two outcomes that are of interest are the event that the ticket is drawn and the event that it isn't.

Your original endowment of wealth—the amount that you would have if you did not purchase the lottery ticket—is \$100 if 13 is drawn, and \$100 if it isn't drawn. But if you buy the lottery ticket for \$5, you will have a wealth distribution consisting of \$295 if the ticket is a winner, and \$95 if it is not a winner. The original endowment of probabilities of wealth in different circumstances has been changed by the purchase of the lottery ticket. Let us examine this point in more detail.

In this discussion we'll restrict ourselves to examining monetary gambles for convenience of exposition. Of course, it is not money alone that matters; it is the consumption that money can buy that is the ultimate "good" being chosen. The same principles apply to gambles over goods, but restricting ourselves to monetary outcomes makes things simpler. Second, we will restrict ourselves to very simple situations where there are only a few possible outcomes. Again, this is only for reasons of simplicity.

Above we described the case of gambling in a lottery; here we'll consider the case of insurance. Suppose that an individual initially has \$35,000 worth of assets, but there is a possibility that he may lose \$10,000. For example, his car may be stolen, or a storm may damage his house. Suppose that the probability of this event happening is $p = .01$. Then the probability distribution the person is facing is a 1 percent probability of having \$25,000 of assets, and a 99 percent probability of having \$35,000.

Insurance offers a way to change this probability distribution. Suppose that there is an insurance contract that will pay the person \$100 if the loss occurs in exchange for a \$1 premium. Of course the premium must be paid whether or not the loss occurs. If the person decides to purchase \$10,000 dollars of insurance, it will cost him \$100. In this case he will have a 1 percent chance of having \$34,900 (\$35,000 of other assets − \$10,000 loss + \$10,000 payment from the insurance payment – \$100 insurance premium) and a 99 percent chance of having \$34,900 (\$35,000 of assets − \$100 insurance premium). Thus the consumer ends up with the same wealth no matter what happens. He is now fully insured against loss.

In general, if this person purchases K dollars of insurance and has to pay a premium γK, then he will face the gamble:[1]

$$\text{probability } .01 \text{ of getting } \$25,000 + K - \gamma K$$

[1] The Greek letter γ, gamma, is pronounced "gam-ma."

and

probability .99 of getting $\$35,000 - \gamma K$.

What kind of insurance will this person choose? Well, that depends on his preferences. He might be very conservative and choose to purchase a lot of insurance, or he might like to take risks and not purchase any insurance at all. People have different preferences over probability distributions in the same way that they have different preferences over the consumption of ordinary goods.

In fact, one very fruitful way to look at decision making under uncertainty is just to think of the money available under different circumstances as different goods. A thousand dollars after a large loss has occurred may mean a very different thing from a thousand dollars when it hasn't. Of course, we don't have to apply this idea just to money: an ice cream cone if it happens to be hot and sunny tomorrow is a very different good from an ice cream cone if it is rainy and cold. In general, consumption goods will be of different value to a person depending upon the circumstances under which they become available.

Let us think of the different outcomes of some random event as being different **states of nature**. In the insurance example given above there were two states of nature: the loss occurs or it doesn't. But in general there could be many different states of nature. We can then think of a **contingent consumption plan** as being a specification of what will be consumed in each different state of nature—each different outcome of the random process. *Contingent* means depending on something not yet certain, so a contingent consumption plan means a plan that depends on the outcome of some event. In the case of insurance purchases, the contingent consumption was described by the terms of the insurance contract: how much money you would have if a loss occurred and how much you would have if it didn't. In the case of the rainy and sunny days, the contingent consumption would just be the *plan* of what would be consumed given the various outcomes of the weather.

People have preferences over different plans of consumption, just like they have preferences over actual consumption. It certainly might make you feel better now to know that you are fully insured. People make choices that reflect their preferences over consumption in different circumstances, and we can use the theory of choice that we have developed to analyze those choices.

If we think about a contingent consumption plan as being just an ordinary consumption bundle, we are right back in the framework described in the previous chapters. We can think of preferences as being defined over different consumption plans, with the "terms of trade" being given by the budget constraint. We can then model the consumer as choosing the best consumption plan he or she can afford, just as we have done all along.

Let's describe the insurance purchase in terms of the indifference-curve analysis we've been using. The two states of nature are the event that the loss occurs and the event that it doesn't. The contingent consumptions are the values of how much money you would have in each circumstance. We can plot this on a graph as in Figure 12.1.

Figure 12.1

Insurance. The budget line associated with the purchase of insurance. The insurance premium γ allows us to give up some consumption in the good outcome (C_g) in order to have more consumption in the bad outcome (C_b).

Your endowment of contingent consumption is \$25,000 in the "bad" state—if the loss occurs—and \$35,000 in the "good" state—if it doesn't occur. Insurance offers you a way to move away from this endowment point. If you purchase K dollars' worth of insurance, you give up γK dollars of consumption possibilities in the good state in exchange for $K - \gamma K$ dollars of consumption possibilities in the bad state. Thus the consumption you lose in the good state, divided by the extra consumption you gain in the bad state, is

$$\frac{\Delta C_g}{\Delta C_b} = -\frac{\gamma K}{K - \gamma K} = -\frac{\gamma}{1-\gamma}.$$

This is the slope of the budget line through your endowment. It is just as if the price of consumption in the good state is $1 - \gamma$ and the price in the bad state is γ.

We can draw in the indifference curves that a person might have for contingent consumption. Here again it is very natural for indifference curves to have a convex shape: this means that the person would rather have a constant amount of consumption in each state than a large amount in one state and a low amount in the other.

Given the indifference curves for consumption in each state of nature, we can look at the choice of how much insurance to purchase. As usual, this will be characterized by a tangency condition: the marginal rate of substitution between consumption in each state of nature should be equal to the price at which you can trade off consumption in those states.

Of course, once we have a model of optimal choice, we can apply all of the machinery developed in early chapters to its analysis. We can examine how the demand for insurance changes as the price of insurance changes, as the wealth of the consumer changes, and so on. The theory of consumer behavior is perfectly adequate to model behavior under uncertainty as well as certainty.

EXAMPLE: Catastrophe Bonds

We have seen that insurance is a way to transfer wealth from good states of nature to bad states of nature. Of course there are two sides to these transactions: those who buy insurance and those who sell it. Here we focus on the sell side of insurance.

The sell side of the insurance market is divided into a retail component, which deals directly with end buyers, and a wholesale component, in which insurers sell risks to other parties. The wholesale part of the market is known as the **reinsurance market.**

Typically, the reinsurance market has relied on large investors such as pension funds to provide financial backing for risks. However, some reinsurers rely on large individual investors. Lloyd's of London, one of the most famous reinsurance consortia, generally uses private investors.

Recently, the reinsurance industry has been experimenting with **catastrophe bonds**, which, according to some, are a more flexible way to provide reinsurance. These bonds, generally sold to large institutions, have typically been tied to natural disasters, like earthquakes or hurricanes.

A financial intermediary, such as a reinsurance company or an investment bank, issues a bond tied to a particular insurable event, such as an earthquake involving, say, at least $500 million in insurance claims. If there is no earthquake, investors are paid a generous interest rate. But if the earthquake occurs and the claims exceed the amount specified in the bond, investors sacrifice their principal and interest.

Catastrophe bonds have some attractive features. They can spread risks widely and can be subdivided indefinitely, allowing each investor to bear

only a small part of the risk. The money backing up the insurance is paid in advance, so there is no default risk to the insured.

From the economist's point of view, "cat bonds" are a form of **state contingent security**, that is, a security that pays off if and only if some particular event occurs. This concept was first introduced by Nobel laureate Kenneth J. Arrow in a paper published in 1952 and was long thought to be of only theoretical interest. But it turned out that all sorts of options and other derivatives could be best understood using contingent securities. Now Wall Street rocket scientists draw on this 60-year-old work when creating exotic new derivatives such as catastrophe bonds.

12.2 Utility Functions and Probabilities

If the consumer has reasonable preferences about consumption in different circumstances, then we will be able to use a utility function to describe these preferences, just as we have done in other contexts. However, the fact that we are considering choice under uncertainty does add a special structure to the choice problem. In general, how a person values consumption in one state as compared to another will depend on the *probability* that the state in question will actually occur. In other words, the rate at which I am willing to substitute consumption if it rains for consumption if it doesn't should have something to do with how likely I think it is to rain. The preferences for consumption in different states of nature will depend on the beliefs of the individual about how likely those states are.

For this reason, we will write the utility function as depending on the probabilities as well as on the consumption levels. Suppose that we are considering two mutually exclusive states such as rain and shine, loss or no loss, or whatever. Let c_1 and c_2 represent consumption in states 1 and 2, and let π_1 and π_2 be the probabilities that state 1 or state 2 actually occurs.

If the two states are mutually exclusive, so that only one of them can happen, then $\pi_2 = 1 - \pi_1$. But we'll generally write out both probabilities just to keep things looking symmetric.

Given this notation, we can write the utility function for consumption in states 1 and 2 as $u(c_1, c_2, \pi_1, \pi_2)$. This is the function that represents the individual's preference over consumption in each state.

EXAMPLE: Some Examples of Utility Functions

We can use nearly any of the examples of utility functions that we've seen up until now in the context of choice under uncertainty. One nice example is the case of perfect substitutes. Here it is natural to weight each

consumption by the probability that it will occur. This gives us a utility function of the form

$$u(c_1, c_2, \pi_1, \pi_2) = \pi_1 c_1 + \pi_2 c_2.$$

In the context of uncertainty, this kind of expression is known as the **expected value**. It is just the average level of consumption that you would get.

Another example of a utility function that might be used to examine choice under uncertainty is the Cobb–Douglas utility function:

$$u(c_1, c_2, \pi, 1 - \pi) = c_1^{\pi} c_2^{1-\pi}.$$

Here the utility attached to any combination of consumption bundles depends on the pattern of consumption in a nonlinear way.

As usual, we can take a monotonic transformation of utility and still represent the same preferences. It turns out that the logarithm of the Cobb-Douglas utility will be very convenient in what follows. This will give us a utility function of the form

$$\ln u(c_1, c_2, \pi_1, \pi_2) = \pi_1 \ln c_1 + \pi_2 \ln c_2.$$

12.3 Expected Utility

One particularly convenient form that the utility function might take is the following:

$$u(c_1, c_2, \pi_1, \pi_2) = \pi_1 v(c_1) + \pi_2 v(c_2).$$

This says that utility can be written as a weighted sum of some function of consumption in each state, $v(c_1)$ and $v(c_2)$, where the weights are given by the probabilities π_1 and π_2.

Two examples of this were given above. The perfect substitutes, or expected value utility function, had this form where $v(c) = c$. The Cobb-Douglas didn't have this form originally, but when we expressed it in terms of logs, it had the linear form with $v(c) = \ln c$.

If one of the states is certain, so that $\pi_1 = 1$, say, then $v(c_1)$ is the utility of certain consumption in state 1. Similarly, if $\pi_2 = 1$, $v(c_2)$ is the utility of consumption in state 2. Thus the expression

$$\pi_1 v(c_1) + \pi_2 v(c_2)$$

represents the average utility, or the expected utility, of the pattern of consumption (c_1, c_2).

For this reason, we refer to a utility function with the particular form described here as an **expected utility function**, or, sometimes, a **von Neumann-Morgenstern utility function**.[2]

When we say that a consumer's preferences can be represented by an expected utility function, or that the consumer's preferences have the expected utility property, we mean that we can choose a utility function that has the additive form described above. Of course we could also choose a different form; any monotonic transformation of an expected utility function is a utility function that describes the same preferences. But the additive form representation turns out to be especially convenient. If the consumer's preferences are described by $\pi_1 \ln c_1 + \pi_2 \ln c_2$ they will also be described by $c_1^{\pi_1} c_2^{\pi_2}$. But the latter representation does not have the expected utility property, while the former does.

On the other hand, the expected utility function can be subjected to some kinds of monotonic transformation and still have the expected utility property. We say that a function $v(u)$ is a **positive affine transformation** if it can be written in the form: $v(u) = au + b$ where $a > 0$. A positive affine transformation simply means multiplying by a positive number and adding a constant. It turns out that if you subject an expected utility function to a positive affine transformation, it not only represents the same preferences (this is obvious since an affine transformation is just a special kind of monotonic transformation) but it also still has the expected utility property.

Economists say that an expected utility function is "unique up to an affine transformation." This just means that you can apply an affine transformation to it and get another expected utility function that represents the same preferences. But any other kind of transformation will destroy the expected utility property.

12.4 Why Expected Utility Is Reasonable

The expected utility representation is a convenient one, but is it a reasonable one? Why would we think that preferences over uncertain choices would have the particular structure implied by the expected utility function? As it turns out there are compelling reasons why expected utility is a reasonable objective for choice problems in the face of uncertainty.

The fact that outcomes of the random choice are consumption goods that will be consumed in different circumstances means that ultimately *only one* of those outcomes is actually going to occur. Either your house

[2] John von Neumann was one of the major figures in mathematics in the twentieth century. He also contributed several important insights to physics, computer science, and economic theory. Oscar Morgenstern was an economist at Princeton who, along with von Neumann, helped to develop mathematical game theory.

will burn down or it won't; either it will be a rainy day or a sunny day. The way we have set up the choice problem means that only one of the many possible outcomes is going to occur, and hence only one of the contingent consumption plans will actually be realized.

This turns out to have a very interesting implication. Suppose you are considering purchasing fire insurance on your house for the coming year. In making this choice you will be concerned about wealth in three situations: your wealth now (c_0), your wealth if your house burns down (c_1), and your wealth if it doesn't (c_2). (Of course, what you really care about are your consumption possibilities in each outcome, but we are simply using wealth as a proxy for consumption here.) If π_1 is the probability that your house burns down and π_2 is the probability that it doesn't, then your preferences over these three different consumptions can generally be represented by a utility function $u(\pi_1, \pi_2, c_0, c_1, c_2)$.

Suppose that we are considering the tradeoff between wealth now and one of the possible outcomes—say, how much money we would be willing to sacrifice now to get a little more money if the house burns down. *Then this decision should be independent of how much consumption you will have in the other state of nature—how much wealth you will have if the house is not destroyed.* For the house will either burn down or it won't. If it happens to burn down, then the value of extra wealth shouldn't depend on how much wealth you would have if it *didn't* burn down. Bygones are bygones—so what *doesn't* happen shouldn't affect the value of consumption in the outcome that *does* happen.

Note that this is an *assumption* about an individual's preferences. It may be violated. When people are considering a choice between two things, the amount of a third thing they have typically matters. The choice between coffee and tea may well depend on how much cream you have. But this is because you consume coffee together with cream. If you considered a choice where you rolled a die and got either coffee, *or* tea, *or* cream, then the amount of cream that you might get shouldn't affect your preferences between coffee and tea. Why? Because you are either getting one thing or the other: if you end up with cream, the fact that you might have gotten either coffee or tea is irrelevant.

Thus in choice under uncertainty there is a natural kind of "independence" between the different outcomes because they must be consumed separately—in different states of nature. The choices that people plan to make in one state of nature should be independent from the choices that they plan to make in other states of nature. This assumption is known as the **independence assumption**. It turns out that this implies that the utility function for contingent consumption will take a very special structure: it has to be additive across the different contingent consumption bundles.

That is, if c_1, c_2, and c_3 are the consumptions in different states of nature, and π_1, π_2, and π_3 are the probabilities that these three different states of

nature materialize, then if the independence assumption alluded to above is satisfied, the utility function must take the form

$$U(c_1, c_2, c_3) = \pi_1 u(c_1) + \pi_2 u(c_2) + \pi_3 u(c_3).$$

This is what we have called an expected utility function. Note that the expected utility function does indeed satisfy the property that the marginal rate of substitution between two goods is independent of how much there is of the third good. The marginal rate of substitution between goods 1 and 2, say, takes the form

$$\begin{aligned} \mathrm{MRS}_{12} &= -\frac{\Delta U(c_1, c_2, c_3)/\Delta c_1}{\Delta U(c_1, c_2, c_3)/\Delta c_2} \\ &= -\frac{\pi_1 \Delta u(c_1)/\Delta c_1}{\pi_2 \Delta u(c_2)/\Delta c_2}. \end{aligned}$$

This MRS depends only on how much you have of goods 1 and 2, not how much you have of good 3.

12.5 Risk Aversion

We claimed above that the expected utility function had some very convenient properties for analyzing choice under uncertainty. In this section we'll give an example of this.

Let's apply the expected utility framework to a simple choice problem. Suppose that a consumer currently has \$10 of wealth and is contemplating a gamble that gives him a 50 percent probability of winning \$5 and a 50 percent probability of losing \$5. His wealth will therefore be random: he has a 50 percent probability of ending up with \$5 and a 50 percent probability of ending up with \$15. The *expected value* of his wealth is \$10, and the expected utility is

$$\frac{1}{2}u(\$15) + \frac{1}{2}u(\$5).$$

This is depicted in Figure 12.2. The expected utility of wealth is the average of the two numbers $u(\$15)$ and $u(\$5)$, labeled $.5u(5) + .5u(15)$ in the graph. We have also depicted the utility of the expected value of wealth, which is labeled $u(\$10)$. Note that in this diagram the expected utility of wealth is less than the utility of the expected wealth. That is,

$$u\left(\frac{1}{2}15 + \frac{1}{2}5\right) = u\,(10) > \frac{1}{2}u\,(15) + \frac{1}{2}u\,(5)\,.$$

Risk aversion. For a risk-averse consumer the utility of the expected value of wealth, $u(10)$, is greater than the expected utility of wealth, $.5u(5) + .5u(15)$.

Figure 12.2

In this case we say that the consumer is **risk averse** since he prefers to have the expected value of his wealth rather than face the gamble. Of course, it could happen that the preferences of the consumer were such that he prefers a a random distribution of wealth to its expected value, in which case we say that the consumer is a **risk lover**. An example is given in Figure 12.3.

Note the difference between Figures 12.2 and 12.3. The risk-averse consumer has a *concave* utility function—its slope gets flatter as wealth is increased. The risk-loving consumer has a *convex* utility function—its slope gets steeper as wealth increases. Thus the curvature of the utility function measures the consumer's attitude toward risk. In general, the more concave the utility function, the more risk averse the consumer will be, and the more convex the utility function, the more risk loving the consumer will be.

The intermediate case is that of a linear utility function. Here the consumer is **risk neutral**: the expected utility of wealth is the utility of its expected value. In this case the consumer doesn't care about the riskiness of his wealth at all—only about its expected value.

EXAMPLE: The Demand for Insurance

Let's apply the expected utility structure to the demand for insurance that we considered earlier. Recall that in that example the person had a wealth

Figure 12.3 **Risk loving**. For a risk-loving consumer the expected utility of wealth, $.5u(5) + .5u(15)$, is greater than the utility of the expected value of wealth, $u(10)$.

of \$35,000 and that he might incur a loss of \$10,000. The probability of the loss was 1 percent, and it cost him γK to purchase K dollars of insurance. By examining this choice problem using indifference curves we saw that the optimal choice of insurance was determined by the condition that the MRS between consumption in the two outcomes—loss or no loss—must be equal to $-\gamma/(1-\gamma)$. Let π be the probability that the loss will occur, and $1-\pi$ be the probability that it won't occur.

Let state 1 be the situation involving no loss, so that the person's wealth in that state is

$$c_1 = \$35,000 - \gamma K,$$

and let state 2 be the loss situation with wealth

$$c_2 = \$35,000 - \$10,000 + K - \gamma K.$$

Then the consumer's optimal choice of insurance is determined by the condition that his MRS between consumption in the two outcomes be equal to the price ratio:

$$\text{MRS} = -\frac{\pi \Delta u(c_2)/\Delta c_2}{(1-\pi)\Delta u(c_1)/\Delta c_1} = -\frac{\gamma}{1-\gamma}. \tag{12.1}$$

Now let us look at the insurance contract from the viewpoint of the insurance company. With probability π they must pay out K, and with

probability $(1-\pi)$ they pay out nothing. No matter what happens, they collect the premium γK. Then the expected profit, P, of the insurance company is

$$P = \gamma K - \pi K - (1-\pi)\cdot 0 = \gamma K - \pi K.$$

Let us suppose that on the average the insurance company just breaks even on the contract. That is, they offer insurance at a "fair" rate, where "fair" means that the expected value of the insurance is just equal to its cost. Then we have

$$P = \gamma K - \pi K = 0,$$

which implies that $\gamma = \pi$.

Inserting this into equation (12.1) we have

$$\frac{\pi \Delta u(c_2)/\Delta c_2}{(1-\pi)\Delta u(c_1)/\Delta c_1} = \frac{\pi}{1-\pi}.$$

Canceling the π's leaves us with the condition that the optimal amount of insurance must satisfy

$$\frac{\Delta u(c_1)}{\Delta c_1} = \frac{\Delta u(c_2)}{\Delta c_2}. \tag{12.2}$$

This equation says that the *marginal utility of an extra dollar of income if the loss occurs should be equal to the marginal utility of an extra dollar of income if the loss doesn't occur.*

Let us suppose that the consumer is risk averse, so that his marginal utility of money is declining as the amount of money he has increases. Then if $c_1 > c_2$, the marginal utility at c_1 would be less than the marginal utility at c_2, and vice versa. Furthermore, if the marginal utilities of income are equal at c_1 and c_2, as they are in equation (12.2), then we must have $c_1 = c_2$. Applying the formulas for c_1 and c_2, we find

$$35,000 - \gamma K = 25,000 + K - \gamma K,$$

which implies that $K = \$10,000$. This means that when given a chance to buy insurance at a "fair" premium, a risk-averse consumer will always choose to fully insure.

This happens because the utility of wealth in each state depends only on the total amount of wealth the consumer has in that state—and not what he *might* have in some other state—so that if the total amounts of wealth the consumer has in each state are equal, the marginal utilities of wealth must be equal as well.

To sum up: if the consumer is a risk-averse, expected utility maximizer and if he is offered fair insurance against a loss, then he will optimally choose to fully insure.

12.6 Diversification

Let us turn now to a different topic involving uncertainty—the benefits of diversification. Suppose that you are considering investing $100 in two different companies, one that makes sunglasses and one that makes raincoats. The long-range weather forecasters have told you that next summer is equally likely to be rainy or sunny. How should you invest your money?

Wouldn't it make sense to hedge your bets and put some money in each? By diversifying your holdings of the two investments, you can get a return on your investment that is more certain, and therefore more desirable if you are a risk-averse person.

Suppose, for example, that shares of the raincoat company and the sunglasses company currently sell for $10 apiece. If it is a rainy summer, the raincoat company will be worth $20 and the sunglasses company will be worth $5. If it is a sunny summer, the payoffs are reversed: the sunglasses company will be worth $20 and the raincoat company will be worth $5. If you invest your entire $100 in the sunglasses company, you are taking a gamble that has a 50 percent chance of giving you $200 and a 50 percent chance of giving you $50. The same magnitude of payoffs results if you invest all your money in the sunglasses company: in either case you have an expected payoff of $125.

But look what happens if you put half of your money in each. Then, if it is sunny you get $100 from the sunglasses investment and $25 from the raincoat investment. But if it is rainy, you get $100 from the raincoat investment and $25 from the sunglasses investment. Either way, you end up with $125 for sure. By diversifying your investment in the two companies, you have managed to reduce the overall risk of your investment, while keeping the expected payoff the same.

Diversification was quite easy in this example: the two assets were perfectly negatively correlated—when one went up, the other went down. Pairs of assets like this can be extremely valuable because they can reduce risk so dramatically. But, alas, they are also very hard to find. Most asset values move together: when GM stock is high, so is Ford stock, and so is Goodrich stock. But as long as asset price movements are not *perfectly* positively correlated, there will be some gains from diversification.

12.7 Risk Spreading

Let us return now to the example of insurance. There we considered the situation of an individual who had $35,000 and faced a .01 probability of a $10,000 loss. Suppose that there were 1000 such individuals. Then, on average, there would be 10 losses incurred, and thus $100,000 lost each year. Each of the 1000 people would face an *expected loss* of .01 times $10,000, or

\$100 a year. Let us suppose that the probability that any person incurs a loss doesn't affect the probability that any of the others incur losses. That is, let us suppose that the risks are *independent.*

Then each individual will have an expected wealth of $.99 \times \$35,000 + .01 \times \$25,000 = \$34,900$. But each individual also bears a large amount of risk: each person has a 1 percent probability of losing \$10,000.

Suppose that each consumer decides to *diversify* the risk that he or she faces. How can they do this? Answer: by selling some of their risk to other individuals. Suppose that the 1000 consumers decide to insure one another. If anybody incurs the \$10,000 loss, each of the 1000 consumers will contribute \$10 to that person. This way, the poor person whose house burns down is compensated for his loss, and the other consumers have the peace of mind that they will be compensated if that poor soul happens to be themselves! This is an example of **risk spreading**: each consumer spreads his risk over all of the other consumers and thereby reduces the amount of risk he bears.

Now on the average, 10 houses will burn down a year, so on the average, each of the 1000 individuals will be paying out \$100 a year. But this is just on the average. Some years there might be 12 losses, and other years there might be 8 losses. The probability is very small that an individual would actually have to pay out more than \$200, say, in any one year, but even so, the risk is there.

But there is even a way to diversify this risk. Suppose that the homeowners agree to pay \$100 a year for certain, whether or not there are any losses. Then they can build up a cash reserve fund that can be used in those years when there are multiple fires. They are paying \$100 a year for certain, and on average that money will be sufficient to compensate homeowners for fires.

As you can see, we now have something very much like a cooperative insurance company. We could add a few more features: the insurance company gets to invest its cash reserve fund and earn interest on its assets, and so on, but the essence of the insurance company is clearly present.

12.8 Role of the Stock Market

The stock market plays a role similar to that of the insurance market in that it allows for risk spreading. Recall from Chapter 11 that we argued that the stock market allowed the original owners of firms to convert their stream of returns over time to a lump sum. Well, the stock market also allows them to convert their risky position of having all their wealth tied up in one enterprise to a situation where they have a lump sum that they can invest in a variety of assets. The original owners of the firm have an incentive to issue shares in their company so that they can spread the risk of that single company over a large number of shareholders.

Similarly, the later shareholders of a company can use the stock market to reallocate their risks. If a company you hold shares in is adopting a policy that is too risky for your taste—or too conservative—you can sell those shares and purchase others.

In the case of insurance, an individual was able to reduce his risk to zero by purchasing insurance. For a flat fee of $100, the individual could purchase full insurance against the $10,000 loss. This was true because there was basically no risk in the aggregate: if the probability of the loss occurring was 1 percent, then on average 10 of the 1000 people would face a loss—we just didn't know which ones.

In the case of the stock market, there is risk in the aggregate. One year the stock market as a whole might do well, and another year it might do poorly. Somebody has to bear that kind of risk. The stock market offers a way to transfer risky investments from people who don't want to bear risk to people who are willing to bear risk.

Of course, few people outside of Las Vegas *like* to bear risk: most people are risk averse. Thus the stock market allows people to transfer risk from people who don't want to bear it to people who are willing to bear it if they are sufficiently compensated for it. We'll explore this idea further in the next chapter.

Summary

1. Consumption in different states of nature can be viewed as consumption goods, and all the analysis of previous chapters can be applied to choice under uncertainty.

2. However, the utility function that summarizes choice behavior under uncertainty may have a special structure. In particular, if the utility function is linear in the probabilities, then the utility assigned to a gamble will just be the expected utility of the various outcomes.

3. The curvature of the expected utility function describes the consumer's attitudes toward risk. If it is concave, the consumer is a risk averter; and if it is convex, the consumer is a risk lover.

4. Financial institutions such as insurance markets and the stock market provide ways for consumers to diversify and spread risks.

REVIEW QUESTIONS

1. How can one reach the consumption points to the left of the endowment in Figure 12.1?

2. Which of the following utility functions have the expected utility property? (a) $u(c_1, c_2, \pi_1, \pi_2) = a(\pi_1 c_1 + \pi_2 c_2)$, (b) $u(c_1, c_2, \pi_1, \pi_2) = \pi_1 c_1 + \pi_2 c_2^2$, (c) $u(c_1, c_2, \pi_1, \pi_2) = \pi_1 \ln c_1 + \pi_2 \ln c_2 + 17$.

3. A risk-averse individual is offered a choice between a gamble that pays \$1000 with a probability of 25% and \$100 with a probability of 75%, or a payment of \$325. Which would he choose?

4. What if the payment was \$320?

5. Draw a utility function that exhibits risk-loving behavior for small gambles and risk-averse behavior for larger gambles.

6. Why might a neighborhood group have a harder time self insuring for flood damage versus fire damage?

APPENDIX

Let us examine a simple problem to demonstrate the principles of expected utility maximization. Suppose that the consumer has some wealth w and is considering investing some amount x in a risky asset. This asset could earn a return of r_g in the "good" outcome, or it could earn a return of r_b in the "bad" outcome. You should think of r_g as being a positive return—the asset increases in value, and r_b being a negative return—a decrease in asset value.

Thus the consumer's wealth in the good and bad outcomes will be

$$W_g = (w - x) + x(1 + r_g) = w + xr_g$$
$$W_b = (w - x) + x(1 + r_b) = w + xr_b.$$

Suppose that the good outcome occurs with probability π and the bad outcome with probability $(1 - \pi)$. Then the expected utility if the consumer decides to invest x dollars is

$$EU(x) = \pi u(w + xr_g) + (1 - \pi)u(w + xr_b).$$

The consumer wants to choose x so as to maximize this expression.

Differentiating with respect to x, we find the way in which utility changes as x changes:

$$EU'(x) = \pi u'(w + xr_g)r_g + (1 - \pi)u'(w + xr_b)r_b. \tag{12.3}$$

The second derivative of utility with respect to x is

$$EU''(x) = \pi u''(w + xr_g)r_g^2 + (1 - \pi)u''(w + xr_b)r_b^2. \tag{12.4}$$

If the consumer is risk averse his utility function will be concave, which implies that $u''(w) < 0$ for every level of wealth. Thus the second derivative of expected utility is unambiguously negative. Expected utility will be a concave function of x.

Consider the change in expected utility for the first dollar invested in the risky asset. This is just equation (12.3) with the derivative evaluated at $x = 0$:

$$\begin{aligned} EU'(0) &= \pi u'(w) r_g + (1-\pi) u'(w) r_b \\ &= u'(w)[\pi r_g + (1-\pi) r_b]. \end{aligned}$$

The expression inside the brackets is the **expected return** on the asset. If the expected return on the asset is negative, then expected utility must decrease when the first dollar is invested in the asset. But since the second derivative of expected utility is negative due to concavity, then utility must continue to decrease as additional dollars are invested.

Hence we have found that if the *expected value* of a gamble is negative, a risk averter will have the highest *expected utility* at $x^* = 0$: he will want no part of a losing proposition.

On the other hand, if the expected return on the asset is positive, then increasing x from zero will increase expected utility. Thus he will always want to invest a little bit in the risky asset, no matter how risk averse he is.

Expected utility as a function of x is illustrated in Figure 12.4. In Figure 12.4A the expected return is negative, and the optimal choice is $x^* = 0$. In Figure 12.4B the expected return is positive over some range, so the consumer wants to invest some positive amount x^* in the risky asset.

Figure 12.4 **How much to invest in the risky asset**. In panel A, the optimal investment is zero, but in panel B the consumer wants to invest a positive amount.

The optimal amount for the consumer to invest will be determined by the condition that the derivative of expected utility with respect to x be equal to zero. Since the second derivative of utility is automatically negative due to concavity, this will be a global maximum.

Setting (12.3) equal to zero we have

$$EU'(x) = \pi u'(w + x r_g) r_g + (1-\pi) u'(w + x r_b) r_b = 0. \tag{12.5}$$

This equation determines the optimal choice of x for the consumer in question.

EXAMPLE: The Effect of Taxation on Investment in Risky Assets

How does the level of investment in a risky asset behave when you tax its return? If the individual pays taxes at rate t, then the after-tax returns will be $(1-t)r_g$ and $(1-t)r_b$. Thus the first-order condition determining his optimal investment, x, will be

$$EU'(x) = \pi u'(w + x(1-t)r_g)(1-t)r_g + (1-\pi)u'(w + x(1-t)r_b)(1-t)r_b = 0.$$

Canceling the $(1-t)$ terms, we have

$$EU'(x) = \pi u'(w + x(1-t)r_g)r_g + (1-\pi)u'(w + x(1-t)r_b)r_b = 0. \qquad (12.6)$$

Let us denote the solution to the maximization problem without taxes—when $t = 0$—by x^* and denote the solution to the maximization problem with taxes by $\hat{x}$. What is the relationship between x^* and $\hat{x}$?

Your first impulse is probably to think that $x^* > \hat{x}$—that taxation of a risky asset will tend to discourage investment in it. But that turns out to be exactly wrong! Taxing a risky asset in the way we described will actually *encourage* investment in it!

In fact, there is an exact relation between x^* and $\hat{x}$. It must be the case that

$$\hat{x} = \frac{x^*}{1-t}.$$

The proof is simply to note that this value of $\hat{x}$ satisfies the first-order condition for the optimal choice in the presence of the tax. Substituting this choice into equation (12.6) we have

$$\begin{aligned} EU'(\hat{x}) &= \pi u'(w + \frac{x^*}{1-t}(1-t)r_g)r_g \\ &\quad + (1-\pi)u'(w + \frac{x^*}{1-t}(1-t)r_b)r_b \\ &= \pi u'(w + x^* r_g)r_g + (1-\pi)u'(w + x^* r_b)r_b = 0, \end{aligned}$$

where the last equality follows from the fact that x^* is the optimal solution when there is no tax.

What is going on here? How can imposing a tax increase the amount of investment in the risky asset? Here is what is happening. When the tax is imposed, the individual will have less of a gain in the good state, but he will also have *less of a loss in the bad state.* By scaling his original investment up by $1/(1-t)$ the consumer can reproduce the same *after-tax* returns that he had before the tax was put in place. The tax reduces his expected return, but it also reduces his risk: by increasing his investment the consumer can get exactly the same pattern of returns he had before and thus completely offset the effect of the tax. A tax on a risky investment represents a tax on the gain when the return is positive—but it represents a subsidy on the loss when the return is negative.

CHAPTER 13

RISKY ASSETS

In the last chapter we examined a model of individual behavior under uncertainty and the role of two economic institutions for dealing with uncertainty: insurance markets and stock markets. In this chapter we will further explore how stock markets serve to allocate risk. In order to do this, it is convenient to consider a simplified model of behavior under uncertainty.

13.1 Mean-Variance Utility

In the last chapter we examined the expected utility model of choice under uncertainty. Another approach to choice under uncertainty is to describe the probability distributions that are the objects of choice by a few parameters and think of the utility function as being defined over those parameters. The most popular example of this approach is the **mean-variance model**. Instead of thinking that a consumer's preferences depend on the entire probability distribution of his wealth over every possible outcome, we suppose that his preferences can be well described by considering just a few summary statistics about the probability distribution of his wealth.

Let us suppose that a random variable w takes on the values w_s for $s = 1, \ldots, S$ with probability π_s. The **mean** of a probability distribution is simply its average value:

$$\mu_w = \sum_{s=1}^{S} \pi_s w_s.$$

This is the formula for an average: take each outcome w_s, weight it by the probability that it occurs, and sum it up over all outcomes.[1]

The **variance** of a probability distribution is the average value of $(w - \mu_w)^2$:

$$\sigma_w^2 = \sum_{s=1}^{S} \pi_s (w_s - \mu_w)^2.$$

The variance measures the "spread" of the distribution and is a reasonable measure of the riskiness involved. A closely related measure is the **standard deviation**, denoted by σ_w, which is the square root of the variance: $\sigma_w = \sqrt{\sigma_w^2}$.

The mean of a probability distribution measures its average value—what the distribution is centered around. The variance of the distribution measures the "spread" of the distribution—how spread out it is around the mean. See Figure 13.1 for a graphical depiction of probability distributions with different means and variances.

The mean-variance model assumes that the utility of a probability distribution that gives the investor wealth w_s with a probability of π_s can be expressed as a function of the mean and variance of that distribution, $u(\mu_w, \sigma_w^2)$. Or, if it is more convenient, the utility can be expressed as a function of the mean and standard deviation, $u(\mu_w, \sigma_w)$. Since both variance and standard deviation are measures of the riskiness of the wealth distribution, we can think of utility as depending on either one.

This model can be thought of as a simplification of the expected utility model described in the preceding chapter. If the choices that are being made can be completely characterized in terms of their mean and variance, then a utility function for mean and variance will be able to rank choices in the same way that an expected utility function will rank them. Furthermore, even if the probability distributions cannot be completely characterized by their means and variances, the mean-variance model may well serve as a reasonable approximation to the expected utility model.

We will make the natural assumption that a higher expected return is good, other things being equal, and that a higher variance is bad. This is simply another way to state the assumption that people are typically averse to risk.

[1] The Greek letter μ, mu, is pronounced "mew." The Greek letter σ, sigma, is pronounced "sig-ma."

Figure 13.1 **Mean and variance**. The probability distribution depicted in panel A has a positive mean, while that depicted in panel B has a negative mean. The distribution in panel A is more "spread out" than the one in panel B, which means that it has a larger variance.

Let us use the mean-variance model to analyze a simple portfolio problem. Suppose that you can invest in two different assets. One of them, the **risk-free asset**, always pays a fixed rate of return, r_f. This would be something like a Treasury bill that pays a fixed rate of interest regardless of what happens.

The other asset is a **risky asset**. Think of this asset as being an investment in a large mutual fund that buys stocks. If the stock market does well, then your investment will do well. If the stock market does poorly, your investment will do poorly. Let m_s be the return on this asset if state s occurs, and let π_s be the probability that state s will occur. We'll use r_m to denote the expected return of the risky asset and σ_m to denote the standard deviation of its return.

Of course you don't have to choose one or the other of these assets; typically you'll be able to divide your wealth between the two. If you hold a fraction of your wealth x in the risky asset, and a fraction $(1-x)$ in the risk-free asset, the expected return on your portfolio will be given by

$$\begin{aligned} r_x &= \sum_{s=1}^{S} (x m_s + (1-x) r_f) \pi_s \\ &= x \sum_{s=1}^{S} m_s \pi_s + (1-x) r_f \sum_{s=1}^{S} \pi_s. \end{aligned}$$

Since $\sum \pi_s = 1$, we have

$$r_x = x r_m + (1-x) r_f.$$

Risk and return. The budget line measures the cost of achieving a larger expected return in terms of the increased standard deviation of the return. At the optimal choice the indifference curve must be tangent to this budget line.

Figure 13.2

Thus the expected return on the portfolio is a weighted average of the two expected returns.

The variance of your portfolio return will be given by

$$\sigma_x^2 = \sum_{s=1}^{S} (xm_s + (1-x)r_f - r_x)^2 \pi_s.$$

Substituting for r_x, this becomes

$$\begin{aligned}\sigma_x^2 &= \sum_{s=1}^{S} (xm_s - xr_m)^2 \pi_s \\ &= \sum_{s=1}^{S} x^2 (m_s - r_m)^2 \pi_s \\ &= x^2 \sigma_m^2.\end{aligned}$$

Thus the standard deviation of the portfolio return is given by

$$\sigma_x = \sqrt{x^2 \sigma_m^2} = x\sigma_m.$$

It is natural to assume that $r_m > r_f$, since a risk-averse investor would never hold the risky asset if it had a lower expected return than the risk-free asset. It follows that if you choose to devote a higher fraction of your wealth to the risky asset, you will get a higher expected return, but you will also incur higher risk. This is depicted in Figure 13.2.

If you set $x = 1$ you will put all of your money in the risky asset and you will have an expected return and standard deviation of (r_m, σ_m). If you set $x = 0$ you will put all of your wealth in the sure asset and you have an expected return and standard deviation of $(r_f, 0)$. If you set x somewhere between 0 and 1, you will end up somewhere in the middle of the line connecting these two points. This line gives us a budget line describing the market tradeoff between risk and return.

Since we are assuming that people's preferences depend only on the mean and variance of their wealth, we can draw indifference curves that illustrate an individual's preferences for risk and return. If people are risk averse, then a higher expected return makes them better off and a higher standard deviation makes them worse off. This means that standard deviation is a "bad." It follows that the indifference curves will have a positive slope, as shown in Figure 13.2.

At the optimal choice of risk and return the slope of the indifference curve has to equal the slope of the budget line in Figure 13.2. We might call this slope the **price of risk** since it measures how risk and return can be traded off in making portfolio choices. From inspection of Figure 13.2 the price of risk is given by

$$p = \frac{r_m - r_f}{\sigma_m}. \tag{13.1}$$

So our optimal portfolio choice between the sure and the risky asset could be characterized by saying that the marginal rate of substitution between risk and return must be equal to the price of risk:

$$\text{MRS} = -\frac{\Delta U/\Delta\sigma}{\Delta U/\Delta\mu} = \frac{r_m - r_f}{\sigma_m}. \tag{13.2}$$

Now suppose that there are many individuals who are choosing between these two assets. Each one of them has to have his marginal rate of substitution equal to the price of risk. Thus in equilibrium all of the individuals' MRSs will be equal: when people are given sufficient opportunities to trade risks, the equilibrium price of risk will be equal across individuals. Risk is like any other good in this respect.

We can use the ideas that we have developed in earlier chapters to examine how choices change as the parameters of the problem change. All of the framework of normal goods, inferior goods, revealed preference, and so on can be brought to bear on this model. For example, suppose that an individual is offered a choice of a new risky asset y that has a mean return of r_y, say, and a standard deviation of σ_y, as illustrated in Figure 13.3.

If offered the choice between investing in x and investing in y, which will the consumer choose? The original budget set and the new budget set are both depicted in Figure 13.3. Note that every choice of risk and return that was possible in the original budget set is possible with the new budget

Preferences between risk and return. The asset with risk-return combination y is preferred to the one with combination x.

Figure 13.3

set since the new budget set contains the old one. Thus investing in the asset y and the risk-free asset is definitely better than investing in x and the risk-free asset, since the consumer can choose a better final portfolio.

The fact that the consumer can choose how much of the risky asset he wants to hold is very important for this argument. If this were an "all or nothing" choice where the consumer was compelled to invest all of his money in either x or y, we would get a very different outcome. In the example depicted in Figure 13.3, the consumer would prefer investing all of his money in x to investing all of his money in y, since x lies on a higher indifference curve than y. But if he can mix the risky asset with the risk-free asset, he would always prefer to mix with y rather than to mix with x.

13.2 Measuring Risk

We have a model above that describes the price of risk . . . but how do we measure the *amount* of risk in an asset? The first thing that you would probably think of is the standard deviation of an asset's return. After all, we are assuming that utility depends on the mean and variance of wealth, aren't we?

In the above example, where there is only one risky asset, that is exactly right: the amount of risk in the risky asset is its standard deviation. But if

there are many risky assets, the standard deviation is not an appropriate measure for the amount of risk in an asset.

This is because a consumer's utility depends on the mean and variance of total wealth—not the mean and variance of any single asset that he might hold. What matters is how the returns of the various assets a consumer holds *interact* to create a mean and variance of his wealth. As in the rest of economics, it is the marginal impact of a given asset on total utility that determines its value, not the value of that asset held alone. Just as the value of an extra cup of coffee may depend on how much cream is available, the amount that someone would be willing to pay for an extra share of a risky asset will depend on how it interacts with other assets in his portfolio.

Suppose, for example, that you are considering purchasing two assets, and you know that there are only two possible outcomes that can happen. Asset A will be worth either \$10 or −\$5, and asset B will be worth either −\$5 or \$10. But when asset A is worth \$10, asset B will be worth −\$5 and vice versa. In other words the values of the two assets will be *negatively correlated:* when one has a large value, the other will have a small value.

Suppose that the two outcomes are equally likely, so that the average value of each asset will be \$2.50. Then if you don't care about risk at all and you must hold one asset or the other, the most that you would be willing to pay for either one would be \$2.50—the expected value of each asset. If you are averse to risk, you would be willing to pay even less than \$2.50.

But what if you can hold both assets? Then if you hold one share of each asset, you will get \$5 whichever outcome arises. Whenever one asset is worth \$10, the other is worth −\$5. Thus, if you can hold both assets, the amount that you would be willing to pay to purchase *both* assets would be \$5.

This example shows in a vivid way that the value of an asset will depend in general on how it is correlated with other assets. Assets that move in opposite directions—that are negatively correlated with each other—are very valuable because they reduce overall risk. In general the value of an asset tends to depend much more on the correlation of its return with other assets than with its own variation. Thus the amount of risk in an asset depends on its correlation with other assets.

It is convenient to measure the risk in an asset relative to the risk in the stock market as a whole. We call the riskiness of a stock relative to the risk of the market the **beta** of a stock, and denote it by the Greek letter β. Thus, if i represents some particular stock, we write β_i for its riskiness relative to the market as a whole. Roughly speaking:

$$\beta_i = \frac{\text{how risky asset } i \text{ is}}{\text{how risky the stock market is}}.$$

If a stock has a beta of 1, then it is just as risky as the market as a whole;

when the market moves up by 10 percent, this stock will, on the average, move up by 10 percent. If a stock has a beta of less than 1, then when the market moves up by 10 percent, the stock will move up by less than 10 percent. The beta of a stock can be estimated by statistical methods to determine how sensitive the movements of one variable are relative to another, and there are many investment advisory services that can provide you with estimates of the beta of a stock.[2]

13.3 Counterparty Risk

Financial institutions loan money not just to individuals but to each other. There is always the chance that one party to a loan may fail to repay the loan, a risk known as **counterparty risk**.

To see how this works, imagine 3 banks, A, B, and C. Bank A owes B a billion dollars, Bank B owes C a billion dollars, and Bank C owes bank A a billion dollars. Now suppose that Bank A runs out of money and defaults on its loan. Bank B is now out a billion dollars and may not be able to pay C. Bank C, in turn, can't pay A, pushing A even further in the hole. This sort of effect is known as **financial contagion** or **systemic risk**. It is a very simplified version of what happened to U.S. financial institutions in the Fall of 2008.

What's the solution? One way to deal with this sort of problem is to have a "lender of last resort," which is typically a central bank, such as the U.S. Federal Reserve System. Bank A can go to the Federal Reserve and request an emergency loan of a billion dollars. It now pays off its loan from Bank B, which in turn pays Bank C, which in turn pays back Bank A. Bank A now has sufficient assets to pay back the loan from the central bank.

This is, of course, an overly simplified example. Initially, there was no net debt among the three banks. If they had gotten together to compare assets and liabilities, they would have certainly discovered that fact. However, when assets and liabilities span thousands of financial institutions, it may be difficult to determine net positions, which is why lenders of last resort may be necessary.

13.4 Equilibrium in a Market for Risky Assets

We are now in a position to state the equilibrium condition for a market with risky assets. Recall that in a market with only certain returns, we

[2] The Greek letter β, beta, is pronounced "bait-uh." For those of you who know some statistics, the beta of a stock is defined to be $\beta_i = \text{cov}(\tilde{r}_i, \tilde{r}_m)/\text{var}(\tilde{r}_m)$. That is, β_i is the covariance of the return on the stock with the market return divided by the variance of the market return.

saw that all assets had to earn the same rate of return. Here we have a similar principle: all assets, after adjusting for risk, have to earn the same rate of return.

The catch is about adjusting for risk. How do we do that? The answer comes from the analysis of optimal choice given earlier. Recall that we considered the choice of an optimal portfolio that contained a riskless asset and a risky asset. The risky asset was interpreted as being a mutual fund—a diversified portfolio including many risky assets. In this section we'll suppose that this portfolio consists of *all* risky assets.

Then we can identify the expected return on this market portfolio of risky assets with the market expected return, r_m, and identify the standard deviation of the market return with the market risk, σ_m. The return on the safe asset is r_f, the risk-free return.

We saw in equation (13.1) that the price of risk, p, is given by

$$p = \frac{r_m - r_f}{\sigma_m}.$$

We said above that the amount of risk in a given asset i relative to the total risk in the market is denoted by β_i. This means that to measure the *total* amount of risk in asset i, we have to multiply by the market risk, σ_m. Thus the total risk in asset i is given by $\beta_i \sigma_m$.

What is the cost of this risk? Just multiply the total amount of risk, $\beta_i \sigma_m$, by the price of risk. This gives us the *risk adjustment:*

$$\begin{aligned} \text{risk adjustment} &= \beta_i \sigma_m p \\ &= \beta_i \sigma_m \frac{r_m - r_f}{\sigma_m} \\ &= \beta_i (r_m - r_f). \end{aligned}$$

Now we can state the equilibrium condition in markets for risky assets: in equilibrium all assets should have the same risk-adjusted rate of return. The logic is just like the logic used in Chapter 12: if one asset had a higher risk-adjusted rate of return than another, everyone would want to hold the asset with the higher risk-adjusted rate. Thus in equilibrium the risk-adjusted rates of return must be equalized.

If there are two assets i and j that have expected returns r_i and r_j and betas of β_i and β_j, we must have the following equation satisfied in equilibrium:

$$r_i - \beta_i (r_m - r_f) = r_j - \beta_j (r_m - r_f).$$

This equation says that in equilibrium the risk-adjusted returns on the two assets must be the same—where the risk adjustment comes from multiplying the total risk of the asset by the price of risk.

Another way to express this condition is to note the following. The risk-free asset, by definition, must have $\beta_f = 0$. This is because it has zero risk,

The market line. The market line depicts the combinations of expected return and beta for assets held in equilibrium.

Figure 13.4

and β measures the amount of risk in an asset. Thus for any asset i we must have

$$r_i - \beta_i(r_m - r_f) = r_f - \beta_f(r_m - r_f) = r_f.$$

Rearranging, this equation says

$$r_i = r_f + \beta_i(r_m - r_f)$$

or that the expected return on any asset must be the risk-free return plus the risk adjustment. This latter term reflects the extra return that people demand in order to bear the risk that the asset embodies. This equation is the main result of the **Capital Asset Pricing Model (CAPM)**, which has many uses in the study of financial markets.

13.5 How Returns Adjust

In studying asset markets under certainty, we showed how prices of assets adjust to equalize returns. Let's look at the same adjustment process here.

According to the model sketched out above, the expected return on any asset should be the risk-free return plus the risk premium:

$$r_i = r_f + \beta_i(r_m - r_f).$$

In Figure 13.4 we have illustrated this line in a graph with the different values of beta plotted along the horizontal axis and different expected returns on the vertical axis. According to our model, all assets that are held in equilibrium have to lie along this line. This line is called the **market line**.

What if some asset's expected return and beta didn't lie on the market line? What would happen?

The expected return on the asset is the expected change in its price divided by its current price:

$$r_i = \text{ expected value of } \frac{p_1 - p_0}{p_0}.$$

This is just like the definition we had before, with the addition of the word "expected." We have to include "expected" now since the price of the asset tomorrow is uncertain.

Suppose that you found an asset whose expected return, adjusted for risk, was higher than the risk-free rate:

$$r_i - \beta_i(r_m - r_f) > r_f.$$

Then this asset is a very good deal. It is giving a higher risk-adjusted return than the risk-free rate.

When people discover that this asset exists, they will want to buy it. They might want to keep it for themselves, or they might want to buy it and sell it to others, but since it is offering a better tradeoff between risk and return than existing assets, there is certainly a market for it.

But as people attempt to buy this asset they will bid up today's price: p_0 will rise. This means that the expected return $r_i = (p_1 - p_0)/p_0$ will fall. How far will it fall? Just enough to lower the expected rate of return back down to the market line.

Thus it is a good deal to buy an asset that lies above the market line. For when people discover that it has a higher return given its risk than assets they currently hold, they will bid up the price of that asset.

This is all dependent on the hypothesis that people agree about the amount of risk in various assets. If they disagree about the expected returns or the betas of different assets, the model becomes much more complicated.

EXAMPLE: Value at Risk

It is sometimes of interest to determine the risk of a certain set of assets. For example, suppose that a bank holds a particular portfolio of stocks. It may want to estimate the probability that the portfolio will fall by more than a million dollars on a given day. If this probability is 5% then we say that the portfolio has a "one-day 5% **value at risk** of $1 million." Typically value at risk is computed for 1 day or 2 week periods, using loss probabilities of 1% or 5%.

The theoretical idea of VaR is attractive. All the challenges lie in figuring out ways to estimate it. But, as financial analyst Philippe Jorion has put it, "[T]he greatest benefit of VaR lies in the imposition of a structured

methodology for critically thinking about risk. Institutions that go through the process of computing their VaR are forced to confront their exposure to financial risks and to set up a proper risk management function. Thus the process of getting to VaR may be as important as the number itself."

The VaR is determined entirely by the probability distribution of the value of the portfolio, and this depends on the correlation of the assets in the portfolio. Typically, assets are positively correlated, so they all move up or down at once. Even worse, the distribution of asset prices tends to have "fat tails" so that there may be a relatively high probability of an extreme price movement. Ideally, one would estimate VaR using a long history of price movements. In practice, this is difficult to do, particularly for new and exotic assets.

In the Fall of 2008 many financial institutions discovered that their VaR estimates were severely flawed since asset prices dropped much more than was anticipated. In part this was due to the fact that statistical estimates were based on very small samples that were gathered during a stable period of economic activity. The estimated values at risk understated the true risk of the assets in question.

EXAMPLE: Ranking Mutual Funds

The Capital Asset Pricing Model can be used to compare different investments with respect to their risk and their return. One popular kind of investment is a mutual fund. These are large organizations that accept money from individual investors and use this money to buy and sell stocks of companies. The profits made by such investments are then paid out to the individual investors.

The advantage of a mutual fund is that you have professionals managing your money. The disadvantage is they charge you for managing it. These fees are usually not terribly large, however, and most small investors are probably well advised to use a mutual fund.

But how do you choose a mutual fund in which to invest? You want one with a high expected return of course, but you also probably want one with a minimum amount of risk. The question is, how much risk are you willing to tolerate to get that high expected return?

One thing that you might do is to look at the historical performance of various mutual funds and calculate the average yearly return and the beta—the amount of risk—of each mutual fund you are considering. Since we haven't discussed the precise definition of beta, you might find it hard to calculate. But there are books where you can look up the historical betas of mutual funds.

If you plotted the expected returns versus the betas, you would get a

diagram similar to that depicted in Figure 13.5.[3] Note that the mutual funds with high expected returns will generally have high risk. The high expected returns are there to compensate people for bearing risk.

One interesting thing you can do with the mutual fund diagram is to compare investing with professional managers to a very simple strategy like investing part of your money in an **index fund**. There are several indices of stock market activity like the Dow-Jones Industrial Average, or the Standard and Poor's Index, and so on. The indices are typically the average returns on a given day of a certain group of stocks. The Standard and Poor's Index, for example, is based on the average performance of 500 large stocks in the United States.

Figure 13.5 **Mutual funds**. Comparing the returns on mutual fund investment to the market line.

An **index fund** is a mutual fund that holds the stocks that make up such an index. This means that you are guaranteed to get the average performance of the stocks in the index, virtually by definition. Since holding the average is not a very difficult thing to do—at least compared to trying to beat the average—index funds typically have low management fees. Since an index fund holds a very broad base of risky assets, it will have a beta

[3] See Michael Jensen, "The Performance of Mutual Funds in the Period 1945–1964," *Journal of Finance*, 23 (May 1968), 389–416, for a more detailed discussion of how to examine mutual fund performance using the tools we have sketched out in this chapter. Mark Grinblatt and Sheridan Titman have examined more recent data in "Mutual Fund Performance: An Analysis of Quarterly Portfolio Holdings," *The Journal of Business*, 62 (July 1989), 393–416.

that is very close to 1—it will be just as risky as the market as a whole, because the index fund holds nearly all the stocks in the market as a whole.

How does an index fund do as compared to the typical mutual fund? Remember the comparison has to be made with respect to both risk and return of the investment. One way to do this is to plot the expected return and beta of a Standard and Poor's Index fund, and draw the line connecting it to the risk-free rate, as in Figure 13.5. You can get any combination of risk and return on this line that you want just by deciding how much money you want to invest in the risk-free asset and how much you want to invest in the index fund.

Now let's count the number of mutual funds that plot below this line. These are mutual funds that offer risk and return combinations that are dominated by those available by the index fund/risk-free asset combinations. When this is done, it turns out that the vast majority of the risk-return combinations offered by mutual funds are below the line. The number of funds that plot above the line is no more than could be expected by chance alone.

But seen another way, this finding might not be too surprising. The stock market is an incredibly competitive environment. People are always trying to find undervalued stocks in order to purchase them. This means that on average, stocks are usually trading for what they're really worth. If that is the case, then betting the averages is a pretty reasonable strategy—since beating the averages is almost impossible.

Summary

1. We can use the budget set and indifference curve apparatus developed earlier to examine the choice of how much money to invest in risky and riskless assets.

2. The marginal rate of substitution between risk and return will have to equal the slope of the budget line. This slope is known as the price of risk.

3. The amount of risk present in an asset depends to a large extent on its correlation with other assets. An asset that moves opposite the direction of other assets helps to reduce the overall risk of your portfolio.

4. The amount of risk in an asset relative to that of the market as a whole is called the **beta** of the asset.

5. The fundamental equilibrium condition in asset markets is that risk-adjusted returns have to be the same.

6. Counterparty risk, which is the risk that the other side of a transaction will not pay, can also be an important risk factor.

REVIEW QUESTIONS

1. If the risk-free rate of return is 6%, and if a risky asset is available with a return of 9% and a standard deviation of 3%, what is the maximum rate of return you can achieve if you are willing to accept a standard deviation of 2%? What percentage of your wealth would have to be invested in the risky asset?

2. What is the price of risk in the above exercise?

3. If a stock has a β of 1.5, the return on the market is 10%, and the risk-free rate of return is 5%, what expected rate of return should this stock offer according to the Capital Asset Pricing Model? If the expected value of the stock is \$100, what price should the stock be selling for today?

CHAPTER 14

CONSUMER'S SURPLUS

In the preceding chapters we have seen how to derive a consumer's demand function from the underlying preferences or utility function. But in practice we are usually concerned with the reverse problem—how to estimate preferences or utility from observed demand behavior.

We have already examined this problem in two other contexts. In Chapter 5 we showed how one could estimate the parameters of a utility function from observing demand behavior. In the Cobb-Douglas example used in that chapter, we were able to estimate a utility function that described the observed choice behavior simply by calculating the average expenditure share of each good. The resulting utility function could then be used to evaluate changes in consumption.

In Chapter 7 we described how to use revealed preference analysis to recover estimates of the underlying preferences that may have generated some observed choices. These estimated indifference curves can also be used to evaluate changes in consumption.

In this chapter we will consider some more approaches to the problem of estimating utility from observing demand behavior. Although some of the methods we will examine are less general than the two methods we

examined previously, they will turn out to be useful in several applications that we will discuss later in the book.

We will start by reviewing a special case of demand behavior for which it is very easy to recover an estimate of utility. Later we will consider more general cases of preferences and demand behavior.

14.1 Demand for a Discrete Good

Let us start by reviewing demand for a discrete good with quasilinear utility, as described in Chapter 6. Suppose that the utility function takes the form $v(x) + y$ and that the x-good is only available in integer amounts. Let us think of the y-good as money to be spent on other goods and set its price to 1. Let p be the price of the x-good.

We saw in Chapter 6 that in this case consumer behavior can be described in terms of the reservation prices, $r_1 = v(1) - v(0)$, $r_2 = v(2) - v(1)$, and so on. The relationship between reservation prices and demand was very simple: if n units of the discrete good are demanded, then $r_n \geq p \geq r_{n+1}$.

To verify this, let's look at an example. Suppose that the consumer chooses to consume 6 units of the x-good when its price is p. Then the utility of consuming $(6, m - 6p)$ must be at least as large as the utility of consuming any other bundle $(x, m - px)$:

$$v(6) + m - 6p \geq v(x) + m - px. \tag{14.1}$$

In particular this inequality must hold for $x = 5$, which gives us

$$v(6) + m - 6p \geq v(5) + m - 5p.$$

Rearranging, we have $v(6) - v(5) = r_6 \geq p$.

Equation (14.1) must also hold for $x = 7$. This gives us

$$v(6) + m - 6p \geq v(7) + m - 7p,$$

which can be rearranged to yield

$$p \geq v(7) - v(6) = r_7.$$

This argument shows that if 6 units of the x-good is demanded, then the price of the x-good must lie between r_6 and r_7. In general, if n units of the x-good are demanded at price p, then $r_n \geq p \geq r_{n+1}$, as we wanted to show. The list of reservation prices contains all the information necessary to describe the demand behavior. The graph of the reservation prices forms a "staircase" as shown in Figure 14.1. This staircase is precisely the demand curve for the discrete good.

14.2 Constructing Utility from Demand

We have just seen how to construct the demand curve given the reservation prices or the utility function. But we can also do the same operation in reverse. If we are given the demand curve, we can construct the utility function—at least in the special case of quasilinear utility.

At one level, this is just a trivial operation of arithmetic. The reservation prices are defined to be the difference in utility:

$$\begin{aligned} r_1 &= v(1) - v(0) \\ r_2 &= v(2) - v(1) \\ r_3 &= v(3) - v(2) \\ &\vdots \end{aligned}$$

If we want to calculate $v(3)$, for example, we simply add up both sides of this list of equations to find

$$r_1 + r_2 + r_3 = v(3) - v(0).$$

It is convenient to set the utility from consuming zero units of the good equal to zero, so that $v(0) = 0$, and therefore $v(n)$ is just the sum of the first n reservation prices.

This construction has a nice geometrical interpretation that is illustrated in Figure 14.1A. The utility from consuming n units of the discrete good is just the area of the first n bars which make up the demand function. This is true because the height of each bar is the reservation price associated with that level of demand and the width of each bar is 1. This area is sometimes called the **gross benefit** or the **gross consumer's surplus** associated with the consumption of the good.

Note that this is only the utility associated with the consumption of good 1. The final utility of consumption depends on the how much the consumer consumes of good 1 *and* good 2. If the consumer chooses n units of the discrete good, then he will have $m - pn$ dollars left over to purchase other things. This leaves him with a total utility of

$$v(n) + m - pn.$$

This utility also has an interpretation as an area: we just take the area depicted in Figure 14.1A, subtract off the expenditure on the discrete good, and add m.

The term $v(n) - pn$ is called **consumer's surplus** or the **net consumer's surplus**. It measures the net benefits from consuming n units of the discrete good: the utility $v(n)$ minus the reduction in the expenditure on consumption of the other good. The consumer's surplus is depicted in Figure 14.1B.

Figure 14.1

Reservation prices and consumer's surplus. The gross benefit in panel A is the area under the demand curve. This measures the utility from consuming the x-good. The consumer's surplus is depicted in panel B. It measures the utility from consuming both goods when the first good has to be purchased at a constant price p.

14.3 Other Interpretations of Consumer's Surplus

There are some other ways to think about consumer's surplus. Suppose that the price of the discrete good is p. Then the value that the consumer places on the first unit of consumption of that good is r_1, but he only has to pay p for it. This gives him a "surplus" of $r_1 - p$ on the first unit of consumption. He values the second unit of consumption at r_2, but again he only has to pay p for it. This gives him a surplus of $r_2 - p$ on that unit. If we add this up over all n units the consumer chooses, we get his total consumer's surplus:

$$CS = r_1 - p + r_2 - p + \cdots + r_n - p = r_1 + \cdots + r_n - np.$$

Since the sum of the reservation prices just gives us the utility of consumption of good 1, we can also write this as

$$CS = v(n) - pn.$$

We can interpret consumer's surplus in yet another way. Suppose that a consumer is consuming n units of the discrete good and paying pn dollars

to do so. How much money would he need to induce him to give up his entire consumption of this good? Let R be the required amount of money. Then R must satisfy the equation

$$v(0) + m + R = v(n) + m - pn.$$

Since $v(0) = 0$ by definition, this equation reduces to

$$R = v(n) - pn,$$

which is just consumer's surplus. Hence the consumer's surplus measures how much a consumer would need to be paid to give up his entire consumption of some good.

14.4 From Consumer's Surplus to Consumers' Surplus

Up until now we have been considering the case of a single consumer. If several consumers are involved we can add up each consumer's surplus across all the consumers to create an aggregate measure of the **consumers' surplus**. Note carefully the distinction between the two concepts: consumer's surplus refers to the surplus of a single consumer; consumers' surplus refers to the sum of the surpluses across a number of consumers.

Consumers' surplus serves as a convenient measure of the aggregate gains from trade, just as consumer's surplus serves as a measure of the individual gains from trade.

14.5 Approximating a Continuous Demand

We have seen that the area underneath the demand curve for a discrete good measures the utility of consumption of that good. We can extend this to the case of a good available in continuous quantities by approximating the continuous demand curve by a staircase demand curve. The area under the continuous demand curve is then approximately equal to the area under the staircase demand.

See Figure 14.2 for an example. In the Appendix to this chapter we show how to use calculus to calculate the exact area under a demand curve.

14.6 Quasilinear Utility

It is worth thinking about the role that quasilinear utility plays in this analysis. In general the price at which a consumer is willing to purchase

Figure 14.2

Approximating a continuous demand. The consumer's surplus associated with a continuous demand curve can be approximated by the consumer's surplus associated with a discrete approximation to it.

some amount of good 1 will depend on how much money he has for consuming other goods. This means that in general the reservation prices for good 1 will depend on how much good 2 is being consumed.

But in the special case of quasilinear utility the reservation prices are independent of the amount of money the consumer has to spend on other goods. Economists say that with quasilinear utility there is "no income effect" since changes in income don't affect demand. This is what allows us to calculate utility in such a simple way. Using the area under the demand curve to measure utility will only be *exactly* correct when the utility function is quasilinear.

But it may often be a good approximation. If the demand for a good doesn't change very much when income changes, then the income effects won't matter very much, and the change in consumer's surplus will be a reasonable approximation to the change in the consumer's utility.[1]

14.7 Interpreting the Change in Consumer's Surplus

We are usually not terribly interested in the absolute level of consumer's surplus. We are generally more interested in the change in consumer's

[1] Of course, the change in consumer's surplus is only one way to represent a change in utility—the change in the square root of consumer's surplus would be just as good. But it is standard to use consumer's surplus as a standard measure of utility.

surplus that results from some policy change. For example, suppose the price of a good changes from p' to p''. How does the consumer's surplus change?

In Figure 14.3 we have illustrated the change in consumer's surplus associated with a change in price. The change in consumer's surplus is the difference between two roughly triangular regions and will therefore have a roughly trapezoidal shape. The trapezoid is further composed of two subregions, the rectangle indicated by R and the roughly triangular region indicated by T.

Change in consumer's surplus. The change in consumer's surplus is sum of the square area R and the roughly triangular area T, and thus has a roughly trapezoidal shape.

Figure 14.3

The rectangle measures the loss in surplus due to the fact that the consumer is now paying more for all the units he continues to consume. After the price increases the consumer continues to consume x'' units of the good, and each unit of the good is now more expensive by $p'' - p'$. This means he has to spend $(p'' - p')x''$ more money than he did before just to consume x'' units of the good.

But this is not the entire welfare loss. Due to the increase in the price of the x-good, the consumer has decided to consume less of it than he was before. The triangle T measures the value of the *lost* consumption of the x-good. The total loss to the consumer is the sum of these two effects: R measures the loss from having to pay more for the units he continues to consume, and T measures the loss from the reduced consumption.

EXAMPLE: The Change in Consumer's Surplus

Question: Consider the linear demand curve $D(p) = 20 - 2p$. When the price changes from 2 to 3 what is the associated change in consumer's surplus?

Answer: When $p = 2$, $D(2) = 16$, and when $p = 3$, $D(3) = 14$. Thus we want to compute the area of a trapezoid with a height of 1 and bases of 14 and 16. This is equivalent to a rectangle with height 1 and base 14 (having an area of 14), plus a triangle of height 1 and base 2 (having an area of 1). The total area will therefore be 15.

14.8 Compensating and Equivalent Variation

The theory of consumer's surplus is very tidy in the case of quasilinear utility. Even if utility is not quasilinear, consumer's surplus may still be a reasonable measure of consumer's welfare in many applications. Usually the errors in measuring demand curves outweigh the approximation errors from using consumer's surplus.

But it may be that for some applications an approximation may not be good enough. In this section we'll outline a way to measure "utility changes" without using consumer's surplus. There are really two separate issues involved. The first has to do with how to estimate utility when we can observe a number of consumer choices. The second has to do with how we can measure utility in monetary units.

We've already investigated the estimation problem. We gave an example of how to estimate a Cobb-Douglas utility function in Chapter 6. In that example we noticed that expenditure shares were relatively constant and that we could use the average expenditure share as estimates of the Cobb-Douglas parameters. If the demand behavior didn't exhibit this particular feature, we would have to choose a more complicated utility function, but the principle would be just the same: if we have enough observations on demand behavior and that behavior is consistent with maximizing something, then we will generally be able to estimate the function that is being maximized.

Once we have an estimate of the utility function that describes some observed choice behavior we can use this function to evaluate the impact of proposed changes in prices and consumption levels. At the most fundamental level of analysis, this is the best we can hope for. All that matters are the consumer's preferences; any utility function that describes the consumer's preferences is as good as any other.

However, in some applications it may be convenient to use certain monetary measures of utility. For example, we could ask how much money we

would have to give a consumer to compensate him for a change in his consumption patterns. A measure of this type essentially measures a change in utility, but it measures it in monetary units. What are convenient ways to do this?

Suppose that we consider the situation depicted in Figure 14.4. Here the consumer initially faces some prices $(p_1^*, 1)$ and consumes some bundle (x_1^*, x_2^*). The price of good 1 then increases from p_1^* to $\hat{p}_1$, and the consumer changes his consumption to $(\hat{x}_1, \hat{x}_2)$. How much does this price change hurt the consumer?

The compensating and the equivalent variations. Panel A shows the compensating variation (CV), and panel B shows the equivalent variation (EV).

Figure 14.4

One way to answer this question is to ask how much money we would have to give the consumer *after* the price change to make him just as well off as he was *before* the price change. In terms of the diagram, we ask how far up we would have to shift the new budget line to make it tangent to the indifference curve that passes through the original consumption point (x_1^*, x_2^*). The change in income necessary to restore the consumer to his original indifference curve is called the **compensating variation** in income, since it is the change in income that will just compensate the consumer for the price change. The compensating variation measures how much extra money the government would have to give the consumer if it wanted to exactly compensate the consumer for the price change.

Another way to measure the impact of a price change in monetary terms is to ask how much money would have to be taken away from the consumer

before the price change to leave him as well off as he would be *after* the price change. This is called the **equivalent variation** in income since it is the income change that is equivalent to the price change in terms of the change in utility. In Figure 14.4 we ask how far down we must shift the original budget line to just touch the indifference curve that passes through the new consumption bundle. The equivalent variation measures the maximum amount of income that the consumer would be willing to pay to avoid the price change.

In general the amount of money that the consumer would be willing to pay to avoid a price change would be different from the amount of money that the consumer would have to be paid to compensate him for a price change. After all, at different sets of prices a dollar is worth a different amount to a consumer since it will purchase different amounts of consumption.

In geometric terms, the compensating and equivalent variations are just two different ways to measure "how far apart" two indifference curves are. In each case we are measuring the distance between two indifference curves by seeing how far apart their tangent lines are. In general this measure of distance will depend on the slope of the tangent lines—that is, on the prices that we choose to determine the budget lines.

However, the compensating and equivalent variation are the same in one important case—the case of quasilinear utility. In this case the indifference curves are parallel, so the distance between any two indifference curves is the same no matter where it is measured, as depicted in Figure 14.5. In the case of quasilinear utility the compensating variation, the equivalent variation, and the change in consumer's surplus all give the same measure of the monetary value of a price change.

EXAMPLE: Compensating and Equivalent Variations

Suppose that a consumer has a utility function $u(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. He originally faces prices $(1, 1)$ and has income 100. Then the price of good 1 increases to 2. What are the compensating and equivalent variations?

We know that the demand functions for this Cobb-Douglas utility function are given by

$$x_1 = \frac{m}{2p_1}$$
$$x_2 = \frac{m}{2p_2}.$$

Using this formula, we see that the consumer's demands change from $(x_1^*, x_2^*) = (50, 50)$ to $(\hat{x}_1, \hat{x}_2) = (25, 50)$.

To calculate the compensating variation we ask how much money would be necessary at prices (2,1) to make the consumer as well off as he was consuming the bundle (50,50)? If the prices were (2,1) and the consumer

Quasilinear preferences. With quasilinear preferences, the distance between two indifference curves is independent of the slope of the budget lines.

Figure 14.5

had income m, we can substitute into the demand functions to find that the consumer would optimally choose the bundle $(m/4, m/2)$. Setting the utility of this bundle equal to the utility of the bundle $(50, 50)$ we have

$$\left(\frac{m}{4}\right)^{\frac{1}{2}}\left(\frac{m}{2}\right)^{\frac{1}{2}} = 50^{\frac{1}{2}}50^{\frac{1}{2}}.$$

Solving for m gives us

$$m = 100\sqrt{2} \approx 141.$$

Hence the consumer would need about $141 - 100 = \$41$ of additional money after the price change to make him as well off as he was before the price change.

In order to calculate the equivalent variation we ask how much money would be necessary at the prices (1,1) to make the consumer as well off as he would be consuming the bundle (25,50). Letting m stand for this amount of money and following the same logic as before,

$$\left(\frac{m}{2}\right)^{\frac{1}{2}}\left(\frac{m}{2}\right)^{\frac{1}{2}} = 25^{\frac{1}{2}}50^{\frac{1}{2}}.$$

Solving for m gives us

$$m = 50\sqrt{2} \approx 70.$$

Thus if the consumer had an income of \$70 at the original prices, he would be just as well off as he would be facing the new prices and having an income of \$100. The equivalent variation in income is therefore about $100 - 70 = \$30$.

EXAMPLE: Compensating and Equivalent Variation for Quasilinear Preferences

Suppose that the consumer has a quasilinear utility function $v(x_1) + x_2$. We know that in this case the demand for good 1 will depend only on the price of good 1, so we write it as $x_1(p_1)$. Suppose that the price changes from p_1^* to $\hat{p}_1$. What are the compensating and equivalent variations?

At the price p_1^*, the consumer chooses $x_1^* = x_1(p_1^*)$ and has a utility of $v(x_1^*) + m - p_1^* x_1^*$. At the price $\hat{p}_1$, the consumer choose $\hat{x}_1 = x_1(\hat{p}_1)$ and has a utility of $v(\hat{x}_1) + m - \hat{p}_1 \hat{x}_1$.

Let C be the compensating variation. This is the amount of extra money the consumer would need after the price change to make him as well off as he would be before the price change. Setting these utilities equal we have

$$v(\hat{x}_1) + m + C - \hat{p}_1 \hat{x}_1 = v(x_1^*) + m - p_1^* x_1^*.$$

Solving for C we have

$$C = v(x_1^*) - v(\hat{x}_1) + \hat{p}_1 \hat{x}_1 - p_1^* x_1^*.$$

Let E be the equivalent variation. This is the amount of money that you could take away from the consumer before the price change that would leave him with the same utility that he would have after the price change. Thus it satisfies the equation

$$v(x_1^*) + m - E - p_1^* x_1^* = v(\hat{x}_1) + m - \hat{p}_1 \hat{x}_1.$$

Solving for E, we have

$$E = v(x_1^*) - v(\hat{x}_1) + \hat{p}_1 \hat{x}_1 - p_1^* x_1^*.$$

Note that for the case of quasilinear utility the compensating and equivalent variation are the same. Furthermore, they are both equal to the change in (net) consumer's surplus:

$$\Delta CS = [v(x_1^*) - p_1^* x_1^*] - [v(\hat{x}_1) - \hat{p}_1 \hat{x}_1].$$

14.9 Producer's Surplus

The demand curve measures the amount that will be demanded at each price; the **supply curve** measures the amount that will be supplied at

each price. Just as the area *under* the demand curve measures the surplus enjoyed by the demanders of a good, the area *above* the supply curve measures the surplus enjoyed by the suppliers of a good.

We've referred to the area under the demand curve as consumer's surplus. By analogy, the area above the supply curve is known as **producer's surplus**. The terms consumer's surplus and producer's surplus are somewhat misleading, since who is doing the consuming and who is doing the producing really doesn't matter. It would be better to use the terms "demander's surplus" and "supplier's surplus," but we'll bow to tradition and use the standard terminology.

Suppose that we have a supply curve for a good. This simply measures the amount of a good that will be supplied at each possible price. The good could be supplied by an individual who owns the good in question, or it could be supplied by a firm that produces the good. We'll take the latter interpretation so as to stick with the traditional terminology and depict the producer's supply curve in Figure 14.6. If the producer is able to sell x^* units of her product in a market at a price p^*, what is the surplus she enjoys?

It is most convenient to conduct the analysis in terms of the producer's *inverse* supply curve, $p_s(x)$. This function measures what the price would have to be to get the producer to supply x units of the good.

Producer's surplus. The net producer's surplus is the triangular area to the left of the supply curve in panel A, and the change in producer's surplus is the trapezoidal area in panel B.

Figure 14.6

Think about the inverse supply function for a discrete good. In this case the producer is willing to sell the first unit of the good at price $p_s(1)$, but

she actually gets the market price p^* for it. Similarly, she is willing to sell the second unit for $p_s(2)$, but she gets p^* for it. Continuing in this way we see that the producer will be just willing to sell the last unit for $p_s(x^*) = p^*$.

The difference between the minimum amount she would be willing to sell the x^* units for and the amount she actually sells the units for is the **net producer's surplus**. It is the triangular area depicted in Figure 14.6A.

Just as in the case of consumer's surplus, we can ask how producer's surplus changes when the price increases from p' to p''. In general, the change in producer's surplus will be the difference between two triangular regions and will therefore generally have the roughly trapezoidal shape depicted in Figure 14.6B. As in the case of consumer's surplus, the roughly trapezoidal region will be composed of a rectangular region R and a roughly triangular region T. The rectangle measures the gain from selling the units previously sold anyway at p' at the higher price p''. The roughly triangular region measures the gain from selling the extra units at the price p''. This is analogous to the change in consumer's surplus considered earlier.

Although it is common to refer to this kind of change as an increase in producer's surplus, in a deeper sense it really represents an increase in consumer's surplus that accrues to the consumers who own the firm that generated the supply curve. Producer's surplus is closely related to the idea of profit, but we'll have to wait until we study firm behavior in more detail to spell out the relationship.

14.10 Benefit-Cost Analysis

We can use the consumer surplus apparatus we have developed to calculate the benefits and costs of various economic policies.

For example, let us examine the impact of a price ceiling. Consider the situation depicted in Figure 14.7. With no intervention, the price would be p_0 and the quantity sold would be q_0.

The authorities believe this price is too high and impose the price ceiling at p_c. This reduces the amount that suppliers are willing to supply to q_c which, in turn, reduces their producer surplus to the shaded area in the diagram.

Now that there is only q_c available for consumers, the question is who will get it?

One assumption is that the output will go to the consumers with the highest willingness to pay. Let p_e, the **effective price**, be the price that would induce consumers to demand q_e. If everyone who is willing to pay more than p_e gets the good, then the producer surplus will be the shaded area in the diagram.

Note that the lost consumer and producer surplus is given by the trapezoidal area in the middle of the diagram. This is the difference between

Figure 14.7

A price ceiling. The price ceiling at p_c reduces supply to q_e. It reduces consumer surplus to CS and producer surplus to PS. The effective price of the good, p_e, is the price that would clear the market. The diagram also shows what happens with rationing, in which case the price of a ration coupon would be $p_e - p_c$.

the consumer plus producer surplus in the competitive market and the difference in the market with the price ceiling.

Assuming that the quantity will go to consumers with the highest willingness to pay is overly optimistic in most situation. Hence, we we would generally expect that this trapezoidal area is a lower bound on the lost consumer plus producer surplus in the case of a price ceiling.

Rationing

The diagram we have just examined can also be used to describe the social losses due to rationing. Instead of fixing a price ceiling of p_c, suppose that the authorities issue ration coupons that allow for only q_c units to be purchased. In order to purchase one unit of the good, a consumer needs to pay p_c to the seller and produce a ration coupon.

If the ration coupons are marketable, then they would sell for a price of $p_e - p_c$. This would make the the total price of the purchase equal to p_e, which is the price that clears the market for the good being sold.

14.11 Calculating Gains and Losses

If we have estimates of the market demand and supply curves for a good, it is not difficult in principle to calculate the loss in consumers' surplus due to changes in government policies. For example, suppose the government decides to change its tax treatment of some good. This will result in a change in the prices that consumers face and therefore a change in the amount of the good that they will choose to consume. We can calculate the consumers' surplus associated with different tax proposals and see which tax reforms generate the smallest loss.

This is often useful information for judging various methods of taxation, but it suffers from two defects. First, as we've indicated earlier, the consumer's surplus calculation is only valid for special forms of preferences—namely, preferences representable by a quasilinear utility function. We argued earlier that this kind of utility function may be a reasonable approximation for goods for which changes in income lead to small changes in demand, but for goods whose consumption is closely related to income, the use of consumer surplus may be inappropriate.

Second, the calculation of this loss effectively lumps together all the consumers and producers and generates an estimate of the "cost" of a social policy only for some mythical "representative consumer." In many cases it is desirable to know not only the average cost across the population, but who bears the costs. The political success or failure of policies often depends more on the *distribution* of gains and losses than on the average gain or loss.

Consumer's surplus may be easy to calculate, but we've seen that it is not that much more difficult to calculate the true compensating or equivalent variation associated with a price change. If we have estimates of the demand functions of each household—or at least the demand functions for a sample of representative households—we can calculate the impact of a policy change on each household in terms of the compensating or equivalent variation. Thus we will have a measure of the "benefits" or "costs" imposed on each household by the proposed policy change.

Mervyn King, an economist at the London School of Economics, has described a nice example of this approach to analyzing the implications of reforming the tax treatment of housing in Britain in his paper "Welfare Analysis of Tax Reforms Using Household Data," *Journal of Public Economics*, 21 (1983), 183–214.

King first examined the housing expenditures of 5,895 households and estimated a demand function that best described their purchases of housing services. Next, he used this demand function to determine a utility function for each household. Finally, he used the estimated utility function to calculate how much each household would gain or lose under certain changes in the taxation of housing in Britain. The measure that he used

was similar to the equivalent variation described earlier in this chapter. The basic nature of the tax reform he studied was to eliminate tax concessions to owner-occupied housing and to raise rents in public housing. The revenues generated by these changes would be handed back to the households in the form of transfers proportional to household income.

King found that 4,888 of the 5,895 households would benefit from this kind of reform. More importantly he could identify explicitly those households that would have significant losses from the tax reform. King found, for example, that 94 percent of the highest income households gained from the reform, while only 58 percent of the lowest income households gained. This kind of information would allow special measures to be undertaken which might help in designing the tax reform in a way that could satisfy distributional objectives.

Summary

1. In the case of a discrete good and quasilinear utility, the utility associated with the consumption of n units of the discrete good is just the sum of the first n reservation prices.

2. This sum is the gross benefit of consuming the good. If we subtract the amount spent on the purchase of the good, we get the consumer's surplus.

3. The change in consumer's surplus associated with a price change has a roughly trapezoidal shape. It can be interpreted as the change in utility associated with the price change.

4. In general, we can use the compensating variation and the equivalent variation in income to measure the monetary impact of a price change.

5. If utility is quasilinear, the compensating variation, the equivalent variation, and the change in consumer's surplus are all equal. Even if utility is not quasilinear, the change in consumer's surplus may serve as a good approximation of the impact of the price change on a consumer's utility.

6. In the case of supply behavior we can define a producer's surplus that measures the net benefits to the supplier from producing a given amount of output.

REVIEW QUESTIONS

1. A good can be produced in a competitive industry at a cost of \$10 per unit. There are 100 consumers are each willing to pay \$12 each to consume

a single unit of the good (additional units have no value to them.) What is the equilibrium price and quantity sold? The government imposes a tax of \$1 on the good. What is the deadweight loss of this tax?

2. Suppose that the demand curve is given by $D(p) = 10 - p$. What is the gross benefit from consuming 6 units of the good?

3. In the above example, if the price changes from 4 to 6, what is the change in consumer's surplus?

4. Suppose that a consumer is consuming 10 units of a discrete good and the price increases from \$5 per unit to \$6. However, after the price change the consumer continues to consume 10 units of the discrete good. What is the loss in the consumer's surplus from this price change?

APPENDIX

Let's use some calculus to treat consumer's surplus rigorously. Start with the problem of maximizing quasilinear utility:

$$\max_{x,y} v(x) + y$$
$$\text{such that } px + y = m.$$

Substituting from the budget constraint we have

$$\max_{x} v(x) + m - px.$$

The first-order condition for this problem is

$$v'(x) = p.$$

This means that the inverse demand function $p(x)$ is defined by

$$p(x) = v'(x). \tag{14.2}$$

Note the analogy with the discrete-good framework described in the text: the price at which the consumer is just willing to consume x units is equal to the marginal utility.

But since the inverse demand curve measures the derivative of utility, we can simply integrate under the inverse demand function to find the utility function.

Carrying out the integration we have:

$$v(x) = v(x) - v(0) = \int_0^x v'(t)\,dt = \int_0^x p(t)\,dt.$$

Hence utility associated with the consumption of the x-good is just the area under the demand curve.

Comparison of CV, CS, and EV. Table 14.1

p_1	CV	CS	EV
1	0.00	0.00	0.00
2	7.18	6.93	6.70
3	11.61	10.99	10.40
4	14.87	13.86	12.94
5	17.46	16.09	14.87

EXAMPLE: A Few Demand Functions

Suppose that the demand function is linear, so that $x(p) = a - bp$. Then the change in consumer's surplus when the price moves from p to q is given by

$$\int_p^q (a - bt)\,dt = at - b\frac{t^2}{2}\Big]_p^q = a(q-p) - b\frac{q^2 - p^2}{2}.$$

Another commonly used demand function, which we examine in more detail in the next chapter, has the form $x(p) = Ap^\epsilon$, where $\epsilon < 0$ and A is some positive constant. When the price changes from p to q, the associated change in consumer's surplus is

$$\int_p^q At^\epsilon\,dt = A\frac{t^{\epsilon+1}}{\epsilon+1}\Big]_p^q = A\frac{q^{\epsilon+1} - p^{\epsilon+1}}{\epsilon+1},$$

for $\epsilon \neq -1$.

When $\epsilon = -1$, this demand function is $x(p) = A/p$, which is closely related to our old friend the Cobb-Douglas demand, $x(p) = am/p$. The change in consumer's surplus for the Cobb-Douglas demand is

$$\int_p^q \frac{am}{t}\,dt = am\ln t\Big]_p^q = am(\ln q - \ln p).$$

EXAMPLE: CV, EV, and Consumer's Surplus

In the text we calculated the compensating and equivalent variations for the Cobb-Douglas utility function. In the preceding example we calculated the change in consumer's surplus for the Cobb-Douglas utility function. Here we compare these three monetary measures of the impact on utility of a price change.

Suppose that the price of good 1 changes from 1 to $2, 3\ldots$ while the price of good 2 stays fixed at 1 and income stays fixed at 100. Table 14.1 shows the equivalent variation (EV), compensating variation (CV), and the change in consumer's surplus (CS) for the Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\frac{1}{10}} x_2^{\frac{9}{10}}$.

Note that the change in consumer's surplus always lies between the CV and the EV and that the difference between the three numbers is relatively small. It is possible to show that both of these facts are true in reasonably general circumstances. See Robert Willig, "Consumer's Surplus without Apology," *American Economic Review*, 66 (1976), 589–597.

CHAPTER 15

MARKET DEMAND

We have seen in earlier chapters how to model individual consumer choice. Here we see how to add up individual choices to get total **market demand**. Once we have derived the market demand curve, we will examine some of its properties, such as the relationship between demand and revenue.

15.1 From Individual to Market Demand

Let us use $x_i^1(p_1, p_2, m_i)$ to represent consumer i's demand function for good 1 and $x_i^2(p_1, p_2, m_i)$ for consumer i's demand function for good 2. Suppose that there are n consumers. Then the **market demand** for good 1, also called the **aggregate demand** for good 1, is the sum of these individual demands over all consumers:

$$X^1(p_1, p_2, m_1, \ldots, m_n) = \sum_{i=1}^{n} x_i^1(p_1, p_2, m_i).$$

The analogous equation holds for good 2.

Since each individual's demand for each good depends on prices and his or her money income, the aggregate demand will generally depend on prices and the *distribution* of incomes. However, it is sometimes convenient to think of the aggregate demand as the demand of some "representative consumer" who has an income that is just the sum of all individual incomes. The conditions under which this can be done are rather restrictive, and a complete discussion of this issue is beyond the scope of this book.

If we do make the representative consumer assumption, the aggregate demand function will have the form $X^1(p_1, p_2, M)$, where M is the sum of the incomes of the individual consumers. Under this assumption, the aggregate demand in the economy is just like the demand of some individual who faces prices (p_1, p_2) and has income M.

If we fix all the money incomes and the price of good 2, we can illustrate the relation between the aggregate demand for good 1 and its price, as in Figure 15.1. Note that this curve is drawn holding all other prices and incomes fixed. If these other prices and incomes change, the aggregate demand curve will shift.

The market demand curve. The market demand curve is the sum of the individual demand curves.

Figure 15.1

For example, if goods 1 and 2 are substitutes, then we know that increasing the price of good 2 will tend to increase the demand for good 1 whatever its price. This means that increasing the price of good 2 will tend to shift the aggregate demand curve for good 1 outward. Similarly,

if goods 1 and 2 are complements, increasing the price of good 2 will shift the aggregate demand curve for good 1 inward.

If good 1 is a normal good for an individual, then increasing that individual's money income, holding everything else fixed, would tend to increase that individual's demand, and therefore shift the aggregate demand curve outward. If we adopt the representative consumer model, and suppose that good 1 is a normal good for the representative consumer, then any economic change that increases aggregate income will increase the demand for good 1.

15.2 The Inverse Demand Function

We can look at the aggregate demand curve as giving us quantity as a function of price or as giving us price as a function of quantity. When we want to emphasize this latter view, we will sometimes refer to the **inverse demand function**, $P(X)$. This function measures what the market price for good 1 would have to be for X units of it to be demanded.

We've seen earlier that the price of a good measures the marginal rate of substitution (MRS) between it and all other goods; that is, the price of a good represents the marginal willingness to pay for an extra unit of the good by anyone who is demanding that good. If all consumers are facing the same prices for goods, then all consumers will have the same marginal rate of substitution at their optimal choices. Thus the inverse demand function, $P(X)$, measures the marginal rate of substitution, or the marginal willingness to pay, of *every* consumer who is purchasing the good.

The geometric interpretation of this summing operation is pretty obvious. Note that we are summing the demand or supply curves *horizontally*: for any given price, we add up the individuals' quantities demanded, which, of course, are measured on the horizontal axis.

EXAMPLE: Adding Up "Linear" Demand Curves

Suppose that one individual's demand curve is $D_1(p) = 20 - p$ and another individual's is $D_2(p) = 10 - 2p$. What is the market demand function? We have to be a little careful here about what we mean by "linear" demand functions. Since a negative amount of a good usually has no meaning, we *really* mean that the individual demand functions have the form

$$D_1(p) = \max\{20 - p, 0\}$$
$$D_2(p) = \max\{10 - 2p, 0\}.$$

What economists call "linear" demand curves actually aren't linear functions! The sum of the two demand curves looks like the curve depicted in Figure 15.2. Note the kink at $p = 5$.

The sum of two "linear" demand curves. Since the demand curves are only linear for positive quantities, there will typically be a kink in the market demand curve.

Figure 15.2

15.3 Discrete Goods

If a good is available only in discrete amounts, then we have seen that the demand for that good for a single consumer can be described in terms of the consumer's reservation prices. Here we examine the market demand for this kind of good. For simplicity, we will restrict ourselves to the case where the good will be available in units of zero or one.

In this case the demand of a consumer is completely described by his reservation price—the price at which he is just willing to purchase one unit. In Figure 15.3 we have depicted the demand curves for two consumers, A and B, and the market demand, which is the sum of these two demand curves. Note that the market demand curve in this case must "slope downward," since a decrease in the market price must increase the number of consumers who are willing to pay at least that price.

15.4 The Extensive and the Intensive Margin

In preceding chapters we have concentrated on consumer choice in which the consumer was consuming positive amounts of each good. When the price changes, the consumer decides to consume more or less of one good or the other, but still ends up consuming some of both goods. Economists sometimes say that this is an adjustment on the **intensive margin**.

In the reservation-price model, the consumers are deciding whether or not to enter the market for one of the goods. This is sometimes called an adjustment on the **extensive margin**. The slope of the aggregate demand curve will be affected by both sorts of decisions.

Figure 15.3 **Market demand for a discrete good**. The market demand curve is the sum of the demand curves of all the consumers in the market, here represented by the two consumers A and B.

We saw earlier that the adjustment on the intensive margin was in the "right" direction for normal goods: when the price went up, the quantity demanded went down. The adjustment on the extensive margin also works in the "right" direction. Thus aggregate demand curves can generally be expected to slope downward.

15.5 Elasticity

In Chapter 6 we saw how to derive a demand function from a consumer's underlying preferences. It is often of interest to have a measure of how "responsive" demand is to some change in price or income. Now the first idea that springs to mind is to use the slope of a demand function as a measure of responsiveness. After all, the definition of the slope of a demand function is the change in quantity demanded divided by the change in price:

$$\text{slope of demand function} = \frac{\Delta q}{\Delta p},$$

and that certainly looks like a measure of responsiveness.

Well, it is a measure of responsiveness—but it presents some problems. The most important one is that the slope of a demand function depends on the units in which you measure price and quantity. If you measure demand in gallons rather than in quarts, the slope becomes four times smaller. Rather than specify units all the time, it is convenient to consider a unit-free measure of responsiveness. Economists have chosen to use a measure known as **elasticity**.

The **price elasticity of demand**, ϵ, is defined to be the percent change in quantity divided by the percent change in price.[1] A 10 percent increase

[1] The Greek letter ϵ, epsilon, is pronounced "eps-i-lon."

in price is the same percentage increase whether the price is measured in American dollars or English pounds; thus measuring increases in percentage terms keeps the definition of elasticity unit-free.

In symbols the definition of elasticity is

$$\epsilon = \frac{\Delta q/q}{\Delta p/p}.$$

Rearranging this definition we have the more common expression:

$$\epsilon = \frac{p}{q}\frac{\Delta q}{\Delta p}.$$

Hence elasticity can be expressed as the ratio of price to quantity multiplied by the slope of the demand function. In the Appendix to this chapter we describe elasticity in terms of the derivative of the demand function. If you know calculus, the derivative formulation is the most convenient way to think about elasticity.

The sign of the elasticity of demand is generally negative, since demand curves invariably have a negative slope. However, it is tedious to keep referring to an elasticity of *minus* something-or-other, so it is common in verbal discussion to refer to elasticities of 2 or 3, rather than -2 or -3. We will try to keep the signs straight in the text by referring to the absolute value of elasticity, but you should be aware that verbal treatments tend to drop the minus sign.

Another problem with negative numbers arises when we compare magnitudes. Is an elasticity of -3 greater or less than an elasticity of -2? From an algebraic point of view -3 is smaller than -2, but economists tend to say that the demand with the elasticity of -3 is "more elastic" than the one with -2. In this book we will make comparisons in terms of absolute value so as to avoid this kind of ambiguity.

EXAMPLE: The Elasticity of a Linear Demand Curve

Consider the linear demand curve, $q = a - bp$, depicted in Figure 15.4. The slope of this demand curve is a constant, $-b$. Plugging this into the formula for elasticity we have

$$\epsilon = \frac{-bp}{q} = \frac{-bp}{a - bp}.$$

When $p = 0$, the elasticity of demand is zero. When $q = 0$, the elasticity of demand is (negative) infinity. At what value of price is the elasticity of demand equal to -1?

Figure 15.4 **The elasticity of a linear demand curve**. Elasticity is infinite at the vertical intercept, one halfway down the curve, and zero at the horizontal intercept.

To find such a price, we write down the equation

$$\frac{-bp}{a - bp} = -1$$

and solve it for p. This gives

$$p = \frac{a}{2b},$$

which, as we see in Figure 15.4, is just halfway down the demand curve.

15.6 Elasticity and Demand

If a good has an elasticity of demand greater than 1 in absolute value we say that it has an **elastic demand**. If the elasticity is less than 1 in absolute value we say that it has an **inelastic demand**. And if it has an elasticity of exactly -1, we say it has **unit elastic demand**.

An elastic demand curve is one for which the quantity demanded is very responsive to price: if you increase the price by 1 percent, the quantity demanded decreases by more than 1 percent. So think of elasticity as the responsiveness of the quantity demanded to price, and it will be easy to remember what elastic and inelastic mean.

In general the elasticity of demand for a good depends to a large extent on how many close substitutes it has. Take an extreme case—our old friend,

the red pencils and blue pencils example. Suppose that everyone regards these goods as perfect substitutes. Then if some of each of them are bought, they must sell for the same price. Now think what would happen to the demand for red pencils if their price rose, and the price of blue pencils stayed constant. Clearly it would drop to zero—the demand for red pencils is very elastic since it has a perfect substitute.

If a good has many close substitutes, we would expect that its demand curve would be very responsive to its price changes. On the other hand, if there are few close substitutes for a good, it can exhibit a quite inelastic demand.

15.7 Elasticity and Revenue

Revenue is just the price of a good times the quantity sold of that good. If the price of a good increases, then the quantity sold decreases, so revenue may increase or decrease. Which way it goes obviously depends on how responsive demand is to the price change. If demand drops a lot when the price increases, then revenue will fall. If demand drops only a little when the price increases, then revenue will increase. This suggests that the direction of the change in revenue has something to do with the elasticity of demand.

Indeed, there is a very useful relationship between price elasticity and revenue change. The definition of revenue is

$$R = pq.$$

If we let the price change to $p + \Delta p$ and the quantity change to $q + \Delta q$, we have a new revenue of

$$\begin{aligned} R' &= (p + \Delta p)(q + \Delta q) \\ &= pq + q\Delta p + p\Delta q + \Delta p \Delta q. \end{aligned}$$

Subtracting R from R' we have

$$\Delta R = q\Delta p + p\Delta q + \Delta p \Delta q.$$

For small values of Δp and Δq, the last term can safely be neglected, leaving us with an expression for the change in revenue of the form

$$\Delta R = q\Delta p + p\Delta q.$$

That is, the change in revenue is roughly equal to the quantity times the change in price plus the original price times the change in quantity. If we want an expression for the rate of change of revenue per change in price, we just divide this expression by Δp to get

$$\frac{\Delta R}{\Delta p} = q + p\frac{\Delta q}{\Delta p}.$$

This is treated geometrically in Figure 15.5. The revenue is just the area of the box: price times quantity. When the price increases, we add a rectangular area on the top of the box, which is approximately $q\Delta p$, but we subtract an area on the side of the box, which is approximately $p\Delta q$. For small changes, this is exactly the expression given above. (The leftover part, $\Delta p\Delta q$, is the little square in the corner of the box, which will be very small relative to the other magnitudes.)

Figure 15.5 **How revenue changes when price changes**. The change in revenue is the sum of the box on the top minus the box on the side.

When will the net result of these two effects be positive? That is, when do we satisfy the following inequality:

$$\frac{\Delta R}{\Delta p} = p\frac{\Delta q}{\Delta p} + q(p) > 0?$$

Rearranging we have

$$\frac{p}{q}\frac{\Delta q}{\Delta p} > -1.$$

The left-hand side of this expression is $\epsilon(p)$, which is a negative number. Multiplying through by -1 reverses the direction of the inequality to give us:

$$|\epsilon(p)| < 1.$$

Thus revenue increases when price increases if the elasticity of demand is less than 1 in absolute value. Similarly, revenue decreases when price increases if the elasticity of demand is greater than 1 in absolute value.

Another way to see this is to write the revenue change as we did above:

$$\Delta R = p\Delta q + q\Delta p > 0$$

and rearrange this to get

$$-\frac{p}{q}\frac{\Delta q}{\Delta p} = |\epsilon(p)| < 1.$$

Yet a third way to see this is to take the formula for $\Delta R/\Delta p$ and rearrange it as follows:

$$\begin{aligned}\frac{\Delta R}{\Delta p} &= q + p\frac{\Delta q}{\Delta p}\\ &= q\left[1 + \frac{p}{q}\frac{\Delta q}{\Delta p}\right]\\ &= q\left[1 + \epsilon(p)\right].\end{aligned}$$

Since demand elasticity is naturally negative, we can also write this expression as

$$\frac{\Delta R}{\Delta p} = q\left[1 - |\epsilon(p)|\right].$$

In this formula it is easy to see how revenue responds to a change in price: if the absolute value of elasticity is greater than 1, then $\Delta R/\Delta p$ must be negative and vice versa.

The intuitive content of these mathematical facts is not hard to remember. If demand is very responsive to price—that is, it is very elastic—then an increase in price will reduce demand so much that revenue will fall. If demand is very unresponsive to price—it is very inelastic—then an increase in price will not change demand very much, and overall revenue will increase. The dividing line happens to be an elasticity of -1. At this point if the price increases by 1 percent, the quantity will decrease by 1 percent, so overall revenue doesn't change at all.

EXAMPLE: Strikes and Profits

In 1979 the United Farm Workers called for a strike against lettuce growers in California. The strike was highly effective: the production of lettuce was cut almost in half. But the reduction in the supply of lettuce inevitably caused an increase in the price of lettuce. In fact, during the strike the price

of lettuce rose by nearly 400 percent. Since production halved and prices quadrupled, the net result of was almost a *doubling* producer profits![2]

One might well ask why the producers eventually settled the strike. The answer involves short-run and long-run supply responses. Most of the lettuce consumed in U.S. during the winter months is grown in the Imperial Valley. When the supply of this lettuce was drastically reduced in one season, there wasn't time to replace it with lettuce from elsewhere so the market price of lettuce skyrocketed. If the strike had held for several seasons, lettuce could be planted in other regions. This increase in supply from other sources would tend reduce the price of lettuce back to its normal level, thereby reducing the profits of the Imperial Valley growers.

15.8 Constant Elasticity Demands

What kind of demand curve gives us a constant elasticity of demand? In a linear demand curve the elasticity of demand goes from zero to infinity, which is not exactly what you would call constant, so that's not the answer.

We can use the revenue calculation described above to get an example. We know that if the elasticity is 1 at price p, then the revenue will not change when the price changes by a small amount. So if the revenue remains constant for all changes in price, we must have a demand curve that has an elasticity of -1 everywhere.

But this is easy. We just want price and quantity to be related by the formula

$$pq = \overline{R},$$

which means that

$$q = \frac{\overline{R}}{p}$$

is the formula for a demand function with constant elasticity of -1. The graph of the function $q = \overline{R}/p$ is given in Figure 15.6. Note that price times quantity is constant along the demand curve.

The general formula for a demand with a constant elasticity of ϵ turns out to be

$$q = Ap^{\epsilon},$$

where A is an arbitrary positive constant and ϵ, being an elasticity, will typically be negative. This formula will be useful in some examples later on.

A convenient way to express a constant elasticity demand curve is to take logarithms and write

$$\ln q = \ln A + \epsilon \ln p.$$

[2] See Colin Carter, et. al., "Agricultural Labor Strikes and Farmers' Incomes," *Economic Inquiry*, 25, 1987,121–133.

Unit elastic demand. For this demand curve price times quantity is constant at every point. Thus the demand curve has a constant elasticity of -1.

Figure 15.6

In this expression, the logarithm of q depends in a linear way on the logarithm of p.

15.9 Elasticity and Marginal Revenue

In section 15.7 we examined how revenue changes when you change the price of a good, but it is often of interest to consider how revenue changes when you change the quantity of a good. This is especially useful when we are considering production decisions by firms.

We saw earlier that for small changes in price and quantity, the change in revenue is given by

$$\Delta R = p\Delta q + q\Delta p.$$

If we divide both sides of this expression by Δq, we get the expression for **marginal revenue**:

$$\text{MR} = \frac{\Delta R}{\Delta q} = p + q\frac{\Delta p}{\Delta q}.$$

There is a useful way to rearrange this formula. Note that we can also write this as

$$\frac{\Delta R}{\Delta q} = p\left[1 + \frac{q\Delta p}{p\Delta q}\right].$$

What is the second term inside the brackets? Nope, it's not elasticity, but you're close. It is the reciprocal of elasticity:

$$\frac{1}{\epsilon} = \frac{1}{\frac{p\Delta q}{q\Delta p}} = \frac{q\Delta p}{p\Delta q}.$$

Thus the expression for marginal revenue becomes

$$\frac{\Delta R}{\Delta q} = p(q)\left[1 + \frac{1}{\epsilon(q)}\right].$$

(Here we've written $p(q)$ and $\epsilon(q)$ to remind ourselves that both price and elasticity will typically depend on the level of output.)

When there is a danger of confusion due to the fact that elasticity is a negative number we will sometimes write this expression as

$$\frac{\Delta R}{\Delta q} = p(q)\left[1 - \frac{1}{|\epsilon(q)|}\right].$$

This means that if elasticity of demand is -1, then marginal revenue is zero—revenue doesn't change when you increase output. If demand is inelastic, then $|\epsilon|$ is less than 1, which means $1/|\epsilon|$ is greater than 1. Thus $1-1/|\epsilon|$ is negative, so that revenue will decrease when you increase output.

This is quite intuitive. If demand isn't very responsive to price, then you have to cut prices a lot to increase output: so revenue goes down. This is all completely consistent with the earlier discussion about how revenue changes as we change price, since an increase in quantity means a decrease in price and vice versa.

EXAMPLE: Setting a Price

Suppose that you were in charge of setting a price for some product that you were producing and that you had a good estimate of the demand curve for that product. Let us suppose that your goal is to set a price that maximizes profits—revenue minus costs. Then you would never want to set it where the elasticity of demand was less than 1—you would never want to set a price where demand was inelastic.

Why? Consider what would happen if you raised your price. Then your revenues would increase—since demand was inelastic—and the quantity you were selling would decrease. But if the quantity sold decreases, then your production costs must also decrease, or at least, they can't increase. So your overall profit must rise, which shows that operating at an inelastic part of the demand curve cannot yield maximal profits.

15.10 Marginal Revenue Curves

We saw in the last section that marginal revenue is given by

$$\frac{\Delta R}{\Delta q} = p(q) + \frac{\Delta p(q)}{\Delta q} q$$

or

$$\frac{\Delta R}{\Delta q} = p(q) \left[1 - \frac{1}{|\epsilon(q)|} \right].$$

We will find it useful to plot these marginal revenue curves. First, note that when quantity is zero, marginal revenue is just equal to the price. For the first unit of the good sold, the extra revenue you get is just the price. But after that, the marginal revenue will be less than the price, since $\Delta p/\Delta q$ is negative.

Think about it. If you decide to sell one more unit of output, you will have to decrease the price. But this reduction in price reduces the revenue you receive on all the units of output that you were selling already. Thus the extra revenue you receive will be less than the price that you get for selling the extra unit.

Let's consider the special case of the linear (inverse) demand curve:

$$p(q) = a - bq.$$

Here it is easy to see that the slope of the inverse demand curve is constant:

$$\frac{\Delta p}{\Delta q} = -b.$$

Thus the formula for marginal revenue becomes

$$\begin{aligned} \frac{\Delta R}{\Delta q} &= p(q) + \frac{\Delta p(q)}{\Delta q} q \\ &= p(q) - bq \\ &= a - bq - bq \\ &= a - 2bq. \end{aligned}$$

This marginal revenue curve is depicted in Figure 15.7A. The marginal revenue curve has the same vertical intercept as the demand curve, but has twice the slope. Marginal revenue is negative when $q > a/2b$. The quantity $a/2b$ is the quantity at which the elasticity is equal to -1. At any larger

Figure 15.7 **Marginal revenue**. (A) Marginal revenue for a linear demand curve. (B) Marginal revenue for a constant elasticity demand curve.

quantity demand will be inelastic, which implies that marginal revenue is negative.

The constant elasticity demand curve provides another special case of the marginal revenue curve. (See Figure 15.7B.) If the elasticity of demand is constant at $\epsilon(q) = \epsilon$, then the marginal revenue curve will have the form

$$MR = p(q)\left[1 - \frac{1}{|\epsilon|}\right].$$

Since the term in brackets is constant, the marginal revenue curve is some constant fraction of the inverse demand curve. When $|\epsilon| = 1$, the marginal revenue curve is constant at zero. When $|\epsilon| > 1$, the marginal revenue curve lies below the inverse demand curve, as depicted. When $|\epsilon| < 1$, marginal revenue is negative.

15.11 Income Elasticity

Recall that the price elasticity of demand is defined as

$$\text{price elasticity of demand} = \frac{\%\text{ change in quantity demanded}}{\%\text{ change in price}}.$$

This gives us a unit-free measure of how the amount demanded responds to a change in price.

The **income elasticity of demand** is used to describe how the quantity demanded responds to a change in income; its definition is

$$\text{income elasticity of demand} = \frac{\%\text{ change in quantity}}{\%\text{ change in income}}.$$

Recall that a **normal good** is one for which an increase in income leads to an increase in demand; so for this sort of good the income elasticity of demand is positive. An **inferior good** is one for which an increase in income leads to a decrease in demand; for this sort of good, the income elasticity of demand is negative. Economists sometimes use the term **luxury goods**. These are goods that have an income elasticity of demand that is greater than 1: a 1 percent increase in income leads to *more* than a 1 percent increase in demand for a luxury good.

As a general rule of thumb, however, income elasticities tend to cluster around 1. We can see the reason for this by examining the budget constraint. Write the budget constraints for two different levels of income:

$$\begin{aligned} p_1 x_1' + p_2 x_2' &= m' \\ p_1 x_1^0 + p_2 x_2^0 &= m^0. \end{aligned}$$

Subtract the second equation from the first and let Δ denote differences, as usual:

$$p_1\,\Delta x_1 + p_2\,\Delta x_2 = \Delta m.$$

Now multiply and divide price i by x_i/x_i and divide both sides by m:

$$\frac{p_1 x_1}{m}\frac{\Delta x_1}{x_1} + \frac{p_2 x_2}{m}\frac{\Delta x_2}{x_2} = \frac{\Delta m}{m}.$$

Finally, divide both sides by $\Delta m/m$, and use $s_i = p_i x_i/m$ to denote the **expenditure share** of good i. This gives us our final equation,

$$s_1\frac{\Delta x_1/x_1}{\Delta m/m} + s_2\frac{\Delta x_2/x_2}{\Delta m/m} = 1.$$

This equation says that the *weighted average of the income elasticities is 1,* where the weights are the expenditure shares. Luxury goods that have an income elasticity greater than 1 must be counterbalanced by goods that have an income elasticity less than 1, so that "on average" income elasticities are about 1.

Summary

1. The market demand curve is simply the sum of the individual demand curves.

2. The reservation price measures the price at which a consumer is just indifferent between purchasing or not purchasing a good.

3. The demand function measures quantity demanded as a function of price. The inverse demand function measures price as a function of quantity. A given demand curve can be described in either way.

4. The elasticity of demand measures the responsiveness of the quantity demanded to price. It is formally defined as the percent change in quantity divided by the percent change in price.

5. If the absolute value of the elasticity of demand is less than 1 at some point, we say that demand is *inelastic* at that point. If the absolute value of elasticity is greater than 1 at some point, we say demand is *elastic* at that point. If the absolute value of the elasticity of demand at some point is exactly 1, we say that the demand has *unitary* elasticity at that point.

6. If demand is inelastic at some point, then an increase in quantity will result in a reduction in revenue. If demand is elastic, then an increase in quantity will result in an increase in revenue.

7. The marginal revenue is the extra revenue one gets from increasing the quantity sold. The formula relating marginal revenue and elasticity is $\mathrm{MR} = p[1 + 1/\epsilon] = p[1 - 1/|\epsilon|]$.

8. If the inverse demand curve is a linear function $p(q) = a - bq$, then the marginal revenue is given by $\mathrm{MR} = a - 2bq$.

9. Income elasticity measures the responsiveness of the quantity demanded to income. It is formally defined as the percent change in quantity divided by the percent change in income.

REVIEW QUESTIONS

1. If the market demand curve is $D(p) = 100 - .5p$, what is the inverse demand curve?

2. An addict's demand function for a drug may be very inelastic, but the market demand function might be quite elastic. How can this be?

3. If $D(p) = 12 - 2p$, what price will maximize revenue?

4. Suppose that the demand curve for a good is given by $D(p) = 100/p$. What price will maximize revenue?

5. True or false? In a two good model if one good is an inferior good the other good must be a luxury good.

APPENDIX

In terms of derivatives the price elasticity of demand is defined by

$$\epsilon = \frac{p}{q}\frac{dq}{dp}.$$

In the text we claimed that the formula for a constant elasticity demand curve was $q = Ap^{\epsilon}$. To verify that this is correct, we can just differentiate it with respect to price:

$$\frac{dq}{dp} = \epsilon A p^{\epsilon - 1}$$

and multiply by price over quantity:

$$\frac{p}{q}\frac{dq}{dp} = \frac{p}{Ap^{\epsilon}}\epsilon A p^{\epsilon-1} = \epsilon.$$

Everything conveniently cancels, leaving us with ϵ as required.

A linear demand curve has the formula $q(p) = a - bp$. The elasticity of demand at a point p is given by

$$\epsilon = \frac{p}{q}\frac{dq}{dp} = \frac{-bp}{a - bp}.$$

When p is zero, the elasticity is zero. When q is zero, the elasticity is infinite.

Revenue is given by $R(p) = pq(p)$. To see how revenue changes as p changes we differentiate revenue with respect to p to get

$$R'(p) = pq'(p) + q(p).$$

Suppose that revenue increases when p increases. Then we have

$$R'(p) = p\frac{dq}{dp} + q(p) > 0.$$

Rearranging, we have

$$\epsilon = \frac{p}{q}\frac{dq}{dp} > -1.$$

Recalling that dq/dp is negative and multiplying through by -1, we find

$$|\epsilon| < 1.$$

Hence if revenue increases when price increases, we must be at an inelastic part of the demand curve.

Figure 15.8 **Laffer curve**. A possible shape for the Laffer curve, which relates tax rates and tax revenues.

EXAMPLE: The Laffer Curve

In this section we'll consider some simple elasticity calculations that can be used to examine an issue of considerable policy interest, namely, how tax revenue changes when the tax rate changes.

Suppose that we graph tax revenue versus the tax rate. If the tax rate is zero, then tax revenues are zero; if the tax rate is 1, nobody will want to demand or supply the good in question, so the tax revenue is also zero. Thus revenue as a function of the tax rate must first increase and eventually decrease. (Of course, it can go up and down several times between zero and 1, but we'll ignore this possibility to keep things simple.) The curve that relates tax rates and tax revenues is known as the **Laffer curve**, depicted in Figure 15.8.

The interesting feature of the Laffer curve is that it suggests that when the tax rate is high enough, an increase in the tax rate will end up *reducing* the revenues collected. The reduction in the supply of the good due to the increase in the tax rate can be so large that tax revenue actually decreases. This is called the Laffer effect, after the economist who popularized this diagram in the early eighties. It has been said that the virtue of the Laffer curve is that you can explain it to a congressman in half an hour and he can talk about it for six months. Indeed, the Laffer curve figured prominently in the debate over the effect of the 1980 tax cuts. The catch in the above argument is the phrase "high enough." Just how high does the tax rate have to be for the Laffer effect to work?

To answer this question let's consider the following simple model of the labor market. Suppose that firms will demand zero labor if the wage is greater than $\overline{w}$ and an arbitrarily large amount of labor if the wage is exactly $\overline{w}$. This means that the demand curve for labor is flat at some wage $\overline{w}$. Suppose that the supply

curve of labor, $S(w)$, has a conventional upward slope. The equilibrium in the labor market is depicted in Figure 15.9.

Labor market. Equilibrium in the labor market with a horizontal demand curve for labor. When labor income is taxed, less will be supplied at each wage rate.

Figure 15.9

If we put a tax on labor at the rate t, then if the firm pays $\overline{w}$, the worker only gets $w = (1-t)\overline{w}$. Thus the supply curve of labor tilts to the left, and the amount of labor sold drops, as in Figure 15.9. The after-tax wage has gone down and this has discouraged the sale of labor. So far so good.

Tax revenue, T, is therefore given by the formula

$$T = t\overline{w}S(w),$$

where $w = (1-t)\overline{w}$ and $S(w)$ is the supply of labor.

In order to see how tax revenue changes as we change the tax rate we differentiate this formula with respect to t to find

$$\frac{dT}{dt} = \left[-t\frac{dS(w)}{dw}\overline{w} + S(w)\right]\overline{w}. \tag{15.1}$$

(Note the use of the chain rule and the fact that $dw/dt = -\overline{w}$.)

The Laffer effect occurs when revenues decline when t increases—that is, when this expression is negative. Now this clearly means that the supply of labor is going to have to be quite elastic—it has to drop a lot when the tax increases. So let's try to see what values of elasticity will make this expression negative.

In order for equation (15.1) to be negative, we must have

$$-t\frac{dS(w)}{dw}\overline{w} + S(w) < 0.$$

Transposing yields

$$t\frac{dS(w)}{dw}\overline{w} > S(w),$$

and dividing both sides by $tS(w)$ gives

$$\frac{dS(w)}{dw}\frac{\overline{w}}{S(w)} > \frac{1}{t}.$$

Multiplying both sides by $(1-t)$ and using the fact that $w = (1-t)\overline{w}$ gives us

$$\frac{dS}{dw}\frac{w}{S} > \frac{1-t}{t}.$$

The left-hand side of this expression is the elasticity of labor supply. We have shown that the Laffer effect can only occur if the elasticity of labor supply is greater than $(1-t)/t$.

Let us take an extreme case and suppose that the tax rate on labor income is 50 percent. Then the Laffer effect can occur only when the elasticity of labor supply is greater than 1. This means that a 1 percent reduction in the wage would lead to more than a 1 percent reduction in the labor supply. This is a very large response.

Econometricians have often estimated labor-supply elasticities, and about the largest value anyone has ever found has been around 0.2. So the Laffer effect seems pretty unlikely for the kinds of tax rates that we have in the United States. However, in other countries, such as Sweden, tax rates go much higher, and there is some evidence that the Laffer phenomenon may have occurred.[3]

EXAMPLE: Another Expression for Elasticity

Here is another expression for elasticity that is sometimes useful. It turns out that elasticity can also be expressed as

$$\frac{d\ln Q}{d\ln P}.$$

The proof involves repeated application of the chain rule. We start by noting that

$$\begin{aligned}\frac{d\ln Q}{d\ln P} &= \frac{d\ln Q}{dQ}\frac{dQ}{d\ln P}\\ &= \frac{1}{Q}\frac{dQ}{d\ln P}.\end{aligned} \tag{15.2}$$

[3] See Charles E. Stuart, "Swedish Tax Rates, Labor Supply, and Tax Revenues," *Journal of Political Economy*, 89, 5 (October 1981), 1020–38.

We also note that

$$\begin{aligned}\frac{dQ}{dP} &= \frac{dQ}{d\ln P}\frac{d\ln P}{dP}\\ &= \frac{dQ}{d\ln P}\frac{1}{P},\end{aligned}$$

which implies that

$$\frac{dQ}{d\ln P} = P\frac{dQ}{dP}.$$

Substituting this into equation (15.2), we have

$$\frac{d\ln Q}{d\ln P} = \frac{1}{Q}\frac{dQ}{dP}P = \epsilon,$$

which is what we wanted to establish.

Thus elasticity measures the slope of the demand curve plotted on log-log paper: how the log of the quantity changes as the log of the price changes.

CHAPTER 16

EQUILIBRIUM

In preceding chapters we have seen how to construct individual demand curves by using information about preferences and prices. In Chapter 15 we added up these individual demand curves to construct market demand curves. In this chapter we will describe how to use these market demand curves to determine the equilibrium market price.

In Chapter 1 we said that there were two fundamental principles of microeconomic analysis. These were the optimization principle and the equilibrium principle. Up until now we have been studying examples of the optimization principle: what follows from the assumption that people choose their consumption optimally from their budget sets. In later chapters we will continue to use optimization analysis to study the profit-maximization behavior of firms. Finally, we combine the behavior of consumers and firms to study the equilibrium outcomes of their interaction in the market.

But before undertaking that study in detail it seems worthwhile at this point to give some examples of equilibrium analysis—how the prices adjust so as to make the demand and supply decisions of economic agents compatible. In order to do so, we will have to briefly consider the other side of the market—the supply side.

16.1 Supply

We have already seen a few examples of supply curves. In Chapter 1 we looked at a vertical supply curve for apartments. In Chapter 9 we considered situations where consumers would choose to be net suppliers or demanders of goods that they owned, and we analyzed labor-supply decisions.

In all of these cases the supply curve simply measured how much the consumer was willing to supply of a good at each possible market price. Indeed, this is the definition of the supply curve: for each p, we determine how much of the good will be supplied, $S(p)$. In the next few chapters we will discuss the supply behavior of firms. However, for many purposes, it is not really necessary to know where the supply curve or the demand curve comes from in terms of the optimizing behavior that generates the curves. For many problems the fact that there is a functional relationship between the price and the quantity that consumers want to demand or supply at that price is enough to highlight important insights.

16.2 Market Equilibrium

Suppose that we have a number of consumers of a good. Given their individual demand curves we can add them up to get a market demand curve. Similarly, if we have a number of independent suppliers of this good, we can add up their individual supply curves to get the **market supply curve**.

The individual demanders and suppliers are assumed to take prices as given—outside of their control—and simply determine their best response given those market prices. A market where each economic agent takes the market price as outside of his or her control is called a **competitive market.**

The usual justification for the competitive-market assumption is that each consumer or producer is a small part of the market as a whole and thus has a negligible effect on the market price. For example, each supplier of wheat takes the market price to be more or less independent of his actions when he determines how much wheat he wants to produce and supply to the market.

Although the market price may be independent of any *one* agent's actions in a competitive market, it is the actions of all the agents together that determine the market price. The **equilibrium price** of a good is that price where the supply of the good equals the demand. Geometrically, this is the price where the demand and the supply curves cross.

If we let $D(p)$ be the market demand curve and $S(p)$ the market supply curve, the equilibrium price is the price p^* that solves the equation

$$D(p^*) = S(p^*).$$

The solution to this equation, p^*, is the price where market demand equals market supply.

Why should this be an equilibrium price? An economic equilibrium is a situation where all agents are choosing the best possible action for themselves and each person's behavior is consistent with that of the others. At any price other than an equilibrium price, some agents' behaviors would be infeasible, and there would therefore be a reason for their behavior to change. Thus a price that is not an equilibrium price cannot be expected to persist since at least some agents would have an incentive to change their behavior.

The demand and supply curves represent the optimal choices of the agents involved, and the fact that they are equal at some price p^* indicates that the behaviors of the demanders and suppliers are compatible. At any price *other* than the price where demand equals supply these two conditions will *not* be met.

For example, suppose that we consider some price $p' < p^*$ where demand is greater than supply. Then some suppliers will realize that they can sell their goods at more than the going price p' to the disappointed demanders. As more and more suppliers realize this, the market price will be pushed up to the point where demand and supply are equal.

Similarly if $p' > p^*$, so that demand is less than supply, then some suppliers will not be able to sell the amount that they expected to sell. The only way in which they will be able to sell more output will be to offer it at a lower price. But if all suppliers are selling the identical goods, and if some supplier offers to sell at a lower price, the other suppliers must match that price. Thus excess supply exerts a downward pressure on the market price. Only when the amount that people want to buy at a given price equals the amount that people want to sell at that price will the market be in equilibrium.

16.3 Two Special Cases

There are two special cases of market equilibrium that are worth mentioning since they come up fairly often. The first is the case of fixed supply. Here the amount supplied is some given number and is independent of price; that is, the supply curve is vertical. In this case the equilibrium *quantity* is determined entirely by the supply conditions and the equilibrium *price* is determined entirely by demand conditions.

The opposite case is the case where the supply curve is completely horizontal. If an industry has a perfectly horizontal supply curve, it means that the industry will supply any amount of a good at a constant price. In this situation the equilibrium *price* is determined by the supply conditions, while the equilibrium *quantity* is determined by the demand curve.

The two cases are depicted in Figure 16.1. In these two special cases the determination of price and quantity can be separated; but in the general case the equilibrium price and the equilibrium quantity are jointly determined by the demand and supply curves.

Special cases of equilibrium. Case A shows a vertical supply curve where the equilibrium price is determined solely by the demand curve. Case B depicts a horizontal supply curve where the equilibrium price is determined solely by the supply curve.

Figure 16.1

16.4 Inverse Demand and Supply Curves

We can look at market equilibrium in a slightly different way that is often useful. As indicated earlier, individual demand curves are normally viewed as giving the optimal quantities demanded as a function of the price charged. But we can also view them as inverse demand functions that measure the *price* that someone is willing to pay in order to acquire some given amount of a good. The same thing holds for supply curves. They can be viewed as measuring the quantity supplied as a function of the price. But we can also view them as measuring the *price* that must prevail in order to generate a given amount of supply.

These same constructions can be used with *market* demand and *market* supply curves, and the interpretations are just those given above. In this framework an equilibrium price is determined by finding that quantity at

which the amount the demanders are willing to pay to consume that quantity is the same as the price that suppliers must receive in order to supply that quantity.

Thus, if we let $P_S(q)$ be the inverse supply function and $P_D(q)$ be the inverse demand function, equilibrium is determined by the condition

$$P_S(q^*) = P_D(q^*).$$

EXAMPLE: Equilibrium with Linear Curves

Suppose that both the demand and the supply curves are linear:

$$D(p) = a - bp$$

$$S(p) = c + dp.$$

The coefficients (a, b, c, d) are the parameters that determine the intercepts and slopes of these linear curves. The equilibrium price can be found by solving the following equation:

$$D(p) = a - bp = c + dp = S(p).$$

The answer is

$$p^* = \frac{a-c}{d+b}.$$

The equilibrium quantity demanded (and supplied) is

$$\begin{aligned} D(p^*) &= a - bp^* \\ &= a - b\frac{a-c}{b+d} \\ &= \frac{ad+bc}{b+d}. \end{aligned}$$

We can also solve this problem by using the inverse demand and supply curves. First we need to find the inverse demand curve. At what price is some quantity q demanded? Simply substitute q for $D(p)$ and solve for p. We have

$$q = a - bp,$$

so

$$P_D(q) = \frac{a-q}{b}.$$

In the same manner we find

$$P_S(q) = \frac{q-c}{d}.$$

Setting the demand price equal to the supply price and solving for the equilibrium quantity we have

$$P_D(q) = \frac{a-q}{b} = \frac{q-c}{d} = P_S(q)$$

$$q^* = \frac{ad+bc}{b+d}.$$

Note that this gives the same answer as in the original problem for both the equilibrium price and the equilibrium quantity.

16.5 Comparative Statics

After we have found an equilibrium by using the demand equals supply condition (or the demand price equals the supply price condition), we can see how it will change as the demand and supply curves change. For example, it is easy to see that if the demand curve shifts to the right in a parallel way—some fixed amount more is demanded at every price—the equilibrium price and quantity must both rise. On the other hand, if the supply curve shifts to the right, the equilibrium quantity rises, but the equilibrium price must fall.

What if both curves shift to the right? Then the quantity will definitely increase while the change in price is ambiguous—it could increase or it could decrease.

EXAMPLE: Shifting Both Curves

Question: Consider the competitive market for apartments described in Chapter 1. Let the equilibrium price in that market be p^* and the equilibrium quantity be q^*. Suppose that a developer converts m of the apartments to condominiums, which are bought by the people who are currently living in the apartments. What happens to the equilibrium price?

Answer: The situation is depicted in Figure 16.2. The demand and supply curves both shift to the left by the *same* amount. Hence the price is unchanged and the quantity sold simply drops by m.

Algebraically the new equilibrium price is determined by

$$D(p) - m = S(p) - m,$$

which clearly has the same solution as the original demand equals supply condition.

Figure 16.2 **Shifting both curves**. Both demand and supply curves shift to the left by the same amount, which implies the equilibrium price will remain unchanged.

16.6 Taxes

Describing a market before and after taxes are imposed presents a very nice exercise in comparative statics, as well as being of considerable interest in the conduct of economic policy. Let us see how it is done.

The fundamental thing to understand about taxes is that when a tax is present in a market, there are *two* prices of interest: the price the demander pays and the price the supplier gets. These two prices—the demand price and the supply price—differ by the amount of the tax.

There are several different kinds of taxes that one might impose. Two examples we will consider here are **quantity taxes** and **value** taxes (also called **ad valorem** taxes).

A quantity tax is a tax levied per unit of quantity bought or sold. Gasoline taxes are a good example of this. The gasoline tax is roughly 12 cents a gallon. If the demander is paying $P_D = \$1.50$ per gallon of gasoline, the supplier is getting $P_S = \$1.50 - .12 = \$1.38$ per gallon. In general, if t is the amount of the quantity tax per unit sold, then

$$P_D = P_S + t.$$

A value tax is a tax expressed in percentage units. State sales taxes are the most common example of value taxes. If your state has a 5 percent

sales tax, then when you pay \$1.05 for something (including the tax), the supplier gets \$1.00. In general, if the tax rate is given by τ, then

$$P_D = (1 + \tau)P_S.$$

Let us consider what happens in a market when a quantity tax is imposed. For our first case we suppose that the supplier is required to pay the tax, as in the case of the gasoline tax. Then the amount supplied will depend on the supply price—the amount the supplier actually gets after paying the tax—and the amount demanded will depend on the demand price—the amount that the demander pays. The amount that the supplier gets will be the amount the demander pays minus the amount of the tax. This gives us two equations:

$$D(P_D) = S(P_S)$$

$$P_S = P_D - t.$$

Substituting the second equation into the first, we have the equilibrium condition:

$$D(P_D) = S(P_D - t).$$

Alternatively we could also rearrange the second equation to get $P_D = P_S + t$ and then substitute to find

$$D(P_S + t) = S(P_S).$$

Either way is equally valid; which one you use will depends on convenience in a particular case.

Now suppose that instead of the supplier paying the tax, the demander has to pay the tax. Then we write

$$P_D - t = P_S,$$

which says that the amount paid by the demander minus the tax equals the price received by the supplier. Substituting this into the demand equals supply condition we find

$$D(P_D) = S(P_D - t).$$

Note that this is the same equation as in the case where the supplier pays the tax. As far as the equilibrium price facing the demanders and the suppliers is concerned, it really doesn't matter who is responsible for paying the tax—it just matters that the tax must be paid by someone.

This really isn't so mysterious. Think of the gasoline tax. There the tax is included in the posted price. But if the price were instead listed as the before-tax price and the gasoline tax were added on as a separate item to

be paid by the demanders, then do you think that the amount of gasoline demanded would change? After all, the final price to the consumers would be the same whichever way the tax was charged. Insofar as the consumers can recognize the net cost to them of goods they purchase, it really doesn't matter which way the tax is levied.

There is an even simpler way to show this using the inverse demand and supply functions. The equilibrium quantity traded is that quantity q^* such that the demand price at q^* *minus the tax being paid* is just equal to the supply price at q^*. In symbols:

$$P_D(q^*) - t = P_S(q^*).$$

If the tax is being imposed on the suppliers, then the condition is that the supply price *plus the amount of the tax* must equal the demand price:

$$P_D(q^*) = P_S(q^*) + t.$$

But these are the same equations, so the same equilibrium prices and quantities must result.

Finally, we consider the geometry of the situation. This is most easily seen by using the inverse demand and supply curves discussed above. We want to find the quantity where the curve $P_D(q)-t$ crosses the curve $P_S(q)$. In order to locate this point we simply shift the demand curve down by t and see where this shifted demand curve intersects the original supply curve. Alternatively we can find the quantity where $P_D(q)$ equals $P_S(q)+t$. To do this, we simply shift the supply curve up by the amount of the tax. Either way gives us the correct answer for the equilibrium quantity. The picture is given in Figure 16.3.

From this diagram we can easily see the qualitative effects of the tax. The quantity sold must decrease, the price paid by the demanders must go up, and the price received by the suppliers must go down.

Figure 16.4 depicts another way to determine the impact of a tax. Think about the definition of equilibrium in this market. We want to find a quantity q^* such that when the supplier faces the price p_s and the demander faces the price $p_d = p_s + t$, the quantity q^* is demanded by the demander and supplied by the supplier. Let us represent the tax t by a vertical line segment and slide it along the supply curve until it just touches the demand curve. That point is our equilibrium quantity!

EXAMPLE: Taxation with Linear Demand and Supply

Suppose that the demand and supply curves are both linear. Then if we impose a tax in this market, the equilibrium is determined by the equations

$$a - bp_D = c + dp_S$$

The imposition of a tax. In order to study the impact of a tax, we can either shift the demand curve down, as in panel A, or shift the supply curve up, as in panel B. The equilibrium prices paid by the demanders and received by the suppliers will be the same either way.

Figure 16.3

and

$$p_D = p_S + t.$$

Substituting from the second equation into the first, we have

$$a - b(p_S + t) = c + dp_S.$$

Solving for the equilibrium supply price, p_S^*, gives

$$p_S^* = \frac{a - c - bt}{d + b}.$$

The equilibrium demand price, p_D^*, is then given by $p_S^* + t$:

$$\begin{aligned} p_D^* &= \frac{a - c - bt}{d + b} + t \\ &= \frac{a - c + dt}{d + b}. \end{aligned}$$

Note that the price paid by the demander increases and the price received by the supplier decreases. The amount of the price change depends on the slope of the demand and supply curves.

Figure 16.4 **Another way to determine the impact of a tax.** Slide the line segment along the supply curve until it hits the demand curve.

16.7 Passing Along a Tax

One often hears about how a tax on producers doesn't hurt profits, since firms can simply pass along a tax to consumers. As we've seen above, a tax really shouldn't be regarded as a tax on firms or on consumers. Rather, taxes are on transactions *between* firms and consumers. In general, a tax will both raise the price paid by consumers and lower the price received by firms. How much of a tax gets passed along will therefore depend on the characteristics of demand and supply.

This is easiest to see in the extreme cases: when we have a perfectly horizontal supply curve or a perfectly vertical supply curve. These are also known as the case of **perfectly elastic** and **perfectly inelastic** supply.

We've already encountered these two special cases earlier in this chapter. If an industry has a horizontal supply curve, it means that the industry will supply any amount desired of the good at some given price, and zero units of the good at any lower price. In this case the price is entirely determined by the supply curve and the quantity sold is determined by demand. If an industry has a vertical supply curve, it means that the quantity of the good is fixed. The equilibrium price of the good is determined entirely by demand.

Let's consider the imposition of a tax in a market with a perfectly elastic supply curve. As we've seen above, imposing a tax is just like shifting the

Special cases of taxation. (A) In the case of a perfectly elastic supply curve the tax gets completely passed along to the consumers. (B) In the case of a perfectly inelastic supply none of the tax gets passed along.

Figure 16.5

supply curve up by the amount of the tax, as illustrated in Figure 16.5A.

In the case of a perfectly elastic supply curve it is easy to see that the price to the consumers goes up by exactly the amount of the tax. The supply price is exactly the same as it was before the tax, and the demanders end up paying the entire tax. When you think about the meaning of the horizontal supply curve, this is not hard to understand. The horizontal supply curve means that the industry is willing to supply any amount of the good at some particular price, p^*, and zero amount at any lower price. Thus, if any amount of the good is going to be sold at all in equilibrium, the suppliers must receive p^* for selling it. This effectively determines the equilibrium supply price, and the demand price is $p^* + t$.

The opposite case is illustrated in Figure 16.5B. If the supply curve is vertical and we "shift the supply curve up," we don't change anything in the diagram. The supply curve just slides along itself, and we still have the same amount of the good supplied, with or without the tax. In this case, the demanders determine the equilibrium price of the good, and they are willing to pay a certain amount, p^*, for the supply of the good that is available, tax or no tax. Thus they end up paying p^*, and the suppliers end up receiving $p^* - t$. The entire amount of the tax is paid by the suppliers.

This case often strikes people as paradoxical, but it really isn't. If the suppliers could raise their prices after the tax is imposed and still sell their entire fixed supply, they would have raised their prices before the tax was imposed and made more money! If the demand curve doesn't move, then

the only way the price can increase is if the supply is reduced. If a policy doesn't change either supply or demand, it certainly can't affect price.

Now that we understand the special cases, we can examine the in-between case where the supply curve has an upward slope but is not perfectly vertical. In this situation, the amount of the tax that gets passed along will depend on the steepness of the supply curve relative to the demand curve. If the supply curve is nearly horizontal, nearly all of the tax gets passed along to the consumers, while if the supply curve is nearly vertical, almost none of the tax gets passed along. See Figure 16.6 for some examples.

Figure 16.6

Passing along a tax. (A) If the supply curve is nearly horizontal, much of the tax can be passed along. (B) If it is nearly vertical, very little of the tax can be passed along.

16.8 The Deadweight Loss of a Tax

We've seen that taxing a good will typically increase the price paid by the demanders and decrease the price received by the suppliers. This certainly represents a cost to the demanders and suppliers, but from the economist's viewpoint, the real cost of the tax is that the output has been reduced.

The lost output is the social cost of the tax. Let us explore the social cost of a tax using the consumers' and producers' surplus tools developed in Chapter 14. We start with the diagram given in Figure 16.7. This depicts the equilibrium demand price and supply price after a tax, t, has been imposed.

Output has been decreased by this tax, and we can use the tools of consumers' and producers' surplus to value the social loss. The loss in consumers' surplus is given by the areas $A + B$, and the loss in producers' surplus is given in areas $C + D$. These are the same kind of losses that we examined in Chapter 14.

The deadweight loss of a tax. The area $B + D$ measures the deadweight loss of the tax.

Figure 16.7

Since we're after an expression for the social cost of the tax, it seems sensible to add the areas $A+B$ and $C+D$ to each other to get the total loss to the consumers and to the producers of the good in question. However, we've still left out one party—namely, the government.

The government *gains* revenue from the tax. And, of course, the consumers who benefit from the government services provided with these tax revenues also gain from the tax. We can't really say how much they gain until we know what the tax revenues will be spent on.

Let us make the assumption that the tax revenues will just be handed back to the consumers and the producers, or equivalently that the services provided by the government revenues will be just equal in value to the revenues spent on them.

Then the net benefit to the government is the area $A + C$—the total revenue from the tax. Since the loss of producers' and consumers' surpluses are net costs, and the tax revenue to the government is a net benefit, the total net cost of the tax is the algebraic sum of these areas: the loss in

consumers' surplus, $-(A+B)$, the loss in producers' surplus, $-(C+D)$, and the gain in government revenue, $+(A+C)$.

The net result is the area $-(B+D)$. This area is known as the **deadweight loss** of the tax or the **excess burden** of the tax. This latter phrase is especially descriptive.

Recall the interpretation of the loss of consumers' surplus. It is how much the consumers would pay to avoid the tax. In terms of this diagram the consumers are willing to pay $A+B$ to avoid the tax. Similarly, the producers are willing to pay $C+D$ to avoid the tax. Together they are willing to pay $A+B+C+D$ to avoid a tax that raises $A+C$ dollars of revenue. The *excess burden* of the tax is therefore $B+D$.

What is the source of this excess burden? Basically it is the lost value to the consumers and producers due to the reduction in the sales of the good. You can't tax what isn't there.[1] So the government doesn't get any revenue on the reduction in sales of the good. From the viewpoint of society, it is a pure loss—a deadweight loss.

We could also derive the deadweight loss directly from its definition, by just measuring the social value of the lost output. Suppose that we start at the old equilibrium and start moving to the left. The first unit lost was one where the price that someone was willing to pay for it was just equal to the price that someone was willing to sell it for. Here there is hardly any social loss since this unit was the marginal unit that was sold.

Now move a little farther to the left. The demand price measures how much someone was willing to pay to receive the good, and the supply price measures the price at which someone was willing to supply the good. The difference is the lost value on that unit of the good. If we add this up over the units of the good that are not produced and consumed because of the presence of the tax, we get the deadweight loss.

EXAMPLE: The Market for Loans

The amount of borrowing or lending in an economy is influenced to a large degree by the interest rate charged. The interest rate serves as a price in the market for loans.

We can let $D(r)$ be the demand for loans by borrowers and $S(r)$ be the supply of loans by lenders. The equilibrium interest rate, r^*, is then determined by the condition that demand equal supply:

$$D(r^*) = S(r^*). \tag{16.1}$$

Suppose we consider adding taxes to this model. What will happen to the equilibrium interest rate?

[1] At least the government hasn't figured out how to do this yet. But they're working on it.

In the U.S. economy individuals have to pay income tax on the interest they earn from lending money. If everyone is in the same tax bracket, t, the after-tax interest rate facing lenders will be $(1-t)r$. Thus the supply of loans, which depends on the after-tax interest rate, will be $S((1-t)r)$.

On the other hand, the Internal Revenue Service code allows many borrowers to deduct their interest charges, so if the borrowers are in the same tax bracket as the lenders, the after-tax interest rate they pay will be $(1-t)r$. Hence the demand for loans will be $D((1-t)r)$. The equation for interest rate determination with taxes present is then

$$D((1-t)r') = S((1-t)r'). \tag{16.2}$$

Now observe that if r^* solves equation (16.1), then $r^* = (1-t)r'$ must solve equation (16.2) so that

$$r^* = (1-t)r',$$

or

$$r' = \frac{r^*}{(1-t)}.$$

Thus the interest rate in the presence of the tax will be higher by $1/(1-t)$. The *after-tax* interest rate $(1-t)r'$ will be r^*, just as it was before the tax was imposed!

Figure 16.8 may make things clearer. Making interest income taxable will tilt the supply curve for loans up by a factor of $1/(1-t)$; but making interest payments tax deductible will also tilt the demand curve for loans up by $1/(1-t)$. The net result is that the market interest rate rises by precisely $1/(1-t)$.

Inverse demand and supply functions provide another way to look at this problem. Let $r_b(q)$ be the inverse demand function for borrowers. This tells us what the after-tax interest rate would have to be to induce people to borrow q. Similarly, let $r_l(q)$ be the inverse supply function for lenders. The equilibrium amount lent will then be determined by the condition

$$r_b(q^*) = r_l(q^*). \tag{16.3}$$

Now introduce taxes into the situation. To make things more interesting, we'll allow borrowers and lenders to be in different tax brackets, denoted by t_b and t_l. If the market interest rate is r, then the after-tax rate facing borrowers will be $(1-t_b)r$, and the quantity they choose to borrow will be determined by the equation

$$(1-t_b)r = r_b(q),$$

or

$$r = \frac{r_b(q)}{1-t_b}. \tag{16.4}$$

Figure 16.8 **Equilibrium in the loan market**. If borrowers and lenders are in the same tax bracket, the after-tax interest rate and the amount borrowed are unchanged.

Similarly, the after-tax rate facing lenders will be $(1-t_l)r$, and the amount they choose to lend will be determined by the equation

$$(1-t_l)r = r_l(q)$$

or

$$r = \frac{r_l(q)}{1-t_l}. \tag{16.5}$$

Combining equations (16.4) and (16.5) gives the equilibrium condition:

$$r = \frac{r_b(\hat{q})}{1-t_b} = \frac{r_l(\hat{q})}{1-t_l}. \tag{16.6}$$

From this equation it is easy to see that if borrowers and lenders are in the same tax bracket, so that $t_b = t_l$, then $\hat{q} = q^*$. What if they are in different tax brackets? It is not hard to see that the tax law is subsidizing borrowers and taxing lenders, but what is the net effect? If the borrowers face a higher price than the lenders, then the system is a net tax on borrowing, but if the borrowers face a lower price than the lenders, then it is a net subsidy. Rewriting the equilibrium condition, equation (16.6), we have

$$r_b(\hat{q}) = \frac{1-t_b}{1-t_l} r_l(\hat{q}).$$

Thus borrowers will face a higher price than lenders if

$$\frac{1-t_b}{1-t_l} > 1,$$

which means that $t_l > t_b$. So if the tax bracket of lenders is greater than the tax bracket of borrowers, the system is a net tax on borrowing, but if $t_l < t_b$, it is a net subsidy.

EXAMPLE: Food Subsidies

In years when there were bad harvests in nineteenth-century England the rich would provide charitable assistance to the poor by buying up the harvest, consuming a fixed amount of the grain, and selling the remainder to the poor at half the price they paid for it. At first thought this seems like it would provide significant benefits to the poor, but on second thought, doubts begin to arise.

The only way that the poor can be made better off is if they end up consuming more grain. But there is a fixed amount of grain available after the harvest. So how can the poor be better off because of this policy?

As a matter of fact they are not; the poor end up paying exactly the same price for the grain with or without the policy. To see why, we will model the equilibrium with and without this program. Let $D(p)$ be the demand curve for the poor, K the amount demanded by the rich, and S the fixed amount supplied in a year with a bad harvest. By assumption the supply of grain and the demand by the rich are fixed. Without the charity provided by the rich, the equilibrium price is determined by total demand equals total supply:

$$D(p^*) + K = S.$$

With the program in place, the equilibrium price is determined by

$$D(\hat{p}/2) + K = S.$$

But now observe: if p^* solves the first equation, then $\hat{p} = 2p^*$ solves the second equation. So when the rich offer to buy the grain and distribute it to the poor, the market price is simply bid up to twice the original price—and the poor pay the same price they did before!

When you think about it this isn't too surprising. If the demand of the rich is fixed and the supply of grain is fixed, then the amount that the poor can consume is fixed. Thus the equilibrium price facing the poor is determined entirely by their own demand curve; the equilibrium price will be the same, regardless of how the grain is provided to the poor.

EXAMPLE: Subsidies in Iraq

Even subsidies that are put in place "for a good reason" can be extremely difficult to dislodge. Why? Because they create a political constituency that comes to rely on them. This is true in every country, but Iraq represents a particularly egregious case. As of 2005, fuel and food subsidies in Iraq consumed nearly one third of the government's budget.[2]

Almost all of the Iraqi government's budget comes from oil exports. There is very little refining capacity in the country, so Iraq imports gasoline at 30 to 35 cents a liter, which it then sells to the public at 1.5 cents. A substantial amount of this gasoline is sold on the black market and smuggled into Turkey, where gas is about one dollar a liter.

Food and fuel oil are also highly subsidized. Politicians are reluctant to remove these subsidies due to the politically unstable environment. When similar subsidies were removed in Yemen, there was rioting in the streets, with dozens of people dying. A World Bank study concluded that more than half of the GDP in Iraq was spent on subsidies. According to the finance minister, Ali Abdulameer Allawi, "They've reached the point where they've become insane. They distort the economy in a grotesque way, and create the worst incentives you can think of."

16.9 Pareto Efficiency

An economic situation is **Pareto efficient** if there is no way to make any person better off without hurting anybody else. Pareto efficiency is a desirable thing—if there is some way to make some group of people better off, why not do it?—but efficiency is not the only goal of economic policy. For example, efficiency has almost nothing to say about income distribution or economic justice.

However, efficiency is an important goal, and it is worth asking how well a competitive market does in achieving Pareto efficiency. A competitive market, or any economic mechanism, has to determine two things. First, how much is produced, and second, who gets it. A competitive market determines how much is produced based on how much people are willing to pay to purchase the good as compared to how much people must be paid to supply the good.

Consider Figure 16.9. At any amount of output less than the competitive amount q^*, there is someone who is willing to supply an extra unit of the

[2] James Glanz, "Despite Crushing Costs, Iraqi Cabinet Lets Big Subsidies Stand," *New York Times*, August 11, 2005.

Pareto efficiency. The competitive market determines a Pareto efficient amount of output because at q^* the price that someone is willing to pay to buy an extra unit of the good is equal to the price that someone must be paid to sell an extra unit of the good.

Figure 16.9

good at a price that is less than the price that someone is willing to pay for an extra unit of the good.

If the good were produced and exchanged between these two people at any price between the demand price and the supply price, they would both be made better off. Thus any amount less than the equilibrium amount cannot be Pareto efficient, since there will be at least two people who could be made better off.

Similarly, at any output larger than q^*, the amount someone would be willing to pay for an extra unit of the good is less than the price that it would take to get it supplied. Only at the market equilibrium q^* would we have a Pareto efficient amount of output supplied—an amount such that the willingness to pay for an extra unit is just equal to the willingness to be paid to supply an extra unit.

Thus the competitive market produces a Pareto efficient amount of output. What about the way in which the good is allocated among the consumers? In a competitive market everyone pays the same price for a good—the marginal rate of substitution between the good and "all other goods" is equal to the price of the good. Everyone who is willing to pay this price is able to purchase the good, and everyone who is not willing to pay this price cannot purchase the good.

What would happen if there were an allocation of the good where the marginal rates of substitution between the good and "all other goods" were not the same? Then there must be at least two people who value a marginal unit of the good differently. Maybe one values a marginal unit at $5 and one values it at $4. Then if the one with the lower value sells a bit of the good to the one with the higher value at any price between $4 and $5, both people would be made better off. Thus any allocation with different marginal rates of substitution cannot be Pareto efficient.

EXAMPLE: Waiting in Line

One commonly used way to allocate resources is by making people wait in line. We can analyze this mechanism for resource allocation using the same tools that we have developed for analyzing the market mechanism. Let us look at a concrete example: suppose that your university is going to distribute tickets to the championship basketball game. Each person who waits in line can get one ticket for free.

The cost of a ticket will then simply be the cost of waiting in line. People who want to see the basketball game very much will camp out outside the ticket office so as to be sure to get a ticket. People who don't care very much about the game may drop by a few minutes before the ticket window opens on the off chance that some tickets will be left. The willingness to pay for a ticket should no longer be measured in dollars but rather in waiting time, since tickets will be allocated according to willingness to wait.

Will waiting in line result in a Pareto efficient allocation of tickets? Ask yourself whether it is possible that someone who waited for a ticket might be willing to sell it to someone who didn't wait in line. Often this will be the case, simply because willingness to wait and willingness to pay differ across the population. If someone is willing to wait in line to buy a ticket and then sell it to someone else, allocating tickets by willingness to wait does not exhaust all the gains to trade—some people would generally still be willing to trade the tickets after the tickets have been allocated. Since waiting in line does not exhaust all of the gains from trade, it does not in general result in a Pareto efficient outcome.

If you allocate a good using a price set in dollars, then the dollars paid by the demanders provide benefits to the suppliers of the good. If you allocate a good using waiting time, the hours spent in line don't benefit anybody. The waiting time imposes a cost on the buyers of the good and provides no benefits at all to the suppliers. Waiting in line is a form of **deadweight loss**—the people who wait in line pay a "price" but no one else receives any benefits from the price they pay.

Summary

1. The supply curve measures how much people will be willing to supply of some good at each price.

2. An equilibrium price is one where the quantity that people are willing to supply equals the quantity that people are willing to demand.

3. The study of how the equilibrium price and quantity change when the underlying demand and supply curves change is another example of comparative statics.

4. When a good is taxed, there will always be two prices: the price paid by the demanders and the price received by the suppliers. The difference between the two represents the amount of the tax.

5. How much of a tax gets passed along to consumers depends on the relative steepness of the demand and supply curves. If the supply curve is horizontal, all of the tax gets passed along to consumers; if the supply curve is vertical, none of the tax gets passed along.

6. The deadweight loss of a tax is the net loss in consumers' surplus plus producers' surplus that arises from imposing the tax. It measures the value of the output that is not sold due to the presence of the tax.

7. A situation is Pareto efficient if there is no way to make some group of people better off without making some other group worse off.

8. The Pareto efficient amount of output to supply in a single market is that amount where the demand and supply curves cross, since this is the only point where the amount that demanders are willing to pay for an extra unit of output equals the price at which suppliers are willing to supply an extra unit of output.

REVIEW QUESTIONS

1. What is the effect of a subsidy in a market with a horizontal supply curve? With a vertical supply curve?

2. Suppose that the demand curve is vertical while the supply curve slopes upward. If a tax is imposed in this market who ends up paying it?

3. Suppose that all consumers view red pencils and blue pencils as perfect substitutes. Suppose that the supply curve for red pencils is upward sloping. Let the price of red pencils and blue pencils be p_r and p_b. What would happen if the government put a tax only on red pencils?

4. The United States imports about half of its petroleum needs. Suppose that the rest of the oil producers are willing to supply as much oil as the United States wants at a constant price of \$25 a barrel. What would happen to the price of domestic oil if a tax of \$5 a barrel were placed on foreign oil?

5. Suppose that the supply curve is vertical. What is the deadweight loss of a tax in this market?

6. Consider the tax treatment of borrowing and lending described in the text. How much revenue does this tax system raise if borrowers and lenders are in the same tax bracket?

7. Does such a tax system raise a positive or negative amount of revenue when $t_l < t_b$?

CHAPTER 17

MEASUREMENT

Up until now we have used simple algebraic expressions to describe utility functions, production functions, demand curves, supply curves, and so on. For actual applications we have to estimate these functions using statistical techniques. The study of how to do this effectively is known as **econometrics**.

When we analyze data we generally are concerned with the following questions.

Summarize. How can we describe the data succinctly? Example: how many cups of coffee are consumed per person per day?

Estimate. How can we estimate some unknown parameters? Example: what is the elasticity of demand for coffee?

Test. How can we determine whether an unknown parameter satisfies some restriction? Example: do men and women drink the same amount of coffee per day on average?

Forecast. How can we forecast what the price of coffee will be next year?

Predict. How can we predict what would happen to some variable of interest if something changes? Example: if the government imposed a 10% tax on coffee what would happen to consumption?

There are a variety of statistical techniques that can be used to answer such questions, which we will explore in this chapter. Our primary focus will be on estimation and prediction, but we will say a few words about the other topics.

17.1 Summarize data

The simplest way to summarize data is with a table. For example, Table 17.1 depicts data from an online survey of 1,000 consumers who were asked, "On average how many cups of coffee do you drink per day?" The table shows that about 45% of those who responded indicated that they drink zero cups of coffee per day. Closer inspection reveals that 16% averaged one cup a day and about the same number drank two cups per day.

Table 17.1 **Coffee consumption from an online survey**

0 cups	1 cup	2 cups	3 cups	4+ cups
0.448	0.163	0.161	0.110	0.119

This information can be presented in a more vivid way using a barplot (or barchart) as in Figure 17.1. In this chart it is clear that about the same fraction of respondents indicated they consumed 1 or 2 cups a day, and roughly the same number of respondents indicated they consumed 3 or 4+ cups per day.

We can also break down the information by category. The same survey reported gender of the respondents, so we could examine how reported coffee consumption varies across gender as in Table 17.2 or Figure 17.2. As before, a barplot summarizes the information in a way that is more readily understood. For example, it appears that a larger fraction of males than females report drinking 0 cups of coffee and that woman appear to drink more coffee than men overall.

It is often useful to compute various **summary statistics** based on the data. The **mean** number of cups of coffee consumed per day turns out to be 1.28. We can also compute **conditional means**, such as the average number of cups of coffee consumed by those who drink coffee, or the average number of cups of coffee consumed by males. Computing the conditional mean just involves computing an average over those consumers who satisfy

Coffee consumption in sample. The vertical distance indicates the fraction of the sample that said they consumed 0 to 4 or more cups of coffee per day.

Figure 17.1

the relevant condition (drink more than 0 cups per day, are males, and so on). In our sample, men drank 1.24 cups per day on average and women drank 1.39 cups per day.

Average coffee consumption by gender

Table 17.2

Cups	Female	Male
0	0.176	0.219
1	0.093	0.057
2	0.079	0.070
3	0.050	0.046
4+	0.057	0.052

Figure 17.2 **Coffee consumption by gender**. The vertical height indicates the fraction of the sample that said they consumed the indicated number of cups of coffee per day.

EXAMPLE: Simpson's paradox

Conditional means can sometimes behave in surprising ways. Suppose that we plot coffee consumption as a function of income among men and women. A hypothetical relationship might look like Figure 17.3. Note that consumption is increasing in income for both men and women, but that overall consumption is decreasing in income. This phenomenon is an example of **Simpson's paradox**.

Simpson's paradox is not uncommon in real life. Table 17.3 shows the application and admission statistics for men and women to graduate school at the University of California, Berkeley in Fall 1973.

Table 17.3 **Applicants and admissions to UC Berkeley, Fall 1973**

Gender	Applicants	Admitted
Men	8442	44%
Women	4321	35%

Simpson's paradox. In this hypothetical example, coffee consumption increases as income increases for both men and women (the two upward-sloping solid lines), but *decreases* as income increases overall.

Figure 17.3

It appears that men are more likely to have been admitted than were women. Is this an example of gender bias? Table 17.4 breaks the data down by department. In this table, it is easy to see that no department was significantly biased in favor of men; in fact, most departments had a small bias towards women.

Admission by department

Table 17.4

Department	Men Applied	Men Admitted	Women Applied	Women Admitted
A	825	62%	108	82%
B	560	63%	25	68%
C	325	37%	593	34%
D	417	33%	375	35%
E	191	28%	393	24%
F	272	6%	341	7%

A report concluded that the explanation for this apparent paradox was that women tended to apply to departments with low admission rates, while men tended to apply to departments with high admission rates. Even

though there was no evidence of bias at a department level, the overall statistics gave an impression of bias.[1]

17.2 Test

In the previous section we saw that men drank 1.23 cups of coffee per day on average and women drank 1.39 cups per day on average. But this is just one particular sample of a thousand consumers. If we took a different sample, we would find different numbers. How confident can we be that the mean consumption of coffee by women exceeds the mean consumption of coffee by men in the entire population?

One way to answer this question is to pose it in the following way. Suppose that men and women actually drank the *same* amount of coffee per day. How likely would it be in a particular sample of 1,000 consumers to observe one group drinking $1.39 - 1.23 = 0.16$ cups more than the other?

In our sample, it turns out that with a few additional assumptions the probability that we would see a difference at least this large is about 9.6%. In other words, if men and women had the same average consumption in the population, we would see an estimated difference of this magnitude or larger in roughly 1 out of 10 samples. Even though our sample shows that coffee consumption is somewhat different between men and women, we can't be confident that this relationship holds for the population as a whole.

17.3 Estimating demand using experimental data

Suppose you work for a company that sells coffee beans via a website. Your coffee currently sells for \$15 a pound, but you are contemplating cutting the price to \$14. You hope that you will sell more coffee at the lower price, but how much more? Is it worth cutting your price to get more sales?

In this case it is natural to run an experiment to see how the demand for coffee changes when the price changes. For example, you could cut the price of the coffee for a few weeks and see how much additional coffee you sell. If your profit goes up, it might make sense to make the sale price permanent.

Another possibility would be to put the coffee on sale in just a few states or cities and see what happens in those locations. If we try this experiment, it is important to recognize that there are other factors that affect the demand for coffee than just the price. For example, the amount

[1] P. J. Bickel, E. A. Hammel and J. W. O'Connell (1975). "Sex Bias in Graduate Admissions: Data from Berkeley." *Science* 187 (4175): 398-404.

of coffee you sell in a given region during a given period could vary with the season of the year or with the weather.

Ideally, you would choose which cities are treated using some random method such as a coin flip. Such **randomized treatment** helps eliminate sources of systematic bias.

It would also be a good idea to think about ways you could control for these systematic effects. For example, you could compare the sales in the cities where you cut the price to sales in those cities where you kept the price constant. Or you could collect data on weather in the cities you examine and use statistical techniques to control for the observed variation in weather.

In the language of statistics, the cities where you reduce the price of coffee is your **treatment group** and the cities where you leave the price of coffee constant is your **control group**. Running the experiment is simply a small-scale version of the policy you are thinking about implementing—namely, cutting the price for everyone. If you make the experiment as much like the proposed policy as possible, then the experiment will probably give you a pretty good idea about what would happen if you scaled the experiment up to the entire country.

17.4 Effect of treatment

Another thing you could do to estimate how the demand for coffee would respond to a price cut is to send out coupons to a randomly chosen set of people and see how many people use these coupons to buy coffee.

The trouble with this procedure is that the people who redeem the coupons may be different from the population at large. It is likely that people who go to the trouble of using the coupons might be more price-sensitive on average than those who don't bother to use the coupons.

In the case of a coupon, some fraction of the population (the coupon users) are *choosing* to receive a lower price rather than simply facing a lower price for coffee. In general, those who *choose* to be treated are those who are more interested in the treatment and may be more likely to respond to it differently than the population as whole. So the impact of the treatment (the coupon) on those who choose to use it (the treated) could be quite different than the impact of a price cut for everyone.

On the other hand, sometimes you might be interested in the "effect of the treatment on the treated," as opposed to the effect of treatment on the population. For example, if the policy you had in mind was sending out coupons to the entire population, then an experiment that involved sending out coupons to a subset of the population would be an appropriate experiment.

The key issue is whether the consumers are making a choice of whether or not to be treated (that is, to get the lower price). Ideally, the experiment

will mimic the proposed policy as closely as possible.

17.5 Estimating demand using observational data

Let us now consider a different situation. Suppose now you are interested in estimating how the *nationwide* demand for coffee in the United States changes as the price changes. In this case there is no obvious way to do an experiment. Since you don't have **experimental data**, you have to use **observational data**.

The statistical tool that economists use most commonly to address problems of this sort is called **regression**. A regression is simply a way to express conditional expectations. For example, a regression could describe the expected consumption of coffee by a randomly chosen consumer, conditional on the consumer being female. When we estimate a regression we are trying to describe the relationship between a variable of interest (in our case coffee consumption) and other characteristics, such as gender, income, age, price, and so on. There are many varieties of regression, but we will focus on the simplest form, which is called **ordinary least squares** or **OLS**.

So suppose we are given some data on prices and quantity sold of coffee at different time periods. How can we use this data to estimate a demand function?

It is important to think about the **data generation process**: how was this data produced? We can apply some of the theory developed in earlier chapters on consumer choice.

Think of a consumer as purchasing two things, coffee (x_1) and "all other goods" (x_2). Good 2 is sometimes referred to as a **composite commodity** or **quantity index**, as described in Chapter 7.

Denote the price of coffee by p_1, the price of "all other goods" by p_2, and the total expenditure by m. The utility maximization problem for a single consumer is

$$\max_{x_1,x_2} u(x_1, x_2)$$

$$\text{such that } p_1x_1 + p_2x_2 = m.$$

We can write the demand function for coffee as

$$x_1 = D(p_1, p_2, m).$$

As mentioned in section 2.4, we can multiply prices and income by any positive constant and demand stays the same. So let us multiply prices and income by $1/p_2$. This gives us

$$x_1 = D(p_1/p_2, 1, m/p_2).$$

This says the demand for coffee is a function of the price of coffee relative to the price of all other goods and income relative to the price of all other goods. In practice, we compute these numbers by using a price index, like the Consumer Price Index (CPI) or Personal Consumption Expenditure price index (PCEPI). (See the discussion in Chapter 7 on index numbers to see how these are constructed.)

Now we can add up demand across all consumers to get aggregate demand. In order to avoid additional notation, we will use the same notation as above to write $x = D(p, m)$, where x is now the aggregate demand for coffee, p is the price of coffee divided by the CPI, m is total consumer expenditure divided by the CPI, and $D(p, m)$ is the aggregate demand function.

Functional form

We now need to pick an algebraic formula for the demand function. There are three forms for demand functions commonly used in practice.

Linear demand. $x = c + bp + dm$.

Log-linear demand. $\log(x) = \log(c) + b\log(p) + d\log(m)$.

Semi-log demand. $\log(x) = c + bp + dm$.

The most popular form is log-linear demand, since it is easy to interpret the coefficients. As we have seen in Chapter 15, section 8, b and d measure the price and income elasticity of demand, respectively. (In these expressions, all logarithms are natural logarithms.)

Statistical model

Of course, we would not expect our model to fit perfectly, so we need to add an **error term**, denoted by e_t. The error term measures the difference between our ideal specification of demand and the actual observed demand. It can be interpreted as the cumulative effect of all the omitted, unobserved variables that affect demand.

So our final specification of the data-generating process is

$$\log(x_t) = \log(c) + b\log(p_t) + d\log(m_t) + e_t,$$

where the error term is interpreted as the aggregate of all the other variables that might be correlated with coffee consumption.

Under certain conditions, ordinary least squares can be used to provide good estimates of the parameters (b, c, d). The most important condition is that the price of coffee and total expenditure are not correlated with the error term.

It is not hard to see intuitively why this condition is necessary. The coefficient b is supposed to measure how the demand for coffee changes as the price changes *holding everything else constant.* But if p_t and e_t are positively correlated in the data, then increases in p_t will tend to be associated with increases in e_t in our sample. So the observed change in x_t will depend on both the change in p_t *and* the change in e_t. In this case, we say that there is a **confounding effect**. We will get a poor estimate of how a change in price affects coffee consumption if other variables are changing systematically as the price of coffee changes.

The ideal way to ensure that the price of coffee is not correlated with the error term is to run an experiment. In this context that would mean choosing different prices of coffee and seeing how demand responds. However, as described above, **experimental data** of this sort would be difficult to collect for total coffee consumption. Often we are stuck with **observational data**.

Given what we know about the coffee market, is it likely that changes in the price of coffee would be correlated with factors that influence the demand for coffee? As it happens coffee beans are grown in dozens of countries and are sold on a world market. The supply of coffee beans varies significantly from year to year, with the important effects things such as weather, political events, changes in transportation costs, and so on.

From the point of view of a particular country, the price of coffee varies exogenously, since it depend on factors that primarily affect the supply of coffee, not the demand for coffee.

Estimation

All that remains is to actually do the estimation. We can use a statistics package such as R or Stata to estimate the regression described above. The estimated price elasticity turns out to be -0.077, and the estimated income elasticity turns out to be 0.34. This says that a 1% increase in price results in a 0.77% drop in coffee consumption, so demand for coffee is pretty inelastic. It turns out that this estimate is rather imprecise, but it is the best we can do with the available data.

17.6 Identification

In estimating the demand for coffee, we argued that the world price of coffee was exogenous from the point of view of a specific country. In terms of

supply and demand, we are saying that supply curve facing a single country is more or less flat at the equilibrium price. The price could shift from year to year, depending on weather and other factors, and the resulting equilibria would trace out the demand curve, as shown in Figure 17.4.

Shift in supply. In this example, the shifts in the supply curve of coffee trace out the demand curve.

Figure 17.4

But suppose we are interested in the *world* demand for coffee. In this case it is unreasonable to assume that the price is determined exogenously; rather it is determined by the interaction of supply and demand.

For example, we might think that the supply of coffee is more or less fixed in a given year but varies from year to year depending on the weather. In this case, the supply curve shifts, but the demand curve remains constant, and the observed prices and quantities would still lie along the demand function. So estimating demand as a function of price would still make sense.

The problematic case is where both supply and demand are shifting, as in Figure 17.5. In this case, it is impossible to estimate either curve. Generally you can estimate a demand function if there is something that shifts supply and not demand, and you can estimate a supply function if there is something that shifts demand but not supply. But if both curves shift in unknown ways, we can't identify what is driving price changes and quantity changes. This is known as the **identification problem**.

Figure 17.5 **Estimating demand**. Here both demand and supply curves shift over time, so we can't estimate either curve without further information.

17.7 What can go wrong?

Let us return to the simple demand estimation problem described earlier, but now consider a situation where the price of a product is set by the seller rather than the price being exogenously determined on world markets. To be specific, suppose a company called KoffeeTime makes a cold drink called Koffeetino. Over the years they have set the price according to market conditions. When the economic activity is slow, due to a recession, they see the sales of Koffeetino drop, so they quickly cut the price in response. When the economy is booming, they see the sales are high, so they raise the price.

This means that in the historical data we will see high prices associated with high sales and vice versa. The observed "demand curve" slopes up!

What is going on? We usually think that high prices cause consumers to purchase less. Here the drop in consumption is causing the prices to be cut. But what is causing the drop in consumption? The answer, in this case, is that income has dropped due to the "bad times." Income, in this case, is a **confounding variable** since it affects both the right-hand side and the left-hand side of the regression—both the price and the quantity.

For a *fixed* level of income, we would expect to see higher prices leading to less demand and lower prices leading to more demand for Koffeetino. If we add income to the regression (as theory tells us to do), then it is possible

that we will get a meaningful estimate of price elasticity In econometric language, this is an example of **missing variable bias**: we failed to include an important variable in the regression and so we got a biased estimate of the effect.

But, in reality, there are always omitted variables—we can never have a complete list of everything that affects demand. For example, it may be that weather affects Koffeetino. In years with particularly cold temperatures sales fall and in warm years sales rise. The company may respond by raising or lowering price in response to the change in sales, leading to the same problem we had before.

As mentioned earlier, omitted variables that are not correlated with price are not much of a problem. But omitted variables that are correlated with price (confounding variables) can result in biased estimates. This will often be the case when the price is chosen, since the choice may depend on lots of things that the econometrician may not subsequently observe.

As it turns out, there are ways to address this problem that are covered in more advanced courses. Experiments are the gold standard, but sometimes observational data can be used to estimate causal effects even without explicit experiments.

17.8 Policy evaluation

A common reason to estimate the magnitude of some effect is that we are contemplating some policy change. Ideally, we would run an experiment on a small scale to estimate the impact of the proposed change. But, as we have seen, it is sometimes difficult or costly to run such an experiment.

Sometimes we can find a **natural experiment** that is similar to the ideal experiment we would run if we could. For example, in 2008 the State of Oregon ran a lottery among low-income adults to determine who would be allowed to apply for Medicare. A year after this lottery, the treatment group—those who were allowed to apply for Medicare—were substantially more likely to be covered by health insurance than those who weren't allowed to apply.[2]

The researchers were able to see how the treated group differed from the control group. In the first year of the study, it turned out that the treated group had higher health care utilization, lower out-of-pocket medical expenditures and medical debt, and better self-reported physical and mental health than the control group. One might reasonably expect that this would carry over to larger populations who were offered the chance to sign up for Medicare.

[2] Amy Finkelstein et. al., "The Oregon Health Insurance Experiment: Evidence from the First Year," http://economics.mit.edu/files/6796.

Of course, offering people the *opportunity* to sign up for Medicare is different from extending it to the entire population. In the first case, people still choose to apply and those who apply could be different from the entire population in relevant ways.

EXAMPLE: Crime and police

It is important to distinguish correlation and causation. A classic example: if we observe more police in precincts with high crime rates, can we conclude that police cause crime? Of course not. A more likely explanation is that the causality runs the other direction: more police were assigned to high crime areas *because* they had high crime.

If we use statistical procedures to estimate the relationship between numbers of police and crime rates, we may well see a positive relationship (more police are associated with more crime). However, that says nothing about what would happen if we deliberately assigned more police to a given precinct.

In order to understand the *causal* impact of police on crime rates, we need to understand (1) how police were assigned to precincts in the historical data and (2) how the assignment of additional police to a given precinct changes crime rates.

Ideally we would use a controlled experiment to determine how the number of police affects crime rates. However, sometimes there might be a "natural experiment" that mimics such a random assignment. For example, the police department in Washington, DC, increases the number of police on the street during periods when there are security alerts concerning elevated risk of terrorist activity. Two economists examined the data on crime reports for these days and found that crime was substantially lower, particularly for auto theft.[3]

Summary

1. Statistics can be used to summarize, estimate, test, and predict.

2. Omitted variable bias occurs when the analyst fails to include an important variable in the regression that is correlated with other variables. In this case the omitted variable is known as a confounding variable.

3. Observational data can only tell us about correlations, but we normally need experiments to determine causality.

[3] Jonathan Klick and Alexander Tababrok, "Using Terror Alert Levels to Estimate the Effect of Police on Crime," *Journal of Law and Economics* 48:1 (April 2005), 267–79.

4. However, in some cases there are natural experiments that can be useful in answering questions of interest.

5. It is important to distinguish between the effect of a policy that applies to the entire population and the effect of a policy that applies only to those who choose to participate.

6. In general, in evaluating a policy proposal, the experiment used should be as close as possible to the policy being considered.

REVIEW QUESTIONS

1. When the *Titanic* sank in 1912, both male and female crew members had a higher survival rate than the third-class passengers. However, overall the third-class passengers had a higher survival rate than the crew. What do we call this phenomenon?

2. Suppose that you want to test the hypothesis that a coin has a probability of 1/2 of coming up heads when you flip it. You flip it 5 times and it comes up heads every time. How likely is it that you would see a pattern of 5 heads in a row if true probability of coming up heads is 1/2?

3. Suppose we estimate a demand function of the form $x = e^{c+bp}$, where p is price, x is the quantity consumed, and b is a parameter. What is this functional form called?

CHAPTER 18

AUCTIONS

Auctions are one of the oldest form of markets, dating back to at least 500 BC. Today, all sorts of commodities, from used computers to fresh flowers, are sold using auctions.

Economists became interested in auctions in the early 1970s when the OPEC oil cartel raised the price of oil. The U.S. Department of the Interior decided to hold auctions to sell the right to drill in coastal areas that were expected to contain vast amounts of oil. The government asked economists how to design these auctions, and private firms hired economists as consultants to help them design a bidding strategy. This effort prompted considerable research in auction design and strategy.

More recently, the Federal Communications Commission (FCC) decided to auction off parts of the radio spectrum for use by cellular phones, personal digital assistants, and other communication devices. Again, economists played a major role in the design of both the auctions and the strategies used by the bidders. These auctions were hailed as very successful public policy, resulting in revenues to the U.S. government of over twenty-three billion dollars to date.

Other countries have also used auctions for privatization projects. For example, Australia sold off several government-owned electricity plants, and New Zealand auctioned off parts of its state-owned telephone system.

Consumer-oriented auctions have also experienced something of a renaissance on the Internet. There are hundreds of auctions on the Internet, selling collectibles, computer equipment, travel services, and other items. OnSale claims to be the largest, reporting over forty-one million dollars worth of merchandise sold in 1997.

18.1 Classification of Auctions

The economic classification of auctions involves two considerations: first, what is the nature of the good that is being auctioned, and second, what are the rules of bidding? With respect to the nature of the good, economists distinguish between **private-value auctions** and **common-value auctions**.

In a private-value auction, each participant has a potentially different value for the good in question. A particular piece of art may be worth $500 to one collector, $200 to another, and $50 to yet another, depending on their taste. In a common-value auction, the good in question is worth essentially the same amount to every bidder, although the bidders may have different estimates of that common value. The auction for off-shore drilling rights described above had this characteristic: a given tract either had a certain amount of oil or not. Different oil companies may have had different estimates about how much oil was there, based on the outcomes of their geological surveys, but the oil had the same market value regardless of who won the auction.

We will spend most of the time in this chapter discussing private-value auctions, since they are the most familiar case. At the end of the chapter, we will describe some of the features of common-value auctions.

Bidding Rules

The most prevalent form of bidding structure for an auction is the **English auction**. The auctioneer starts with a **reserve price**, which is the lowest price at which the seller of the good will part with it.[1] Bidders successively offer higher prices; generally each bid must exceed the previous bid by some minimal **bid increment**. When no participant is willing to increase the bid further, the item is awarded to the highest bidder.

Another form of auction is known as a **Dutch auction**, due to its use in the Netherlands for selling cheese and fresh flowers. In this case the auctioneer starts with a high price and gradually lowers it by steps until someone is willing to buy the item. In practice, the "auctioneer" is often

[1] See the footnote about "reservation price" in Chapter 6, which is essentially the same concept.

a mechanical device like a dial with a pointer which rotates to lower and lower values as the auction progresses. Dutch auctions can proceed very rapidly, which is one of their chief virtues.

Yet a third form of auctions is a **sealed-bid auction**. In this type of auction, each bidder writes down a bid on a slip of paper and seals it in an envelope. The envelopes are collected and opened, and the good is awarded to the person with the highest bid who then pays the auctioneer the amount that he or she bid. If there is a reserve price, and all bids are lower than the reserve price, then no one may receive the item.

Sealed-bid auctions are commonly used for construction work. The person who wants the construction work done requests bids from several contractors with the understanding that the job will be awarded to the contractor with the lowest bid.

Finally, we consider a variant on the sealed bid-auction that is known as the **philatelist auction** or **Vickrey auction**. The first name is due to the fact that this auction form was originally used by stamp collectors; the second name is in honor of William Vickrey, who received the 1996 Nobel prize for his pioneering work in analyzing auctions. The Vickrey auction is like the sealed-bid auction, with one critical difference: the good is awarded to the highest bidder, but at the *second-highest* price. In other words, the person who bids the most gets the good, but he or she only has to pay the bid made by the second-highest bidder. Though at first this sounds like a rather strange auction form, we will see below that it has some very nice properties.

18.2 Auction Design

Let us suppose that we have a single item to auction off and that there are n bidders with (private) values $v_1, \ldots, v_n$. For simplicity, we assume that the values are all positive and that the seller has a zero value. Our goal is to choose an auction form to sell this item.

This is a special case of an **economic mechanism design** problem. In the case of the auction there are two natural goals that we might have in mind:

- **Pareto efficiency.** Design an auction that results in a Pareto efficient outcome.
- **Profit maximization.** Design an auction that yields the highest expected profit to the seller.

Profit maximization seems pretty straightforward, but what does Pareto efficiency mean in this context? It is not hard to see that Pareto efficiency requires that the good be assigned to the person with the highest value.

To see this, suppose that person 1 has the highest value and person 2 has some lower value for the good. If person 2 receives the good, then there is an easy way to make both 1 and 2 better off: transfer the good from person 2 to person 1 and have person 1 pay person 2 some price p that lies between v_1 and v_2. This shows that assigning the good to anyone but the person who has the highest value cannot be Pareto efficient.

If the seller knows the values $v_1, \ldots, v_n$ the auction design problem is pretty trivial. In the case of profit maximization, the seller should just award the item to the person with the highest value and charge him or her that value. If the desired goal is Pareto efficiency, the person with the highest value should still get the good, but the price paid could be any amount between that person's value and zero, since the distribution of the surplus does not matter for Pareto efficiency.

The more interesting case is when the seller does not know the buyers' values. How can one achieve efficiency or profit maximization in this case?

First consider Pareto efficiency. It is not hard to see that an English auction achieves the desired outcome: the person with the highest value will end up with the good. It requires only a little more thought to determine the price that this person will pay: it will be the value of the *second-highest* bidder plus, perhaps, the minimal bid increment.

Think of a specific case where the highest value is, say \$100, the second-highest value is \$80, and the bid increment is, say, \$5. Then the person with the \$100 valuation would be willing to bid \$85, while the person with the \$80 value would not. Just as we claimed, the person with the highest valuation gets the good, at the second highest price (plus, perhaps, the bid increment). (We keep saying "perhaps" since if both players bid \$80 there would be a tie and the exact outcome would depend on the rule used for tie-breaking.)

What about profit maximization? This case turns out to be more difficult to analyze since it depends on the *beliefs* that the seller has about the buyers' valuations. To see how this works, suppose that there are just two bidders either of whom could have a value of \$10 or \$100 for the item in question. Assume these two cases are equally likely, so that there are four equally probable arrangements for the values of bidders 1 and 2: (10,10), (10,100), (100,10), (100,100). Finally, suppose that the minimal bid increment is \$1 and that ties are resolved by flipping a coin.

In this example, the winning bids in the four cases described above will be (10,11,11,100) and the bidder with the highest value will always get the good. The expected revenue to the seller is $\$33 = \frac{1}{4}(10 + 11 + 11 + 100)$.

Can the seller do better than this? Yes, if he sets an appropriate reservation price. In this case, the profit-maximizing reservation price is \$100. Three-quarters of the time, the seller will sell the item for this price, and one-quarter of the time there will be no winning bid. This yields an expected revenue of \$75, much higher than the expected revenue yielded by the English auction with no reservation price.

Note that this policy is *not* Pareto efficient, since one-quarter of the time no one gets the good. This is analogous to the deadweight loss of monopoly and arises for exactly the same reason.

The addition of the reservation price is very important if you are interested in profit maximization. In 1990, the New Zealand government auctioned off some of the spectrum for use by radio, television, and cellular telephones, using a Vickrey auction. In one case, the winning bid was NZ$100,000, but the second-highest bid was only NZ$6! This auction may have led to a Pareto efficient outcome, but it was certainly not revenue maximizing!

We have seen that the English auction with a zero reservation price guarantees Pareto efficiency. What about the Dutch auction? The answer here is not necessarily. To see this, consider a case with two bidders who have values of $100 and $80. If the high-value person believes (erroneously!) that the second-highest value is $70, he or she would plan to wait until the auctioneer reached, say, $75 before bidding. But, by then, it would be too late—the person with the second-highest value would have already bought the good at $80. In general, there is no guarantee that the good will be awarded to the person with the highest valuation.

The same holds for the case of a sealed-bid auction. The optimal bid for each of the agents depends on their *beliefs* about the values of the other agents. If those beliefs are inaccurate, the good may easily end up being awarded to someone who does not have the highest valuation.[2]

Finally, we consider the Vickrey auction—the variant on the sealed-bid auction where the highest bidder gets the item, but only has to pay the second-highest price.

First we observe that *if* everyone bids their true value for the good in question, the item will end up being awarded to the person with the highest value, who will pay a price equal to that of the person with the second-highest value. This is essentially the same as the outcome of the English auction (up to the bid increment, which can be arbitrarily small).

But is it optimal to state your true value in a Vickrey auction? We saw that for the standard sealed-bid auction, this is not generally the case. But the Vickrey auction is different: the surprising answer is that it is always in each player's interest to write down his or her true value.

To see why, let us look at the special case of two bidders, who have values v_1 and v_2 and write down bids of b_1 and b_2. The expected payoff to bidder 1 is:

$$\text{Prob}(b_1 \geq b_2)[v_1 - b_2],$$

[2] On the other hand, if all players' beliefs are accurate, on average, and all bidders play optimally, the various auction forms described above turn out to yield the same allocation and the same expected price in equilibrium. For a detailed analysis, see P. Milgrom, "Auctions and Bidding: a Primer," *Journal of Economic Perspectives*, **3**(3), 1989, 3–22, and P. Klemperer, "Auction Theory: A Guide to the Literature," *Economic Surveys*, **13**(3), 1999, 227–286.

where "Prob" stands for "probability."

The first term in this expression is the probability that bidder 1 has the highest bid; the second term is the consumer surplus that bidder 1 enjoys if he wins. (If $b_1 < b_2$, then bidder 1 gets a surplus of 0, so there is no need to consider the term containing $\text{Prob}(b_1 \leq b_2)$.)

Suppose that $v_1 > b_2$. Then bidder 1 wants to make the probability of winning as large as possible, which he can do by setting $b_1 = v_1$. Suppose, on the other hand, that $v_1 < b_2$. Then bidder 1 wants to make the probability of winning as small as possible, which he can do by setting $b_1 = v_1$. In either case, an optimal strategy for bidder 1 is to set his bid equal to his true value! Honesty is the best policy ... at least in a Vickrey auction!

The interesting feature of the Vickrey auction is that it achieves essentially the same outcome as an English auction, but without the iteration. This is apparently why it was used by stamp collectors. They sold stamps at their conventions using English auctions and via their newsletters using sealed-bid auctions. Someone noticed that the sealed-bid auction would mimic the outcome of the English auctions if they used the second-highest bid rule. But it was left to Vickrey to conduct the full-fledged analysis of the philatelist auction and show that truth-telling was the optimal strategy and that the philatelist auction was equivalent to the English auction.

EXAMPLE: Goethe's auction

In 1797 the German poet Johann Wolfgang von Goethe completed a poem that he wanted to offer to a publisher. He sent one of the possible publishers a letter that contained this passage:

> I am inclined to offer Mr. Vieweg from Berlin an epic poem, *Hermann and Dorothea,* which will have approximately 2000 hexameters ... Concerning the royalty we will proceed as follows: I will hand over to Mr. Counsel Böttiger a sealed note which contains my demand, and I wait for what Mr. Vieweg will suggest to offer for my work. If his offer is lower than my demand, then I take my note back, unopened, and the negotiation is broken. If, however, his offer is higher, then I will not ask for more than what is written in the note to be opened by Mr. Böttiger.

In essence this is a Vickrey auction. The publisher's dominant strategy is to name his true value, which will allow him to acquire the book only if Goethe's reserve price is less than his true value.

It was a great plan, but Goethe's lawyer, Böttiger, leaked the price in the sealed envelope, which was 1,000 thalers.[3] The publisher therefore bid the

[3] This refers to currency used at the time in Germany and is the ancestor of the term "dollars."

minimal amount and ended up making an estimated profit of 2,600 thalers.

Goethe was apparently suspicious that something was amiss, so the next time he wanted to sell a work, he set up a competitive auction involving 36 publishers and ended up doing much better.[4]

18.3 Other Auction Forms

The Vickrey auction was thought to be only of limited interest until online auctions became popular. The world's largest online auction house, eBay, claims to have almost 30 million registered users who, in 2000, traded $5 billion worth of merchandise.

Auctions run by eBay last for several days, or even weeks, and it is inconvenient for users to monitor the auction process continually. In order to avoid constant monitoring, eBay introduced an automated **bidding agent**, which they call a **proxy bidder**. Users tell their bidding agent the most they are willing to pay for an item and an initial bid. As the bidding progresses, the agent automatically increases a participant's bid by the minimal bid increment when necessary, as long as this doesn't raise the participant's bid over his or her maximum.

Essentially this is a Vickrey auction: each user reveals to their bidding agent the maximum price he or she is willing to pay. In theory, the participant who enters the highest bid will win the item but will only have to pay the second-highest bid (plus a minimal bid increment to break the tie.) According to the analysis in the text, each bidder has an incentive to reveal his or her true value for the item being sold.

In practice, bidder behavior is a bit different than that predicted by the Vickrey model. Often bidders wait until close to the end of the auction to enter their bids. This behavior appears to be for two distinct reasons: a reluctance to reveal interest too early in the game, and the hope to snatch up a bargain in an auction with few participants. Nevertheless, the bidding agent model seems to serve users very well. The Vickrey auction, which was once thought to be only of theoretical interest, is now the preferred method of bidding for the world's largest online auction house!

There are even more exotic auction designs in use. One peculiar example is the **escalation auction**. In this type of auction, the highest bidder wins the item, but the highest and the second-highest bidders *both* have to pay the amount they bid.

Suppose, for example, that you auction off 1 dollar to a number of bidders under the escalation auction rules. Typically a few people bid 10 or 15 cents, but eventually most of the bidders drop out. When the highest bid

[4] See the entire story in Benny Moldovanu and Manfred Tietzel, "Goethe's Second-Price Auction," *The Journal of Political Economy*, Vol. 106, No. 4 (Aug.1998), pp. 854–859.

approaches 1 dollar, the remaining bidders begin to catch on to the problem they face. If one has bid 90 cents, and the other 85 cents, the low bidder realizes that if he stays put, he will pay 85 cents and get nothing but, if he escalates to 95 cents, he will walk away with a nickel.

But once he has done this, the bidder who was at 90 cents can reason the same way. In fact, it is in her interest to bid *over* a dollar. If, for example, she bids $1.05 (and wins), she will lose only 5 cents rather than 90 cents! It's not uncommon to see the winning bid end up at $5 or $6.

A somewhat related auction is the **everyone pays auction**. Think of a crooked politician who announces that he will sell his vote under the following conditions: all the lobbyists contribute to his campaign, but he will vote for the appropriations favored by the highest contributor. This is essentially an auction where everyone pays but only the high bidder gets what she wants!

EXAMPLE: Late Bidding on eBay

According to standard auction theory eBay's proxy bidder should induce people to bid their true value for an item. The highest bidder wins at (essentially) the second highest bid, just as in a Vickrey auction. But it doesn't work quite like that in practice. In many auctions, participants wait until virtually the last minute to place their bids. In one study, 37 percent of the auctions had bids in the last minute and 12 percent had bids in the last 10 seconds. Why do we see so many "late bids"?

There are at least two theories to explain this phenomenon. Patrick Bajari and Ali Hortaçsu, two auction experts, argue that for certain sorts of auctions, people don't want to bid early to avoid driving up the selling price. EBay typically displays the bidder identification and actual bids (not the maximum bids) for items being sold. If you are an expert on rare stamps, with a well-known eBay member name, you may want to hold back placing your bid so as not to reveal that you are interested in a particular stamp.

This explanation makes a lot of sense for collectibles such as stamps and coins, but late bidding also occurs in auctions for generic items, such as computer parts. Al Roth and Axel Ockenfels suggest that late bidding is a way to avoiding bidding wars.

Suppose that you and someone else are bidding for a Pez dispenser with a seller's reserve price of $2. It happens that you each value the dispenser at $10. If you both bid early, stating your true maximum value of $10, then even if the tie is resolved in your favor you end up paying $10—since that is also the other bidder's maximum value. You may "win" but you don't get any consumer surplus!

Alternatively, suppose that each of you waits until the auction is almost over and then bids $10 in the last possible seconds of the auction. (At

eBay, this is called "sniping.") In this case, there's a good chance that one of the bids won't get through, so the winner ends up paying only the seller's reserve price of $2.

Bidding high at the last minute introduces some randomness into the outcome. One of the players gets a great deal and the other gets nothing. But that's not necessarily so bad: if they both bid early, one of the players ends up paying his full value and the other gets nothing.

In this analysis, the late bidding is a form of "implicit collusion." By waiting to bid, and allowing chance to play a role, bidders can end up doing substantially better on average than they do by bidding early.

18.4 Position Auctions

A **position auction** is a way to auction off positions, such as a position in a line or a position on a web page. The defining characteristic is that all players rank the positions in the same way, but they may value the positions differently. Everybody would agree that it is better to be in the front of the line than further back, but they could be willing to pay different amounts to be first in line.

One prominent example of a position auction is the auction used by search engine providers such as Google, Microsoft, and Yahoo to sell ads. In this case all advertisers agree that being in the top position is best, the second from the top position is second best, and so on. However, the advertisers are often selling different things, so the expected profit that they will get from a visitor to their web page will differ.

Here we describe a simplified version of these online ad auctions. Details differ across search engines, but the model below captures the general behavior.

We suppose that there are $s = 1, \ldots, S$ slots where ads can be displayed. Let x_s denote the number of clicks that an ad can expect to receive in slot s. We assume that slots are ordered with respect to the number of clicks they are likely to receive, so $x_1 > x_2 > \cdots > x_S$.

Each of the advertisers has a value per click, which is related to the expected profit it can get from a visitor to its web site. Let v_s be the value per click of the advertiser whose ad is shown in slot s.

Each advertiser states a bid, b_s, which is interpreted as the amount it is willing to pay for slot s. The best slot (slot 1) is awarded to the advertiser with the highest bid, the second-best slot (slot 2) is awarded to the advertiser with the second highest bid, and so on.

The price that an advertiser pays for a bid is determined by the bid of the advertiser below him. This is a variation on the Vickrey auction model described earlier and is sometimes known as a **generalized second price auction** or GSP.

In the GSP, advertiser 1 pays b_2 per click, advertiser 2 pays b_3 per click, and so on. The rationale for this arrangement is that if an advertiser paid the price it bid, it would have an incentive to cut its bid until it just beat the advertiser below it. By setting the payment of the advertiser in slot s to be the bid of the advertiser in slot $s+1$, each advertiser ends up paying the minimum bid necessary to retain its position.

Putting these pieces together, we see that the profit of the advertiser in slot s is $(v_s - b_{s+1})x_s$. This is just the value of the clicks minus the cost of the clicks that an advertiser receives.

What is the equilibrium of this auction? Extrapolating from the Vickrey auction, one might speculate that each advertiser should bid its true value. This is true if there is only one slot being auctioned, but is false in general.

Two Bidders

Let us look at the case of 2 slots and 2 bidders. We assume that the high bidder gets x_1 clicks and pays the bid of the second highest bidder b_2. The second highest bidder gets slot 2 and pays a reserve price r.

Suppose your value is v and you bid b. If $b > b_2$ you get a payoff of $(v - b_2)x_1$ and if $b \leq b_2$ you get a payoff of $(v - r)x_2$. Your expected payoff is then

$$\text{Prob}(b > b_2)(v - b_2)x_1 + [1 - \text{Prob}(b > b_2)](v - r)x_2.$$

We can rearrange your expected payoff to be

$$(v - r)x_2 + \text{Prob}(b > b_2)[v(x_1 - x_2) + rx_2 - b_2x_1] \tag{18.1}$$

Note that when the term in the brackets is positive (i.e., you make a profit), you want the probability that $b > b_2$ to be as large as possible, and when the term is negative (you make a loss) you want the probability that $b > b_2$ to be as small as possible.

However, this can easily be arranged. Simply choose a bid according to this formula:

$$bx_1 = v(x_1 - x_2) + rx_2.$$

Now it is easy to check that when $b > b_2$, the bracketed term in expression (18.1) is positive and when $b \leq b_2$ the bracketed term in (18.1) is negative or zero. Hence this bid will win the auction exactly when you want to win and lose it exactly when you want to lose.

Note that this bidding rule is a dominant strategy: each bidder wants to bid according to this formula, regardless of what the other player bids. This means, of course, that the auction ends up putting the bidder with the highest value in first place.

It is also easy to interpret the bid. If there are two bidders and two slots, the second highest bidder will always get the second slot and end up paying rx_2. The contest is about the *extra* clicks that the highest bidder gets. The bidder who has the highest value will win those clicks, but that bidder only has to pay the minimum amount necessary to beat the second highest bidder.

We see that in this auction, you don't want to bid your true value per click, but you do want to bid an amount that reflects your true value of the *incremental clicks* you are getting.

More Than Two Bidders

What happens if there are more than two bidders? In this case, there will typically not be a dominant strategy equilibrium, but there will be a equilibrium in prices. Let us look at a situation with 3 slots and 3 bidders.

The bidder in slot 3 pays a reservation price r. In equilibrium, the bidder won't want to move up to slot 2, so

$$(v_3 - r)x_3 \geq (v_3 - p_2)x_2$$

or

$$v_3(x_2 - x_3) \leq p_2x_2 - rx_3.$$

This inequality says that if the bidder prefers position 3 to position 2, the value of the extra clicks it gets in position 2 must be less than the cost of those extra clicks.

This inequality gives us a bound on the cost of clicks in position 2:

$$p_2x_2 \leq rx_3 + v_3(x_2 - x_3). \tag{18.2}$$

Applying the same argument to the bidder in position 2, we have

$$p_1x_1 \leq p_2x_2 + v_2(x_1 - x_2). \tag{18.3}$$

Substituting inequality (18.2) into inequality (18.3) we have

$$p_1x_1 \leq rx_3 + v_3(x_2 - x_3) + v_2(x_1 - x_2). \tag{18.4}$$

The total revenue in the auction is $p_1x_1 + p_2x_2 + p_3x_3$. Adding together inequality (18.2), inequality (18.3), and the revenue for slot 3 we have a lower bound on the total revenue in the auction

$$R_L \leq v_2(x_1 - x_2) + 2v_3(x_2 - x_3) + 3rx_3.$$

So far, we have looked at 3 bidders for 3 slots. What happens if there are 4 bidders for the 3 slots? In this case the reserve price is replaced by

the value of the fourth bidder. The logic is that the fourth bidder is willing to buy any clicks that exceed its value, just as with the standard Vickrey auction. This gives us a revenue expression of

$$R_L \leq v_2(x_1 - x_2) + 2v_3(x_2 - x_3) + 3v_4x_3.$$

We note a few things about this expression. First, the competition in the search engine auction is about incremental clicks: how many clicks you get if you bid for a higher position. Second, the bigger the gap between clicks the larger the revenue. Third, when $v_4 > r$ the revenue will be larger. This simply says that competition tends to increase revenue.

Quality Scores

In practice, the bids are multiplied by a **quality score** to get an auction ranking score. The ad with the highest bid times quality gets first position, the second-highest ranking ad gets the second position, and so on. Each ad pays the minimum price per click necessary to retain its position. If we let q_s be the quality of the ad in slot s, the ads are ordered by $b_1q_1 > b_2q_s > b_3q_3 \cdots$ and so on.

The price that the ad in slot 1 pays is just enough to retain its position, so $p_1q_1 = b_2q_2$, or $p_1 = b_2q_2/q_1$. (There may be some rounding to break ties.)

There are several components of ad quality. However, the major component is typically the historical clickthrough rate that an ad gets. This means that ad rank is basically determined by

$$\frac{\text{cost}}{\text{clicks}} \times \frac{\text{clicks}}{\text{impressions}} = \frac{\text{cost}}{\text{impressions}}$$

Hence the ad that gets first place will be the one that is willing to pay the most per impression (i.e., ad view) rather than price per click.

When you think about it, this makes a lot of sense. Suppose one advertiser is willing to pay \$10 per click but is likely to get only 1 click in a day. Another advertiser is willing to pay \$1 per click will get 100 clicks in a day. Which ad should be shown in the most prominent position?

Ranking ads in this way also helps the users. If two ads have the same bid, then the one that users tend to click on more will get a higher position. Users can "vote with their clicks" for the ads that they find the most useful.

18.5 Should you advertise on your brand?

One question that sometimes arises in online ad auctions is whether advertisers should advertise on their own brand? This is particularly important

for advertisers who have strong, well-known brands since they tend to be shown high up in the organic search results. Why should a well-known brand pay for ad clicks when it would get organic clicks anyway?

Let us examine this question using a little algebra. As above, we let v be the value of a click (a visit to the website), which we will assume is the same for an organic click and an ad click. Let x_a be the number of ad clicks, x_{oa} be the number of organic clicks when an ad is present, and x_{on} be the number of organic clicks when the ad is not present. Finally, let $c(x_a)$ be the cost of x_a ad clicks.

If the website owner chooses to advertise, it has a profit of $vx_a + vx_{oa} - c(x_a)$. Note that advertiser gets both ad clicks and search results clicks, but only pays for the ad clicks. If the website owner chooses not to advertise, it gets vx_{on}. Putting these two expressions together, we see that the website owner will find it profitable to advertise when:

$$vx_a + vx_{oa} - c(x_a) > vx_{on},$$

where we have assumed the numerator is positive. Collecting terms, we see that the website owner will want to advertise when

$$v > \frac{c(x_a)}{x_a - (x_{on} - x_{oa})}.$$

The important piece of this expression is $x_{on} - x_{oa}$, which measures how the ad "cannibalizes" the organic clicks. If there is no cannibalization, so that $x_{on} = x_{oa}$, then the expression just reduces to "value greater than average cost." On the other hand, if there is a large amount of cannibalization, the value of a visitor would have to be high to overcome the reduction in organic clicks.

18.6 Auction revenue and number of bidders

It is interesting to see how the auction revenue changes as the number of bidders increases. Suppose that there is some distribution of values by the buyers, and we end up with a random draw of n bidders for the auction, who have values $(v_1, \ldots, v_n)$. To keep things simple, suppose the reserve price is 0. If we have only one bidder, his value is v_1 and he gets the item for free. If we draw another bidder from the population, he has probability 1/2 of having a larger bid than v_1 and the expected revenue is $\min(v_1, v_2)$. If we draw a third bidder, there is a 1/3 probability that he has the maximum bid, and so it goes.

The general principle is that the expected revenue will keep increasing as the number of bidders increases, but it will do so at a slower rate. The expected revenue will be the expected value of the second-largest valuation in a sample of size n, a number known as the **second order statistic**. If

we specify a particular distribution of values, we can see how this evolves as we add more participants to the auction.

Figure 18.1 depicts an example of what the expected revenue looks like if the values are distributed uniformly on the interval $[0, 1]$. As you can see, by the time there are 10 or so bidders, the expected value is pretty close to 1, illustrating that auctions are a pretty good way to generate revenue.

Auction revenue. This shows how auction revenue changes as the number of bidders changes.

Figure 18.1

18.7 Problems with Auctions

We've seen above that English auctions (or Vickrey auctions) have the desirable property of achieving Pareto efficient outcomes. This makes them attractive candidates for resource allocation mechanisms. In fact, most of the airwave auctions used by the FCC were variants on the English auction.

But English auctions are not perfect. They are still susceptible to collusion. The example of pooling in auction markets, described in Chapter 25, shows how antique dealers in Philadelphia colluded on their bidding strategies in auctions.

There are also various ways to manipulate the outcome of auctions. In the analysis described earlier, we assumed that a bid *committed* the bidder to pay. However, some auction designs allow bidders to drop out once

the winning bids are revealed. Such an option allows for manipulation. For example, in 1993 the Australian government auctioned off licenses for satellite-television services using a standard sealed-bid auction. The winning bid for one of the licenses, A$212 million, was made by a company called Ucom. Once the government announced Ucom had won, they proceeded to default on their bid, leaving the government to award the license to the second-highest bidder—which was also Ucom! They defaulted on this bid as well; four months later, after several more defaults, they paid A$117 million for the license, which was A$95 million less than their initial winning bid! The license ended up being awarded to the highest bidder at the second-highest price—but the poorly designed auction caused at least a year delay in bringing pay-TV to Australia.[5]

EXAMPLE: Taking Bids Off the Wall

One common method for manipulating auctions is for the seller to take fictitious bids, a practice known as "taking bids off the wall." Such manipulation has found its way to online auctions as well, even where no walls are involved.

According to a recent news story,[6] a New York jeweler sold large quantities of diamonds, gold, and platinum jewelry online. Though the items were offered on eBay with no reserve price, the seller distributed spreadsheets to his employees which instructed them to place bids in order to increase the final sales price. According to the lawsuit, the employees placed over 232,000 bids in a one-year period, inflating the selling prices by 20% on average.

When confronted with the evidence, the jeweler agreed to pay a $400,000 fine to settle the civil fraud complaint.

18.8 The Winner's Curse

We turn now to the examination of **common-value auctions**, where the good that is being awarded has the *same* value to all bidders. However, each of the bidders may have different estimates of that value. To emphasize this, let us write the (estimated) value of bidder i as $v + \epsilon_i$ where v is the

[5] See John McMillan, "Selling Spectrum Rights," *Journal of Economic Perspectives*, **8**(3), 145–152, for details of this story and how its lessons were incorporated into the design of the U.S. spectrum auction. This article also describes the New Zealand example mentioned earlier.

[6] Barnaby J. Feder, "Jeweler to Pay $400,000 in Online Auction Fraud Settlement," *New York Times,* June 9, 2007.

true, common value and ϵ_i is the "error term" associated with bidder i's estimate.

Let's examine a sealed-bid auction in this framework. What bid should bidder i place? To develop some intuition, let's see what happens if each bidder bids their estimated value. In this case, the person with the highest value of ϵ_i, ϵ_{max}, gets the good. But as long as $\epsilon_{max} > 0$, this person is paying more than v, the true value of the good. This is the so-called **Winner's Curse**. If you win the auction, it is because you have overestimated the value of the good being sold. In other words, you have won only because you were too optimistic!

The *optimal* strategy in a common-value auction like this is to bid less than your estimated value—and the more bidders there are, the lower you want your own bid to be. Think about it: if you are the highest bidder out of five bidders you may be overly optimistic, but if you are the highest bidder out of twenty bidders you must be *super* optimistic. The more bidders there are, the more humble you should be about your own estimates of the "true value" of the good in question.

The Winner's Curse seemed to be operating in the FCC's May 1996 spectrum auction for personal communications services. The largest bidder in that auction, NextWave Personal Communications Inc., bid $4.2 billion for sixty-three licenses, winning them all. However, in January 1998 the company filed for Chapter Eleven bankruptcy protection, after finding itself unable to pay its bills.

18.9 Stable Marriage Problem

There are many examples of **two-sided matching models** where consumers are matched up with each other. Men may be matched with women by a dating service or matchmaker, students may be matched with colleges, pledges may be matched with sororities, interns matched with hospitals, and so on.

What are good algorithms for making such matches? Do "stable" outcomes always exist? Here we examine a simple mechanism for making matches that are stable in a precisely defined sense.

Let us suppose that there are n men and an equal number of women and we need to match them up as dancing partners. Each woman can rank the men according to her preferences and the same goes for the men. For simplicity, let us suppose that there are no ties in these rankings and that everyone would prefer to dance than to sit on the sidelines.

What is a good way to arrange for dancing partners? One attractive criterion is to find a way to produce a "stable" matching. The definition of stable, in this context, is that there is no couple that would prefer each other to their current partner. Said another way, if a man prefers another

woman to his current partner, that woman wouldn't want him—she would prefer the partner she currently had.

Does a stable matching always exist? If so, how can one be found? The answer is that, contrary to the impression one would get from soap operas and romance novels, there always are stable matchings and they are relatively easy to construct.

The most famous algorithm, known as the **deferred acceptance algorithm**, goes like this.[7]

Step 1. Each man proposes to his most preferred woman.

Step 2. Each woman records the list of proposals she receives on her dance card.

Step 3. After all men have proposed to their most-preferred choice, each woman (gently) rejects all of the suitors except for her most preferred.

Step 4. The rejected suitors propose to the next woman on their lists.

Step 5. Continue to step 2 or terminate the algorithm when every woman has received an offer.

This algorithm always produce a stable matching. Suppose, to the contrary, that there is some man that prefers another woman to his present partner. Then he would have invited her to dance before his current partner. If she preferred him to her current partner, she would have rejected her current partner earlier in the process.

It turns out that this algorithm yields the best possible stable matching for the men in the sense that each man prefers the outcome of this matching process to any other stable matching. Of course, if we flipped the roles of men and women, we would find the woman-optimal stable matching.

Though the example described is slightly frivolous, processes like the deferred acceptance algorithm are used to match students to schools in Boston and New York, residents to hospitals nationwide, and even organ donors to recipients.

18.10 Mechanism Design

Auctions and the two-sided matching model that we have discussed in this chapter are examples of **economic mechanisms**. The idea of an economic

[7] Gale, David, and Lloyd Shapley [1962], "College Admissions and the Stability of Marriage," *American Mathematical Monthly,* 69, 9-15.

mechanism is to define a "game" or "market" that will yield some desired outcome.

For example, one might want to design a mechanism to sell a painting. A natural mechanism here would be an auction. But even with an auction, there are many design choices. Should it be designed to maximize efficiency (i.e., to ensure that the painting goes to the person who values it most highly) or should it be designed to maximize expected revenue for the seller, even if there is a risk that the painting may not be sold?

We've seen earlier that there are several different types of auctions, each with advantages and disadvantages. Which one is best in a particular circumstance?

Mechanism design is essentially the inverse of **game theory**. With game theory, we are given a description of the rules of the game and want to determine what the outcome will be. With mechanism design, we are given a description of the outcome that we want to reach and try to design a game that will reach it.[8]

Mechanism design is not limited to auctions or matching problems. It also includes **voting mechanisms** and **public goods** mechanisms, such as those described in Chapter 37, or **externality** mechanisms, such as those described in Chapter 35.

In a general mechanism, we think of a number of agents (i.e., consumers or firms) who each have some private information. In the case of an auction, this private information might be their value for the item being auctioned. In a problem involving firms, the private information might be their cost functions.

The agents report some message about their private information to the "center," which we might think of as an auctioneer. The center examines the messages and reports some outcome: who receives the item in question, what output firms should produce, how much various parties have to pay or be paid, and so on.

The major design decisions are 1) what sort of messages should be sent to the center and 2) what rule the center should use to determine the outcome. The constraints on the problem are the usual sort of resource constraints (i.e., there is only one item to be sold) and the constraints that the individuals will act in their own self-interest. This latter constraint is known as the **incentive compatibility constraint**.

There may be other constraints as well. For example, we may want the agents to participate voluntarily in the mechanism, which would require that they get at least as high a payoff from participating as not participating. We will ignore this constraint for simplicity.

To get a flavor of what mechanism design looks like, let us consider a simple problem of awarding an indivisible good to one of two different

[8] The 2007 Nobel Prize in Economics was awarded to Leo Hurwicz, Roger Myerson, and Eric Maskin for their contributions to economic mechanism design.

agents. Let $(x_1, x_2) = (1, 0)$ if agent 1 gets the good and $(x_1, x_2) = (0, 1)$ if agent 2 gets the good. Let p be the price paid for the good.

We suppose that the message that each agent sends to the center is just a reported value for the good. This is known as a **direct revelation mechanism**. The center will then award the good to the agent with the highest reported value and charge that agent some price p.

What are the constraints on p? Suppose agent 1 has the highest value. Then his message to the center should be such that the payoff he gets in response to that message is at least as large as the payoff he would get if he sent the same message as agent 2 (who gets a zero payoff). This says

$$v_1 - p \geq 0.$$

By the same token, agent 2 must get at least as large a payoff from his message as he would get if he sent the message sent by agent 1 (which resulted in agent 1 getting the good). This says

$$0 \geq v_2 - p.$$

Putting these two conditions together, we have $v_1 \geq p \geq v_2$, which says that the price charged by the center must lie between the highest and second-highest value.

In order to determine which price the center must charge, we need to consider its objects and its information. If the center believes that the v_1 can be arbitrarily close to v_2 and it always wants to award the item to the highest bidder, then it has to set a price of v_2.

This is just the **Vickrey auction** described earlier, in which each party submits a bid and the item is awarded to the highest bidder at the second-highest bid. This is clearly an attractive mechanism for this particular problem.

Summary

1. Auctions have been used for thousands of years to sell things.

2. If each bidder's value is independent of the other bidders, the auction is said to be a private-value auction. If the value of the item being sold is essentially the same for everyone, the auction is said to be a common-value auction.

3. Common auction forms are the English auction, the Dutch auction, the sealed-bid auction, and the Vickrey auction.

4. English auctions and Vickrey auctions have the desirable property that their outcomes are Pareto efficient.

5. Profit-maximizing auctions typically require a strategic choice of the reservation price.

6. Despite their advantages as market mechanisms, auctions are vulnerable to collusion and other forms of strategic behavior.

REVIEW QUESTIONS

1. Consider an auction of antique quilts to collectors. Is this a private-value or a common-value auction?

2. Suppose that there are only two bidders with values of \$8 and \$10 for an item with a bid increment of \$1. What should the reservation price be in a profit-maximizing English auction?

3. Suppose that we have two copies of *Intermediate Microeconomics* to sell to three (enthusiastic) students. How can we use a sealed-bid auction that will guarantee that the bidders with the two highest values get the books?

4. Consider the Ucom example in the text. Was the auction design efficient? Did it maximize profits?

5. A game theorist fills a jar with pennies and auctions it off on the first day of class using an English auction. Is this a private-value or a common-value auction? Do you think the winning bidder usually makes a profit?

CHAPTER 19

TECHNOLOGY

In this chapter we begin our study of firm behavior. The first thing to do is to examine the constraints on a firm's behavior. When a firm makes choices it faces many constraints. These constraints are imposed by its customers, by its competitors, and by nature. In this chapter we're going to consider the latter source of constraints: nature. Nature imposes the constraint that there are only certain feasible ways to produce outputs from inputs: there are only certain kinds of technological choices that are possible. Here we will study how economists describe these technological constraints.

If you understand consumer theory, production theory will be very easy since the same tools are used. In fact, production theory is much simpler than consumption theory because the output of a production process is generally observable, whereas the "output" of consumption (utility) is not directly observable.

19.1 Inputs and Outputs

Inputs to production are called **factors of production**. Factors of production are often classified into broad categories such as land, labor, capital,

and raw materials. It is pretty apparent what labor, land, and raw materials mean, but capital may be a new concept. **Capital goods** are those inputs to production that are themselves produced goods. Basically capital goods are machines of one sort or another: tractors, buildings, computers, or whatever.

Sometimes capital is used to describe the money used to start up or maintain a business. We will always use the term **financial capital** for this concept and use the term capital goods, or **physical capital**, for produced factors of production.

We will usually want to think of inputs and outputs as being measured in *flow* units: a certain amount of labor per week and a certain number of machine hours per week will produce a certain amount of output a week.

We won't find it necessary to use the classifications given above very often. Most of what we want to describe about technology can be done without reference to the *kind* of inputs and outputs involved—just with the amounts of inputs and outputs.

19.2 Describing Technological Constraints

Nature imposes **technological constraints** on firms: only certain combinations of inputs are feasible ways to produce a given amount of output, and the firm must limit itself to technologically feasible production plans.

The easiest way to describe feasible production plans is to list them. That is, we can list all combinations of inputs and outputs that are technologically feasible. The set of all combinations of inputs and outputs that comprise a technologically feasible way to produce is called a **production set.**

Suppose, for example, that we have only one input, measured by x, and one output, measured by y. Then a production set might have the shape indicated in Figure 19.1. To say that some point (x, y) is in the production set is just to say that it is technologically possible to produce y amount of output if you have x amount of input. The production set shows the *possible* technological choices facing a firm.

As long as the inputs to the firm are costly it makes sense to limit ourselves to examining the *maximum possible output* for a given level of input. This is the boundary of the production set depicted in Figure 19.1. The function describing the boundary of this set is known as the **production function.** It measures the maximum possible output that you can get from a given amount of input.

Of course, the concept of a production function applies equally well if there are several inputs. If, for example, we consider the case of two inputs, the production function $f(x_1, x_2)$ would measure the maximum amount of output y that we could get if we had x_1 units of factor 1 and x_2 units of factor 2.

Figure 19.1 **A production set**. Here is a possible shape for a production set.

In the two-input case there is a convenient way to depict production relations known as the **isoquant**. An isoquant is the set of all possible combinations of inputs 1 and 2 that are just sufficient to produce a given amount of output.

Isoquants are similar to indifference curves. As we've seen earlier, an indifference curve depicts the different consumption bundles that are just sufficient to produce a certain level of utility. But there is one important difference between indifference curves and isoquants. Isoquants are labeled with the amount of output they can produce, not with a utility level. Thus the labeling of isoquants is fixed by the technology and doesn't have the kind of arbitrary nature that the utility labeling has.

19.3 Examples of Technology

Since we already know a lot about indifference curves, it is easy to understand how isoquants work. Let's consider a few examples of technologies and their isoquants.

Fixed Proportions

Suppose that we are producing holes and that the only way to get a hole is to use one man and one shovel. Extra shovels aren't worth anything, and neither are extra men. Thus the total number of holes that you can produce will be the minimum of the number of men and the number of shovels that you have. We write the production function as $f(x_1, x_2) = \min\{x_1, x_2\}$.

Fixed proportions. Isoquants for the case of fixed proportions.

Figure 19.2

The isoquants look like those depicted in Figure 19.2. Note that these isoquants are just like the case of perfect complements in consumer theory.

Perfect Substitutes

Suppose now that we are producing homework and the inputs are red pencils and blue pencils. The amount of homework produced depends only on the total number of pencils, so we write the production function as $f(x_1, x_2) = x_1 + x_2$. The resulting isoquants are just like the case of perfect substitutes in consumer theory, as depicted in Figure 19.3.

Cobb-Douglas

If the production function has the form $f(x_1, x_2) = Ax_1^a x_2^b$, then we say that it is a **Cobb-Douglas production function**. This is just like the functional form for Cobb-Douglas preferences that we studied earlier. The numerical magnitude of the utility function was not important, so we set $A = 1$ and usually set $a + b = 1$. But the magnitude of the production function does matter so we have to allow these parameters to take arbitrary values. The parameter A measures, roughly speaking, the scale of production: how much output we would get if we used one unit of each input. The parameters a and b measure how the amount of output responds to

Figure 19.3 **Perfect substitutes**. Isoquants for the case of perfect substitutes.

changes in the inputs. We'll examine their impact in more detail later on. In some of the examples, we will choose to set $A = 1$ in order to simplify the calculations.

The Cobb-Douglas isoquants have the same nice, well-behaved shape that the Cobb-Douglas indifference curves have; as in the case of utility functions, the Cobb-Douglas production function is about the simplest example of well-behaved isoquants.

19.4 Properties of Technology

As in the case of consumers, it is common to assume certain properties about technology. First we will generally assume that technologies are **monotonic**: if you increase the amount of at least one of the inputs, it should be possible to produce at least as much output as you were producing originally. This is sometimes referred to as the property of **free disposal**: if the firm can costlessly dispose of any inputs, having extra inputs around can't hurt it.

Second, we will often assume that the technology is **convex**. This means that if you have two ways to produce y units of output, (x_1, x_2) and (z_1, z_2), then their weighted average will produce *at least* y units of output.

One argument for convex technologies goes as follows. Suppose that you have a way to produce 1 unit of output using a_1 units of factor 1 and a_2

units of factor 2 and that you have another way to produce 1 unit of output using b_1 units of factor 1 and b_2 units of factor 2. We call these two ways to produce output **production techniques**.

Furthermore, let us suppose that you are free to scale the output up by arbitrary amounts so that $(100a_1, 100a_2)$ and $(100b_1, 100b_2)$ will produce 100 units of output. But now note that if you have $25a_1 + 75b_1$ units of factor 1 and $25a_2 + 75b_2$ units of factor 2 you can still produce 100 units of output: just produce 25 units of the output using the "a" technique and 75 units of the output using the "b" technique.

This is depicted in Figure 19.4. By choosing the level at which you operate each of the two activities, you can produce a given amount of output in a variety of different ways. In particular, every input combination along the line connecting $(100a_1, 100a_2)$ and $(100b_1, 100b_2)$ will be a feasible way to produce 100 units of output.

Convexity. If you can operate production activities independently, then weighted averages of production plans will also be feasible. Thus the isoquants will have a convex shape.

Figure 19.4

In this kind of technology, where you can scale the production process up and down easily and where separate production processes don't interfere with each other, convexity is a very natural assumption.

19.5 The Marginal Product

Suppose that we are operating at some point, (x_1, x_2), and that we consider using a little bit more of factor 1 while keeping factor 2 fixed at the level x_2. How much more output will we get per additional unit of factor 1? We have to look at the change in output per unit change of factor 1:

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1}.$$

We call this the **marginal product of factor 1**. The marginal product of factor 2 is defined in a similar way, and we denote them by $MP_1(x_1, x_2)$ and $MP_2(x_1, x_2)$, respectively.

Sometimes we will be a bit sloppy about the concept of marginal product and describe it as the extra output we get from having "one" more unit of factor 1. As long as "one" is small relative to the total amount of factor 1 that we are using, this will be satisfactory. But we should remember that a marginal product is a *rate*: the extra amount of output per unit of extra input.

The concept of marginal product is just like the concept of marginal utility that we described in our discussion of consumer theory, except for the ordinal nature of utility. Here, we are discussing physical output: the marginal product of a factor is a specific number, which can, in principle, be observed.

19.6 The Technical Rate of Substitution

Suppose that we are operating at some point (x_1, x_2) and that we consider giving up a little bit of factor 1 and using just enough more of factor 2 to produce the same amount of output y. How much extra of factor 2, Δx_2, do we need if we are going to give up a little bit of factor 1, Δx_1? This is just the slope of the isoquant; we refer to it as the **technical rate of substitution (TRS)**, and denote it by $\text{TRS}(x_1, x_2)$.

The technical rate of substitution measures the tradeoff between two inputs in production. It measures the rate at which the firm will have to substitute one input for another in order to keep output constant.

To derive a formula for the TRS, we can use the same idea that we used to determine the slope of the indifference curve. Consider a change in our use of factors 1 and 2 that keeps output fixed. Then we have

$$\Delta y = MP_1(x_1, x_2)\Delta x_1 + MP_2(x_1, x_2)\Delta x_2 = 0,$$

which we can solve to get

$$\text{TRS}(x_1, x_2) = \frac{\Delta x_2}{\Delta x_1} = -\frac{MP_1(x_1, x_2)}{MP_2(x_1, x_2)}.$$

Note the similarity with the definition of the marginal rate of substitution.

19.7 Diminishing Marginal Product

Suppose that we have certain amounts of factors 1 and 2 and we consider adding more of factor 1 while holding factor 2 fixed at a given level. What might happen to the marginal product of factor 1?

As long as we have a monotonic technology, we know that the total output will go up as we increase the amount of factor 1. But it is natural to expect that it will go up at a decreasing rate. Let's consider a specific example, the case of farming.

One man on one acre of land might produce 100 bushels of corn. If we add another man and keep the same amount of land, we might get 200 bushels of corn, so in this case the marginal product of an extra worker is 100. Now keep adding workers to this acre of land. Each worker may produce more output, but eventually the extra amount of corn produced by an extra worker will be less than 100 bushels. After 4 or 5 people are added the additional output per worker will drop to 90, 80, 70 ... or even fewer bushels of corn. If we get hundreds of workers crowded together on this one acre of land, an extra worker may even cause output to go down! As in the making of broth, extra cooks *can* make things worse.

Thus we would typically expect that the marginal product of a factor will diminish as we get more and more of that factor. This is called the **law of diminishing marginal product**. It isn't really a "law"; it's just a common feature of most kinds of production processes.

It is important to emphasize that the law of diminishing marginal product applies only when all *other* inputs are being held fixed. In the farming example, we considered changing only the labor input, holding the land and raw materials fixed.

19.8 Diminishing Technical Rate of Substitution

Another closely related assumption about technology is that of **diminishing technical rate of substitution**. This says that as we increase the amount of factor 1, and adjust factor 2 so as to stay on the same isoquant, the technical rate of substitution declines. Roughly speaking, the assumption of diminishing TRS means that the slope of an isoquant must decrease in absolute value as we move along the isoquant in the direction of increasing x_1, and it must increase as we move in the direction of increasing x_2. This means that the isoquants will have the same sort of convex shape that well-behaved indifference curves have.

The assumptions of a diminishing technical rate of substitution and diminishing marginal product are closely related but are not exactly the same. Diminishing marginal product is an assumption about how the marginal product changes as we increase the amount of one factor, *holding the*

other factor fixed. Diminishing TRS is about how the *ratio* of the marginal products—the slope of the isoquant—changes as we increase the amount of one factor and *reduce the amount of the other factor so as to stay on the same isoquant.*

19.9 The Long Run and the Short Run

Let us return now to the original idea of a technology as being just a list of the feasible production plans. We may want to distinguish between the production plans that are *immediately* feasible and those that are *eventually* feasible.

In the **short run**, there will be some factors of production that are fixed at predetermined levels. Our farmer described above might only consider production plans that involve a fixed amount of land, if that is all he has access to. It may be true that if he had more land, he could produce more corn, but in the short run he is stuck with the amount of land that he has.

On the other hand, in the long run the farmer is free to purchase more land, or to sell some of the land he now owns. He can adjust the level of the land input so as to maximize his profits.

The economist's distinction between the long run and the short run is this: in the short run there is at least one factor of production that is fixed: a fixed amount of land, a fixed plant size, a fixed number of machines, or whatever. In the **long run**, *all* the factors of production can be varied.

There is no specific time interval implied here. What is the long run and what is the short run depends on what kinds of choices we are examining. In the short run at least some factors are fixed at given levels, but in the long run the amount used of these factors can be changed.

Let's suppose that factor 2, say, is fixed at $\overline{x}_2$ in the short run. Then the relevant production function for the short run is $f(x_1, \overline{x}_2)$. We can plot the functional relation between output and x_1 in a diagram like Figure 19.5.

Note that we have drawn the short-run production function as getting flatter and flatter as the amount of factor 1 increases. This is just the law of diminishing marginal product in action again. Of course, it can easily happen that there is an initial region of increasing marginal returns where the marginal product of factor 1 increases as we add more of it. In the case of the farmer adding labor, it might be that the first few workers added increase output more and more because they would be able to divide up jobs efficiently, and so on. But given the fixed amount of land, eventually the marginal product of labor will decline.

19.10 Returns to Scale

Now let's consider a different kind of experiment. Instead of increasing the amount of one input while holding the other input fixed, let's increase the

Production function. This is a possible shape for a short-run production function.

Figure 19.5

amount of *all* inputs to the production function. In other words, let's scale the amount of all inputs up by some constant factor: for example, use twice as much of both factor 1 and factor 2.

If we use twice as much of each input, how much output will we get? The most likely outcome is that we will get twice as much output. This is called the case of **constant returns to scale**. In terms of the production function, this means that two times as much of each input gives two times as much output. In the case of two inputs we can express this mathematically by

$$2f(x_1, x_2) = f(2x_1, 2x_2).$$

In general, if we scale all of the inputs up by some amount t, constant returns to scale implies that we should get t times as much output:

$$tf(x_1, x_2) = f(tx_1, tx_2).$$

We say that this is the likely outcome for the following reason: it should typically be possible for the firm to *replicate* what it was doing before. If the firm has twice as much of each input, it can just set up two plants side by side and thereby get twice as much output. With three times as much of each input, it can set up three plants, and so on.

Note that it is perfectly possible for a technology to exhibit constant returns to scale and diminishing marginal product to each factor. **Returns to scale** describes what happens when you increase *all* inputs, while diminishing marginal product describes what happens when you increase *one* of the inputs and hold the others fixed.

Constant returns to scale is the most "natural" case because of the replication argument, but that isn't to say that other things might not happen. For example, it could happen that if we scale up both inputs by some factor t, we get *more* than t times as much output. This is called the case of **increasing returns to scale**. Mathematically, increasing returns to scale means that

$$f(tx_1, tx_2) > tf(x_1, x_2).$$

for all $t > 1$.

What would be an example of a technology that had increasing returns to scale? One nice example is that of an oil pipeline. If we double the diameter of a pipe, we use twice as much materials, but the cross section of the pipe goes up by a factor of 4. Thus we will likely be able to pump more than twice as much oil through it.

(Of course, we can't push this example too far. If we keep doubling the diameter of the pipe, it will eventually collapse of its own weight. Increasing returns to scale usually just applies over some range of output.)

The other case to consider is that of **decreasing returns to scale**, where

$$f(tx_1, tx_2) < tf(x_1, x_2)$$

for all $t > 1$.

This case is somewhat peculiar. If we get less than twice as much output from having twice as much of each input, we must be doing something wrong. After all, we could just replicate what we were doing before!

The usual way in which diminishing returns to scale arises is because we forgot to account for some input. If we have twice as much of every input but one, we won't be able to exactly replicate what we were doing before, so there is no reason that we have to get twice as much output. Diminishing returns to scale is really a short-run phenomenon, with something being held fixed.

Of course, a technology can exhibit different kinds of returns to scale at different levels of production. It may well happen that for low levels of production, the technology exhibits increasing returns to scale—as you scale all the inputs by some small amount t, the output increases by *more* than t. Later on, for larger levels of output, increasing scale by t may just increase output by the same factor t.

EXAMPLE: Datacenters

Datacenters are large buildings that house thousands of computers used to perform tasks such as serving web pages. Internet companies such as Google, Yahoo, Microsoft, Amazon, and many others have built thousands of datacenters around the world.

A typical datacenter consists of hundreds of racks which hold computer motherboards that are similar to the motherboard in your desktop computer. Generally these systems are designed to be easily scalable so that the computational power of the data center can scale up or down just by adding or removing racks of computers.

The replication argument implies that the production function for computing services is effectively constant returns to scale: to double output, you simply double all inputs.

EXAMPLE: Copy Exactly!

Intel operates dozens of "fab plants" that fabricate, assemble, sort, and test advanced computer chips. Chip fabrication is such a delicate process that Intel found it difficult to manage quality in a heterogeneous environment. Even minor variations in plant design, such as cleaning procedures or the length of cooling hoses, could have a large impact on the yield of the fab process.

In order to manage these very subtle effects, Intel moved to its Copy Exactly! process. According to Intel, the Copy Exactly directive is: "everything which might affect the process, or how it is run, is to be copied down to the finest detail, unless it is either physically impossible to do so, or there is an overwhelming competitive benefit to introducing a change."

This means that one Intel plant is very much like another, and deliberately so. As the replication argument suggests, the easiest way to scale up production at Intel is to replicate current operating procedures as closely as possible.

Summary

1. The technological constraints of the firm are described by the production set, which depicts all the technologically feasible combinations of inputs and outputs, and by the production function, which gives the maximum amount of output associated with a given amount of the inputs.

2. Another way to describe the technological constraints facing a firm is through the use of isoquants—curves that indicate all the combinations of inputs capable of producing a given level of output.

3. We generally assume that isoquants are convex and monotonic, just like well–behaved preferences.

4. The marginal product measures the extra output per extra unit of an input, holding all other inputs fixed. We typically assume that the marginal product of an input diminishes as we use more and more of that input.

5. The technical rate of substitution (TRS) measures the slope of an isoquant. We generally assume that the TRS diminishes as we move out along an isoquant—which is another way of saying that the isoquant has a convex shape.

6. In the short run some inputs are fixed, while in the long run all inputs are variable.

7. Returns to scale refers to the way that output changes as we change the *scale* of production. If we scale all inputs up by some amount t and output goes up by the same factor, then we have constant returns to scale. If output scales up by more that t, we have increasing returns to scale; and if it scales up by less than t, we have decreasing returns to scale.

REVIEW QUESTIONS

1. Consider the production function $f(x_1, x_2) = x_1^2 x_2^2$. Does this exhibit constant, increasing, or decreasing returns to scale?

2. Consider the production function $f(x_1, x_2) = 4x_1^{\frac{1}{2}} x_2^{\frac{1}{3}}$. Does this exhibit constant, increasing, or decreasing returns to scale?

3. The Cobb-Douglas production function is given by $f(x_1, x_2) = Ax_1^a x_2^b$. It turns out that the type of returns to scale of this function will depend on the magnitude of $a + b$. Which values of $a + b$ will be associated with the different kinds of returns to scale?

4. The technical rate of substitution between factors x_2 and x_1 is -4. If you desire to produce the same amount of output but cut your use of x_1 by 3 units, how many more units of x_2 will you need?

5. True or false? If the law of diminishing marginal product did not hold, the world's food supply could be grown in a flowerpot.

6. In a production process is it possible to have decreasing marginal product in an input and yet increasing returns to scale?

CHAPTER 20

PROFIT MAXIMIZATION

In the last chapter we discussed ways to describe the technological choices facing the firm. In this chapter we describe a model of how the firm chooses the amount to produce and the method of production to employ. The model we will use is the model of profit maximization: the firm chooses a production plan so as to maximize its profits.

In this chapter we will assume that the firm faces fixed prices for its inputs and outputs. We said earlier that economists call a market where the individual producers take the prices as outside their control a **competitive market**. So in this chapter we want to study the profit-maximization problem of a firm that faces competitive markets for the factors of production it uses and the output goods it produces.

20.1 Profits

Profits are defined as revenues minus cost. Suppose that the firm produces n outputs $(y_1, \ldots, y_n)$ and uses m inputs $(x_1, \ldots, x_m)$. Let the prices of the output goods be $(p_1, \ldots, p_n)$ and the prices of the inputs be $(w_1, \ldots, w_m)$.

The profits the firm receives, π, can be expressed as

$$\pi = \sum_{i=1}^{n} p_i y_i - \sum_{i=1}^{m} w_i x_i.$$

The first term is revenue, and the second term is cost.

In the expression for cost we should be sure to include *all* of the factors of production used by the firm, valued at their market price. Usually this is pretty obvious, but in cases where the firm is owned and operated by the same individual, it is possible to forget about some of the factors.

For example, if an individual works in his own firm, then his labor is an input and it should be counted as part of the costs. His wage rate is simply the market price of his labor—what he *would* be getting if he sold his labor on the open market. Similarly, if a farmer owns some land and uses it in his production, that land should be valued at its market value for purposes of computing the economic costs.

We have seen that economic costs like these are often referred to as **opportunity costs**. The name comes from the idea that if you are using your labor, for example, in one application, you forgo the opportunity of employing it elsewhere. Therefore those lost wages are part of the cost of production. Similarly with the land example: the farmer has the opportunity of renting his land to someone else, but he chooses to forgo that rental income in favor of renting it to himself. The lost rents are part of the opportunity cost of his production.

The economic definition of profit requires that we value all inputs and outputs at their opportunity cost. Profits as determined by accountants do not necessarily accurately measure economic profits, as they typically use historical costs—what a factor was purchased for originally—rather than economic costs—what a factor would cost if purchased now. There are many variations on the use of the term "profit," but we will always stick to the economic definition.

Another confusion that sometimes arises is due to getting time scales mixed up. We usually think of the factor inputs as being measured in terms of *flows*. So many labor hours per week and so many machine hours per week will produce so much output per week. Then the factor prices will be measured in units appropriate for the purchase of such flows. Wages are naturally expressed in terms of dollars per hour. The analog for machines would be the **rental rate**—the rate at which you can rent a machine for the given time period.

In many cases there isn't a very well-developed market for the rental of machines, since firms will typically buy their capital equipment. In this case, we have to compute the implicit rental rate by seeing how much it would cost to buy a machine at the beginning of the period and sell it at the end of the period.

20.2 The Organization of Firms

In a capitalist economy, firms are owned by individuals. Firms are only legal entities; ultimately it is the owners of firms who are responsible for the behavior of the firm, and it is the owners who reap the rewards or pay the costs of that behavior.

Generally speaking, firms can be organized as proprietorships, partnerships, or corporations. A **proprietorship** is a firm that is owned by a single individual. A **partnership** is owned by two or more individuals. A **corporation** is usually owned by several individuals as well, but under the law has an existence separate from that of its owners. Thus a partnership will last only as long as both partners are alive and agree to maintain its existence. A corporation can last longer than the lifetimes of any of its owners. For this reason, most large firms are organized as corporations.

The owners of each of these different types of firms may have different goals with respect to managing the operation of the firm. In a proprietorship or a partnership the owners of the firm usually take a direct role in actually managing the day-to-day operations of the firm, so they are in a position to carry out whatever objectives they have in operating the firm. Typically, the owners would be interested in maximizing the profits of their firm, but, if they have nonprofit goals, they can certainly indulge in these goals instead.

In a corporation, the owners of the corporation are often distinct from the managers of the corporation. Thus there is a separation of ownership and control. The owners of the corporation must define an objective for the managers to follow in their running of the firm, and then do their best to see that they actually pursue the goals the owners have in mind. Again, profit maximization is a common goal. As we'll see below, this goal, properly interpreted, is likely to lead the managers of the firm to choose actions that are in the interests of the owners of the firm.

20.3 Profits and Stock Market Value

Often the production process that a firm uses goes on for many periods. Inputs put in place at time t pay off with a whole flow of services at later times. For example, a factory building erected by a firm could last for 50 or 100 years. In this case an input at one point in time helps to produce output at other times in the future.

In this case we have to value a flow of costs and a flow of revenues over time. As we've seen in Chapter 10, the appropriate way to do this is to use the concept of present value. When people can borrow and lend in financial markets, the interest rate can be used to define a natural price of consumption at different times. Firms have access to the same sorts of

financial markets, and the interest rate can be used to value investment decisions in exactly the same way.

Consider a world of perfect certainty where a firm's flow of future profits is publicly known. Then the present value of those profits would be the **present value of the firm**. It would be how much someone would be willing to pay to purchase the firm.

As we indicated above, most large firms are organized as corporations, which means that they are jointly owned by a number of individuals. The corporation issues stock certificates to represent ownership of shares in the corporation. At certain times the corporation issues dividends on these shares, which represent a share of the profits of the firm. The shares of ownership in the corporation are bought and sold in the **stock market**. The price of a share represents the present value of the stream of dividends that people expect to receive from the corporation. The total stock market value of a firm represents the present value of the stream of profits that the firm is expected to generate. Thus the objective of the firm—maximizing the present value of the stream of profits the firm generates—could also be described as the goal of maximizing stock market value. In a world of certainty, these two goals are the same thing.

The owners of the firm will generally want the firm to choose production plans that maximize the stock market value of the firm, since that will make the value of the shares they hold as large as possible. We saw in Chapter 10 that whatever an individual's tastes for consumption at different times, he or she will always prefer an endowment with a higher present value to one with a lower present value. By maximizing stock market value, a firm makes its shareholders' budget sets as large as possible, and thereby acts in the best interests of all of its shareholders.

If there is uncertainty about a firm's stream of profits, then instructing managers to maximize profits has no meaning. Should they maximize expected profits? Should they maximize the expected utility of profits? What attitude toward risky investments should the managers have? It is difficult to assign a meaning to profit maximization when there is uncertainty present. However, in a world of uncertainty, maximizing *stock market value* still has meaning. If the managers of a firm attempt to make the value of the firm's shares as large as possible then they make the firm's owners—the shareholders—as well-off as possible. Thus maximizing stock market value gives a well-defined objective function to the firm in nearly all economic environments.

Despite these remarks about time and uncertainty, we will generally limit ourselves to the examination of much simpler profit-maximization problems, namely, those in which there is a single, certain output and a single period of time. This simple story still generates significant insights and builds the proper intuition to study more general models of firm behavior. Most of the ideas that we will examine carry over in a natural way to these more general models.

20.4 The Boundaries of the Firm

One question that constantly confronts managers of firms is whether to "make or buy." That is, should a firm make something internally or buy it from an external supplier? The question is broader than it sounds, as it can refer not only to physical goods, but also services of one sort or another. Indeed, in the broadest interpretation, "make or buy" applies to almost every decision a firm makes.

Should a company provide its own cafeteria? Janitorial services? Photocopying services? Travel assistance? Obviously, many factors enter into such decisions. One important consideration is size. A small mom-and-pop video store with 12 employees is probably not going to provide a cafeteria. But it might outsource janitorial services, depending on cost, capabilities, and staffing.

Even a large organization, which could easily afford to operate food services, may or may not choose to do so, depending on availability of alternatives. Employees of an organization located in a big city have access to many places to eat; if the organization is located in a remote area, choices may be fewer.

One critical issue is whether the goods or services in question are externally provided by a monopoly or by a competitive market. By and large, managers prefer to buy goods and services on a competitive market, if they are available. The second-best choice is dealing with an internal monopolist. The worse choice of all, in terms of price and quality of service, is dealing with an external monopolist.

Think about photocopying services. The ideal situation is to have dozens of competitive providers vying for your business; that way you will get cheap prices and high-quality service. If your school is large, or in an urban area, there may be many photocopying services vying for your business. On the other hand, small rural schools may have less choice and often higher prices.

The same is true of businesses. A highly competitive environment gives lots of choices to users. By comparison, an internal photocopying division may be less attractive. Even if prices are low, the service could be sluggish. But the least attractive option is surely to have to submit to a single external provider. An internal monopoly provider may have bad service, but at least the money stays inside the firm.

As technology changes, what is typically inside the firm changes. Forty years ago, firms managed many services themselves. Now they tend to outsource as much as possible. Food service, photocopying service, and janitorial services are often provided by external organizations that specialize in such activities. Such specialization often allows these companies to provide higher quality and less expensive services to the organizations that use their services.

20.5 Fixed and Variable Factors

In a given time period, it may be very difficult to adjust some of the inputs. Typically a firm may have contractual obligations to employ certain inputs at certain levels. An example of this would be a lease on a building, where the firm is legally obligated to purchase a certain amount of space over the period under examination. We refer to a factor of production that is in a fixed amount for the firm as a **fixed factor**. If a factor can be used in different amounts, we refer to it as a **variable factor**.

As we saw in Chapter 19, the short run is defined as that period of time in which there are some fixed factors—factors that can only be used in fixed amounts. In the long run, on the other hand, the firm is free to vary all of the factors of production: all factors are variable factors.

There is no rigid boundary between the short run and the long run. The exact time period involved depends on the problem under examination. The important thing is that some of the factors of production are fixed in the short run and variable in the long run. Since all factors are variable in the long run, a firm is always free to decide to use zero inputs and produce zero output—that is, to go out of business. Thus the least profits a firm can make in the long run are zero profits.

In the short run, the firm is obligated to employ some factors, even if it decides to produce zero output. Therefore it is perfectly possible that the firm could make *negative* profits in the short run.

By definition, fixed factors are factors of production that must be paid for even if the firm decides to produce zero output: if a firm has a long-term lease on a building, it must make its lease payments each period whether or not it decides to produce anything that period. But there is another category of factors that only need to be paid for if the firm decides to produce a positive amount of output. One example is electricity used for lighting. If the firm produces zero output, it doesn't have to provide any lighting; but if it produces any positive amount of output, it has to purchase a fixed amount of electricity to use for lighting.

Factors such as these are called **quasi-fixed factors**. They are factors of production that must be used in a fixed amount, independent of the output of the firm, as long as the output is positive. The distinction between fixed factors and quasi-fixed factors is sometimes useful in analyzing the economic behavior of the firm.

20.6 Short-Run Profit Maximization

Let's consider the short-run profit-maximization problem when input 2 is fixed at some level $\overline{x}_2$. Let $f(x_1, x_2)$ be the production function for the firm, let p be the price of output, and let w_1 and w_2 be the prices of the

two inputs. Then the profit-maximization problem facing the firm can be written as

$$\max_{x_1} \; pf(x_1, \overline{x}_2) - w_1 x_1 - w_2 \overline{x}_2.$$

The condition for the optimal choice of factor 1 is not difficult to determine.

If x_1^* is the profit-maximizing choice of factor 1, then the output price times the marginal product of factor 1 should equal the price of factor 1. In symbols,

$$pMP_1(x_1^*, \overline{x}_2) = w_1.$$

In other words, the *value of the marginal product of a factor should equal its price.*

In order to understand this rule, think about the decision to employ a little more of factor 1. As you add a little more of it, Δx_1, you produce $\Delta y = MP_1 \Delta x_1$ more output that is worth $pMP_1 \Delta x_1$. But this marginal output costs $w_1 \Delta x_1$ to produce. If the value of marginal product exceeds its cost, then profits can be increased by *increasing* input 1. If the value of marginal product is less than its cost, then profits can be increased by *decreasing* the level of input 1.

If the profits of the firm are as large as possible, then profits should not increase when we increase or decrease input 1. This means that at a profit-maximizing choice of inputs and outputs, the value of the marginal product, $pMP_1(x_1^*, \overline{x}_2)$, should equal the factor price, w_1.

We can derive the same condition graphically. Consider Figure 20.1. The curved line represents the production function holding factor 2 fixed at $\overline{x}_2$. Using y to denote the output of the firm, profits are given by

$$\pi = py - w_1 x_1 - w_2 \overline{x}_2.$$

This expression can be solved for y to express output as a function of x_1:

$$y = \frac{\pi}{p} + \frac{w_2}{p}\overline{x}_2 + \frac{w_1}{p}x_1. \tag{20.1}$$

This equation describes **isoprofit lines**. These are just all combinations of the input goods and the output good that give a constant level of profit, π. As π varies we get a family of parallel straight lines each with a slope of w_1/p and each having a vertical intercept of $\pi/p + w_2\overline{x}_2/p$, which measures the profits plus the fixed costs of the firm.

The fixed costs are fixed, so the only thing that really varies as we move from one isoprofit line to another is the level of profits. Thus higher levels of profit will be associated with isoprofit lines with higher vertical intercepts.

The profit-maximization problem is then to find the point on the production function that has the highest associated isoprofit line. Such a point is illustrated in Figure 20.1. As usual it is characterized by a tangency condition: the slope of the production function should equal the slope of

Figure 20.1 **Profit maximization**. The firm chooses the input and output combination that lies on the highest isoprofit line. In this case the profit-maximizing point is (x_1^*, y^*).

the isoprofit line. Since the slope of the production function is the marginal product, and the slope of the isoprofit line is w_1/p, this condition can also be written as

$$MP_1 = \frac{w_1}{p},$$

which is equivalent to the condition we derived above.

20.7 Comparative Statics

We can use the geometry depicted in Figure 20.1 to analyze how a firm's choice of inputs and outputs varies as the prices of inputs and outputs vary. This gives us one way to analyze the **comparative statics** of firm behavior.

For example: how does the optimal choice of factor 1 vary as we vary its factor price w_1? Referring to equation (20.1), which defines the isoprofit line, we see that increasing w_1 will make the isoprofit line steeper, as shown in Figure 20.2A. When the isoprofit line is steeper, the tangency must occur further to the left. Thus the optimal level of factor 1 must decrease. This simply means that as the price of factor 1 increases, the demand for factor 1 must decrease: factor demand curves must slope downward.

Similarly, if the output price decreases the isoprofit line must become steeper, as shown in Figure 20.2B. By the same argument as given in the

Figure 20.2

Comparative statics. Panel A shows that increasing w_1 will reduce the demand for factor 1. Panel B shows that increasing the price of output will increase the demand for factor 1 and therefore increase the supply of output.

last paragraph the profit-maximizing choice of factor 1 will decrease. If the amount of factor 1 decreases and the level of factor 2 is fixed in the short run by assumption, then the supply of output must decrease. This gives us another comparative statics result: a reduction in the output price must decrease the supply of output. In other words, the supply function must slope upwards.

Finally, we can ask what will happen if the price of factor 2 changes? Because this is a short-run analysis, changing the price of factor 2 will not change the firm's choice of factor 2—in the short run, the level of factor 2 is fixed at $\overline{x}_2$. Changing the price of factor 2 has no effect on the *slope* of the isoprofit line. Thus the optimal choice of factor 1 will not change, nor will the supply of output. All that changes are the profits that the firm makes.

20.8 Profit Maximization in the Long Run

In the long run the firm is free to choose the level of all inputs. Thus the long-run profit-maximization problem can be posed as

$$\max_{x_1, x_2} \; pf(x_1, x_2) - w_1 x_1 - w_2 x_2.$$

This is basically the same as the short-run problem described above, but now both factors are free to vary.

The condition describing the optimal choices is essentially the same as before, but now we have to apply it to *each* factor. Before we saw that the value of the marginal product of factor 1 must be equal to its price, whatever the level of factor 2. The same sort of condition must now hold for *each* factor choice:

$$pMP_1(x_1^*, x_2^*) = w_1$$

$$pMP_2(x_1^*, x_2^*) = w_2.$$

If the firm has made the optimal choices of factors 1 and 2, the value of the marginal product of each factor should equal its price. At the optimal choice, the firm's profits cannot increase by changing the level of either input.

The argument is the same as used for the short-run profit-maximizing decisions. If the value of the marginal product of factor 1, for example, exceeded the price of factor 1, then using a little more of factor 1 would produce MP_1 more output, which would sell for pMP_1 dollars. If the value of this output exceeds the cost of the factor used to produce it, it clearly pays to expand the use of this factor.

These two conditions give us two equations in two unknowns, x_1^* and x_2^*. If we know how the marginal products behave as a function of x_1 and x_2, we will be able to solve for the optimal choice of each factor as a function of the prices. The resulting equations are known as the **factor demand curves**.

20.9 Inverse Factor Demand Curves

The **factor demand curves** of a firm measure the relationship between the price of a factor and the profit-maximizing choice of that factor. We saw above how to find the profit-maximizing choices: for any prices, (p, w_1, w_2), we just find those factor demands, (x_1^*, x_2^*), such that the value of the marginal product of each factor equals its price.

The **inverse factor demand curve** measures the same relationship, but from a different point of view. It measures what the factor prices must be for some given quantity of inputs to be demanded. Given the optimal choice of factor 2, we can draw the relationship between the optimal choice of factor 1 and its price in a diagram like that depicted in Figure 20.3. This is simply a graph of the equation

$$pMP_1(x_1, x_2^*) = w_1.$$

This curve will be downward sloping by the assumption of diminishing marginal product. For any level of x_1, this curve depicts what the factor price must be in order to induce the firm to demand that level of x_1, holding factor 2 fixed at x_2^*.

The inverse factor demand curve. This measures what the price of factor 1 must be to get x_1 units demanded if the level of the other factor is held fixed at x_2^*.

Figure 20.3

20.10 Profit Maximization and Returns to Scale

There is an important relationship between competitive profit maximization and returns to scale. Suppose that a firm has chosen a long-run profit-maximizing output $y^* = f(x_1^*, x_2^*)$, which it is producing using input levels (x_1^*, x_2^*).

Then its profits are given by

$$\pi^* = py^* - w_1x_1^* - w_2x_2^*.$$

Suppose that this firm's production function exhibits constant returns to scale and that it is making positive profits in equilibrium. Then consider what would happen if it doubled the level of its input usage. According to the constant returns to scale hypothesis, it would double its output level. What would happen to profits?

It is not hard to see that its profits would also double. But this contradicts the assumption that its original choice was profit maximizing! We derived this contradiction by assuming that the original profit level was positive; if the original level of profits were zero there would be no problem: two times zero is still zero.

This argument shows that the only reasonable long-run level of profits for a competitive firm that has constant returns to scale at all levels of output is a zero level of profits. (Of course if a firm has negative profits in the long run, it should go out of business.)

Most people find this to be a surprising statement. Firms are out to maximize profits aren't they? How can it be that they can only get zero profits in the long run?

Think about what would happen to a firm that did try to expand indefinitely. Three things might occur. First, the firm could get so large that it could not really operate effectively. This is just saying that the firm *really* doesn't have constant returns to scale at all levels of output. Eventually, due to coordination problems, it might enter a region of decreasing returns to scale.

Second, the firm might get so large that it would totally dominate the market for its product. In this case there is no reason for it to behave competitively—to take the price of output as given. Instead, it would make sense for such a firm to try to use its size to influence the market price. The model of competitive profit maximization would no longer be a sensible way for the firm to behave, since it would effectively have no competitors. We'll investigate more appropriate models of firm behavior in this situation when we discuss monopoly.

Third, if one firm can make positive profits with a constant returns to scale technology, so can any other firm with access to the same technology. If one firm wants to expand its output, so would other firms. But if all firms expand their outputs, this will certainly push down the price of output and lower the profits of all the firms in the industry.

20.11 Revealed Profitability

When a profit-maximizing firm makes its choice of inputs and outputs it reveals two things: first, that the inputs and outputs used represent a *feasible* production plan, and second, that these choices are more profitable than other feasible choices that the firm could have made. Let us examine these points in more detail.

Suppose that we observe two choices that the firm makes at two different sets of prices. At time t, it faces prices (p^t, w_1^t, w_2^t) and makes choices (y^t, x_1^t, x_2^t). At time s, it faces prices (p^s, w_1^s, w_2^s) and makes choices (y^s, x_1^s, x_2^s). If the production function of the firm hasn't changed between times s and t and if the firm is a profit maximizer, then we must have

$$p^t y^t - w_1^t x_1^t - w_2^t x_2^t \geq p^t y^s - w_1^t x_1^s - w_2^t x_2^s \tag{20.2}$$

and

$$p^s y^s - w_1^s x_1^s - w_2^s x_2^s \geq p^s y^t - w_1^s x_1^t - w_2^s x_2^t. \tag{20.3}$$

That is, the profits that the firm achieved facing the t period prices must be larger than if they used the s period plan and vice versa. If either of these inequalities were violated, the firm could not have been a profit-maximizing firm (with an unchanging technology).

Thus if we ever observe two time periods where these inequalities are violated we would know that the firm was not maximizing profits in at least one of the two periods. The satisfaction of these inequalities is virtually an axiom of profit-maximizing behavior, so it might be referred to as the **Weak Axiom of Profit Maximization (WAPM)**.

If the firm's choices satisfy WAPM, we can derive a useful comparative statics statement about the behavior of factor demands and output supplies when prices change. Transpose the two sides of equation (20.3) to get

$$-p^s y^t + w_1^s x_1^t + w_2^s x_1^t \geq -p^s y^s + w_1^s x_1^s + w_2^s x_2^s \tag{20.4}$$

and add equation (20.4) to equation (20.2) to get

$$\begin{aligned}&(p^t - p^s)y^t - (w_1^t - w_1^s)x_1^t - (w_2^t - w_2^s)x_2^t \\ &\geq (p^t - p^s)y^s - (w_1^t - w_1^s)x_1^s - (w_2^t - w_2^s)x_2^s.\end{aligned} \tag{20.5}$$

Now rearrange this equation to yield

$$(p^t - p^s)(y^t - y^s) - (w_1^t - w_1^s)(x_1^t - x_1^s) - (w_2^t - w_2^s)(x_2^t - x_2^s) \geq 0. \tag{20.6}$$

Finally define the change in prices, $\Delta p = (p^t - p^s)$, the change in output, $\Delta y = (y^t - y^s)$, and so on to find

$$\Delta p \Delta y - \Delta w_1 \Delta x_1 - \Delta w_2 \Delta x_2 \geq 0. \tag{20.7}$$

This equation is our final result. It says that the change in the price of output times the change in output minus the change in each factor price times the change in that factor must be nonnegative. This equation comes solely from the definition of profit maximization. Yet it contains all of the comparative statics results about profit-maximizing choices!

For example, suppose that we consider a situation where the price of output changes, but the price of each factor stays constant. If $\Delta w_1 = \Delta w_2 = 0$, then equation (20.7) reduces to

$$\Delta p \Delta y \geq 0.$$

Thus if the price of output goes up, so that $\Delta p > 0$, then the change in output must be nonnegative as well, $\Delta y \geq 0$. This says that the profit-maximizing supply curve of a competitive firm must have a positive (or at least a zero) slope.

Similarly, if the price of output and of factor 2 remain constant, equation (20.7) becomes

$$-\Delta w_1 \Delta x_1 \geq 0,$$

which is to say

$$\Delta w_1 \Delta x_1 \leq 0.$$

Thus if the price of factor 1 goes up, so that $\Delta w_1 > 0$, then equation (20.7) implies that the demand for factor 1 will go down (or at worst stay the same), so that $\Delta x_1 \leq 0$. This means that the factor demand curve must be a decreasing function of the factor price: factor demand curves have a negative slope.

The simple inequality in WAPM, and its implication in equation (20.7), give us strong observable restrictions about how a firm will behave. It is natural to ask whether these are all of the restrictions that the model of profit maximization imposes on firm behavior. Said another way, if we observe a firm's choices, and these choices satisfy WAPM, can we construct an estimate of the technology for which the observed choices are profit-maximizing choices? It turns out that the answer is yes. Figure 20.4 shows how to construct such a technology.

Figure 20.4

Construction of a possible technology. If the observed choices are maximal profit choices at each set of prices, then we can estimate the shape of the technology that generated those choices by using the isoprofit lines.

In order to illustrate the argument graphically, we suppose that there is one input and one output. Suppose that we are given an observed choice in period t and in period s, which we indicate by (p^t, w_1^t, y^t, x_1^t) and (p^s, w_1^s, y^s, x_1^s). In each period we can calculate the profits π_s and π_t and plot all the combinations of y and x_1 that yield these profits.

That is, we plot the two isoprofit lines

$$\pi_t = p^t y - w_1^t x_1$$

and

$$\pi_s = p^s y - w_1^s x_1.$$

The points above the isoprofit line for period t have higher profits than π_t at period t prices, and the points above the isoprofit line for period s have higher profits than π_s at period s prices. WAPM requires that the choice in period t must lie below the period s isoprofit line and that the choice in period s must lie below the period t isoprofit line.

If this condition is satisfied, it is not hard to generate a technology for which (y^t, x_1^t) and (y^s, x_1^s) are profit-maximizing choices. Just take the shaded area beneath the two lines. These are all of the choices that yield lower profits than the observed choices at both sets of prices.

The proof that this technology will generate the observed choices as profit-maximizing choices is clear geometrically. At the prices (p^t, w_1^t), the choice (y^t, x_1^t) is on the highest isoprofit line possible, and the same goes for the period s choice.

Thus, when the observed choices satisfy WAPM, we can "reconstruct" an estimate of a technology that might have generated the observations. In this sense, any observed choices consistent with WAPM could be profit-maximizing choices. As we observe more choices that the firm makes, we get a tighter estimate of the production function, as illustrated in Figure 20.5.

This estimate of the production function can be used to forecast firm behavior in other environments or for other uses in economic analysis.

Estimating the technology. As we observe more choices we get a tighter estimate of the production function.

Figure 20.5

EXAMPLE: How Do Farmers React to Price Supports?

The U.S. government currently spends between \$40 and \$60 billion a year in aid to farmers. A large fraction of this amount is used to subsidize the production of various products including milk, wheat, corn, soybeans, and cotton. Occasionally, attempts are made to reduce or eliminate these subsidies. The effect of elimination of these subsidies would be to reduce the price of the product received by the farmers.

Farmers sometimes argue that eliminating the subsidies to milk, for example, would not reduce the total supply of milk, since dairy farmers would choose to *increase* their herds and their supply of milk so as to keep their standard of living constant.

If farmers are behaving so as to maximize profits, this is impossible. As we've seen above, the logic of profit maximization *requires* that a decrease in the price of an output leads to a reduction in its supply: if Δp is negative, then Δy must be negative as well.

It is certainly possible that small family farms have goals other than simple maximization of profits, but larger "agribusiness" farms are more likely to be profit maximizers. Thus the perverse response to the elimination of subsidies alluded to above could only occur on a limited scale, if at all.

20.12 Cost Minimization

If a firm is maximizing profits and if it chooses to supply some output y, then it must be minimizing the cost of producing y. If this were not so, then there would be some cheaper way of producing y units of output, which would mean that the firm was not maximizing profits in the first place.

This simple observation turns out to be quite useful in examining firm behavior. It turns out to be convenient to break the profit-maximization problem into two stages: first we figure out how to minimize the costs of producing any desired level of output y, then we figure out which level of output is indeed a profit-maximizing level of output. We begin this task in the next chapter.

Summary

1. Profits are the difference between revenues and costs. In this definition it is important that all costs be measured using the appropriate market prices.

2. Fixed factors are factors whose amount is independent of the level of output; variable factors are factors whose amount used changes as the level of output changes.

3. In the short run, some factors must be used in predetermined amounts. In the long run, all factors are free to vary.

4. If the firm is maximizing profits, then the value of the marginal product of each factor that it is free to vary must equal its factor price.

5. The logic of profit maximization implies that the supply function of a competitive firm must be an increasing function of the price of output and that each factor demand function must be a decreasing function of its price.

6. If a competitive firm exhibits constant returns to scale, then its long-run maximum profits must be zero.

REVIEW QUESTIONS

1. In the short run, if the price of the fixed factor is increased, what will happen to profits?

2. If a firm had everywhere increasing returns to scale, what would happen to its profits if prices remained fixed and if it doubled its scale of operation?

3. If a firm had decreasing returns to scale at all levels of output and it divided up into two equal-size smaller firms, what would happen to its overall profits?

4. A gardener exclaims: "For only \$1 in seeds I've grown over \$20 in produce!" Besides the fact that most of the produce is in the form of zucchini, what other observations would a cynical economist make about this situation?

5. Is maximizing a firm's profits always identical to maximizing the firm's stock market value?

6. If $pMP_1 > w_1$, then should the firm increase or decrease the amount of factor 1 in order to increase profits?

7. Suppose a firm is maximizing profits in the short run with variable factor x_1 and fixed factor x_2. If the price of x_2 goes down, what happens to the firm's use of x_1? What happens to the firm's level of profits?

8. A profit-maximizing competitive firm that is making positive profits in long-run equilibrium (may/may not) have a technology with constant returns to scale.

APPENDIX

The profit-maximization problem of the firm is

$$\max_{x_1,x_2} pf(x_1,x_2) - w_1x_1 - w_2x_2,$$

which has first-order conditions

$$p\frac{\partial f(x_1^*,x_2^*)}{\partial x_1} - w_1 = 0$$
$$p\frac{\partial f(x_1^*,x_2^*)}{\partial x_2} - w_2 = 0.$$

These are just the same as the marginal product conditions given in the text. Let's see how profit-maximizing behavior looks using the Cobb-Douglas production function.

Suppose the Cobb-Douglas function is given by $f(x_1,x_2) = x_1^a x_2^b$. Then the two first-order conditions become

$$pax_1^{a-1}x_2^b - w_1 = 0$$
$$pbx_1^a x_2^{b-1} - w_2 = 0.$$

Multiply the first equation by x_1 and the second equation by x_2 to get

$$pax_1^a x_2^b - w_1x_1 = 0$$
$$pbx_1^a x_2^b - w_2x_2 = 0.$$

Using $y = x_1^a x_2^b$ to denote the level of output of this firm we can rewrite these expressions as

$$pay = w_1x_1$$
$$pby = w_2x_2.$$

Solving for x_1 and x_2 we have

$$x_1^* = \frac{apy}{w_1}$$
$$x_2^* = \frac{bpy}{w_2}.$$

This gives us the demands for the two factors as a function of the optimal output choice. But we still have to solve for the optimal choice of output. Inserting the optimal factor demands into the Cobb-Douglas production function, we have the expression

$$\left(\frac{pay}{w_1}\right)^a \left(\frac{pby}{w_2}\right)^b = y.$$

Factoring out the y gives

$$\left(\frac{pa}{w_1}\right)^a \left(\frac{pb}{w_2}\right)^b y^{a+b} = y.$$

Or

$$y = \left(\frac{pa}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{pb}{w_2}\right)^{\frac{b}{1-a-b}} .$$

This gives us the supply function of the Cobb-Douglas firm. Along with the factor demand functions derived above it gives us a complete solution to the profit-maximization problem.

Note that when the firm exhibits constant returns to scale—when $a + b = 1$—this supply function is not well defined. As long as the output and input prices are consistent with zero profits, a firm with a Cobb-Douglas technology is indifferent about its level of supply.

CHAPTER 21

COST MINIMIZATION

Our goal is to study the behavior of profit-maximizing firms in both competitive and noncompetitive market environments. In the last chapter we began our investigation of profit-maximizing behavior in a competitive environment by examining the profit-maximization problem directly.

However, some important insights can be gained through a more indirect approach. Our strategy will be to break up the profit-maximization problem into two pieces. First, we will look at the problem of how to minimize the costs of producing any given level of output, and then we will look at how to choose the most profitable level of output. In this chapter we'll look at the first step—minimizing the costs of producing a given level of output.

21.1 Cost Minimization

Suppose that we have two factors of production that have prices w_1 and w_2, and that we want to figure out the cheapest way to produce a given level of output, y. If we let x_1 and x_2 measure the amounts used of the

two factors and let $f(x_1, x_2)$ be the production function for the firm, we can write this problem as

$$\min_{x_1,x_2} w_1x_1 + w_2x_2$$

$$\text{such that } f(x_1, x_2) = y.$$

The same warnings apply as in the preceding chapter concerning this sort of analysis: make sure that you have included *all* costs of production in the calculation of costs, and make sure that everything is being measured on a compatible time scale.

The solution to this cost-minimization problem—the minimum costs necessary to achieve the desired level of output—will depend on w_1, w_2, and y, so we write it as $c(w_1, w_2, y)$. This function is known as the **cost function** and will be of considerable interest to us. The cost function $c(w_1, w_2, y)$ measures the minimal costs of producing y units of output when factor prices are (w_1, w_2).

In order to understand the solution to this problem, let us depict the costs and the technological constraints facing the firm on the same diagram. The isoquants give us the technological constraints—all the combinations of x_1 and x_2 that can produce y.

Suppose that we want to plot all the combinations of inputs that have some given level of cost, C. We can write this as

$$w_1x_1 + w_2x_2 = C,$$

which can be rearranged to give

$$x_2 = \frac{C}{w_2} - \frac{w_1}{w_2}x_1.$$

It is easy to see that this is a straight line with a slope of $-w_1/w_2$ and a vertical intercept of C/w_2. As we let the number C vary we get a whole family of **isocost lines**. Every point on an isocost curve has the same cost, C, and higher isocost lines are associated with higher costs.

Thus our cost-minimization problem can be rephrased as: find the point on the isoquant that has the lowest possible isocost line associated with it. Such a point is illustrated in Figure 21.1.

Note that if the optimal solution involves using some of each factor, and if the isoquant is a nice smooth curve, then the cost-minimizing point will be characterized by a tangency condition: the slope of the isoquant must be equal to the slope of the isocost curve. Or, using the terminology of Chapter 19, the *technical rate of substitution must equal the factor price ratio:*

$$-\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)} = \text{TRS}(x_1^*, x_2^*) = -\frac{w_1}{w_2}. \tag{21.1}$$

Figure 21.1 **Cost minimization**. The choice of factors that minimize production costs can be determined by finding the point on the isoquant that has the lowest associated isocost curve.

(If we have a boundary solution where one of the two factors isn't used, this tangency condition need not be met. Similarly, if the production function has "kinks," the tangency condition has no meaning. These exceptions are just like the situation with the consumer, so we won't emphasize these cases in this chapter.)

The algebra that lies behind equation (21.1) is not difficult. Consider any change in the pattern of production $(\Delta x_1, \Delta x_2)$ that keeps output constant. Such a change must satisfy

$$MP_1(x_1^*, x_2^*)\Delta x_1 + MP_2(x_1^*, x_2^*)\Delta x_2 = 0. \tag{21.2}$$

Note that Δx_1 and Δx_2 must be of opposite signs; if you increase the amount used of factor 1 you must decrease the amount used of factor 2 in order to keep output constant.

If we are at the cost minimum, then this change cannot lower costs, so we have

$$w_1\Delta x_1 + w_2\Delta x_2 \geq 0. \tag{21.3}$$

Now consider the change $(-\Delta x_1, -\Delta x_2)$. This also produces a constant level of output, and it too cannot lower costs. This implies that

$$-w_1\Delta x_1 - w_2\Delta x_2 \geq 0. \tag{21.4}$$

Putting expressions (21.3) and (21.4) together gives us

$$w_1 \Delta x_1 + w_2 \Delta x_2 = 0. \tag{21.5}$$

Solving equations (21.2) and (21.5) for $\Delta x_2 / \Delta x_1$ gives

$$\frac{\Delta x_2}{\Delta x_1} = -\frac{w_1}{w_2} = -\frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)},$$

which is just the condition for cost minimization derived above by a geometric argument.

Note that Figure 21.1 bears a certain resemblance to the solution to the consumer-choice problem depicted earlier. Although the solutions look the same, they really aren't the same kind of problem. In the consumer problem, the straight line was the budget constraint, and the consumer moved along the budget constraint to find the most-preferred position. In the producer problem, the isoquant is the technological constraint and the producer moves along the isoquant to find the optimal position.

The choices of inputs that yield minimal costs for the firm will in general depend on the input prices and the level of output that the firm wants to produce, so we write these choices as $x_1(w_1, w_2, y)$ and $x_2(w_1, w_2, y)$. These are called the **conditional factor demand functions**, or **derived factor demands**. They measure the relationship between the prices and output and the optimal factor choice of the firm, *conditional* on the firm producing a given level of output, y.

Note carefully the difference between the *conditional* factor demands and the profit-maximizing factor demands discussed in the last chapter. The conditional factor demands give the cost-minimizing choices for a given *level* of output; the profit-maximizing factor demands give the profit-maximizing choices for a given *price* of output.

Conditional factor demands are usually not directly observed; they are a hypothetical construct. They answer the question of how much of each factor *would* the firm use if it wanted to produce a given level of output in the cheapest way. However, the conditional factor demands are useful as a way of separating the problem of determining the optimal level of output from the problem of determining the most cost-effective method of production.

EXAMPLE: Minimizing Costs for Specific Technologies

Suppose that we consider a technology where the factors are perfect complements, so that $f(x_1, x_2) = \min\{x_1, x_2\}$. Then if we want to produce y units of output, we clearly need y units of x_1 and y units of x_2. Thus the minimal costs of production will be

$$c(w_1, w_2, y) = w_1 y + w_2 y = (w_1 + w_2) y.$$

What about the perfect substitutes technology, $f(x_1, x_2) = x_1 + x_2$? Since goods 1 and 2 are perfect substitutes in production it is clear that the firm will use whichever is cheaper. Thus the minimum cost of producing y units of output will be $w_1 y$ or $w_2 y$, whichever is less. In other words:

$$c(w_1, w_2, y) = \min\{w_1 y, w_2 y\} = \min\{w_1, w_2\} y.$$

Finally, we consider the Cobb-Douglas technology, which is described by the formula $f(x_1, x_2) = x_1^a x_2^b$. In this case we can use calculus techniques to show that the cost function will have the form

$$c(w_1, w_2, y) = K w_1^{\frac{a}{a+b}} w_2^{\frac{b}{a+b}} y^{\frac{1}{a+b}},$$

where K is a constant that depends on a and b. The details of the calculation are presented in the Appendix.

21.2 Revealed Cost Minimization

The assumption that the firm chooses factors to minimize the cost of producing output will have implications for how the observed choices change as factor prices change.

Suppose that we observe two sets of prices, (w_1^t, w_2^t) and (w_1^s, w_2^s), and the associated choices of the firm, (x_1^t, x_2^t) and (x_1^s, x_2^s). Suppose that each of these choices produces the same output level y. Then if each choice is a cost-minimizing choice at its associated prices, we must have

$$w_1^t x_1^t + w_2^t x_2^t \leq w_1^t x_1^s + w_2^t x_2^s$$

and

$$w_1^s x_1^s + w_2^s x_2^s \leq w_1^s x_1^t + w_2^s x_2^t.$$

If the firm is always choosing the cost-minimizing way to produce y units of output, then its choices at times t and s must satisfy these inequalities. We will refer to these inequalities as the **Weak Axiom of Cost Minimization (WACM)**.

Write the second equation as

$$-w_1^s x_1^t - w_2^s x_2^t \leq -w_1^s x_1^s - w_2^s x_2^s$$

and add it to the first equation to get

$$(w_1^t - w_1^s) x_1^t + (w_2^t - w_2^s) x_2^t \leq (w_1^t - w_1^s) x_1^s + (w_2^t - w_2^s) x_2^s,$$

which can be rearranged to give us

$$(w_1^t - w_1^s)(x_1^t - x_1^s) + (w_2^t - w_2^s)(x_2^t - x_2^s) \leq 0.$$

Using the delta notation to depict the *changes* in the factor demands and factor prices, we have

$$\Delta w_1 \Delta x_1 + \Delta w_2 \Delta x_2 \leq 0.$$

This equation follows solely from the assumption of cost-minimizing behavior. It implies restrictions on how the firm's behavior can change when input prices change and output remains constant.

For example, if the price of the first factor increases and the price of the second factor stays constant, then $\Delta w_2 = 0$, so the inequality becomes

$$\Delta w_1 \Delta x_1 \leq 0.$$

If the price of factor 1 increases, then this inequality implies that the demand for factor 1 must decrease; thus the conditional factor demand functions must slope down.

What can we say about how the minimal costs change as we change the parameters of the problem? It is easy to see that costs must increase if either factor price increases: if one good becomes more expensive and the other stays the same, the minimal costs cannot go down and in general will increase. Similarly, if the firm chooses to produce more output and factor prices remain constant, the firm's costs will have to increase.

21.3 Returns to Scale and the Cost Function

In Chapter 19 we discussed the idea of returns to scale for the production function. Recall that a technology is said to have increasing, decreasing, or constant returns to scale as $f(tx_1, tx_2)$ is greater, less than, or equal to $tf(x_1, x_2)$ for all $t > 1$. It turns out that there is a nice relation between the kind of returns to scale exhibited by the production function and the behavior of the cost function.

Suppose first that we have the natural case of constant returns to scale. Imagine that we have solved the cost-minimization problem to produce 1 unit of output, so that we know the **unit cost function**, $c(w_1, w_2, 1)$. Now what is the cheapest way to produce y units of output? Simple: we just use y times as much of every input as we were using to produce 1 unit of output. This would mean that the minimal cost to produce y units of output would just be $c(w_1, w_2, 1)y$. In the case of constant returns to scale, the cost function is linear in output.

What if we have increasing returns to scale? In this case it turns out that costs increase less than linearly in output. If the firm decides to produce twice as much output, it can do so at *less* than twice the cost, as long as the factor prices remain fixed. This is a natural implication of the idea of increasing returns to scale: if the firm doubles its inputs, it will more than

double its output. Thus if it wants to produce double the output, it will be able to do so by using less than twice as much of every input.

But using twice as much of every input will exactly double costs. So using less than twice as much of every input will make costs go up by less than twice as much: this is just saying that the cost function will increase less than linearly with respect to output.

Similarly, if the technology exhibits decreasing returns to scale, the cost function will increase more than linearly with respect to output. If output doubles, costs will more than double.

These facts can be expressed in terms of the behavior of the **average cost function**. The average cost function is simply the cost *per unit* to produce y units of output:

$$AC(y) = \frac{c(w_1, w_2, y)}{y}.$$

If the technology exhibits constant returns to scale, then we saw above that the cost function had the form $c(w_1, w_2, y) = c(w_1, w_2, 1)y$. This means that the average cost function will be

$$AC(w_1, w_2, y) = \frac{c(w_1, w_2, 1)y}{y} = c(w_1, w_2, 1).$$

That is, the cost per unit of output will be constant no matter what level of output the firm wants to produce.

If the technology exhibits increasing returns to scale, then the costs will increase less than linearly with respect to output, so the average costs will be declining in output: as output increases, the average costs of production will tend to fall.

Similarly, if the technology exhibits decreasing returns to scale, then average costs will rise as output increases.

As we saw earlier, a given technology can have *regions* of increasing, constant, or decreasing returns to scale—output can increase more rapidly, equally rapidly, or less rapidly than the scale of operation of the firm at different levels of production. Similarly, the cost function can increase less rapidly, equally rapidly, or more rapidly than output at different levels of production. This implies that the average cost function may decrease, remain constant, or increase over different levels of output. In the next chapter we will explore these possibilities in more detail.

From now on we will be most concerned with the behavior of the cost function with respect to the output variable. For the most part we will regard the factor prices as being fixed at some predetermined levels and only think of costs as depending on the output choice of the firm. Thus for the remainder of the book we will write the cost function as a function of output alone: $c(y)$.

21.4 Long-Run and Short-Run Costs

The cost function is defined as the minimum cost of achieving a given level of output. Often it is important to distinguish the minimum costs if the firm is allowed to adjust all of its factors of production from the minimum costs if the firm is only allowed to adjust some of its factors.

We have defined the short run to be a time period where some of the factors of production must be used in a fixed amount. In the long run, all factors are free to vary. The **short-run cost function** is defined as the minimum cost to produce a given level of output, only adjusting the variable factors of production. The **long-run cost function** gives the minimum cost of producing a given level of output, adjusting *all* of the factors of production.

Suppose that in the short run factor 2 is fixed at some predetermined level $\overline{x}_2$, but in the long run it is free to vary. Then the short-run cost function is defined by

$$c_s(y, \overline{x}_2) = \min_{x_1} w_1 x_1 + w_2 \overline{x}_2$$

$$\text{such that } f(x_1, \overline{x}_2) = y.$$

Note that in general the minimum cost to produce y units of output in the short run will depend on the amount and cost of the fixed factor that is available.

In the case of two factors, this minimization problem is easy to solve: we just find the smallest amount of x_1 such that $f(x_1, \overline{x}_2) = y$. However, if there are many factors of production that are variable in the short run the cost-minimization problem will involve more elaborate calculation.

The short-run factor demand function for factor 1 is the amount of factor 1 that minimizes costs. In general it will depend on the factor prices and on the levels of the fixed factors as well, so we write the short-run factor demands as

$$\begin{aligned} x_1 &= x_1^s(w_1, w_2, \overline{x}_2, y) \\ x_2 &= \overline{x}_2. \end{aligned}$$

These equations just say, for example, that if the building size is fixed in the short run, then the number of workers that a firm wants to hire at any given set of prices and output choice will typically depend on the size of the building.

Note that by definition of the short-run cost function

$$c_s(y, \overline{x}_2) = w_1 x_1^s(w_1, w_2, \overline{x}_2, y) + w_2 \overline{x}_2.$$

This just says that the minimum cost of producing output y is the cost associated with using the cost-minimizing choice of inputs. This is true by definition but turns out to be useful nevertheless.

The long-run cost function in this example is defined by

$$c(y) = \min_{x_1, x_2} w_1 x_1 + w_2 x_2$$

$$\text{such that } f(x_1, x_2) = y.$$

Here both factors are free to vary. Long-run costs depend only on the level of output that the firm wants to produce along with factor prices. We write the long-run cost function as $c(y)$, and write the long-run factor demands as

$$x_1 = x_1(w_1, w_2, y)$$
$$x_2 = x_2(w_1, w_2, y).$$

We can also write the long-run cost function as

$$c(y) = w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y).$$

Just as before, this simply says that the minimum costs are the costs that the firm gets by using the cost-minimizing choice of factors.

There is an interesting relation between the short-run and the long-run cost functions that we will use in the next chapter. For simplicity, let us suppose that factor prices are fixed at some predetermined levels and write the long-run factor demands as

$$x_1 = x_1(y)$$
$$x_2 = x_2(y).$$

Then the long-run cost function can also be written as

$$c(y) = c_s(y, x_2(y)).$$

To see why this is true, just think about what it means. The equation says that the minimum costs when all factors are variable is just the minimum cost when factor 2 is fixed *at the level that minimizes long-run costs.* It follows that the long-run demand for the variable factor—the cost-minimizing choice—is given by

$$x_1(w_1, w_2, y) = x_1^s(w_1, w_2, x_2(y), y).$$

This equation says that the cost-minimizing amount of the variable factor in the long run is that amount that the firm would choose in the short run—if it happened to have the long-run cost-minimizing amount of the fixed factor.

21.5 Fixed and Quasi-Fixed Costs

In Chapter 20 we made the distinction between fixed factors and quasi-fixed factors. Fixed factors are factors that must receive payment whether or not any output is produced. Quasi-fixed factors must be paid only if the firm decides to produce a positive amount of output.

It is natural to define fixed costs and quasi-fixed costs in a similar manner. **Fixed costs** are costs associated with the fixed factors: they are independent of the level of output, and, in particular, they must be paid whether or not the firm produces output. **Quasi-fixed costs** are costs that are also independent of the level of output, but only need to be paid if the firm produces a positive amount of output.

There are no fixed costs in the long run, by definition. However, there may easily be quasi-fixed costs in the long run. If it is necessary to spend a fixed amount of money before any output at all can be produced, then quasi-fixed costs will be present.

21.6 Sunk Costs

Sunk costs are another kind of fixed costs. The concept is best explained by example. Suppose that you have decided to lease an office for a year. The monthly rent that you have committed to pay is a fixed cost, since you are obligated to pay it regardless of the amount of output you produce. Now suppose that you decide to refurbish the office by painting it and buying furniture. The cost for paint is a fixed cost, but it is also a **sunk cost** since it is a payment that is made and cannot be recovered. The cost of buying the furniture, on the other hand, is not entirely sunk, since you can resell the furniture when you are done with it. It's only the *difference* between the cost of new and used furniture that is sunk.

To spell this out in more detail, suppose that you borrow $20,000 at the beginning of the year at, say, 10 percent interest. You sign a lease to rent an office and pay $12,000 in advance rent for next year. You spend $6,000 on office furniture and $2,000 to paint the office. At the end of the year you pay back the $20,000 loan plus the $2,000 interest payment and sell the used office furniture for $5,000.

Your total sunk costs consist of the $12,000 rent, the $2,000 of interest, the $2,000 of paint, but only $1,000 for the furniture, since $5,000 of the orginal furniture expenditure is recoverable.

The difference between sunk costs and recoverable costs can be quite significant. A $100,000 expenditure to purchase five light trucks sounds like a lot of money, but if they can later be sold on the used truck market for $80,000, the actual sunk cost is only $20,000. A $100,000 expenditure

on a custom-made press for stamping out gizmos that has a zero resale value is quite different; in this case the entire expenditure is sunk.

The best way to keep these issues straight is to make sure to treat all expenditures on a flow basis: how much does it cost to do business for a year? That way, one is less likely to forget the resale value of capital equipment and more likely to keep the distinction between sunk costs and recoverable costs clear.

Summary

1. The cost function, $c(w_1, w_2, y)$, measures the minimum costs of producing a given level of output at given factor prices.

2. Cost-minimizing behavior imposes observable restrictions on choices that firms make. In particular, conditional factor demand functions will be negatively sloped.

3. There is an intimate relationship between the returns to scale exhibited by the technology and the behavior of the cost function. *Increasing* returns to scale implies *decreasing* average cost, *decreasing* returns to scale implies *increasing* average cost, and *constant* returns to scale implies *constant* average cost.

4. Sunk costs are costs that are not recoverable.

REVIEW QUESTIONS

1. Prove that a profit-maximizing firm will always minimize costs.

2. If a firm is producing where $MP_1/w_1 > MP_2/w_2$, what can it do to reduce costs but maintain the same output?

3. Suppose that a cost-minimizing firm uses two inputs that are perfect substitutes. If the two inputs are priced the same, what do the conditional factor demands look like for the inputs?

4. The price of paper used by a cost-minimizing firm increases. The firm responds to this price change by changing its demand for certain inputs, but it keeps its output constant. What happens to the firm's use of paper?

5. If a firm uses n inputs ($n > 2$), what inequality does the theory of revealed cost minimization imply about changes in factor prices (Δw_i) and the changes in factor demands (Δx_i) for a given level of output?

APPENDIX

Let us study the cost-minimization problem posed in the text using the optimization techniques introduced in Chapter 5. The problem is a constrained-minimization problem of the form

$$\min_{x_1,x_2} w_1x_1 + w_2x_2$$

$$\text{such that } f(x_1,x_2) = y.$$

Recall that we had several techniques to solve this kind of problem. One way was to substitute the constraint into the objective function. This can still be used when we have a specific functional form for $f(x_1,x_2)$, but isn't much use in the general case.

The second method was the method of Lagrange multipliers and that works fine. To apply this method we set up the Lagrangian

$$L = w_1x_1 + w_2x_2 - \lambda(f(x_1,x_2) - y)$$

and differentiate with respect to x_1, x_2 and λ. This gives us the first-order conditions:

$$\begin{aligned} w_1 - \lambda\frac{\partial f(x_1,x_2)}{\partial x_1} &= 0 \\ w_2 - \lambda\frac{\partial f(x_1,x_2)}{\partial x_2} &= 0 \\ f(x_1,x_2) - y &= 0. \end{aligned}$$

The last condition is simply the constraint. We can rearrange the first two equations and divide the first equation by the second equation to get

$$\frac{w_1}{w_2} = \frac{\partial f(x_1,x_2)/\partial x_1}{\partial f(x_1,x_2)/\partial x_2}.$$

Note that this is the same first-order condition that we derived in the text: the technical rate of substitution must equal the factor price ratio.

Let's apply this method to the Cobb-Douglas production function:

$$f(x_1,x_2) = x_1^a x_2^b.$$

The cost-minimization problem is then

$$\min_{x_1,x_2} w_1x_1 + w_2x_2$$

$$\text{such that } x_1^a x_2^b = y.$$

Here we have a specific functional form, and we can solve it using either the substitution method or the Lagrangian method. The substitution method would involve first solving the constraint for x_2 as a function of x_1:

$$x_2 = \left(yx_1^{-a}\right)^{1/b}$$

and then substituting this into the objective function to get the unconstrained minimization problem

$$\min_{x_1} w_1 x_1 + w_2 \left(y x_1^{-a}\right)^{1/b}.$$

We could now differentiate with respect to x_1 and set resulting derivative equal to zero, as usual. The resulting equation can be solved to get x_1 as a function of w_1, w_2, and y, to get the conditional factor demand for x_1. This isn't hard to do, but the algebra is messy, so we won't write down the details.

We will, however, solve the Lagrangian problem. The three first-order conditions are

$$\begin{aligned} w_1 &= \lambda a x_1^{a-1} x_2^b \\ w_2 &= \lambda b x_1^a x_2^{b-1} \\ y &= x_1^a x_2^b. \end{aligned}$$

Multiply the first equation by x_1 and the second equation by x_2 to get

$$w_1 x_1 = \lambda a x_1^a x_2^b = \lambda a y$$

$$w_2 x_2 = \lambda b x_1^a x_2^b = \lambda b y,$$

so that

$$x_1 = \lambda \frac{ay}{w_1} \tag{21.6}$$

$$x_2 = \lambda \frac{by}{w_2}. \tag{21.7}$$

Now we use the third equation to solve for λ. Substituting the solutions for x_1 and x_2 into the third first-order condition, we have

$$\left(\frac{\lambda a y}{w_1}\right)^a \left(\frac{\lambda b y}{w_2}\right)^b = y.$$

We can solve this equation for λ to get the rather formidable expression

$$\lambda = (a^{-a} b^{-b} w_1^a w_2^b y^{1-a-b})^{\frac{1}{a+b}},$$

which, along with equations (21.6) and (21.7), gives us our final solutions for x_1 and x_2. These factor demand functions will take the form

$$x_1(w_1, w_2, y) = \left(\frac{a}{b}\right)^{\frac{b}{a+b}} w_1^{\frac{-b}{a+b}} w_2^{\frac{b}{a+b}} y^{\frac{1}{a+b}}$$

$$x_2(w_1, w_2, y) = \left(\frac{a}{b}\right)^{-\frac{a}{a+b}} w_1^{\frac{a}{a+b}} w_2^{\frac{-a}{a+b}} y^{\frac{1}{a+b}}.$$

The cost function can be found by writing down the costs when the firm makes the cost-minimizing choices. That is,

$$c(w_1, w_2, y) = w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y).$$

Some tedious algebra shows that

$$c(w_1, w_2, y) = \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{a}{b}\right)^{\frac{-a}{a+b}} \right] w_1^{\frac{a}{a+b}} w_2^{\frac{b}{a+b}} y^{\frac{1}{a+b}}.$$

(Don't worry, this formula won't be on the final exam. It is presented only to demonstrate how to get an explicit solution to the cost-minimization problem by applying the method of Lagrange multipliers.)

Note that costs will increase more than, equal to, or less than linearly with output as $a + b$ is less than, equal to, or greater than 1. This makes sense since the Cobb-Douglas technology exhibits decreasing, constant, or increasing returns to scale depending on the value of $a + b$.

CHAPTER 22

COST CURVES

In the last chapter we described the cost-minimizing behavior of a firm. Here we continue that investigation through the use of an important geometric construction, the **cost curve**. Cost curves can be used to depict graphically the cost function of a firm and are important in studying the determination of optimal output choices.

22.1 Average Costs

Consider the cost function described in the last chapter. This is the function $c(w_1, w_2, y)$ that gives the minimum cost of producing output level y when factor prices are (w_1, w_2). In the rest of this chapter we will take the factor prices to be fixed so that we can write cost as a function of y alone, $c(y)$.

Some of the costs of the firm are independent of the level of output of the firm. As we've seen in Chapter 21, these are the fixed costs. Fixed costs are the costs that must be paid regardless of what level of output the firm produces. For example, the firm might have mortgage payments that are required no matter what its level of output.

Other costs change when output changes: these are the variable costs. The total costs of the firm can always be written as the sum of the variable costs, $c_v(y)$, and the fixed costs, F:

$$c(y) = c_v(y) + F.$$

The **average cost function** measures the cost per unit of output. The **average variable cost function** measures the variable costs per unit of output, and the **average fixed cost function** measures the fixed costs per unit output. By the above equation:

$$AC(y) = \frac{c(y)}{y} = \frac{c_v(y)}{y} + \frac{F}{y} = AVC(y) + AFC(y)$$

where $AVC(y)$ stands for average variable costs and $AFC(y)$ stands for average fixed costs. What do these functions look like? The easiest one is certainly the average fixed cost function: when $y = 0$ it is infinite, and as y increases the average fixed cost decreases toward zero. This is depicted in Figure 22.1A.

Figure 22.1

Construction of the average cost curve. (A) The average fixed costs decrease as output is increased. (B) The average variable costs eventually increase as output is increased. (C) The combination of these two effects produces a U-shaped average cost curve.

Consider the variable cost function. Start at a zero level of output and consider producing one unit. Then the average variable costs at $y = 1$ is just the variable cost of producing this one unit. Now increase the level of production to 2 units. We would expect that, at worst, variable costs would double, so that average variable costs would remain constant. If

we can organize production in a more efficient way as the scale of output is increased, the average variable costs might even decrease initially. But eventually we would expect the average variable costs to rise. Why? If fixed factors are present, they will eventually constrain the production process.

For example, suppose that the fixed costs are due to the rent or mortgage payments on a building of fixed size. Then as production increases, average variable costs—the per-unit production costs—may remain constant for a while. But as the capacity of the building is reached, these costs will rise sharply, producing an average variable cost curve of the form depicted in Figure 22.1B.

The average cost curve is the sum of these two curves; thus it will have the U-shape indicated in Figure 22.1C. The initial decline in average costs is due to the decline in average fixed costs; the eventual increase in average costs is due to the increase in average variable costs. The combination of these two effects yields the U-shape depicted in the diagram.

22.2 Marginal Costs

There is one more cost curve of interest: the **marginal cost curve**. The marginal cost curve measures the *change* in costs for a given change in output. That is, at any given level of output y, we can ask how costs will change if we change output by some amount Δy:

$$MC(y) = \frac{\Delta c(y)}{\Delta y} = \frac{c(y + \Delta y) - c(y)}{\Delta y}.$$

We could just as well write the definition of marginal costs in terms of the variable cost function:

$$MC(y) = \frac{\Delta c_v(y)}{\Delta y} = \frac{c_v(y + \Delta y) - c_v(y)}{\Delta y}.$$

This is equivalent to the first definition, since $c(y) = c_v(y) + F$ and the fixed costs, F, don't change as y changes.

Often we think of Δy as being one unit of output, so that marginal cost indicates the change in our costs if we consider producing one more discrete unit of output. If we are thinking of the production of a discrete good, then marginal cost of producing y units of output is just $c(y) - c(y - 1)$. This is often a convenient way to think about marginal cost, but is sometimes misleading. Remember, marginal cost measures a *rate of change*: the change in costs divided by a change in output. If the change in output is a single unit, then marginal cost looks like a simple change in costs, but it is really a rate of change as we increase the output by one unit.

How can we put this marginal cost curve on the diagram presented above? First we note the following. The variable costs are zero when zero units of output are produced, by definition. Thus for the first unit of output produced

$$MC(1) = \frac{c_v(1) + F - c_v(0) - F}{1} = \frac{c_v(1)}{1} = AVC(1).$$

Thus the marginal cost for the first small unit of amount equals the average variable cost for a single unit of output.

Now suppose that we are producing in a range of output where *average* variable costs are decreasing. Then it must be that the *marginal* costs are less than the average variable costs in this range. For the way that you push an average down is to add in numbers that are less than the average.

Think about a sequence of numbers representing average costs at different levels of output. If the average is decreasing, it must be that the cost of each additional unit produced is less than average up to that point. To make the average go down, you have to be adding additional units that are less than the average.

Similarly, if we are in a region where average variable costs are rising, then it must be the case that the marginal costs are greater than the average variable costs—it is the higher marginal costs that are pushing the average up.

Thus we know that the marginal cost curve must lie below the average variable cost curve to the left of its minimum point and above it to the right. This implies that the marginal cost curve must intersect the average variable cost curve at its minimum point.

Exactly the same kind of argument applies for the average cost curve. If average costs are falling, then marginal costs must be less than the average costs and if average costs are rising the marginal costs must be larger than the average costs. These observations allow us to draw in the marginal cost curve as in Figure 22.2.

To review the important points:

- The average variable cost curve may initially slope down but need not. However, it will eventually rise, as long as there are fixed factors that constrain production.

- The average cost curve will initially fall due to declining fixed costs but then rise due to the increasing average variable costs.

- The marginal cost and average variable cost are the same at the first unit of output.

- The marginal cost curve passes through the minimum point of both the average variable cost and the average cost curves.

Figure 22.2 **Cost curves**. The average cost curve (AC), the average variable cost curve (AVC), and the marginal cost curve (MC).

22.3 Marginal Costs and Variable Costs

There are also some other relationships between the various curves. Here is one that is not so obvious: it turns out that the area beneath the marginal cost curve up to y gives us the variable cost of producing y units of output. Why is that?

The marginal cost curve measures the cost of producing each additional unit of output. If we add up the cost of producing each unit of output we will get the total costs of production—except for fixed costs.

This argument can be made rigorous in the case where the output good is produced in discrete amounts. First, we note that

$$c_v(y) = [c_v(y) - c_v(y-1)] + [c_v(y-1) - c_v(y-2] + \\ \cdots + [c_v(1) - c_v(0)].$$

This is true since $c_v(0) = 0$ and all the middle terms cancel out; that is, the second term cancels the third term, the fourth term cancels the fifth term, and so on. But each term in this sum is the marginal cost at a different level of output:

$$c_v(y) = MC(y-1) + MC(y-2) + \cdots + MC(0).$$

Thus each term in the sum represents the area of a rectangle with height $MC(y)$ and base of 1. Summing up all these rectangles gives us the area under the marginal cost curve as depicted in Figure 22.3.

Marginal cost and variable costs. The area under the marginal cost curve gives the variable costs.

Figure 22.3

EXAMPLE: Specific Cost Curves

Let's consider the cost function $c(y) = y^2 + 1$. We have the following derived cost curves:

- variable costs: $c_v(y) = y^2$
- fixed costs: $c_f(y) = 1$
- average variable costs: $AVC(y) = y^2/y = y$
- average fixed costs: $AFC(y) = 1/y$
- average costs: $AC(y) = \dfrac{y^2+1}{y} = y + \dfrac{1}{y}$
- marginal costs: $MC(y) = 2y$

These are all obvious except for the last one, which is also obvious if you know calculus. If the cost function is $c(y) = y^2 + F$, then the marginal cost function is given by $MC(y) = 2y$. If you don't know this fact already, memorize it, because you'll use it in the exercises.

What do these cost curves look like? The easiest way to draw them is first to draw the average variable cost curve, which is a straight line with slope 1. Then it is also simple to draw the marginal cost curve, which is a straight line with slope 2.

The average cost curve reaches its minimum where average cost equals marginal cost, which says

$$y + \frac{1}{y} = 2y,$$

which can be solved to give $y_{\text{min}} = 1$. The average cost at $y = 1$ is 2, which is also the marginal cost. The final picture is given in Figure 22.4.

Figure 22.4 **Cost curves**. The cost curves for $c(y) = y^2 + 1$.

EXAMPLE: Marginal Cost Curves for Two Plants

Suppose that you have two plants that have two different cost functions, $c_1(y_1)$ and $c_2(y_2)$. You want to produce y units of output in the cheapest

way. In general, you will want to produce some amount of output in each plant. The question is, how much should you produce in each plant?

Set up the minimization problem:

$$\min_{y_1, y_2} c_1(y_1) + c_2(y_2)$$
$$\text{such that } y_1 + y_2 = y.$$

Now how do you solve it? It turns out that at the optimal division of output between the two plants we must have the marginal cost of producing output at plant 1 equal to the marginal cost of producing output at plant 2. In order to prove this, suppose the marginal costs were not equal; then it would pay to shift a small amount of output from the plant with higher marginal costs to the plant with lower marginal costs. If the output division is optimal, then switching output from one plant to the other can't lower costs.

Let $c(y)$ be the cost function that gives the cheapest way to produce y units of output—that is, the cost of producing y units of output given that you have divided output in the best way between the two plants. The marginal cost of producing an extra unit of output must be the same no matter which plant you produce it in.

We depict the two marginal cost curves, $MC_1(y_1)$ and $MC_2(y_2)$, in Figure 22.5. The marginal cost curve for the two plants taken together is just the horizontal sum of the two marginal cost curves, as depicted in Figure 22.5C.

Figure 22.5

Marginal costs for a firm with two plants. The overall marginal cost curve on the right is the horizontal sum of the marginal cost curves for the two plants shown on the left.

For any fixed level of marginal costs, say c, we will produce y_1^* and y_2^* such that $MC_1(y_1^*) = MC(y_2^*) = c$, and we will thus have $y_1^* + y_2^*$ units of output produced. Thus the amount of output produced at any marginal cost c is just the sum of the outputs where the marginal cost of plant 1 equals c and the marginal cost of plant 2 equals c: the horizontal sum of the marginal cost curves.

22.4 Cost Curves for Online Auctions

We explored an auction model of search engine advertising in Chapter 18. Recall the setup. When a user enters a query into a search engine, the query is matched with keywords chosen by advertisers. Those advertisers whose keywords match the query are entered into an auction. The highest bidder gets the most prominent position, the second-highest bidder gets the second most prominent position and so on. The more prominent the position, the more clicks the ad tends to get, other things (such as ad quality) being equal.

In the auction examined earlier, it was assumed that each advertiser could choose a separate bid for each keyword. In practice, an advertiser chooses a single bid that is used in all auctions in which they participate. The fact that prices are determined by an auction is not all that important from an advertiser's point of view. What matters is the relationship between the number of clicks the ad gets, x, and the cost of those clicks, $c(x)$.

This is just our old friend the total cost function. Once an advertiser knows the cost function, it can determine how many clicks it wants to buy. Letting v represent the value of a click, the profit maximization problem is

$$\max_x vx - c(x).$$

As we have seen, the optimal solution entails setting value equal to marginal cost. Once the advertiser determines the profit-maximizing number of clicks, it can choose a bid that will yield that many clicks.

This process is shown in Figure 22.6, which is a standard plot of average cost and marginal cost, with the addition of a new line illustrating the bid.

How does the advertising discover its cost curve? One answer is that the advertiser can experiment with different bids and record the resulting number of clicks and cost. Or, the search engine can provide an estimate of the cost function by using the information from the auctions.

Suppose, for example, we want to estimate what would happen if an advertiser increases its bid per click from 50 cents to 80 cents. The search engine can look at each auction in which the advertiser participates to how its position changes and how many new clicks it could be expected to receive in the new position.

Click-cost curves. The profit-maximizing number of clicks is where value equals marginal cost, which determines the appropriate bid and average cost per click.

Figure 22.6

22.5 Long-Run Costs

In the above analysis, we have regarded the firm's fixed costs as being the costs that involve payments to factors that it is unable to adjust in the short run. In the long run a firm can choose the level of its "fixed" factors—they are no longer fixed.

Of course, there may still be quasi-fixed factors in the long run. That is, it may be a feature of the technology that some costs have to be paid to produce any positive level of output. But in the long run there are no fixed costs, in the sense that it is always possible to produce zero units of output at zero costs—that is, it is always possible to go out of business. If quasi-fixed factors are present in the long run, then the average cost curve will tend to have a U-shape, just as in the short run. But in the long run it will always be possible to produce zero units of output at a zero cost, by definition of the long run.

Of course, what constitutes the long run depends on the problem we are analyzing. If we are considering the fixed factor to be the size of the plant, then the long run will be how long it would take the firm to change the size of its plant. If we are considering the fixed factor to be the contractual obligations to pay salaries, then the long run would be how long it would take the firm to change the size of its work force.

Just to be specific, let's think of the fixed factor as being plant size and

denote it by k. The firm's short-run cost function, given that it has a plant of k square feet, will be denoted by $c_s(y, k)$, where the s subscript stands for "short run." (Here k is playing the role of $\overline{x}_2$ in Chapter 21.)

For any given level of output, there will be some plant size that is the optimal size to produce that level of output. Let us denote this plant size by $k(y)$. This is the firm's conditional factor demand for plant size as a function of output. (Of course, it also depends on the prices of plant size and other factors of production, but we have suppressed these arguments.) Then, as we've seen in Chapter 21, the long-run cost function of the firm will be given by $c_s(y, k(y))$. This is the total cost of producing an output level y, given that the firm is allowed to adjust its plant size optimally. The long-run cost function of the firm is just the short-run cost function evaluated at the optimal choice of the fixed factors:

$$c(y) = c_s(y, k(y)).$$

Let us see how this looks graphically. Pick some level of output y^*, and let $k^* = k(y^*)$ be the optimal plant size for that level of output. The short-run cost function for a plant of size k^* will be given by $c_s(y, k^*)$, and the long-run cost function will be given by $c(y) = c_s(y, k(y))$, just as above.

Now, note the important fact that the short-run cost to produce output y must always be at least as large as the long-run cost to produce y. Why? In the short run the firm has a fixed plant size, while in the long run the firm is free to adjust its plant size. Since one of its long-run choices is always to choose the plant size k^*, its optimal choice to produce y units of output must have costs at least as small as $c(y, k^*)$. This means that the firm must be able to do at least as well by adjusting plant size as by having it fixed. Thus

$$c(y) \leq c_s(y, k^*)$$

for all levels of y.

In fact, at one particular level of y, namely y^*, we know that

$$c(y^*) = c_s(y^*, k^*).$$

Why? Because at y^* the *optimal* choice of plant size is k^*. So at y^*, the long-run costs and the short-run costs are the same.

If the short-run cost is always greater than the long-run cost and they are equal at one level of output, then this means that the short-run and the long-run average costs have the same property: $AC(y) \leq AC_s(y, k^*)$ and $AC(y^*) = AC_s(y^*, k^*)$. This implies that the short-run average cost curve always lies above the long-run average cost curve and that they touch at one point, y^*. Thus the long-run average cost curve (LAC) and the short-run average cost curve (SAC) must be tangent at that point, as depicted in Figure 22.7.

Short-run and long-run average costs. The short-run average cost curve must be tangent to the long-run average cost curve.

Figure 22.7

We can do the same sort of construction for levels of output other than y^*. Suppose we pick outputs $y_1, y_2, \ldots, y_n$ and accompanying plant sizes $k_1 = k(y_1), k_2 = k(y_2), \ldots, k_n = k(y_n)$. Then we get a picture like that in Figure 22.8. We summarize Figure 22.8 by saying that the long-run average cost curve is the **lower envelope** of the short-run average cost curves.

22.6 Discrete Levels of Plant Size

In the above discussion we have implicitly assumed that we can choose a continuous number of different plant sizes. Thus each different level of output has a unique optimal plant size associated with it. But we can also consider what happens if there are only a few different levels of plant size to choose from.

Suppose, for example, that we have four different choices, k_1, k_2, k_3, and k_4. We have depicted the four different average cost curves associated with these plant sizes in Figure 22.9.

How can we construct the long-run average cost curve? Well, remember the long-run average cost curve is the cost curve you get by adjusting k optimally. In this case that isn't hard to do: since there are only four different plant sizes, we just see which one has the lowest costs associated with it and pick that plant size. That is, for any level of output y, we just

Figure 22.8 **Short-run and long-run average costs**. The long-run average cost curve is the envelope of the short-run average cost curves.

choose the plant size that gives us the minimum cost of producing that output level.

Thus the long-run average cost curve will be the lower envelope of the short-run average costs, as depicted in Figure 22.9. Note that this figure has qualitatively the same implications as Figure 22.8: the short-run average costs always are at least as large as the long-run average costs, and they are the same at the level of output where the long-run demand for the fixed factor equals the amount of the fixed factor that you have.

22.7 Long-Run Marginal Costs

We've seen in the last section that the long-run average cost curve is the lower envelope of the short-run average cost curves. What are the implications of this for marginal costs? Let's first consider the case where there are discrete levels of plant size. In this situation the long-run marginal cost curve consists of the appropriate pieces of the short-run marginal cost curves, as depicted in Figure 22.10. For each level of output, we see which short-run average cost curve we are operating on and then look at the marginal cost associated with that curve.

Discrete levels of plant size. The long-run cost curve is the lower envelope of the short-run curves, just as before.

Figure 22.9

This has to hold true no matter how many different plant sizes there are, so the picture for the continuous case looks like Figure 22.11. The long-run marginal cost at any output level y has to equal the short-run marginal cost associated with the optimal level of plant size to produce y.

Summary

1. Average costs are composed of average variable costs plus average fixed costs. Average fixed costs always decline with output, while average variable costs tend to increase. The net result is a U-shaped average cost curve.

2. The marginal cost curve lies below the average cost curve when average costs are decreasing, and above when they are increasing. Thus marginal costs must equal average costs at the point of minimum average costs.

3. The area under the marginal cost curve measures the variable costs.

4. The long-run average cost curve is the lower envelope of the short-run average cost curves.

Figure 22.10

Long-run marginal costs. When there are discrete levels of the fixed factor, the firm will choose the amount of the fixed factor to minimize average costs. Thus the long-run marginal cost curve will consist of the various segments of the short-run marginal cost curves associated with each different level of the fixed factor.

REVIEW QUESTIONS

1. Which of the following are true? (1) Average fixed costs never increase with output; (2) average total costs are always greater than or equal to average variable costs; (3) average cost can never rise while marginal costs are declining.

2. A firm produces identical outputs at two different plants. If the marginal cost at the first plant exceeds the marginal cost at the second plant, how can the firm reduce costs and maintain the same level of output?

3. True or false? In the long run a firm always operates at the minimum level of average costs for the optimally sized plant to produce a given amount of output.

Long-run marginal costs. The relationship between the long-run and the short-run marginal costs with continuous levels of the fixed factor.

Figure 22.11

APPENDIX

In the text we claimed that average variable cost equals marginal cost for the first unit of output. In calculus terms this becomes

$$\lim_{y \to 0} \frac{c_v(y)}{y} = \lim_{y \to 0} c'(y).$$

The left-hand side of this expression is not defined at $y = 0$. But its limit is defined, and we can compute it using l'Hôpital's rule, which states that the limit of a fraction whose numerator and denominator both approach zero is given by the limit of the derivatives of the numerator and the denominator. Applying this rule, we have

$$\lim_{y \to 0} \frac{c_v(y)}{y} = \frac{\lim_{y \to 0} dc_v(y)/dy}{\lim_{y \to 0} dy/dy} = \frac{c'(0)}{1},$$

which establishes the claim.

We also claimed that the area under the marginal cost curve gave us variable cost. This is easy to show using the fundamental theorem of calculus. Since

$$MC(y) = \frac{dc_v(y)}{dy},$$

we know that the area under the marginal cost curve is

$$c_v(y) = \int_0^y \frac{dc_v(x)}{dx}\, dx = c_v(y) - c_v(0) = c_v(y).$$

The discussion of long-run and short-run marginal cost curves is all pretty clear geometrically, but what does it mean economically? It turns out that the calculus argument gives the nicest intuition. The argument is simple. The marginal cost of production is just the change in cost that arises from changing output. In the short run we have to keep plant size (or whatever) fixed, while in the long run we are free to adjust it. So the long-run marginal cost will consist of two pieces: how costs change holding plant size fixed plus how costs change when plant size adjusts. But if the plant size is chosen optimally, this last term has to be zero! Thus the long-run and the short-run marginal costs have to be the same.

The mathematical proof involves the chain rule. Using the definition from the text:

$$c(y) \equiv c_s(y, k(y)).$$

Differentiating with respect to y gives

$$\frac{dc(y)}{dy} = \frac{\partial c_s(y,k)}{\partial y} + \frac{\partial c_s(y,k)}{\partial k}\frac{\partial k(y)}{\partial y}.$$

If we evaluate this at a specific level of output y^* and its associated optimal plant size $k^* = k(y^*)$, we know that

$$\frac{\partial c_s(y^*, k^*)}{\partial k} = 0$$

because that is the necessary first-order condition for k^* to be the cost-minimizing plant size at y^*. Thus the second term in the expression cancels out and all that we have left is the short-run marginal cost:

$$\frac{dc(y^*)}{dy} = \frac{\partial c_s(y^*, k^*)}{\partial y}.$$

CHAPTER 23

FIRM SUPPLY

In this chapter we will see how to derive the supply curve of a competitive firm from its cost function using the model of profit maximization. The first thing we have to do is to describe the market environment in which the firm operates.

23.1 Market Environments

Every firm faces two important decisions: choosing how much it should produce and choosing what price it should set. If there were no constraints on a profit-maximizing firm, it would set an arbitrarily high price and produce an arbitrarily large amount of output. But no firm exists in such an unconstrained environment. In general, the firm faces two sorts of constraints on its actions.

First, it faces the **technological constraints** summarized by the production function. There are only certain feasible combinations of inputs and outputs, and even the most profit-hungry firm has to respect the realities of the physical world. We have already discussed how we can summarize the technological constraints, and we've seen how the technological

constraints lead to the **economic constraints** summarized by the cost function.

But now we bring in a new constraint—or at least an old constraint from a different perspective. This is the **market constraint**. A firm can produce whatever is physically feasible, and it can set whatever price it wants . . . but it can only sell as much as people are willing to buy.

If it sets a certain price p it will sell a certain amount of output x. We call the relationship between the price a firm sets and the amount that it sells the **demand curve facing the firm**.

If there were only one firm in the market, the demand curve facing the firm would be very simple to describe: it is just the market demand curve described in earlier chapters on consumer behavior. For the market demand curve measures how much of the good people want to buy at each price. Thus the demand curve summarizes the market constraints facing a firm that has a market all to itself.

But if there are other firms in the market, the constraints facing an individual firm will be different. In this case, the firm has to guess how the *other* firms in the market will behave when it chooses its price and output.

This is not an easy problem to solve, either for firms or for economists. There are a lot of different possibilities, and we will try to examine them in a systematic way. We'll use the term **market environment** to describe the ways that firms respond to each other when they make their pricing and output decisions.

In this chapter we'll examine the simplest market environment, that of **pure competition**. This is a good comparison point for many other environments, and it is of considerable interest in its own right. First let's give the economist's definition of pure competition, and then we'll try to justify it.

23.2 Pure Competition

To a lay person, "competition" has the connotation of intense rivalry. That's why students are often surprised that the economist's definition of competition seems so passive: we say that a market is **purely competitive** if each firm assumes that the market price is independent of its own level of output. Thus, in a competitive market, each firm only has to worry about how much output it wants to produce. Whatever it produces can only be sold at one price: the going market price.

In what sort of environment might this be a reasonable assumption for a firm to make? Well, suppose that we have an industry composed of many firms that produce an identical product, and that each firm is a small part of the market. A good example would be the market for wheat. There are thousands of wheat farmers in the United States, and even the largest of them produces only an infinitesimal fraction of the total supply. It is

reasonable in this case for any one firm in the industry to take the market price as being predetermined. A wheat farmer doesn't have to worry about what price to set for his wheat—if he wants to sell any at all, he has to sell it at the market price. He is a **price taker**: the price is given as far as he is concerned; all he has to worry about is how much to produce.

This kind of situation—an identical product and many small firms—is a classic example of a situation where price-taking behavior is sensible. But it is not the only case where price-taking behavior is possible. Even if there are only a few firms in the market, they may still treat the market price as being outside their control.

Think of a case where there is a fixed supply of a perishable good: say fresh fish or cut flowers in a marketplace. Even if there are only 3 or 4 firms in the market, each firm may have to take the *other* firms' prices as given. If the customers in the market only buy at the lowest price, then the lowest price being offered is the market price. If one of the other firms wants to sell anything at all, it will have to sell at the market price. So in this sort of situation competitive behavior—taking the market price as outside of your control—seems plausible as well.

We can describe the relationship between price and quantity perceived by a competitive firm in terms of a diagram as in Figure 23.1. As you can see, this demand curve is very simple. A competitive firm believes that it will sell nothing if it charges a price higher than the market price. If it sells at the market price, it can sell whatever amount it wants, and if it sells below the market price, it will get the entire market demand at that price.

As usual we can think of this kind of demand curve in two ways. If we think of quantity as a function of price, this curve says that you can sell any amount you want at or below the market price. If we think of price as a function of quantity, it says that no matter how much you sell, the market price will be independent of your sales.

(Of course, this doesn't have to be true for literally *any* amount. Price has to be independent of your output for any amount you might consider selling. In the case of the cut-flower seller, the price has to be independent of how much she sells for any amount up to her stock on hand—the maximum that she could consider selling.)

It is important to understand the difference between the "demand curve facing a firm" and the "market demand curve." The market demand curve measures the relationship between the market price and the total amount of output sold. The demand curve facing a firm measures the relationship between the market price and the output of *that particular firm.*

The market demand curve depends on consumers' behavior. The demand curve facing a firm not only depends on consumers' behavior but it also depends on the behavior of the other firms. The usual justification for the competitive model is that when there are many small firms in the market, each one faces a demand curve that is essentially flat. But even if there are only two firms in the market, and one insists on charging a fixed price

no matter what, then the other firm in the market will face a competitive demand curve like the one depicted in Figure 23.1. Thus the competitive model may hold in a wider variety of circumstances than is apparent at first glance.

Figure 23.1 **The demand curve facing a competitive firm**. The firm's demand is horizontal at the market price. At higher prices, the firm sells nothing, and below the market price it faces the entire market demand curve.

23.3 The Supply Decision of a Competitive Firm

Let us use the facts we have discovered about cost curves to figure out the supply curve of a competitive firm. By definition a competitive firm ignores its influence on the market price. Thus the maximization problem facing a competitive firm is

$$\max_y py - c(y).$$

This just says that the competitive firm wants to maximize its profits: the difference between its revenue, py, and its costs, $c(y)$.

What level of output will a competitive firm choose to produce? Answer: it will operate where marginal revenue equals marginal cost—where the extra revenue gained by one more unit of output just equals the extra cost

of producing another unit. If this condition did not hold, the firm could always increase its profits by changing its level of output.

In the case of a competitive firm, marginal revenue is simply the price. To see this, ask how much extra revenue a competitive firm gets when it increases its output by Δy. We have

$$\Delta R = p\Delta y$$

since by hypothesis p doesn't change. Thus the extra revenue per unit of output is given by

$$\frac{\Delta R}{\Delta y} = p,$$

which is the expression for marginal revenue.

Thus a competitive firm will choose a level of output y where the marginal cost that it faces at y is just equal to the market price. In symbols:

$$p = MC(y).$$

For a given market price, p, we want to find the level of output where profits are maximal. If price is greater than marginal cost at some level of output y, then the firm can increase its profits by producing a little more output. For price greater than marginal costs means

$$p - \frac{\Delta c}{\Delta y} > 0.$$

So increasing output by Δy means that

$$p\Delta y - \frac{\Delta c}{\Delta y}\Delta y > 0.$$

Simplifying we find that

$$p\Delta y - \Delta c > 0,$$

which means that the increase in revenues from the extra output exceeds the increase in costs. Thus profits must increase.

A similar argument can be made when price is less than marginal cost. Then reducing output will increase profits, since the lost revenues are more than compensated for by the reduced costs.

So at the optimal level of output, a firm must be producing where price equals marginal costs. Whatever the level of the market price p, the firm will choose a level of output y where $p = MC(y)$. Thus the marginal cost curve of a competitive firm is precisely its supply curve. Or put another way, the market price is precisely marginal cost—as long as each firm is producing at its profit-maximizing level.

Figure 23.2 **Marginal cost and supply**. Although there are two levels of output where price equals marginal cost, the profit-maximizing quantity supplied can lie only on the upward-sloping part of the marginal cost curve.

23.4 An Exception

Well . . . maybe not *precisely*. There are two troublesome cases. The first case is when there are several levels of output where price equals marginal cost, such as the case depicted in Figure 23.2. Here there are two levels of output where price equals marginal cost. Which one will the firm choose?

It is not hard to see the answer. Consider the first intersection, where the marginal cost curve is sloping down. Now if we increase output a little bit here, the costs of each additional unit of output will decrease. That's what it means to say that the marginal cost curve is decreasing. But the market price will stay the same. Thus profits must definitely go up.

So we can rule out levels of output where the marginal cost curve slopes downward. At those points an increase in output must always increase profits. The supply curve of a competitive firm must lie along the upward-sloping part of the marginal cost curve. This means that the supply curve itself must always be upward sloping. The "Giffen good" phenomenon cannot arise for supply curves.

Price equals marginal cost is a *necessary* condition for profit maximization. It is not in general a *sufficient* condition. Just because we find a

point where price equals marginal cost doesn't mean that we've found the maximum profit point. But if we find the maximum profit point, we know that price must equal marginal cost.

23.5 Another Exception

This discussion is assuming that it is profitable to produce something. After all it could be that the best thing for a firm to do is to produce zero output. Since it is always possible to produce a zero level of output, we have to compare our candidate for profit maximization with the choice of doing nothing at all.

If a firm produces zero output it still has to pay its fixed costs, F. Thus the profits from producing zero units of output are just $-F$. The profits from producing a level of output y are $py - c_v(y) - F$. The firm is better off going out of business when

$$-F > py - c_v(y) - F,$$

that is, when the "profits" from producing nothing, and just paying the fixed costs, exceed the profits from producing where price equals marginal cost. Rearranging this equation gives us the **shutdown condition**:

$$AVC(y) = \frac{c_v(y)}{y} > p.$$

If average variable costs are greater than p, the firm would be better off producing zero units of output. This makes good sense, since it says that the revenues from selling the output y don't even cover the *variable* costs of production, $c_v(y)$. In this case the firm might as well go out of business. If it produces nothing it will lose its fixed costs, but it would lose even more if it continued to produce.

This discussion indicates that only the portions of the marginal cost curve that lie above the average variable cost curve are possible points on the supply curve. If a point where price equals marginal cost is beneath the average variable cost curve, the firm would optimally choose to produce zero units of output.

We now have a picture for the supply curve like that in Figure 23.3. The competitive firm produces along the part of the marginal cost curve that is upward sloping and lies above the average variable cost curve.

EXAMPLE: Pricing Operating Systems

A computer requires an operating system in order to run, and most hardware manufacturers sell their computers with the operating systems already

Figure 23.3

Average variable cost and supply. The supply curve is the upward-sloping part of the marginal cost curve that lies above the average variable cost curve. The firm will not operate on those points on the marginal cost curve below the average cost curve since it could have greater profits (less losses) by shutting down.

installed. In the early 1980s several operating system producers were fighting for supremacy in the IBM-PC-compatible microcomputer market. The common practice at that time was for the producer of the operating system to charge the computer manufacturer for each copy of the operating system that was *installed* on a microcomputer that it sold.

Microsoft Corporation offered an alternative plan in which the charge to the manufacturer was based on the number of microcomputers that were *built* by the manufacturer. Microsoft set their licensing fee low enough that this plan was attractive to the producers.

Note the clever nature of Microsoft's pricing strategy: once the contract with a manufacturer was signed, the marginal cost of installing MS-DOS on an already-built computer was zero. Installing a competing operating system, on the other hand, could cost $50 to $100. The hardware manufacturer (and ultimately the user) paid Microsoft for the operating system, but the structure of the pricing contract made MS-DOS very attractive relative to the competition. As a result, Microsoft ended up being the default operating system installed on microcomputers and achieved a market penetration of over 90 percent.

23.6 The Inverse Supply Function

We have seen that the supply curve of a competitive firm is determined by the condition that price equals marginal cost. As before we can express this relation between price and output in two ways: we can either think of output as a function of price, as we usually do, or we can think of the "inverse supply function" that gives price as a function of output. There is a certain insight to be gained by looking at it in the latter way. Since price equals marginal cost at each point on the supply curve, the market price must be a measure of marginal cost for every firm operating in the industry. A firm that produces a lot of output and a firm that produces only a little output must have the *same marginal cost,* if they are both maximizing profits. The total cost of production of each firm can be very different, but the marginal cost of production must be the same.

The equation $p = MC(y)$ gives us the inverse supply function: price as a function of output. This way of expressing the supply curve can be very useful.

23.7 Profits and Producer's Surplus

Given the market price we can now compute the optimal operating position for the firm from the condition that $p = MC(y)$. Given the optimal operating position we can compute the profits of the firm. In Figure 23.4 the area of the box is just p^*y^*, or total revenue. The area $y^*AC(y^*)$ is total costs since

$$yAC(y) = y\frac{c(y)}{y} = c(y).$$

Profits are simply the difference between these two areas.

Recall our discussion of **producer's surplus** in Chapter 14. We defined producer's surplus to be the area to the left of the supply curve, in analogy to consumer's surplus, which was the area to the left of the demand curve. It turns out that producer's surplus is closely related to the profits of a firm. More precisely, producer's surplus is equal to revenues minus variable costs, or equivalently, profits plus the fixed costs:

$$\text{profits} = py - c_v(y) - F$$

$$\text{producer's surplus} = py - c_v(y).$$

The most direct way to measure producer's surplus is to look at the difference between the revenue box and the box $y^*AVC(y^*)$, as in Figure 23.5A. But there are other ways to measure producer's surplus by using the marginal cost curve itself.

Figure 23.4 **Profits**. Profits are the difference between total revenue and total costs, as shown by the colored rectangle.

We know from Chapter 22 that the area under the marginal cost curve measures the total variable costs. This is true because the area under the marginal cost curve is the cost of producing the first unit plus the cost of producing the second unit, and so on. So to get producer's surplus, we can subtract the area under the marginal cost curve from the revenue box and get the area depicted in Figure 23.5B.

Finally, we can combine the two ways of measuring producer's surplus. Use the "box" definition up to the point where marginal cost equals average variable cost, and then use the area above the marginal cost curve, as shown in Figure 23.5C. This latter way is the most convenient for most applications since it is just the area to the left of the supply curve. Note that this is consistent with definition of producer's surplus given in Chapter 14.

We are seldom interested in the *total* amount of producer's surplus; more often it is the *change* in producer's surplus that is of interest. The change in producer's surplus when the firm moves from output y^* to output y' will generally be a trapezoidal shaped region like that depicted in Figure 23.6.

Note that the change in producer's surplus in moving from y^* to y' is just the change in profits in moving from y^* to y', since by definition the fixed costs don't change. Thus we can measure the impact on profits of a change in output from the information contained in the marginal cost curve, without having to refer to the average cost curve at all.

Producer's surplus. Three equivalent ways to measure producer's surplus. Panel A depicts a box measuring revenue minus variable cost. Panel B depicts the area above the marginal cost curve. Panel C uses the box up until output z (area R) and then uses the area above the marginal cost curve (area T).

Figure 23.5

EXAMPLE: The Supply Curve for a Specific Cost Function

What does the supply curve look like for the example given in the last chapter where $c(y) = y^2 + 1$? In that example the marginal cost curve was always above the average variable cost curve, and it always sloped upward. So "price equals marginal costs" gives us the supply curve directly. Substituting $2y$ for marginal cost we get the formula

$$p = 2y.$$

This gives us the inverse supply curve, or price as a function of output. Solving for output as a function of price we have

$$S(p) = y = \frac{p}{2}$$

as our formula for the supply curve. This is depicted in Figure 23.7.

Figure 23.6 **The change in producer's surplus**. Since the supply curve coincides with the upward-sloping part of the marginal cost curve, the change in producer's surplus will typically have a roughly trapezoidal shape.

If we substitute this supply function into the definition of profits, we can calculate the maximum profits for each price p. Performing the calculation we have:

$$\begin{aligned}\pi(p) &= py - c(y)\\ &= p\frac{p}{2} - \left(\frac{p}{2}\right)^2 - 1\\ &= \frac{p^2}{4} - 1.\end{aligned}$$

How do the maximum profits relate to producer's surplus? In Figure 23.7 we see that producer's surplus—the area to the left of the supply curve between a price of zero and a price of p—will be a triangle with a base of $y = p/2$ and a height of p. The area of this triangle is

$$A = \left(\frac{1}{2}\right)\left(\frac{p}{2}\right)p = \frac{p^2}{4}.$$

Comparing this with the profits expression, we see that producer's surplus equals profits plus fixed costs, as claimed.

A specific example of a supply curve. The supply curve and producer's surplus for the cost function $c(y) = y^2 + 1$.

Figure 23.7

23.8 The Long-Run Supply Curve of a Firm

The long-run supply function for the firm measures how much the firm would optimally produce when it is allowed to adjust plant size (or whatever factors are fixed in the short run). That is, the long-run supply curve will be given by

$$p = MC_l(y) = MC(y, k(y)).$$

The short-run supply curve is given by price equals marginal cost at some fixed level of k:

$$p = MC(y, k).$$

Note the difference between the two expressions. The short-run supply curve involves the marginal cost of output holding k fixed at a given level of output, while the long-run supply curve involves the marginal cost of output when you adjust k optimally.

Now, we know something about the relationship between short-run and long-run marginal costs: the short-run and the long-run marginal costs coincide at the level of output y^* where the fixed factor choice associated with the short-run marginal cost is the optimal choice, k^*. Thus the short-run and the long-run supply curves of the firm coincide at y^*, as in Figure 23.8.

In the short run the firm has some factors in fixed supply; in the long run these factors are variable. Thus, when the price of output changes, the

Figure 23.8 **The short-run and long-run supply curves**. Typically the long-run supply curve will be more elastic than the short-run supply curve.

firm has more choices to adjust in the long run than in the short run. This suggests that the long-run supply curve will be more responsive to price—more elastic—than the short-run supply curve, as illustrated in Figure 23.8.

What else can we say about the long-run supply curve? The long run is defined to be that time period in which the firm is free to adjust all of its inputs. One choice that the firm has is the choice of whether to remain in business. Since in the long run the firm can always get zero profits by going out of business, the profits that the firm makes in long-run equilibrium have to be at least zero:

$$py - c(y) \geq 0,$$

which means

$$p \geq \frac{c(y)}{y}.$$

This says that in the long run price has to be at least as large as average cost. Thus the relevant part of the long-run supply curve is the upward-sloping part of the marginal cost curve that lies above the long-run average cost curve, as depicted in Figure 23.9.

This is completely consistent with the short-run story. In the long run all costs are variable costs, so the short-run condition of having price above average variable cost is equivalent to the long-run condition of having price above average cost.

The long-run supply curve. The long-run supply curve will be the upward-sloping part of the long-run marginal cost curve that lies above the average cost curve.

Figure 23.9

23.9 Long-Run Constant Average Costs

One particular case of interest occurs when the long-run technology of the firm exhibits constant returns to scale. Here the long-run supply curve will be the long-run marginal cost curve, which, in the case of constant average cost, coincides with the long-run average cost curve. Thus we have the situation depicted in Figure 23.10, where the long-run supply curve is a horizontal line at $c_{\min}$, the level of constant average cost.

This supply curve means that the firm is willing to supply any amount of output at $p = c_{\min}$, an arbitrarily large amount of output at $p > c_{\min}$, and zero output at $p < c_{\min}$. When we think about the replication argument for constant returns to scale this makes perfect sense. Constant returns to scale implies that if you can produce 1 unit for $c_{\min}$ dollars, you can produce n units for $nc_{\min}$ dollars. Therefore you will be willing to supply any amount of output at a price equal to $c_{\min}$, and an arbitrarily large amount of output at any price greater than $c_{\min}$.

On the other hand, if $p < c_{\min}$, so that you cannot break even supplying even one unit of output, you will certainly not be able to break even supplying n units of output. Hence, for any price less than $c_{\min}$, you will want to supply zero units of output.

Figure 23.10 **Constant average costs**. In the case of constant average costs, the long-run supply curve will be a horizontal line.

Summary

1. The relationship between the price a firm charges and the output that it sells is known as the demand curve facing the firm. By definition, a competitive firm faces a horizontal demand curve whose height is determined by the market price—the price charged by the other firms in the market.

2. The (short-run) supply curve of a competitive firm is that portion of its (short-run) marginal cost curve that is upward sloping and lies above the average variable cost curve.

3. The change in producer's surplus when the market price changes from p_1 to p_2 is the area to the left of the marginal cost curve between p_1 and p_2. It also measures the firm's change in profits.

4. The long-run supply curve of a firm is that portion of its long-run marginal cost curve that is upward sloping and that lies above its long-run average cost curve.

REVIEW QUESTIONS

1. A firm has a cost function given by $c(y) = 10y^2 + 1000$. What is its supply curve?

2. A firm has a cost function given by $c(y) = 10y^2 + 1000$. At what output is average cost minimized?

3. If the supply curve is given by $S(p) = 100 + 20p$, what is the formula for the inverse supply curve?

4. A firm has a supply function given by $S(p) = 4p$. Its fixed costs are 100. If the price changes from 10 to 20, what is the change in its profits?

5. If the long-run cost function is $c(y) = y^2 + 1$, what is the long-run supply curve of the firm?

6. Classify each of the following as either technological or market constraints: the price of inputs, the number of other firms in the market, the quantity of output produced, and the ability to produce more given the current input levels.

7. What is the major assumption that characterizes a purely competitive market?

8. In a purely competitive market a firm's marginal revenue is always equal to what? A profit-maximizing firm in such a market will operate at what level of output?

9. If average variable costs exceed the market price, what level of output should the firm produce? What if there are no fixed costs?

10. Is it ever better for a perfectly competitive firm to produce output even though it is losing money? If so, when?

11. In a perfectly competitive market what is the relationship between the market price and the cost of production for all firms in the industry?

APPENDIX

The discussion in this chapter is very simple if you speak calculus. The profit-maximization problem is

$$\max_y \; py - c(y)$$

$$\text{such that } y \geq 0.$$

The necessary conditions for the optimal supply, y^*, are the first-order condition

$$p - c'(y^*) = 0$$

and the second-order condition

$$-c''(y^*) \leq 0.$$

The first-order condition says price equals marginal cost, and the second-order condition says that the marginal cost must be increasing. Of course this is presuming that $y^* > 0$. If price is less than average variable cost at y^*, it will pay the firm to produce a zero level of output. To determine the supply curve of a competitive firm, we must find all the points where the first- and second-order conditions are satisfied and compare them to each other—and to $y = 0$—and pick the one with the largest profits. That's the profit-maximizing supply.

CHAPTER 24

INDUSTRY SUPPLY

We have seen how to derive a firm's supply curve from its marginal cost curve. But in a competitive market there will typically be many firms, so the supply curve the industry presents to the market will be the sum of the supplies of all the individual firms. In this chapter we will investigate the **industry supply curve**.

24.1 Short-Run Industry Supply

We begin by studying an industry with a fixed number of firms, n. We let $S_i(p)$ be the supply curve of firm i, so that the **industry supply curve**, or the **market supply curve** is

$$S(p) = \sum_{i=1}^{n} S_i(p),$$

which is the sum of the individual supply curves. Geometrically we take the sum of the quantities supplied by each firm at each price, which gives us a *horizontal* sum of supply curves, as in Figure 24.1.

Figure 24.1 **The industry supply curve**. The industry supply curve ($S_1 + S_2$) is the sum of the individual supply curves (S_1 and S_2).

24.2 Industry Equilibrium in the Short Run

In order to find the industry equilibrium we take this market supply curve and find the intersection with the market demand curve. This gives us an equilibrium price, p^*.

Given this equilibrium price, we can go back to look at the individual firms and examine their output levels and profits. A typical configuration with three firms, A, B, and C, is illustrated in Figure 24.2. In this example, firm A is operating at a price and output combination that lies on its average cost curve. This means that

$$p = \frac{c(y)}{y}.$$

Cross multiplying and rearranging, we have

$$py - c(y) = 0.$$

Thus firm A is making zero profits.

Firm B is operating at a point where price is greater than average cost: $p > c(y)/y$, which means it is making a profit in this short-run equilibrium.

Figure 24.2

Short-run equilibrium. An example of a short-run equilibrium with three firms. Firm A is making zero profits, firm B is making positive profits, and firm C is making negative profits, that is, making a loss.

Firm C is operating where price is less than average cost, so it is making negative profits, that is, making a loss.

In general, combinations of price and output that lie above the average cost curve represent positive profits, and combinations that lie below represent negative profits. Even if a firm is making negative profits, it will still be better for it to stay in business in the short run if the price and output combination lie above the average *variable* cost curve. For in this case, it will make less of a loss by remaining in business than by producing a zero level of output.

24.3 Industry Equilibrium in the Long Run

In the long run, firms are able to adjust their fixed factors. They can choose the plant size, or the capital equipment, or whatever to maximize their long-run profits. This just means that they will move from their short-run to their long-run cost curves, and this adds no new analytical difficulties: we simply use the long-run supply curves as determined by the long-run marginal cost curve.

However, there is an additional long-run effect that may occur. If a firm is making losses in the long run, there is no reason to stay in the industry, so we would expect to see such a firm *exit* the industry, since by exiting from the industry, the firm could reduce its losses to zero. This is just another way of saying that the only relevant part of a firm's supply curve in the long run is that part that lies *on or above* the average cost curve—since these are locations that correspond to nonnegative profits.

Similarly, if a firm is making profits we would expect *entry* to occur. After all, the cost curve is supposed to include the cost of all factors necessary to produce output, measured at their market price (i.e., their opportunity cost). If a firm is making profits in the long run it means that *anybody* can go to market, acquire those factors, and produce the same amount of output at the same cost.

In most competitive industries there are no restrictions against new firms entering the industry; in this case we say the industry exhibits **free entry**. However, in some industries there are **barriers to entry**, such as licenses or legal restrictions on how many firms can be in the industry. For example, regulations on the sales of alcohol in many states prevent free entry to the retail liquor industry.

The two long-run effects—acquiring different fixed factors and the entry and exit phenomena—are closely related. An existing firm in an industry can decide to acquire a new plant or store and produce more output. Or a new firm may enter the industry by acquiring a new plant and producing output. The only difference is in who owns the new production facilities.

Of course as more firms enter the industry—and firms that are losing money exit the industry—the total amount produced will change and lead to a change in the market price. This in turn will affect profits and the incentives to exit and enter. What will the final equilibrium look like in an industry with free entry?

Let's examine a case where all firms have identical long-run cost functions, say, $c(y)$. Given the cost function we can compute the level of output where average costs are minimized, which we denote by y^*. We let $p^* = c(y^*)/y^*$ be the minimum value of average cost. This cost is significant because it is the lowest price that could be charged in the market and still allow firms to break even.

We can now graph the industry supply curves for each different number of firms that can be in the market. Figure 24.3 illustrates the industry supply curves if there are $1, \ldots, 4$ firms in the market. (We are using 4 firms only for purposes of an example; in reality, one would expect there to be many more firms in a competitive industry.) Note that since all firms have the same supply curve, the total amount supplied if 2 firms are in the market is just twice as much as when 1 firm is the market, the supply when 3 firms are in the market is just three times as much, and so on.

Now add two more lines to the diagram: a horizontal line at p^*, the minimum price consistent with nonnegative profits, and the market demand curve. Consider the intersections of the demand curve and the supply curves for $n = 1, 2, \ldots$ firms. If firms enter the industry when positive profits are being made, then the relevant intersection is the *lowest price consistent with nonnegative profits.* This is denoted by p' in Figure 24.3, and it happens to occur when there are three firms in the market. If one

Industry supply curves with free entry. Supply curves for $1, \ldots, 4$ firms. The equilibrium price, p', occurs at the lowest possible intersection of demand and supply such that $p' \geq p^*$.

Figure 24.3

more firm enters the market, profits are pushed to be negative. In this case, the maximum number of competitive firms this industry can support is three.

24.4 The Long-Run Supply Curve

The construction given in the last section—draw the industry supply curves for each possible number of firms that could be in the market and then look for the largest number of firms consistent with nonnegative profits—is perfectly rigorous and easy to apply. However, there is a useful approximation that usually gives something very close to the right answer.

Let's see if there is some way to construct *one* industry supply curve out of the n curves we have above. The first thing to note is that we can rule out all of the points on the supply curve that are below p^*, since those can never be long-run operating positions. But we can also rule out some of the points on the supply curves *above* p^*.

We typically assume that the market demand curve is downward sloping. The steepest possible demand curve is therefore a vertical line. This implies that points like A in Figure 24.3 would never be observed—for any downward-sloping demand curve that passed through A would also have to intersect a supply curve associated with a larger number of firms, as

shown by the hypothetical demand curve D'' passing through the point A in Figure 24.3.

Thus we can eliminate a portion of each supply curve from being a possible long-run equilibrium position. Every point on the one-firm supply curve that lies to the right of the intersection of the two-firm supply curve and the line determined by p^* cannot be consistent with long-run equilibrium. Similarly, every point on the two-firm supply curve that lies to the right of the intersection of the three-firm supply curve with the p^* line cannot be consistent with long-run equilibrium . . . and every point on the n-firm supply curve that lies to the right of the intersection of the $n + 1$-firm supply curve with the p^* line cannot be consistent with equilibrium.

The parts of the supply curves on which the long-run equilibrium can actually occur are indicated by the black line segments in Figure 24.4. The nth black line segment shows all the combinations of prices and industry output that are consistent with having n firms in long-run equilibrium. Note that these line segments get flatter and flatter as we consider larger and larger levels of industry output, involving more and more firms in the industry.

Figure 24.4

The long-run supply curve. We can eliminate portions of the supply curves that can never be intersections with a downward-sloping market demand curve in the long run, such as the points on each supply curve to the right of the dotted lines.

Why do these curves get flatter? Think about it. If there is one firm in the market and the price goes up by Δp, it will produce, say, Δy more output. If there are n firms in the market and the price goes up by Δp, *each* firm will produce Δy more output, so we will get $n\Delta y$ more output in total. This means that the supply curve will be getting flatter and flatter as there are more and more firms in the market, since the supply of output will be more and more sensitive to price.

By the time we get a reasonable number of firms in the market, the slope of the supply curve will be very flat indeed. Flat enough so that it is reasonable to take it as having a slope of zero—that is, as taking the long-run industry supply curve to be a flat line at price equals minimum average cost. This will be a poor approximation if there are only a few firms in the industry in the long run. But the assumption that a small number of firms behave competitively will also probably be a poor approximation! If there are a reasonable number of firms in the long run, the equilibrium price cannot get far from minimum average cost. This is depicted in Figure 24.5.

Approximate long-run supply curve. The long-run supply curve will be approximately flat at price equals minimum average cost.

Figure 24.5

This result has the important implication that in a competitive industry with free entry, profits cannot get very far from zero. If there are significant levels of profits in an industry with free entry, it will induce other firms to

enter that industry and thereby push profits toward zero.

Remember, the correct calculation of economic costs involves measuring all factors of production at their market prices. As long as *all* factors are being measured and properly priced, a firm earning positive profits can be exactly duplicated by anyone. Anyone can go to the open market and purchase the factors of production necessary to produce the same output in the same way as the firm in question.

In an industry with free entry and exit, the long-run average cost curve should be essentially flat at a price equal to the minimum average cost. This is just the kind of long-run supply curve that a single firm with constant returns to scale would have. This is no accident. We argued that constant returns to scale was a reasonable assumption since a firm could always replicate what it was doing before. But another firm could replicate it as well! Expanding output by building a duplicate plant is just like a new firm entering the market with duplicate production facilities. Thus the long-run supply curve of a competitive industry with free entry will look like the long-run supply curve of a firm with constant returns to scale: a flat line at price equals minimum average cost.

EXAMPLE: Taxation in the Long Run and in the Short Run

Consider an industry that has free entry and exit. Suppose that initially it is in a long-run equilibrium with a fixed number of firms, and zero profits, as depicted in Figure 24.6. In the short run, with a fixed number of firms, the supply curve of the industry is upward sloping, while in the long run, with a variable number of firms, the supply curve is flat at price equals minimum average cost.

What happens when we put a tax on this industry? We use the geometric analysis discussed in Chapter 16: in order to find the new price paid by the demanders, we shift the supply curve up by the amount of the tax.

In general, the consumers will face a higher price and the producers will receive a lower price after the tax is imposed. But the producers were just breaking even before the tax was imposed; thus they must be losing money at any lower price. These economic losses will encourage some firms to leave the industry. Thus the supply of output will be reduced, and the price to the consumers will rise even further.

In the long run, the industry will supply along the horizontal long-run supply curve. In order to supply along this curve, the firms will have to receive a price equal to the minimum average cost—just what they were receiving before the tax was imposed. Thus the price to the consumers will have to rise by the entire amount of the tax.

In Figure 24.6, the equilibrium is initially at $P_D = P_S$. Then the tax is imposed, shifting the short-run supply curve up by the amount of the tax, and the equilibrium price paid by the demanders increases to P'_D. The

Figure 24.6

Taxation in the short run and long run. In the short run, with a fixed number of firms, the industry supply curve will have an upward slope, so that part of the tax falls on the consumers and part on the firms. In the long run, the industry supply curve will be horizontal so all of the tax falls on the consumers.

equilibrium price received by the suppliers falls to $P'_S = P'_D - t$. But this is only in the short run—when there are a fixed number of firms in the industry. Because of free entry and exit, the *long-run* supply curve in the industry is horizontal at $P_D = P_S =$ minimum average cost. Hence, in the long run, shifting up the supply curve implies that the entire amount of the tax gets passed along to the consumers.

To sum up: in an industry with free entry, a tax will initially raise the price to the consumers by less than the amount of the tax, since some of the incidence of the tax will fall on the producers. But in the long run the tax will induce firms to exit from the industry, thereby reducing supply, so that consumers will eventually end up paying the entire burden of the tax.

24.5 The Meaning of Zero Profits

In an industry with free entry, profits will be driven to zero by new entrants: whenever profits are positive, there will be an incentive for a new firm to come in to acquire some of those profits. When profits are zero it doesn't

mean that the industry disappears; it just means that it stops growing, since there is no longer an inducement to enter.

In a long-run equilibrium with zero profits, all of the factors of production are being paid their market price—the same market price that these factors could earn elsewhere. The owner of the firm, for example, is still collecting a payment for her labor time, or for the amount of money she invested in the firm, or for whatever she contributes to the operation of the firm. The same goes for all other factors of production. The firm is still making money—it is just that all the money that it makes is being paid out to purchase the inputs that it uses. Each factor of production is earning the same amount in this industry that it could earn elsewhere, so there are no extra rewards—no pure profits—to attract new factors of production to this industry. But there is nothing to cause them to leave either. Industries in long-run equilibrium with zero profits are mature industries; they're not likely to appear as the cover story in *Business Week*, but they form the backbone of the economy.

Remember, economic profits are defined using the market prices of all factors of production. The market prices measure the opportunity cost of those factors—what they could earn elsewhere. Any amount of money earned in excess of the payments to the factors of production is a pure economic profit. But whenever someone finds a pure economic profit, other people will try to enter the industry and acquire some of that profit for themselves. It is this attempt to capture economic profits that eventually drives them to zero in a competitive industry with free entry.

In some quarters, the profit motive is regarded with some disdain. But when you think about it purely on economic grounds, profits are providing exactly the right signals as far as resource allocation is concerned. If a firm is making positive profits, it means that people value the output of the firm more highly than they value the inputs. Doesn't it make sense to have more firms producing that kind of output?

24.6 Fixed Factors and Economic Rent

If there is free entry, profits are driven to zero in the long run. But not every industry has free entry. In some industries the number of firms in the industry is fixed.

A common reason for this is that there are some factors of production that are available in fixed supply. We said that in the long run the fixed factors could be bought or sold by an individual firm. But there are some factors that are fixed for the *economy as a whole* even in the long run.

The most obvious example of this is in resource-extraction industries: oil in the ground is a necessary input to the oil-extraction industry, and there is only so much oil around to be extracted. A similar statement could be made for coal, gas, precious metals, or any other such resource.

Agriculture gives another example. There is only a certain amount of land that is suitable for agriculture.

A more exotic example of such a fixed factor is talent. There are only a certain number of people who possess the necessary level of talent to be professional athletes or entertainers. There may be "free entry" into such fields—but only for those who are good enough to get in!

There are other cases where the fixed factor is fixed not by nature, but by law. In many industries it is necessary to have a license or permit, and the number of these permits may be fixed by law. The taxicab industry in many cities is regulated in this way. Liquor licenses are another example.

If there are restrictions such as the above on the number of firms in the industry, so that firms cannot enter the industry freely, it may appear that it is possible to have an industry with positive profits in the long run, with no economic forces to drive those profits to zero.

This appearance is wrong. There is an economic force that pushes profits to zero. If a firm is operating at a point where its profits appear to be positive in the long run, it is probably because we are not appropriately measuring the market value of whatever it is that is preventing entry.

Here it is important to remember the economic definition of costs: we should value each factor of production at its *market price*—its opportunity cost. If it appears that a farmer is making positive profits after we have subtracted his costs of production, it is probably because we have forgotten to subtract the cost of his land.

Suppose that we manage to value all of the inputs to farming except for the land cost, and we end up with π dollars per year for profits. How much would the land be worth on a free market? How much would someone pay to rent that land for a year?

The answer is: they would be willing to rent it for π dollars per year, the "profits" that it brings in. You wouldn't even have to know anything about farming to rent this land and earn π dollars—after all, we valued the farmer's labor at its market price as well, and that means that you can hire a farmer and still make π dollars of profit. So the market value of that land—its competitive rent—is just π. The economic profits to farming are zero.

Note that the rental rate determined by this procedure may have nothing whatsoever to do with the historical cost of the farm. What matters is not what you bought it for, but what you can sell it for—that's what determines opportunity cost.

Whenever there is some fixed factor that is preventing entry into an industry, there will be an equilibrium rental rate for that factor. Even with fixed factors, you can always enter an industry by buying out the position of a firm that is currently in the industry. Every firm in the industry has the option of selling out—and the opportunity cost of not doing so is a cost of production that it has to consider.

Thus in one sense it is always the *possibility* of entry that drives profits to

zero. After all, there are two ways to enter an industry: you can form a new firm, or you can buy out an existing firm that is currently in the industry. If a new firm can buy everything necessary to produce in an industry and still make a profit, it will do so. But if there are some factors that are in fixed supply, then competition for those factors among potential entrants will bid the prices of these factors up to a point where the profit disappears.

EXAMPLE: Taxi Licenses in New York City

Earlier we said that licenses to operate New York City taxicabs sell for about $100,000. Yet in 1986 taxicab drivers made only about $400 for a 50-hour week; this translated into less than an $8 hourly wage. The New York Taxi and Limosine Commission argued that this wage was too low to attract skilled drivers and that taxi fares should be raised in order to attract better drivers.

An economist would argue that allowing the fares to increase would have virtually no effect on the take-home pay of the drivers; all that would happen is that the value of the taxicab license would increase. We can see why by examining the commission's figures for the costs of operating a taxi. In 1986, the lease rate was $55 for a day shift and $65 for a night shift. The driver who leased the taxi paid for the gasoline and netted about $80 a day in income.

But note how much the owner of the taxicab license made. Assuming that the cab could be rented for two shifts for 320 days a year, the lease income comes to $38,400. Insurance, depreciation, maintenance, and so on amounted to about $21,100 a year; this leaves a net profit of $17,300 per year. Since the license cost about $100,000, this indicates a total return of about 17 percent.

An increase in the rate that taxis were allowed to charge would be reflected directly in the value of the license. A fare increase that brought in an extra $10,000 a year would result in a license's value increasing by about $60,000. The wage rate for the cab drivers—which is set in the labor market—would not be affected by such a change.[1]

24.7 Economic Rent

The examples in the last section are instances of **economic rent**. Economic rent is defined as those payments to a factor of production that are in excess of the minimum payment necessary to have that factor supplied.

[1] Figures are taken from an unsigned editorial in the *New York Times,* August 17, 1986.

Consider, for example, the case of oil discussed earlier. In order to produce oil you need some labor, some machinery, and, most importantly, some oil in the ground! Suppose that it costs \$1 a barrel to pump oil out of the ground from an existing well. Then any price in excess of \$1 a barrel will induce firms to supply oil from existing wells. But the actual price of oil is much higher than \$1 a barrel. People want oil for various reasons, and they are willing to pay more than its cost of production to get it. The excess of the price of oil over its cost of production is economic rent.

Why don't firms enter this industry? Well, they try. But there is only a certain amount of oil available. Oil will sell for more than its cost of production because of the limited supply.

Now consider taxicab licenses. Viewed as pieces of paper, these cost almost nothing to produce. But in New York City a taxicab license can sell for \$100,000! Why don't people enter this industry and produce more taxicab licenses? The reason is that entry is illegal—the supply of taxicab licenses is controlled by the city.

Farmland is yet another example of economic rent. In the aggregate, the total amount of land is fixed. There would be just as much land supplied at zero dollars an acre as at \$1000 an acre. Thus in the aggregate, the payments to land constitute economic rent.

From the viewpoint of the economy as a whole, it is the price of agricultural products that determines the value of agricultural land. But from the viewpoint of the individual farmer, the value of his land is a cost of production that enters into the pricing of his product.

This is depicted in Figure 24.7. Here AVC represents the average cost curve for all factors of production *excluding* land costs. (We are assuming that land is the only fixed factor.) If the price of the crop grown on this land is p^*, then the "profits" attributable to the land are measured by the area of the box: these are the economic rents. This is how much the land would rent for in a competitive market—whatever it took to drive the profits to zero.

The average cost curve *including* the value of the land is labeled AC. If we measure the value of the land correctly, the economic profits to operating the farm will be exactly zero. Since the equilibrium rent for the land will be whatever it takes to drive profits to zero, we have

$$p^* y^* - c_v(y^*) - \text{rent} = 0$$

or

$$\text{rent} = p^* y^* - c_v(y^*). \tag{24.1}$$

This is precisely what we referred to as producer's surplus earlier. Indeed, it is the same concept, simply viewed in a different light. Thus we can also measure rent by taking the area to the left of the marginal cost curve, as we saw earlier.

Figure 24.7 **Economic rent for land**. The area of the box represents the economic rent on the land.

Given the definition of rent in equation (24.1), it is now easy to see the truth of what we said earlier: it is the equilibrium price that determines rent, not the reverse. The firm supplies along its marginal cost curve—which is independent of the expenditures on the fixed factors. The rent will adjust to drive profits to zero.

24.8 Rental Rates and Prices

Since we are measuring output in flow units—so much output per unit of time, we should be careful to measure profits and rents in dollars per unit of time. Thus in the above discussion we talked about the rent per year for land or for a taxicab license.

If the land or the license is to be sold outright rather than rented, the equilibrium price would be the present value of the stream of rental payments. This is a simple consequence of the usual argument that assets generating a stream of payments should sell for their present values in a competitive market.

EXAMPLE: Liquor Licenses

In the United States, each state sets its own policy with respect to sales of alcohol. Some states have a liquor monopoly; other states issue licenses to

those who wish to sell alcohol. In some cases, licenses are issued on payment of a fee; in other cases, the number of licenses is fixed. In Michigan, for example, the number of licenses for sales of beer and wine for consumption on premises is limited to one for every 1,500 residents.

After each Federal census, a state liquor control board allocates licenses to communities whose populations have grown. (Licenses are not taken away from communities whose populations have fallen, however.) This artificial scarcity of licenses has created a vibrant market for licenses to serve liquor in many fast-growing communities. For example, in 1983 Ann Arbor, Michigan, had sixty-six existing liquor licenses. Six new licenses were allowed to be issued as a result of the 1980 census, and 33 applicants lined up to lobby for these licenses. At the time, the market value of a liquor license was about $80,000. The local newspaper ran a story asserting that "demand exceeds supply for liquor licenses." It was hardly surprising to the local economists that giving away an $80,000 asset for a zero price resulted in excess demand!

There have been many proposals to relax the liquor control laws in Michigan by allowing the state to issue new licenses. However, these proposals have never been enacted into law due to the opposition of various political groups. Some of these groups are opposed to the consumption of alcohol on grounds of public health or religion. Others have somewhat different motives. For example, one of the most vociferous opponents of relaxed liquor laws is the Michigan Licensed Beverage Association, a group that represents the sellers of alcoholic beverages in Michigan. Though at first glance it appears paradoxical that this group would oppose liberalization of the liquor laws, a little reflection indicates a possible reason: issuing more liquor licenses would undoubtably lower the resale value of *existing* licenses—imposing significant capital losses on current holders of such licenses.

24.9 The Politics of Rent

Often economic rent exists because of legal restrictions on entry into the industry. We mentioned two examples above: taxicab licenses and liquor licenses. In each of these cases the number of licenses is fixed by law, thus restricting entry to the industry and creating economic rents.

Suppose that the New York City government wants to increase the number of operating taxicabs. What will happen to the market value of the existing taxicab licenses? Obviously they will fall in value. This reduction in value hits the industry right in the pocketbook, and it is sure to create a lobbying force to oppose any such move.

The federal government also artificially restricts output of some products in such a way as to create a rent. For example, the federal government has declared that tobacco can only be grown on certain lands. The value of

this land is then determined by the demand for tobacco products. Any attempt to eliminate this licensing system has to contend with a serious lobby. Once the government creates artificial scarcity, it is very hard to eliminate it. The beneficiaries of the artificial scarcity—the people who have acquired the right to operate in the industry—will vigorously oppose any attempts to enlarge the industry.

The incumbents in an industry in which entry is legally restricted may well devote considerable resources to maintaining their favored position. Lobbying expenses, lawyers' fees, public relations costs, and so on can be substantial. From the viewpoint of society these kinds of expenses represent pure social waste. They aren't true costs of production; they don't lead to any *more* output being produced. Lobbying and public relations efforts just determine who gets the money associated with existing output.

Efforts directed at keeping or acquiring claims to factors in fixed supplies are sometimes referred to as **rent seeking**. From the viewpoint of society they represent a pure deadweight loss since they don't create any more output, they just change the market value of existing factors of production.

EXAMPLE: Farming the Government

There is only one good thing to say about the U.S. program of farm subsidies: it produces a never-ending source of examples for economics textbooks. Every new reform of the farm program brings new problems. "If you want to find the holes in a program, just toss them out to farmers. No one is more innovative in finding ways to use them," says Terry Bar, the vice president of the National Council of Farm Cooperatives.[2]

Up until 1996 the basic structure of farm subsidies in the U.S. involved price supports: the Federal government guaranteed a support price for a crop and would make up the difference if the price fell below the support price. In order to qualify for this program, a farmer had to agree not to farm a certain fraction of his land.

By the very nature of this plan, most of the benefits accrued to the large farmers. According to one calculation, 13 percent of the direct Federal subsidies were going to the 1 percent of the farmers who had sales over $500,000 a year. The Food Security Act of 1985 significantly restricted the payments to large farmers. As a result, the farmers broke up their holdings by leasing the land to local investors. The investors would acquire parcels large enough to take advantage of the subsidies, but too small to run into the restrictions aimed at large farmers. Once the land was acquired the investor would register it with a government program that would pay the

[2] Quoted in William Robbins, "Limits on Subsidies to Big Farms Go Awry, Sending Costs Climbing," *New York Times,* June 15, 1987, A1.

investor *not* to plant the land. This practice became known as "farming the government."

According to one study, the restriction on payments to the large farmers in the 1985 farm act resulted in the creation of 31,000 new applicants for farm subsidies. The cost of these subsidies was in the neighborhood of $2.3 billion.

Note that the ostensible goal of the program—restricting the amount of government subsidies paid to large farmers—has not been achieved. When the large farmers rent their land to small farmers, the market price of the rents depends on the generosity of the Federal subsidies. The higher the subsidies, the higher the equilibrium rent the large farmers receive. The benefits from the subsidy program still falls on those who initially own the land, since it is ultimately the value of what the land can earn—either from growing crops or farming the government—that determines its market value.

The Farm Act of 1996 promised a phaseout of most agricultural subsidies by 2002. However, the 1998 federal budget restored over 6 billion dollars of federal farm subsidies, illustrating once again how hard it is to reconcile politics and economics.

24.10 Energy Policy

We end this chapter with an extended example that uses some of the concepts we have developed.

In 1974 the Organization of Petroleum Exporting Countries (OPEC) levied a significant increase in the price of oil. Countries that had no domestically produced petroleum had little choice about energy policy—the price of oil and goods produced using oil had to rise.

At that time the United States produced about half of its domestic oil consumption, and Congress felt that it was unfair that the domestic producers should receive "windfall profits" from an uncontrolled increase in price. (The term windfall profits refers to an increase in profits due to some outside event, as opposed to an increase in profits due to production decisions.) Consequently, Congress devised a bizarre plan to attempt to hold down the price of products that used oil. The most prominent of these products is gasoline, so we will analyze the effect of the program for that market.

Two-Tiered Oil Pricing

The policy adopted by Congress was known as "two-tiered" oil pricing, and it went something like this. Imported oil would sell for whatever its market price was, but domestic oil—oil produced from wells that were in

place before 1974—would sell for its old price: the price that it sold for before OPEC. Roughly speaking, we'll say that imported oil sold for about $15 a barrel, while domestic oil sold for around $5. The idea was that the average price of oil would then be about $10 a barrel and this would help hold down the price of gasoline.

Could such a scheme work? Let's think about it from the viewpoint of the gasoline producers. What would the supply curve of gasoline look like? In order to answer this question we have to ask what the marginal cost curve for gasoline looked like.

What would you do if you were a gasoline refiner? Obviously you would try to use the cheap domestic oil first. Only after you had exhausted your supplies of domestic oil would you turn to the more expensive imported oil. Thus the aggregate marginal cost curve—the industry supply curve—for gasoline would have to look something like that depicted in Figure 24.8. The curve takes a jump at the point where the U.S. production of domestic oil is exhausted and the imported oil begins to be used. Before that point, the domestic price of oil measures the relevant factor price for producing gasoline. After that point, it is the price of foreign oil that is the relevant factor price.

Figure 24.8 depicts the supply curve for gasoline if all oil were to sell for the world price of $15 a barrel, and if all oil were to sell for the domestic price of $5 a barrel. If domestic oil actually sells for $5 a barrel and foreign oil sells for $15 a barrel, then the supply curve for gasoline will coincide with the $5-a-barrel supply curve until the cheaper domestic oil is used up, and then coincide with the $15-a-barrel supply curve.

Now let's find the intersection of this supply curve with the market demand curve to find the equilibrium price in Figure 24.8. The diagram reveals an interesting fact: the price of gasoline is exactly the same in the two-tiered system as it would be if all oil sold at the price of foreign oil! The price of gasoline is determined by the *marginal* cost of production, and the *marginal* cost is determined by the cost of the imported oil.

If you think about it a minute, this makes perfectly good sense. The gasoline companies will sell their product at the price the market will bear. Just because you were lucky enough to get some cheap oil doesn't mean you won't sell your gasoline for the same price that other firms are selling theirs for.

Suppose for the moment that all oil did sell for one price, and that equilibrium was reached at the price p^*. Then the government comes along and lowers the price of the first 100 barrels of oil that each refiner used. Will this affect their supply decision? No way—in order to affect supply you have to change the incentives at the margin. The only way to get a lower price of gasoline is to increase the supply, which means that you have to make the marginal cost of oil cheaper.

The two-tiered oil pricing policy was simply a transfer from the domestic oil producers to the domestic oil refiners. The domestic producers got $10

The supply curve for gasoline. Under the two-tiered oil pricing policy, the supply curve of gasoline would be discontinuous, jumping from the lower supply curve to the upper supply curve when the cheaper oil was exhausted.

Figure 24.8

less for their oil than they would have otherwise, and the profits they would have gotten went to the gasoline refiners. It had no effect on the supply of gasoline, and thus it could have no effect on the price of gasoline.

Price Controls

The economic forces inherent in this argument didn't take long to make themselves felt. The Department of Energy soon realized that it couldn't allow market forces to determine the price of gasoline under the two-tiered system—since market forces alone would imply one price of gasoline, which would be the same price that would prevail in the absence of the two-tiered system.

So they instituted price controls on gasoline. Each refiner was required to charge a price for gasoline that was based on the costs of producing the gasoline—which in turn was primarily determined by the cost of the oil that the refiner was able to purchase.

The availability of cheap domestic oil varied with location. In Texas the refiners were close to the major source of production and thus were able to

purchase large supplies of cheap oil. Due to the price controls, the price of Texas gasoline was relatively cheap. In New England, virtually all oil had to be imported, and thus the price of gasoline in New England was quite high.

When you have different prices for the same product, it is natural for firms to try to sell at the higher price. Again, the Department of Energy had to intervene to prevent the uncontrolled shipping of gasoline from low-price regions to high-price regions. The result of this intervention was the famous gasoline shortages of the mid-seventies. Periodically, the supply of gasoline in a region of the country would dry up, and there would be little available at any price. The free market system of supplying petroleum products had never exhibited such behavior; the shortages were entirely due to the two-tiered oil pricing system coupled with price controls.

Economists pointed this out at the time, but it didn't have much effect on policy. What did have an effect was lobbying by the gasoline refiners. Much of the domestic oil was sold on long-term contracts, and some refiners were able to buy a lot of it, while others could only buy the expensive foreign oil. Naturally they objected that this was unfair, so Congress figured out another scheme to allocate the cheap domestic oil more equitably.

The Entitlement Program

This program was known as the "entitlement program," and it went something like this. Each time a refiner bought a barrel of expensive foreign oil he got a coupon that allowed him to buy a certain amount of cheap domestic oil. The amount that the refiner was allowed to buy depended on supply conditions, but let's say that it was one for one: each barrel of foreign oil that he bought for $15 allowed him to buy one barrel of domestic oil for $5.

What did this do to the marginal price of oil? Now the marginal price of oil was just a weighted average of the domestic price and the foreign price of oil; in the one-for-one case described above, the price would be $10. The effect on the supply curve of gasoline is depicted in Figure 24.9.

The marginal cost of oil was reduced all right, and that meant that the price of gasoline was reduced as well. But look who is paying for it: the domestic oil producers! The United States was buying foreign oil that cost $15 a barrel in real dollars and pretending that it only cost $10. The domestic oil producers were required to sell their oil for less than the market price on the world oil market. We were subsidizing the importation of foreign oil and forcing the domestic oil producers to pay the subsidy!

Eventually this program was abandoned as well, and the U.S. imposed a tax on the domestic production of oil so that the U.S. oil producers wouldn't reap windfall profits due to OPEC's action. Of course, such a tax

Figure 24.9

The entitlement program. Under the entitlement program the supply curve of gasoline would lie between the supply curve if all oil were provided at the imported price and the supply curve if all oil were provided at the domestic price.

discouraged production of domestic oil, and thereby increases the price of gasoline, but this was apparently acceptable to Congress at the time.

24.11 Carbon Tax Versus Cap and Trade

Motivated by concerns about global warming, several climatologists have urged governments to institute policies to reduce carbon emissions. Two of these reduction policies are particularly interesting from an economic point of view: **carbon taxes** and **cap and trade**.

A carbon tax imposes a tax on carbon emissions, while a cap and trade system grants licenses to emit carbon that can be traded on an organized market. To see how these systems compare, let us examine a simple model.

Optimal Production of Emissions

We begin by examining the problem of producing a target amount of emissions in the least costly way. Suppose that there are two firms that have

Figure 24.10 **Cost function for emissions**. The curve shows the cost associated with emission reductions.

current levels of carbon emissions denoted by $(\overline{x}_1, \overline{x}_2)$. Firm i can reduce its level of emissions by x_i at a cost of $c_i(x_i)$. Figure 24.10 shows a possible shape for this cost function.

The goal is to reduce emissions by some target amount, T, in the least costly way. This minimization problem can be written as

$$\min_{x_1,x_2} c_1(x_1) + c_2(x_2)$$
$$\text{such that } x_1 + x_2 = T.$$

If it knew the cost functions, the government could, in principle, solve this optimization problem and assign a specific amount of emission reductions to each firm. However, this is impractical if there are thousands of carbon emitters. The challenge is to find a decentralized, market-based way of achieving the optimal solution.

Let us examine the structure of the optimization problem. It is clear that at the optimal solution the marginal cost of reducing emissions must be the same for each firm. Otherwise it would pay to increase emissions in the firm with the lower marginal cost and decrease emissions in the firm with the higher marginal cost. This would keep the total output at the target level while reducing costs.

Hence we have a simple principle: at the optimal solution, the marginal cost of emissions reduction should be the same for every firm. In the two-firm case we are examining, we can find this optimal point using a simple diagram. Let $MC_1(x_1)$ be the marginal cost of reducing emissions by x_1

for firm 1 and write the marginal cost of emission-reduction for firm 2 as a function of firm 1's output: $MC_2(T - x_1)$, assuming the target is met. We plot these two curves in Figure 24.11. The point where they intersect determines the optimal division of emission reductions between the two firms given that T emission reductions are to be produced in total.

Equilibrium in the cap and trade market. The point t^* gives the optimal carbon tax and emissions license price.

Figure 24.11

A Carbon Tax

Instead of solving for the cost-minimizing solution directly, let us instead consider a decentralized solution using a carbon tax. In this framework, the government sets a tax rate t that it charges for carbon emissions.

If firm 1 starts with $\overline{x}_1$ and reduces its emissions by x_1, then it ends up with $\overline{x}_1 - x_1$ emissions. If it pays t per unit emitted, its carbon tax bill would be $t(\overline{x}_1 - x_1)$.

Faced with this tax, firm 1 would want to choose that level of emission reductions that minimized its total cost of operation: the cost of reducing emissions plus the cost of paying the carbon tax on the emissions that remain. This leads to the cost minimization problem

$$\min_{x_1} c_1(x_1) + t(\overline{x}_1 - x_1).$$

Clearly the firm will want to reduce emissions up to the point where the marginal cost of further reductions just equals the carbon tax, i.e., where $t = MC_1(x_1)$.

If the carbon tax is set to be the rate t^*, as determined in Figure 24.11, then the total amount of carbon emissions will be the targeted amount, T. Thus the carbon tax gives a decentralized way to achieve the optimal outcome.

Cap and Trade

Suppose, alternatively that there is no carbon tax, but that the government issues tradable **emissions licenses**. Each license allows the firm that holds it to produce a certain amount of carbon emissions. The government chooses the number of emissions licenses to achieve the target reduction.

We imagine a market in these licenses so each firm can buy a license to emit x units of carbon at a price of p per unit. The cost to firm 1 of reducing its emissions by x_1 is $c_1(x_1) + p(\overline{x}_1 - x_1)$. Clearly the firm will want to operate where the price of an emissions license equals the marginal cost, $p = MC_1(x_1)$. That is, it will choose the level of emissions at the point where the cost of reducing carbon emissions by one unit would just equal the cost saved by not having to purchase a license.

Hence the marginal cost curve gives us the supply of emissions as a function of the price. The equilibrium price is the price where the total supply of emissions equals the target amount T. The associated price is the same as the optimal carbon tax rate t^* in Figure 24.11.

The question that remains is how to distribute the licenses. One way would be to have the government sell the licenses to firms. This is essentially the same as the carbon tax system. The government could pick a price and sell however many licenses are demanded at that price. Alternatively, it could pick a target level of emissions and auction off permits, letting the firms themselves determine a price. This is one type of "cap and trade" system. Both of these policies should lead to essentially the same market-clearing price.

Another possibility would be for the government to hand out the licenses to the firms according to some formula. This formula could be based on a variety of criteria, but presumably an important reason to award these valuable permits would be building political support for the program. Permits might be handed out based on objective criteria, such as which firms have the most employees, or they might be handed out based on which firms have donated the most to some political causes.

From the economic point of view, it doesn't matter whether the government owns the licenses and sells them to the firms (which is basically a carbon tax system) or whether the firms are given the licenses and sell them to each other (which is basically cap and trade).

If a cap and trade system is created, firms will find it attractive to invest in ways to acquire the emission permits. For example, they would want to lobby Congress for such licenses. These lobbying expenditures should

be counted as part of the cost of the system, as described in our earlier discussion of **rent seeking**. Of course, the carbon tax system would also be subject to similar lobbying. Firms would undoubtedly seek special carbon tax exemptions for one reason or another, but it has been argued that the carbon tax system is less susceptible to political manipulation than a cap and trade system.

Summary

1. The short-run supply curve of an industry is just the horizontal sum of the supply curves of the individual firms in that industry.

2. The long-run supply curve of an industry must take into account the exit and entry of firms in the industry.

3. If there is free entry and exit, then the long-run equilibrium will involve the maximum number of firms consistent with nonnegative profits. This means that the long-run supply curve will be essentially horizontal at a price equal to the minimum average cost.

4. If there are forces preventing the entry of firms into a profitable industry, the factors that prevent entry will earn economic rents. The rent earned is determined by the price of the output of the industry.

REVIEW QUESTIONS

1. If $S_1(p) = p - 10$ and $S_2(p) = p - 15$, then at what price does the industry supply curve have a kink in it?

2. In the short run the demand for cigarettes is totally inelastic. In the long run, suppose that it is perfectly elastic. What is the impact of a cigarette tax on the price that consumers pay in the short run and in the long run?

3. True or false? Convenience stores near the campus have high prices because they have to pay high rents.

4. True or false? In long-run industry equilibrium no firm will be losing money.

5. According to the model presented in this chapter, what determines the amount of entry or exit a given industry experiences?

6. The model of entry presented in this chapter implies that the more firms in a given industry, the (steeper, flatter) is the long-run industry supply curve.

7. A New York City cab operator appears to be making positive profits in the long run after carefully accounting for the operating and labor costs. Does this violate the competitive model? Why or why not?

CHAPTER 25

MONOPOLY

In the preceding chapters we have analyzed the behavior of a competitive industry, a market structure that is most likely when there are a large number of small firms. In this chapter we turn to the opposite extreme and consider an industry structure when there is only *one* firm in the industry—a **monopoly**.

When there is only one firm in a market, that firm is very unlikely to take the market price as given. Instead, a monopoly would recognize its influence over the market price and choose that level of price and output that maximized its overall profits.

Of course, it can't choose price and output independently; for any given price, the monopoly will be able to sell only what the market will bear. If it chooses a high price, it will be able to sell only a small quantity. The demand behavior of the consumers will constrain the monopolist's choice of price and quantity.

We can view the monopolist as choosing the price and letting the consumers choose how much they wish to buy at that price, or we can think of the monopolist as choosing the quantity, and letting the consumers decide what price they will pay for that quantity. The first approach is probably more natural, but the second turns out to be analytically more convenient. Of course, both approaches are equivalent when done correctly.

25.1 Maximizing Profits

We begin by studying the monopolist's profit-maximization problem. Let us use $p(y)$ to denote the market inverse demand curve and $c(y)$ to denote the cost function. Let $r(y) = p(y)y$ denote the revenue function of the monopolist. The monopolist's profit-maximization problem then takes the form

$$\max_y \; r(y) - c(y).$$

The optimality condition for this problem is straightforward: at the optimal choice of output we must have marginal revenue equal to marginal cost. If marginal revenue were less than marginal cost it would pay the firm to decrease output, since the savings in cost would more than make up for the loss in revenue. If the marginal revenue were greater than the marginal cost, it would pay the firm to increase output. The only point where the firm has no incentive to change output is where marginal revenue equals marginal cost.

In terms of algebra, we can write the optimization condition as

$$MR = MC$$

or

$$\frac{\Delta r}{\Delta y} = \frac{\Delta c}{\Delta y}.$$

The same $MR = MC$ condition has to hold in the case of a competitive firm; in that case, marginal revenue is equal to the price and the condition reduces to price equals marginal cost.

In the case of a monopolist, the marginal revenue term is slightly more complicated. If the monopolist decides to increase its output by Δy, there are two effects on revenues. First it sells more output and receives a revenue of $p\Delta y$ from that. But second, the monopolist pushes the price down by Δp and it gets this lower price on *all* the output it has been selling.

Thus the total effect on revenues of changing output by Δy will be

$$\Delta r = p\Delta y + y\Delta p,$$

so that the change in revenue divided by the change in output—the marginal revenue—is

$$\frac{\Delta r}{\Delta y} = p + \frac{\Delta p}{\Delta y} y.$$

(This is exactly the same derivation we went through in our discussion of marginal revenue in Chapter 15. You might want to review that material before proceeding.)

Another way to think about this is to think of the monopolist as choosing its output and price simultaneously—recognizing, of course, the constraint imposed by the demand curve. If the monopolist wants to sell more output it has to lower its price. But this lower price will mean a lower price for all of the units it is selling, not just the new units. Hence the term $y\Delta p$.

In the competitive case, a firm that could lower its price below the price charged by other firms would immediately capture the entire market from its competitors. But in the monopolistic case, the monopoly already has the entire market; when it lowers its price, it has to take into account the effect of the price reduction on all the units it sells.

Following the discussion in Chapter 15, we can also express marginal revenue in terms of elasticity via the formula

$$MR(y) = p(y)\left[1 + \frac{1}{\epsilon(y)}\right]$$

and write the "marginal revenue equals marginal costs" optimality condition as

$$p(y)\left[1 + \frac{1}{\epsilon(y)}\right] = MC(y). \tag{25.1}$$

Since elasticity is naturally negative, we could also write this expression as

$$p(y)\left[1 - \frac{1}{|\epsilon(y)|}\right] = MC(y).$$

From these equations it is easy to see the connection with the competitive case: in the competitive case, the firm faces a flat demand curve—an infinitely elastic demand curve. This means that $1/|\epsilon| = 1/\infty = 0$, so the appropriate version of this equation for a competitive firm is simply price equals marginal cost.

Note that a monopolist will never choose to operate where the demand curve is *inelastic*. For if $|\epsilon| < 1$, then $1/|\epsilon| > 1$, and the marginal revenue is negative, so it can't possibly equal marginal cost. The meaning of this becomes clear when we think of what is implied by an inelastic demand curve: if $|\epsilon| < 1$, then reducing output will increase revenues, and reducing output must reduce total cost, so profits will necessarily increase. Thus any point where $|\epsilon| < 1$ cannot be a profit maximum for a monopolist, since it could increase its profits by producing less output. It follows that a point that yields maximum profits can only occur where $|\epsilon| \geq 1$.

25.2 Linear Demand Curve and Monopoly

Suppose that the monopolist faces a linear demand curve

$$p(y) = a - by.$$

Then the revenue function is

$$r(y) = p(y)y = ay - by^2,$$

and the marginal revenue function is

$$MR(y) = a - 2by.$$

(This follows from the formula given at the end of Chapter 15. It is easy to derive using simple calculus. If you don't know calculus, just memorize the formula, since we will use it quite a bit.)

Note that the marginal revenue function has the same vertical intercept, a, as the demand curve, but it is twice as steep. This gives us an easy way to draw the marginal revenue curve. We know that the vertical intercept is a. To get the horizontal intercept, just take half of the horizontal intercept of the demand curve. Then connect the two intercepts with a straight line. We have illustrated the demand curve and the marginal revenue curve in Figure 25.1.

Figure 25.1 **Monopoly with a linear demand curve**. The monopolist's profit-maximizing output occurs where marginal revenue equals marginal cost.

The optimal output, y^*, is where the marginal revenue curve intersects the marginal cost curve. The monopolist will then charge the maximum price it can get at this output, $p(y^*)$. This gives the monopolist a revenue of $p(y^*)y^*$ from which we subtract the total cost $c(y^*) = AC(y^*)y^*$, leaving a profit area as illustrated.

25.3 Markup Pricing

We can use the elasticity formula for the monopolist to express its optimal pricing policy in another way. Rearranging equation (25.1) we have

$$p(y) = \frac{MC(y^*)}{1 - 1/|\epsilon(y)|}. \tag{25.2}$$

This formulation indicates that the market price is a markup over marginal cost, where the amount of the markup depends on the elasticity of demand. The markup is given by

$$\frac{1}{1 - 1/|\epsilon(y)|}.$$

Since the monopolist always operates where the demand curve is elastic, we are assured that $|\epsilon| > 1$, and thus the markup is greater than 1.

In the case of a constant-elasticity demand curve, this formula is especially simple since $\epsilon(y)$ is a constant. A monopolist who faces a constant-elasticity demand curve will charge a price that is a *constant* markup on marginal cost. This is illustrated in Figure 25.2. The curve labeled $MC/(1 - 1/|\epsilon|)$ is a constant fraction higher than the marginal cost curve; the optimal level of output occurs where $p = MC/(1 - 1/|\epsilon|)$.

EXAMPLE: The Impact of Taxes on a Monopolist

Let us consider a firm with constant marginal costs and ask what happens to the price charged when a quantity tax is imposed. Clearly the marginal costs go up by the amount of the tax, but what happens to the market price?

Let's first consider the case of a linear demand curve, as depicted in Figure 25.3. When the marginal cost curve, MC, shifts up by the amount of the tax to $MC+t$, the intersection of marginal revenue and marginal cost moves to the left. Since the demand curve is half as steep as the marginal revenue curve, the price goes up by half the amount of the tax.

This is easy to see algebraically. The marginal revenue equals marginal cost plus the tax condition is

$$a - 2by = c + t.$$

Figure 25.2 **Monopoly with constant elasticity demand**. To locate the profit-maximizing output level we find the output level where the curve $MC/(1-1/|\epsilon|)$ crosses the demand curve.

Solving for y yields

$$y = \frac{a-c-t}{2b}.$$

Thus the change in output is given by

$$\frac{\Delta y}{\Delta t} = -\frac{1}{2b}.$$

The demand curve is

$$p(y) = a - by,$$

so price will change by $-b$ times the change in output:

$$\frac{\Delta p}{\Delta t} = -b \times -\frac{1}{2b} = \frac{1}{2}.$$

In this calculation the factor 1/2 occurs because of the assumptions of the linear demand curve and constant marginal costs. Together these assumptions imply that the price rises by less than the tax increase. Is this likely to be true in general?

The answer is no—in general a tax may increase the price by more or less than the amount of the tax. For an easy example, consider the case of a monopolist facing a constant-elasticity demand curve. Then we have

$$p = \frac{c+t}{1-1/|\epsilon|},$$

Linear demand and taxation. Imposition of a tax on a monopolist facing a linear demand. Note that the price will rise by half the amount of the tax.

Figure 25.3

so that

$$\frac{\Delta p}{\Delta t} = \frac{1}{1 - 1/|\epsilon|},$$

which is certainly bigger than 1. In this case, the monopolist passes on *more* than the amount of the tax.

Another kind of tax that we might consider is the case of a profits tax. In this case the monopolist is required to pay some fraction τ of its profits to the government. The maximization problem that it faces is then

$$\max_y \ (1 - \tau)[p(y)y - c(y)].$$

But the value of y that maximizes profits will also maximize $(1 - \tau)$ times profits. Thus a pure profits tax will have no effect on a monopolist's choice of output.

25.4 Inefficiency of Monopoly

A competitive industry operates at a point where price equals marginal cost. A monopolized industry operates where price is greater than marginal cost. Thus in general the price will be higher and the output lower

if a firm behaves monopolistically rather than competitively. For this reason, consumers will typically be worse off in an industry organized as a monopoly than in one organized competitively.

But, by the same token, the firm will be better off! Counting both the firm and the consumer, it is not clear whether competition or monopoly will be a "better" arrangement. It appears that one must make a value judgment about the relative welfare of consumers and the owners of firms. However, we will see that one can argue against monopoly on grounds of efficiency alone.

Consider a monopoly situation, as depicted in Figure 25.4. Suppose that we could somehow costlessly force this firm to behave as a competitor and take the market price as being set exogenously. Then we would have (p_c, y_c) for the competitive price and output. Alternatively, if the firm recognized its influence on the market price and chose its level of output so as to maximize profits, we would see the monopoly price and output (p_m, y_m).

Figure 25.4 **Inefficiency of monopoly**. A monopolist produces less than the competitive amount of output and is therefore Pareto inefficient.

Recall that an economic arrangement is Pareto efficient if there is no way to make anyone better off without making somebody else worse off. Is the monopoly level of output Pareto efficient?

Remember the definition of the inverse demand curve. At each level of output, $p(y)$ measures how much people are willing to pay for an additional unit of the good. Since $p(y)$ is greater than $MC(y)$ for all the output levels between y_m and y_c, there is a whole range of output where people are willing to pay more for a unit of output than it costs to produce it. Clearly there is a potential for Pareto improvement here!

For example, consider the situation at the monopoly level of output y_m. Since $p(y_m) > MC(y_m)$ we know that there is someone who is willing to pay more for an extra unit of output than it costs to produce that extra unit. Suppose that the firm produces this extra output and sells it to this person at any price p where $p(y_m) > p > MC(y_m)$. Then this consumer is made better off because he or she was just willing to pay $p(y_m)$ for that unit of consumption, and it was sold for $p < p(y_m)$. Similarly, it cost the monopolist $MC(y_m)$ to produce that extra unit of output and it sold it for $p > MC(y_m)$. All the other units of output are being sold for the same price as before, so nothing has changed there. But in the sale of the extra unit of output, each side of the market gets some extra surplus—each side of the market is made better off and no one else is made worse off. We have found a Pareto improvement.

It is worthwhile considering the reason for this inefficiency. The efficient level of output is when the willingness to pay for an extra unit of output just equals the cost of producing this extra unit. A competitive firm makes this comparison. But a monopolist also looks at the effect of increasing output on the revenue received from the **inframarginal** units, and these inframarginal units have nothing to do with efficiency. A monopolist would always be ready to sell an additional unit at a lower price than it is currently charging if it did not have to lower the price of all the other inframarginal units that it is currently selling.

25.5 Deadweight Loss of Monopoly

Now that we know that a monopoly is inefficient, we might want to know just how inefficient it is. Is there a way to measure the total loss in efficiency due to a monopoly? We know how to measure the loss to the consumers from having to pay p_m rather than p_c—we just look at the change in consumers' surplus. Similarly, for the firm we know how to measure the gain in profits from charging p_m rather than p_c—we just use the change in producer's surplus.

The most natural way to combine these two numbers is to treat the firm—or, more properly, the owners of the firm—and the consumers of the firm's output symmetrically and add together the profits of the firm and the consumers' surplus. The change in the profits of the firm—the change in producer's surplus—measures how much the owners would be willing to pay to get the higher price under monopoly, and the change in

consumers' surplus measures how much the consumers would have to be paid to compensate them for the higher price. Thus the difference between these two numbers should give a sensible measure of the net benefit or cost of the monopoly.

The changes in the producer's and consumers' surplus from a movement from monopolistic to competitive output are illustrated in Figure 25.5. The monopolist's surplus goes down by A due to the lower price on the units he was already selling. It goes up by C due to the profits on the extra units it is now selling.

Figure 25.5 **Deadweight loss of monopoly**. The deadweight loss due to the monopoly is given by the area $B + C$.

The consumers' surplus goes up by A, since the consumers are now getting all the units they were buying before at a cheaper price; and it goes up by B, since they get some surplus on the extra units that are being sold. The area A is just a transfer from the monopolist to the consumer; one side of the market is made better off and one side is made worse off, but the total surplus doesn't change. The area $B + C$ represents a true increase in surplus—this area measures the value that the consumers and the producers place on the extra output that has been produced.

The area $B + C$ is known as the **deadweight loss** due to the monopoly. It provides a measure of how much worse off people are paying the mon-

opoly price than paying the competitive price. The deadweight loss due to monopoly, like the deadweight loss due to a tax, measures the value of the lost output by valuing each unit of lost output at the price that people are willing to pay for that unit.

To see that the deadweight loss measures the value of the lost output, think about starting at the monopoly point and providing one additinal unit of output. The value of that marginal unit of output is the market price. The cost of producing the additional unit of output is the marginal cost. Thus the "social value" of producing an extra unit will be simply the price minus the marginal cost. Now consider the value of the next unit of output; again its social value will be the gap between price and marginal cost at that level of output. And so it goes. As we move from the monopoly level of output to the competitive level of output, we "sum up" the distances between the demand curve and the marginal cost curve to generate the value of the lost output due to the monopoly behavior. The total area between the two curves from the monopoly output to the competitive output is the deadweight loss.

EXAMPLE: The Optimal Life of a Patent

A **patent** offers inventors the exclusive right to benefit from their inventions for a limited period of time. Thus a patent offers a kind of limited monopoly. The reason for offering such patent protection is to encourage innovation. In the absence of a patent system, it is likely that individuals and firms would be unwilling to invest much in research and development, since any new discoveries that they would make could be copied by competitors.

In the United States the life of a patent is 17 years. During that period, the holders of the patent have a monopoly on the invention; after the patent expires, anyone is free to utilize the technology described in the patent. The longer the life of a patent, the more gains can be accrued by the inventors, and thus the more incentive they have to invest in research and development. However, the longer the monopoly is allowed to exist, the more deadweight loss will be generated. The benefit from a long patent life is that it encourages innovation; the cost is that it encourages monopoly. The "optimal" patent life is the period that balances these two conflicting effects.

The problem of determining the optimal patent life has been examined by William Nordhaus of Yale University.[1] As Nordhaus indicates, the problem is very complex and there are many unknown relationships involved. Nevertheless, some simple calculations can give some insight as to whether

[1] William Nordhaus, *Invention, Growth, and Welfare* (Cambridge, Mass.: M.I.T. Press, 1969).

the current patent life is wildly out of line with the estimated benefits and costs described above.

Nordhaus found that for "run-of-the-mill" inventions, a patent life of 17 years was roughly 90 percent efficient—meaning that it achieved 90 percent of the maximum possible consumers' surplus. On the basis of these figures, it does not seem like there is a compelling reason to make drastic changes in the patent system.

EXAMPLE: Patent Thickets

The intellectual property protection offered by patents provides incentives to innovate, but this right can be abused. Some observers have argued that the extensions of intellectual property rights to business processes, software, and other domains has resulted in lower patent quality.

One might think of patents as having three dimensions: length, width, and height. The "length" is the time that the patent protection applies. The "width" is how broadly the claims in the patent are interpreted. The "height" is the standard of novelty applied in determining whether the patent really represents a new idea. Unfortunately, only the length is easily quantified. The other aspects of patent quality, breadth, and novelty, can be quite subjective.

Since it has become so easy to acquire patents in recent years, many firms have invested in acquiring patent portfolios on nearly every aspect of their business. Any company that wants to enter a business and compete with an incumbent who owns a broad range of patents may find itself encumbered in a **patent thicket**.

Even firms that are already well established find it important to invest in acquiring a patent portfolio. In 2004, Microsoft paid $440 million to InterTrust Technology to license a portfolio of patents related to computer security, and signed a 10-year pact with Sun Microsystems in which it paid $900 million to resolve patent issues. During 2003–04, Microsoft was granted over 1,000 patents.

Why the emphasis on patent portfolios? For large companies like Microsoft, their primary value is to be used as bargaining chips in cross-license agreements.

The patent thickets that each company sets up operate like the nuclear missiles held by the U.S. and USSR during the Cold War. Each had enough missiles pointed at the other to create "mutually assured destruction" in the case that one side attacked. Hence, neither side could risk an attack.

It's the same issue with patent thickets. If IBM tries to sue HP for patent infringement, HP would pull out a collection of its own patents and countersue IBM for infringement in some other technology. Even companies that don't particularly want to patent aspects of their business are forced

to do so in order to acquire the ammunition necessary for defense against other suits.

The "nuclear bomb" option in patent thickets is a "preliminary injunction." In certain circumstances, a judge might compel a company to stop selling an item that may be infringing on someone else's patent. This can be exceedingly costly. In 1986, Kodak had to completely shut down its instant photography business due to a court-ordered injunction. Eventually Kodak had to pay a billion-dollar judgment for patent infringement.

An injunction to stop production can be a huge threat, but it has no force against companies that don't produce anything. InterTrust, for example, didn't sell any products—all of its income came from licensing patents. Hence, it could threaten to sue other companies for patent infringement without much worry about the threat of countersuits.

EXAMPLE: Managing the Supply of Potatoes

Everyone is familiar with the Organization of Petroleum Exporting Countries (OPEC), the international oil cartel that attempts to influence the price of oil by setting production quota. Normally, coordinating production to push up prices is illegal in the United States, but there are some industries that are exempt from antitrust rules.

A notable example is agricultural producers. The 1922 Capper-Volstead Act specifically exempts farmers from federal antitrust rules. The result has been the creation of a number of "agricultural marketing boards" that attempt to voluntarily regulate the supply of agricultural products.

For example, the United Potato Growers of America, formed in March 2005, has signed up potato farmers that represent over 60% of the potato acreage in the United States. In 2005 it claimed to reduce production of potatoes by 6.8 million sacks of potatoes, each weighing about 100 pounds a piece. According to the *Wall Street Journal* this is equivalent to about 1.3 billion orders of french fries.[2]

25.6 Natural Monopoly

We have seen earlier that the Pareto efficient amount of output in an industry occurs where price equals marginal cost. A monopolist produces where marginal revenue equals marginal cost and thus produces too little output. It would seem that regulating a monopoly to eliminate the inefficiency is pretty easy—all the regulator has to do is to set price equal to marginal

[2] Timothy W. Martin, "This Spud's Not for You," *Wall Street Journal,* September 26, 2009.

cost, and profit maximization will do the rest. Unfortunately, this analysis leaves out one important aspect of the problem: it may be that the monopolist would make negative profits at such a price.

An example of this is shown in Figure 25.6. Here the minimum point of the average cost curve is to the right of the demand curve, and the intersection of demand and marginal cost lies underneath the average cost curve. Even though the level of output y_{MC} is efficient, it is not profitable. If a regulator set this level of output, the monopolist would prefer to go out of business.

Figure 25.6

A natural monopoly. If a natural monopolist operates where price equals marginal cost, then it will produce an efficient level of output, y_{MC}, but it will be unable to cover its costs. If it is required to produce an output where price equals average cost, y_{AC}, then it will cover its costs, but will produce too little output relative to the efficient amount.

This kind of situation often arises with public utilities. Think of a gas company, for example. Here the technology involves very large fixed costs—creating and maintaining the gas delivery pipes—and a very small marginal cost to providing extra units of gas—once the pipe is laid, it costs very little to pump more gas down the pipe. Similarly, a local telephone company involves very large fixed costs for providing the wires and switching network, while the marginal costs of an extra unit of telephone service is very

low. When there are large fixed costs and small marginal costs, you can easily get the kind of situation described in Figure 25.6. Such a situation is referred to as a **natural monopoly**.

If allowing a natural monopolist to set the monopoly price is undesirable due to the Pareto inefficiency, and forcing the natural monopoly to produce at the competitive price is infeasible due to negative profits, what is left? For the most part natural monopolies are regulated or operated by governments. Different countries have adopted different approaches. In some countries the telephone service is provided by the government and in others it is provided by private firms that are regulated by the government. Both of these approaches have their advantages and disadvantages.

For example, let us consider the case of government regulation of a natural monopoly. If the regulated firm is to require no subsidy, it must make nonnegative profits, which means it must operate on or above the average cost curve. If it is to provide service to all who are willing to pay for it, it must also operate on the demand curve. Thus the natural operating position for a regulated firm is a point like (p_{AC}, y_{AC}) in Figure 25.6. Here the firm is selling its product at the average cost of production, so it covers its costs, but it is producing too little output relative to the efficient level of output.

This solution is often adopted as a reasonable pricing policy for a natural monopolist. Government regulators set the prices that the public utility is allowed to charge. Ideally these prices are supposed to be prices that just allow the firm to break even—produce at a point where price equals average costs.

The problem facing the regulators is to determine just what the true costs of the firm are. Usually there is a public utility commission that investigates the costs of the monopoly in an attempt to determine the true average cost and then sets a price that will cover costs. (Of course, one of these costs is the payment that the firm has to make to its shareholders and other creditors in exchange for the money they have loaned to the firm.)

In the United States these regulatory boards operate at the state and local level. Typically electricity, natural gas, and telephone service operate in this way. Other natural monopolies like cable TV are usually regulated at the local level.

The other solution to the problem of natural monopoly is to let the government operate it. The ideal solution here in this case is to operate the service at price equals marginal cost and provide a lump-sum subsidy to keep the firm in operation. This is often the practice for local public transportation systems such as buses and subways. The lump-sum subsidies may not reflect inefficient operation *per se* but rather, simply reflect the large fixed costs associated with such public utilities.

Then again, the subsidies may just represent inefficiency! The problem with government-run monopolies is that it is almost as difficult to mea-

sure their costs as it is to measure the costs of regulated public utilities. Government regulatory commissions that oversee the operations of public utilities often subject them to probing hearings to require them to justify cost data whereas an internal government bureaucracy may escape such intense scrutiny. The government bureaucrats who run such government monopolies may turn out to be less accountable to the public than those who run the regulated monopolies.

25.7 What Causes Monopolies?

Given information on costs and demand, when would we predict that an industry would be competitive and when would we predict that it would be monopolized? In general the answer depends on the relationship between the average cost curve and the demand curve. The crucial factor is the size of the **minimum efficient scale (MES)**, the level of output that minimizes average cost, relative to the size of demand.

Consider Figure 25.7 where we have illustrated the average cost curves and the market demand curves for two goods. In the first case there is room in the market for many firms, each charging a price close to p^* and each operating at a relatively small scale. In the second market, only one firm can make positive profits. We would expect that the first market might well operate as a competitive market and that the second would operate as a monopolist.

Figure 25.7

Demand relative to minimum efficient scale. (A) If demand is large relative to the minimum efficient scale, a competitive market is likely to result. (B) If it is small, a monopolistic industry structure is possible.

Thus the shape of the average cost curve, which in turn is determined by the underlying technology, is one important aspect that determines whether a market will operate competitively or monopolistically. If the minimum efficient scale of production—the level of output that minimizes average costs—is small relative to the size of the market, we might expect that competitive conditions will prevail.

Note that this is a *relative* statement: what matters is the scale relative to the market size. We can't do too much about the minimum efficient scale—that is determined by the technology. But economic policy can influence the size of the market. If a country chooses nonrestrictive foreign-trade policies, so that domestic firms face foreign competition, then the domestic firms' ability to influence prices will be much less. Conversely, if a country adopts restrictive trade policies, so that the size of the market is limited only to that country, then monopolistic practices are more likely to take hold.

If monopolies arise because the minimum efficient scale is large relative to the size of the market, and it is infeasible to increase the size of the market, then the industry is a candidate for regulation or other sorts of government intervention. Of course such regulation and intervention are costly too. Regulatory boards cost money, and the efforts of the firm to satisfy the regulatory boards can be quite expensive. From society's point of view, the question should be whether the deadweight loss of the monopoly exceeds the costs of regulation.

A second reason why monopoly might occur is that several different firms in an industry might be able to collude and restrict output in order to raise prices and thereby increase their profits. When firms collude in this way and attempt to reduce output and increase price, we say the industry is organized as a **cartel**.

Cartels are illegal. The Antitrust Division of the Justice Department and the Bureau of Competition of the Federal Trade Commission are charged with searching for evidence of noncompetitive behavior on the part of firms. If the government can establish that a group of firms attempted to restrict output or engaged in certain other anticompetitive practices, the firms in question can be forced to pay heavy fines.

On the other hand, an industry may have one dominant firm purely by historical accident. If one firm is first to enter some market, it may have enough of a cost advantage to be able to discourage other firms from entering the industry. Suppose, for example, that there are very large "tooling-up" costs to entering an industry. Then the incumbent—the firm already in the industry—may under certain conditions be able to convince potential entrants that it will cut its prices drastically if they attempt to enter the industry. By preventing entry in this manner, a firm can eventually dominate a market. We will study an example of pricing to prevent entry in Chapter 29.

EXAMPLE: Diamonds Are Forever

The De Beers diamond cartel was formed by Sir Ernest Oppenheimer, a South African mine operator, in 1930. It has since grown into one of the world's most successful cartels. De Beers handles over 80% of the world's yearly production of diamonds and has managed to maintain this near-monopoly for several decades. Over the years, De Beers has developed several mechanism to maintain control of the diamond market.

First, it maintains considerable stocks of diamonds of all types. If a producer attempts to sell outside the cartel, De Beers can quickly flood the market with the same type of diamond, thereby punishing the defector from the cartel. Second, large producers' quotas are based on the *proportion* of total sales. When the market is weak, everyone's production quota is reduced proportionally, thereby automatically increasing scarcity and raising prices.

Third, De Beers is involved at both the mining and wholesaling levels of diamond production. In the wholesale market diamonds are sold to cutters in boxes of assorted diamonds: buyers take a whole box or nothing—they cannot choose individual stones. If the market is weak for a certain size of diamond, De Beers can reduce the number of those diamonds offered in the boxes, thereby making them more scarce.

Finally, De Beers can influence the direction of final demand for diamonds by the $110 million a year it spends on advertising. Again, this advertising can be adjusted to encourage demand for the types and sizes of diamonds that are in relatively scarce supply.[3]

EXAMPLE: Pooling in Auction Markets

Adam Smith once said "People of the same trade seldom meet together, even for merriment and diversion, but the conversation ends in a conspiracy against the public, or in some contrivance to raise prices." Bidding pools in auctions provide an illustrative example of Smith's observation. In 1988 the Justice Department charged 12 Philadelphia antique dealers with antitrust violations for their participation in this particular kind of "conspiracy against the public."[4]

[3] A short description of the diamond market can be found in "The cartel lives to face another threat," *The Economist,* January 10, 1987, 58–60. A more detailed description can be found in Edward J. Epstein, *Cartel* (New York: Putnam, 1978).

[4] See Meg Cox, "At Many Auctions, Illegal Bidding Thrives As a Longtime Practice Among Dealers," *Wall Street Journal*, February 19, 1988, which served as the source for this example.

The dealers were accused of participating in "bidding rings," or "pools," at antique furniture auctions. The members of a pool would appoint one member to bid on certain items. If this bidder succeeded in acquiring an item, the participating dealers would then hold a subsequent private auction, called a "knockout," in which the members of the pool bid among themselves for the item. This practice allowed the members of the pool to acquire the items at much lower prices than would have prevailed if they had bid separately; in many cases the prices in the knockout auctions were 50 to 100 percent greater than the prices paid to the original sellers of the goods.

The dealers were surprised by the Justice Department suit; they considered pooling a common business practice in their trade and did not think it was illegal. They thought of the pools as a tradition of cooperation among themselves; being invited to join a pool was considered a "mark of distinction." According to one dealer, "The day I was allowed to go into the pool was a banner day. If you weren't in the pool, you weren't considered much of a dealer." The dealers were so naive that they kept careful records of their payments in the knockout auctions, which were later used by the Justice Department in the suits against the dealers.

The Justice Department argued "if they are joining together to hold down the price [received by the seller] that is illegal." The Justice Department view prevailed over that of the dealers: 11 of the 12 dealers pleaded guilty and settled the matter with fines of $1,000 to $50,000 and probation. The dealer who held out for a jury trial was found guilty and sentenced to 30 days of house arrest and a fine of $30,000.

EXAMPLE: Price Fixing in Computer Memory Markets

DRAM chips are the "dynamic random access memory" chips that go in your computer. They are pretty much an undifferentiated commodity product and the market for DRAMs is (usually) highly competitive. However, there are allegations that several DRAM producers conspired to fix prices and charge computer makers a higher price than would have obtained under purely competitive conditions. Apple Computer, Compaq, Dell, Gateway, HP, and IBM were apparently affected by this conspiracy.

The Department of Justice started investigating these allegations in 2002. In September 2004, Infineon, a German DRAM manufacturer, pleaded guilty to charges of price fixing, and agreed to pay a $160 million fine. This was the third largest criminal fine ever imposed by the Department of Justice's antitrust division.

According to the court documents, Infineon was charged with "Participating in meetings, conversations, and communications with competitors to discuss the prices of DRAM to be sold to certain customers; Agreeing to

price levels of DRAM to be sold to certain customers; Exchanging information on sales of DRAM to certain customers, for the purpose of monitoring and enforcing the agreed-upon prices."

Subsequently, four executives at Infineon were sentenced to prison terms and had to pay hefty fines. The antitrust authorities take price fixing very seriously, and the consequences to companies and individuals that engage in such activities can be severe.

Summary

1. When there is only a single firm in an industry, we say that it is a monopoly.

2. A monopolist operates at a point where marginal revenue equals marginal cost. Hence a monopolist charges a price that is a markup on marginal cost, where the size of the markup depends on the elasticity of demand.

3. Since a monopolist charges a price in excess of marginal cost, it will produce an inefficient amount of output. The size of the inefficiency can be measured by the deadweight loss—the net loss of consumers' and the producer's surplus.

4. A natural monopoly occurs when a firm cannot operate at an efficient level of output without losing money. Many public utilities are natural monopolies of this sort and are therefore regulated by the government.

5. Whether an industry is competitive or monopolized depends in part on the nature of technology. If the minimum efficient scale is large relative to demand, then the market is likely to be monopolized. But if the minimum efficient scale is small relative to demand, there is room for many firms in the industry, and there is a hope for a competitive market structure.

REVIEW QUESTIONS

1. The market demand curve for heroin is said to be highly inelastic. Heroin supply is also said to be monopolized by the Mafia, which we assume to be interested in maximizing profits. Are these two statements consistent?

2. The monopolist faces a demand curve given by $D(p) = 100 - 2p$. Its cost function is $c(y) = 2y$. What is its optimal level of output and price?

3. The monopolist faces a demand curve given by $D(p) = 10p^{-3}$. Its cost function is $c(y) = 2y$. What is its optimal level of output and price?

4. If $D(p) = 100/p$ and $c(y) = y^2$, what is the optimal level of output of the monopolist? (Be careful.)

5. A monopolist with constant marginal cost is producing where $|\epsilon| = 3$. The government imposes a quantity tax of \$6 per unit of output. If the demand curve facing the monopolist is linear, how much does the price rise?

6. What is the answer to the above question if the demand curve facing the monopolist has constant elasticity?

7. If the demand curve facing the monopolist has a constant elasticity of 2, then what will be the monopolist's markup on marginal cost?

8. The government is considering subsidizing the marginal costs of the monopolist described in the question above. What level of subsidy should the government choose if it wants the monopolist to produce the socially optimal amount of output?

9. Show mathematically that a monopolist always sets its price above marginal cost.

10. True or false? Imposing a quantity tax on a monopolist will always cause the market price to increase by the amount of the tax.

11. What problems face a regulatory agency attempting to force a monopolist to charge the perfectly competitive price?

12. What kinds of economic and technological conditions are conducive to the formation of monopolies?

APPENDIX

Define the revenue function by $r(y) = p(y)y$. Then the monopolist's profit-maximization problem is

$$\max\ r(y) - c(y).$$

The first-order condition for this problem is simply

$$r'(y) - c'(y) = 0,$$

which implies that marginal revenue should equal marginal cost at the optimal choice of output.

Differentiating the definition of the revenue function gives $r'(y) = p(y) + p'(y)y$, and substituting this into the monopolist's first-order condition yields the alternative form

$$p(y) + p'(y)y = c'(y).$$

The second-order condition for the monopolist's profit-maximization problem is

$$r''(y) - c''(y) \leq 0.$$

This implies that

$$c''(y) \geq r''(y)$$

or that the slope of the marginal cost curve exceeds the slope of the marginal revenue curve.

CHAPTER 26

MONOPOLY BEHAVIOR

In a competitive market there are typically several firms selling an identical product. Any attempt by one of the firms to sell its product at more than the market price leads consumers to desert the high-priced firm in favor of its competitors. In a monopolized market there is only one firm selling a given product. When a monopolist raises its price it loses some, but not all, of its customers.

In reality most industries are somewhere in between these two extremes. If a gas station in a small town raises the price at which it sells gasoline and it loses most of its customers, it is reasonable to think that this firm must behave as a competitive firm. If a restaurant in the same town raises its price and loses only a few of its customers, then it is reasonable to think that this restaurant has some degree of monopoly power.

If a firm has some degree of monopoly power it has more options open to it than a firm in a perfectly competitive industry. For example, it can use more complicated pricing and marketing strategies than a firm in a competitive industry. Or it can try to differentiate its product from the products sold by its competitors to enhance its market power even further. In this chapter we will examine how firms can enhance and exploit their market power.

26.1 Price Discrimination

We have argued earlier that a monopoly operates at an inefficient level of output since it restricts output to a point where people are willing to pay more for extra output than it costs to produce it. The monopolist doesn't want to produce this *extra* output, because it would force down the price that it would be able to get for *all* of its output.

But if the monopolist could sell different units of output at different prices, then we have another story. Selling different units of output at different prices is called **price discrimination**. Economists generally consider the following three kinds of price discrimination:

First-degree price discrimination means that the monopolist sells different units of output for different prices *and* these prices may differ from person to person. This is sometimes known as the case of **perfect price discrimination**.

Second-degree price discrimination means that the monopolist sells different units of output for different prices, but every individual who buys the same amount of the good pays the same price. Thus prices differ across the units of the good, but not across people. The most common example of this is bulk discounts.

Third-degree price discrimination occurs when the monopolist sells output to different people for different prices, but every unit of output sold to a given person sells for the same price. This is the most common form of price discrimination, and examples include senior citizens' discounts, student discounts, and so on.

Let us look at each of these to see what economics can say about how price discrimination works.

26.2 First-Degree Price Discrimination

Under **first-degree price discrimination**, or **perfect price discrimination**, each unit of the good is sold to the individual who values it most highly, at the maximum price that this individual is willing to pay for it.

Consider Figure 26.1, which illustrates two consumers' demand curves for a good. Think of a reservation price model for demand where the individuals choose integer amounts of the goods and each step in the demand curve represents a change in the willingness to pay for additional units of the good. We have also illustrated (constant) marginal cost curves for the good.

A producer who is able to perfectly price discriminate will sell each unit of the good at the highest price it will command, that is, at each consumer's reservation price. Since each unit is sold to each consumer at his or her reservation price for that unit, there is no consumers' surplus generated in

First-degree price discrimination. Here are two consumers' demand curves for a good along with the constant marginal cost curve. The producer sells each unit of the good at the maximum price it will command, which yields it the maximum possible profit.

Figure 26.1

this market; all the surplus goes to the producer. In Figure 26.1 the colored areas indicate the *producer's surplus* accruing to the monopolist. In an ordinary competitive market setting these areas would represent *consumers' surplus*, but in the case of perfect price discrimination, the monopolist is able to appropriate this surplus for itself.

Since the producer gets all the surplus in the market, it wants to make sure that the surplus is as large as possible. Put another way, the producer's goal is to maximize its profits (producer's surplus) subject to the constraint that the consumers are just willing to purchase the good. This means that the outcome will be Pareto efficient, since there will be no way to make both the consumers and the producer better off: the producer's profit can't be increased, since it is already the maximal possible profit, and the consumers' surplus can't be increased without reducing the profit of the producer.

If we move to the smooth demand curve approximation, as in Figure 26.2, we see that a perfectly price-discriminating monopolist must produce at an output level where price equals marginal cost: if price were greater than marginal cost, that would mean that there is someone who is willing to pay more than it costs to produce an extra unit of output. So why not produce that extra unit and sell it to that person at his or her reservation price, and thus increase profits?

Just as in the case of a competitive market, the sum of producer's and consumers' surpluses is maximized. However, in the case of perfect price discrimination the producer ends up getting *all* the surplus generated in the market!

We have interpreted first-degree price discrimination as selling each unit at the maximum price it will command. But we could also think of it as selling a fixed amount of the good at a "take it or leave it" price. In the

Figure 26.2

First-degree price discrimination with smooth demand curves. Here are two consumers' smoothed demand curves for a good along with the constant marginal cost curve. Here the producer maximizes profits by producing where price equals marginal cost, just as in the case of a competitive market.

case illustrated in Figure 26.2, the monopolist would offer to sell x_1^0 units of the good to person 1 at a price equal to the area under person 1's demand curve and offer to sell x_2^0 units of the good to person 2 at a price equal to the area under person 2's demand curve B. As before, each person would end up with zero consumer's surplus, and the entire surplus of $A+B$ would end up in the hands of the monopolist.

Perfect price discrimination is an idealized concept—as the word "perfect" might suggest—but it is interesting theoretically since it gives us an example of a resource allocation mechanism other than a competitive market that achieves Pareto efficiency. There are very few real-life examples of perfect price discrimination. The closest example would be something like a small-town doctor who charges his patients different prices, based on their ability to pay.

EXAMPLE: First-degree Price Discrimination in Practice

As mentioned earlier, first-degree price discrimination is primarily a theoretical concept. It's hard to find real-world examples in which every individual is charged a different price. One possible example would be cases where prices are set by bargaining, as in automobile sales or in antique markets. However, these are not ideal examples.

Southwest Airlines recently introduced a system called Ding that attempts something rather close to first-degree price discrimination.[1] The

[1] See Christopher Elliott, "Your Very Own Personal Air Fare," *New York Times,* August 9, 2005.

system uses the Internet in a clever way. The user installs a program on her computer and the airline sends special fare offers to the user periodically. The fares are announced with a "ding" sound, hence the system name. According to one analyst, the fares offered by Ding were about 30 percent lower than comparable fares.

But will these low fares persist? One might also use such a system to offer higher fares. However, that possibility seems unlikely given the intensely competitive nature of the airline industry. It's easy to switch back to standard ways of buying tickets if prices start creeping up.

26.3 Second-Degree Price Discrimination

Second-degree price discrimination is also known as the case of **nonlinear pricing**, since it means that the price per unit of output is not constant but depends on how much you buy. This form of price discrimination is commonly used by public utilities; for example, the price per unit of electricity often depends on how much is bought. In other industries bulk discounts for large purchases are sometimes available.

Let us consider the case depicted earlier in Figure 26.2. We saw that the monopolist would *like* to sell an amount x_1^0 to person 1 at price $A+$ cost and an amount x_2^0 to person 2 at price $B+$ cost. To set the right prices, the monopolist has to *know* the demand curves of the consumers; that is, the monopolist has to know the exact willingness to pay of each person. Even if the monopolist knows something about the statistical distribution of willingness to pay—for example, that college students are willing to pay less than yuppies for movie tickets—it might be hard to tell a yuppie from a college student when they are standing in line at the ticket booth.

Similarly, an airline ticket agent may know that business travelers are willing to pay more than tourists for their airplane tickets, but it is often difficult to tell whether a particular person is a business traveler or a tourist. If switching from a grey flannel suit to Bermuda shorts would save $500 on travel expenses, corporate dress codes could change quickly!

The problem with the first-degree price discrimination example depicted in Figure 26.2 is that person 1—the high-willingess-to-pay person—can *pretend* to be person 2, the low-willingess-to-pay person. The seller may have no effective way to tell them apart.

One way to get around this problem is to offer two different price-quantity packages in the market. One package will be targeted toward the high-demand person, the other package toward the low-demand person. It can often happen that the monopolist can construct price-quantity packages that will induce the consumers to choose the package meant for them; in economics jargon, the monopolist constructs price-quantity packages that give the consumers an incentive to **self select**.

In order to see how this works, Figure 26.3 illustrates the same kind of demand curves used in Figure 26.2, but now laid on top of each other. We've also set marginal cost equal to zero in this diagram to keep the argument simple.

Figure 26.3

Second-degree price discrimination. These are the demand curves of two consumers; the producer has zero marginal cost by assumption. Panel A illustrates the self-selection problem. Panel B shows what happens if the monpolist reduces the output targeted for consumer 1, and panel C illustrates the profit-maximizing solution.

As before, the monopolist would like to offer x_1^0 at price A and to offer x_2^0 at price $A + B + C$. This would capture all the surplus for the monopolist and generate the most possible profit. Unfortunately for the monopolist, these price-quantity combinations are not compatible with self-selection. The high-demand consumer would find it optimal to choose the quantity x_1^0 and pay price A; this would leave him with a surplus equal to area B, which is better than the zero surplus he would get if he chose x_2^0.

One thing the monopolist can do is to offer x_2^0 at a price of $A+C$. In this case the high-demand consumer finds it optimal to choose x_2^0 and receive a gross surplus of $A+B+C$. He pays the monopolist $A+C$, which yields a net surplus of B for consumer 2—just what he would get if he chose x_1^0. This generally yields more profit to the monopolist than it would get by offering only one price-quantity combination.

But the story doesn't end here. There's yet a further thing the monopolist can do to increase profits. Suppose that instead of offering x_1^0 at price A to the low-demand consumer, the monopolist offers a bit less than that at a price slightly less than A. This reduces the monopolist's profits on person 1 by the small colored triangle illustrated in Figure 26.3B. But note that since person 1's package is now less attractive to person 2, the

monopolist can now charge *more* to person 2 for x_2^0! By reducing x_1^0, the monopolist makes area A a little smaller (by the dark triangle) but makes area C bigger (by the triangle plus the light trapezoid area). The net result is that the monopolist's profits increase.

Continuing in this way, the monopolist will want to reduce the amount offered to person 1 up to the point where the profit lost on person 1 due to a further reduction in output just equals the profit gained on person 2. At this point, illustrated in Figure 26.3C, the marginal benefits and costs of quantity reduction just balance. Person 1 chooses x_1^m and is charged A; person 2 chooses x_2^0 and is charged $A + C + D$. Person 1 ends up with a zero surplus and person 2 ends up with a surplus of B—just what he would get if he chose to consume x_1^m.

In practice, the monopolist often encourages this self-selection not by adjusting the *quantity* of the good, as in this example, but rather by adjusting the *quality* of the good. The quantities in the model just examined can be re-interpreted as qualities, and everything works as before. In general, the monopolist will want to reduce the quality offered to the low end of its market so as not to cannibalize sales at the high end. Without the high-end consumers, the low-end consumers would be offered higher quality, but they would still end up with zero surplus. Without the low-end consumers, the high-end consumers would have zero surplus, so it is beneficial to the high-end consumers to have the low-end consumers present. This is because the monopolist has to cut the price to the high-end consumers to discourage them from choosing the product targeted to the low-end consumers.

EXAMPLE: Price Discrimination in Airfares

The airline industry has been very successful at price discrimination (although industry representatives prefer to use the term "yield management.") The model described above applies reasonably well to the problem faced by airlines: there are essentially two types of consumers, business travelers and individual travelers, who generally have quite different willingnesses to pay. Although there are several competing airlines in the U.S. market, it is quite common to see only one or two airlines serving specific city pairs. This gives the airlines considerable freedom in setting prices.

We have seen that the optimal pricing policy for a monopolist dealing with two groups of consumers is to sell to the high-willingness-to-pay market at a high price and offer a reduced-quality product to the market with the lower willingness to pay. The point of the reduced-quality product is to dissuade those with a high willingness to pay from purchasing the lower priced good.

The way the airlines implement this is to offer an "unrestricted fare" for business travel and a "restricted fare" for non-business travel. The

restricted fare often requires advanced purchase, a Saturday-night stayover, or other such impositions. The point of these impositions, of course, is to be able to discriminate between the high-demand business travelers and the more price sensitive individual travelers. By offering a "degraded" product—the restricted fares—the airlines can charge the customers who require flexible travel arrangements considerably more for their tickets.

Such arrangements may well be socially useful; without the ability to price discriminate, a firm may decide that it is optimal to sell *only* to the high-demand markets.

Another way that airlines price discriminate is with first-class and coach-class travel. First-class travelers pay substantially more for their tickets, but they receive an enhanced level of service: more space, better food, and more attention. Coach-class travelers, on the other hand, receive a lower level of service on all these dimensions. This sort of quality discrimination has been a feature of transportation services for hundreds of years. Witness, for example, this commentary on railroad pricing by Emile Dupuit, a nineteenth century French economist:

> It is not because of the few thousand francs which would have to be spent to put a roof over the third-class carriage or to upholster the third-class seats that some company or other has open carriages with wooden benches . . . What the company is trying to do is prevent the passengers who can pay the second-class fare from traveling third class; it hits the poor, not because it wants to hurt them, but to frighten the rich . . . And it is again for the same reason that the companies, having proved almost cruel to the third-class passengers and mean to the second-class ones, become lavish in dealing with first-class customers. Having refused the poor what is necessary, they give the rich what is superfluous.[2]

The next time you fly coach class, perhaps it will be of some solace to know that rail travel in nineteenth century France was even more uncomfortable!

EXAMPLE: Prescription Drug Prices

A month's supply of the antidepressant Zoloft sells for $29.74 in Austria, $32.91 in Luxembourg, $40.97 in Mexico, and $64.67 in the United States. Why the difference? Drug makers, like other firms, charge what the market

[2] Translation by R. B. Ekelund in "Price Discrimination and Product Differentiation in Economic Theory: An Early Analysis," *Quarterly Journal of Economics*, 84 (1970), 268–78.

will bear. Poorer countries can't pay as much as richer ones, so drug prices tend to be lower.

But that's not the whole story. Bargaining power also differs dramatically from country to country. Canada, which has a national health plan, often has lower drug prices than the United States, where there is no centralized provider of health care.

It has been proposed that drug companies be forced to charge a single price worldwide. Leaving aside the thorny question of enforcement, we might well ask what the consequences of such a policy would be. Would the world overall end up with lower prices or higher prices?

The answer depends on the relative size of the market. A drug for malaria would find most of its demand in poor countries. If forced to charge a single price, drug companies would likely sell such a drug at a low price. But a drug for diseases that afflicted those in wealthy countries would likely sell for a high price, making it too expensive for those in poorer areas.

Typically, moving from price discrimination to a single-price regime will raise some prices and lower others, making some people better off and some people worse off. In some cases, a product may not be supplied at all to some markets if a seller is forced to apply uniform pricing.

26.4 Third-Degree Price Discrimination

Recall that this means that the monopolist sells to different people at different prices, but every unit of the good sold to a given group is sold at the same price. Third-degree price discrimination is the most common form of price discrimination. Examples of this might be student discounts at the movies, or senior citizens' discounts at the drugstore. How does the monopolist determine the optimal prices to charge in each market?

Let us suppose that the monopolist is able to identify two groups of people and can sell an item to each group at a different price. We suppose that the consumers in each market are not able to resell the good. Let us use $p_1(y_1)$ and $p_2(y_2)$ to denote the inverse demand curves of groups 1 and 2, respectively, and let $c(y_1 + y_2)$ be the cost of producing output. Then the profit-maximization problem facing the monopolist is

$$\max_{y_1, y_2} \; p_1(y_1)y_1 + p_2(y_2)y_2 - c(y_1 + y_2).$$

The optimal solution must have

$$MR_1(y_1) = MC(y_1 + y_2)$$

$$MR_2(y_2) = MC(y_1 + y_2).$$

That is, the marginal cost of producing an extra unit of output must be equal to the marginal revenue in *each* market. If the marginal revenue in

market 1 exceeded marginal cost, it would pay to expand output in market 1, and similarly for market 2. Since marginal cost is the same in each market, this means of course that marginal revenue in each market must also be the same. Thus a good should bring the same increase in revenue whether it is sold in market 1 or in market 2.

We can use the standard elasticity formula for marginal revenue and write the profit-maximization conditions as

$$p_1(y_1)\left[1 - \frac{1}{|\epsilon_1(y_1)|}\right] = MC(y_1 + y_2)$$

$$p_2(y_2)\left[1 - \frac{1}{|\epsilon_2(y_2)|}\right] = MC(y_1 + y_2),$$

where $\epsilon_1(y_1)$ and $\epsilon_2(y_2)$ represent the elasticities of demand in the respective markets, evaluated at the profit-maximizing choices of output.

Now note the following. If $p_1 > p_2$, then we must have

$$1 - \frac{1}{|\epsilon_1(y_1)|} < 1 - \frac{1}{|\epsilon_2(y_2)|},$$

which in turn implies that

$$\frac{1}{|\epsilon_1(y_1)|} > \frac{1}{|\epsilon_2(y_2)|}.$$

This means that

$$|\epsilon_2(y_2)| > |\epsilon_1(y_1)|.$$

Thus the market with the higher price must have the lower elasticity of demand. Upon reflection, this is quite sensible. An elastic demand is a price-sensitive demand. A firm that price discriminates will therefore set a low price for the price-sensitive group and a high price for the group that is relatively price insensitive. In this way it maximizes its overall profits.

We suggested that senior citizens' discounts and student discounts were good examples of third-degree price discrimination. Now we can see why they have discounts. It is likely that students and senior citizens are more sensitive to price than the average consumer and thus have more elastic demands for the relevant region of prices. Therefore a profit-maximizing firm will price discriminate in their favor.

EXAMPLE: Linear Demand Curves

Let us consider a problem where the firm faces two markets with linear demand curves, $x_1 = a - bp_1$ and $x_2 = c - dp_2$. Suppose for simplicity that marginal costs are zero. If the firm is allowed to price discriminate,

it will produce where marginal revenue equals zero in each market—at a price and output combination that is halfway down each demand curve, with outputs $x_1^* = a/2$ and $x_2^* = c/2$ and prices $p_1^* = a/2b$ and $p_2^* = c/2d$.

Suppose that the firm were forced to sell in both markets at the same price. Then it would face a demand curve of $x = (a + c) - (b + d)p$ and would produce halfway down this demand curve, resulting in an output of $x^* = (a + c)/2$ and price of $p^* = (a + c)/2(b + d)$. Note that the total output is the same whether or not price discrimination is allowed. (This is a special feature of the linear demand curve and does not hold in general.)

However, there is an important exception to this statement. We have assumed that when the monopolist chooses the optimal single price it will sell a positive amount of output in each market. It may very well happen that at the profit-maximizing price, the monopolist will sell output to only one of the markets, as illustrated in Figure 26.4.

Price discrimination with linear demands. If the monopolist can charge only one price, it will charge p_1^*, and sell only to market 1. But if price discrimination is allowed, it will also sell at price p_2^* to market 2.

Figure 26.4

Here we have two linear demand curves; since marginal cost is assumed to be zero, the monopolist will want to operate at a point where the elasticity of demand is -1, which we know to be halfway down the market demand curve. Thus the price p_1^* is a profit-maximizing price—lowering the price any further would reduce revenues in market 1. If the demand in market 2 is very small, the monopolist may not want to lower its price any further in order to sell to this market: it will end up selling only to the larger market.

In this case, allowing price discrimination will unambiguously increase total output, since the monopolist will find it in its interest to sell to both markets if it can charge a different price in each one.

EXAMPLE: Calculating Optimal Price Discrimination

Suppose that a monopolist faces two markets with demand curves given by

$$D_1(p_1) = 100 - p_1$$
$$D_2(p_2) = 100 - 2p_2.$$

Assume that the monopolist's marginal cost is constant at \$20 a unit. If it can price discriminate, what price should it charge in each market in order to maximize profits? What if it can't price discriminate? Then what price should it charge?

To solve the price-discrimination problem, we first calculate the inverse demand functions:

$$p_1(y_1) = 100 - y_1$$
$$p_2(y_2) = 50 - y_2/2.$$

Marginal revenue equals marginal cost in each market yields the two equations:

$$100 - 2y_1 = 20$$
$$50 - y_2 = 20.$$

Solving we have $y_1^* = 40$ and $y_2^* = 30$. Substituting back into the inverse demand functions gives us the prices $p_1^* = 60$ and $p_2^* = 35$.

If the monopolist must charge the same price in each market, we first calculate the total demand:

$$D(p) = D_1(p_1) + D_2(p_2) = 200 - 3p.$$

The inverse demand curve is

$$p(y) = \frac{200}{3} - \frac{y}{3}.$$

Marginal revenue equals marginal cost gives us

$$\frac{200}{3} - \frac{2}{3}y = 20,$$

which can be solved to give $y^* = 70$ and $p^* = 43\frac{1}{3}$.

In accord with the discussion in the previous section, it is important to check that this price generates non-negative demands in each market. However, it is easily checked that this is the case.

EXAMPLE: Price Discrimination in Academic Journals

Most written scholarly communication takes place in academic journals. These journals are sold by subscription to libraries and to individual scholars. It is very common to see different subscription prices being charged to libraries and individuals. In general, we would expect that the demand by libraries would be much more inelastic than demand by individuals, and, just as economic analysis would predict, the prices for library subscriptions are typically much higher than the prices for individual subscriptions. Often library subscriptions are 2 to 3 times more expensive than subscriptions to individuals.

More recently, some publishers have begun to price discriminate by geography. During 1984, when the U.S. dollar was at an all-time high as compared to the English pound, many British publishers began to charge different prices to U.S. subscribers than to European subscribers. It would be expected that the U.S. demand would be more inelastic. Since the dollar price of British journals was rather low due to the exchange rate, a 10 percent increase in the U.S. price would result in a smaller percentage drop in demand than a similar increase in the British price. Thus, on grounds of profit maximization, it made sense for the British publishers to raise the prices of their journals to the group with the lower elasticity of demand—the U.S. subscribers. According to a 1984 study, North American libraries were charged an average of 67 percent more for their journals than U.K. libraries, and 34 percent more than anyone else in the world.[3]

Further evidence for price discrimination can be found by examining the pattern of price increases. According to a study by the University of Michigan Library, "... publishers have carefully considered their new pricing strategy. There seems to be a direct correlation ... between patterns of library usage and the magnitude of the pricing differential. The greater the use, the larger the differential."[4]

By 1986 the exchange rate had turned in favor of the pound, and the dollar prices of the British journals had increased significantly. Along with the price increase came some serious resistance to the higher prices. The concluding sentences of the Michigan report are illustrative: "One expects that a vendor with a monopoly on a product will charge according to demand. What the campus as a customer must determine is whether it will continue to pay up to 114% more than its British counterparts for the identical product."

[3] Hamaker, C. and Astle, D., "Recent Pricing Patterns in British Journal Publishing," *Library Acquisitions: Practice and Theory*, 8, 4 (Spring 1984), 225–32.

[4] The study was conducted by Robert Houbeck for the University of Michigan Library, and published in Vol. 2, No. 1 of the *University Library Update*, April 1986.

26.5 Bundling

Firms often choose to sell goods in **bundles**: packages of related goods offered for sale together. A noteworthy example is a bundle of software, sometimes known as a "software suite." Such a bundle might consist of several different software tools—a word processor, a spreadsheet, and a presentation tool—that are sold together in one set. Another example is a magazine: this consists of a bundle of articles that could, in principle, be sold separately. Similarly, magazines are often sold via subscription—which is just a way of bundling separate issues together.

Bundling can be due to cost savings: it is often less expensive to sell several articles stapled together than it is to sell each of them separately. Or it may be due to complementarities among the goods involved: software programs sold in bundles often work together more effectively than off-the-shelf programs.

But there can also be reasons involving consumer behavior. Let's consider a simple example. Suppose that there are two classes of consumers and two different software programs, a word processor and a spreadsheet. Type A consumers are willing to pay $120 for the word processor and $100 for the spreadsheet. Type B consumers have the opposite preferences: they are willing to pay $120 for the spreadsheet and $100 for the word processor. This information is summarized in Table 26.1.

Table 26.1 **Willingness to pay for software components.**

Type of consumer	Word processor	Spreadsheet
Type A consumers	120	100
Type B consumers	100	120

Suppose that you are selling these products. For simplicity, let us assume that the marginal cost is negligible so that you only want to maximize revenue. Furthermore, make the conservative assumption that the willingess to pay for the bundle consisting of the word processor and the spreadsheet is just the sum of the willingesses to pay for each component.

Now consider the profits from two different marketing policies. First, suppose that you sell each item separately. The revenue maximizing policy is to set a price of $100 for each piece of software. If you do this, you will sell two copies of the word processor and two copies of the spreadsheet, and receive a total revenue of $400.

But what if you bundle the items together? In this case, you could sell *each* bundle for $220, and receive a net revenue of $440. The bundling strategy is clearly more attractive!

What is going on in this example? Recall that when you sell an item to several different people, the price is determined by the purchaser who has the *lowest* willingess to pay. The more diverse the valuations of the individuals, the lower the price you have to charge to sell a given number of items. In this case bundling the word processor and the spreadsheet reduces the dispersion of willingess to pay—allowing the monopolist to set a higher price for the bundle of goods.

EXAMPLE: Software Suites

Microsoft, Lotus, and other software manufacturers have taken to bundling much of their applications software. For example, in 1993 Microsoft offered a spreadsheet, word processor, presentation tool, and database as the "Microsoft Office" package at a suggested retail price of $750. (The discounted "street price" was about $450.) If bought separately, the individual software applications would total $1,565! Lotus offered its "Smart Suite" at essentially the same price; its separate components sold for a total of $1,730.

According to an article by Steve Lohr in the October 15, 1993, *New York Times*, 50 percent of Microsoft's applications software was sold in bundles, and generated revenue of over $1 billion a year.

These software suites fit the bundling model well. Tastes for software are often very heterogeneous. Some people use a word processor every day and use a spreadsheet only occasionally. Other people have the reverse pattern of software use. If you wish to sell a spreadsheet to a large number of users, you have to sell it at a price that will be attractive to an occasional user. Similarly with the word processor: it is the willingness to pay of the *marginal* user that sets the market price. By bundling the two products together, the dispersion of willingnesses to pay is reduced and total profits can increase.

This is not to say that bundling is the whole story in software suites; other phenomena are also at work. The individual components of the suites are guaranteed to work well together; they are complementary goods in this respect. Furthermore, the success of a piece of software tends to depend strongly on how many people use it, and bundling software helps to build market share. We will investigate this phenomenon of **network externalities** in Chapter 36.

26.6 Two-Part Tariffs

Consider the pricing problem facing the owners of an amusement park. They can set one price for tickets to get into the park and another price for

the rides. How should they set these two prices if they want to maximize profits? Note that the demand for access and the demand for rides are interrelated: the price that people are willing to pay to get into the park will depend on the price that they have to pay for the rides. This kind of two-part pricing scheme is known as a **two-part tariff**.[5]

Other applications of two-part tariffs abound: Polaroid sells its camera for one price and its film for another. People who are deciding whether or not to purchase the camera presumably consider the price of the film. A company that makes razor blades sells the razor for one price and the blades for another—again the price they set for the blades influences the demand for razors and vice versa.

Let us consider how to solve this pricing problem in the context of the original example: the so-called Disneyland Dilemma. As usual we will make some simplifying assumptions. First, we assume that there is only one kind of ride in Disneyland. Second, we assume that people only desire to go to Disneyland for the rides. Finally, we assume that everyone has the same tastes for rides.

In Figure 26.5 we have depicted the demand curve and the (constant) marginal cost curve for rides. As usual the demand curve slopes down—if Disney sets a high price for each ride, fewer rides will be taken. Suppose that they set a price of p^*, as in Figure 26.5, that leads to a demand for x^* rides. How much will they be able to charge for admission to the park, given that the rides cost p^*?

The total willingness to pay for x^* rides is measured by the consumers' surplus. Hence the most that the owners of the park can charge for admission is the area labeled "consumer's surplus" in Figure 26.5. The total profits to the monopolist is this area plus the profit on the rides, $(p^* - MC)x^*$.

It is not hard to see that total profits are maximized when price equals marginal cost: we've seen before that this price gives the largest possible consumer plus producer surplus. Since the monopolist gets to charge people their consumers' surplus, setting price equal to marginal cost and the entry fee to the resulting consumer's surplus is the profit-maximizing policy.

Indeed, this is the policy that Disneyland, and most other amusement parks follow. There is one price for admission, but then the attractions inside are free. It appears that the marginal cost of the rides is less than the transactions cost of collecting a separate payment for them.

26.7 Monopolistic Competition

We have described a monopolistic industry as being one in which there is a single large producer. But we've been somewhat vague about exactly what

[5] See the classic article by Walter Oi, "A Disneyland Dilemma: Two-Part Tariffs for a Mickey Mouse Monopoly," *Quarterly Journal of Economics,* 85 (1971), 77–96.

Disneyland Dilemma. If the owners of the park set a price of p^*, then x^* rides will be demanded. The consumers' surplus measures the price that they can charge for admission to the park. The total profits of the firm are maximized when the owners set price equal to marginal cost.

Figure 26.5

comprises an industry. One definition of an industry is that it consists of all firms that produce a given product. But then what do we mean by product? After all, there is only one firm that produces Coca-Cola—does that mean that this firm is a monopolist?

Clearly the answer is no. The Coca-Cola firm still has to compete with other producers of soft drinks. We should really think of an industry as being the set of firms that produce products that are viewed as close substitutes by consumers. Each firm in the industry can produce a unique product—a unique brand name, say—but consumers view each of the brands as being substitutes to some degree.

Even though a firm may have a legal monopoly on its trademarks, and brand names, so that other firms can't produce *exactly* the same product, it is usually possible for other firms to produce *similar* products. From the viewpoint of a given firm, the production decisions of its competitors will be a very important consideration in deciding exactly how much it will produce and what price it can charge.

Thus the demand curve facing a firm will usually depend on the output decisions and the prices charged by other firms that produce similar products. The slope of the demand curve facing the firm will depend on

how similar the other firms' products are. If a large number of the firms in the industry produce *identical* products, then the demand curve facing any one of them will be essentially flat. Each firm must sell its product for whatever price the other firms are charging. Any firm that tried to raise its price above the prices of the other firms selling identical products would soon lose all of its customers.

On the other hand, if one firm has the exclusive rights to sell a particular product, then it may be able to raise its price without losing all of its customers. Some, but not all, of its customers may switch to competitors' products. Just how many customers switch depends on how similar the customers think the products are—that is, on the elasticity of the demand curve facing the firm.

If a firm is making a profit selling a product in an industry, and other firms are not allowed to perfectly reproduce that product, they still may find it profitable to enter that industry and produce a similar but distinctive product. Economists refer to this phenomenon as **product differentiation**—each firm attempts to differentiate its product from the other firms in the industry. The more successful it is at differentiating its product from other firms selling similar products, the more monopoly power it has—that is, the less elastic is the demand curve for the product. For example, consider the soft drink industry. In this industry there are a number of firms producing similar, but not identical products. Each product has its following of consumers, and so has some degree of market power.

An industry structure such as that described above shares elements of both competition and monopoly; it is therefore referred to as **monopolistic competition**. The industry structure is monopolistic in that each firm faces a downward-sloping demand curve for its product. It therefore has some market power in the sense that it can set its own price, rather than passively accept the market price as does a competitive firm. On the other hand the firms must compete for customers in terms of both price and the kinds of products they sell. Furthermore, there are no restrictions against new firms entering into a monopolistically competitive industry. In these aspects the industry is like a competitive industry.

Monopolistic competition is probably the most prevalent form of industry structure. Unfortunately, it is also the most difficult form to analyze. The extreme cases of pure monopoly and pure competition are much simpler and can often be used as first approximations to more elaborate models of monopolistic competition. In a detailed model of a monopolistically competitive industry, much depends on the specific details of the products and technology, as well as on the nature of the strategic choices available to firms. It is unreasonable to model a monopolistically competitive industry in the abstract, as we have done with the simpler cases of pure competition and pure monopoly. Rather, the institutional details of the particular industry under consideration must be examined. We will describe some methods that economists use to analyze strategic choice in the next two

chapters, but a detailed study of monopolistic competition will have to wait for more advanced courses.

We can, however, describe an interesting feature of the free entry aspect of monopolistic competition. As more and more firms enter the industry for a particular kind of product, how would we expect the demand curve of an incumbent firm to change? First, we would expect the demand curve to shift inward since we would expect that at each price, it would sell fewer units of output as more firms enter the industry. Second, we would expect that the demand curve facing a given firm would become more elastic as more firms produced more and more similar products. Thus entry into an industry by new firms with similar products will tend to shift the demand curves facing existing firms to the left and make them flatter.

If firms continue to enter the industry as long as they expect to make a profit, equilibrium must satisfy the following three conditions:

1. Each firm is selling at a price and output combination on its demand curve.

2. Each firm is maximizing its profits, given the demand curve facing it.

3. Entry has forced the profits of each firm down to zero.

These facts imply a very particular geometrical relationship between the demand curve and the average cost curve: the demand curve and the average cost curve must be tangent to each other.

The argument is illustrated in Figure 26.6. Fact 1 says that the output and price combination must be somewhere on the demand curve, and fact 3 says that the output and price combination must also be on the average cost curve. Thus the operating position of the firm must be at a point that lies on both curves. Could the demand curve cross the average cost curve? No, because then there would be some point on the demand curve above the average cost curve—but this would be a point yielding *positive* profits.[6] And by fact 2, the zero profit point is a profit maximum.

Another way to see this is to examine what would happen if the firm depicted in Figure 26.6 charged any price other than the break-even price. At any other price, higher or lower, the firm would lose money, while at the break-even price, the firm makes zero profits. Thus the break-even price is the profit-maximizing price.

There are two worthwhile observations about the monopolistically competitive equilibrium. First, although profits are zero, the situation is still Pareto inefficient. Profits have nothing to do with the efficiency question: when price is greater than marginal cost, there is an efficiency argument for expanding output.

[6] If $p > c(y)/y$, then simple algebra shows that $py - c(y) > 0$.

Figure 26.6 **Monopolistic competition**. In a monopolistically competitive equilibrium with zero profits, the demand curve and the average cost curve must be tangent.

Second, it is clear that firms will typically be operating to the left of the level of output where average cost is minimized. This has sometimes been interpreted as saying that in monopolistic competition there is "excess capacity." If there were fewer firms, each could operate at a more efficient scale of operation, which would be better for consumers. However, if there were fewer firms there would also be less product variety, and this would tend to make consumers worse off. Which of these effects dominates is a difficult question to answer.

26.8 A Location Model of Product Differentiation

In Atlantic City there is a boardwalk that stretches along the beach. Some ice cream vendors with pushcarts want to sell ice cream on the boardwalk. If one vendor is going to be given the concession to sell ice cream on the boardwalk, where should he locate?[7]

Suppose that consumers are distributed evenly along the beach. From a social point of view, it makes sense to locate the ice cream vendor so that

[7] The discussion here is based on the classic model of Harold Hotelling, "Stability in Competition," *Economic Journal*, March 1929.

the total distance walked by all the consumers is minimized. It is not hard to see that this optimal location is halfway along the boardwalk.

Now suppose that two ice cream vendors are allowed. Suppose that we fix the price that they are able to charge for their ice cream and just ask where they should locate in order to minimize the total distance walked. If each consumer walks to the ice cream vendor nearest him, we should put one vendor a quarter of the way along the boardwalk and one vendor three-quarters of the way along the boardwalk. The consumer halfway along the boardwalk will be indifferent between the two ice cream vendors; each has a market share of one-half of the consumers. (See Figure 26.7A.)

But do the ice cream vendors have an incentive to stay in these locations? Put yourself in the position of vendor L. If you move a little bit to the right, you will steal some of the other vendor's customers and you won't lose any of your own. By moving to the right, you will still be the closest vendor to all the customers to your left and you will still be closer to the customers on your right. You will therefore increase your market share and your profits.

Figure 26.7

Competition in location. Panel A shows the socially optimal location pattern; L locates one-quarter of the way along the line and R locates three-quarters of the way along. But each vendor will find it in its private interest to move toward the middle. The only equilibrium location is for both vendors to be in the middle, as shown in Panel B.

But vendor R can reason the same way—by moving to the left, he will steal some of the other vendor's customers and not lose any of his own! This shows that the socially optimal location patterns are not an equi-

librium. The only equilibrium is for both vendors to sell in the middle of the boardwalk, as shown in Figure 26.7B. In this case, competition for customers has resulted in an *inefficient* location pattern.

The boardwalk model can serve as a metaphor for other sorts of product-differentiation problems. Instead of the boardwalk, think of the choice of music varieties by two radio stations. At one extreme we have classical music and at the other we have heavy metal rock. Each listener chooses the station that appeals more to his tastes. If the classical station plays music that is a bit more toward the middle of the taste spectrum, it won't lose the classical clients, but it will gain a few of the middlebrow listeners. If the rock station moves a bit toward the middle, it won't lose any of its rock lovers but will get a few of the middlebrow listeners. In equilibrium, both stations play the same sort of music and the people with more extreme tastes are unhappy with both of them!

26.9 Product Differentiation

The boardwalk model suggest that monopolistic competition will result in too little product differentiation: each firm will want to make its product similar to that of the other firm in order to steal the other firm's customers. Indeed, we can think of markets in which there is too much imitation relative to what seems to be optimal.

However, it doesn't always work this way. Suppose that the boardwalk is *very* long. Then each ice cream vendor would be perfectly happy sitting near each end of the boardwalk. If their market areas don't overlap, nothing is to be gained from moving closer to the middle of the boardwalk. In this case, neither monopolist has an incentive to imitate the other, and the products are about as different as they can get.

It is possible to produce models of monopolistic competition where there is *excessive* product differentiation. In such models, each firm attempts to make consumers think that its product is different from the products of its competitors so as to create some degree of market power. If the firms succeed in convincing the consumers that their product has no close substitutes, they will be able to charge a higher price for it than they would otherwise be able to do.

This leads each producer to invest heavily in creating a distinctive brand identity. Laundry soap, for example, is a pretty standardized commodity. Yet manufacturers invest huge amounts of money in advertisements that claim cleaner clothes, better smell, a better marriage, and and a generally happier life if you choose their brand rather than a competitor's. This "product positioning" is much like the ice cream vendors locating far away from each other in order to avoid head-to-head competition.

There are critics who have argued that such excessive investment in product positioning is wasteful. Perhaps this is true in some cases, but then

again, "excessive variety" may simpley be a consequence of encouraging firms to provide consumers with a variety of products from which to choose.

26.10 More Vendors

We have shown that if there are two vendors whose market areas overlap, and each seller sells the same price, they will both end up located at the "middle" of the boardwalk. What happens if there are more than two vendors who compete in their location?

The next easiest case is that of three vendors. This case gives rise to a rather peculiar outcome: there may be *no* equilibrium location pattern! To see this, look at Figure 26.8. If there are three vendors located on the boardwalk, there must be one located between the other two. As before, it pays each of the "outside" vendors to move towards the middle vendor since they can steal some of its customers without losing any of their own. But if they get *too* close to the other vendor, it pays it to jump immediately to the right of its right-hand competitor or immediately to the left of its left-hand competitor to steal *its* market. No matter what the location pattern, it pays someone to move!

No equilibrium. There is no pure strategy equilibrium in the Hotelling model with 3 firms since for any configuration, at least one firm wants to change location.

Figure 26.8

Luckily, this "perverse" result only holds in the case of three competitors. If there are four or more competitors, an equilibrium location pattern will generally emerge.

Summary

1. There will typically be an incentive for a monopolist to engage in price discrimination of some sort.

2. Perfect price discrimination involves charging each customer a different take-it-or-leave-it price. This will result in an efficient level of output.

3. If a firm can charge different prices in two different markets, it will tend to charge the lower price in the market with the more elastic demand.

4. If a firm can set a two-part tariff, and consumers are identical, then it will generally want to set price equal to marginal cost and make all of its profits from the entry fee.

5. The industry structure known as monopolistic competition refers to a situation in which there is product differentiation, so each firm has some degree of monopoly power, but there is also free entry so that profits are driven to zero.

6. Monopolistic competition can result in too much or too little product differentiation in general.

REVIEW QUESTIONS

1. Will a monopoly ever provide a Pareto efficient level of output on its own?

2. Suppose that a monopolist sells to two groups that have constant elasticity demand curves, with elasticity ϵ_1 and ϵ_2. The marginal cost of production is constant at c. What price is charged to each group?

3. Suppose that the amusement park owner can practice perfect first-degree price discrimination by charging a different price for each ride. Assume that all rides have zero marginal cost and all consumers have the same tastes. Will the monopolist do better charging for rides and setting a zero price for admission, or better by charging for admission and setting a zero price for rides?

4. Disneyland also offers a discount on admissions to residents of Southern California. (You show them your zip code at the gate.) What kind of price discrimination is this? What does this imply about the elasticity of demand for Disney attractions by Southern Californians?

CHAPTER 27

FACTOR MARKETS

In our examination of factor demands in Chapter 20 we only considered the case of a firm that faced a competitive output market and a competitive factor market. Now that we have studied monopoly behavior, we can examine some alternative specifications of factor demand behavior. For example, what happens to factor demands if a firm behaves as a monopolist in its output market? Or what happens to factor demands if a firm is the sole demander for the use of some factors? We investigate these questions and some related questions in this chapter.

27.1 Monopoly in the Output Market

When a firm determines its profit-maximizing demand for a factor, it will always want to choose a quantity such that the marginal revenue from hiring a little more of that factor just equals the marginal cost of doing so. This follows from the standard logic: if the marginal revenue of some action didn't equal the marginal cost of that action, then it would pay for the firm to change the action.

This general rule takes various special forms depending on our assumptions about the environment in which the firm operates. For example, suppose that the firm has a monopoly for its output. For simplicity we will suppose that there is only one factor of production and write the production function as $y = f(x)$. The revenue that the firm receives depends on its production of output so we write $R(y) = p(y)y$, where $p(y)$ is the inverse demand function. Let us see how a marginal increase in the amount of the input affects the revenues of the firm.

Suppose that we increase the amount of the input a little bit, Δx. This will result in a small increase in output, Δy. The ratio of the increase in output to the increase in the input is the **marginal product** of the factor:

$$MP_x = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{27.1}$$

This increase in output will cause revenue to change. The change in revenue is called the **marginal revenue**.

$$MR_y = \frac{\Delta R}{\Delta y} = \frac{R(y + \Delta y) - R(y)}{\Delta y}. \tag{27.2}$$

The effect on revenue due to the marginal increase in the input is called the **marginal revenue product**. Examining equations (27.1) and (27.2) we see that it is given by

$$\begin{aligned} MRP_x &= \frac{\Delta R}{\Delta x} = \frac{\Delta R}{\Delta y}\frac{\Delta y}{\Delta x} \\ &= MR_y \times MP_x. \end{aligned}$$

We can use our standard expression for marginal revenue to write this as

$$\begin{aligned} MRP_x &= \left[p(y) + \frac{\Delta p}{\Delta y} y\right] MP_x \\ &= p(y)\left[1 + \frac{1}{\epsilon}\right] MP_x \\ &= p(y)\left[1 - \frac{1}{|\epsilon|}\right] MP_x. \end{aligned}$$

The first expression is the usual expression for marginal revenue. The second and third expressions use the elasticity form of marginal revenue, which was discussed in Chapter 15.

Now it is easy to see how this generalizes the competitive case we examined earlier in Chapter 20. The elasticity of the demand curve facing an individual firm in a competitive market is infinite; consequently the marginal revenue for a competitive firm is just equal to price. Hence the

"marginal revenue product" of an input for a firm in a competitive market is just the **value of the marginal product** of that input, pMP_x.

How does the marginal revenue product (in the case of a monopoly) compare to the value of the marginal product? Since the demand curve has a negative slope, we see that the marginal revenue product will always be less than the value of the marginal product:

$$MRP_x = p\left[1 - \frac{1}{|\epsilon|}\right] MP_x \leq pMP_x.$$

As long as the demand function is not perfectly elastic, the MRP_x will be strictly less than pMP_x. This means that at any level of employment of the factor, the marginal value of an additional unit is less for a monopolist than for a competitive firm. In the rest of this section we will assume that we are dealing with this case—the case where the monopolist actually has some monopoly power.

At first encounter this statement seems paradoxical since a monopolist makes higher profits than a competitive firm. In this sense the total factor input is "worth more" to a monopolist than to a competitive firm.

The resolution of this "paradox" is to note the difference between total value and marginal value. The total amount employed of the factor is indeed worth more to the monopolist than to the competitive firm since the monopolist will make more profits from the factor than the competitive firm. However, at a *given* level of output an increase in the employment of the factor will increase output and *reduce* the price that a monopolist is able to charge. But an increase in a competitive firm's output will not change the price it can charge. Thus on the margin, a small *increase* in the employment of the factor is worth less to the monopolist than to the competitive firm.

Since increases in the factor employment are worth less to a monopolist than to a competitive firm on the margin in the short run, it makes sense that the monopolist would usually want to employ less of the input. Indeed this is generally true: the monopolist increases its profits by reducing its output, and so it will usually hire lower amounts of inputs than a competitive firm.

In order to determine how much of the factor a firm employs, we have to compare the marginal revenue of an additional unit of the factor to the marginal cost of hiring that factor. Let us assume that the firm operates in a competitive factor market, so that it can hire as much of the factor as it wants at a constant price of w. In this case, the competitive firm wants to hire x_c units of the factor, where

$$pMP(x_c) = w.$$

The monopolist, on the other hand, wants to hire x_m units of the factor, where

$$MRP(x_m) = w.$$

We have illustrated this in Figure 27.1. Since $MRP(x) < pMP(x)$, the point where $MRP(x_m) = w$ will always be to the left of the point where $pMP(x_c) = w$. Hence the monopolist will hire less than the competitive firm.

Figure 27.1 **Factor demand by a monopolist**. Since the marginal revenue product curve (MRP) lies beneath the curve measuring the value of the marginal product (pMP), the factor demand by a monopolist must be less than the factor demand by the same firm if it behaves competitively.

27.2 Monopsony

In a monopoly there is a single seller of a commodity. In a **monopsony** there is a single buyer. The analysis of a monopsonist is similar to that of a monopolist. For simplicity, we suppose that the buyer produces output that will be sold in a competitive market.

As above, we will suppose that the firm produces output using a single factor according to the production function $y = f(x)$. However, unlike the discussion above, we suppose that the firm dominates the factor market in which it operates and recognizes the amount of the factor that it demands will influence the price that it has to pay for this factor.

We summarize this relationship by the (inverse) supply curve $w(x)$. The interpretation of this function is that if the firm wants to hire x units of the factor it must pay a price of $w(x)$. We assume that $w(x)$ is an increasing function: the more of the x-factor the firm wants to employ, the higher must be the factor price it offers.

A firm in a competitive factor market by definition faces a flat factor supply curve: it can hire as much as it wants at the going factor price. A monopsonist faces an upward-sloping factor supply curve: the more it wants to hire, the higher a factor price it must offer. A firm in a competitive factor market is a **price taker**. A monopsonist is a **price maker**.

The profit-maximization problem facing the monopsonist is

$$\max_x \; pf(x) - w(x)x.$$

The condition for profit maximization is that the marginal revenue from hiring an extra unit of the factor should equal the marginal cost of that unit. Since we have assumed a competitive output market the marginal revenue is simply pMP_x. What about the marginal cost?

The total change in costs from hiring Δx more of the factor will be

$$\Delta c = w\Delta x + x\Delta w,$$

so that the change in costs per unit change in Δx is

$$\frac{\Delta c}{\Delta x} = MC_x = w + \frac{\Delta w}{\Delta x}x.$$

The interpretation of this expression is similar to the interpretation of the marginal revenue expression: when the firm increases its employment of the factor it has to pay $w\Delta x$ more in payment to the factor. But the increased demand for the factor will push the factor price up by Δw, and the firm has to pay this higher price on all of the units it was previously employing.

We can also write the marginal cost of hiring additional units of the factor as

$$\begin{aligned} MC_x &= w\left[1 + \frac{x}{w}\frac{\Delta w}{\Delta x}\right] \\ &= w\left[1 + \frac{1}{\eta}\right] \end{aligned}$$

where η is the *supply* elasticity of the factor. Since supply curves typically slope upward, η will be a positive number. If the supply curve is *perfectly* elastic, so that η is infinite, this reduces to the case of a firm facing a competitive factor market. Note the similarity of these observations with the analogous case of a monopolist.

Let's analyze the case of a monopsonist facing a linear supply curve for the factor. The inverse supply curve has the form

$$w(x) = a + bx,$$

so that total costs have the form

$$C(x) = w(x)x = ax + bx^2,$$

and thus the marginal cost of an additional unit of the input is

$$MC_x(x) = a + 2bx.$$

The construction of the monopsony solution is given in Figure 27.2. We find the position where the value of the marginal product equals marginal cost to determine x^* and then see what the factor price must be at that point.

Figure 27.2

Monopsony. The firm operates where the marginal revenue from hiring an extra unit of the factor equals the marginal cost of that extra unit.

Since the marginal cost of hiring an extra unit of the factor exceeds the factor price, the factor price will be lower than if the firm had faced a competitive factor market. Too little of the factor will be hired relative to the competitive market. Just as in the case of the monopoly, a monopsonist operates at a Pareto inefficient point. But the inefficiency now lies in the factor market rather than in the output market.

EXAMPLE: The Minimum Wage

Suppose that the labor market is competitive and that the government sets a minimum wage that is higher than the prevailing equilibrium wage. Since demand equals supply at the equilibrium wage, the supply of labor will exceed the demand for labor at the higher minimum wage. This is depicted in Figure 27.3A.

Figure 27.3

Minimum wage. Panel A shows the effect of a minimum wage in a competitive labor market. At the competitive wage, w_c, employment would be L_c. At the minimum wage, $\overline{w}$, employment is only L_{mw}. Panel B shows the effect of a minimum wage in a monopsonized labor market. Under monopsony, the wage is w_m and employment is L_m, which is less than the employment in the competitive labor market. If the minimum wage is set to w_c, employment will increase to L_c.

Things are very different if the labor market is dominated by a monopsonist. In this case, it is possible that imposing a minimum wage may actually *increase* employment. This is depicted in Figure 27.3B. If the government sets the minimum wage equal to the wage that would prevail in a competitive market, the "monopsonist" now perceives that it can hire workers at a constant wage of w_c. Since the wage rate it faces is now independent of how many workers it hires, it will hire until the value of the marginal product equals w_c. That is, it will hire just as many workers as if it faced a competitive labor market.

Setting a wage floor for a monopsonist is just like setting a price ceiling for a monopolist; each policy makes the firm behave as though it faced a competitive market.

27.3 Upstream and Downstream Monopolies

We have now examined two cases involving imperfect competition and factor markets: the case of a firm with a monopoly in the output market but facing a competitive factor market, and the case of a firm with a competitive output market that faces a monopolized factor market. Other variations are possible. The firm could face a monopoly seller in its factor market for example. Or it could face a monopsony buyer in its output market. It doesn't make much sense to plod through each possible case; they quickly become repetitive. However, we will examine one interesting market structure in which a monopoly produces output that is used as a factor of production by another monopolist.

Suppose then that one monopolist produces output x at a constant marginal cost of c. We call this monopolist the **upstream monopolist**. It sells the x-factor to another monopolist, the **downstream monopolist** at a price of k. The downstream monopolist uses the x-factor to produce output y according to the production function $y = f(x)$. This output is then sold in a monopolist market in which the inverse demand curve is $p(y)$. For purposes of this example, we consider a linear inverse demand curve $p(y) = a - by$.

To make things simple, think of the production function as just being $y = x$, so that for each unit of the x-input, the monopolist can produce one unit of the y-output. We further suppose that the downstream monopolist has no costs of production other than the unit price k that it must pay to the upstream monopolist.

In order to see how this market works, start with the downstream monopolist. Its profit-maximization problem is

$$\max_y \; p(y)y - ky = [a - by]y - ky.$$

Setting marginal revenue equal to marginal cost, we have

$$a - 2by = k,$$

which implies that

$$y = \frac{a-k}{2b}.$$

Since the monopolist demands one unit of the x-input for each y-output that it produces, this expression also determines the factor demand function

$$x = \frac{a-k}{2b}. \tag{27.3}$$

This function tells us the relationship between the factor price k and the amount of the factor that the downstream monopolist will demand.

Turn now to the problem of the upstream monopolist. Presumably it understands this process and can determine how much of the x-good it will sell if it sets various prices k; this is simply the factor demand function given in equation (27.3). The upstream monopolist wants to choose x to maximize its profit.

We can determine this level easily enough. Solving equation (27.3) for k as a function of x we have

$$k = a - 2bx.$$

The marginal revenue associated with this factor demand function is

$$MR = a - 4bx.$$

Setting marginal revenue equal to marginal cost we have

$$a - 4bx = c,$$

or

$$x = \frac{a - c}{4b}.$$

Since the production function is simply $y = x$, this also gives us the total amount of the final product that is produced:

$$y = \frac{a - c}{4b}. \tag{27.4}$$

It is of interest to compare this to the amount that would be produced by a single integrated monopolist. Suppose that the upstream and the downstream firms merged so that we had one monopolist who faced an output inverse demand function $p = a - by$ and faced a constant marginal cost of c per unit produced. The marginal revenue equals marginal cost equation is

$$a - 2by = c,$$

which implies that the profit-maximizing output is

$$y = \frac{a - c}{2b}. \tag{27.5}$$

Comparing equation (27.4) to equation (27.5) we see that the integrated monopolist produces *twice* as much output as the nonintegrated monopolists.

This is depicted in Figure 27.4. The final demand curve facing the downstream monopolist $p(y)$, and the marginal revenue curve associated with this demand function is itself the demand function facing the upstream monopolist. The marginal revenue curve associated with this demand function

Figure 27.4 **Upstream and downstream monopoly**. The downstream monopolist faces the (inverse) demand curve $p(y)$. The marginal revenue associated with this demand curve is $MR_D(y)$. This in turn is the demand curve facing the upstream monopolist, and the associated marginal revenue curve is $MR_U(y)$. The integrated monopolist produces at y_i^*; the nonintegrated monopolist produces at y_m^*.

is therefore *four* times as steep as the final demand curve—which is why the output in this market is half what it would be in the integrated market.

Of course the fact that the final marginal revenue curve is exactly four times as steep is particular to the linear demand case. However, it is not hard to see that an integrated monopolist will always produce more than an upstream-downstream pair of monopolists. In the latter case the upstream monopolist raises its price above its marginal cost and then the downstream monopolist raises its price above this already marked-up cost. There is a **double markup**. The price is not only too high from a social point of view, it is too high from the viewpoint of maximizing total monopoly profits! If the two monopolists merged, price would go down and profits would go up.

Summary

1. A profit-maximizing firm always wants to set the marginal revenue of each action it takes equal to the marginal cost of that action.

2. In the case of a monopolist, the marginal revenue associated with an

increase in the employment of a factor is called the marginal revenue product.

3. For a monopolist, the marginal revenue product will always be smaller than the value of the marginal product due to the fact that the marginal revenue from increasing output is always less than price.

4. Just as a monopoly consists of a market with a single seller, a monopsony consists of a market with a single buyer.

5. For a monopsonist the marginal cost curve associated with a factor will be steeper than the supply curve of that factor.

6. Hence a monopsonist will hire an inefficiently small amount of the factor of production.

7. If an upstream monopolist sells a factor to a downstream monopolist, then the final price of output will be too high due to the double markup phenomenon.

REVIEW QUESTIONS

1. We saw that a monopolist never produced where the demand for output was inelastic. Will a monopsonist produce where a factor is inelastically supplied?

2. In our example of the minimum wage, what would happen if the labor market was dominated by a monopsonist and the government set a wage that was above the competitive wage?

3. In our examination of the upstream and downstream monopolists we derived expressions for the total output produced. What are the appropriate expressions for the equilibrium prices, p and k?

APPENDIX

We can calculate marginal revenue product by using the chain rule. Let $y = f(x)$ be the production function and $p(y)$ be the inverse demand function. Revenue as a function of the factor employment is just

$$R(x) = p(f(x))f(x).$$

Differentiating this expression with respect to x we have

$$\begin{aligned}\frac{dR(x)}{dx} &= p(y)f'(x) + f(x)p'(y)f'(x)\\ &= [p(y) + p'(y)y]f'(x)\\ &= MR \times MP.\end{aligned}$$

Let us examine the behavior of a firm that is a competitor in its output market and a monopsonist in its factor market. Letting $w(x)$ be the inverse factor supply function, the profit-maximization problem is

$$\max_x \; pf(x) - w(x)x.$$

Differentiating with respect to x, we have

$$pf'(x) = w(x) + w'(x)x = w(x)\left[1 + \frac{x}{w}\frac{dw}{dx}\right] = w(x)\left[1 + \frac{1}{\eta}\right].$$

Since the factor supply curve slopes upward, the right-hand side of this expression will be larger than w. Hence the monopsonist will choose to employ less of the factor than would a firm that behaves competitively in the factor market.

CHAPTER 28

OLIGOPOLY

We have now investigated two important forms of market structure: pure competition, where there are typically many small competitors, and pure monopoly, where there is only one large firm in the market. However, much of the world lies between these two extremes. Often there are a number of competitors in the market, but not so many as to regard each of them as having a negligible effect on price. This is the situation known as **oligopoly**.

The model of monopolistic competition described in Chapter 25 is a special form of oligopoly that emphasizes issues of product differentiation and entry. However, the models of oligopoly that we will study in this chapter are more concerned with the strategic interactions that arise in an industry with a small number of firms.

There are several models that are relevant since there are several different ways for firms to behave in an oligopolistic environment. It is unreasonable to expect one grand model since many different behavior patterns can be observed in the real world. What we want is a guide to some of the possible patterns of behavior and some indication of what factors might be important in deciding when the various models are applicable.

For simplicity, we will usually restrict ourselves to the case of two firms; this is called a situation of **duopoly**. The duopoly case allows us to capture many of the important features of firms engaged in strategic interaction without the notational complications involved in models with a larger number of firms. Also, we will limit ourselves to investigation of cases in which each firm is producing an identical product. This allows us to avoid the problems of product differentiation and focus only on strategic interactions.

28.1 Choosing a Strategy

If there are two firms in the market and they are producing a homogeneous product, then there are four variables of interest: the price that each firm charges and the quantities that each firm produces.

When one firm decides about its choices for prices and quantities it may already know the choices made by the other firm. If one firm gets to set its price before the other firm, we call it the **price leader** and the other firm the **price follower**. Similarly, one firm may get to choose its quantity first, in which case it is a **quantity leader** and the other is a **quantity follower**. The strategic interactions in these cases form a **sequential game**.[1]

On the other hand, it may be that when one firm makes its choices it doesn't know the choices made by the other firm. In this case, it has to guess about the other firm's choice in order to make a sensible decision itself. This is a **simultaneous game**. Again there are two possibilities: the firms could each simultaneously choose prices or each simultaneously choose quantities.

This classification scheme gives us four possibilities: quantity leadership, price leadership, simultaneous quantity setting, and simultaneous price setting. Each of these types of interaction gives rise to a different set of strategic issues.

There is also another possible form of interaction that we will examine. Instead of the firms competing against each other in one form or another they may be able to **collude**. In this case the two firms can jointly agree to set prices and quantities that maximize the sum of their profits. This sort of collusion is called a **cooperative game**.

EXAMPLE: Pricing Matching

It is common to see advertisements where the vendor offers to "meet or beat" any price. These are generally considered to be a sign of intensely

[1] We will examine game theory in more detail in the next chapter, but it seems appropriate to introduce these specific examples here.

competitive market. However, such offers can also be used as a way to dampen competition.

Suppose there are two tire stores, East Side Tires and West Side Tires, that are advertising the same brand tire for $50.

If East Side Tires cuts its advertised price to $45 while the West Side price stays at $50, we would expect that some of those customers on the west side of town would be willing to travel a few extra minutes in order to save $5. East Side Tires would then sell more tires at a lower price. If the increase in sales was large enough to overcome the price reduction, its profits would increase.

That, in a nutshell, is the basic logic of competition: if customers are sufficiently sensitive to price, then a seller that cuts its price enjoys a surge in sales and an increase in profit.

But instead of actually cutting its price, suppose instead that West Side Tires continued to charge $50 but added a promise to match any lower price. What happens now if East Side cuts its advertised price?

In this case, those who find West Side Tires more convenient can just bring in the East Side ad and get the discounted price. Then, East Side Tires attracts no new customers from its price cut. In fact, it loses revenue since it sells essentially the same number of tires at a lower price.

The moral: a vendor that offers a low-price guarantee takes away much of its competitors' motivation for cutting prices.

28.2 Quantity Leadership

In the case of quantity leadership, one firm makes a choice before the other firm. This is sometimes called the **Stackelberg model** in honor of the first economist who systematically studied leader-follower interactions.[2]

The Stackelberg model is often used to describe industries in which there is a dominant firm, or a natural leader. For example, IBM is often considered to be a dominant firm in the computer industry. A commonly observed pattern of behavior is for smaller firms in the computer industry to wait for IBM's announcements of new products and then adjust their own product decisions accordingly. In this case we might want to model the computer industry with IBM playing the role of a Stackelberg leader, and the other firms in the industry being Stackelberg followers.

Let us turn now to the details of the theoretical model. Suppose that firm 1 is the leader and that it chooses to produce a quantity y_1. Firm 2 responds by choosing a quantity y_2. Each firm knows that the equilibrium price in the market depends on the total output produced. We use the

[2] Heinrich von Stackelberg was a German economist who published his influential work on market organization, *Marktform und Gleichgewicht*, in 1934.

inverse demand function $p(Y)$ to indicate the equilibrium price as a function of industry output, $Y = y_1 + y_2$.

What output should the leader choose to maximize its profits? The answer depends on how the leader thinks that the follower will react to its choice. Presumably the leader should expect that the follower will attempt to maximize profits as well, given the choice made by the leader. In order for the leader to make a sensible decision about its own production, it has to consider the follower's profit-maximization problem.

The Follower's Problem

We assume that the follower wants to maximize its profits

$$\max_{y_2} \; p(y_1 + y_2)y_2 - c_2(y_2).$$

The follower's profit depends on the output choice of the leader, but from the viewpoint of the follower the leader's output is predetermined—the production by the leader has already been made, and the follower simply views it as a constant.

The follower wants to choose an output level such that marginal revenue equals marginal cost:

$$MR_2 = p(y_1 + y_2) + \frac{\Delta p}{\Delta y_2} y_2 = MC_2.$$

The marginal revenue has the usual interpretation. When the follower increases its output, it increases its revenue by selling more output at the market price. But it also pushes the price down by Δp, and this lowers its profits on all the units that were previously sold at the higher price.

The important thing to observe is that the profit-maximizing choice of the follower will depend on the choice made by the leader. We write this relationship as

$$y_2 = f_2(y_1).$$

The function $f_2(y_1)$ tells us the profit-maximizing output of the follower as a function of the leader's choice. This function is called the **reaction function** since it tells us how the follower will react to the leader's choice of output.

Let's derive a reaction curve in the simple case of linear demand. In this case the (inverse) demand function takes the form $p(y_1+y_2) = a-b(y_1+y_2)$. For convenience we'll take costs to be zero.

Then the profit function for firm 2 is

$$\pi_2(y_1, y_2) = [a - b(y_1 + y_2)]y_2$$

or

$$\pi_2(y_1, y_2) = ay_2 - by_1y_2 - by_2^2.$$

We can use this expression to draw the **isoprofit lines** in Figure 28.1. These are lines depicting those combinations of y_1 and y_2 that yield a constant level of profit to firm 2. That is, the isoprofit lines are comprised of all points (y_1, y_2) that satisfy equations of the form

$$ay_2 - by_1y_2 - by_2^2 = \overline{\pi}_2.$$

Note that profits to firm 2 will increase as we move to isoprofit lines that are further to the left. This is true since if we fix the output of firm 2 at some level, firm 2's profits will increase as firm 1's output decreases. Firm 2 will make its maximum possible profits when it is a monopolist; that is, when firm 1 chooses to produce zero units of output.

Figure 28.1

Derivation of a reaction curve. This reaction curve gives the profit-maximizing output for the follower, firm 2, for each output choice of the leader, firm 1. For each choice of y_1 the follower chooses the output level $f_2(y_1)$ associated with the isoprofit line farthest to the left.

For each possible choice of firm 1's output, firm 2 wants to choose its own output to make its profits as large as possible. This means that for each

choice of y_1, firm 2 will pick the value of y_2 that puts it on the isoprofit line furthest to the left, as illustrated in Figure 28.1. This point will satisfy the usual sort of tangency condition: the slope of the isoprofit line must be vertical at the optimal choice. The locus of these tangencies describes firm 2's reaction curve, $f_2(y_1)$.

To see this result algebraically, we need an expression for the marginal revenue associated with the profit function for firm 2. It turns out that this expression is given by

$$MR_2(y_1, y_2) = a - by_1 - 2by_2.$$

(This is easy to derive using calculus. If you don't know calculus, you'll just have to take this statement on faith.) Setting the marginal revenue equal to marginal cost, which is zero in this example, we have

$$a - by_1 - 2by_2 = 0,$$

which we can solve to derive firm 2's reaction curve:

$$y_2 = \frac{a - by_1}{2b}.$$

This reaction curve is the straight line depicted in Figure 28.1.

The Leader's Problem

We have now examined how the follower will choose its output *given* the choice of the leader. We turn now to the leader's profit-maximization problem.

Presumably, the leader is also aware that its actions influence the output choice of the follower. This relationship is summarized by the reaction function $f_2(y_1)$. Hence when making its output choice it should recognize the influence that it exerts on the follower.

The profit-maximization problem for the leader therefore becomes

$$\begin{aligned} &\max_{y_1} \; p(y_1 + y_2)y_1 - c_1(y_1) \\ &\text{such that } y_2 = f_2(y_1). \end{aligned}$$

Substituting the second equation into the first gives us

$$\max_{y_1} \; p[y_1 + f_2(y_1)]y_1 - c_1(y_1).$$

Note that the leader recognizes that when it chooses output y_1, the total output produced will be $y_1 + f_2(y_1)$: its own output *plus* the output produced by the follower.

When the leader contemplates changing its output it has to recognize the influence it exerts on the follower. Let's examine this in the context of the linear demand curve described above. There we saw that the reaction function was given by

$$f_2(y_1) = y_2 = \frac{a - by_1}{2b}. \tag{28.1}$$

Since we've assumed that marginal costs are zero, the leader's profits are

$$\pi_1(y_1, y_2) = p(y_1 + y_2)y_1 = ay_1 - by_1^2 - by_1y_2. \tag{28.2}$$

But the output of the follower, y_2, will depend on the leader's choice via the reaction function $y_2 = f_2(y_1)$.

Substituting from equation (28.1) into equation (28.2) we have

$$\begin{aligned}\pi_1(y_1, y_2) &= ay_1 - by_1^2 - by_1 f_2(y_1)\\ &= ay_1 - by_1^2 - by_1\frac{a - by_1}{2b}.\end{aligned}$$

Simplifying this expression gives us

$$\pi_1(y_1, y_2) = \frac{a}{2}y_1 - \frac{b}{2}y_1^2.$$

The marginal revenue for this function is

$$MR = \frac{a}{2} - by_1.$$

Setting this equal to marginal cost, which is zero in this example, and solving for y_1 gives us

$$y_1^* = \frac{a}{2b}.$$

In order to find the follower's output, we simply substitute y_1^* into the reaction function,

$$\begin{aligned}y_2^* &= \frac{a - by_1^*}{2b}\\ &= \frac{a}{4b}.\end{aligned}$$

These two equations give a total industry output of $y_1^* + y_2^* = 3a/4b$.

The Stackelberg solution can also be illustrated graphically using the isoprofit curves depicted in Figure 28.2. (This figure also illustrates the Cournot equilibrium which will be described in section 28.5.) Here we have illustrated the reaction curves for both firms and the isoprofit curves for firm 1. The isoprofit curves for firm 1 have the same general shape as the isoprofit curves for firm 2; they are simply rotated 90 degrees. Higher

Figure 28.2 **Stackelberg equilibrium**. Firm 1, the leader, chooses the point on firm 2's reaction curve that touches firm 1's lowest possible isoprofit line, thus yielding the highest possible profits for firm 1.

profits for firm 1 are associated with isoprofit curves that are lower down since firm 1's profits will increase as firm 2's output decreases.

Firm 2 is behaving as a follower, which means that it will choose an output along its reaction curve, $f_2(y_1)$. Thus firm 1 wants to choose an output combination on the reaction curve that gives it the highest possible profits. But the highest possible profits means picking that point on the reaction curve that touches the *lowest* isoprofit line, as illustrated in Figure 28.2. It follows by the usual logic of maximization that the reaction curve must be tangent to the isoprofit curve at this point.

28.3 Price Leadership

Instead of setting quantity, the leader may instead set price. In order to make a sensible decision about how to set its price, the leader must forecast how the follower will behave. Accordingly, we must first investigate the profit-maximization problem facing the follower.

The first thing we observe is that in equilibrium the follower must always set the same price as the leader. This follows from our assumption that the two firms are selling identical products. If one charged a different price from

the other, all of the consumers would prefer the producer with the lower price, and we couldn't have an equilibrium with both firms producing.

Suppose that the leader has set a price p. We will suppose that the follower takes this price as given and chooses its profit-maximizing output. This is essentially the same as the competitive behavior we investigated earlier. In the competitive model, each firm takes the price as being outside of its control because it is such a small part of the market; in the price-leadership model, the follower takes the price as being outside of its control since it has already been set by the leader.

The follower wants to maximize profits:

$$\max_{y_2} \; py_2 - c_2(y_2).$$

This leads to the familiar condition that the follower will want to choose an output level where price equals marginal cost. This determines a supply curve for the follower, $S(p)$, which we have illustrated in Figure 28.3.

Price leader. The demand curve facing the leader is the market demand curve minus the follower's supply curve. The leader equates marginal revenue and marginal cost to find the optimal quantity to supply, y_L^*. The total amount supplied to the market is y_T^* and the equilibrium price is p^*.

Figure 28.3

Turn now to the problem facing the leader. It realizes that if it sets a price p, the follower will supply $S(p)$. That means that the amount of

output the leader will sell will be $R(p) = D(p) - S(p)$. This is called the **residual demand curve** facing the leader.

Suppose that the leader has a constant marginal cost of production c. Then the profits that it achieves for any price p are given by:

$$\pi_1(p) = (p - c)[D(p) - S(p)] = (p - c)R(p).$$

In order to maximize profits the leader wants to choose a price and output combination where marginal revenue equals marginal cost. However, the marginal revenue should be the marginal revenue for the *residual* demand curve—the curve that actually measures how much output it will be able to sell at each given price. In Figure 28.3 the residual demand curve is linear; therefore the marginal revenue curve associated with it will have the same vertical intercept and be twice as steep.

Let's look at a simple algebraic example. Suppose that the inverse demand curve is $D(p) = a - bp$. The follower has a cost function $c_2(y_2) = y_2^2/2$, and the leader has a cost function $c_1(y_1) = cy_1$.

For any price p the follower wants to operate where price equals marginal cost. If the cost function is $c_2(y_2) = y_2^2/2$, it can be shown that the marginal cost curve is $MC_2(y_2) = y_2$. Setting price equal to marginal cost gives us

$$p = y_2.$$

Solving for the follower's supply curve gives $y_2 = S(p) = p$.

The demand curve facing the leader—the residual demand curve—is

$$R(p) = D(p) - S(p) = a - bp - p = a - (b + 1)p.$$

From now on this is just like an ordinary monopoly problem. Solving for p as a function of the leader's output y_1, we have

$$p = \frac{a}{b+1} - \frac{1}{b+1}y_1. \tag{28.3}$$

This is the inverse demand function facing the leader. The associated marginal revenue curve has the same intercept and is twice as steep. This means that it is given by

$$MR_1 = \frac{a}{b+1} - \frac{2}{b+1}y_1.$$

Setting marginal revenue equal to marginal cost gives us the equation

$$MR_1 = \frac{a}{b+1} - \frac{2}{b+1}y_1 = c = MC_1.$$

Solving for the leader's profit-maximizing output, we have

$$y_1^* = \frac{a - c(b+1)}{2}.$$

We could go on and substitute this into equation (28.3) to get the equilibrium price, but the equation is not particularly interesting.

28.4 Comparing Price Leadership and Quantity Leadership

We've seen how to calculate the equilibrium price and output in the case of quantity leadership and price leadership. Each model determines a different equilibrium price and output combination; each model is appropriate in different circumstances.

One way to think about quantity setting is to think of the firm as making a capacity choice. When a firm sets a quantity it is in effect determining how much it is able to supply to the market. If one firm is able to make an investment in capacity first, then it is naturally modeled as a quantity leader.

On the other hand, suppose that we look at a market where capacity choices are not important but one of the firms distributes a catalog of prices. It is natural to think of this firm as a price setter. Its rivals may then take the catalog price as given and make their own pricing and supply decision accordingly.

Whether the price-leadership or the quantity-leadership model is appropriate is not a question that can be answered on the basis of pure theory. We have to look at how the firms actually make their decisions in order to choose the most appropriate model.

28.5 Simultaneous Quantity Setting

One difficulty with the leader-follower model is that it is necessarily asymmetric: one firm is able to make its decision before the other firm. In some situations this is unreasonable. For example, suppose that two firms are *simultaneously* trying to decide what quantity to produce. Here each firm has to forecast what the other firm's output will be in order to make a sensible decision itself.

In this section we will examine a one-period model in which each firm has to forecast the other firm's output choice. Given its forecast, each firm then chooses a profit-maximizing output for itself. We then seek an equilibrium in forecasts—a situation where each firm finds its beliefs about the other firm to be confirmed. This model is known as the **Cournot model**, after the nineteenth-century French mathematician who first examined its implications.[3]

We begin by assuming that firm 1 expects that firm 2 will produce y_2^e units of output. (The e stands for *expected* output.) If firm 1 decides to produce y_1 units of output, it expects that the total output produced will

[3] Augustin Cournot (pronounced "core-no") was born in 1801. His book, *Researches into the Mathematical Principles of the Theory of Wealth*, was published in 1838.

be $Y = y_1 + y_2^e$, and output will yield a market price of $p(Y) = p(y_1 + y_2^e)$. The profit-maximization problem of firm 1 is then

$$\max_{y_1} p(y_1 + y_2^e)y_1 - c(y_1).$$

For any given belief about the output of firm 2, y_2^e, there will be some optimal choice of output for firm 1, y_1. Let us write this functional relationship between the *expected output* of firm 2 and the *optimal choice* of firm 1 as

$$y_1 = f_1(y_2^e).$$

This function is simply the reaction function that we investigated earlier in this chapter. In our original treatment the reaction function gave the follower's output as a function of the leader's choice. Here the reaction function gives one firm's optimal choice as a function of its *beliefs* about the other firm's choice. Although the interpretation of the reaction function is different in the two cases, the mathematical definition is exactly the same.

Similarly, we can derive firm 2's reaction curve:

$$y_2 = f_2(y_1^e),$$

which gives firm 2's optimal choice of output for a given expectation about firm 1's output, y_1^e.

Now, recall that each firm is choosing its output level *assuming* that the other firm's output will be at y_1^e or y_2^e. For arbitrary values of y_1^e and y_2^e this won't happen—in general firm 1's *optimal* level of output, y_1, will be different from what firm 2 *expects* the output to be, y_1^e.

Let us seek an output combination (y_1^*, y_2^*) such that the optimal output level for firm 1, assuming firm 2 produces y_2^*, is y_1^* and the optimal output level for firm 2, assuming that firm 1 stays at y_1^*, is y_2^*. In other words, the output choices (y_1^*, y_2^*) satisfy

$$y_1^* = f_1(y_2^*)$$

$$y_2^* = f_2(y_1^*).$$

Such a combination of output levels is known as a **Cournot equilibrium**. In a Cournot equilibrium, each firm is maximizing its profits, given its beliefs about the other firm's output choice, and, furthermore, those beliefs are confirmed in equilibrium: each firm optimally chooses to produce the amount of output that the other firm expects it to produce. In a Cournot equilibrium neither firm will find it profitable to change its output once it discovers the choice actually made by the other firm.

An example of a Cournot equilibrium is given in Figure 28.2. The Cournot equilibrium is simply the pair of outputs at which the two reaction curves cross. At such a point, each firm is producing a profit-maximizing level of output given the output choice of the other firm.

28.6 An Example of Cournot Equilibrium

Recall the case of the linear demand function and zero marginal costs that we investigated earlier. We saw that in this case the reaction function for firm 2 took the form

$$y_2 = \frac{a - by_1^e}{2b}.$$

Since in this example firm 1 is exactly the same as firm 2, its reaction curve has the same form:

$$y_1 = \frac{a - by_2^e}{2b}.$$

Figure 28.4 depicts this pair of reaction curves. The intersection of the two lines gives us the Cournot equilibrium. At this point each firm's choice is the profit-maximizing choice, given its beliefs about the other firm's behavior, and each firm's beliefs about the other firm's behavior are confirmed by its *actual* behavior.

Cournot equilibrium. Each firm is maximizing its profits, given its beliefs about the other firm's output decision. The Cournot equilibrium is at (y_1^*, y_2^*), where the two reaction curves cross.

Figure 28.4

In order to calculate the Cournot equilibrium algebraically, we look for the point (y_1, y_2) where each firm is doing what the other firm expects it to do. We set $y_1 = y_1^e$ and $y_2 = y_2^e$, which gives us the following two equations in two unknowns:

$$y_1 = \frac{a - by_2}{2b}$$

$$y_2 = \frac{a - by_1}{2b}.$$

In this example, both firms are identical, so each will produce the same level of output in equilibrium. Hence we can substitute $y_1 = y_2$ into one of the above equations to get

$$y_1 = \frac{a - by_1}{2b}.$$

Solving for y_1^*, we get

$$y_1^* = \frac{a}{3b}.$$

Since the two firms are identical, this implies that

$$y_2^* = \frac{a}{3b}$$

as well, and the total industry output is

$$y_1^* + y_2^* = \frac{2a}{3b}.$$

28.7 Adjustment to Equilibrium

We can use Figure 28.4 to describe a process of adjustment to equilibrium. Suppose that at time t the firms are producing outputs (y_1^t, y_2^t), which are not necessarily equilibrium outputs. If firm 1 expects that firm 2 is going to continue to keep its output at y_2^t, then next period firm 1 would want to choose the profit-maximizing output given that expectation, namely $f_1(y_2^t)$. Thus firm 1's choice in period $t + 1$ will be given by

$$y_1^{t+1} = f_1(y_2^t).$$

Firm 2 can reason the same way, so firm 2's choice next period will be

$$y_2^{t+1} = f_2(y_1^t).$$

These equations describe how each firm adjusts its output in the face of the other firm's choice. Figure 28.4 illustrates the movement of the

outputs of the firms implied by this behavior. Here is the way to interpret the diagram. Start with some operating point (y_1^t, y_2^t). Given firm 2's level of output, firm 1 optimally chooses to produce $y_1^{t+1} = f_1(y_2^t)$ next period. We find this point in the diagram by moving horizontally to the left until we hit firm 1's reaction curve.

If firm 2 expects firm 1 to continue to produce y_1^{t+1}, its optimal response is to produce y_2^{t+1}. We find this point by moving vertically upward until we hit firm 2's reaction function. We continue to move along the "staircase" to determine the sequence of output choices of the two firms. In the example illustrated, this adjustment process converges to the Cournot equilibrium. We say that in this case the Cournot equilibrium is a **stable equilibrium**.

Despite the intuitive appeal of this adjustment process, it does present some difficulties. Each firm is assuming that the other's output will be fixed from one period to the next, but as it turns out, both firms keep changing their output. Only in equilibrium is one firm's expectation about the other firm's output choice actually satisfied. For this reason, we will generally ignore the question of how the equilibrium is reached and focus only on the issue of how the firms behave in the equilibrium.

28.8 Many Firms in Cournot Equilibrium

Suppose now that we have several firms involved in a Cournot equilibrium, not just two. In this case we suppose that each firm has an expectation about the output choices of the other firms in the industry and seek to describe the equilibrium output.

Suppose that there are n firms and let $Y = y_1 + \cdots + y_n$ be the total industry output. Then the "marginal revenue equals marginal cost condition" for firm i is

$$p(Y) + \frac{\Delta p}{\Delta Y} y_i = MC(y_i).$$

If we factor out $P(Y)$ and multiply the second term by Y/Y, we can write this equation as

$$p(Y)\left[1 + \frac{\Delta p}{\Delta Y}\frac{Y}{p(Y)}\frac{y_i}{Y}\right] = MC(y_i).$$

Using the definition of elasticity of the aggregate demand curve and letting $s_i = y_i/Y$ be firm i's share of total market output, this reduces to

$$p(Y)\left[1 - \frac{s_i}{|\epsilon(Y)|}\right] = MC(y_i). \tag{28.4}$$

We can also write this expression as

$$p(Y)\left[1 - \frac{1}{|\epsilon(Y)|/s_i}\right] = MC(y_i).$$

This looks just like the expression for the monopolist except for the s_i term. We can think of $|\epsilon(Y)|/s_i$ as being the elasticity of the demand curve facing the firm: the smaller the market share of the firm, the more elastic the demand curve it faces.

If its market share is 1—the firm is a monopolist—the demand curve facing the firm is the market demand curve, so the condition just reduces to that of the monopolist. If the firm is a very small part of a large market, its market share is effectively zero, and the demand curve facing the firm is effectively flat. Thus the condition reduces to that of the pure competitor: price equals marginal cost.

This is one justification for the competitive model described in Chapter 23. If there are a large number of firms, then each firm's influence on the market price is negligible, and the Cournot equilibrium is effectively the same as pure competition.

28.9 Simultaneous Price Setting

In the Cournot model described above we have assumed that firms were choosing their quantities and letting the market determine the price. Another approach is to think of firms as setting their prices and letting the market determine the quantity sold. This model is known as **Bertrand competition**.[4]

When a firm chooses its price, it has to forecast the price set by the other firm in the industry. Just as in the case of Cournot equilibrium we want to find a pair of prices such that each price is a profit-maximizing choice given the choice made by the other firm.

What does a Bertrand equilibrium look like? When firms are selling identical products, as we have been assuming, the Bertrand equilibrium has a very simple structure indeed. It turns out to be the competitive equilibrium, where price equals marginal cost!

First we note that price can never be less than marginal cost since then either firm would increase its profits by producing less. So let us consider the case where price is greater than marginal cost. Suppose that both firms are selling output at some price $\hat{p}$ greater than marginal cost. Consider the position of firm 1. If it lowers its price by any small amount ϵ and if the other firm keeps its price fixed at $\hat{p}$, all of the consumers will prefer to purchase from firm 1. By cutting its price by an arbitrarily small amount, it can steal all of the customers from firm 2.

If firm 1 really believes that firm 2 will charge a price $\hat{p}$ that is greater than marginal cost, it will always pay firm 1 to cut its price to $\hat{p} - \epsilon$. But firm 2 can reason the same way! Thus any price higher than marginal

[4] Joseph Bertrand, also a French mathematician, presented his model in a review of Cournot's work.

cost cannot be an equilibrium; the only equilibrium is the competitive equilibrium.

This result seems paradoxical when you first encounter it: how can we get a competitive price if there are only two firms in the market? If we think of the Bertrand model as a model of competitive bidding it makes more sense. Suppose that one firm "bids" for the consumers' business by quoting a price above marginal cost. Then the other firm can always make a profit by undercutting this price with a lower price. It follows that the only price that each firm cannot rationally expect to be undercut is a price equal to marginal cost.

It is often observed that competitive bidding among firms that are unable to collude can result in prices that are much lower than can be achieved by other means. This phenomenon is simply an example of the logic of Bertrand competition.

28.10 Collusion

In the models we have examined up until now the firms have operated independently. But if the firms collude so as to jointly determine their output, these models are not very reasonable. If collusion is possible, the firms would do better to choose the output that maximizes total industry profits and then divide up the profits among themselves. When firms get together and attempt to set prices and outputs so as to maximize total industry profits, they are known as a **cartel**. As we saw in Chapter 25, a cartel is simply a group of firms that jointly collude to behave like a single monopolist and maximize the sum of their profits.

Thus the profit-maximization problem facing the two firms is to choose their outputs y_1 and y_2 so as to maximize total industry profits:

$$\max_{y_1, y_2} \; p(y_1 + y_2)[y_1 + y_2] - c_1(y_1) - c_2(y_2).$$

This will have the optimality conditions

$$p(y_1^* + y_2^*) + \frac{\Delta p}{\Delta Y}[y_1^* + y_2^*] = MC_1(y_1^*)$$

$$p(y_1^* + y_2^*) + \frac{\Delta p}{\Delta Y}[y_1^* + y_2^*] = MC_2(y_2^*).$$

The interpretation of these conditions is interesting. When firm 1 considers expanding its output by Δy_1, it will contemplate the usual two effects: the extra profits from selling more output and the reduction in profits from forcing the price down. But in the second effect, it now takes into account the effect of the lower price on both its own output and the output of the

other firm. This is because it is now interested in maximizing total industry profits, not just its own profits.

The optimality conditions imply that the marginal revenue of an extra unit of output must be the same no matter where it is produced. It follows that $MC_1(y_1^*) = MC_2(y_2^*)$, so that the two marginal costs will be equal in equilibrium. If one firm has a cost advantage, so that its marginal cost curve always lies below that of the other firm, then it will necessarily produce more output in equilibrium in the cartel solution.

The problem with agreeing to join a cartel in real life is that there is always a temptation to cheat. Suppose, for example, that the two firms are operating at the outputs that maximize industry profits (y_1^*, y_2^*) and firm 1 considers producing a little more output, Δy_1. The marginal profits accruing to firm 1 will be

$$\frac{\Delta \pi_1}{\Delta y_1} = p(y_1^* + y_2^*) + \frac{\Delta p}{\Delta Y} y_1^* - MC_1(y_1^*). \tag{28.5}$$

We saw earlier that the optimality condition for the cartel solution is

$$p(y_1^* + y_2^*) + \frac{\Delta p}{\Delta Y} y_1^* + \frac{\Delta p}{\Delta Y} y_2^* - MC_1(y_1^*) = 0.$$

Rearranging this equation gives us

$$p(y_1^* + y_2^*) + \frac{\Delta p}{\Delta Y} y_1^* - MC_1(y_1^*) = -\frac{\Delta p}{\Delta Y} y_2^* > 0. \tag{28.6}$$

The last inequality follows since $\Delta p / \Delta Y$ is negative, since the market demand curve has a negative slope.

Inspecting equations (28.5) and (28.6) we see that

$$\frac{\Delta \pi_1}{\Delta y_1} > 0.$$

Thus, if firm 1 believes that firm 2 will keep its output fixed, then it will believe that it can increase profits by increasing its own production. In the cartel solution, the firms act together to restrict output so as not to "spoil" the market. They recognize the effect on joint profits from producing more output in either firm. But if each firm believes that the other firm will stick to its output quota, then each firm will be tempted to increase its own profits by unilaterally expanding its output. At the output levels that maximize joint profits, it will always be profitable for each firm to unilaterally increase its output—if each firm expects that the other firm will keep its output fixed.

The situation is even worse than that. If firm 1 believes that firm 2 will keep its output fixed, then it will find it profitable to increase its own output. But if it thinks that firm 2 will increase its output, then

firm 1 would want increase its output first and make its profits while it can!

Thus, in order to maintain an effective cartel, the firms need a way to detect and punish cheating. If they have no way to observe each other's output, the temptation to cheat may break the cartel. We'll return to this point a little later.

To make sure that we understand the cartel solution, let's calculate it for the case of zero marginal costs and the linear demand curve we used in the Cournot case.

The aggregate profit function will be

$$\pi(y_1, y_2) = [a - b(y_1 + y_2)](y_1 + y_2) = a(y_1 + y_2) - b(y_1 + y_2)^2,$$

so the marginal revenue equals marginal cost conditions will be

$$a - 2b(y_1^* + y_2^*) = 0,$$

which implies that

$$y_1^* + y_2^* = \frac{a}{2b}.$$

Since marginal costs are zero, the division of output between the two firms doesn't matter. All that is determined is the total level of industry output.

This solution is shown in Figure 28.5. Here we have illustrated the isoprofit curves for each of the firms and have highlighted the locus of common tangents. Why is this line of interest? Since the cartel is trying to maximize total industry profits, it follows that the marginal profits from having either firm produce more output must be the same—otherwise it would pay to have the more profitable firm produce more output. This in turn implies that the slopes of the isoprofit curves must be the same for each firm; that is, that the isoprofit curves must be tangent to each other. Hence the output combinations that maximize total industry profits—the cartel solution—are those that lie along the line illustrated in Figure 28.5.

Figure 28.5 also illustrates the temptation to cheat that is present at the cartel solution. Consider, for example, the point where the two firms split the market equally. Think about what would happen if firm 1 believed that firm 2 would keep its output constant. If firm 1 increased its output and firm 2 kept constant output, then firm 1 would move to a lower isoprofit curve—which means that firm 1 would increase its profits. This is exactly the story told in the algebra above. If one firm thinks that the other's output will remain constant, it will be tempted to increase its own output and thereby make higher profits.

28.11 Punishment Strategies

We have seen that a cartel is fundamentally unstable in the sense that it is always in the interest of each of the firms to increase their production

Figure 28.5

A cartel. If industry profits are maximized, then the marginal profit from producing more output in either firm must be the same. This implies that the isoprofit curves must be tangent to one another at the profit-maximizing levels of output.

above that which maximizes aggregate profit. If the cartel is to operate successfully, some way must be found to "stabilize" the behavior. One way to do this is for firms to threaten to punish each other for cheating on the cartel agreement. In this section, we investigate the size of punishments necessary to stabilize a cartel.

Consider a duopoly composed of two identical firms. If each firm produces half the monopoly amount of output, total profits will be maximized and each firm will get a payoff of, say, π_m. In an effort to make this outcome stable, one firm announces to the other: "If you stay at the production level that maximizes joint industry projects, fine. But if I discover you cheating by producing more than this amount, I will punish you by producing the Cournot level of output forever." This is known as a **punishment strategy**.

When will this sort of threat be adequate to stabilize the cartel? We have to look at the benefits and costs of cheating as compared to those of cooperating. Suppose that cheating occurs, and the punishment is carried out. Since the optimal response to Cournot behavior is Cournot behavior (by definition), this results in each firm receiving a per-period profit of, say, π_c. Of course, the Cournot payoff, π_c is less than the cartel payoff, π_m.

Let us suppose that the two firms are each producing at the collusive,

monopoly level of production. Put yourself in the place of one of the firms trying to decide whether to continue to produce at your quota. If you produce more output, deviating from your quota, you make profit π_d, where $\pi_d > \pi_m$. This is the standard temptation facing a cartel member described above: if each firm restricts output and pushes the price up, then each firm has an incentive to capitalize on the high price by increasing its production.

But this isn't the end of the story because of the punishment for cheating. By producing at the cartel amount, each firm gets a steady stream of payments of π_m. The present value of this stream starting today is given by

$$\text{Present value of cartel behavior} = \pi_m + \frac{\pi_m}{r}.$$

If the firm produces more than the cartel amount, it gets a one-time benefit of profits π_d, but then has to live with the breakup of the cartel and the reversion to Cournot behavior:

$$\text{Present value of cheating} = \pi_d + \frac{\pi_c}{r}.$$

When will the present value of remaining at the cartel output be greater than the present value of cheating on the cartel agreement? Obviously when

$$\pi_m + \frac{\pi_m}{r} > \pi_d + \frac{\pi_c}{r},$$

which can also be written as

$$r < \frac{\pi_m - \pi_c}{\pi_d - \pi_m}.$$

Note that the numerator of this fraction is positive, since the monopoly profits are larger than the Cournot profits, and the denominator is positive, since deviation is even more profitable than sticking with the monopoly quota.

The inequality says that as long as the interest rate is sufficiently small, so that the prospect of future punishment is sufficiently important, it will pay the firms to stick to their quotas.

The weakness of this model is that the threat to revert to Cournot behavior forever is not very believable. One firm certainly may believe that the other will punish it for deviating, but "forever" is a long time. A more realistic model would consider shorter periods of retaliation, but the analysis then becomes much more complex. In the next chapter, we discuss some models of "repeated games" that illustrate some of the possible behaviors.

EXAMPLE: Price Matching and Competition

We have seen that there is always a temptation for each member of a cartel to produce more than its quota. In order to maintain a successful

cartel, some way must be found to police members' behavior by some form of punishment for deviations from the joint profit-maximizing output. In particular this means that firms must be able to keep track of the prices and production levels of the other firms in the cartel.

One easy way to acquire information about what the other firms in your industry are charging is to use your customers to spy on the other firms. It is common to see retail firms announce that they will "beat any price." In some cases, such an offer may indicate a highly competitive retail environment. But in other cases, this same policy can be used to gather information about other firms' prices in order to maintain a cartel.

Suppose, for example, that two firms agree, either explicitly or implicitly to sell a certain model of refrigerator for $700. How can either of the stores be sure that the other firm isn't cheating on their agreement and selling the refrigerator for $675? One way is to offer to beat any price a customer can find. That way, the customers report any attempts to cheat on the collusive arrangement.

EXAMPLE: Voluntary Export Restraints

During the 1980s, the Japanese automobile companies agreed to a "voluntary export restraint (VER)." This meant that they would "voluntarily" reduce the exports of their automobiles to the United States. The typical U.S. consumer thought that this was a great victory for U.S. trade negotiators.

But if you think about this for a minute, things look quite different. In our examination of oligopoly we have seen that the problem facing firms in an industry is how to *restrict* output in order to support higher prices and discourage competition. As we've seen, there will always be a temptation to cheat on production agreements; every cartel must find a way to detect and prevent this cheating. It is especially convenient for the firms if a third party, such as the government, can serve this role. This is exactly the role that the U.S. government played for the Japanese auto makers!

According to one estimate Japanese imported cars were about $2500 more expensive in 1984 than they would have been without the VERs. Furthermore, the higher prices of imported cars allowed American producers to sell their automobiles at about $1000 more than they would have otherwise.[5]

Due to these higher prices the U.S. consumers paid about $10 billion more for Japanese cars in 1985–86 than they would have otherwise. This money has gone directly into the pockets of the Japanese automobile producers. Much of this additional profit appears to have been invested in

[5] Robert Crandall, "Import Quotas and the Automobile Industry: the Costs of Protectionism," *The Brookings Review,* Summer, 1984.

increasing productive capabilities, which allowed the Japanese auto producers to reduce the cost of producing new cars in subsequent years. The VERs did succeed in saving American jobs; however, it appears that the cost per job saved was about $160,000 per year.

If the goal of the VER policy was simply to increase the health of the American automobile industry, there was a much simpler way to do this: just impose a $2500 tariff on each imported Japanese car. This way the revenues due to the restriction of trade would accrue to the U.S. government rather than to the Japanese automobile industry. Rather than send $10 billion abroad during 1985–86, the U.S. government could have spent the money on projects designed to increase the long-term health of the U.S. auto industry.

28.12 Comparison of the Solutions

We have now examined several models of duopoly behavior: quantity leadership (Stackelberg), price leadership, simultaneous quantity setting (Cournot), simultaneous price setting (Bertrand), and the collusive solution. How do they compare?

In general, collusion results in the smallest industry output and the highest price. Bertrand equilibrium—the competitive equilibrium—gives us the highest output and the lowest price. The other models give results that are in between these two extremes.

A variety of other models are possible. For example, we could look at a model with differentiated products where the two goods produced were not perfect substitutes for each other. Or we could look at a model where the firms make a sequence of choices over time. In this framework, the choices that one firm makes at one time can influence the choices that the other firm makes later on.

We have also assumed that each firm knows the demand function and the cost functions of the other firms in the industry. In reality these functions are never known for sure. Each firm needs to estimate the demand and cost conditions facing its rivals when it makes its own decisions. All of these phenomena have been modeled by economists, but the models become much more complex.

Summary

1. An oligopoly is characterized by a market with a few firms that recognize their strategic interdependence. There are several possible ways for oligopolies to behave depending on the exact nature of their interaction.

2. In the quantity-leader (Stackelberg) model one firm leads by setting its output, and the other firm follows. When the leader chooses an output, it will take into account how the follower will respond.

3. In the price-leader model, one firm sets its price, and the other firm chooses how much it wants to supply at that price. Again the leader has to take into account the behavior of the follower when it makes its decision.

4. In the Cournot model, each firm chooses its output so as to maximize its profits given its beliefs about the other firm's choice. In equilibrium each firm finds that its expectation about the other firm's choice is confirmed.

5. A Cournot equilibrium in which each firm has a small market share implies that price will be very close to marginal cost—that is, the industry will be nearly competitive.

6. In the Bertrand model each firm chooses its price given its beliefs about the price that the other firm will choose. The only equilibrium price is the competitive equilibrium.

7. A cartel consists of a number of firms colluding to restrict output and to maximize industry profit. A cartel will typically be unstable in the sense that each firm will be tempted to sell more than its agreed upon output if it believes that the other firms will not respond.

REVIEW QUESTIONS

1. Suppose that we have two firms that face a linear demand curve $p(Y) = a - bY$ and have constant marginal costs, c, for each firm. Solve for the Cournot equilibrium output.

2. Consider a cartel in which each firm has identical and constant marginal costs. If the cartel maximizes total industry profits, what does this imply about the division of output between the firms?

3. Can the leader ever get a lower profit in a Stackelberg equilibrium than he would get in the Cournot equilibrium?

4. Suppose there are n identical firms in a Cournot equilibrium. Show that the absolute value of the elasticity of the market demand curve must be greater than $1/n$. (Hint: in the case of a monopolist, $n = 1$, and this simply says that a monopolist operates at an elastic part of the demand curve. Apply the logic that we used to establish that fact to this problem.)

5. Draw a set of reaction curves that result in an unstable equilibrium.

6. Do oligopolies produce an efficient level of output?

CHAPTER 29

GAME THEORY

The previous chapter on oligopoly theory presented the classical economic theory of strategic interaction among firms. But that is really just the tip of the iceberg. Economic agents can interact strategically in a variety of ways, and many of these have been studied by using the apparatus of **game theory**. Game theory is concerned with the general analysis of strategic interaction. It can be used to study parlor games, political negotiation, and economic behavior. In this chapter we will briefly explore this fascinating subject to give you a flavor of how it works and how it can be used to study economic behavior in oligopolistic markets.

29.1 The Payoff Matrix of a Game

Strategic interaction can involve many players and many strategies, but we'll limit ourselves to two-person games with a finite number of strategies. This will allow us to depict the game easily in a **payoff matrix**. It is simplest to examine this in the context of a specific example.

Suppose that two people are playing a simple game. Person A will write one of two words on a piece of paper, "top" or "bottom." Simultaneously,

person B will independently write "left" or "right" on a piece of paper. After they do this, the papers will be examined and they will each get the payoff depicted in Table 29.1. If A says top and B says left, then we examine the top left-hand corner of the matrix. In this matrix the payoff to A is the first entry in the box, 1, and the payoff to B is the second entry, 2. Similarly, if A says bottom and B says right, then A will get a payoff of 1 and B will get a payoff of 0.

Person A has two strategies: he can choose top or he can choose bottom. These strategies could represent economic choices like "raise price" or "lower price." Or they could represent political choices like "declare war" or "don't declare war." The payoff matrix of a game simply depicts the payoffs to each player for each combination of strategies that are chosen.

What will be the outcome of this sort of game? The game depicted in Table 29.1 has a very simple solution. From the viewpoint of person A, it is always better for him to say bottom since his payoffs from that choice (2 or 1) are always greater than their corresponding entries in top (1 or 0). Similarly, it is always better for B to say left since 2 and 1 dominate 1 and 0. Thus we would expect that the equilibrium strategy is for A to play bottom and B to play left.

In this case, we have a **dominant strategy**. There is one optimal choice of strategy for each player no matter what the other player does. Whichever choice B makes, player A will get a higher payoff if he plays bottom, so it makes sense for A to play bottom. And whichever choice A makes, B will get a higher payoff if he plays left. Hence, these choices dominate the alternatives, and we have an equilibrium in dominant strategies.

Table 29.1

A payoff matrix of a game.

		Player B	
		Left	Right
Player A	Top	1, 2	0, 1
	Bottom	2, 1	1, 0

If there is a dominant strategy for each player in some game, then we would predict that it would be the equilibrium outcome of the game. For a dominant strategy is a strategy that is best no matter what the other player does. In this example, we would expect an equilibrium outcome in

which A plays bottom, receiving an equilibrium payoff of 2, and B plays left, receiving an equilibrium payoff of 1.

29.2 Nash Equilibrium

Dominant strategy equilibria are nice when they happen, but they don't happen all that often. For example, the game depicted in Table 29.2 doesn't have a dominant strategy equilibrium. Here when B chooses left the payoffs to A are 2 or 0. When B chooses right, the payoffs to A are 0 or 1. This means that when B chooses left, A would want to choose top; and when B chooses right, A would want to choose bottom. Thus A's optimal choice depends on what he thinks B will do.

Table 29.2 **A Nash equilibrium.**

		Player B	
		Left	Right
Player A	Top	2, 1	0, 0
	Bottom	0, 0	1, 2

However, perhaps the dominant strategy equilibrium is too demanding. Rather than require that A's choice be optimal for *all* choices of B, we can just require that it be optimal for the *optimal* choices of B. For if B is a well-informed intelligent player, he will only want to choose optimal strategies. (Although, what is optimal for B will depend on A's choice as well!)

We will say that a pair of strategies is a **Nash equilibrium** if A's choice is optimal, given B's choice, *and* B's choice is optimal given A's choice.[1] Remember that neither person knows what the other person will do when he has to make his own choice of strategy. But each person may have

[1] John Nash is an American mathematician who formulated this fundamental concept of game theory in 1951. In 1994 he received the Nobel Prize in economics, along with two other game theory pioneers, John Harsanyi and Reinhard Selten. The 2002 film *A Beautiful Mind* is loosely based on John Nash's life; it won the Academy Award for best movie.

some expectation about what the other person's choice will be. A Nash equilibrium can be interpreted as a pair of expectations about each person's choice such that, when the other person's choice is revealed, neither individual wants to change his behavior.

In the case of Table 29.2, the strategy (top, left) is a Nash equilibrium. To prove this note that if A chooses top, then the best thing for B to do is to choose left, since the payoff to B from choosing left is 1 and from choosing right is 0. And if B chooses left, then the best thing for A to do is to choose top since then A will get a payoff of 2 rather than of 0.

Thus if A chooses top, the optimal choice for B is to choose left; and if B chooses left, then the optimal choice for A is top. So we have a Nash equilibrium: each person is making the optimal choice, *given* the other person's choice.

The Nash equilibrium is a generalization of the Cournot equilibrium described in the last chapter. There the choices were output levels, and each firm chose its output level taking the other firm's choice as being fixed. Each firm was supposed to do the best for itself, assuming that the other firm continued to produce the output level it had chosen—that is, it continued to play the strategy it had chosen. A Cournot equilibrium occurs when each firm is maximizing profits given the other firm's behavior; this is precisely the definition of a Nash equilibrium.

The Nash equilibrium notion has a certain logic. Unfortunately, it also has some problems. First, a game may have more than one Nash equilibrium. In fact, in Table 29.2 the choices (bottom, right) also comprise a Nash equilibrium. You can either verify this by the kind of argument used above, or just note that the structure of the game is symmetric: B's payoffs are the same in one outcome as A's payoffs are in the other, so that our proof that (top, left) is an equilibrium is also a proof that (bottom, right) is an equilibrium.

The second problem with the concept of a Nash equilibrium is that there are games that have no Nash equilibrium of the sort we have been describing at all. Consider, for example, the case depicted in Table 29.3. Here a Nash equilibrium of the sort we have been examining does not exist. If player A plays top, then player B wants to play left. But if player B plays left, then player A wants bottom. Similarly, if player A plays bottom, then player B will play right. But if player B plays right, then player A will play top.

29.3 Mixed Strategies

However, if we enlarge our definition of strategies, we can find a new sort of Nash equilibrium for this game. We have been thinking of each agent as choosing a strategy once and for all. That is, each agent is making one choice and sticking to it. This is called a **pure strategy**.

Table 29.3 **A game with no Nash equilibrium (in pure strategies).**

		Player B	
		Left	Right
Player A	Top	0, 0	0, –1
	Bottom	1, 0	–1, 3

Another way to think about it is to allow the agents to *randomize* their strategies—to assign a probability to each choice and to play their choices according to those probabilities. For example, A might choose to play top 50 percent of the time and bottom 50 percent of the time, while B might choose to play left 50 percent of the time and right 50 percent of the time. This kind of strategy is called a **mixed strategy**.

If A and B follow the mixed strategies given above, of playing each of their choices half the time, then they will have a probability of 1/4 of ending up in each of the four cells in the payoff matrix. Thus the average payoff to A will be 0, and the average payoff to B will be 1/2.

A Nash equilibrium in mixed strategies refers to an equilibrium in which each agent chooses the optimal frequency with which to play his strategies given the frequency choices of the other agent.

It can be shown that for the sort of games we are analyzing in this chapter, there will always exist a Nash equilibrium in mixed strategies. Because a Nash equilibrium in mixed strategies always exists, and because the concept has a certain inherent plausibility, it is a very popular equilibrium notion in analyzing game behavior. In the example in Table 29.3 it can be shown that if player A plays top with probability 3/4 and bottom with probability 1/4, and player B plays left with probability 1/2 and right with probability 1/2, this will constitute a Nash equilibrium.

EXAMPLE: Rock Paper Scissors

But enough of this theory. Let's look at an example that really matters: the well-known pastime of "rock paper scissors." In this game, each player simultaneously chooses to display a fist (rock), a palm (paper), or his first two fingers (scissors). The rules: rock breaks scissors, scissors cuts paper, paper wraps rock.

Throughout history, countless hours have been spent in playing this game. There is even a professional society, the RPS Society, that pro-

motes the game. It offers both a web site and a movie documenting the 2003 championships in Toronto.

Of course, game theorists recognize that the equilibrium strategy in rock paper scissors is to randomly choose one of the three outcomes. But humans are not necessarily so good at choosing totally random outcomes. If you can predict your opponent's choices to some degree, you can have an edge in making your own choices.

According to the somewhat tongue-in-cheek account of Jennifer 8. Lee, psychology is paramount.[2] In her article she writes that "most people have a go-to throw, reflective of their character, when they are caught off guard. Paper, considered a refined, even passive, throw, is apparently favored by literary types and journalists."

What is the go-to throw of economists, I wonder? Perhaps it is scissors, since we like to cut to the essential forces at work in human behavior. Should you play rock against an economist, then? Perhaps, but I wouldn't rely on it . . .

29.4 The Prisoner's Dilemma

Another problem with the Nash equilibrium of a game is that it does not necessarily lead to Pareto efficient outcomes. Consider, for example, the game depicted in Table 29.4. This game is known as the **prisoner's dilemma**. The original discussion of the game considered a situation where two prisoners who were partners in a crime were being questioned in separate rooms. Each prisoner had a choice of confessing to the crime, and thereby implicating the other, or denying that he had participated in the crime. If only one prisoner confessed, then he would go free, and the authorities would throw the book at the other prisoner, requiring him to spend 6 months in prison. If both prisoners denied being involved, then both would be held for 1 month on a technicality, and if both prisoners confessed they would both be held for 3 months. The payoff matrix for this game is given in Table 29.4. The entries in each cell in the matrix represent the utility that each of the agents assigns to the various prison terms, which for simplicity we take to be the negative of the length of their prison terms.

Put yourself in the position of player A. If player B decides to deny committing the crime, then you are certainly better off confessing, since then you'll get off free. Similarly, if player B confesses, then you'll be better off confessing, since then you get a sentence of 3 months rather than a sentence of 6 months. Thus *whatever* player B does, player A is better off confessing.

[2] Yes, "8" really is her middle name. "Rock, Paper, Scissors: High Drama in the Tournament Ring" was published in the *New York Times* on September 5, 2004.

Table 29.4

The prisoner's dilemma.

		Player B	
		Confess	Deny
Player A	Confess	–3, –3	0, –6
	Deny	–6, 0	–1, –1

The same thing goes for player B—he is better off confessing as well. Thus the unique Nash equilibrium for this game is for both players to confess. In fact, both players confessing is not only a Nash equilibrium, it is a dominant strategy equilibrium, since each player has the same optimal choice independent of the other player.

But if they could both just hang tight, they would each be better off! If they both could be sure the other would hold out, and both could agree to hold out themselves, they would each get a payoff of -1, which would make each of them better off. The strategy (deny, deny) is Pareto efficient—there is no other strategy choice that makes both players better off—while the strategy (confess, confess) is Pareto inefficient.

The problem is that there is no way for the two prisoners to coordinate their actions. If each could trust the other, then they could both be made better off.

The prisoner's dilemma applies to a wide range of economic and political phenomena. Consider, for example, the problem of arms control. Interpret the strategy of "confess" as "deploy a new missile" and the strategy of "deny" as "don't deploy." Note that the payoffs are reasonable. If my opponent deploys his missile, I certainly want to deploy, even though the best strategy for both of us is to agree not to deploy. But if there is no way to make a binding agreement, we each end up deploying the missile and are both made worse off.

Another good example is the problem of cheating in a cartel. Now interpret confess as "produce more than your quota of output" and interpret deny as "stick to the original quota." If you think the other firm is going to stick to its quota, it will pay you to produce more than your own quota. And if you think that the other firm will overproduce, then you might as well, too!

The prisoner's dilemma has provoked a lot of controversy as to what is the "correct" way to play the game—or, more precisely, what is a reasonable way to play the game. The answer seems to depend on whether you are playing a one–shot game or whether the game is to be repeated an indefinite

number of times.

If the game is going to be played just one time, the strategy of defecting—in this example, confessing—seems to be a reasonable one. After all, whatever the other fellow does, you are better off, and you have no way of influencing the other person's behavior.

29.5 Repeated Games

In the preceding section, the players met only once and played the prisoner's dilemma game a single time. However, the situation is different if the game is to be played repeatedly by the same players. In this case there are new strategic possibilities open to each player. If the other player chooses to defect on one round, then you can choose to defect on the next round. Thus your opponent can be "punished" for "bad" behavior. In a repeated game, each player has the opportunity to establish a reputation for cooperation, and thereby encourage the other player to do the same.

Whether this kind of strategy will be viable depends on whether the game is going to be played a *fixed* number of times or an *indefinite* number of times.

Let us consider the first case, where both players know that the game is going to be played 10 times, say. What will the outcome be? Suppose we consider round 10. This is the last time the game will be played, by assumption. In this case, it seems likely that each player will choose the dominant strategy equilibrium, and defect. After all, playing the game for the last time is just like playing it once, so we should expect the same outcome.

Now consider what will happen on round 9. We have just concluded that each player will defect on round 10. So why cooperate on round 9? If you cooperate, the other player might as well defect now and exploit your good nature. Each player can reason the same way, and thus each will defect.

Now consider round 8. If the other person is going to defect on round 9 . . . and so it goes. If the game has a known, fixed number of rounds, then each player will defect on every round. If there is no way to enforce cooperation on the last round, there will be no way to enforce cooperation on the next to the last round, and so on.

Players cooperate because they hope that cooperation will induce further cooperation in the future. But this requires that there will always be the possibility of future play. Since there is no possibility of future play in the last round, no one will cooperate then. But then why should anyone cooperate on the next to the last round? Or the one before that? And so it goes—the cooperative solution "unravels" from the end in a prisoner's dilemma with a known, fixed number of plays.

But if the game is going to be repeated an indefinite number of times, then you *do* have a way of influencing your opponent's behavior: if he

refuses to cooperate this time, you can refuse to cooperate next time. As long as both parties care enough about future payoffs, the threat of non-cooperation in the future may be sufficient to convince people to play the Pareto efficient strategy.

This has been demonstrated in a convincing way in a series of experiments run by Robert Axelrod.[3] He asked dozens of experts on game theory to submit their favorite strategies for the prisoner's dilemma and then ran a "tournament" on a computer to pit these strategies against each other. Every strategy was played against every other strategy on the computer, and the computer kept track of the total payoffs.

The winning strategy—the one with the highest overall payoff—turned out to be the simplest strategy. It is called "tit for tat" and goes like this. On the first round, you cooperate—play the "deny" strategy. On every round thereafter, if your opponent cooperated on the previous round, you cooperate. If your opponent defected on the previous round, you defect. In other words, do whatever the other player did in the last round.

The tit-for-tat strategy does very well because it offers an immediate punishment for defection. It is also a forgiving strategy: it punishes the other player only once for each defection. If he falls into line and starts to cooperate, then tit for tat will reward the other player with cooperation. It appears to be a remarkably good mechanism for achieving the efficient outcome in a prisoner's dilemma that will be played an indefinite number of times.

29.6 Enforcing a Cartel

In Chapter 28 we discussed the behavior of duopolists playing a price-setting game. We argued there that if each duopolist could choose his price, then the equilibrium outcome would be the competitive equilibrium. If each firm thought that the other firm would keep its price fixed, then each firm would find it profitable to undercut the other. The only place where this would not be true was if each firm were charging the lowest possible price, which in the case we examined was a price of zero, since the marginal costs were zero. In the terminology of this chapter, each firm charging a zero price is a Nash equilibrium in pricing strategies—what we called a Bertrand equilibrium in Chapter 28.

The payoff matrix for the duopoly game in pricing strategies has the same structure as the prisoner's dilemma. If each firm charges a high price, then they both get large profits. This is the situation where they are both cooperating to maintain the monopoly outcome. But if one firm is charging

[3] Robert Axelrod is a political scientist from the University of Michigan. For an extended discussion, see his book *The Evolution of Cooperation* (New York: Basic Books, 1984).

a high price, then it will pay the other firm to cut its price a little, capture the other fellow's market, and thereby get even higher profits. But if both firms cut their prices, they both end up making lower profits. Whatever price the other fellow is charging, it will always pay you to shave your price a little bit. The Nash equilibrium occurs when each fellow is charging the lowest possible price.

However, if the game is repeated an indefinite number of times, there may be other possible outcomes. Suppose that you decide to play tit for tat. If the other fellow cuts his price this week, you will cut yours next week. If each player knows that the other player is playing tit for tat, then each player would be fearful of cutting his price and starting a price war. The threat implicit in tit for tat may allow the firms to maintain high prices.

Real-life cartels sometimes appear to employ tit-for-tat strategies. For example, the Joint Executive Committee was a famous cartel that set the price of railroad freight in the United States in the late 1800s. The formation of this cartel preceded antitrust regulation in the United States, and at the time was perfectly legal.[4]

The cartel determined what market share each railroad could have of the freight shipped. Each firm set its rates individually, and the JEC kept track of how much freight each firm shipped. However, there were several occasions during 1881, 1884, and 1885 where some members of the cartel thought that other member firms were cutting rates so as to increase their market share, despite their agreement. During these periods, there were often price wars. When one firm tried to cheat, all firms would cut their prices so as to "punish" the defectors. This kind of tit-for-tat strategy was apparently able to support the cartel arrangement for some time.

EXAMPLE: Tit for Tat in Airline Pricing

Airline pricing provides an interesting example of tit-for-tat behavior. Airlines often offer special promotional fares of one sort or another; many observers of the airline industry claim that these promotions can be used to signal competitors to refrain from cutting prices on key routes.

A senior director of marketing for a major U.S. airline described a case in which Northwest lowered fares on night flights from Minneapolis to various West Coast cities in an effort to fill empty seats. Continental Airlines interpreted this as an attempt to gain market share at its expense and responded by cutting *all* its Minneapolis fares to Northwest's night-fare

[4] For a detailed analysis, see Robert Porter, "A Study of Cartel Stability: the Joint Executive Committee, 1880–1886," *The Bell Journal of Economics*, 14, 2 (Autumn 1983), 301–25.

level. However, the Continental fare cuts were set to expire one or two days after they were introduced.

Northwest interpreted this as a signal from Continental that it was not serious about competing in this market, but simply wanted Northwest to retract its night-fare cuts. But Northwest decided to send a message of its own to Continental: it instituted a set of cheap fares to the West Coast for its flights departing from Houston, Continental's home base! Northwest thereby signaled that it felt its cuts were justified, while Continental's response was inappropriate.

All these fare cuts had very short expiration dates; this feature seems to indicate that they were meant more as messages to the competition than as bids for larger market share. As the analyst explained, fares that an airline doesn't want to offer "should almost always have an expiration date on them in the hopes that the competition will eventually wake up and match."

The implicit rules of competition in duopoly airline markets seem to be the following: if the other firm keeps its prices high, I will maintain my high prices; but if the other firm cuts its prices, I will play tit for tat and cut my prices in response. In other words, both firms "live by the Golden Rule": do unto others as you would have them do unto you. This threat of retaliation then serves to keep all prices high.[5]

29.7 Sequential Games

Up until now we have been thinking about games in which both players act simultaneously. But in many situations one player gets to move first, and the other player responds. An example of this is the Stackelberg model described in Chapter 28, where one player is a leader and the other player is a follower.

Let's describe a game like this. In the first round, player A gets to choose top or bottom. Player B gets to observe the first player's choice and then chooses left or right. The payoffs are illustrated in a game matrix in Table 29.5.

Note that when the game is presented in this form it has two Nash equilibria: (top, left) and (bottom, right). However, we'll show below that one of these equilibria isn't really reasonable. The payoff matrix hides the fact that one player gets to know what the other player has chosen before he makes his choice. In this case it is more useful to consider a diagram that illustrates the asymmetric nature of the game.

Figure 29.1 is a picture of the game in **extensive form**—a way to represent the game that shows the time pattern of the choices. First, player A

[5] Facts taken from A. Nomani, "Fare Warning: How Airlines Trade Price Plans," *Wall Street Journal,* October 9, 1990, B1.

The payoff matrix of a sequential game. **Table 29.5**

		Player B	
		Left	Right
Player A	Top	1, 9	1, 9
	Bottom	0, 0	2, 1

Extensive form of the game. This way of depicting a game indicates the order in which the players move. **Figure 29.1**

has to choose top or bottom, and then player B has to choose left or right. But when B makes his choice, he will know what A has done.

The way to analyze this game is to go to the end and work backward. Suppose that player A has already made his choice and we are sitting in one branch of the game tree. If player A has chosen top, then it doesn't matter what player B does, and the payoff is (1,9). If player A has chosen bottom, then the sensible thing for player B to do is to choose right, and the payoff is (2,1).

Now think about player A's initial choice. If he chooses top, the outcome will be (1,9) and thus he will get a payoff of 1. But if he chooses bottom, he

gets a payoff of 2. So the sensible thing for him to do is to choose bottom. Thus the equilibrium choices in the game will be (bottom, right), so that the payoff to player A will be 2 and to player B will be 1.

The strategies (top, left) are not a reasonable equilibrium in this sequential game. That is, they are not an equilibrium given the order in which the players actually get to make their choices. It is true that if player A chooses top, player B could choose left—but it would be silly for player A to ever choose top!

From player B's point of view this is rather unfortunate, since he ends up with a payoff of 1 rather than 9! What might he do about it?

Well, he can *threaten* to play left if player A plays bottom. If player A thought that player B would actually carry out this threat, he would be well advised to play top. For top gives him 1, while bottom—if player B carries out his threat—will only give him 0.

But is this threat credible? After all, once player A makes his choice, that's it. Player B can get either 0 or 1, and he might as well get 1. Unless player B can somehow convince player A that he will really carry out his threat—even when it hurts him to do so—he will just have to settle for the lower payoff.

Player B's problem is that once player A has made his choice, player A expects player B to do the rational thing. Player B would be better off if he could *commit* himself to play left if player A plays bottom.

One way for B to make such a commitment is to allow someone else to make his choices. For example, B might hire a lawyer and instruct him to play left if A plays bottom. If A is aware of these instructions, the situation is radically different from his point of view. If he knows about B's instructions to his lawyer, then he knows that if he plays bottom he will end up with a payoff of 0. So the sensible thing for him to do is to play top. In this case B has done better for himself by *limiting* his choices.

29.8 A Game of Entry Deterrence

In our examination of oligopoly we took the number of firms in the industry as fixed. But in many situations, entry is possible. Of course, it is in the interest of the firms in the industry to try to prevent such entry. Since they are already in the industry, they get to move first and thus have an advantage in choosing ways to keep their opponents out.

Suppose, for example, that we consider a monopolist who is facing a threat of entry by another firm. The entrant decides whether or not to come into the market, and then the incumbent decides whether or not to cut its price in response. If the entrant decides to stay out, it gets a payoff of 1 and the incumbent gets a payoff of 9.

If the entrant decides to come in, then its payoff depends on whether the incumbent fights—by competing vigorously—or not. If the incumbent

fights, then we suppose that both players end up with 0. On the other hand, if the incumbent decides not to fight, we suppose that the entrant gets 2 and the incumbent gets 1. See Figure 29.2.

Note that this is exactly the structure of the sequential game we studied earlier, and thus it has a structure identical to that depicted in Figure 29.1. The incumbent is player B, while the potential entrant is player A. The top strategy is to stay out, and the bottom strategy is to enter. The left strategy is to fight and the right strategy is not to fight. As we've seen in this game, the equilibrium outcome is for the potential entrant to enter and the incumbent *not* to fight.

The new entry game. This figure depicts the entry game with the changed payoffs.

Figure 29.2

The incumbent's problem is that he cannot precommit himself to fighting if the other firm enters. If the other firm enters, the damage is done and the rational thing for the incumbent to do is to live and let live. Insofar as the potential entrant recognizes this, he will correctly view any threats to fight as empty.

But suppose that the incumbent can purchase some extra production capacity that will allow him to produce more output at his current marginal cost. Of course, if he remains a monopolist, he won't want to actually use this capacity since he is already producing the profit-maximizing monopoly output.

But, if the other firm enters, the incumbent will now be able to produce so much output that he may well be able to compete much more successfully against the new entrant. By investing in the extra capacity, he will lower his costs of fighting if the other firm tries to enter. Let us assume that if he purchases the extra capacity and if he chooses to fight, he will make a profit of 2. This changes the game tree to the form depicted in .

Now, because of the increased capacity, the threat of fighting is credible. If the potential entrant comes into the market, the incumbent will get a payoff of 2 if he fights and 1 if he doesn't; thus the incumbent will rationally choose to fight. The entrant will therefore get a payoff of 0 if he enters, and if he stays out he will get a payoff of 1. The sensible thing for the potential entrant to do is to stay out.

But this means that the incumbent will remain a monopolist and never have to use his extra capacity! Despite this, it is worthwhile for the monopolist to invest in the extra capacity in order to make credible the *threat* of fighting if a new firm tries to enter the market. By investing in "excess" capacity, the monopolist has signaled to the potential entrant that he will be able to successfully defend his market.

Summary

1. A game can be described by indicating the payoffs to each of the players for each configuration of strategic choices they make.

2. A dominant strategy equilibrium is a set of choices for which each player's choices are optimal *regardless* of what the other players choose.

3. A Nash equilibrium is a set of choices for which each player's choice is optimal, given the choices of the other players.

4. The prisoner's dilemma is a particular game in which the Pareto efficient outcome is strategically dominated by an inefficient outcome.

5. If a prisoner's dilemma is repeated an indefinite number of times, then it is possible that the Pareto efficient outcome may result from rational play.

6. In a sequential game, the time pattern of choices is important. In these games, it can often be advantageous to find a way to precommit to a particular line of play.

REVIEW QUESTIONS

1. Consider the tit-for-tat strategy in the repeated prisoner's dilemma. Suppose that one player makes a mistake and defects when he meant to cooperate. If both players continue to play tit for tat after that, what happens?

2. Are dominant strategy equilibria always Nash equilibria? Are Nash equilibria always dominant strategy equilibria?

3. Suppose your opponent is *not* playing her Nash equilibrium strategy. Should you play your Nash equilibrium strategy?

4. We know that the single-shot prisoner's dilemma game results in a dominant Nash equilibrium strategy that is Pareto inefficient. Suppose we allow the two prisoners to retaliate after their respective prison terms. Formally, what aspect of the game would this affect? Could a Pareto efficient outcome result?

5. What is the dominant Nash equilibrium strategy for the repeated prisoner's dilemma game when both players know that the game will end after one million repetitions? If you were going to run an experiment with human players for such a scenario, would you predict that players would use this strategy?

6. Suppose that player B rather than player A gets to move first in the sequential game described in this chapter. Draw the extensive form of the new game. What is the equilibrium for this game? Does player B prefer to move first or second?

CHAPTER 30

GAME APPLICATIONS

In the last chapter we described a number of important concepts in game theory and illustrated them using a few examples. In this chapter we examine four important issues in game theory—cooperation, competition, coexistence, and commitment—and see how they work in various strategic interactions.

In order to do this, we first develop an important analytic tool, **best response curves**, which can be used to solve for equilibria in games.

30.1 Best Response Curves

Consider a two-person game, and put yourself in the position of one of the players. For any choice the other player can make, your **best response** is the choice that maximizes your payoff. If there are several choices that maximize your payoff, then your best response will be the set of all such choices.

For example, consider the game depicted in Table 30.1, which we used to illustrate the concept of a Nash equilibrium. If the column player chooses left, row's best response is to choose top; if column chooses right, then

A simple game **Table 30.1**

		Column	
		Left	Right
Row	Top	2, 1	0, 0
	Bottom	0, 0	1, 2

row's best response is to choose bottom. Similarly, the best responses for column are to play left in response to top and to play right in response to bottom.

We can write this out in a little table:

Column's choice:	Left	Right
Row's best response:	Top	Bottom
Row's choice:	Top	Bottom
Column's best response:	Left	Right

Notice that if column thinks that row will play top, then column will want to play left, *and* if row thinks that column will play left, row will want to play top. So the pair of choices (top, left) are mutually consistent in the sense that each player is making an optimal response to the other player's choice.

Consider a general two-person game in which row has choices $r_1, \ldots, r_R$ and column has choices $c_1, \ldots, c_C$. For each choice r that row makes, let $b_c(r)$ be a best response for column, and for each choice c that column makes, let $b_r(c)$ be a best response for row. Then a **Nash equilibrium** is a pair of strategies (r^*, c^*) such that

$$c^* = b_c(r^*)$$
$$r^* = b_r(c^*).$$

The concept of Nash equilibrium formalizes the idea of "mutual consistency." If row expects column to play left, then row will choose to play top, and if column expects row to play top, column will want to play left. So it is the *beliefs* and the *actions* of the players that are mutually consistent in a Nash equilibrium.

Note that in some cases one of the players may be indifferent among several best responses. This is why we only require that c^* be *one* of column's best responses, and r^* be *one* of row's best responses. If there is

a unique best response for each choice then the best response *curves* can be represented as best response *functions*.

This way of looking at the concept of a Nash equilibrium makes it clear that it is simply a generalization of the Cournot equilibrium described in Chapter 28. In the Cournot case, the choice variable is the amount of output produced, which is a continuous variable. The Cournot equilibrium has the property that each firm is choosing its profit-maximizing output, given the choice of the other firm.

The Bertrand equilibrium, also described in Chapter 28, is a Nash equilibrium in pricing strategies. Each firm chooses the price that maximizes its profit, given the choice that it thinks the other firm will make.

These examples show how the best response curve generalizes the earlier models, and allows for a relatively simple way to solve for Nash equilibrium. These properties make best response curves a very helpful tool to solve for an equilibrium of a game.

30.2 Mixed Strategies

Let us use best response functions to analyze the game shown in Table 30.2.

Table 30.2 **Solving for Nash equilibrium.**

		Ms. Column	
		Left	Right
Mr. Row	Top	2, 1	0, 0
	Bottom	0, 0	1, 2

We are interested in looking for mixed strategy equilibria as well as pure strategy equilibria, so we let r be the probability that row plays top, and $(1 - r)$ the probability that he plays bottom. Similarly, let c be the probability that column plays left, and $(1-c)$ the probability that she plays right. The pure strategies occur when r and c equal 0 or 1.

Let us calculate row's expected payoff if he chooses probability r of playing top and column chooses probability c of playing left. Look at the following array

Combination	Probability	Payoff to Row
Top, Left	rc	2
Bottom, Left	$(1-r)c$	0
Top, Right	$r(1-c)$	0
Bottom, Right	$(1-r)(1-c)$	1

To calculate the expected payoff to row, we weight row's payoffs in the third column by the probability that they occur, given in the second column, and add these up. The answer is

$$\text{Row's payoff} = 2rc + (1-r)(1-c),$$

which we can multiply out to be

$$\text{Row's payoff} = 2rc + 1 - r - c + rc.$$

Now suppose that row contemplates increasing r by Δr. How will his payoff change?

$$\begin{aligned}\Delta\text{payoff to row} &= 2c\,\Delta r - \Delta r + c\,\Delta r \\ &= (3c-1)\Delta r.\end{aligned}$$

This expression will be positive when $3c > 1$ and negative when $3c < 1$. Hence, row will want to increase r whenever $c > 1/3$, decrease r when $c < 1/3$, and be happy with any value of $0 \le r \le 1$ when $c = 1/3$.

Similarly, the payoff to column is given by

$$\text{Column's payoff} = cr + 2(1-c)(1-r).$$

Column's payoff will change when c changes by Δc according to

$$\begin{aligned}\Delta\,\text{payoff to column} &= r\,\Delta c + 2r\,\Delta c - 2\Delta c \\ &= (3r-2)\Delta c.\end{aligned}$$

Hence column will want to increase c whenever $r > 2/3$, decrease c when $r < 2/3$, and be happy with any value of $0 \le c \le 1$ when $r = 2/3$.

We can use this information to plot the best response curves. Start with row. If column choses $c = 0$, row will want to make r as small as possible, so $r = 0$ is the best response to $c = 0$. This choice will continue to be the best response up until $c = 1/3$, at which point *any* value of r between 0 and 1 is a best response. For all $c > 1/3$, the best response row can make is $r = 1$.

These curves are depicted in Figure 30.1. It is easy to see that they cross in three places: $(0,0)$, $(2/3, 1/3)$, and $(1,1)$, which correspond to the three Nash equilibria of this game. Two of these strategies are pure strategies, and one is a mixed strategy.

Figure 30.1 **Best response curves**. The two curves depict the best response of row and column to each other's choices. The intersections of the curves are Nash equilibria. In this case there are three equilibria, two with pure strategies and one with mixed strategies.

30.3 Games of Coordination

Armed with the tools of the last section we can examine our first class of games, **coordination games**. These are games where the payoffs to the players are highest when they can coordinate their strategies. The problem, in practice, is to develop mechanisms that enable this coordination.

Battle of the Sexes

The classic example of a coordination game is the so-called battle of the sexes. In this game, a boy and a girl want to meet at a movie but haven't had a chance to arrange which one. Alas, they forgot their cell phones, so they have no way to coordinate their meeting and have to guess which movie the other will want to attend.

The boy wants to see the latest action flick, while the girl would rather go to an art film, but they would both rather go to the same movie than not meet up at all. Payoffs consistent with these preferences are shown in

Table 30.3

The battle of the sexes.

		Girl	
		Action	Art
Boy	Action	2, 1	0, 0
	Art	0, 0	1, 2

Table 30.3. Note the defining feature of coordination games: the payoffs are higher when the players coordinate their actions than when they don't.

What are the Nash equilibria of this game? Luckily, this is just the game we used in the last section to illustrate best response curves. We saw there that there are three equilibria: both choose action, both choose art, or each chooses his or her preferred choice with probability 2/3.

Since all of these are possible equilibria, it is hard to say what will happen from this description alone. Generally, we would look to considerations outside the formal description of the game to resolve the problem. For example, suppose that the art film was a closer destination for one of the two players. Then both players might reasonably suppose that would be the equilibrium choice.

When players have good reasons to believe that one of the equilibria is more "natural" than the others, it is called a **focal point** of the game.

Prisoner's Dilemma

The prisoner's dilemma, which we discussed extensively in the last chapter, is also a coordination game. Recall the story: two prisoners can either confess, thereby implicating the other, or deny committing a crime. The payoffs are shown in Table 30.4.

The striking feature of the prisoner's dilemma is that confessing is a dominant strategy, even though coordination (both choose deny) is far superior in terms of the total payoff. Coordination would allow the prisoners to choose the best payoff, but the problem is that there is no easy way to make it happen in a single-shot game.

One way out of the prisoner's dilemma is to enlarge the game by adding new choices. We saw in the last chapter that an indefinitely repeated prisoner's dilemma game could achieve the cooperative outcome via strategies like tit for tat, in which players rewarded cooperation and punished lack of cooperation through their future actions. The extra strategic consideration

Table 30.4

The prisoner's dilemma.

		Player B	
		Confess	Deny
Player A	Confess	–3, –3	0, –6
	Deny	–6, 0	–1, –1

here is that refusing to cooperate today may result in extended punishment later on.

Another way to "solve" the prisoner's dilemma is to add the possibility of contracting. For example, both players could sign a contract saying that they will stick with the cooperative strategy. If either of them reneges on the contract, he or she will have to pay a fine or be punished in some way. Contracts are very helpful in achieving all sorts of outcomes, but they rely on the existence of a legal system that will enforce such contracts. This makes sense for business negotiations but is not an appropriate assumption in other contexts, such as military games or international negotiations.

Assurance Games

Consider the U.S.-U.S.S.R. arms race of the 1950s in which each country could build nuclear missiles or refrain from building them. The payoffs to these strategies might look like those shown in Table 30.5. The best outcome for both parties is to refrain from building the missiles, giving a payoff of (4, 4). But if one refrains while the other builds, the payoff will be 3 to the builder and 1 to the refrainer. The payoff if they both build missile sites is (2, 2).

It is not hard to see that there are two pure strategy Nash equilibria, (refrain, refrain) and (build, build). However, (refrain, refrain) is better for both parties. The trouble is, neither party knows which choice the other will make. Before committing to refrain, each party wants some *assurance* that the other will refrain.

One way to achieve this assurance is for one of the players to move first, by opening itself to inspection, say. Note that this can be unilateral, at least as long as one believes the payoffs in the game. If one player announces that it is refraining from deploying nuclear missiles and gives the other player sufficient evidence of its choice, it can rest assured that the other player will also refrain.

An arms race. **Table 30.5**

		U.S.S.R.	
		Refrain	Build
U.S.	Refrain	4, 4	1, 3
	Build	3, 1	2, 2

Chicken

Our last coordination game is based on an automobile game popularized in the movies. Two teenagers start at opposite ends of the street and drive in a straight line toward each other. The first to swerve loses face; if neither swerves, they both crash into each other. Some possible payoffs are shown in Table 30.6.

There are two pure strategy Nash equilibria, (row swerves, column doesn't) and (column swerves, row doesn't). Column prefers the first equilibrium and row the second, but each equilibrium is better than a crash. Note the difference between this and the assurance game; there, both players were better off doing the same thing (building or refraining) than doing different things. Here, both players are worse off doing the same thing (driving straight or swerving) than if they did different things.

Each player knows that if he can commit himself to driving straight, the other will chicken out. But of course, each player also knows that it would be crazy to crash into each other. So how can one of the players enforce his preferred equilibrium?

One important strategy is commitment. Suppose that row ostentatiously fastened a steering wheel lock on his car before starting out. Column, recognizing that row now has no choice but to go straight, would choose to swerve. Of course if both players put on a lock, the outcome would be disastrous!

How to Coordinate

If you are a player in a coordination game, you may want to get the other player to cooperate at an equilibrium that you both like (the assurance game), cooperate at an equilibrium one of you likes (battle of the sexes), play something other than the equilibrium strategy (the prisoner's dilemma), or make a choice leading to your preferred outcome (chicken).

In the assurance game, the battle of the sexes, and chicken, this can be accomplished by one player's moving first, and committing herself to a

particular choice. The other player can then observe the choice and respond accordingly. In the prisoner's dilemma, this strategy doesn't work: if one player chooses not to confess, it is in the other's interest to do so. Instead of sequential moves, repetition and contracting are major ways to "solve" the prisoner's dilemma.

Table 30.6

Chicken.

		Column	
		Swerve	Straight
Row	Swerve	0, 0	–1, 1
	Straight	1, –1	–2, –2

30.4 Games of Competition

The opposite pole from cooperation is competition. This is the famous case of **zero-sum games**, so called because the payoff to one player is equal to the losses of the other.

Most sports are effectively zero-sum games: a point awarded to one team is equivalent to a point subtracted from the other team. Competition is fierce in such games because the players' interests are diametrically opposed.

Let us illustrate a zero-sum game by looking at soccer, known as football in most of the world. Row is kicking a penalty shot and column is defending. Row can kick to the left or kick to the right; column can favor one side and defend to the left or defend to the right in order to deflect the kick.

We will express the payoffs to these strategies in terms of expected points. Obviously row will be more successful if column jumps the wrong way. On the other hand, the game may not be perfectly symmetric since row may be better at kicking in one direction than another and column may be better at defending one direction or the other.

Let us assume that row will score 80 percent of the time if he kicks to the left and column jumps to the right but only 50 percent of the time if column jumps to the left. If row kicks to the right, we will assume that he succeeds 90 percent of the time if column jumps to the left but 20 percent

Penalty point in soccer. **Table 30.7**

		Column	
		Defend left	Defend right
Row	Kick left	50, –50	80, –80
	Kick right	90, –90	20, –20

of the time if column jumps to the right. These payoffs are illustrated in Table 30.7.

Note that the payoffs in each entry sum to zero, indicating that the players have diametrically opposed goals. Row wants to maximize his expected payoff, and column wants to maximize her expected payoff—which means she wants to minimize row's payoff.

Obviously, if column knows which way row will kick she will have a tremendous advantage. Row, recognizing this, will therefore try to keep column guessing. In particular, he will kick sometimes to his strong side and sometimes to his weak side. That is, he will pursue a **mixed strategy**.

If row kicks left with probability p, he will get an expected payoff of $50p + 90(1 - p)$ when column jumps left and $80p + 20(1 - p)$ when column jumps right. Row wants to make this expected payoff as big as possible, and column wants to make it as small as possible.

For example, suppose that row chooses to kick left half the time. If column jumps left, row will have an expected payoff of $50 \times 1/2 + 90 \times 1/2 = 70$, and if column jumps right, row will have an expected payoff of $80 \times 1/2 + 20 \times 1/2 = 50$.

Column, of course, can carry through this same reasoning. If column believes that row will kick to the left half the time, then column will want to jump to the right, since this is the choice that minimizes row's expected payoff (thereby maximizing column's expected payoff).

Figure 30.2 shows row's expected payoffs for different choices of p. This simply involves graphing the two functions $50p+90(1-p)$ and $80p+20(1-p)$. Since these two expressions are linear functions of p, the graphs are straight lines.

Row recognizes that column will always try to minimize his expected payoff. Thus, for any p, the best payoff he can hope for is the *minimum* of the payoffs given by the two strategies. We've illustrated this by the colored line in Figure 30.2.

Where does the maximum of these minimum payoffs occur? Obviously, it occurs at the peak of the colored line, or, equivalently, where the two

Figure 30.2

Row's strategy. The two curves show row's expected payoff as a function of p, the probability that he kicks to the left. Whatever p he chooses, column will try to minimize row's payoff.

lines intersect. We can calculate this value algebraically by solving

$$50p + 90(1 - p) = 80p + 20(1 - p)$$

for p. You should verify that the solution is $p = .7$.

Hence, if row kicks to the left 70 percent of the time and column responds optimally, row will have an expected payoff of $50 \times .7 + 90 \times .3 = 62$.

What about column? We can perform a similar analysis for her choices. Suppose column decides to jump to the left with probability q and jump to the right with probability $(1 - q)$. Then row's expected payoff will be $50q + 80(1 - q)$ if column jumps to the left and $90q + 20(1 - q)$ if column jumps to the right. For each q, column will want to *minimize* row's payoff. But column recognizes that row wants to *maximize* this same payoff.

Hence, if column chooses to jump to the left with probability 1/2, she recognizes that row will get an expected payoff of $50 \times 1/2 + 80 \times 1/2 = 65$ if row kicks left and $90 \times 1/2 + 20 \times 1/2 = 55$ if row kicks right. In this case row will, of course, choose to kick left.

We can plot the two payoffs in Figure 30.3, which is analogous to the previous diagram. From column's viewpoint, it is the maximum of the two lines that is relevant, since this reflects row's optimal choice for each choice

of q. Hence, the diagram depicts these lines in color. Just as before we can find the best q for column—the point where row's maximum payoff is minimized. This occurs where

$$50q + 80(1 - q) = 90q + 20(1 - q),$$

which implies $q = .6$.

Column's strategy. The two lines show row's expected payoff as a function of q, the probability that column jumps to the left. Whatever q column chooses, row will try to maximize his own payoff.

Figure 30.3

We have now calculated the equilibrium strategies for each of the two players. Row should kick to the left with probability .7, and column should jump to the left with probability .6. These values were chosen so that row's payoffs and column's payoffs will be the same, whatever the other player does, since we found the values by equating the payoffs from the two strategies the opposing player could choose.

So when row chooses .7, column is indifferent between jumping left and jumping right, or, for that matter jumping left with any probability q. In particular, column is perfectly happy jumping left with probability .6.

Similarly, if column jumps left with probability .6, then row is indifferent between kicking left and kicking right, or any mixture of the two. In

particular, he is happy to kick left with probability .7. Hence these choices are a Nash equilibrium: each player is optimizing, given the choices of the other.

In equilibrium row scores 62 percent of the time and fails to score 38 percent of the time. This is the best he can do, if the other player responds optimally.

What if column responds nonoptimally? Can row do better? To answer this question, we can use the best response curves introduced at the beginning of this chapter. We have already seen that when p is less than .7, column will want to jump left, and when p is greater than .7, column will want to jump right. Similarly when q is less than .6, row will want to kick left, and when q is greater than .6, row will want to kick right.

Figure 30.4 depicts these best response curves. Note that they intersect at the point where $p = .7$ and $q = .6$. The nice thing about the best response curves is that they tell each player what to do for every choice the other player makes, optimal or not. The only choice that is an optimal response to an optimal choice is where the two curves cross—the Nash equilibrium.

Figure 30.4

Best response curves. These are the best response curves for row and column, as a function of p, the probability that row kicks to the left, and q, the probability that column jumps to the left.

30.5 Games of Coexistence

We have interpreted mixed strategies as randomization by the players. In the penalty kick game, if row's strategy is to play left with probability .7 and right with probability .3, then we think that row will "mix it up" and play left 70 percent of the time and right 30 percent of the time.

But there is another interpretation. Suppose that kickers and goalies are matched up at random and that 70 percent of the kickers always kick left and 30 percent always kick right. Then, from the goalie's point of view, it is just like facing a single player who randomizes with those probabilities.

This isn't all that compelling as a story for soccer games, but it is a reasonable story for animal behavior. The idea is that various kinds of behavior are genetically programmed and that evolution selects the mixtures of the population that are stable with respect to evolutionary forces. In recent years, biologists have come to regard game theory as an indispensible tool to study animal behavior.

The most famous game of animal interaction is the **hawk-dove game**. This doesn't refer to a game between hawks and doves (which would have a pretty predictable outcome) but rather to a game involving a single species that exhibits two kinds of behavior.

Think of a wild dog. When two wild dogs come across a piece of food, they have to decide whether to fight or to share. Fighting is the hawkish strategy: one will win and one will lose. Sharing is a dovish strategy: it works well when the other player is also dovish, but if the other player is hawkish, the offer to share is rejected and the dovish player will get nothing.

A possible set of payoffs is given in Table 30.8.

Table 30.8 **Hawk-dove game.**

		Column	
		Hawk	Dove
Row	Hawk	–2, –2	4, 0
	Dove	0, 4	2, 2

If both wild dogs play dove, they end up with (2, 2). If one plays hawk and the other plays dove, the hawkish player wins everything. But if both play hawk, each dog will be seriously injured.

It obviously can't be an equilibrium if everyone plays hawk, since if some dog played dove, it would end up with 0 rather than -2. And if all dogs played dove, it would pay someone to deviate and play hawk. So there will have to be some mixture of hawk types and dove types in equilibrium. What sort of mixture should we expect?

Suppose that the fraction playing hawk is p. Then a hawk will meet another hawk with probability p and meet a dove with probability $1 - p$. The expected payoff to the hawk type will be

$$H = -2p + 4(1 - p).$$

The expected payoff to the dove type will be

$$D = 2(1 - p).$$

Suppose that the type that has the higher payoff reproduces more rapidly, passing its tendency to play hawk or dove on to its offspring. So if $H > D$, we would see the fraction of hawk types in the population increase, and if $H < D$, we would expect to see the number of dove types increase.

The only way the population can be in equilibrium is if the payoffs to each type are the same. This requires

$$H = -2p + 4(1 - p) = 2(1 - p) = D,$$

which solves for $p = 1/2$.

We have found that a 50-50 mixture of doves and hawks is an equilibrium. Is it stable, in some sense? We plot the payoffs to hawk and dove as a function of p, the fraction of the population playing hawk in Figure 30.5. Note that when $p > 1/2$, the payoff to playing hawk is less than that of playing dove, so we would expect to see the doves reproduce more rapidly, moving us back to the equilibrium 50-50 ratio. Similarly, when $p < 1/2$, the payoff to hawk is greater than the payoff to dove, leading the hawks to reproduce more rapidly.

This argument shows that not only is $p = 1/2$ an equilibrium but it is also stable under evolutionary forces. Considerations of this sort lead to a concept known as an **evolutionarily stable strategy** or an **ESS**.[1] Remarkably, an ESS turns out to be a Nash equilibrium, even though it was derived from quite different considerations.

The Nash equilibrium concept was designed to deal with calculating, rational individuals, each of whom is trying to devise a strategy appropriate for the best strategy the other player might choose. The ESS was designed to model animal behavior under evolutionary forces, where strategies that had greater fitness payoffs would reproduce more rapidly. But the ESS equilibria are also Nash equilibria, giving another argument for why this particular concept in game theory is so compelling.

[1] See John Maynard Smith, *Evolution and the Theory of Games*, (Cambridge University Press, 1982).

Figure 30.5

Payoffs in the hawk-dove game. The payoff to hawk is depicted in color; the payoff to dove is in black. When $p > 1/2$, the payoff to hawk is less than dove and vice versa, showing that the equilibrium is stable.

30.6 Games of Commitment

The previous examples involving games of cooperation and competition have been concerned with games with **simultaneous moves**. Each player had to make his or her choice without knowing what the other player was choosing (or had chosen). Indeed, games of coordination or competition can be quite trivial if one player knows the other's choices.

In this section we turn our attention to games with **sequential moves**. An important strategic issue that arises in such games is **commitment**. To see how this works, look back at the game of chicken described earlier in this chapter. We saw there that if one player could force himself to choose straight, the other player would optimally choose to swerve. In the assurance game, the outcome would be better for both players if one of them moved first.

Note that this committed choice must be both irreversible and observable by the other player. Irreversibility is part of what it means to be committed, while observability is crucial if the other player is going to be persuaded to change his or her behavior.

The Frog and the Scorpion

We begin with the fable of the frog and the scorpion. They were standing on the bank of the river, trying to figure out a way across. "I know," said

the scorpion "I will climb on your back and you can swim across the river." The frog said, "But what if you sting me with your stinger?" The scorpion said, "Why would I do that? Then we would both die."

The frog found this convincing, so the scorpion climbed on his back and they started across the river. Halfway across, at the deepest point, the scorpion stung the frog. Writhing in pain, the frog cried out, "Why did you do that? Now we are both doomed!" "Alas," said the scorpion, as he sank into the river, "it is my nature."

Let's look at the frog and the scorpion from the viewpoint of game theory. Figure 30.6 depicts a sequential game with payoffs consistent with the story. Start at the bottom of the game tree. If the frog refuses the scorpion, both get nothing. Looking up one line, we see that if the frog carries the scorpion, he receives utility 5, for doing a good deed, and the scorpion receives a payoff of 3, for getting across the river. In line where the frog is stung, he receives a payoff of −10, and the scorpion gets a payoff of 5, representing the satisfaction from fulfilling his natural instincts.

Figure 30.6 **The frog and the scorpion**. If the frog chooses to carry the scorpion, the scorpion will choose to sting him and both will die.

It is best to start with the final move of the game: the scorpion's choice of sting or refrain. Stinging has a higher payoff to the scorpion because "it is his nature" to sting. Hence the frog should rationally choose to refuse to carry the scorpion. Unfortunately, the frog didn't understand the scorpion's payoffs; apparently, he thought that the scorpion's payoffs

looked something like those in Figure 30.7. Alas, this mistake was fatal for the frog.

A smart frog would figure out some way to make the scorpion commit to not stinging. He could, for example, tie his tail. Or he could hire a hit frog, who would retaliate against the scorpion's family. Whatever the strategy, the critical thing for the frog to do is to change the payoffs to the scorpion by making stinging more costly or refraining more rewarding.

The frog and the scorpion. With these payoffs, if the frog chooses to carry the scorpion, the scorpion will not choose to sting him, and both will make it across the river safely.

Figure 30.7

The Kindly Kidnapper

Kidnapping for ransom is a big business in some parts of the world. In Columbia, it is estimated that there are over 2,000 kidnappings for ransom per year. In the former Soviet Union, kidnappings rose from 5 in 1992 to 105 in 1999. Many of the victims are Western businesspeople.

Some countries, such as Italy, have laws against paying ransom. The reasoning is that if the victim's family or employers can commit themselves not to pay ransom, then the kidnappers will have no motive to abduct the victim in the first place.

The problem is, of course, once a kidnapping has taken place, a victim's family will prefer to pay the kidnappers, even if it is illegal to do so. Hence penalties for paying ransom may not be effective as a commitment device.

Suppose some kidnappers abduct a hostage and then discover that they can't get paid. Should they release the hostage? The hostage, of course, promises not to reveal the identity of the kidnappers. But will he keep this promise? Once he is released, he has no incentive to do so—and every incentive to try to punish the kidnappers. Even if the kidnappers want to let the hostage go, they can't do so for fear of being identified.

Figure 30.8 depicts some possible payoffs. The kidnapper would feel bad about killing the hostage, receiving a payoff of −3. Of course, the hostage would feel even worse, receiving a payoff of −10. If the hostage is released, and refrains from identifying the kidnapper, the hostage gets a payoff of 3 and the kidnapper gets a payoff of 5. But if the hostage does identify the kidnapper, he gets a payoff of 5, leaving the kidnapper with a payoff of −5.

Figure 30.8 **Kidnap game**. The kidnapper would like to release the hostage, but if he does, the hostage will identify him.

Now it is the hostage who has the commitment problem: how can he convince the kidnappers that he won't renege on his promise and reveal their identity?

The hostage needs to figure out a way to change the payoffs of the game. In particular, he needs to find a way to impose a cost on himself if he identifies the kidnappers.

Thomas Schelling, an economist at the University of Maryland who has worked extensively on strategic analysis in dynamic games, suggests that the hostage might have the kidnappers photograph him in some embarrassing act and leave them with the photos. This effectively changes the payoffs

from his subsequently revealing the identity of the kidnappers, since they then have the option of revealing the embarrassing photograph.

This sort of strategy is known as an "exchange of hostages." In the Middle Ages, when two kings wanted to ensure a contract wouldn't be broken, they would exchange hostages such as family members. If either king broke the agreement, the hostages would be sacrificed. Neither wanted to sacrifice their family members, so each king would have an incentive to respect the terms of their contract.

In the case of the kidnapping, the embarrassing photo would impose costs on the hostage if it were released, thereby ensuring that he will stick to his agreement not to reveal the identity of the kidnappers.

When Strength Is Weakness

Our next example comes from the world of animal psychology. It turns out that pigs quickly establish dominance-subordinateness relations, in which the dominant pig tends to boss the subordinate pig around.

Some psychologists put two pigs, one dominant, one subordinate, in a long pen.[2] At one end of the pen was a lever that would release a portion of food to a trough located at the other end of the pen. The question of interest was this: which pig would push the lever and which would eat the food?

Somewhat surprisingly the outcome of the experiment was that the dominant pig pressed the lever, while the subordinate pig waited for the food. The subordinate pig then ate most of the food, while the dominant pig rushed as fast as it could to the trough end of the pen, ending up with only a few scraps. Table 30.9 depicts a game that illustrates the problem.

Table 30.9

Pigs pressing levers.

		Dominant Pig	
		Don't press lever	Press lever
Subordinate Pig	Don't press lever	0, 0	4, 1
	Press lever	0, 5	2, 3

[2] The original reference is Baldwin and Meese, "Social Behavior in Pigs Studied by Means of Operant Conditioning," (*Animal Behavior,* (1979)). I draw on the description of John Maynard Smith, *Evolution and the Theory of Games* (Cambridge University Press, 1982).

The subordinate pig compares a payoff of (0, 4) to (0, 2) and concludes, sensibly enough, that pressing the lever is dominated by not pressing it. Given that the subordinate pig doesn't press the lever, the dominant pig has no choice but to do so.

If the dominant pig could refrain from eating all the food and reward the subordinate pig for pressing the lever, it could achieve a better outcome. The problem is that pigs have no contracts, and the dominant pig can't help being a hog!

As in the case of the kindly kidnapper, the dominant pig has a commitment problem. If he could only commit to not eating all the food, he would end up much better off.

Savings and Social Security

Commitment problems aren't limited to the animal world. They also show up in economic policy.

Saving for retirement is an interesting and timely example. Everyone gives lip service to the fact that saving is a good idea. Unfortunately, few people actually do it. Part of the reason for the reluctance to save is that individuals recognize that society won't let them starve, so there is a good chance they will be bailed out later on.

To formulate this in a game between the generations, let's consider two strategies for the older generation: save or squander. The younger generation likewise has two strategies: support their elders or save for their own retirement. A possible game matrix is shown in Table 30.10.

Table 30.10 **Intergenerational conflict over savings.**

		Younger Generation	
		Support	Refrain
Older Generation	Save	2, –1	1, 0
	Squander	3, –1	–2, –2

If the older generation saves and the younger generation also supports them, the old folks end up with a utility level of 2 and the young folks end up with −1. If the older generation squanders and the younger generation supports them, the elders end up with a utility of 3 and the young folks end up with −1.

If the younger generation refrains from providing support to their elders and the older generation saves, the old folks get 1 and the young folks get 0. Finally, if the old folks squander and the young folks neglect them, each ends up with utility of -2, the old folks from starving and the young folks from having to watch.

It is not hard to see that there are two Nash equilibria in this game. If the old folks choose to save, then the young folks will choose optimally to neglect them. But if the old folks choose to squander, then it is optimal for the younger generation to support them. And of course, given that the younger generation will support their elders, it is optimal for their elders to squander!

However, this analysis ignores the time structure of the game: one of the (few) advantages of being old is that you get to move first. If we draw out the game tree, the payoffs become those in Figure 30.9.

Figure 30.9

The savings game in extended form. Knowing that the younger generation will support them, the older generation chooses to squander. The subgame perfect equilibrium is (support, squander).

If the oldsters save, the youngsters will choose to neglect them, so the oldsters end up with a payoff of 1. If the oldsters squander, they know that the youngsters won't be able to bear watching them starve, so the oldsters end up with a payoff of 3. Hence the sensible thing for the oldsters to do is to squander, knowing they will be bailed out later on.

Of course, most developed countries now have a program like the U.S. Social Security program that forces each generation to save for retirement.

EXAMPLE: Dynamic inefficiency of price discrimination

Recall the definition of first-degree price discrimination: the seller prices the good so that it extracts the entire consumer surplus from the buyer. We argued in Chapter 14 that this was efficient since all valuable trades were carried out.

Despite this, there are cases where first-degree price discrimination can lead to inefficiency. Suppose a salesman is such a good bargainer that he can extract the entire consumer surplus from each customer—he is an expert at first-degree price discrimination. But here's the problem: if everyone knows that he can extract all the consumer surplus, why would they ever want to buy something from him? At best, they would just break even.

This is a bit like the example of the dominant pig in the previous section. The dominant pig ended up in a worse position because it could not commit to sharing. Well, people are (usually) more intelligent than pigs, so they realize that if they want some repeat business they should make sure their customers get some surplus out of the transaction.

To do this it is necessary to find a way to commit to offering a good deal to the customers, perhaps by offering a coupon that can be used after the negotiation has taken place or by having a posted price that is available to everyone.

Hold Up

Consider the following strategic interaction. You hire a contractor to build a warehouse. After the plans are approved and the construction is almost done, you realize that the color is bad, so you ask the contractor to change the paint, which involves a trivial expense. The contractor comes back and says: "That change order will be $1500, please."

You recognize that it will cost you at least that much to delay completion until you can find a painter, and you really do want the new color, so, muttering under your breath, you pay the cost. Congratulations, you have been held up!

Of course, contractors are not the only party at fault in this sort of game. The clients can "hold up" their payment as well, causing lots of grief for the contractor.

The game tree for the hold-up problem is depicted in Figure 30.10. We suppose that the value the owner places on having the new paint is $1500 and that the actual cost of painting is $200. Starting at the top of leaves

of the tree, if the contractor charges $1500, it will realize a profit of $1300, and the client gets a net utility of zero.

If the client looks for another painter, it will cost him $200 to pay the painter and, say, $1400 in lost time. He gets the color he wants which is worth $1500, buy has to pay $1600 in direct costs and delay costs, leaving him with a net loss of $100.

If the contractor charges the client the actual cost of $200, he breaks even and the client gets a $1500 value for $200, leaving him with a net payoff of $1300.

As can be seen, the optimal choice for the contractor is to extort the payment, and the optimal choice for the customer is to give in. But a sensible client will recognize that change orders will occur in any project. Because of this, the client will be reluctant to hire contractors with a reputation for extortion which is, of course, bad for the contractor.

The hold-up problem. The contractor charges a high price for the change since the client has no alternative.

Figure 30.10

How do firms solve the hold-up problem? The basic answer is contracts. Normally, contractors negotiate a contract specifying what kinds of change orders are appropriate and how their costs will be determined. Sometimes there are even arbitration or other dispute resolution procedures built into the contracts. A lot of time, energy, and money goes into writing contracts just to make certain that hold up won't occur.

But contracts aren't the only solution. Another way to solve the problem is through commitment. For example, the contractor might post a bond

guaranteeing timely completion of the project. Again, there will generally be some objectively specified terms about what constitutes completion.

Another important factor is reputation. Obviously, a contractor who persistently tries to extort his customers will get a bad reputation. He won't be hired again by this customer, and he certainly won't get good recommendations. This reputation effect can be examined in a repeated game context in which hold up today will cost the contractor in the future.

30.7 Bargaining

The classical bargaining problem is divide the dollar. Two players have a dollar that they want to divide between them. How do they do it?

The problem, as stated, has no answer since there is too little information to construct a reasonable model. The challenge in modeling bargaining is to find some other dimensions on which the players can negotiate.

One solution, the **Nash bargaining model**, takes an axiomatic approach by specifying certain properties that a reasonable bargaining solution should have and then proving that there is only one outcome that satisfies these axioms.

The outcome ends up depending on how risk averse the players are and what will happen if no bargain is made. Unfortunately, a full treatment of this model is beyond the scope of this book.

An alternative approach, the **Rubinstein bargaining model**, looks at a sequence of choices and then solves for the subgame perfect equilibrium. Luckily the basic insight of this model is easy to illustrate in simple cases.

Two players, Alice and Bob, have \$1 to divide between them. They agree to spend at most three days negotiating over the division. The first day, Alice will make an offer, Bob either accepts or comes back with a counteroffer the next day, and on the third day Alice gets to make one final offer. If they cannot reach an agreement in three days, both players get zero.

We assume Alice and Bob differ in their degree of impatience: Alice discounts payoffs in the future at a rate of α per day, and Bob discounts payoffs at a rate of β per day. Finally, we assume that if a player is indifferent between two offers, he will accept the one that is most preferred by his opponent. This idea is that the opponent could offer some arbitrarily small amount that would make the player strictly prefer one choice and that this assumption allows us to approximate such an "arbitrarily small amount" by zero. It turns out that there is a unique subgame perfect equilibrium of this bargaining game.

We start our analysis at the end of the game, right before the last day. At this point Alice can make a take-it-or-leave-it offer to Bob. Clearly, the optimal thing for Alice to do at this point is to offer Bob the smallest possible amount that he would accept, which, by assumption, is zero. So if

the game actually lasts three days, Alice would get \$1 and Bob would get zero (i.e., an arbitrarily small amount).

Now go back to the previous move, when Bob gets to propose a division. At this point Bob should realize that Alice can guarantee herself \$1 on the next move by simply rejecting his offer. A dollar next period is worth α to Alice this period, so any offer less than α would be sure to be rejected. Bob certainly prefers $1-\alpha$ now to zero next period, so he should rationally offer α to Alice, which Alice will then accept. So if the game ends on the second move, Alice gets α and Bob gets $1-\alpha$.

Now move to the first day. At this point Alice gets to make the offer and she realizes that Bob can get $1-\alpha$ if he simply waits until the second day. Hence Alice must offer a payoff that has at least this present value to Bob in order to avoid delay. Thus she offers $\beta(1-\alpha)$ to Bob. Bob finds this (just) acceptable and the game ends. The final outcome is that the game ends on the first move with Alice receiving $1-\beta(1-\alpha)$ and Bob receiving $\beta(1-\alpha)$.

The first panel in Figure 30.11 illustrates this process for the case where $\alpha = \beta < 1$. The outermost diagonal line shows the possible payoff patterns on the first day, namely, all payoffs of the form $x_A + x_B = 1$. The next diagonal line moving toward the origin shows the present value of the payoffs if the game ends in the second period: $x_A + x_B = \alpha$. The diagonal line closest to the origin shows the present value of the payoffs if the game ends in the third period; the equation for this line is $x_A + x_B = \alpha^2$. The right-angled path depicts the minimum acceptable divisions each period, leading up to the final subgame perfect equilibrium. The second panel in Figure 30.11 shows how the same process might look with more stages in the negotiation.

It is natural to let the horizon go to infinity and ask what happens in the infinite game. It turns out that the subgame perfect equilibrium division is

$$\text{Payoff to Alice} = \frac{1-\beta}{1-\alpha\beta}$$

$$\text{Payoff to Bob} = \frac{\beta(1-\alpha)}{1-\alpha\beta}.$$

Note that if $\alpha = 1$ and $\beta < 1$, then Alice receives the entire payoff.

The Ultimatum Game

The Rubinstein bargaining model is so elegant that economists rushed to test it in the laboratory. They found, alas, that elegance does not imply accuracy. Naive subjects (i.e., noneconomics majors) aren't very good at looking ahead more than one or two steps, if that.

In addition, there are other factors that cause problems. To see this, let us examine a one-step version of the bargaining model described above.

Figure 30.11 **A bargaining game**. The heavy line connects together the equilibrium outcomes in the subgames. The point on the line that is furthest out is the subgame perfect equilibrium.

Alice and Bob still have \$1 to divide between them. Alice proposes a division, and, if Bob agrees, the game ends. The question is, what should Alice say?

According to the theory, she should propose something like 99 cents for Alice, 1 cent for Bob. Bob, figuring that 1 cent is better than nothing, accepts, and Alice goes home happy that she studied economics.

Unfortunately, it doesn't work out like that. A more likely outcome is that Bob, disgusted by the paltry 1 cent, says "No way," and Alice ends up with nothing. Alice, recognizing this possibility, will tend to sweeten the offer. In actual experiments, the average offer for U.S. undergraduates is about 45 cents, and this offer tends to be accepted most of the time.

The offering players are behaving rationally, in the sense that the 45 cent offer is pretty close to maximizing the expected payoff, given the observed frequency of rejection. It is the receiving players who behave differently than the theory predicts, since they reject small offers, even though this makes them worse off.

There are many proposed explanations for this. One view is that too small an offer violates **social norms** of behavior. Indeed, economists have found quite significant cross-cultural differences in behavior in ultimatum games. Another, not inconsistent view, is that receivers get some utility payoff from hurting the offerers, in retaliation for the small offer. After all, if all you are losing is a penny, the satisfaction of striking back at the other player is pretty attractive by comparison. We will the ultimatum game in more detail in the next chapter.

Summary

1. A player's best response function gives the optimal choice for him as a function of the choices the other player(s) might make.

2. A Nash equilibrium in a two-person game is a pair of strategies, one for each player, each of which is a best response to the other.

3. A mixed strategy Nash equilibrium involves randomizing among several strategies.

4. Common games of coordination are the battle of the sexes, where both players want to do the same thing rather than different things; the prisoner's dilemma, where the dominant strategy ends up hurting both players; the assurance game, where both players want to cooperate as long as they think the other will cooperate; and chicken, where players want to avoid doing the same thing.

5. A two-person zero-sum game is one where the payoffs to one player are the negative of the payoffs to the other.

6. Evolutionary games are concerned with outcomes that are stable under population reproduction.

7. In sequential games, players move in turn. Each player therefore has to reason about what the other will do in response to his or her choices.

8. In many sequential games, commitment is an important issue. Finding ways to force commitment to play particular strategies can be important.

REVIEW QUESTIONS

1. In a two-person Nash equilibrium, each player is making a best response to what? In a dominant strategy equilibrium, each player is making a best response to what?

2. Look at the best responses for row and column in the section on mixed strategies. Do these give rise to best response functions?

3. If both players make the same choice in a coordination game, all will be well.

4. The text claims that row scores 62 percent of the time in equilibrium. Where does this number come from?

5. A contractor says that he intends to "low-ball the bid and make up for it on change orders." What does he mean?

CHAPTER 31

BEHAVIORAL ECONOMICS

The economic model of consumer choice that we have studied is simple and elegant, and is a reasonable starting place for many sorts of analysis. However, it is most definitely not the whole story, and in many cases a deeper model of consumer behavior is necessary to accurately describe choice behavior.

The field of **behavioral economics** is devoted to studying how consumers actually make choices. It uses some of the insights from psychology to develop predictions about choices people will make and many of these predictions are at odds with the conventional economic model of "rational" consumers.

In this chapter we will look at some of the most important phenomena that have been identified by behavioral economists, and contrast the predictions of these behavioral theories with those presented earlier in this book.[1]

[1] In writing this chapter, I have found Colin F. Camerer, George Loewenstein, and Matthew Rabin's book *Advances in Behavioral Economics,* Princeton University Press, 2003, to be very useful, particularly the introductory survey by Camerer and Loewenstein. Other works will be noted as the relevant topics are discussed.

31.1 Framing Effects in Consumer Choice

In the basic model of consumer behavior, the choices were described in the abstract: red pencils or blue pencils, hamburgers and french fries, and so on. However, in real life, people are strongly affected by how choices are presented to them or **framed**.

A faded pair of jeans in a thrift shop may be perceived very differently than the same jeans sold in an exclusive store. The decision to buy a stock may feel quite different than the decision to sell a stock, even if both transactions end up with the same portfolio. A store might sell dozens of copies of a book priced at $29.95, whereas the same book priced at $29.00 would have substantially fewer sales.

These are all examples of **framing effects**, and they are clearly a powerful force in choice behavior. Indeed, much of marketing practice is based on understanding and utilizing such biases in consumer choice.

The Disease Dilemma

Framing effects are particularly common in choices involving uncertainty. For example, consider the following decision problem:[2]

A serious disease threatens 600 people. You are offered a choice between two treatments, A and B, which will yield the following outcomes.

Treatment A. Saving 200 lives for certain.

Treatment B. A 1/3 chance of saving 600 lives and a 2/3 chance of saving no one.

Which would you choose? Now consider the choices between these treatments.

Treatment C. Having 400 people die for certain.

Treatment D. A 2/3 chance of 600 people dying and a 1/3 chance of no one dying.

Now which treatment would you choose?

[2] A. Tversky and D. Kahneman, 1981, "The framing of decisions and the psychology of choice," *Science,* 211, 453–458.

In the **positive framing** comparison—which describes how many people will live—most individuals choose A over B, but in the **negative framing** comparison most people choose D over C even though the outcomes in A-C and B-D are exactly the same. Apparently, framing the question positively (in terms of lives saved) makes a treatment much more attractive than framing the choice negatively (in terms of lives lost).

Even expert decisions makers can fall into this trap. When psychologists tried this question on a group of physicians, 72 percent of them chose the safe treatment A over the risky treatment B. But when the question was framed negatively, only 22 percent chose the risky treatment C while 72 percent chose the safe treatment.

Though few of us are faced with life-or-death decisions, there are similar examples for more mundane choices, such as buying or selling stocks. A rational choice of an investment portfolio would, ideally, depend on an assessment of the possible outcomes of the investments rather than how one acquired those investments.

For example, suppose that you are given 100 shares of stock in ConcreteBlocks.com (whose slogan is "We give away the blocks, you pay for packing and shipping"). You might be reluctant to sell shares you received as a gift despite the fact that you would never consider buying them yourself.

People are often reluctant to sell losing stocks, thinking that they will "come back." Maybe they will, maybe they won't. But ultimately you shouldn't let history determine your investment portfolio—the right question to ask is whether you have the portfolio choices today that you want.

Anchoring Effects

The hypothetical ConcreteBlocks.com example described above is related to the so-called **anchoring effect.** The idea here is that people's choices can be influenced by completely spurious information. In a classic study the experimenter spun a wheel of fortune and pointed out the number that came up to a subject.[3] The subject was then asked whether the number of African countries in the United Nations was greater or less than the number on the wheel of fortune.

After they responded, the subjects were asked for their best guess about how many African countries were in the United Nations. Even though the number shown on the wheel of fortune was obviously random, it exerted a significant influence on the subjects' reported guesses.

In a similar experimental design, MBA students were given an expensive bottle of wine and then asked if they would pay an amount for that bottle equal to the last two digits of their Social Security number. For example,

[3] D. Kahneman and A. Tversky, 1974, "Judgment under uncertainty: Heuristics and biases," *Science*, 185: 1124–1131.

if the last two digits were 29, the question was "Would you pay $29 for this bottle of wine?"

After answering that question, the students were asked what the maximum amount is that they *were* willing to pay for the wine. Their answers to this latter question were strongly influenced by the price determined by the last two digits of their Social Security number. For example, those with Social Security digits of 50 or under were willing to pay $11.62 on average, while those with digits in the upper half of the distribution were willing to pay $19.95 on average.

Again, these choices seem like mere laboratory games. However, there are very serious economic decisions that can also be influenced by minor variations in the way the choice is framed.

Consider, for example, choices of pension plans.[4]

Some economists looked at data from three employers that offered automatic enrollment in 401(k) plans. Employees could opt out, but they had to make an explicit choice to do so. The economists found that the participation rate in these programs with automatic enrollment was spectacularly high, with over 85 percent of workers accepting the default choice of enrolling in the 401(k) plans.

That's the good news. The bad news is that almost all of these workers also chose the default investment, typically a money market fund with very low returns and a low monthly contribution. Presumably, the employers made the default investment highly conservative to eliminate downside risk and possible employee lawsuits.

In subsequent work, these economists examined the experience at a company where there was no default choice of pension plan: within a month of starting work, employees were required to choose either to enroll in the 401(k) plan or to postpone enrollment.

By eliminating the standard default choices of non-enrollment, and of enrollment in a fund that had low rates of return, this "active decision" approach raised participation rates from 35 percent to 70 percent for newly hired employees. Moreover, employees who enrolled in the 401(k) plan overwhelmingly chose high savings rates.

As this example illustrates, careful design of human resources benefits programs can make a striking difference in which programs are chosen, potentially having a large effect on consumer savings behavior.

Bracketing

People often have trouble understanding their own behavior, finding it too difficult to predict what they will actually choose in different circumstances.

[4] James Choi, David Laibson, Brigitte Madrian, and Andrew Metrick, "For Better or for Worse: Default Effects and 401(k) Savings Behavior," NBER working paper, W8651, 2001.

For example, a marketing professor gave students a choice of six different snacks that they could consume in each of three successive weeks during class.[5] (You should be so lucky!) In one treatment, the students had to choose the snacks in advance; in the other treatment, they chose the snacks on each day then immediately consumed them.

When the students had to choose in advance, they chose a much more diverse set of snacks. In fact, 64 percent chose a different snack each week in this treatment compared to only 9 percent in the other group. When faced with making the choices all at once, people apparently preferred variety to exclusivity. But when it came down to actually choosing, they made the choice with which they were most comfortable. We are all creatures of habit, even in our choice of snacks.

Too Much Choice

Conventional theory argues that more choice is better. However, this claim ignores the costs of making choices. In affluent countries, consumers can easily become overwhelmed with choices, making it difficult for them to arrive at a decision.

In one experiment, two marketing researchers set up sampling booths for jam in a supermarket.[6] One booth offered 24 flavors and one offered only 6. More people stopped at the larger display, but substantially more people actually bought jam at the smaller display. More choice seemed to be attractive to shoppers, but the profusion of choices in the larger display appeared to make it more difficult for the shoppers to reach a decision.

Two experts in behavioral finance wondered whether the same problem with "excessive choice" showed up in investor decisions. They found that people who designed their own retirement portfolios tended to be just as happy with the average portfolio chosen by their co-workers as they were with their own choice. Having the flexibility to construct their own retirement portfolios didn't seem to make investors feel better off.[7]

Constructed Preferences

How are we to interpret these examples? Psychologists and behavioral economists argue that preferences are not a guide to choice; rather, preferences are "discovered" in part through the experiences of choice.

[5] I. Simonson, 1990, "The effect of purchase quantity and timing on variety-seeking behavior," *Journal of Marketing Research*, 17: 150–164.

[6] Sheena S. Iyengar and Mark R. Lepper, "When choice is demotivating: can one desire too much of a good thing?" *Journal of Personality and Social Psychology*, 2000.

[7] Shlomo Benartzi and Richard Thaler, "How Much Is Investor Autonomy Worth?" UCLA working paper, 2001.

Imagine watching someone in the supermarket picking up a tomato, putting it down, then picking it up again. Do they want it or not? Is the price-quality combination offered acceptable? When you watch such behavior, you are seeing someone who is "on the margin" in terms of making the choice. They are, in the psychologists' interpretation, *discovering* their preferences.

Conventional theory treats preferences as preexisting. In this view, preferences *explain* behavior. Psychologists instead think of preferences as being constructed—people develop or create preferences through the act of choosing and consuming.

It seems likely that the psychological model is a better description of what actually happens. However, the two viewpoints are not entirely incompatible. As we have seen, once preferences have been discovered, albeit by some mysterious process, they tend to become built-in to choices. Choices, once made, tend to anchor decisions. If you tried to buy that tomato from that consumer once they have finally decided to choose it, you would likely have to pay more than it cost them.

31.2 Uncertainty

Ordinary choice is complicated enough, but choice under uncertainty tends to be particularly tricky. We've already seen that people's decisions may depend on how choice alternatives are phrased. But there are many other biases in behavior in this domain.

Law of Small Numbers

If you have taken a course in statistics, you might be familiar with the Law of Large Numbers. This is a mathematical principle that says (roughly) that the average of a large sample from a population tends to be close to the mean of that population.

The Law of Small Numbers is a psychological statement that says that people tend to be overly influenced by small samples, particularly if they experience them themselves.[8]

Consider the following question:[9]

[8] The term originated with A. Tversky and D. Kahneman, 1971, "Belief in the law of small numbers," *Psychological Bulletin*,76, 2: 105–110. Much of the following discussion is based on a working paper by Matthew Rabin of the University of California at Berkeley entitled "Inference by Believers in the Law of Small Numbers."

[9] A. Tversky and D. Kahneman, 1982, "Judgments of and by Representativeness," in *Judgment under Uncertainty: Heuristics and Biases,* D. Kahneman, P. Slovic, and A. Tversky, Cambridge University Press, 84–98.

"A certain town is served by two hospitals. In the larger hospital about 45 babies are born each day, and in the smaller hospital about 15 babies are born each day. As you know, about 50 percent of all babies are boys. However, the exact percentage varies from day to day. Sometimes it may be higher than 50 percent, sometimes lower. For a period of 1 year, each hospital recorded the days on which more than 60 percent of the babies born were boys. Which hospital do you think recorded more such days?"

In a survey of college students, 22 percent of the subjects said that they thought that it was more likely that the larger hospital recorded more such days, while 56 percent said that they thought the number of days would be about the same. Only 22 percent correctly said that the smaller hospital would report more days.

If the correct account seems peculiar to you, suppose the smaller hospital recorded 2 births per day and the larger hospital 100 births per day. Roughly 25 percent of the time the smaller hospital would have 100 percent male births, while this would be very rare for the large hospital.

It appears that people expect samples to look like the distribution from which they are drawn. Or, saying this another way, people underestimate the actual magnitude of the fluctuations in a sample.

A related issue is that people find it difficult to recognize randomness. In one experiment, subjects were asked to write down a series of 150 "random" coin tosses. About 15 percent of the sequences they produced had heads or tails three times in a row, but this pattern would occur randomly about 25 percent of the time. Only 3 percent of the subjects' sequences had 4 heads or 4 tails in a row, while probability theory says that this should occur about 12 percent of the time.

This has important implications for game theory, for example. We saw that in many cases people should try to randomize their strategy choices so as to keep their opponents guessing. But, as the psychological literature shows, people aren't very good at randomizing. On the other hand, people aren't very good at detecting non-random behavior either, at least without some training in statistics. The point of mixed strategy equilibria is not that choices are *mathematically* unpredictable, but rather that they should be unpredictable by the players in the game.

Some economic researchers studied final and semi-final tennis matches at Wimbledon.[10] Ideally, tennis players should switch their serves from side to side so that their opponent can't guess which side the serve is coming from. However, even very accomplished players can't do this quite as well as one might expect. According to the authors:

"Our tests indicate that the tennis players are not quite playing ran-

[10] M. Walker and J. Wooders, 1999, "Minimax Play at Wimbledon," University of Arizona working paper.

domly: they switch their serves from left to right and vice versa somewhat too often to be consistent with random play. This is consistent with extensive experimental research in psychology and economics which indicates that people who are attempting to behave truly randomly tend to "switch too often."

Asset Integration and Loss Aversion

In our study of expected utility we made an implicit assumption that what individuals cared about was the total amount of wealth that they ended up with in various outcomes. This is known as the **asset integration hypothesis**.

Even though most people would accept this as a reasonable thing to do, it is hard to put into practice (even for economists). In general, people tend to avoid too many small risks and accept too many large risks.

Suppose that you make $100,000 a year and that you are offered a coin flip. If heads comes up you get $14 and if tails comes up you lose $10. This bet has an expected value of $12 and has a minuscule effect on your total income in a given year. Unless you have moral scruples about gambling, this would be a very attractive bet and you should almost certainly take it. However, a surprisingly large number of people won't take such a bet.

This **excess risk aversion** shows up in insurance markets where people tend to over-insure themselves against various small events. For example, people buy insurance against loosing their cell phone, even though they can often replace it at quite a low cost. People also buy auto insurance with deductibles that are much too low to make economic sense.

In general, when making insurance decisions you should look at the "house odds." If cell phone insurance costs you $3 a month, or $36 a year, and a new cell phone costs $180, then the house odds are 36/180, or 20 percent. The cell phone insurance would pay off in expected value only if you have more than a 20 percent chance of losing your phone or if it would be an extreme financial hardship to replace it.

It appears that people aren't really **risk averse** as much as they are **loss averse**. That is, people put seemingly excessive weight on the status quo—where they start—as opposed to where they end up.

In an experiment that has been replicated many times, two researchers gave half of the subjects in a group coffee mugs.[11] They asked this group to report the lowest price at which they would sell the mugs. Then they asked the group that didn't have mugs the highest price at which they would buy a mug. Since the groups were chosen randomly, the buying and selling prices should be about equal. However, in the experiment, the median

[11] D. Kahneman, J. L. Kitsch, and R. Thaler, 1990, "Experimental tests of the endowment effect and the Coase theorem," *Journal of Political Economy,* 98, 1325–1348.

selling price was \$5.79 and the median buying price was \$2.25, a substantial difference. Apparently, the subjects with coffee mugs were more reluctant to part with them than subjects without mugs. Their preferences seemed to be influenced by their endowment, contrary to standard consumer theory.

A similar effect shows up in what is known as the **sunk cost fallacy**. Once you have bought something, the amount you paid is "sunk," or no longer recoverable. So future behavior should not be influenced by sunk costs.

But, alas, real people tend to care about how much they paid for something. Researchers have found that the price at which owners listed condominiums in Boston was highly correlated with the buying price.[12] As pointed out earlier, owners of stock are very reluctant to realize losses, even when it would be advantageous for tax reasons.

The fact that ordinary people are subject to the sunk cost fallacy is interesting, but perhaps it is even more interesting that professionals are less susceptible to this problem. For example, the authors of the condominium example mentioned above found that individuals who bought condos for investment purposes were less likely to be influenced by sunk costs than individuals who lived in the condos.

Similarly, financial advisers are seldom reluctant to realize losses, particularly when there is a tax advantage to do so. It appears that one reason to hire professional advisers is to draw on their dispassionate analysis of decisions.

31.3 Time

Just as behavior involving uncertainty is subject to various forms of anomalous behavior, behavior involving time has its own set of anomalies.

Discounting

Consider, for example, time discounting. A standard model in economics, **exponential discounting**, posits that people discount the future at a constant fraction. If $u(c)$ is the utility of consumption today, then the utility of consumption t years in the future looks like $\delta^t u(c)$, where $\delta < 1$.

This is a mathematically convenient specification, but there are other forms of discounting that seem to fit the data better.

One economist auctioned off bonds that paid off at various times in the future and found that people valued payment at future times less than the

[12] David Genesove and Christopher Mayer, 2001, "Loss aversion and seller behavior: Evidence from the housing market," *Quarterly Journal of Economics,* 116, 4, 1233–1260.

exponential discounting theory would predict. An alternative theory, called **hyperbolic discounting**, suggests that the discount factor does not take the form δ^t but rather takes the form $1/(1+kt)$.

One particularly attractive feature of exponential discounting is that behavior is "time consistent." Think about a person with a three-period planning horizon with utility function of the form

$$u(c_1) + \delta u(c_2) + \delta^2 u(c_3).$$

The marginal rate of substitution between periods 1 and 2 is

$$MRS_{12} = \frac{\delta MU(c_2)}{MU(c_1)},$$

while the MRS between periods 2 and 3 is

$$MRS_{23} = \frac{\delta^2 MU(c_3)}{\delta MU(c_2)} = \frac{\delta MU(c_3)}{MU(c_2)}.$$

This last expression shows that the rate at which the individual is willing to substitute consumption in period 2 for consumption in period 3 is the *same* whether viewed from the perspective of period 1 or of period 2. This is not true for hyperbolic discounting. An individual with hyperbolic discounting discounts the long-term future more heavily than he discounts the short-term future.

Such a person will exhibit **time inconsistency:** he may make a plan today about his future behavior, but when the future arrives he will want to do something different. Think of a couple who decide to spend $5,000 on a trip to Europe rather than save their money. They rationalize their decision on the grounds that they will start saving *next* summer. But when next summer arrives, they decide to spend their money on a cruise.

Self-control

A closely related issue to the time consistency problem is the problem of **self-control**. Almost everyone faces this issue to some degree. We might vow to count our calories and eat less while standing on the bathroom scale, but our resolve can easily vanish when we sit down to a nice meal. Rational people are apparently slim and healthy, unlike the rest of us.

One important question is whether people are aware of their own difficulties with self-control. If I know that I have a tendency to procrastinate, perhaps I should realize that when an important task comes along I should do it right away. Or if I have a tendency to overcommit myself, perhaps I should learn to say no more often.

But there is the other possibility. If I know that I am likely to yield to the temptation to have another desert tomorrow, I may as well have another desert today. The flesh is weak, but the spirit may be weak too.

One way to deal with self-control is to find ways to commit yourself to future actions. That is, you can try to find a way to make it more costly to deviate from the desired action in the future. For example, people who make a public pronouncement about their future behavior might be less likely to deviate from their intended behavior. There are pills for alcoholics that make them violently sick if they drink alcohol. There are also **commitment devices** for dieters: someone who has his stomach stapled will be less likely to overeat.

Contracts between individuals are there to ensure that people carry out their future intentions—even when it might not be attractive for them to do so due to changed conditions. In a similar way, people can hire others to impose costs on them if they deviate from intended actions, making, in effect, a contract with themselves. Dieting spas, exercise instructors, and tutors are forms of "purchased self-control."

EXAMPLE: Overconfidence

An interesting variation on self-control is the phenomenon of **overconfidence.** Two financial economists, Brad Barber and Terrance Odean, studied the performance of 66,465 households with discount brokerage accounts. During the period they studied, households that traded infrequently received an 18 percent return on their investments, while the return for the households that traded most actively was 11.3 percent.

One of the most important factors that apparently influenced this excessive trading was gender: the men traded a lot more than women. Psychologists commonly find that men tend to have excessive confidence in their own abilities, while women, for the most part, tend to be more realistic. Psychologists refer to men's behavior as self-serving attribution bias. Basically, men (or at least some men) tend to think their successes are a result of their own skill, rather than dumb luck, and so become overconfident.

This overconfidence can have financial repercussions. In the sample of brokerage accounts, men traded 45 percent more than women. This excessive trading resulted in the average return to men that was a full percentage point lower than the return to women. As Barber and Odean put it, "trading can be hazardous to your wealth."

31.4 Strategic Interaction and Social Norms

A particularly interesting set of psychological, or perhaps sociological, behaviors arise in strategic interaction. We have studied game theory, which

attempts to predict how rational players should interact. But there is also a subject known as **behavioral game theory** that examines how actual people interact. Indeed, there are systematic and strong deviations from the pure theory.

Ultimatum Game

Consider the **ultimatum game**, which was discussed briefly in the last chapter. As you will recall, this is a game with two players, the proposer and the responder. The proposer is given $10 and asked to propose a division between himself and the responder. The responder is then shown the division and asked whether or not he wishes to accept it. If he accepts, the division is carried out; if he refuses the division, both people walk away with nothing.

Let's first think about how fully rational players might act. Once the responder sees the division, he has a dominant strategy: accept the money as long as he gets anything at all. After all, suppose I offer you the choice between 10 cents and nothing. Wouldn't you rather have 10 cents than nothing at all?

Given that a rational responder will choose any amount, the divider should choose the minimal amount to give him—say, a penny. So the outcome predicted by game theory is an extreme split: the divider will end up with almost everything.

This isn't the way things turn out when the game is actually played. In fact, responders tend to reject offers that they perceive as unfair. Offers that give the responder less than 30 percent of the amount to be divided are rejected more than 50 percent of the time.

Of course, if the divider recognizes that the responder will reject "unfair" offers, the divider will rationally want to make a division that is closer to equal. The average division tends to be about 45 percent to the responder and 55 percent to the divider, with about 16 percent of the offers being rejected.

There has been a considerable amount of literature examining how the characteristics of the players affect the outcome of the game. One example is gender differences: it appears that men tend to receive more favorable divisions, particularly when the divisions are made by women.

Cultural differences can also be important. It appears that some cultures value fairness more than others, inducing people to reject offers that are perceived as unfair.[13] Interestingly enough, the offered amounts don't vary much from region to region and culture to culture, while there are

[13] See Swee-Hoon Chuah, Robert Hoffman, Martin Jones, and Geoffrey Williams, "Do Cultures Clash? Evidence from Cross-National Ultimatum Game Experiments," Nottingham University Business School working paper.

systematic differences in the divisions that are acceptable. The size of the pie is also important. If the size of the pie is $10, you might be reluctant to accept $1. But if the size of the pie is $1,000, would you be willing to reject $100? Apparently, responders do find it difficult to turn down larger amounts of money.

Another variation is in the design of the game. In one variant, the so-called **strategy method**, the responders are asked to name the minimal division that they will accept *before* seeing the amount they are offered. The proposers are aware that the decision will be made in advance but, of course, don't know what the minimum acceptable division is. This experimental design tends to increase the amounts that the proposers offer; that is, it tends to make the divisions more equal.

Fairness

One effect at work in the ultimatum game seems to be a concern for fairness. Most people seem to have a natural bias towards equal (or at least not too unequal) division. This is not simply an individual phenomenon, but a social phenomenon. People will enforce **fairness norms** even when it is not directly in their interest to do so.

Consider, for example, **punishment games**, which are a generalization of ultimatum games with a third party who observes the choices made by the proposer/divider. The third party can choose, at some cost to himself, to deduct some of the proposer's profits.[14]

Experimenters have found that around 60 percent of these third-party observers will actually punish those who make unfair divisions. There seems to be something in the human makeup—whether innate or learned—that finds unfair behavior objectionable.

Indeed, there are differences across cultures with respect to social norms for fairness; individuals in some societies seem to value it highly, while in other societies fairness is less strongly valued. However, the urge to punish those who are unfair is widely felt. It has been suggested that a predilection towards "fair" outcomes is part of human nature, perhaps because individuals that behaved fairly towards each other had higher chances of surviving and reproducing.

31.5 Assessment of Behavioral Economics

Psychologists, marketers, and behavioral economists have amassed a variety of examples showing how the basic theory of economic choice is wrong, or, at least, incomplete.

[14] See Ernst Fehr and Urs Fischbacher, 2004, "Third-party punishment and social norms," *Evolution and Human Behavior*, 25, 63–87.

Some of these examples appear to be "optical illusions." For example, the fact that framing a choice problem differently can affect decisions is similar to the fact that human judgment of sizes and distances can be affected by how figures are drawn. If people took the time to consider the choices carefully—applying a measuring stick of dispassionate reasoning—they would reach the right conclusion.

Though it is undoubtedly true that people don't behave completely in accord with the simplest theories of economic behavior, one still might respond that no theory is 100 percent correct. Psychologists have also documented that people don't really understand simple principles of physics. Example: If you tie a weight to the end of a rope, swing it around your head in a circle and then let go, which way will the weight fly?

Many people say that the weight will fly radially outward rather than the correct response that the weight will move tangentially to the circle.[15] Of course, people have lived in the physical world their entire lives. If they occasionally misunderstand how it works, we shouldn't be too surprised when people misunderstand the economic world.

Apparently our intuitive understanding of physics is good enough for everyday life, and even the demands of amateur and professional sports: a baseball player may not be able to describe how a ball will travel, despite the fact that he can throw it well. Similarly, one might argue that people tend to be pretty good at the sorts of day-to-day decisions they are forced to make, even if they aren't very good at abstract reasoning about them.

Another reaction to behavioral anomalies is that markets tend to reward rational behavior, while punishing irrationality. Even if many participants do not behave rationally, those who *do* behave sensibly will have the biggest effect on prices and outcomes. There is likely some truth to this view as well. Recall the example that real estate investors seemed to be less influenced by sunk costs than ordinary individuals.

In addition, you can hire experts to help you make better decisions. Diet consultants and financial advisers can offer objective advice about how to eat and how to invest. If you are worried about being too fair, you can always hire a tough negotiator.

Returning to the optical illusion example, the reason that we use rulers and yardsticks is that we learn not to trust our own eyes. Similarly, in making important decisions it is prudent to consult the views of objective experts.

[15] See M. McCloskey, 1983, "Intuitive Physics," *Scientific American,* April, 114–123.

Summary

1. Behavioral economics is concerned with how consumers make choices in reality.

2. In many cases, actual consumer behavior is different from that predicted by the simple model of the rational consumer.

3. Consumers make different choices depending on how a problem is framed or presented.

4. The default matters a lot.

5. People find it difficult to predict their own choice behavior.

6. Too many choices may be overwhelming and make it difficult to make a decision.

7. Choice behavior can be particularly problematic in choices involving uncertainty.

8. Peole tend to exhibit excess risk aversion in experimental settings.

9. People may discount the future more heavily than conventional theory assumes.

10. Time inconsistency means that actual choices may end up being different than planned choices.

11. The ultimatum game involves one player proposing a division of some money and the other player either accepting that division or ending the game. Conventional game theory predicts very unfairm divisions.

12. However, consumers seem to have a preference for "fair" divisions and will punish those who behave unfairly, even if harms themselves.

REVIEW QUESTIONS

1. Subjects are allowed to buy tickets in a lottery. One group is told that they have a 55 percent chance of winning, the other group is told that they have a 45 percent chance of not winning. Which group is more likely to buy lottery tickets? What is the name for this effect?

2. Mary plans the entire week's meals for her family, while Fred shops each day. Which is likely to produce more varied meals? What is this effect called?

3. You are the human resources director for a medium-size company and are trying to decide how many mutual funds to offer in your employees' pension plan. Would it be better to offer 10 choices or 50 choices?

4. What is the probability that a fair coin will come up heads three times in a row when tossed?

5. John decides that he will save $5 this week and $10 next week. But when next week arrives, he decides to save only $8. What is the term used to describe this sort of inconsistent behavior?

CHAPTER 32

EXCHANGE

Up until now we have generally considered the market for a single good in isolation. We have viewed the demand and supply functions for a good as depending on its price alone, disregarding the prices of other goods. But in general the prices of other goods *will* affect people's demands and supplies for a particular good. Certainly the prices of substitutes and complements for a good will influence the demand for it, and, more subtly, the prices of goods that people sell will affect the amount of income they have and thereby influence how much of other goods they will be able to buy.

Up until now we have been ignoring the effect of these other prices on the market equilibrium. When we discussed the equilibrium conditions in a particular market, we only looked at part of the problem: how demand and supply were affected by the price of the particular good we were examining. This is called **partial equilibrium** analysis.

In this chapter we will begin our study of **general equilibrium** analysis: how demand and supply conditions interact in several markets to determine the prices of many goods. As you might suspect, this is a complex problem, and we will have to adopt several simplifications in order to deal with it.

First, we will limit our discussion to the behavior of competitive markets, so that each consumer or producer will take prices as given and optimize

accordingly. The study of general equilibrium with imperfect competition is very interesting but too difficult to examine at this point.

Second, we will adopt our usual simplifying assumption of looking at the smallest number of goods and consumers that we possibly can. In this case, it turns out that many interesting phenomena can be depicted using only two goods and two consumers. All of the aspects of general equilibrium analysis that we will discuss can be generalized to arbitrary numbers of consumers and goods, but the exposition is simpler with two of each.

Third, we will look at the general equilibrium problem in two stages. We will start with an economy where people have fixed endowments of goods and examine how they might trade these goods among themselves; no production will be involved. This case is naturally known as the case of **pure exchange**. Once we have a clear understanding of pure exchange markets we will examine production behavior in the general equilibrium model.

32.1 The Edgeworth Box

There is a convenient graphical tool known as the **Edgeworth box** that can be used to analyze the exchange of two goods between two people.[1] The Edgeworth box allows us to depict the endowments and preferences of two individuals in one convenient diagram, which can be used to study various outcomes of the trading process. In order to understand the construction of an Edgeworth box it is necessary to examine the indifference curves and the endowments of the people involved.

Let us call the two people involved A and B and the two goods involved 1 and 2. We will denote A's consumption bundle by $X_A = (x_A^1, x_A^2)$, where x_A^1 represents A's consumption of good 1 and x_A^2 represents A's consumption of good 2. Then B's consumption bundle is denoted by $X_B = (x_B^1, x_B^2)$. A *pair* of consumption bundles, X_A and X_B, is called an **allocation**. An allocation is a **feasible allocation** if the total amount of each good consumed is equal to the total amount available:

$$x_A^1 + x_B^1 = \omega_A^1 + \omega_B^1$$

$$x_A^2 + x_B^2 = \omega_A^2 + \omega_B^2.$$

A particular feasible allocation that is of interest is the **initial endowment allocation**, (ω_A^1, ω_A^2) and (ω_B^1, ω_B^2). This is the allocation that the consumers start with. It consists of the amount of each good that consumers bring to the market. They will exchange some of these goods with each other in the course of trade to end up at a **final allocation**.

[1] The Edgeworth box is named in honor of Francis Ysidro Edgeworth (1845–1926), an English economist who was one of the first to use this analytical tool.

The Edgeworth box shown in Figure 32.1 can be used to illustrate these concepts graphically. We first use a standard consumer theory diagram to illustrate the endowment and preferences of consumer A. We can also mark off on these axes the *total* amount of each good in the economy—the amount that A has plus the amount that B has of each good. Since we will only be interested in feasible allocations of goods between the two consumers, we can draw a box that contains the set of possible bundles of the two goods that A can hold.

An Edgeworth box. The width of the box measures the total amount of good 1 in the economy and the height measures the total amount of good 2. Person A's consumption choices are measured from the lower left-hand corner while person B's choices are measured from the upper right.

Figure 32.1

Note that the bundles in this box also indicate the amount of the goods that B can hold. If there are 10 units of good 1 and 20 units of good 2, then if A holds (7,12), B must be holding (3,8). We can depict how much A holds of good 1 by the distance along the horizontal axis from the origin in the lower left-hand corner of the box and the amount B holds of good 1 by measuring the distance along the horizontal axis from the upper right-hand corner. Similarly, distances along the vertical axes give the amounts of good 2 that A and B hold. Thus the points in this box give us both the

bundles that A can hold and the bundles that B can hold—just measured from different origins. The points in the Edgeworth box can represent all feasible allocations in this simple economy.

We can depict A's indifference curves in the usual manner, but B's indifference curves take a somewhat different form. To construct them we take a standard diagram for B's indifference curves, turn it upside down, and "overlay" it on the Edgeworth box. This gives us B's indifference curves on the diagram. If we start at A's origin in the lower left-hand corner and move up and to the right, we will be moving to allocations that are more preferred by A. As we move down and to the left we will be moving to allocations that are more preferred by B. (If you rotate your book and look at the diagram, this discussion may seem clearer.)

The Edgeworth box allows us to depict the possible consumption bundles for both consumers—the feasible allocations—and the preferences of both consumers. It thereby gives a complete description of the economically relevant characteristics of the two consumers.

32.2 Trade

Now that we have both sets of preferences and endowments depicted we can begin to analyze the question of how trade takes place. We start at the original endowment of goods, denoted by the point W in Figure 32.1. Consider the indifference curves of A and B that pass through this allocation. The region where A is better off than at her endowment consists of all the bundles above her indifference curve through W. The region where B is better off than at his endowment consists of all the allocations that are above—from his point of view—*his* indifference curve through W. (This is *below* his indifference curve from *our* point of view . . . unless you've still got your book upside down.)

Where is the region of the box where A and B are *both* made better off? Clearly it is in the intersection of these two regions. This is the lens-shaped region illustrated in Figure 32.1. Presumably in the course of their negotiations the two people involved will find some mutually advantageous trade—some trade that will move them to some point inside the lens-shaped area such as the point M in Figure 32.1.

The particular movement to M depicted in Figure 32.1 involves person A giving up $|x_A^1 - \omega_A^1|$ units of good 1 and acquiring in exchange $|x_A^2 - \omega_A^2|$ units of good 2. This means that B acquires $|x_B^1 - \omega_B^1|$ units of good 1 and gives up $|x_B^2 - \omega_B^2|$ units of good 2.

There is nothing particularly special about the allocation M. Any allocation inside the lens-shaped region would be possible—for every allocation of goods in this region is an allocation that makes each consumer better off than he or she was at the original endowment. We only need to suppose that the consumers trade to *some* point in this region.

Now we can repeat the same analysis at the point M. We can draw the two indifference curves through M, construct a new lens-shaped "region of mutual advantage," and imagine the traders moving to some new point N in this region. And so it goes . . . the trade will continue until there are no more trades that are preferred by both parties. What does such a position look like?

32.3 Pareto Efficient Allocations

The answer is given in Figure 32.2. At the point M in this diagram the set of points above A's indifference curve doesn't intersect the set of points above B's indifference curve. The region where A is made better off is disjoint from the region where B is made better off. This means that any movement that makes one of the parties better off necessarily makes the other party worse off. Thus there are no exchanges that are advantageous for both parties. There are no mutually improving trades at such an allocation.

An allocation such as this is known as a **Pareto efficient** allocation. The idea of Pareto efficiency is a very important concept in economics that arises in various guises.

A Pareto efficient allocation. At a Pareto efficient allocation such as M, each person is on his highest possible indifference curve, given the indifference curve of the other person. The line connecting such points is known as the contract curve.

Figure 32.2

A Pareto efficient allocation can be described as an allocation where:

1. There is no way to make all the people involved better off; or
2. there is no way to make some individual better off without making someone else worse off; or
3. all of the gains from trade have been exhausted; or
4. there are no mutually advantageous trades to be made, and so on.

Indeed we have mentioned the concept of Pareto efficiency several times already in the context of a single market: we spoke of the Pareto efficient level of output in a single market as being that amount of output where the marginal willingness to buy equaled the marginal willingness to sell. At any level of output where these two numbers differed, there would be a way to make both sides of the market better off by carrying out a trade. In this chapter we will examine more deeply the idea of Pareto efficiency involving many goods and many traders.

Note the following simple geometry of Pareto efficient allocations: the indifference curves of the two agents must be tangent at any Pareto efficient allocation in the interior of the box. It is easy to see why. If the two indifference curves are not tangent at an allocation in the interior of the box, then they must cross. But if they cross, then there must be some mutually advantageous trade—so that point cannot be Pareto efficient. (It is possible to have Pareto efficient allocations on the sides of the box—where one consumer has zero consumption of some good—in which the indifference curves are not tangent. These boundary cases are not important for the current discussion.)

From the tangency condition it is easy to see that there are a lot of Pareto efficient allocations in the Edgeworth box. In fact, given any indifference curve for person A, for example, there is an easy way to find a Pareto efficient allocation. Simply move along A's indifference curve until you find the point that is the best point for B. This will be a Pareto efficient point, and thus both indifference curves must be tangent at this point.

The set of *all* Pareto efficient points in the Edgeworth box is known as the **Pareto set**, or the **contract curve**. The latter name comes from the idea that all "final contracts" for trade must lie on the Pareto set—otherwise they wouldn't be final because there would be some improvement that could be made!

In a typical case the contract curve will stretch from A's origin to B's origin across the Edgeworth box, as shown in Figure 32.2. If we start at A's origin, A has none of either good and B holds everything. This is Pareto efficient since the only way A can be made better off is to take something away from B. As we move up the contract curve A is getting more and more well-off until we finally get to B's origin.

The Pareto set describes all the possible outcomes of mutually advantageous trade from starting anywhere in the box. If we are given the starting point—the initial endowments for each consumer—we can look at the subset of the Pareto set that each consumer prefers to his initial endowment. This is simply the subset of the Pareto set that lies in the lens-shaped region depicted in Figure 32.1. The allocations in this lens-shaped region are the possible outcomes of mutual trade starting from the particular initial endowment depicted in that diagram. But the Pareto set itself doesn't depend on the initial endowment, except insofar as the endowment determines the total amounts of both goods that are available and thereby determines the dimensions of the box.

32.4 Market Trade

The equilibrium of the trading process described above—the set of Pareto efficient allocations—is very important, but it still leaves a lot of ambiguity about where the agents end up. The reason is that the trading process we have described is very general. Essentially we have only assumed that the two parties will move to *some* allocation where they are both made better off.

If we have a *particular* trading process, we will have a more precise description of equilibrium. Let's try to describe a trading process that mimics the outcome of a competitive market.

Suppose that we have a third party who is willing to act as an "auctioneer" for the two agents A and B. The auctioneer chooses a price for good 1 and a price for good 2 and presents these prices to the agents A and B. Each agent then sees how much his or her endowment is worth at the prices (p_1, p_2) and decides how much of each good he or she would want to buy at those prices.

One warning is in order here. If there are really only two people involved in the transaction, then it doesn't make much sense for them to behave in a competitive manner. Instead they would probably attempt to bargain over the terms of trade. One way around this difficulty is to think of the Edgeworth box as depicting the average demands in an economy with only two *types* of consumers, but with many consumers of each type. Another way to deal with this is to point out that the behavior is implausible in the two-person case, but it makes perfect sense in the many-person case, which is what we are really concerned with.

Either way, we know how to analyze the consumer-choice problem in this framework—it is just the standard consumer-choice problem we described in Chapter 5. In Figure 32.3 we illustrate the two demanded bundles of the two agents. (Note that the situation depicted in Figure 32.3 is not an equilibrium configuration since the demand by one agent is not equal to the supply of the other agent.)

Figure 32.3 **Gross demands and net demands**. Gross demands are the amounts the person wants to consume; net demands are the amounts the person wants to purchase.

As in Chapter 9 there are two relevant concepts of "demand" in this framework. The **gross demand** of agent A for good 1, say, is the total amount of good 1 that he wants at the going prices. The **net demand** of agent A for good 1 is the difference between this total demand and the initial endowment of good 1 that agent A holds. In the context of general equilibrium analysis, net demands are sometimes called **excess demands**. We will denote the excess demand of agent A for good 1 by e_A^1. By definition, if A's gross demand is x_A^1, and his endowment is ω_A^1, we have

$$e_A^1 = x_A^1 - \omega_A^1.$$

The concept of excess demand is probably more natural, but the concept of gross demand is generally more useful. We will typically use the word "demand" to mean gross demand and specifically say "net demand" or "excess demand" if that is what we mean.

For arbitrary prices (p_1, p_2) there is no guarantee that supply will equal demand—in either sense of demand. In terms of net demand, this means that the amount that A wants to buy (or sell) will not necessarily equal the amount that B wants to sell (or buy). In terms of gross demand, this means that the total amount that the two agents want hold of the goods is not equal to the total amount of that goods available. Indeed, this is true in the example depicted in Figure 32.3. In this example the agents will not

be able to complete their desired transactions: the markets will not clear.

We say that in this case the market is in **disequilibrium**. In such a situation, it is natural to suppose that the auctioneer will change the prices of the goods. If there is excess demand for one of the goods, the auctioneer will raise the price of that good, and if there is excess supply for one of the goods, the auctioneer will lower its price.

Suppose that this adjustment process continues until the demand for each of the goods equals the supply. What will the final configuration look like?

The answer is given in Figure 32.4. Here the amount that A wants to buy of good 1 just equals the amount that B wants to sell of good 1, and similarly for good 2. Said another way, the total amount that each person wants to buy of each good at the current prices is equal to the total amount available. We say that the market is in **equilibrium**. More precisely, this is called a **market equilibrium**, a **competitive equilibrium**, or a **Walrasian equilibrium**.[2] Each of these terms refers to the same thing: a set of prices such that each consumer is choosing his or her most-preferred affordable bundle, and all consumers' choices are compatible in the sense that demand equals supply in every market.

We know that if each agent is choosing the best bundle that he can afford, then his marginal rate of substitution between the two goods must be equal to the ratio of the prices. But if all consumers are facing the same prices, then all consumers will have to have the *same* marginal rate of substitution between each of the two goods. In terms of Figure 32.4, an equilibrium has the property that each agent's indifference curve is tangent to his budget line. But since each agent's budget line has the slope $-p_1/p_2$, this means that the two agents' indifference curves must be tangent to each other.

32.5 The Algebra of Equilibrium

If we let $x_A^1(p_1, p_2)$ be agent A's demand function for good 1 and $x_B^1(p_1, p_2)$ be agent B's demand function for good 1, and define the analogous expressions for good 2, we can describe this equilibrium as a set of prices (p_1^*, p_2^*) such that

$$x_A^1(p_1^*, p_2^*) + x_B^1(p_1^*, p_2^*) = \omega_A^1 + \omega_B^1$$

$$x_A^2(p_1^*, p_2^*) + x_B^2(p_1^*, p_2^*) = \omega_A^2 + \omega_B^2.$$

These equations say that in equilibrium the total demand for each good should be equal to the total supply.

[2] Leon Walras (1834–1910) was a French economist at Lausanne who was an early investigator of general equilibrium theory.

Figure 32.4 **Equilibrium in the Edgeworth box**. In equilibrium, each person is choosing the most-preferred bundle in his budget set, and the choices exhaust the available supply.

Another way to describe the equilibrium is to rearrange these two equations to get

$$[x_A^1(p_1^*, p_2^*) - \omega_A^1] + [x_B^1(p_1^*, p_2^*) - \omega_B^1] = 0$$
$$[x_A^2(p_1^*, p_2^*) - \omega_A^2] + [x_B^2(p_1^*, p_2^*) - \omega_B^2] = 0.$$

These equations say that the sum of *net demands* of each agent for each good should be zero. Or, in other words, the net amount that A chooses to demand (or supply) must be equal to the net amount that B chooses to supply (or demand).

Yet another formulation of these equilibrium equations comes from the concept of the **aggregate excess demand function**. Let us denote the net demand function for good 1 by agent A by

$$e_A^1(p_1, p_2) = x_A^1(p_1, p_2) - \omega_A^1$$

and define $e_B^1(p_1, p_2)$ in a similar manner.

The function $e_A^1(p_1, p_2)$ measures agent A's **net demand** or his **excess demand**—the difference between what she wants to consume of good 1 and what she initially has of good 1. Now let us add together agent A's net demand for good 1 and agent B's net demand for good 1. We get

$$\begin{aligned} z_1(p_1, p_2) &= e_A^1(p_1, p_2) + e_B^1(p_1, p_2) \\ &= x_A^1(p_1, p_2) + x_B^1(p_1, p_2) - \omega_A^1 - \omega_B^1, \end{aligned}$$

which we call the **aggregate excess demand** for good 1. There is a similar aggregate excess demand for good 2, which we denote by $z_2(p_1, p_2)$.

Then we can describe an equilibrium (p_1^*, p_2^*) by saying that the aggregate excess demand for each good is zero:

$$z_1(p_1^*, p_2^*) = 0$$
$$z_2(p_1^*, p_2^*) = 0.$$

Actually, this definition is stronger than necessary. It turns out that if the aggregate excess demand for good 1 is zero, then the aggregate excess demand for good 2 must necessarily be zero. In order to prove this, it is convenient to first establish a property of the aggregate excess demand function known as **Walras' law**.

32.6 Walras' Law

Using the notation established above, Walras' law states that

$$p_1 z_1(p_1, p_2) + p_2 z_2(p_1, p_2) \equiv 0.$$

That is, *the value of aggregate excess demand is identically zero.* To say that the value of aggregate demand is identically zero means that it is zero for *all* possible choices of prices, not just equilibrium prices.

The proof of this follows from adding up the two agents' budget constraints. Consider first agent A. Since her demand for each good satisfies her budget constraint, we have

$$p_1 x_A^1(p_1, p_2) + p_2 x_A^2(p_1, p_2) \equiv p_1 \omega_A^1 + p_2 \omega_A^2$$

or

$$p_1[x_A^1(p_1, p_2) - \omega_A^1] + p_2[x_A^2(p_1, p_2) - \omega_A^2] \equiv 0$$
$$p_1 e_A^1(p_1, p_2) + p_2 e_A^2(p_1, p_2) \equiv 0.$$

This equation says that the *value of agent A's net demand is zero.* That is, the value of how much A wants to buy of good 1 plus the value of how much she wants to buy of good 2 must equal zero. (Of course the amount that she wants to buy of *one* of the goods must be negative—that is, she intends to sell some of one of the goods to buy more of the other.)

We have a similar equation for agent B:

$$p_1[x_B^1(p_1, p_2) - \omega_B^1] + p_2[x_B^2(p_1, p_2) - \omega_B^2] \equiv 0$$
$$p_1 e_B^1(p_1, p_2) + p_2 e_B^2(p_1, p_2) \equiv 0.$$

Adding the equations for agent A and agent B together and using the definition of aggregate excess demand, $z_1(p_1, p_2)$ and $z_2(p_1, p_2)$, we have

$$p_1[e_A^1(p_1, p_2) + e_B^1(p_1, p_2)] + p_2[e_A^2(p_1, p_2) + e_B^2(p_1, p_2)] \equiv 0$$
$$p_1 z_1(p_1, p_2) + p_2 z_2(p_1, p_2) \equiv 0.$$

Now we can see where Walras' law comes from: since the value of each agent's excess demand equals zero, the value of the sum of the agents' excess demands must equal zero.

We can now demonstrate that if demand equals supply in one market, demand must also equal supply in the other market. Note that Walras' law must hold for all prices, since each agent must satisfy his or her budget constraint for all prices. Since Walras' law holds for all prices, in particular, it holds for a set of prices where the excess demand for good 1 is zero:

$$z_1(p_1^*, p_2^*) = 0.$$

According to Walras' law it must also be true that

$$p_1^* z_1(p_1^*, p_2^*) + p_2^* z_2(p_1^*, p_2^*) = 0.$$

It easily follows from these two equations that if $p_2 > 0$, then we must have

$$z_2(p_1^*, p_2^*) = 0.$$

Thus, as asserted above, if we find a set of prices (p_1^*, p_2^*) where the demand for good 1 equals the supply of good 1, we are guaranteed that the demand for good 2 must equal the supply of good 2. Alternatively, if we find a set of prices where the demand for good 2 equals the supply of good 2, we are guaranteed that market 1 will be in equilibrium.

In general, if there are markets for k goods, then we only need to find a set of prices where $k-1$ of the markets are in equilibrium. Walras' law then implies that the market for good k will automatically have demand equal to supply.

32.7 Relative Prices

As we've seen above, Walras' law implies that there are only $k-1$ independent equations in a k-good general equilibrium model: if demand equals supply in $k-1$ markets, demand must equal supply in the final market. But if there are k goods, there will be k prices to be determined. How can you solve for k prices with only $k-1$ equations?

The answer is that there are really only $k-1$ *independent* prices. We saw in Chapter 2 that if we multiplied all prices and income by a positive number t, then the budget set wouldn't change, and thus the demanded bundle wouldn't change either. In the general equilibrium model, each consumer's income is just the value of his or her endowment at the market prices. If we multiply all prices by $t > 0$, we will automatically multiply each consumer's income by t. Thus, if we find some equilibrium set of prices (p_1^*, p_2^*), then (tp_1^*, tp_2^*) are equilibrium prices as well, for any $t > 0$.

This means that we are free to choose one of the prices and set it equal to a constant. In particular it is often convenient to set one of the prices equal to 1 so that all of the other prices can be interpreted as being measured relative to it. As we saw in Chapter 2, such a price is called a **numeraire** price. If we choose the first price as the numeraire price, then it is just like multiplying all prices by the constant $t = 1/p_1$.

The requirement that demand equal supply in every market can only be expected to determine the equilibrium relative prices, since multiplying all prices by a positive number will not change anybody's demand and supply behavior.

EXAMPLE: An Algebraic Example of Equilibrium

The Cobb-Douglas utility function described in Chapter 6 has the form $u_A(x_A^1, x_A^2) = (x_A^1)^a (x_A^2)^{1-a}$ for person A, and a similar form for person B. We saw there that this utility function gave rise to the following demand functions:

$$\begin{aligned}
x_A^1(p_1, p_2, m_A) &= a\frac{m_A}{p_1} \\
x_A^2(p_1, p_2, m_A) &= (1-a)\frac{m_A}{p_2} \\
x_B^1(p_1, p_2, m_B) &= b\frac{m_B}{p_1} \\
x_B^2(p_1, p_2, m_B) &= (1-b)\frac{m_B}{p_2},
\end{aligned}$$

where a and b are the parameters of the two consumers' utility functions.

We know that in equilibrium, the money income of each individual is given by the value of his or her endowment:

$$\begin{aligned}
m_A &= p_1\omega_A^1 + p_2\omega_A^2 \\
m_B &= p_1\omega_B^1 + p_2\omega_B^2.
\end{aligned}$$

Thus the aggregate excess demands for the two goods are

$$\begin{aligned}
z_1(p_1, p_2) &= a\frac{m_A}{p_1} + b\frac{m_B}{p_1} - \omega_A^1 - \omega_B^1 \\
&= a\frac{p_1\omega_A^1 + p_2\omega_A^2}{p_1} + b\frac{p_1\omega_B^1 + p_2\omega_B^2}{p_1} - \omega_A^1 - \omega_B^1
\end{aligned}$$

and

$$\begin{aligned}
z_2(p_1, p_2) &= (1-a)\frac{m_A}{p_2} + (1-b)\frac{m_B}{p_2} - \omega_A^2 - \omega_B^2 \\
&= (1-a)\frac{p_1\omega_A^1 + p_2\omega_A^2}{p_2} + (1-b)\frac{p_1\omega_B^1 + p_2\omega_B^2}{p_2} - \omega_A^2 - \omega_B^2.
\end{aligned}$$

You should verify that these aggregate demand functions satisfy Walras' law.

Let us choose p_2 as the numeraire price, so that these equations become

$$z_1(p_1, 1) = a\frac{p_1\omega_A^1 + \omega_A^2}{p_1} + b\frac{p_1\omega_B^1 + \omega_B^2}{p_1} - \omega_A^1 - \omega_B^1$$

$$z_2(p_1, 1) = (1-a)(p_1\omega_A^1 + \omega_A^2) + (1-b)(p_1\omega_B^1 + \omega_B^2) - \omega_A^2 - \omega_B^2.$$

All we've done here is set $p_2 = 1$.

We now have an equation for the excess demand for good 1, $z_1(p_1, 1)$, and an equation for the excess demand for good 2, $z_2(p_1, 1)$, with each equation expressed as a function of the relative price of good 1, p_1. In order to find the *equilibrium* price, we set either of these equations equal to zero and solve for p_1. According to Walras' law, we should get the same equilibrium price, no matter which equation we solve.

The equilibrium price turns out to be

$$p_1^* = \frac{a\omega_A^2 + b\omega_B^2}{(1-a)\omega_A^1 + (1-b)\omega_B^1}.$$

(Skeptics may want to insert this value of p_1 into the demand equals supply equations to verify that the equations are satisfied.)

32.8 The Existence of Equilibrium

In the example given above, we had specific equations for each consumer's demand function and we could explicitly solve for the equilibrium prices. But in general, we don't have explicit algebraic formulas for each consumer's demands. We might well ask how do we know that there is *any* set of prices such that demand equals supply in every market? This is known as the question of the **existence of a competitive equilibrium**.

The existence of a competitive equilibrium is important insofar as it serves as a "consistency check" for the various models that we have examined in previous chapters. What use would it be to build up elaborate theories of the workings of a competitive equilibrium if such an equilibrium commonly did not exist?

Early economists noted that in a market with k goods there were $k-1$ relative prices to be determined, and there were $k-1$ equilibrium equations stating that demand should equal supply in each market. Since the number of equations equaled the number of unknowns, they asserted that there would be a solution where all of the equations were satisfied.

Economists soon discovered that such arguments were fallacious. Merely counting the number of equations and unknowns is not sufficient to prove that an equilibrium solution will exist. However, there are mathematical

tools that can be used to establish the existence of a competitive equilibrium. The crucial assumption turns out to be that the aggregate excess demand function is a **continuous function**. This means, roughly speaking, that small changes in prices should result in only small changes in aggregate demand: a small change in prices should not result in a big jump in the quantity demanded.

Under what conditions will the aggregate demand functions be continuous? Essentially there are two kinds of conditions that will guarantee continuity. One is that each individual's demand function be continuous—that small changes in prices will lead to only small changes in demand. This turns out to require that each consumer have convex preferences, which we discussed in Chapter 3. The other condition is more general. Even if consumers themselves have discontinuous demand behavior, as long as all consumers are small relative to the size of the market, the aggregate demand function will be continuous.

This latter condition is quite nice. After all, the assumption of competitive behavior only makes sense when there are a lot of consumers who are small relative to the size of the market. This is exactly the condition that we need in order to get the aggregate demand functions to be continuous. And continuity is just the ticket to ensure that a competitive equilibrium exists. Thus the very assumptions that make the postulated behavior reasonable will ensure that the equilibrium theory will have content.

32.9 Equilibrium and Efficiency

We have now analyzed market trade in a pure exchange model. This gives us a specific model of trade that we can compare to the general model of trade that we discussed in the beginning of this chapter. One question that might arise about the use of a competitive market is whether this mechanism can really exhaust all of the gains from trade. After we have traded to a competitive equilibrium where demand equals supply in every market, will there be any more trades that people will desire to carry out? This is just another way to ask whether the market equilibrium is Pareto efficient: will the agents desire to make any more trades after they have traded at the competitive prices?

We can see the answer by inspecting Figure 32.4: it turns out that the market equilibrium allocation *is* Pareto efficient. The proof is this: an allocation in the Edgeworth box is Pareto efficient if the set of bundles that A prefers doesn't intersect the set of bundles that B prefers. But at the market equilibrium, the set of bundles preferred by A must lie above her budget set, and the same thing holds for B, where "above" means "above from B's point of view." Thus the two sets of preferred allocations can't intersect. This means that there are no allocations that both agents prefer to the equilibrium allocation, so the equilibrium is Pareto efficient.

32.10 The Algebra of Efficiency

We can also show this algebraically. Suppose that we have a market equilibrium that is *not* Pareto efficient. We will show that this assumption leads to a logical contradiction.

To say that the market equilibrium is not Pareto efficient means that there is some other feasible allocation $(y_A^1, y_A^2, y_B^1, y_B^2)$ such that

$$y_A^1 + y_B^1 = \omega_A^1 + \omega_B^1 \tag{32.1}$$

$$y_A^2 + y_B^2 = \omega_A^2 + \omega_B^2 \tag{32.2}$$

and

$$(y_A^1, y_A^2) \succ_A (x_A^1, x_A^2) \tag{32.3}$$

$$(y_B^1, y_B^2) \succ_B (x_B^1, x_B^2). \tag{32.4}$$

The first two equations say that the y–allocation is feasible, and the next two equations say that it is preferred by each agent to the x–allocation. (The symbols $\succ_A$ and $\succ_B$ refer to the preferences of agents A and B.)

But by hypothesis, we have a market equilibrium where each agent is purchasing the best bundle he or she can afford. If (y_A^1, y_A^2) is better than the bundle that A is choosing, then it must cost more than A can afford, and similarly for B:

$$p_1 y_A^1 + p_2 y_A^2 > p_1 \omega_A^1 + p_2 \omega_A^2$$

$$p_1 y_B^1 + p_2 y_B^2 > p_1 \omega_B^1 + p_2 \omega_B^2.$$

Now add these two equations together to get

$$p_1(y_A^1 + y_B^1) + p_2(y_A^2 + y_B^2) > p_1(\omega_A^1 + \omega_B^1) + p_2(\omega_A^2 + \omega_B^2).$$

Substitute from equations (32.1) and (32.2) to get

$$p_1(\omega_A^1 + \omega_B^1) + p_2(\omega_A^2 + \omega_B^2) > p_1(\omega_A^1 + \omega_B^1) + p_2(\omega_A^2 + \omega_B^2),$$

which is clearly a contradiction, since the left-hand side and the right-hand side are the same.

We derived this contradiction by assuming that the market equilibrium was not Pareto efficient. Therefore, this assumption must be wrong. It follows that all market equilibria are Pareto efficient: a result known as the **First Theorem of Welfare Economics**.

The First Welfare Theorem guarantees that a competitive market will exhaust all of the gains from trade: an equilibrium allocation achieved by a

set of competitive markets will necessarily be Pareto efficient. Such an allocation may not have any other desirable properties, but it will necessarily be efficient.

In particular, the First Welfare Theorem says nothing about the distribution of economic benefits. The market equilibrium might not be a "just" allocation—if person A owned everything to begin with, then she would own everything after trade. That would be efficient, but it would probably not be very fair. But, after all, efficiency does count for something, and it is reassuring to know that a simple market mechanism like the one we have described is capable of achieving an efficient allocation.

EXAMPLE: Monopoly in the Edgeworth Box

In order to understand the First Welfare Theorem better, it is useful to consider another resource allocation mechanism that does not lead to efficient outcomes. A nice example of this occurs when one consumer attempts to behave as a monopolist. Suppose now that there is no auctioneer and that instead, agent A is going to quote prices to agent B, and agent B will decide how much he wants to trade at the quoted prices. Suppose further that A knows B's "demand curve" and will attempt to choose the set of prices that makes A as well-off as possible, given the demand behavior of B.

In order to examine the equilibrium in this process, it is appropriate to recall the definition of a consumer's **price offer curve**. The price offer curve, which we discussed in Chapter 6, represents all of the optimal choices of the consumer at different prices. B's offer curve represents the bundles that he will purchase at different prices; that is, it describes B's demand behavior. If we draw a budget line for B, then the point where that budget line intersects his offer curve represents B's optimal consumption.

Thus, if agent A wants to choose the prices to offer to B that make A as well-off as possible, she should find that point on B's offer curve where A has the highest utility. Such a choice is depicted in Figure 32.5.

This optimal choice will be characterized by a tangency condition as usual: A's indifference curve will be tangent to B's offer curve. If B's offer curve cut A's indifference curve, there would be some point on B's offer curve that A preferred—so we couldn't be at the optimal point for A.

Once we have identified this point—denoted by X in Figure 32.5—we just draw a budget line to that point from the endowment. At the prices that generate this budget line, B will choose the bundle X, and A will be as well-off as possible.

Is this allocation Pareto efficient? In general the answer is no. To see this simply note that A's indifference curve will not be tangent to the budget line at X, and therefore A's indifference curve will not be tangent to B's indifference curve. A's indifference curve is tangent to B's *offer curve,*

Figure 32.5 **Monopoly in the Edgeworth box**. A chooses the point on B's offer curve that gives her the highest utility.

but it cannot then be tangent to B's indifference curve. The monopoly allocation is Pareto inefficient.

In fact, it is Pareto inefficient in exactly the same way as described in the discussion of monopoly in Chapter 25. At the margin A would like to sell more at the equilibrium prices, but she can only do so by lowering the price at which she sells—and this will lower her income received from all her inframarginal sales.

We saw in Chapter 26 that a perfectly discriminating monopolist would end up producing an efficient level of output. Recall that a discriminating monopolist was one who was able to sell each unit of a good to the person who was willing to pay the most for that unit. What does a perfectly discriminating monopolist look like in the Edgeworth box?

The answer is depicted in Figure 32.6. Let us start at the initial endowment, W, and imagine A selling each unit of good 1 to B at a different price—the price at which B is just indifferent between buying or not buying that unit of the good. Thus, after A sells the first unit, B will remain on the same indifference curve through W. Then A sells the second unit of good 1 to B for the maximum price he is willing to pay. This means that the allocation moves further to the left, but remains on B's indifference curve through W. Agent A continues to sell units to B in this manner, thereby moving up B's indifference curve to find her—A's—most preferred point, denoted by an X in Figure 32.6.

Figure 32.6

A perfectly discriminating monopolist. Person A chooses the point X on person B's indifference curve through the endowment that gives her the highest utility. Such a point must be Pareto efficient.

It is easy to see that such a point must be Pareto efficient. Agent A will be as well-off as possible given B's indifference curve. At such a point, A has managed to extract all of B's consumer's surplus: B is no better off than he was at his endowment.

These two examples provide useful benchmarks with which to think about the First Welfare Theorem. The ordinary monopolist gives an example of a resource allocation mechanism that results in inefficient equilibria, and the discriminating monopolist gives another example of a mechanism that results in efficient equilibria.

32.11 Efficiency and Equilibrium

The First Welfare Theorem says that the equilibrium in a set of competitive markets is Pareto efficient. What about the other way around? Given a Pareto efficient allocation, can we find prices such that it is a market equilibrium? It turns out that the answer is yes, under certain conditions. The argument is illustrated in Figure 32.7.

Let us pick a Pareto efficient allocation. Then we know that the set of allocations that A prefers to her current assignment is disjoint from the set that B prefers. This implies of course that the two indifference curves are

Figure 32.7 **The Second Theorem of Welfare Economics**. When preferences are convex, a Pareto efficient allocation is an equilibrium for some set of prices.

tangent at the Pareto efficient allocation. So let us draw in the straight line that is their common tangent, as in Figure 32.7.

Suppose that the straight line represents the agents' budget sets. Then if each agent chooses the best bundle on his or her budget set, the resulting equilibrium will be the original Pareto efficient allocation.

Thus the fact that the original allocation is efficient automatically determines the equilibrium prices. The endowments can be any bundles that give rise to the appropriate budget set—that is, bundles that lie somewhere on the constructed budget line.

Can the construction of such a budget line always be carried out? Unfortunately, the answer is no. Figure 32.8 gives an example. Here the illustrated point X is Pareto efficient, but there are no prices at which A and B will want to consume at point X. The most obvious candidate is drawn in the diagram, but the optimal demands of agents A and B don't coincide for that budget. Agent A wants to demand the bundle Y, but agent B wants the bundle X—demand does not equal supply at these prices.

The difference between Figure 32.7 and Figure 32.8 is that the preferences in Figure 32.7 are convex while the ones in Figure 32.8 are not. If the preferences of both agents are convex, then the common tangent will not intersect either indifference curve more than once, and everything will work out fine. This observation gives us the **Second Theorem of Welfare**

A Pareto efficient allocation that is not an equilibrium. It is possible to find Pareto efficient allocations such as X in this diagram that cannot be achieved by competitive markets if preferences are nonconvex.

Figure 32.8

Economics: if all agents have convex preferences, then there will always be a set of prices such that each Pareto efficient allocation is a market equilibrium for an appropriate assignment of endowments.

The proof is essentially the geometric argument we gave above. At a Pareto efficient allocation, the bundles preferred by agent A and by agent B must be disjoint. Thus if both agents have convex preferences we can draw a straight line between the two sets of preferred bundles that separates one from the other. The slope of this line gives us the relative prices, and any endowment that puts the two agents on this line will lead to the final market equilibrium being the original Pareto efficient allocation.

32.12 Implications of the First Welfare Theorem

The two theorems of welfare economics are among the most fundamental results in economics. We have demonstrated the theorems only in the simple Edgeworth box case, but they are true for much more complex models with arbitrary numbers of consumers and goods. The welfare theorems have profound implications for the design of ways to allocate resources.

Let us consider the First Welfare Theorem. This says that any competitive equilibrium is Pareto efficient. There are hardly any explicit assump-

tions in this theorem—it follows almost entirely from the definitions. But there are some implicit assumptions. One major assumption is that agents only care about their own consumption of goods, and not about what other agents consume. If one agent does care about another agent's consumption, we say that there is a **consumption externality**. We shall see that when consumption externalities are present, a competitive equilibrium need not be Pareto efficient.

To take a simple example, suppose that agent A cares about agent B's consumption of cigars. Then there is no particular reason why each agent choosing his or her own consumption bundle at the market prices will result in a Pareto efficient allocation. After each person has purchased the best bundle he or she can afford, there may still be ways to make both of them better off—such as A paying B to smoke fewer cigars. We will discuss externalities in more detail in Chapter 35.

Another important implicit assumption in the First Welfare Theorem is that agents actually behave competitively. If there really were only two agents, as in the Edgeworth box example, then it is unlikely that they would each take price as given. Instead, the agents would probably recognize their market power and would attempt to use their market power to improve their own positions. The concept of competitive equilibrium only makes sense when there are enough agents to ensure that each behaves competitively.

Finally, the First Welfare Theorem is only of interest if a competitive equilibrium actually exists. As we have argued above, this will be the case if the consumers are sufficiently small relative to the size of the market.

Given these provisos, the First Welfare Theorem is a pretty strong result: a private market, with each agent seeking to maximize his or her own utility, will result in an allocation that achieves Pareto efficiency.

The importance of the First Welfare Theorem is that it gives a general mechanism—the competitive market—that we can use to ensure Pareto efficient outcomes. If there are only two agents involved, this doesn't matter very much; it is easy for two people to get together and examine the possibilities for mutual trades. But if there are thousands, or even millions, of people involved there must be some kind of structure imposed on the trading process. The First Welfare Theorem shows that the particular structure of competitive markets has the desirable property of achieving a Pareto efficient allocation.

If we are dealing with a resource problem involving many people, it is important to note that the use of competitive markets economizes on the information that any one agent needs to possess. The only things that a consumer needs to know to make his consumption decisions are the prices of the goods he is considering consuming. Consumers don't need to know anything about how the goods are produced, or who owns what goods, or where the goods come from in a competitive market. If each consumer knows only the prices of the goods, he can determine his demands, and if

the market functions well enough to determine the competitive prices, we are guaranteed an efficient outcome. The fact that competitive markets economize on information in this way is a strong argument in favor of their use as a way to allocate resources.

32.13 Implications of the Second Welfare Theorem

The Second Theorem of Welfare Economics asserts that under certain conditions, every Pareto efficient allocation can be achieved as a competitive equilibrium.

What is the meaning of this result? The Second Welfare Theorem implies that the problems of distribution and efficiency can be separated. Whatever Pareto efficient allocation you want can be supported by the market mechanism. The market mechanism is distributionally neutral; whatever your criteria for a good or a just distribution of welfare, you can use competitive markets to achieve it.

Prices play two roles in the market system: an *allocative* role and a *distributive* role. The allocative role of prices is to indicate relative scarcity; the distributive role is to determine how much of different goods different agents can purchase. The Second Welfare Theorem says that these two roles can be separated: we can redistribute endowments of goods to determine how much wealth agents have, and then use prices to indicate relative scarcity.

Policy discussions often become confused on this point. One often hears arguments for intervening in pricing decisions on grounds of distributional equity. However, such intervention is typically misguided. As we have seen above, a convenient way to achieve efficient allocations is for each agent to face the true social costs of his or her actions and to make choices that reflect those costs. Thus in a perfectly competitive market the marginal decision of whether to consume more or less of some good will depend on the price—which measures how everyone else values this good on the margin. The considerations of efficiency are inherently marginal decisions—each person should face the correct marginal tradeoff in making his or her consumption decisions.

The decision about *how much* different agents should consume is a totally different issue. In a competitive market this is determined by the value of the resources that a person has to sell. From the viewpoint of the pure theory, there is no reason why the state can't transfer purchasing power—endowments—among consumers in any way that is seen fit.

In fact the state doesn't need to transfer the physical endowments themselves. All that is necessary is to transfer the purchasing power of the endowment. The state could tax one consumer on the basis of the value of his endowment and transfer this money to another. As long as the taxes are based on the value of the consumer's *endowment* of goods there will

be no loss of efficiency. It is only when taxes depend on the *choices* that a consumer makes that inefficiencies result, since in this case, the taxes will affect the consumer's marginal choices.

It is true that a tax on endowments will generally change people's behavior. But, according to the First Welfare Theorem, trade from any initial endowments will result in a Pareto efficient allocation. Thus no matter how one redistributes endowments, the equilibrium allocation as determined by market forces will still be Pareto efficient.

However, there are practical matters involved. It would be easy to have a lump-sum tax on consumers. We could tax all consumers with blue eyes, and redistribute the proceeds to consumers with brown eyes. As long as eye color can't be changed, there would be no loss in efficiency. Or we could tax consumers with high IQs and redistribute the funds to consumers with low IQs. Again, as long as IQ can be measured, there is no efficiency loss in this kind of tax.

But there's the problem. How do we measure people's endowment of goods? For most people, the bulk of their endowment consists of their own labor power. People's endowments of labor consist of the labor that they *could* consider selling, not the amount of labor that they actually end up selling. Taxing labor that people decide to sell to the market is a **distortionary tax**. If the sale of labor is taxed, the labor supply decision of consumers will be distorted—they will likely supply less labor than they would have supplied in the absence of a tax. Taxing the potential value of labor—the endowment of labor—is not distortionary. The potential value of labor is, by definition, something that is not changed by taxation. Taxing the value of the endowment sounds easy until we realize that it involves identifying and taxing something that *might* be sold, rather than taxing something that is sold.

We could *imagine* a mechanism for levying this kind of tax. Suppose that we considered a society where each consumer was required to give the money earned in 10 hours of his labor time to the state each week. This kind of tax would be independent of how much the person actually worked—it would only depend on the endowment of labor, not on how much was actually sold. Such a tax is basically transferring some part of each consumer's endowment of labor time to the state. The state could then use these funds to provide various goods, or it could simply transfer these funds to other agents.

According to the Second Welfare Theorem, this kind of lump-sum taxation would be nondistortionary. Essentially any Pareto efficient allocation could be achieved by such lump-sum redistribution.

However, no one is advocating such a radical restructuring of the tax system. Most people's labor supply decisions are relatively insensitive to variations in the wage rate, so the efficiency loss from taxing labor may not be too large anyway. But the message of the Second Welfare Theorem is important. Prices should be used to reflect scarcity. Lump-sum transfers of

wealth should be used to adjust for distributional goals. To a large degree, these two policy decisions can be separated.

People's concern about the distribution of welfare can lead them to advocate various forms of manipulation of prices. It has been argued, for example, that senior citizens should have access to less expensive telephone service, or that small users of electricity should pay lower rates than large users. These are basically attempts to redistribute income through the price system by offering some people lower prices than others.

When you think about it this is a terribly inefficient way to redistribute income. If you want to redistribute income, why don't you simply redistribute income? If you give a person an extra dollar to spend, then he can choose to consume more of any of the goods that he wants to consume—not necessarily just the good being subsidized.

Summary

1. General equilibrium refers to the study of how the economy can adjust to have demand equal supply in all markets at the same time.

2. The Edgeworth box is a graphical tool to examine such a general equilibrium with 2 consumers and 2 goods.

3. A Pareto efficient allocation is one in which there is no feasible reallocation of the goods that would make all consumers at least as well-off and at least one consumer strictly better off.

4. Walras' law states that the value of aggregate excess demand is zero for all prices.

5. A general equilibrium allocation is one in which each agent chooses a most preferred bundle of goods from the set of goods that he or she can afford.

6. Only relative prices are determined in a general equilibrium system.

7. If the demand for each good varies continuously as prices vary, then there will always be some set of prices where demand equals supply in every market; that is, a competitive equilibrium.

8. The First Theorem of Welfare Economics states that a competitive equilibrium is Pareto efficient.

9. The Second Theorem of Welfare Economics states that as long as preferences are convex, then every Pareto efficient allocation can be supported as a competitive equilibrium.

REVIEW QUESTIONS

1. Is it possible to have a Pareto efficient allocation where someone is worse off than he is at an allocation that is not Pareto efficient?

2. Is it possible to have a Pareto efficient allocation where everyone is worse off than they are at an allocation that is not Pareto efficient?

3. True or false? If we know the contract curve, then we know the outcome of any trading.

4. Can some individual be made better off if we are at a Pareto efficient allocation?

5. If the value of excess demand in 8 out of 10 markets is equal to zero, what must be true about the remaining two markets?

APPENDIX

Let us examine the calculus conditions describing Pareto efficient allocations. By definition, a Pareto efficient allocation makes each agent as well-off as possible, given the utility of the other agent. So let us pick $\overline{u}$ as the utility level for agent B, say, and see how we can make agent A as well-off as possible.

The maximization problem is

$$\max_{x_A^1, x_A^2, x_B^1, x_B^2} u_A(x_A^1, x_A^2)$$

$$\text{such that } u_B(x_B^1, x_B^2) = \overline{u}$$

$$x_A^1 + x_B^1 = \omega^1$$

$$x_A^2 + x_B^2 = \omega^2.$$

Here $\omega^1 = \omega_A^1 + \omega_B^1$ is the total amount of good 1 available and $\omega^2 = \omega_A^2 + \omega_B^2$ is the total amount of good 2 available. This maximization problem asks us to find the allocation $(x_A^1, x_A^2, x_B^1, x_B^2)$ that makes person A's utility as large as possible, given a fixed level for person B's utility, and given that the total amount of each good used is equal to the amount available.

We can write the Lagrangian for this problem as

$$\begin{aligned} L = {} & u_A(x_A^1, x_A^2) - \lambda(u_B(x_B^1, x_B^2) - \overline{u}) \\ & - \mu_1(x_A^1 + x_B^1 - \omega^1) - \mu_2(x_A^2 + x_B^2 - \omega^2). \end{aligned}$$

Here λ is the Lagrange multiplier on the utility constraint, and the μ's are the Lagrange multipliers on the resource constraints. When we differentiate with

respect to each of the goods, we have four first-order conditions that must hold at the optimal solution:

$$\begin{aligned}
\frac{\partial L}{\partial x_A^1} &= \frac{\partial u_A}{\partial x_A^1} - \mu_1 = 0 \\
\frac{\partial L}{\partial x_A^2} &= \frac{\partial u_A}{\partial x_A^2} - \mu_2 = 0 \\
\frac{\partial L}{\partial x_B^1} &= -\lambda \frac{\partial u_B}{\partial x_B^1} - \mu_1 = 0 \\
\frac{\partial L}{\partial x_B^2} &= -\lambda \frac{\partial u_B}{\partial x_B^2} - \mu_2 = 0.
\end{aligned}$$

If we divide the first equation by the second, and the third equation by the fourth, we have

$$MRS_A = \frac{\partial u_A / \partial x_A^1}{\partial u_A / \partial x_A^2} = \frac{\mu_1}{\mu_2} \tag{32.5}$$

$$MRS_B = \frac{\partial u_B / \partial x_B^1}{\partial u_B / \partial x_B^2} = \frac{\mu_1}{\mu_2}. \tag{32.6}$$

The interpretation of these conditions is given in the text: at a Pareto efficient allocation, the marginal rates of substitution between the two goods must be the same. Otherwise, there would be some trade that would make each consumer better off.

Let us recall the conditions that must hold for optimal choice by consumers. If consumer A is maximizing utility subject to her budget constraint and consumer B is maximizing utility subject to his budget constraint, and both consumers face the same prices for goods 1 and 2, we must have

$$\frac{\partial u_A / \partial x_A^1}{\partial u_A / \partial x_A^2} = \frac{p_1}{p_2} \tag{32.7}$$

$$\frac{\partial u_B / \partial x_B^1}{\partial u_B / \partial x_B^2} = \frac{p_1}{p_2}. \tag{32.8}$$

Note the similarity with the efficiency conditions. The Lagrange multipliers in the efficiency conditions, μ_1 and μ_2, are just like the prices p_1 and p_2 in the consumer choice conditions. In fact the Lagrange multipliers in this kind of problem are sometimes known as **shadow prices** or **efficiency prices**.

Every Pareto efficient allocation has to satisfy conditions like those in equations (32.5) and (32.6). Every competitive equilibrium has to satisfy conditions like those in equations (32.7) and (32.8). The conditions describing Pareto efficiency and the conditions describing individual maximization in a market environment are virtually the same.

CHAPTER 33

PRODUCTION

In the last chapter we described a general equilibrium model of a pure exchange economy and discussed issues of resource allocation when a fixed amount of each good was available. In this chapter we want to describe how production fits into the general equilibrium framework. When production is possible, the amounts of the goods are not fixed but will respond to market prices.

If you thought the two-consumer two-good assumption was a restrictive framework in which to examine trade, imagine what production is going to look like! The minimal set of players that we can have to make an interesting problem is one consumer, one firm, and two goods. The traditional name for this economic model is the **Robinson Crusoe economy**, after Defoe's shipwrecked hero.

33.1 The Robinson Crusoe Economy

In this economy Robinson Crusoe plays a dual role: he is both a consumer and a producer. Robinson can spend his time loafing on the beach thereby consuming leisure, or he can spend time gathering coconuts. The more

coconuts he gathers the more he has to eat, but the less time he has to improve his tan.

Robinson's preferences for coconuts and leisure are depicted in Figure 33.1. They are just like the preferences for leisure and consumption depicted in Chapter 9, except we are measuring labor on the horizontal axis rather than leisure. So far nothing new has been added.

Figure 33.1

The Robinson Crusoe economy. The indifference curves depict Robinson's preferences for coconuts and leisure. The production function depicts the technological relationship between the amount he works and the amount of coconuts he produces.

Now let's draw in the **production function**, the function that illustrates the relationship between how much Robinson works and how many coconuts he gets. This will typically have the shape depicted in Figure 33.1. The more Robinson works, the more coconuts he will get; but, due to diminishing returns to labor, the marginal product of his labor declines: the number of extra coconuts that he gets from an extra hour's labor decreases as the hours of labor increase.

How much does Robinson work and how much does he consume? To answer these questions, look for the highest indifference curve that just touches the production set. This will give the most-preferred combination

of labor and consumption that Robinson can get, given the technology for gathering coconuts that he is using.

At this point, the slope of the indifference curve must equal the slope of the production function by the standard argument: if they crossed, there would be some other feasible point that was preferred. This means that the marginal product of an extra hour of labor must equal the marginal rate of substitution between leisure and coconuts. If the marginal product were greater than the marginal rate of substitution, it would pay for Robinson to give up a little leisure in order to get the extra coconuts. If the marginal product were less than the marginal rate of substitution, it would pay for Robinson to work a little less.

33.2 Crusoe, Inc.

So far this story is only a slight extension of models we have already seen. But now let's add a new feature. Suppose that Robinson is tired of simultaneously being a producer and consumer and that he decides to alternate roles. One day he will behave entirely as a producer, and the next day he will behave entirely as a consumer. In order to coordinate these activities, he decides to set up a labor market and a coconut market.

He also sets up a firm, Crusoe, Inc., and becomes its sole shareholder. The firm is going to look at the prices for labor and coconuts and decide how much labor to hire and how many coconuts to produce, guided by the principle of profit maximization. Robinson, in his role as a worker, is going to collect income from working at the firm; in his role as shareholder in the firm he will collect profits; and, in his role as consumer he will decide how much to purchase of the firm's output. (No doubt this sounds peculiar, but there really isn't that much else to do on a desert island.)

In order to keep track of his transactions, Robinson invents a currency he calls "dollars," and he chooses, somewhat arbitrarily, to set the price of coconuts at one dollar apiece. Thus coconuts are the numeraire good for this economy; as we've seen in Chapter 2, a numeraire good is one whose price has been set to one. Since the price of coconuts is normalized at one, we have only to determine the wage rate. What should his wage rate be in order to make this market work?

We're going to think about this problem first from the viewpoint of Crusoe, Inc., and then from the viewpoint of Robinson, the consumer. The discussion is a little schizophrenic at times, but that's what you have to put up with if you want to have an economy with only one person. We're going to look at the economy after it has been running along for some time, and everything is in equilibrium. In equilibrium, the demand for coconuts will equal the supply of coconuts and the demand for labor will equal the

supply of labor. Both Crusoe, Inc. and Robinson the consumer will be making optimal choices given the constraints they face.

33.3 The Firm

Each evening, Crusoe, Inc. decides how much labor it wants to hire the next day, and how many coconuts it wants to produce. Given a price of coconuts of 1 and a wage rate of labor of w, we can solve the firm's profit-maximization problem in Figure 33.2. We first consider all combinations of coconuts and labor that yield a constant level of profits, π. This means that

$$\pi = C - wL.$$

Solving for C, we have

$$C = \pi + wL.$$

Just as in Chapter 20, this formula describes the isoprofit lines—all combinations of labor and coconuts that yield profits of π. Crusoe, Inc. will choose a point where the profits are maximized. As usual, this implies a tangency condition: the slope of the production function—the marginal product of labor—must equal w, as illustrated in Figure 33.2.

Profit maximization. Crusoe, Inc. chooses a production plan that maximizes profits. At the optimal point the production function must be tangent to an isoprofit line.

Figure 33.2

Thus the vertical intercept of the isoprofit line measures the maximal level of profits measured in units of coconuts: if Robinson generates π^* dollars of profit, this money can buy π^* coconuts, since the price of coconuts has been chosen to be 1. There we have it. Crusoe, Inc. has done its job. Given the wage w, it has determined how much labor it wants to hire, how many coconuts it wants to produce, and what profits it will generate by following this plan. So Crusoe, Inc. declares a stock dividend of π^* dollars and mails it off to its sole shareholder, Robinson.

33.4 Robinson's Problem

The next day Robinson wakes up and receives his dividend of π^* dollars. While eating his coconut breakfast, he contemplates how much he wants to work and consume. He may consider just consuming his endowment—spend his profits on π^* coconuts and consume his endowment of leisure. But listening to his stomach growl is not so pleasant, and it might make sense to work for a few hours instead. So Robinson trudges down to Crusoe, Inc. and starts to gather coconuts, just as he has done every other day.

We can describe Robinson's labor-consumption choice using standard indifference curve analysis. Plotting labor on the horizontal axis and coconuts on the vertical axis, we can draw in an indifference curve as illustrated in Figure 33.3.

Since labor is a bad, by assumption, and coconuts are a good, the indifference curve has a positive slope as shown in the diagram. If we indicate the maximum amount of labor by $\overline{L}$, then the distance from $\overline{L}$ to the chosen supply of labor gives Robinson's demand for leisure. This is just like the supply of labor model examined in Chapter 9, except we have reversed the origin on the horizontal axis.

Robinson's budget line is also illustrated in Figure 33.3. It has a slope of w and passes through his endowment point $(\pi^*, 0)$. (Robinson has a zero endowment of labor and a π^* endowment of coconuts since that would be his bundle if he engaged in no market transactions.) Given the wage rate, Robinson chooses optimally how much he wants to work and how many coconuts he wants to consume. At his optimal consumption, the marginal rate of substitution between consumption and leisure must equal the wage rate, just as in a standard consumer choice problem.

33.5 Putting Them Together

Now we superimpose Figures 33.2 and 33.3 to get Figure 33.4. Look at what has happened! Robinson's bizarre behavior has worked out all right after all. He ends up consuming at exactly the same point as he would have if he had made all the decisions at once. Using the market system

Robinson's maximization problem. Robinson the consumer decides how much to work and consume given the prices and wages. The optimal point will occur where the indifference curve is tangent to the budget line.

Figure 33.3

results in the same outcome as choosing the consumption and production plans directly.

Since the marginal rate of substitution between leisure and consumption equals the wage, and the marginal product of labor equals the wage, we are assured that the marginal rate of substitution between labor and consumption equals the marginal product—that is, that the slopes of the indifference curve and the production set are the same.

In the case of a one-person economy, using the market is pretty silly. Why should Robinson bother to break up his decision into two pieces? But in an economy with many people, breaking up decisions no longer seems so odd. If there are many firms, then questioning each person about how much they want of each good is simply impractical. In a market economy the firms simply have to look at the prices of goods in order to make their production decisions. For the prices of goods measure how much the consumers value *extra* units of consumption. And the decision that the firms face, for the most part, is whether they should produce more or less output.

The market prices reflect the marginal values of the goods that the firms use as inputs and outputs. If firms use the change in profits as a guide to production, where the profits are measured at market prices, then their decisions will reflect the marginal values that consumers place on the goods.

Figure 33.4 **Equilibrium in both consumption and production**. The amount of coconuts demanded by the consumer Robinson equals the amount of coconuts supplied by Crusoe, Inc.

33.6 Different Technologies

In the above discussion we have assumed that the technology available to Robinson exhibited diminishing returns to labor. Since labor was the only input to production, this was equivalent to decreasing returns to scale. (This is not necessarily true if there is more than one input!)

It is useful to consider some other possibilities. Suppose, for example, that the technology exhibited constant returns to scale. Recall that constant returns to scale means that using twice as much of all inputs produces twice as much output. In the case of a one-input production function, this means that the production function must be a straight line through the origin as depicted in Figure 33.5.

Since the technology has constant returns to scale, the argument in Chapter 20 implies that the only reasonable operating position for a competitive firm is at zero profits. This is because if the profits were ever greater than zero, it would pay for the firm to expand output indefinitely, and if profits were ever less than zero, it would pay the firm to produce zero output.

Thus Robinson's endowment involves zero profits and $\bar{L}$, his initial endowment of labor time. His budget set coincides with the production set, and the story is much the same as before.

The situation is somewhat different with an increasing returns to scale

Constant returns to scale. If the technology exhibits constant returns to scale, Crusoe, Inc., makes zero profits.

Figure 33.5

technology, as depicted in Figure 33.6. There is no difficulty in this simple example in exhibiting the optimal choice of consumption and leisure for Robinson. The indifference curve will be tangent to the production set as usual. The problem arises in trying to support this point as a profit-maximizing point. For if the firm were faced with the prices given by Robinson's marginal rate of substitution, it would want to produce more output than Robinson would demand.

If the firm exhibits increasing returns to scale at the optimal choice, then the average costs of production will exceed the marginal costs of production—and that means that the firm will be making negative profits. The goal of profit maximization would lead the firm to want to increase its output—but this would be incompatible with the demands for output and the supplies of inputs from the consumers. In the case depicted, there is *no* price at which the utility-maximizing demand by the consumer will equal the profit-maximizing supply from the firm.

Increasing returns to scale is an example of a **nonconvexity**. In this case the production set—the set of coconuts and labor that are feasible for the economy—is not a convex set. Thus the common tangent to the indifference curve and the production function at the point (L^*, C^*) in Figure 33.6 will not separate the preferred points from the feasible points as it does in Figure 33.4.

Nonconvexities such as this pose grave difficulties for the functioning of competitive markets. In a competitive market consumers and firms look

Figure 33.6

Increasing returns to scale. The production set exhibits increasing returns to scale and the Pareto efficient allocation cannot be achieved by a competitive market.

at just one set of numbers—the market prices—to determine their consumption and production decisions. If the technology and the preferences are convex, then the only things that the economic agents need to know to make efficient decisions are the relationship between the prices and the marginal rates of substitution near the points where the economy is currently producing: the prices tell the agents everything that is necessary in order to determine an efficient allocation of resources.

But if the technology and/or the preferences are nonconvex, then the prices do not convey all the information necessary in order to choose an efficient allocation. Information about the slopes of the production function and indifference curves far away from the current operating position is also necessary.

However, these observations apply only when the returns to scale are large relative to the size of the market. Small regions of increasing returns to scale do not pose undue difficulties for a competitive market.

33.7 Production and the First Welfare Theorem

Recall that in the case of a pure exchange economy, a competitive equilibrium is Pareto efficient. This fact is known as the First Theorem of Welfare Economics. Does the same result hold in an economy with production? The diagrammatic approach used above is not adequate to answer

this question, but a generalization of the algebraic argument we provided in Chapter 32 does nicely. It turns out that the answer is yes: if all firms act as competitive profit maximizers, then a competitive equilibrium will be Pareto efficient.

This result has the usual caveats. First, it has nothing to do with distribution. Profit maximization only ensures efficiency, not justice! Second, this result only makes sense when a competitive equilibrium actually exists. In particular, this will rule out large areas of increasing returns to scale. Third, the theorem implicitly assumes that the choices of any one firm do not affect the production possibilities of other firms. That is, it rules out the possibility of **production externalities**. Similarly, the theorem requires that firms' production decisions do not directly affect the consumption possibilities of consumers; that is, that there are no **consumption externalities**. More precise definitions of externalities will be given in Chapter 35, where we will examine their effect on efficient allocations in more detail.

33.8 Production and the Second Welfare Theorem

In the case of a pure exchange economy, every Pareto efficient allocation is a possible competitive equilibrium, as long as consumers exhibit convex preferences. In the case of an economy involving production, the same result is true, but now we require not only that consumers' preferences are convex, but also that firms' production sets are convex. As discussed above, this requirement effectively rules out the possibility of increasing returns to scale: if firms have increasing returns to scale at the equilibrium level of production, they would want to produce more output at the competitive prices.

However, with constant or decreasing returns to scale, the Second Welfare Theorem works fine. Any Pareto efficient allocation can be achieved through the use of competitive markets. Of course in general it will be necessary to redistribute endowments among the consumers to support different Pareto efficient allocations. In particular, both the income from endowments of labor and ownership shares of the firm will have to be redistributed. As indicated in the last chapter, there may be significant practical difficulties involved with this sort of redistribution.

33.9 Production Possibilities

We have now seen how production and consumption decisions can be made in a one-input, one-output economy. In this section we want to explore how this model can be generalized to an economy with several inputs and outputs. Although we will deal only with the two-good case, the concepts will generalize naturally to many goods.

So let us suppose that there is some other good that Robinson might produce—say fish. He can devote his time to gathering coconuts or to fishing. In Figure 33.7 we have depicted the various combinations of coconuts and fish that Robinson can produce from devoting different amounts of time to each activity. This set is known as a **production possibilities set**. The boundary of the production possibilities set is called the **production possibilities frontier**. This should be contrasted with the production function discussed earlier that depicts the relationship between the input good and the output good; the production possibilities set depicts only the set of *output* goods that is feasible. (In more advanced treatments, both inputs and outputs can be considered part of the production possibilities set, but these treatments cannot easily be handled with two-dimensional diagrams.)

Figure 33.7

A production possibilities set. The production possibilities set measures the set of outputs that are feasible given the technology and the amounts of inputs.

The shape of the production possibilities set will depend on the nature of the underlying technologies. If the technologies for producing coconuts and fish exhibit constant returns to scale the production possibilities set will take an especially simple form. Since by assumption there is only one input to production—Robinson's labor—the production functions for fish and coconuts will be simply *linear* functions of labor.

For example, suppose that Robinson can produce 10 pounds of fish per

hour or 20 pounds of coconuts per hour. Then if he devotes L_f hours to fish production and L_c hours to coconut production, he will produce $10L_f$ pounds of fish and $20L_c$ pounds of coconuts. Suppose that Robinson decides to work 10 hours a day. Then the production possibilities set will consist of all combinations of coconuts, C, and fish, F, such that

$$\begin{aligned} F &= 10L_f \\ C &= 20L_c \\ L_c + L_f &= 10. \end{aligned}$$

The first two equations measure the production relationships, and the third measures the resource constraint. To determine the production possibilities frontier solve the first two equations for L_f and L_c to get

$$\begin{aligned} L_f &= \frac{F}{10} \\ L_c &= \frac{C}{20}. \end{aligned}$$

Now add these two equations together, and use the fact that $L_f + L_c = 10$ to find

$$\frac{F}{10} + \frac{C}{20} = 10.$$

This equation gives us all the combinations of fish and coconuts that Robinson can produce if he works 10 hours a day. It is depicted in Figure 33.8A.

The slope of this production possibilities set measures the **marginal rate of transformation**—how much of one good Robinson can get if he decides to sacrifice some of the other good. If Robinson gives up enough labor to produce 1 less pound of fish, he will be able to get 2 more pounds of coconuts. Think about it: if Robinson works one hour less on fish production, he will get 10 pounds less fish. But then if he devotes that time to coconuts, he will get 20 pounds more coconuts. The tradeoff is at a ratio of 2 to 1.

33.10 Comparative Advantage

The construction of the production possibilities set given above was quite simple since there was only one way to produce fish and one way to produce coconuts. What if there is more than one way to produce each good? Suppose that we add another worker to our island economy, who has different skills in producing fish and coconuts.

To be specific, let us call the new worker Friday, and suppose that he can produce 20 pounds of fish per hour, or 10 pounds of coconuts per hour.

Thus if Friday works for 10 hours, his production possibilities set will be determined by

$$F = 20L_f$$
$$C = 10L_c$$
$$L_c + L_f = 10.$$

Doing the same sort of calculations as we did for Robinson, Friday's production possibilities set is given by

$$\frac{F}{20} + \frac{C}{10} = 10.$$

This is depicted in Figure 33.8B. Note that the marginal rate of transformation between coconuts and fish for Friday is $\Delta C/\Delta F = -1/2$, whereas for Robinson the marginal rate of transformation is -2. For every pound of coconuts that Friday gives up, he can get two pounds of fish; for every pound of fish that Robinson gives up, he can get two pounds of coconuts. In this circumstance we say that Friday has a **comparative advantage** in fish production, and Robinson has a comparative advantage in coconut production. In Figure 33.8 we have depicted three production possibilities sets: Panel A shows Robinson's, panel B shows Friday's, and panel C depicts the joint production possibilities set—how much of each good could be produced in total by both people.

Figure 33.8 **Joint production possibilities**. Robinson's and Friday's production possibilities sets and the joint production possibilities set.

The joint production possibilities set combines the best of both workers. If both workers are used entirely to produce coconuts, we will get 300

coconuts—100 from Friday and 200 from Robinson. If we want to get more fish, it makes sense to shift the person who is most productive at fish—Friday—out of coconut production and into fish production. For each pound of coconuts that Friday doesn't produce we get 2 pounds of fish; thus the slope of the joint production possibilities set is $-1/2$—which is exactly Friday's marginal rate of transformation.

When Friday is producing 200 pounds of fish, he is fully occupied. If we want even more fish, we have to switch to using Robinson. From this point on the joint production possibilities set will have a slope of -2, since we will be operating along Robinson's production possibilities set. Finally, if we want to produce as much fish as possible, both Robinson and Friday concentrate on fish production and we get 300 pounds of fish, 200 from Friday, and 100 from Robinson.

Since the workers each have a comparative advantage in different goods, the joint production possibilities set will have a "kink," as shown in Figure 33.8C. There is only one kink in this example since there are just two different ways to produce output—Crusoe's way and Friday's way. If there are many different ways to produce output, the production possibilities set will have the more typical "rounded" structure, as depicted in Figure 33.7.

33.11 Pareto Efficiency

In the last two sections we saw how to construct the production possibilities set, the set that describes the feasible consumption bundles for the economy as a whole. Here we consider Pareto efficient ways to choose among the feasible consumption bundles.

We will indicate aggregate consumption bundles by (X^1, X^2). This indicates that there are X^1 units of good 1 and X^2 units of good 2 that are available for consumption. In the Crusoe/Friday economy, the two goods are coconuts and fish, but we will use the (X^1, X^2) notation in order to emphasize the similarities with the analysis in Chapter 32. Once we know the total amount of each good, we can draw an Edgeworth box as in Figure 33.9.

Given (X^1, X^2), the set of Pareto efficient consumption bundles will be the same sort as those examined in the last chapter: the Pareto efficient consumption levels will lie along the Pareto set—the line of mutual tangencies of the indifference curves, as illustrated in Figure 33.9. These are the allocations in which each consumer's marginal rate of substitution—the rate at which he or she is just willing to trade—equals that of the other.

These allocations are Pareto efficient as far as the consumption decisions are concerned. If people can simply trade one good for another, the Pareto set describes the set of bundles that exhausts the gains from trade. But in an economy with production, there is another way to "exchange" one good for another—namely, to produce less of one good and more of another.

Figure 33.9 **Production and the Edgeworth box**. At each point on the production possibilities frontier, we can draw an Edgeworth box to illustrate the possible consumption allocations.

The Pareto set describes the set of Pareto efficient bundles *given* the amounts of goods 1 and 2 available, but in an economy with production those amounts can themselves be chosen out of the production possibilities set. Which choices from the production possibilities set will be Pareto efficient choices?

Let us think about the logic underlying the marginal rate of substitution condition. We argued that in a Pareto efficient allocation, the MRS of consumer A had to be equal to the MRS of consumer B: the rate at which consumer A would just be willing to trade one good for the other should be equal to the rate at which consumer B would just be willing to trade one good for the other. If this were not true, then there would be some trade that would make both consumers better off.

Recall that the marginal rate of transformation (MRT) measures the rate at which one good can be "transformed" into the other. Of course, one good really isn't being literally *transformed* into the other. Rather the factors of production are being moved around so as to produce less of one good and more of the other.

Suppose that the economy were operating at a position where the marginal rate of substitution of one of the consumers was not equal to the marginal rate of transformation between the two goods. Then such a position cannot be Pareto efficient. Why? Because at this point, the rate at

which the consumer is willing to trade good 1 for good 2 is different from the rate at which good 1 can be transformed into good 2—there is a way to make the consumer better off by rearranging the pattern of production.

Suppose, for example, that the consumer's MRS is 1; the consumer is just willing to substitute good 1 for good 2 on a one-to-one basis. Suppose that the MRT is 2, which means that giving up one unit of good 1 will allow society to produce two units of good 2. Then clearly it makes sense to reduce the production of good 1 by one unit; this will generate two extra units of good 2. Since the consumer was just indifferent between giving up one unit of good 1 and getting one unit of the other good in exchange, he or she will now certainly be better off by getting *two* extra units of good 2.

The same argument can be made whenever one of the consumers has a MRS that is different from the MRT—there will always be a rearrangement of consumption and production that will make that consumer better off. We have already seen that for Pareto efficiency each consumer's MRS should be the same, and the argument given above implies that each consumer's MRS should in fact be equal to the MRT.

Figure 33.9 illustrates a Pareto efficient allocation. The MRSs of each consumer are the same, since their indifference curves are tangent in the Edgeworth box. And each consumer's MRS is equal to the MRT—the slope of the production possibilities set.

33.12 Castaways, Inc.

In the last section we derived the necessary conditions for Pareto efficiency: the MRS of each consumer must equal the MRT. Any way of distributing resources that results in Pareto efficiency must satisfy this condition. Earlier in this chapter, we claimed that a competitive economy with profit-maximizing firms and utility-maximizing consumers would result in a Pareto efficient allocation. In this section we explore the details of how this works.

Our economy now contains two individuals, Robinson and Friday. There are four goods: two factors of production (Robinson's labor and Friday's labor) and two output goods (coconuts and fish). Let us suppose that Robinson and Friday are both shareholders of the firm, which we will now refer to as Castaways, Inc. Of course, they are also the sole employees and customers, but as usual we shall examine each role in turn, and not allow the participants to see the wider picture. After all, the object of the analysis is to understand how a *decentralized* resource allocation system works—one in which each person only has to determine his or her own decisions, without regard for the functioning of the economy as a whole.

Start first with Castaways, Inc., and consider the profit-maximization problem. Castaways, Inc., produces two outputs, coconuts (C) and fish (F), and it uses two kinds of labor, Crusoe's labor (L_C) and Friday's labor

(L_F). Given the price of coconuts (p_C), the price of fish (p_F), and the wage rates of Crusoe and Friday (w_C and w_F), the profit-maximization problem is

$$\max_{C,F,L_F,L_C} p_C C + p_F F - w_C L_C - w_F L_F$$

subject to the technological constraints described by the production possibilities set.

Let us suppose that the firm finds it optimal in equilibrium to hire L_F^* units of Friday's labor and L_C^* units of Crusoe's labor. The question we want to focus on here is how profit maximization determines the pattern of output to produce. Let $L^* = w_C L_C^* + w_F L_F^*$ represent the labor costs of production, and write the profits of the firm, π, as

$$\pi = p_C C + p_F F - L^*.$$

Rearranging this equation, we have

$$C = \frac{\pi + L^*}{p_C} - \frac{p_F F}{p_C}.$$

This equation describes the **isoprofit lines** of the firm, as depicted in Figure 33.10, with a slope of $-p_F/p_C$ and a vertical intercept of $(\pi+L^*)/p_C$. Since L^* is fixed by assumption, higher profits will be associated with isoprofit lines that have higher vertical intercepts.

If the firm wants to maximize its profits, it will choose a point on the production possibilities set such that the isoprofit line through that point has the highest possible vertical intercept. By this stage, it should be clear that this implies that the isoprofit line must be tangent to the production possibilities set; that is, that the slope of the production possibilities set (the MRT) should be equal to the slope of the isoprofit line, $-p_F/p_C$:

$$\text{MRT} = -\frac{p_F}{p_C}.$$

We've described this profit-maximization problem in the case of one firm, but it holds for an arbitrary number of firms: each firm that chooses the most profitable way to produce coconuts and fish will operate where the marginal rate of transformation between any two goods equals the price ratio between those two goods. This holds true even if the firms have quite different production possibilities sets, as long as they face the same prices for the two goods.

This means that in equilibrium the prices of the two goods will measure the marginal rate of transformation—the opportunity cost of one good in terms of the other. If you want more coconuts, you will have to give up some fish. How much fish? Just look at the price ratio of fish to coconuts: the ratio of these economic variables tells us what the technological tradeoff must be.

Profit maximization. At the point yielding maximum profits, the marginal rate of transformation must equal the slope of the isoprofit line, $-p_F/p_C$.

Figure 33.10

33.13 Robinson and Friday as Consumers

We've seen how Castaways, Inc., determines its profit-maximizing production plan. In order to do this, it must hire some labor and it may generate some profits. When it hires labor, it pays wages to the labor; when it makes profits, it pays dividends to its shareholders. Either way the money made by Castaways, Inc., gets paid back to Robinson and Friday, either in the form of wages or profits.

Since the firm pays out all of its receipts to its workers and its shareholders, this means that they must necessarily have enough income to purchase its output. This is just a variation on Walras' law discussed in Chapter 32: people get their income from selling their endowments, so they must always have enough income to purchase those endowments. Here people get income from selling their endowments and also receive profits from the firm. But since money never disappears from or is added to the system, people will always have exactly enough money to purchase what is produced.

What do the consumers do with the money from the firm? As usual, they use the money to purchase consumption goods. Each person chooses the best bundle of goods that he can afford at the prices p_F and p_C. As

we've seen earlier, the optimal consumption bundle for each consumer must satisfy the condition that the marginal rate of substitution between the two goods must be equal to the common price ratio. But this price ratio is also equal to the marginal rate of transformation, due to profit maximization by the firm. Thus the necessary conditions for Pareto efficiency are met: the MRS of each consumer equals the MRT.

In this economy, the prices of the goods serve as a signal of relative scarcity. They indicate the technological scarcity—how much the production of one good must be reduced in order to produce more of the other; and they indicate the consumption scarcity—how much people are willing to reduce their consumption of one good in order to acquire some of the other good.

33.14 Decentralized Resource Allocation

The Crusoe-Friday economy is a drastically simplified picture. In order to make a start on a larger model of the functioning of an economy, one needs to use substantially more elaborate mathematics. However, even this simple model contains some useful insights.

The most important of these is the relationship between individuals' *private* goals of utility maximization and the *social* goals of efficient use of resources. Under certain conditions, the individuals' pursuit of private goals will result in an allocation that is Pareto efficient overall. Furthermore, any Pareto efficient allocation can be supported as an outcome of a competitive market, if initial endowments—including the ownership of firms—can be suitably redistributed.

The great virtue of a competitive market is that each individual and each firm only has to worry about its own maximization problem. The only facts that need to be communicated among the firms and the consumers are the prices of the goods. Given these signals of relative scarcity, consumers and firms have enough information to make decisions that achieve an efficient allocation of resources. In this sense, the social problems involved in efficiently utilizing resources can be decentralized, and solved at the individual level.

Each individual can solve his or her own problem of what to consume. The firms face the prices of the goods the consumers consume and decide how much to produce of each of them. In making this decision, they are guided by profit signals. In this context, profits serve as exactly the right guide. To say that a production plan is profitable is to say that people are willing to pay more for some good than it costs to produce it—so it is natural to expand the production of such goods. If all firms pursue a competitive profit-maximizing policy, and all consumers choose consumption bundles to maximize their own utility, then resulting competitive equilibrium must be a Pareto efficient allocation.

Summary

1. The general equilibrium framework can be extended by allowing competitive, profit-maximizing firms to produce goods destined for exchange in the economy.

2. Under certain conditions there exists a set of prices for all of the input and output goods in the economy such that the profit-maximizing actions of firms along with the utility-maximizing behavior of individuals results in the demand for each good equaling the supply in all markets—that is, a competitive equilibrium exists.

3. Under certain conditions the resulting competitive equilibrium will be Pareto efficient: the First Welfare Theorem holds in an economy with production.

4. With the addition of convex production sets, the Second Welfare Theorem also holds in the case of production.

5. When goods are being produced as efficiently as possible, the marginal rate of transformation between two goods indicates the number of units of one good the economy must give up to obtain additional units of the other good.

6. Pareto efficiency requires that each individual's marginal rate of substitution be equal to the marginal rate of transformation.

7. The virtue of competitive markets is that they provide a way to achieve an efficient allocation of resources by decentralizing production and consumption decisions.

REVIEW QUESTIONS

1. The competitive price of coconuts is \$6 per pound and the price of fish is \$3 per pound. If society were to give up 1 pound of coconuts, how many more pounds of fish could be produced?

2. What would happen if the firm depicted in Figure 33.2 decided to pay a higher wage?

3. In what sense is a competitive equilibrium a good or bad thing for a given economy?

4. If Robinson's marginal rate of substitution between coconuts and fish is -2 and the marginal rate of transformation between the two goods is -1, what should he do if he wants to increase his utility?

5. Suppose that Robinson and Friday both want 60 pounds of fish and 60 pounds of coconuts per day. Using the production rates given in the chapter, how many hours must Robinson and Friday work per day if they don't help each other? Suppose they decide to work together in the most efficient manner possible. Now how many hours each day do they have to work? What is the economic explanation for the reduction in hours?

APPENDIX

Let us derive the calculus conditions for Pareto efficiency in an economy with production. We let X^1 and X^2 represent the total amount of good 1 and good 2 produced and consumed, as in the body of this chapter:

$$X^1 = x_A^1 + x_B^1$$
$$X^2 = x_A^2 + x_B^2.$$

The first thing we need is a convenient way to describe the production possibilities frontier—all the combinations of X^1 and X^2 that are technologically feasible. The most useful way to do this for our purposes is by use of the **transformation function**. This is a function of the aggregate amounts of the two goods $T(X^1, X^2)$, such that the combination (X^1, X^2) is on the production possibilities frontier (the boundary of the production possibilities set) if and only if

$$T(X^1, X^2) = 0.$$

Once we have described the technology, we can calculate the marginal rate of transformation: the rate at which we have to sacrifice good 2 in order to produce more of good 1. Although the name evokes an image of one good being "transformed" into another, that is a somewhat misleading picture. What really happens is that other resources are moved from producing good 2 to producing good 1. Thus, by devoting fewer resources to good 2 and more to good 1, we move from one point on the production possibilities frontier to another. The marginal rate of transformation is just the slope of the production possibilities set, which we denote by dX^2/dX^1.

Consider a small change in production (dX^1, dX^2) that remains feasible. Thus we have

$$\frac{\partial T(X^1, X^2)}{\partial X^1} dX^1 + \frac{\partial T(X^1, X^2)}{\partial X^2} dX^2 = 0.$$

Solving for the marginal rate of transformation:

$$\frac{dX^2}{dX^1} = -\frac{\partial T/\partial X^1}{\partial T/\partial X^2}.$$

We'll use this formula in a moment.

A Pareto efficient allocation is one that maximizes any one person's utility, given the level of the other people's utility. In the two-person case, we can write this maximization problem as

$$\begin{aligned} \max_{x_A^1, x_A^2, x_B^1, x_B^2} \quad & u_A(x_A^1, x_A^2) \\ \text{such that } & u_B(x_B^1, x_B^2) = \overline{u} \\ & T(X^1, X^2) = 0. \end{aligned}$$

The Lagrangian for this problem is

$$\begin{aligned} L = u_A(x_A^1, x_A^2) - \lambda(u_B(x_B^1, x_B^2) - \overline{u}) \\ - \mu(T(X_1, X_2) - 0), \end{aligned}$$

and the first-order conditions are

$$\begin{aligned} \frac{\partial L}{\partial x_A^1} &= \frac{\partial u_A}{\partial x_A^1} - \mu \frac{\partial T}{\partial X^1} = 0 \\ \frac{\partial L}{\partial x_A^2} &= \frac{\partial u_A}{\partial x_A^2} - \mu \frac{\partial T}{\partial X^2} = 0 \\ \frac{\partial L}{\partial x_B^1} &= -\lambda \frac{\partial u_B}{\partial x_B^1} - \mu \frac{\partial T}{\partial X^1} = 0 \\ \frac{\partial L}{\partial x_B^2} &= -\lambda \frac{\partial u_B}{\partial x_B^2} - \mu \frac{\partial T}{\partial X^2} = 0. \end{aligned}$$

Rearranging and dividing the first equation by the second gives

$$\frac{\partial u_A / \partial x_A^1}{\partial u_A / \partial x_A^2} = \frac{\partial T / \partial X^1}{\partial T / \partial X^2}.$$

Performing the same operation on the third and fourth equations gives

$$\frac{\partial u_B / \partial x_B^1}{\partial u_B / \partial x_B^2} = \frac{\partial T / \partial X^1}{\partial T / \partial X^2}.$$

The left-hand sides of these equations are our old friends, the marginal rates of substitution. The right-hand side is the marginal rate of transformation. Thus the equations require that each person's marginal rate of substitution between the goods must equal the marginal rate of transformation: the rate at which each person is just willing to substitute one good for the other must be the same as the rate at which it is technologically feasible to transform one good into the other.

The intuition behind this result is straightforward. Suppose that the MRS for some individual was not equal to the MRT. Then the rate at which the individual would be willing to sacrifice one good to get more of the other would be different than the rate that was technologically feasible—but this means that there would be some way to increase that individual's utility while not affecting anyone else's consumption.

CHAPTER 34

WELFARE

Up until now we have focused on considerations of Pareto efficiency in evaluating economic allocations. But there are other important considerations. It must be remembered that Pareto efficiency has nothing to say about the distribution of welfare across people; giving everything to one person will typically be Pareto efficient. But the rest of us might not consider this a reasonable allocation. In this chapter we will investigate some techniques that can be used to formalize ideas related to the distribution of welfare.

Pareto efficiency is in itself a desirable goal—if there is some way to make some group of people better off without hurting other people, why not do it? But there will usually be many Pareto efficient allocations; how can society choose among them?

The major focus of this chapter will be the idea of a **welfare function**, which provides a way to "add together" different consumers' utilities. More generally, a welfare function provides a way to rank different distributions of utility among consumers. Before we investigate the implications of this concept, it is worthwhile considering just how one might go about "adding together" the individual consumers' preferences to construct some kind of "social preferences."

34.1 Aggregation of Preferences

Let us return to our early discussion of consumer preferences. As usual, we will assume that these preferences are transitive. Originally, we thought of a consumer's preferences as being defined over his own bundle of goods, but now we want to expand on that concept and think of each consumer as having preferences over the entire allocation of goods among the consumers. Of course, this includes the possibility that the consumer might not care about what other people have, just as we had originally assumed.

Let us use the symbol $\mathbf{x}$ to denote a particular allocation—a description of what every individual gets of every good. Then given two allocations, $\mathbf{x}$ and $\mathbf{y}$, each individual i can say whether or not he or she prefers $\mathbf{x}$ to $\mathbf{y}$.

Given the preferences of all the agents, we would like to have a way to "aggregate" them into one **social preference**. That is, if we know how all the individuals rank various allocations, we would like to be able to use this information to develop a social ranking of the various allocations. This is the problem of social decision making at its most general level. Let's consider a few examples.

One way to aggregate individual preferences is to use some kind of voting. We could agree that $\mathbf{x}$ is "socially preferred" to $\mathbf{y}$ if a majority of the individuals prefer $\mathbf{x}$ to $\mathbf{y}$. However, there is a problem with this method—it may not generate a transitive social preference ordering. Consider, for example, the case illustrated in Table 34.1.

Preferences that lead to intransitive voting. Table 34.1

Person A	Person B	Person C
$\mathbf{x}$	$\mathbf{y}$	$\mathbf{z}$
$\mathbf{y}$	$\mathbf{z}$	$\mathbf{x}$
$\mathbf{z}$	$\mathbf{x}$	$\mathbf{y}$

Here we have listed the rankings for three alternatives, $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, by three people. Note that a majority of the people prefer $\mathbf{x}$ to $\mathbf{y}$, a majority prefer $\mathbf{y}$ to $\mathbf{z}$, and a majority prefer $\mathbf{z}$ to $\mathbf{x}$. Thus aggregating individual preferences by majority vote won't work since, in general, the social preferences resulting from majority voting aren't well-behaved preferences, since they are not transitive. Since the preferences aren't transitive, there will be no "best" alternative from the set of alternatives $(\mathbf{x}, \mathbf{y}, \mathbf{z})$. Which outcome society chooses will depend on the order in which the vote is taken.

To see this suppose that the three people depicted in Table 34.1 decide to vote first on **x** versus **y**, and then vote on the winner of this contest versus **z**. Since a majority prefer **x** to **y**, the second contest will be between **x** and **z**, which means that **z** will be the outcome.

But what if they decide to vote on **z** versus **x** and then pit the winner of this vote against **y**? Now **z** wins the first vote, but **y** beats **z** in the second vote. Which outcome is the overall winner depends crucially on the order in which the alternatives are presented to the voters.

Another kind of voting mechanism that we might consider is rank-order voting. Here each person ranks the goods according to his preferences and assigns a number that indicates its rank in his ordering: for example, a 1 for the best alternative, 2 for the second best, and so on. Then we sum up the scores of each alternative across the people to determine an aggregate score for each alternative and say that one outcome is socially preferred to another if it has a lower score.

In Table 34.2 we have illustrated a possible preference ordering for three allocations **x**, **y**, and **z** by two people. Suppose first that only alternatives **x** and **y** were available. Then in this example **x** would be given a rank of 1 by person A and 2 by person B. The alternative **y** would be given just the reverse ranking. Thus the outcome of the voting would be a tie with each alternative having an aggregate rank of 3.

Table 34.2 **The choice between x and y depends on z.**

Person A	Person B
x	**y**
y	**z**
z	**x**

But now suppose that **z** is introduced to the ballot. Person A would give **x** a score of 1, **y** a score of 2, and **z** a rank of 3. Person B would give **y** a score of 1, **z** a score of 2, and **x** a score of 3. This means that **x** would now have an aggregate rank of 4, and **y** would have an aggregate rank of 3. In this case **y** would be preferred to **x** by rank-order voting.

The problem with both majority voting and rank-order voting is that their outcomes can be manipulated by astute agents. Majority voting can be manipulated by changing the order on which things are voted so as to yield the desired outcome. Rank-order voting can be manipulated by introducing new alternatives that change the final ranks of the relevant alternatives.

The question naturally arises as to whether there are social decision mechanisms—ways of aggregating preferences—that are immune to this kind of manipulation? Are there ways to "add up" preferences that don't have the undesirable properties described above?

Let's list some things that we would want our social decision mechanism to do:

1. Given any set of complete, reflexive, and transitive individual preferences, the social decision mechanism should result in social preferences that satisfy the same properties.

2. If everybody prefers alternative **x** to alternative **y**, then the social preferences should rank **x** ahead of **y**.

3. The preferences between **x** and **y** should depend only on how people rank **x** versus **y**, and not on how they rank other alternatives.

All three of these requirements seem eminently plausible. Yet it can be quite difficult to find a mechanism that satisfies all of them. In fact, Kenneth Arrow has proved the following remarkable result:[1]

Arrow's Impossibility Theorem. *If a social decision mechanism satisfies properties 1, 2, and 3, then it must be a dictatorship: all social rankings are the rankings of one individual.*

Arrow's Impossibility Theorem is quite surprising. It shows that three very plausible and desirable features of a social decision mechanism are inconsistent with democracy: there is no "perfect" way to make social decisions. There is no perfect way to "aggregate" individual preferences to make one social preference. If we want to find a way to aggregate individual preferences to form social preferences, we will have to give up one of the properties of a social decision mechanism described in Arrow's theorem.

34.2 Social Welfare Functions

If we were to drop any of the desired features of a social welfare function described above, it would probably be property 3—that the social preference between two alternatives only depends on the ranking of those two alternatives. If we do that, certain kinds of rank-order voting become possibilities.

[1] See Kenneth Arrow, *Social Choice and Individual Values* (New York: Wiley, 1963). Arrow, a professor at Stanford University, was awarded the Nobel Prize in economics for his work in this area.

Given the preferences of each individual i over the allocations, we can construct utility functions, $u_i(\mathbf{x})$, that summarize the individuals' value judgments: person i prefers $\mathbf{x}$ to $\mathbf{y}$ if and only if $u_i(\mathbf{x}) > u_i(\mathbf{y})$. Of course, these are just like all utility functions—they can be scaled in any way that preserves the underlying preference ordering. There is no *unique* utility representation.

But let us pick some utility representation and stick with it. Then one way of getting social preferences from individuals' preferences is to add up the individual utilities and use the resulting number as a kind of social utility. That is, we will say that allocation $\mathbf{x}$ is socially preferred to allocation $\mathbf{y}$ if

$$\sum_{i=1}^{n} u_i(\mathbf{x}) > \sum_{i=1}^{n} u_i(\mathbf{y}),$$

where n is the number of individuals in the society.

This works—but of course it is totally arbitrary, since our choice of utility representation is totally arbitrary. The choice of using the sum is also arbitrary. Why not use a weighted sum of utilities? Why not use the product of utilities, or the sum of the squares of utilities?

One reasonable restriction that we might place on the "aggregating function" is that it be increasing in each individual's utility. That way we are assured that if everybody prefers $\mathbf{x}$ to $\mathbf{y}$, then the social preferences will prefer $\mathbf{x}$ to $\mathbf{y}$.

There is a name for this kind of aggregating function; it is called a **social welfare function**. A social welfare function is just some function of the individual utility functions: $W(u_1(\mathbf{x}), \ldots, u_n(\mathbf{x}))$. It gives a way to rank different allocations that depends only on the individual preferences, and it is an increasing function of each individual's utility.

Let's look at some examples. One special case mentioned above is the *sum* of the individual utility functions

$$W(u_1, \ldots, u_n) = \sum_{i=1}^{n} u_i.$$

This is sometimes referred to as a **classical utilitarian** or **Benthamite** welfare function.[2] A slight generalization of this form is the **weighted-sum-of-utilities welfare function**:

$$W(u_1, \ldots, u_n) = \sum_{i=1}^{n} a_i u_i.$$

[2] Jeremy Bentham (1748–1832) was the founder of the utilitarian school of moral philosophy, a school that considers the highest good to be the greatest happiness for the greatest number.

Here the weights, $a_1, \ldots, a_n$, are supposed to be numbers indicating how important each agent's utility is to the overall social welfare. It is natural to take each a_i as being positive.

Another interesting welfare function is the **minimax** or **Rawlsian** social welfare function:

$$W(u_1, \ldots, u_n) = \min\{u_1, \ldots, u_n\}.$$

This welfare function says that the social welfare of an allocation depends only on the welfare of the worst off agent—the person with the minimal utility.[3]

Each of these is a possible way to compare individual utility functions. Each of them represents different ethical judgments about the comparison between different agents' welfares. About the only restriction that we will place on the structure of the welfare function at this point is that it be increasing in each consumer's utility.

34.3 Welfare Maximization

Once we have a welfare function we can examine the problem of welfare maximization. Let us use the notation x_i^j to indicate how much individual i has of good j, and suppose that there are n consumers and k goods. Then the allocation $\mathbf{x}$ consists of the list of how much each of the agents has of each of the goods.

If we have a total amount $X^1, \ldots, X^k$ of goods $1, \ldots, k$ to distribute among the consumers, we can pose the welfare maximization problem:

$$\begin{aligned} \max\ & W(u_1(\mathbf{x}), \ldots, u_n(\mathbf{x})) \\ \text{such that } & \sum_{i=1}^n x_i^1 = X^1 \\ & \vdots \\ & \sum_{i=1}^n x_i^k = X^k. \end{aligned}$$

Thus we are trying to find the feasible allocation that maximizes social welfare. What properties does such an allocation have?

The first thing that we should note is that a maximal welfare allocation must be a Pareto efficient allocation. The proof is easy: suppose that

[3] John Rawls (1931–2002) was a philosopher at Harvard who has argued for this principle of justice.

it were not. Then there would be some other feasible allocation that gave everyone at least as large a utility, and someone strictly greater utility. But the welfare function is an increasing function of each agent's utility. Thus this new allocation would have to have higher welfare, which contradicts the assumption that we originally had a welfare maximum.

We can illustrate this situation in Figure 34.1, where the set U indicates the set of possible utilities in the case of two individuals. This set is known as the **utility possibilities set**. The boundary of this set—the **utility possibilities frontier**—is the set of utility levels associated with Pareto efficient allocations. If an allocation is on the boundary of the utility possibilities set, then there are no other feasible allocations that yield higher utilities for both agents.

Figure 34.1 **Welfare maximization**. An allocation that maximizes a welfare function must be Pareto efficient.

The "indifference curves" in this diagram are called **isowelfare curves** since they depict those distributions of utility that have constant welfare. As usual, the optimal point is characterized by a tangency condition. But for our purposes, the notable thing about this maximal welfare point is that it is Pareto efficient—it must occur on the boundary of the utility possibilities set.

The next observation we can make from this diagram is that *any* Pareto efficient allocation must be a welfare maximum for some welfare function.

Figure 34.2

Maximization of the weighted-sum-of-utilities welfare function. If the utility possibility set is convex, then every Pareto efficient point is a maximum for a weighted-sum-of-utilities welfare function.

An example is given in Figure 34.2.

In Figure 34.2 we have picked a Pareto efficient allocation and found a set of isowelfare curves for which it yields maximal welfare. Actually, we can say a bit more than this. If the set of possible utility distributions is a convex set, as illustrated, then every point on its frontier is a welfare maximum for a weighted-sum-of-utilities welfare function, as illustrated in Figure 34.2. The welfare function thus provides a way to single out Pareto efficient allocations: every welfare maximum is a Pareto efficient allocation, and every Pareto efficient allocation is a welfare maximum.

34.4 Individualistic Social Welfare Functions

Up until now we have been thinking of individual preferences as being defined over entire allocations rather than over each individual's bundle of goods. But, as we remarked earlier, individuals might only care about their own bundles. In this case, we can use x_i to denote individual $i's$ consumption bundle, and let $u_i(x_i)$ be individual $i's$ utility level using some fixed representation of utility. Then a social welfare function will

have the form

$$W = W(u_1(x_1), \ldots, u_n(x_n)).$$

The welfare function is directly a function of the individuals' utility levels, but it is indirectly a function of the individual agents' consumption bundles. This special form of welfare function is known as an **individualistic welfare function** or a **Bergson-Samuelson welfare function**.[4]

If each agent's utility depends only on his or her own consumption, then there are no consumption externalities. Thus the standard results of Chapter 32 apply and we have an intimate relationship between Pareto efficient allocations and market equilibria: all competitive equilibria are Pareto efficient, and, under appropriate convexity assumptions, all Pareto efficient allocations are competitive equilibria.

Now we can carry this categorization one step further. Given the relationship between Pareto efficiency and welfare maxima described above, we can conclude that all welfare maxima are competitive equilibria and that all competitive equilibria are welfare maxima for some welfare function.

34.5 Fair Allocations

The welfare function approach is a very general way to describe social welfare. But because it is so general it can be used to summarize the properties of many kinds of moral judgments. On the other hand, it isn't much use in deciding what kinds of ethical judgments might be reasonable ones.

Another approach is to start with some specific moral judgments and then examine their implications for economic distribution. This is the approach taken in the study of **fair allocations**. We start with a definition of what might be considered a fair way to divide a bundle of goods, and then use our understanding of economic analysis to investigate its implications.

Suppose that you were given some goods to divide fairly among n equally deserving people. How would you do it? It is probably safe to say that in this problem most people would divide the goods equally among the n agents. Given that they are by hypothesis equally deserving, what else could you do?

What is appealing about this idea of equal division? One appealing feature is that it is *symmetric*. Each agent has the same bundle of goods; no agent prefers any other agent's bundle of goods to his or her own, since they all have exactly the same thing.

Unfortunately, an equal division will not necessarily be Pareto efficient. If agents have different tastes they will generally desire to trade away from

[4] Abram Bergson (1914–2002) and Paul Samuelson (1915–2009) were economists who investigated properties of this kind of welfare function in the early 1940s. Samuelson was awarded a Nobel Prize in economics for his many contributions.

equal division. Let us suppose that this trade takes place and that it moves us to a Pareto efficient allocation.

The question arises: is this Pareto efficient allocation still fair in any sense? Does trade from equal division inherit any of the symmetry of the starting point?

The answer is: not necessarily. Consider the following example. We have three people, A, B, and C. A and B have the same tastes, and C has different tastes. We start from an equal division and suppose that A and C get together and trade. Then they will typically both be made better off. Now B, who didn't have the opportunity to trade with C, will **envy** A—that is, he would prefer A's bundle to his own. Even though A and B started with the same allocation, A was luckier in her trading, and this destroyed the symmetry of the original allocation.

This means that arbitrary trading from an equal division will not necessarily preserve the symmetry of the starting point of equal division. We might well ask if there is any allocation that preserves this symmetry? Is there any way to get an allocation that is both Pareto efficient and equitable at the same time?

34.6 Envy and Equity

Let us now try to formalize some of these ideas. What do we mean by "symmetric" or "equitable" anyway? One possible set of definitions is as follows.

We say an allocation is **equitable** if no agent prefers any other agent's bundle of goods to his or her own. If some agent i does prefer some other agent $j's$ bundle of goods, we say that i **envies** j. Finally, if an allocation is both equitable and Pareto efficient, we will say that it is a **fair** allocation.

These are ways of formalizing the idea of symmetry alluded to above. An equal division allocation has the property that no agent envies any other agent—but there are many other allocations that have this same property.

Consider Figure 34.3. To determine whether any allocation is equitable or not, just look at the allocation that results if the two agents swap bundles. If this swapped allocation lies "below" each agent's indifference curve through the original allocation, then the original allocation is an equitable allocation. (Here "below" means below from the point of view of each agent; from our point of view the swapped allocation must lie between the two indifference curves.)

Note also that the allocation in Figure 34.3 is also Pareto efficient. Thus it is not only equitable, in the sense that we defined the term, but it is also efficient. By our definition, it is a fair allocation. Is this kind of allocation a fluke, or will fair allocations typically exist?

It turns out that fair allocations *will* generally exist, and there is an easy way to see that this is so. We start as we did in the last section, where

Figure 34.3 **Fair allocations**. A fair allocation in an Edgeworth box. Each person prefers the fair allocation to the swapped allocation.

we had an equal division allocation and considered trading to a Pareto efficient allocation. Instead of using just any old way to trade, let us use the special mechanism of the competitive market. This will move us to a new allocation where each agent is choosing the best bundle of goods he or she can afford at the equilibrium prices (p_1, p_2), and we know from Chapter 32 that such an allocation must be Pareto efficient.

But is it still equitable? Well, suppose not. Suppose that one of the consumers, say consumer A, envies consumer B. This means that A prefers what B has to her own bundle. In symbols:

$$(x_A^1, x_A^2) \prec_A (x_B^1, x_B^2).$$

But, if A prefers B's bundle to her own, and if her own bundle is the best bundle she can afford at the prices (p_1, p_2), this means that B's bundle must cost more than A can afford. In symbols:

$$p_1\omega_A^1 + p_2\omega_A^2 < p_1x_B^1 + p_2x_B^2.$$

But this is a contradiction! For by hypothesis, A and B started with exactly the same bundle, since they started from an equal division. If A can't afford B's bundle, then B can't afford it either.

Thus we can conclude that it is impossible for A to envy B in these circumstances. A competitive equilibrium from equal division must be a

fair allocation. Thus the market mechanism will preserve certain kinds of equity: if the original allocation is equally divided, the final allocation must be fair.

Summary

1. Arrow's Impossibility Theorem shows that there is no ideal way to aggregate individual preferences into social preferences.

2. Nevertheless, economists often use welfare functions of one sort or another to represent distributional judgments about allocations.

3. As long as the welfare function is increasing in each individual's utility, a welfare maximum will be Pareto efficient. Furthermore, every Pareto efficient allocation can be thought of as maximizing some welfare function.

4. The idea of fair allocations provides an alternative way to make distributional judgments. This idea emphasizes the idea of symmetric treatment.

5. Even when the initial allocation is symmetric, arbitrary methods of trade will not necessarily produce a fair allocation. However, it turns out that the market mechanism will provide a fair allocation.

REVIEW QUESTIONS

1. Suppose that we say that an allocation $\mathbf{x}$ is socially preferred to an allocation $\mathbf{y}$ only if *everyone* prefers $\mathbf{x}$ to $\mathbf{y}$. (This is sometimes called the Pareto ordering, since it is closely related to the idea of Pareto efficiency.) What shortcoming does this have as a rule for making social decisions?

2. A Rawlsian welfare function counts only the welfare of the worst off agent. The opposite of the Rawlsian welfare function might be called the "Nietzschean" welfare function—a welfare function that says the value of an allocation depends only on the welfare of the *best off* agent. What mathematical form would the Nietzschean welfare function take?

3. Suppose that the utility possibilities set is a convex set and that consumers care only about their own consumption. What kind of allocations represent welfare maxima of the Nietzschean welfare function?

4. Suppose that an allocation is Pareto efficient, and that each individual only cares about his own consumption. Prove that there must be some individual that envies no one, in the sense described in the text. (This puzzle requires some thought, but it is worth it.)

5. The ability to set the voting agenda can often be a powerful asset. Assuming that social preferences are decided by pair-wise majority voting and that the preferences given in Table 34.1 hold, demonstrate this fact by producing a voting agenda that results in allocation **y** winning. Find an agenda that has **z** as the winner. What property of the social preferences is responsible for this agenda-setting power?

APPENDIX

Here we consider the problem of welfare maximization, using an individualistic welfare function. Using the transformation function described in Chapter 33 to describe the production possibilities frontier, we write the welfare maximization problem as

$$\max_{x_A^1, x_A^2, x_B^1, x_B^2} W(u_A(x_A^1, x_A^2), u_B(x_B^1, x_B^2))$$
$$\text{such that } T(X^1, X^2) = 0,$$

where we use X^1 and X^2 to denote the total amount of good 1 and good 2 produced and consumed.

The Lagrangian for this problem is

$$L = W(u_A(x_A^1, x_A^2), u_B(x_B^1, x_B^2)) - \lambda(T(X^1, X^2) - 0).$$

Differentiating with respect to each of the choice variables gives us the first-order conditions

$$\frac{\partial L}{\partial x_A^1} = \frac{\partial W}{\partial u_A}\frac{\partial u_A(x_A^1, x_A^2)}{\partial x_A^1} - \lambda\frac{\partial T(X^1, X^2)}{\partial X^1} = 0$$
$$\frac{\partial L}{\partial x_A^2} = \frac{\partial W}{\partial u_A}\frac{\partial u_A(x_A^1, x_A^2)}{\partial x_A^2} - \lambda\frac{\partial T(X^1, X^2)}{\partial X^2} = 0$$
$$\frac{\partial L}{\partial x_B^1} = \frac{\partial W}{\partial u_B}\frac{\partial u_B(x_B^1, x_B^2)}{\partial x_B^1} - \lambda\frac{\partial T(X^1, X^2)}{\partial X^1} = 0$$
$$\frac{\partial L}{\partial x_B^2} = \frac{\partial W}{\partial u_B}\frac{\partial u_B(x_B^1, x_B^2)}{\partial x_B^2} - \lambda\frac{\partial T(X^1, X^2)}{\partial X^2} = 0.$$

Rearranging and dividing the first equation by the second, and the third by the fourth, we have

$$\frac{\partial u_A/\partial x_A^1}{\partial u_A/\partial x_A^2} = \frac{\partial T/\partial X^1}{\partial T/\partial X^2}$$
$$\frac{\partial u_B/\partial x_B^1}{\partial u_B/\partial x_B^2} = \frac{\partial T/\partial X^1}{\partial T/\partial X^2}.$$

Note that these are exactly the same equations that we encountered in the Appendix to Chapter 33. Thus the welfare maximization problem gives us the same first-order conditions as the Pareto efficiency problem.

This is obviously no accident. According to the discussion in the text, the allocation resulting from the maximization of a Bergson-Samuelson welfare function is Pareto efficient, and every Pareto efficient allocation maximizes some welfare function. Thus welfare maxima and Pareto efficient allocations have to satisfy the same first-order conditions.

CHAPTER 35

EXTERNALITIES

We say that an economic situation involves a **consumption externality** if one consumer cares directly about another agent's production or consumption. For example, I have definite preferences about my neighbor playing loud music at 3 in the morning, or the person next to me in a restaurant smoking a cheap cigar, or the amount of pollution produced by local automobiles. These are all examples of *negative* consumption externalities. On the other hand, I may get pleasure from observing my neighbor's flower garden—this is an example of a *positive* consumption externality.

Similarly, a **production externality** arises when the production possibilities of one firm are influenced by the choices of another firm or consumer. A classic example is that of an apple orchard located next to a beekeeper, where there are mutual positive production externalities—each firm's production positively affects the production possibilities of the other firm. Similarly, a fishery cares about the amount of pollutants dumped into its fishing area, since this will negatively influence its catch.

The crucial feature of externalities is that there are goods people care about that are not sold on markets. There is no market for loud music at 3 in the morning, or drifting smoke from cheap cigars, or a neighbor who

keeps a beautiful flower garden. It is this lack of markets for externalities that causes problems.

Up until now we have implicitly assumed that each agent could make consumption or production decisions without worrying about what other agents were doing. All interactions between consumers and producers took place via the market, so that all the economic agents needed to know were the market prices and their own consumption or production possibilities. In this chapter we will relax this assumption and examine the economic consequences of externalities.

In earlier chapters we saw that the market mechanism was capable of achieving Pareto efficient allocations when externalities were *not* present. If externalities are present, the market will not necessarily result in a Pareto efficient provision of resources. However, there are other social institutions such as the legal system, or government intervention, that can "mimic" the market mechanism to some degree and thereby achieve Pareto efficiency. In this chapter we'll see how these institutions work.

35.1 Smokers and Nonsmokers

It is convenient to start with an example to illustrate some of the main considerations. We'll imagine two roommates, A and B, who have preferences over "money" and "smoke." We suppose that both consumers like money, but that A likes to smoke and B likes clean air.

We can depict the consumption possibilities for the two consumers in an Edgeworth box. The length of the horizontal axis will represent the total amount of money the two agents have, and the height of the vertical axis will represent the total amount of smoke that can be generated. The preferences of agent A are increasing in both money and smoke, while agent B's preferences are increasing in money and clean air—the absence of smoke. We'll measure smoke on a scale from 0 to 1, where 0 is no smoke at all, and 1 is the proverbial smoke-filled room.

This setup gives us a diagram like that depicted in Figure 35.1. Note that the picture looks very much like the standard Edgeworth box, but the interpretation is quite different. The amount of smoke is a good for A and a bad for B, so that B is moved to a more preferred position as A consumes less smoke. Be sure to note the difference in the way things are measured on the horizontal and vertical axes. We measure A's money horizontally from the lower left-hand corner of the box, and B's money horizontally from the upper right-hand corner. But the total amount of smoke is measured vertically from the lower left-hand corner. The difference occurs because money can be divided between the two consumers, so there will always be two amounts of money to measure, but there is only one amount of smoke that they must both consume.

In the ordinary Edgeworth box diagram B is made better off when A reduces his consumption of good 2—but that is because B then gets to consume more of good 2. In the Edgeworth box in Figure 35.1 B is also better off when A reduces his consumption of good 2 (smoke), but for a very different reason. In this example, B is better off when A reduces his consumption of smoke since both agents must consume the same amount of smoke and smoke is a bad for agent B.

We've now illustrated the consumption possibilities of the two roommates and their preferences. What about their endowments? Let's assume that they both have the same amount of money, say $100 apiece, so that their endowments will lie somewhere on the vertical line EE' in Figure 35.1. In order to determine exactly where on this line the endowments lie, we must determine the initial "endowment" of smoke/clean air.

Preferences for money and smoke. Smoke is a good for person A but a bad for person B. Which equilibrium we end up at depends on which endowment we start at.

Figure 35.1

The answer to this question depends on the legal rights of smokers and nonsmokers. It may be that A has a right to smoke as much as he wants, and B just has to put up with it. Or, it could be that B has a right to

clean air. Or the legal right to smoke and clean air could be somewhere between these two extremes.

The initial endowment of smoke depends on the legal system. This is not so different from the initial endowment of ordinary sorts of goods. To say that A has an initial endowment of \$100 means that A can decide to consume the \$100 himself, or he can give it away or trade it to any other individual. There is a legal definition of property involved in saying that a person "owns" or "has a right to" \$100. Similarly if a person has a property right to clean air, it means that he can consume clean air if he wants to, or he can give it away or sell that right to someone else. In this way, having a property right to clean air is no different from having a property right to \$100.

Let's start by considering a legal situation where person B has a legal right to clean air. Then the initial endowment in Figure 35.1 is labeled E; it is where A has $(100, 0)$ and B has $(100, 0)$. This means that both A and B have \$100, and that the initial endowment—what there would be in the absence of trade—is clean air.

Just as before, in the case with no externalities, there is no reason why the initial endowment is Pareto efficient. One of the aspects of having a property right to clean air is having the right to trade some of it away for other desirable goods—in this case, for money. It can easily happen that B would prefer to trade some of his right to clean air for some more money. The point labeled X in Figure 35.1 is an example of such a case.

As before, a Pareto efficient allocation is one where neither consumer can be made better off without the other being made worse off. Such an allocation will be characterized by the usual tangency condition that the marginal rates of substitution between smoke and money should be the same between the two agents, as illustrated in Figure 35.1. It is easy to imagine A and B trading to such a Pareto efficient point. In effect, B has the right to clean air, but he can allow himself to be "bribed" to consume some of A's smoke.

Of course, other assignments of property rights are possible. We could imagine a legal system where A had a right to smoke as much as he wanted, and B would have to bribe A to reduce his consumption of smoke. This would correspond to the endowment labeled E' in Figure 35.1. Just as before, this would typically not be Pareto efficient, so we could imagine the agents trading to a mutually preferred point such as the one labeled X'.

Both X and X' are Pareto efficient allocations; they just come from different initial endowments. Certainly the smoker, A, is better off at X' than at X, and the nonsmoker, B, is better off at X than at X'. The two points have different distributional consequences, but on grounds of efficiency they are equally satisfactory.

In fact, there is no reason to limit ourselves to just these two efficient points. As usual there will be a whole contract curve of Pareto efficient allocations of smoke and money. If agents are free to trade both of these

goods, we know that they will end up somewhere on this contract curve. The exact position will depend on their property rights involving smoke and money and on the precise mechanism that they use to trade.

One mechanism that they could use to trade is the price mechanism. Just as before we could imagine an auctioneer calling out prices and asking how much each agent would be willing to buy at those prices. If the initial endowment point gave A the property rights to smoke, he could consider selling some of his smoking rights to B in exchange for B's money. Similarly, if the property rights for clean air were given to B, he could sell some of his clean air to A.

When the auctioneer manages to find a set of prices where supply equals demand everything is fine: we have a nice Pareto efficient outcome. If there is a market for smoke, a competitive equilibrium will be Pareto efficient. Furthermore, the competitive prices will measure the marginal rate of substitution between the two goods, just as in the standard case.

This is just like the usual Edgeworth box analysis, but described in a slightly different framework. As long as we have well-defined property rights in the good involving the externality—no matter who holds the property rights—the agents can trade from their initial endowment to a Pareto efficient allocation. If we want to set up a market in the externality to encourage trade, that will work as well.

The only problem arises if the property rights are *not* well defined. If A believes that he has the right to smoke and B believes that he has the right to clean air, we have difficulties. *The practical problems with externalities generally arise because of poorly defined property rights.*

My neighbor may believe that he has the right to play his trumpet at 3 in the morning, and I may believe that I have the right to silence. A firm may believe that it has the right to dump pollutants into the atmosphere that I breathe, while I may believe that it doesn't. Cases where property rights are poorly defined can lead to an inefficient production of externalities—which means that there would be ways to make both parties involved better off by changing the production of externalities. If property rights are well defined, and mechanisms are in place to allow for negotiation between people, then people can trade their rights to produce externalities in the same way that they trade rights to produce and consume ordinary goods.

35.2 Quasilinear Preferences and the Coase Theorem

We argued above that as long as property rights were well defined, trade between agents would result in an efficient allocation of the externality. In general, the amount of the externality that will be generated in the efficient solution will depend on the assignment of property rights. In the case of the two roommates, the amount of smoke generated will depend on whether the smoker has the property rights or the nonsmoker has them.

But there is a special case where the outcome of the externality is independent of the assignment of property rights. If the agents' preferences are **quasilinear**, then every efficient solution must have the same amount of the externality.

This case is illustrated in Figure 35.2 for the Edgeworth box case of the smoker versus the nonsmoker. Since the indifference curves are all horizontal translates of each other, the locus of mutual tangencies—the set of Pareto efficient allocations—will be a horizontal line. This means that the amount of smoke is the same in every Pareto efficient allocation; only the dollar amounts held by the agents differ across the efficient allocations.

Figure 35.2

Quasilinear preferences and the Coase theorem. If each consumer's preferences are quasilinear, so that they are all horizontal translates of each other, the set of Pareto efficient allocations will be a horizontal line. Thus there will be a unique amount of the externality, in this case smoke, at each Pareto efficient allocation.

The result that under certain conditions the efficient amount of the good involved in the externality is independent of the distribution of property rights is sometimes known as the **Coase Theorem**. However, it should be emphasized just how special these conditions are. The quasilinear preference assumption implies that the demands for the good causing the exter-

nality doesn't depend on the distribution of income. Therefore a reallocation of endowments doesn't affect the efficient amount of the externalities. This is sometimes expressed by saying that the Coase theorem is valid if there are no "income effects."[1]

In this case, the Pareto efficient allocations will involve a unique amount of the externality being generated. The different Pareto efficient allocations will involve different amounts of money being held by the consumers; but the amount of the externality—the amount of smoke—will be independent of the distribution of wealth.

35.3 Production Externalities

Let us now consider a situation involving production externalities. Firm S produces some amount of steel, s, and also produces a certain amount of pollution, x, which it dumps into a river. Firm F, a fishery, is located downstream and is adversely affected by S's pollution.

Suppose that firm S's cost function is given by $c_s(s, x)$, where s is the amount of steel produced and x is the amount of pollution produced. Firm F's cost function is given by $c_f(f, x)$, where f indicates the production of fish and x is the amount of pollution. Note that F's costs of producing a given amount of fish depend on the amount of pollution produced by the steel firm. We will suppose that pollution increases the cost of providing fish $\Delta c_f/\Delta x > 0$, and that pollution *decreases* the cost of steel production, $\Delta c_s/\Delta x \leq 0$. This last assumption says that increasing the amount of pollution will decrease the cost of producing steel—that reducing pollution will increase the cost of steel production, at least over some range.

The steel firm's profit-maximization problem is

$$\max_{s,x} \; p_s s - c_s(s, x)$$

and the fishery's profit-maximization problem is

$$\max_{f} \; p_f f - c_f(f, x).$$

Note that the steel mill gets to choose the amount of pollution that it generates, but the fishery must take the level of pollution as outside of its control.

[1] Ronald Coase is an emeritus professor at the University of Chicago Law School. His famous paper, "The Problem of Social Costs," *The Journal of Law & Economics*, 3 (October 1960), has been given a variety of interpretations. Some authors suggest that Coase only asserted that costless bargaining over externalities achieves a Pareto efficient outcome, not that the outcome will be independent of the assignment of property rights. Coase received the 1991 Nobel Prize in Economics for this work.

The conditions characterizing profit maximization will be

$$p_s = \frac{\Delta c_s(s^*, x^*)}{\Delta s}$$
$$0 = \frac{\Delta c_s(s^*, x^*)}{\Delta x}$$

for the steel firm and

$$p_f = \frac{\Delta c_f(f^*, x^*)}{\Delta f}$$

for the fishery. These conditions say that at the profit-maximizing point, the price of each good—steel and pollution—should equal its marginal cost. In the case of the steel firm, one of its products is pollution, which, by assumption, has a zero price. So the condition determining the profit-maximizing supply of pollution says to produce pollution until the cost of an extra unit is zero.

It is not hard to see the externality here: the fishery cares about the production of pollution but has no control over it. The steel firm looks only at the cost of producing steel when it makes its profit-maximizing calculation; it doesn't consider the cost it imposes on the fishery. The increase in the cost of fishing associated with an increase in pollution is part of the **social cost** of steel production, and it is being ignored by the steel firm. In general, we expect that the steel firm will produce too much pollution from a social point of view since it ignores the impact of that pollution on the fishery.

What does a Pareto efficient production plan for steel and fish look like? There is an easy way to see what it should be. Suppose that the fishery and the steel firm merged and formed one firm that produced both fish and steel (and possibly pollution). Then there is no externality! For a production externality only arises when one firm's actions affect another firm's production possibilities. If there is only one firm, then it will take the interactions between its different "divisions" into account when it chooses the profit-maximizing production plan. We say that the externality has been **internalized** by this reassignment of property rights. Before the merger, each firm had the right to produce whatever amount of steel or fish or pollution that it wanted, regardless of what the other firm did. After the merger, the combined firm has the right to control the production of both the steel mill and the fishery.

The merged firm's profit-maximization problem is

$$\max_{s,f,x} \; p_s s + p_f f - c_s(s, x) - c_f(f, x),$$

which yields optimality conditions of

$$
\begin{aligned}
p_s &= \frac{\Delta c_s(\hat{s}, \hat{x})}{\Delta s} \\
p_f &= \frac{\Delta c_f(\hat{f}, \hat{x})}{\Delta f} \\
0 &= \frac{\Delta c_s(\hat{s}, \hat{x})}{\Delta x} + \frac{\Delta c_f(\hat{f}, \hat{x})}{\Delta x}.
\end{aligned}
$$

The crucial term is the last one. This shows that the merged firm will take into account the effect of pollution on the marginal costs of both the steel firm and the fishery. When the steel division decides how much pollution to produce, it considers the effect of this action on the profits of the fish division; that is, it takes the social cost of its production plan into account.

What does this imply about the amount of pollution produced? When the steel firm acted independently, the amount of pollution was determined by the condition

$$\frac{\Delta c_s(s^*, x^*)}{\Delta x} = 0. \tag{35.1}$$

That is, the steel mill produced pollution until the marginal cost was zero:

$$MC_S(s^*, x^*) = 0.$$

In the merged firm, the amount of pollution is determined by the condition

$$\frac{\Delta c_s(\hat{s}, \hat{x})}{\Delta x} + \frac{\Delta c_f(\hat{f}, \hat{x})}{\Delta x} = 0. \tag{35.2}$$

That is, the merged firm produces pollution until the *sum* of the marginal cost to the steel mill and the marginal cost to the fishery is zero. This condition can also be written as

$$-\frac{\Delta c_s(\hat{s}, \hat{x})}{\Delta x} = \frac{\Delta c_f(\hat{f}, \hat{x})}{\Delta x} > 0 \tag{35.3}$$

or

$$-MC_S(\hat{s}, \hat{x}) = MC_F(\hat{f}, \hat{x}).$$

In this latter expression $MC_F(\hat{f}, \hat{x})$ is positive, since more pollution increases the cost of producing a given amount of fish. Hence the merged firm will want to produce where $-MC_S(\hat{s}, \hat{x})$ is positive; that is, it will want to produce *less* pollution than the independent steel firm. When the true social cost of the externality involved in the steel production is taken into account, the optimal production of pollution will be reduced.

When the steel firm considers minimizing its **private costs** of producing steel, it produces where the marginal cost of extra pollution equals zero;

but the Pareto efficient level of pollution requires minimizing the **social costs** of the pollution. At the Pareto efficient level of pollution, the *sum* of the two firm's marginal costs of pollution must be equal to zero.

This argument is illustrated in Figure 35.3. In this diagram $-MC_S$ measures the marginal cost to the steel firm from producing more pollution. The curve labeled MC_F measures the marginal cost to the fishery of more pollution. The profit-maximizing steel firm produces pollution up to the point where its marginal cost from generating more pollution equals zero.

Figure 35.3

Social cost and private cost. The steel firm produces pollution up to the point where the marginal cost of extra pollution equals zero. But the Pareto efficient production of pollution is at the point where price equals marginal social cost, which includes the cost of pollution borne by the fishery.

But at the Pareto efficient level of pollution, the steel firm pollutes up to the point where the effect of a marginal increase in pollution is equal to the marginal social cost, which counts the impact of pollution on the costs of both firms. At the efficient level of pollution production, the amount that the steel firm is willing to pay for an extra unit of pollution should equal the social costs generated by that extra pollution—which include the costs it imposes on the fishery.

This is perfectly consistent with the efficiency arguments given in earlier

chapters. There we assumed that there were no externalities, so that private costs and social costs coincided. In this case the free market will determine a Pareto efficient amount of output of each good. But if the private costs and the social costs diverge, the market alone may not be sufficient to achieve Pareto efficiency.

EXAMPLE: Pollution Vouchers

Everyone wants a clean environment ... as long as someone else pays for it. Even if we reach a consensus on how much we should reduce pollution, there is still the problem of determining the most cost-effective way to achieve the targeted reduction.

Take the case of nitrogen oxide emissions. One emitter may find it relatively inexpensive to reduce its emissions of this pollutant, whereas another may find it very expensive. Should they both be required to reduce their emission of pollutants by the same physical amount, by the same proportional amount, or by some other rule?

Let's look at a simple economic model. Suppose that there are only two firms. Firm 1's emission quota is x_1 and firm 2's is x_2. The cost of achieving an emission quota x_1 is $c_1(x_1)$ and similarly for firm 2. The total amount of emission is fixed at some target level X. If we want to minimize the total costs of achieving the emissions target, subject to the aggregate constraint, we need to solve the following problem:

$$\min_{x_1,x_2} c_1(x_1) + c_2(x_2)$$
$$\text{such that } x_1 + x_2 = X.$$

A by now standard economic argument shows that the marginal cost of emission control must be equalized across the firms. If one firm had a higher marginal cost of emission control than the other, then we could lower total costs by reducing its quota and increasing the quota of the other firm.

How can we achieve this outcome? If the government regulators had information on the cost of emissions for all firms, they could calculate the appropriate pattern of production and impose it on all the relevant parties. But the cost of gathering all this information, and keeping it up-to-date, is staggering. It is much easier to characterize the optimal solution than to actually implement it!

Many economists have argued that the best way to implement the efficient solution to the emission control problem is to use a market. It appears that such a market based emissions control system will soon be put into effect in Southern California. Here is how the California plan works.[2]

[2] See Richard Stevenson, "Trying a Market Approach to Smog," *New York Times*, March 25, 1992, C1.

Each of the 2700 largest polluters in Southern California is assigned a quota for their emissions of nitrogen oxide. This quota is initially set to be 8 percent less than their previous year's emission. If the firm exactly meets its emissions quota it faces no fines or penalties. However, if it reduces its emissions by *more* than its emissions quota, it can sell the extra "right to emit" on the open market.

Suppose that a firm's quota is 95 tons of nitrogen oxide emissions per year. If it manages to produce only 90 tons in a given year, then it can sell the right to emit 5 tons of nitrogen oxide to some other firm. Each firm can compare the market price of an emission credit to the cost of reducing its emissions and decide whether it was more cost-effective to reduce emissions further or purchase emission credits from other firms.

Firms that find it easy to reduce emissions will sell credits to firms that find it costly to reduce emissions. In equilibrium, the market price of the right to emit one ton of pollution should just equal the marginal cost of reducing emissions by one ton. But this is exactly the condition characterizing the optimal pattern of emissions! The market for emission permits produces the efficient pattern of emissions automatically.

35.4 Interpretation of the Conditions

There are several useful interpretations of the conditions for Pareto efficiency derived above. Each of these interpretations suggests a scheme to correct the efficiency loss created by the production externality.

The first interpretation is that the steel firm faces the wrong price for pollution. As far as the steel firm is concerned, its production of pollution costs it nothing. But that neglects the costs that the pollution imposes on the fishery. According to this view, the situation can be rectified by making sure that the polluter faces the correct social cost of its actions.

One way to do this is to place a tax on the pollution generated by the steel firm. Suppose that we put a tax of t dollars per unit of pollution generated by the steel firm. Then the profit-maximization problem of the steel firm becomes

$$\max_{s,x} \; p_s s - c_s(s,x) - tx.$$

The profit-maximization conditions for this problem will be

$$p_s - \frac{\Delta c_s(s,x)}{\Delta s} = 0$$
$$-\frac{\Delta c_s(s,x)}{\Delta x} - t = 0.$$

Comparing these conditions to equation (35.3), we see that setting

$$t = \frac{\Delta c_f(\hat{f},\hat{x})}{\Delta x}$$

will make these conditions the same as the conditions characterizing the Pareto efficient level of pollution.

This kind of a tax is known as a **Pigouvian tax**.[3] The problem with Pigouvian taxes is that we need to know the optimal level of pollution in order to impose the tax. But if we knew the optimal level of pollution we could just tell the steel firm to produce exactly that much and not have to mess with this taxation scheme at all.

Another interpretation of the problem is that there is a missing market—the market for the pollutant. The externality problem arises because the polluter faces a zero price for an output good that it produces, even though people would be willing to pay money to have that output level reduced. From a social point of view, the output of pollution should have a *negative* price.

We could imagine a world where the fishery had the right to clean water, but could sell the right to allow pollution. Let q be the price per unit of pollution, and let x be the amount of pollution that the steel mill produces. Then the steel mill's profit-maximization problem is

$$\max_{s,x} \; p_s s - qx - c_s(s,x),$$

and the fishery's profit-maximization problem is

$$\max_{f,x} \; p_f f + qx - c_f(f,x).$$

The term qx enters with a negative sign in the profit expression for the steel firm since it represents a cost—the steel firm must buy the right to generate x units of pollution. But it enters with a positive sign in the expression for the profits of the fishery, since the fishery gets revenue from selling this right.

The profit-maximization conditions are

$$p_s = \frac{\Delta c_s(s,x)}{\Delta s} \tag{35.4}$$

$$q = -\frac{\Delta c_s(s,x)}{\Delta x} \tag{35.5}$$

$$p_f = \frac{\Delta c_f(f,x)}{\Delta f} \tag{35.6}$$

$$q = \frac{\Delta c_f(f,x)}{\Delta x}. \tag{35.7}$$

[3] Arthur Pigou (1877–1959), an economist at Cambridge University, suggested such taxes in his influential book *The Economics of Welfare*.

Thus each firm is facing the social marginal cost of each of its actions when it chooses how much pollution to buy or sell. If the price of pollution is adjusted until the demand for pollution equals the supply of pollution, we will have an efficient equilibrium, just as with any other good.

Note that at the optimal solution, equations (35.5) and (35.7) imply that

$$-\frac{\Delta c_s(s,x)}{\Delta x} = \frac{\Delta c_f(f,x)}{\Delta x}.$$

This says that the marginal cost to the steel firm of reducing pollution should equal the marginal benefit to the fishery of that pollution reduction. If this condition were not satisfied, we couldn't have the optimal level of pollution. This is, of course, the same condition we encountered in equation (35.3).

In analyzing this problem we have stated that the fishery had a right to clean water and that the steel mill had to purchase the right to pollute. But we could have assigned the property rights in the opposite way: the steel mill could have the right to pollute and the fishery would have to pay to induce the steel mill to pollute less. Just as in the case of the smoker and nonsmoker, this would also give an efficient outcome. In fact, it would give precisely the *same* outcome, since exactly the same equations would have to be satisfied.

To see this, we now suppose that the steel mill has the right to pollute up to some amount $\overline{x}$, say, but the fishery is willing to pay it to reduce its pollution. The profit-maximization problem for the steel mill is then

$$\max_{s,x}\ p_s s + q(\overline{x} - x) - c_s(s,x).$$

Now the steel mill has two sources of income: it can sell steel, and it can sell pollution relief. The price equals marginal cost conditions become

$$p_s - \frac{\Delta c_s(s,x)}{\Delta s} = 0 \tag{35.8}$$

$$-q - \frac{\Delta c_s(s,x)}{\Delta x} = 0. \tag{35.9}$$

The fishery's maximization problem is now

$$\max_{f,x}\ p_f f - q(\overline{x} - x) - c_f(f,x),$$

which has optimality conditions

$$p_f - \frac{\Delta c_f(f,x)}{\Delta f} = 0 \tag{35.10}$$

$$q - \frac{\Delta c_f(f,x)}{\Delta x} = 0. \tag{35.11}$$

Now observe: the four equations (35.8)–(35.11) are precisely the same as the four equations (35.4)–(35.7). In the case of production externalities, the optimal pattern of production is independent of the assignment of property rights. Of course, the distribution of profits will generally depend on the assignment of property rights. Even though the social outcome will be independent of the distribution of property rights, the owners of the firms in question may have strong views about what is an appropriate distribution.

35.5 Market Signals

Finally we turn to the third interpretation of externalities, which in some respects is the most profound. In the case of the steel mill and the fishery there is no problem if both firms merge—so why don't they merge? In fact, when you think about it, there is a definite incentive for the two firms to merge: if the actions of one affect the other, then they can make higher profits together by coordinating their behavior than by each going alone. *The objective of profit maximization itself should encourage the internalization of production externalities.*

Said another way: if the joint profits of the firms with coordination exceed the sum of the profits without coordination, then the current owners could each be bought out for an amount equal to the present value of the stream of profits for their firm, the two firms could be coordinated, and the buyer could retain the excess profits. The new buyer could be either of the old firms, or anybody else for that matter.

The market itself provides a signal to internalize production externalities, which is one reason this kind of production externality is rarely observed. Most firms have *already* internalized the externalities between units that affect each other's production. The case of the apple orchard and the beekeeper mentioned earlier is a case in point. Here there *would* be an externality if the two firms ignored their interaction . . . but why would they be so foolish as to do so? It is more likely that one or both of the firms would realize that more profits could be made by coordinating their activities, either by mutual agreement or by the sale of one of the firms to the other. Indeed, it is very common for apple orchards to keep honey bees for the purpose of fertilizing the trees. That particular externality is easily internalized.

EXAMPLE: Bees and Almonds

Many varieties of fruit and nut trees need bees to pollinate their blossoms, thereby allowing the trees to produce crops.

According to the Carl Hayden Bee Research Center in Tucson, Arizona, honeybees pollinate about one-third of the human diet and more than 50

different agricultural crops valued at more than $20 billion a year in the United States.[4]

Some owners of orchards keep their own bees; some rely on their neighbors' bees or wild bees. However, as the theory of externalities suggests, the most natural solution to the problem of inadequate bee supply is a market for bee services.

Consider, for example, the California almond market. There are 530,000 acres of almond trees in California, and every year, more than 1 million honeybee hives are needed to pollinate the trees. But California only has 440,000 resident bee hives. There aren't enough California bees to pollinate all those almond trees!

The solution is to import bees from other nearby states. There is, in fact, a ready market for such services, with beekeepers bringing hives from North Dakota, Washington, and Colorado to supplement the native California bees. The almond growers pay well for these services: in 2004, bee pollination services sold for $54 per hive.

35.6 The Tragedy of the Commons

We have argued above that if property rights are well defined, there will be no problem with production externalities. But if property rights are not well defined, the outcome of the economic interactions will undoubtedly involve inefficiencies.

In this section we will examine a particularly well-known inefficiency called "the tragedy of the commons."[5] We will pose this problem in the original context of a common grazing land, although there are many other possible illustrations.

Consider an agricultural village in which the villagers graze their cows on a common field. We want to compare two allocation mechanisms: the first is the private ownership solution where someone owns the field and decides how many cows should graze there; the second is the solution where the field is owned in common by the villagers and access to it is free and unrestricted.

Suppose that it costs a dollars to buy a cow. How much milk the cow produces will depend on how many other cows are grazed on the common land. We'll let $f(c)$ be the value of the milk produced if there are c cows grazed on the common. Thus the value of the milk per cow is just the average product, $f(c)/c$.

[4] Anna Oberthur, "Almond Growers Face Need for Bees," Associated Press, February 29, 2004.

[5] See G. Hardin, "The Tragedy of the Commons," *Science*, 1968, 1243–47.

How many cows would be grazed on the common if we wanted to maximize the total wealth of the village? In order to maximize the total amount of wealth, we set up the following problem:

$$\max_c \; f(c) - ac.$$

It should be clear by now that the maximal production will occur when the marginal product of a cow equals its cost, a:

$$MP(c^*) = a.$$

If the marginal product of a cow were greater than a, it would pay to put another cow on the commons; and if it were less than a, it would pay to take one off.

If the common grazing ground were owned by someone who could restrict access to it, this is indeed the solution that would result. For in this case, the owner of the grazing grounds would purchase just the right amount of cows to maximize his profits.

Now what would happen if the individual villagers decided whether or not to use the common field? Each villager has a choice of grazing a cow or not grazing one, and it will be profitable to graze a cow as long as the output generated by the cow is greater than the cost of a cow. Suppose that there are c cows currently being grazed, so that the current output per cow is $f(c)/c$. When a villager contemplates adding a cow, the total output will be $f(c+1)$, and the total number of cows will be $c+1$. Thus the revenue that the cow generates for the villager will be $f(c+1)/(c+1)$. He must compare this revenue to the cost of the cow, a. If $f(c+1)/(c+1) > a$, it is profitable to add the cow since the value of the output exceeds the cost. Hence the villagers will choose to graze cows until the average product of a cow is driven to a. It follows that the total number of cows grazed will be $\hat{c}$, where

$$\frac{f(\hat{c})}{\hat{c}} = a.$$

Another way to derive this result is to appeal to free entry. If it is profitable to graze a cow on the common field, villagers will purchase cows. They will stop adding cows to the common only when the profits have been driven to zero, that is, when

$$f(\hat{c}) - a\hat{c} = 0,$$

which is just a rearrangement of the condition in the last paragraph.

When an individual decides whether or not to purchase a cow, he looks at the extra value he will get $f(c)/c$ and compares this to the cost of the cow, a. This is fine for him, but what has been left out of this calculation is the fact that his extra cow will reduce the output of milk from all the *other*

cows. Since he is ignoring this **social cost** of his purchase, too many cows will be grazed on the common ground. (We assume that each individual has a number of cows that is negligible relative to the total number grazed on the common.)

This argument is illustrated in Figure 35.4. Here we have depicted a falling average product curve, since it is reasonable to suppose that the output per cow declines as more and more cows are grazed on the common land.

Figure 35.4

The tragedy of the commons. If the grazing area is privately owned, the number of cows will be chosen so that the marginal product of a cow equals its cost. But if grazing area is common property, cows will be grazed until the profits are driven to zero; thus the area will be overgrazed.

Since the average product is falling, it must be that the marginal product curve always lies below the average product curve. Thus the number of cows where the marginal product equals a must be less than where the average product equals a. The field will be overgrazed in the absence of a mechanism to restrict use.

Private property provides such a mechanism. Indeed, we have seen that if everything that people care about is owned by someone who can control its use and, in particular, can exclude others from overusing it, then there are by definition no externalities. The market solution leads to a Pareto

efficient outcome. Inefficiencies can only result from situations where there is no way to exclude others from using something, a topic that we will investigate in the next chapter.

Of course, private property is not the only social institution that can encourage efficient use of resources. For example, rules could be formulated about how many cows can be grazed on the village common. If there is a legal system to enforce those rules, this may be a cost-effective solution to providing an efficient use of the common resource. However, in situations where the law is ambiguous or nonexistent, the tragedy of the commons can easily arise. Overfishing in international waters and the extermination of several species of animals due to overhunting are sobering examples of this phenomenon.

EXAMPLE: Overfishing

According to a report in the *New York Times,* "...overfishing has decimated the stocks of cod, haddock and flounder that have sustained New Englanders for centuries."[6] According to one expert, fishermen in New England are taking 50 to 70 percent of the available stock, over twice the sustainable amount.

This overfishing is a prime example of the problem of the commons: each fisherman has a negligible impact on the total stock of fish, but the accumulated efforts of thousands of fishermen results in serious depletion. The New England Fisheries Management Council is attempting to alleviate the problem by banning new entry to the industry, requiring fishermen to limit their days at sea, and increasing the mesh size of their nets.

It appears that the supplies of fish could be restocked in as little as 5 years if conservation measures were undertaken. The present value of profits to the industry as a whole would be larger with regulation to prevent overfishing. However, such measures would almost certainly imply a substantial reduction in the number of fishing boats in the industry, which is highly unpopular with the small fishermen, who would likely be forced to leave the industry.

EXAMPLE: New England Lobsters

Some fishing industries have already applied stringent rules to avoid overfishing. For example, lobster fishermen work under carefully designed rules to ensure that they do not fish themselves out of a livelihood. For example, they are required to toss back any female lobster bearing eggs, any lobster

[6] "Plenty of Fish in the Sea? Not Anymore," *New York Times,* March 25, 1992, A15.

shorter than a minimum size, and any lobsters bigger than a maximum size.

The "eggers" give birth to more lobsters and the small "tiddlers" grow up to mate. But why throw back the big lobsters? According to marine biologists, large lobsters produce more offspring and larger offspring. If fishermen always took the largest lobsters, the remaining small lobsters would pass their genes onto their progeny, resulting in smaller and smaller lobsters in each generation.

With lobsters there is good news and bad news. First the good news. The 2003 Maine lobster harvest was 5.4 million pounds, more than 2.5 times the 1945–85 average. This suggests that the careful husbandry practiced by the industry has yielded a significant growth in the lobster population.

However, it appears that conservation isn't the only factor. There have also been considerable changes in the population of other species of marine life off the Maine coast, such as sea urchins, and some observers believe that these changes are the primary driver of change in the lobster population.[7]

This leads to the bad news. Further south, in Massachusetts and New York, the lobster catch has fallen dramatically. No one is quite sure why one region is doing so well and the other so poorly. Ironically, Maine may be doing well due to increased harvesting of finned fish and of sea urchins, both of which eat young lobsters. Massachusetts' problems may be due to specific factors, such as a large oil spill and a disfiguring shell disease. Another culprit is warming water: Narragansett Bay temperatures have risen almost two degrees Celsius in the last 20 years.

Ecologies can be very complex and can change rapidly. The efforts to avoid overfishing are to be applauded, but they are only part of the story.

35.7 Automobile Pollution

As suggested above, pollution is a prime example of an economic externality. The activity of one consumer operating an automobile will typically lower the quality of the air that other consumers breathe. It seems unlikely that an unregulated free market would generate the optimal amount of pollution; more likely, if the consumer bears no cost in generating pollution, too much pollution would be produced.

One approach to controlling the amount of automobile pollution is to require that automobiles meet certain standards in the amount of pollution that they generate. This has been the basic thrust of U.S. antipollution policy since the Clean Air Act of 1963. That act, or, more properly, the subsequent amendments, set automobile emission standards for the manufacturers of vehicles in the United States.

[7] See *The Economist*, "Claws!" August 19, 2004, and Cornelia Dean, "Lobster Boom and Bust," *New York Times*, August 9, 2004.

Lawrence White has examined the benefits and costs of this program; most of the following discussion is drawn from this work.[8]

White estimates that the cost of emission control equipment is about $600 per car, the extra maintenance costs are about $180 per car, and the costs of the reduced gasoline mileage and the necessity for unleaded gasoline come to about $670 per car. Thus the total cost per car of the emission control standards is about $1450 over the lifetime of the car. (All figures are in 1981 dollars.)

He argues that there are several problems with the current approach to the regulation of automobile emissions. First, it requires that all automobiles meet the same standards. (California is the only state with different standards for emission control.) This means that *everyone* who buys a car must pay an extra $1450 whether they live in a high pollution area or not. A 1974 National Academy of Sciences study concluded that 63 percent of all U.S. cars did not require the stringent standards now in effect. According to White, "almost two-thirds of car buyers are spending . . . substantial sums for unnecessary systems."

Secondly, most of the responsibility for meeting the standards falls on the manufacturer, and little falls on the user. Owners of cars have little incentive to keep their pollution control equipment in working order unless they live in a state with required inspections.

More significantly, motorists have no incentive to economize on their driving. In cities such as Los Angeles, where pollution is a significant hazard, it makes good economic sense to encourage people to drive less. Under the current system, people who drive 2000 miles a year in North Dakota pay exactly the same amount for pollution control as people who drive 50,000 miles a year in Los Angeles.

An alternative solution to pollution would be *effluent fees*. As described by White, effluent fees would require an annual inspection of all vehicles along with an odometer reading and tests that would estimate the likely emissions of the vehicle during the past year. Different communities could then levy fees based on the estimated amount of pollution that had actually been generated by the operation of the vehicle. This method would ensure that people would face the true cost of generating pollution and would encourage them to choose to generate the socially optimal amount of pollution.

Such a system of effluent fees would encourage the vehicle owners themselves to find low-cost ways of reducing their emissions—investing in pollution control equipment, changing their driving habits, and changing the kinds of vehicles that they operate. A system of effluent fees could impose even higher standards than are now in effect in communities where pollution is a serious problem. Any desired level of pollution control can

[8] See Lawrence White, *The Regulation of Air Pollutant Emissions from Motor Vehicles* (Washington, D.C.: American Enterprise Institute for Public Policy Research, 1982).

be achieved by appropriate effluent fees ... and it can be achieved at a substantially lower cost than the current system of mandated standards.

Of course, there is no reason why there might not also be some federally mandated standards for the two-thirds of the vehicles that are operated in localities where pollution is not a serious problem. If it is cheaper to impose standards than to require inspections, then by all means that should be the proper choice. The appropriate method of pollution control for automobiles should depend on a rational analysis of benefits and costs—as should all social policies of this nature.

Summary

1. The First Theorem of Welfare Economics shows that a free, competitive market will provide an efficient outcome in the absence of externalities.

2. However, if externalities are present, the outcome of a competitive market is unlikely to be Pareto efficient.

3. However, in this case, the state can sometimes "mimic" the role of the market by using prices to provide correct signals about the social cost of individual actions.

4. More importantly, the legal system can ensure that property rights are well defined, so that efficiency-enhancing trades can be made.

5. If preferences are quasilinear, the efficient amount of a consumption externality will be independent of the assignment of property rights.

6. Cures for production externalities include the use of Pigouvian taxes, setting up a market for the externality, simply allowing firms to merge, or transferring property rights in other ways.

7. The tragedy of the commons refers to the tendency for common property to be overused. This is a particularly prevalent form of externality.

REVIEW QUESTIONS

1. True or false? An explicit delineation of property rights usually eliminates the problem of externalities.

2. True or false? The distributional consequences of the delineation of property rights are eliminated when preferences are quasilinear.

3. List some other examples of positive and negative consumption and production externalities.

4. Suppose that the government wants to control the use of the commons, what methods exist for achieving the efficient level of use?

CHAPTER 36

INFORMATION TECHNOLOGY

One of the most radical changes in the economy in the last 15 years has been the emergence of the **information economy**. The popular press is filled with stories about advances in computer technology, the Internet, and new software. Not surprisingly, many of these stories are on the business pages of the newspaper, for this *technological* revolution is also an *economic* revolution.

Some observers have gone so far as to put the Information Revolution on a par with the Industrial Revolution. Just as the Industrial Revolution transformed the way *goods* were produced, distributed, and consumed, the Information Revolution is transforming the way *information* is produced, distributed, and consumed.

It has been claimed that these dramatically new technologies will require a fundamentally different form of economics. Bits, it is argued, are fundamentally different than atoms. Bits can be reproduced costlessly and distributed around the world at the speed of light, and they never deteriorate. Material goods, made of atoms, have none of these properties: they are costly to produce and transport, and they inevitably deteriorate.

It is true that the unusual properties of bits require new economic analysis, but I would argue that they do not require a new *kind* of economic

analysis. After all, economics is primarily about *people* not *goods*. The models we have analyzed in this book have had to do with how people make choices and interact with each other. We have rarely had occasion to refer to the specific goods that were involved in the transactions. The fundamental concerns were the tastes of the individuals, the technology of production, and the structure of the market, and *these* same factors will determine how markets for information will work . . . or not work.

In this chapter we will investigate a few economic models relevant to the information revolution. The first has to do with the economics of networks, the second with switching costs, and the third with rights management for information goods. These examples will illustrate how the fundamental tools of economic analysis can help us to understand the world of bits as well as the world of atoms.

36.1 Systems Competition

Information technology is generally used in *systems*. Such systems involve several components, often provided by different firms, that only have value if they work together. Hardware is useless without software, a DVD player is useless without DVD disks, an operating system is worthless without applications, and a web browser is useless without web servers. All of these are examples of **complements**: goods where the value of one component is significantly enhanced by the presence of another component.

In our discussion of consumer theory, we described left shoes and right shoes as complements. The cases above are equally extreme: the best computer hardware in the world can't function unless there is software written for it. But unlike shoes, the more software that is available for it, the more valuable it becomes.

Competition among the providers of these components often have to worry just as much about their "complementors" as their competitors. A key part of Apple's competitive strategy has to involve their relations with software developers. This gives competitive strategy in information technology (IT) industries a different flavor than strategy in traditional industries.[1]

36.2 The Problem of Complements

To illustrate these points, let us consider the case of a Central Processing Unit (CPU) and an Operating System (OS). A CPU is an integrated

[1] See Shapiro, Carl and Hal R. Varian, *Information Rules: A Strategic Guide to the Network Economy*, Harvard Business School Press, 1998, for a guide to competitive strategy in IT industries.

circuit that is the "brain" of a computer. Two familiar manufacturers of CPUs are Intel and Motorola. An OS is the software that allows users and applications to access the functions of the CPU. Apple and Microsoft both make operating systems. Normally, a special version of an operating system has to be created for each CPU.

From the viewpoint of the end user, the CPU can only be used if there is a compatible operating system. The CPU and the OS are complements, just as left shoes and right shoes are complements.

The most popular CPUs and OSs in the world today are made by Intel and Microsoft, respectively. These are, of course, two separate companies that set the prices of their products independently. The PowerPC, another popular CPU, was designed by a consortium consisting of IBM, Motorola, and Apple. Two commercial operating systems for the PowerPC are the Apple OS and IBM's AIX. In addition to these commercial operating systems, there are free systems like BSD and GNU-Linux that are provided by groups of programmers working on a volunteer basis.

Let us consider the pricing problem facing sellers of complementary products. The critical feature is that the demand for *either* product depends on the price of *both* products. If p_1 is the price of the CPU and p_2 is the price of the OS, the cost to the end user depends on $p_1 + p_2$. Of course, you need more than just a CPU and an OS to make a useful system, but that just adds more prices to the sum; we'll keep things simple by sticking with two components.

The demand for CPUs depends on the price of the total system, so we write $D(p_1+p_2)$. If we let c_1 be the marginal cost of a CPU and F the fixed cost, the profit-maximization problem of the CPU maker can be written

$$\max_{p_1} \ (p_1 - c_1)D(p_1 + p_2) - F_1.$$

Similarly, the profit-maximization problem of the OS maker can be written

$$\max_{p_2} \ (p_2 - c_2)D(p_1 + p_2) - F_2.$$

In order to analyze this problem, let us assume that the demand function has the linear form

$$D(p) = a - bp.$$

Let us also assume, for simplicity, that the marginal costs are so small that they can be ignored. Then the CPU profit-maximization problem becomes

$$\max_{p_1} \ p_1[a - b(p_1 + p_2)] - F_1,$$

or

$$\max_{p_1} \ ap_1 - bp_1^2 - bp_1p_2 - F_1.$$

It turns out that the marginal revenue from a price increase Δp_1 is

$$(a - 2bp_1 - bp_2)\,\Delta p_1.$$

If profit is maximized, then the change in revenue from an increase in p_1 must be zero:

$$a - 2bp_1 - bp_2 = 0.$$

Solving this equation we have

$$p_1 = \frac{a - bp_2}{2b}.$$

In exactly the same way, we can solve for the profit-maximizing choice of the OS price:

$$p_2 = \frac{a - bp_1}{2b}.$$

Note that the optimal choice of each firm's price depends on what it expects the other firm to charge for its component. As usual, we are interested in a **Nash equilibrium**, where each firm's expectations about the other's behavior are satisfied.

Solving the system of two equations in two unknowns, we have

$$p_1 = p_2 = \frac{a}{3b}.$$

This gives us the profit-maximizing prices if each firm unilaterally and independently sets the price of its component of the system. The price of the total system is

$$p_1 + p_2 = \frac{2a}{3b}.$$

Now let us consider the following experiment. Suppose that the two firms merge to form an integrated firm. Instead of setting the prices of the components, the integrated firm sets the price of the final system, which we denote by p. Its profit-maximization problem is therefore

$$\max_p \; p(a - bp).$$

The marginal revenue from increasing the system price by Δp is

$$(a - 2bp)\,\Delta p.$$

Setting this equal to zero and solving, we find that the price that the integrated firm will set for the final system is

$$p = \frac{a}{2b}.$$

Note the following interesting fact: the profit-maximizing price set by the integrated firm is *less* than the profit-maximizing price set by the two independent firms. Since the price of the system is lower, consumers will buy more of them and be better off. Furthermore, the profits of the integrated firm are larger than the sum of the equilibrium profits of the two independent firms. Everyone has been made better off by coordinating the pricing decision!

This turns out to be true in general: a merger of two monopolies that produce complementary products results in lower prices and higher profits than if the two firms set their prices independently.[2]

The intuition is not hard to see. When firm 1 contemplates a price decrease for the CPU, it will increase demand for CPUs *and* OSs. But it only takes into account the impact on its own profit from cutting price, ignoring the profits that will accrue to the other firm. This leads it to cut prices less than it would if it were interested in maximizing joint profit. The same analysis applies to firm 2, leading to prices that are "too high" from the viewpoint of both profit-maximization and consumer surplus.

Relationships among Complementors

The "merger of complementors" analysis is provocative, but we shouldn't immediately leap to the conclusion that mergers of OS and CPU manufacturers are a good idea. What the result says is that *independent* price setting will lead to prices that are too high from the viewpoint of joint profitability, but there are lots of intermediate cases between totally independent and fully integrated.

For example, one of the firms can negotiate prices for components and then sell an integrated bundle. This is, more or less, what Apple does. They buy PowerPC CPUs in bulk from Motorola, build them into computers, and then bundle the operating system and computers together for sale to the end customers.

Another model for dealing with the systems pricing problem is to use revenue sharing. Boeing builds airplane bodies and GE builds airplane engines. The end user generally wants both a body and an engine. If GE and Boeing each set their prices independently, they could decide to set their prices too high. So what they do instead is to negotiate a deal in which GE will receive a fraction of the revenue from the sale of the assembled aircraft. Then GE is happy to have Boeing negotiate to get as high a price as possible for the package, confident that it will receive its specified share.

[2] This rather remarkable fact was discovered by Augustin Cournot, whom we previously met in Chapter 28.

There are other mechanisms that work in different industries. Consider, for example, the DVD industry mentioned in the introduction. This has been a very successful new product, but making it work was tricky. Consumer electronics firms didn't want to produce players unless they were assured that there would be plenty of content available, and content providers didn't want to produce content unless they were sure that would be lots of DVD players out there.

On top of this, both the consumer electronics firms and the content producers would have to worry about the pricing of complements problem: if there were only a few providers of players and only a few providers of content, then they would each want to price their products "too high," reducing the total profit available in the industry and making consumers worse off.

Sony and Philips, who held the basic patents on the DVD technology, helped solve this problem by licensing the technology widely at attractive prices. They also realized that there had to be a lot of competition to keep the prices down and kick start the industry. They recognized that it was much better to have a small share of a large, successful industry than to have a large share of a nonexistent industry.

Yet another model for relationships among complementors might be called "commoditize the complement." Look back at firm 1's profit maximization problem:

$$\max_{p_1} p_1 D(p_1 + p_2) - F_1.$$

At any given configuration of prices, reducing p_1 may or may not increase firm 1's revenues, depending on the demand elasticity. But lowering p_2 will *always* increase firm 1's revenue. The challenge facing firm 1 is then: how can I get firm 2 to cut its price?

One way is to try to make competition for firm 2 more intense. Various strategies are possible here, depending on the nature of the industry. In technology-intensive industries, standardization becomes an important tool. An OS producer, for example, would want to encourage standardized hardware. This not only makes its job easier, but it also ensures that the hardware industry will be highly competitive. This will ensure that competitive forces push down the price of hardware and reduce the total system price to end users, thereby increasing the demand for operating systems.[3]

EXAMPLE: Apple's iPod and iTunes

Apple's iPod music player is hugely popular. As of January 2009, Apple had sold 6 billion songs, accounting for an estimated 70% of online music sales and an 88% market share in the United States.

[3] See Brandenburger, Adam and Barry Nalebuff, *Co-opetition*, Doubleday, 1997 for further analysis of strategy for complementors.

There is an obvious complementary relationship between the music player and the music. The classic business model for complements comes from Gillette: "Give away the razor and sell the blades." But in this case the model is reversed: most of Apple's profit comes from selling the iPod, with only a small fraction coming from selling the music.

This is primarily due to the fact that Apple does not own the music, so the revenue from music sold on iTunes must be shared between the producers of the music and Apple. Since Apple makes most of its money from the player, it wants to have cheap music. Since the studios make most of their money from the songs, they want to have expensive music. This has led to some conflicts between Apple and the music studios.

Originally, all songs on iTunes sold for 99 cents. Some music publishers felt that prices should be higher for new releases. After much back and forth, Apple announced a new policy in March of 2009, where some new releases would sell for $1.29. This is a form of differential pricing, or "versioning," which is common in media markets. Those who are enthused and impatient pay the higher price, while those who are more patient can wait for the price reduction.

EXAMPLE: Who Makes an iPod?

Hint: it's not Apple. In fact, iPods are assembled in a number of Asian countries, by a variety of assemblers, including Asustek, Inventec Appliances, and Foxconn.

But that's not the end of the story. These companies merely assemble the parts that are purchased from other companies. In 2009, some economists tried to track down the origin of the 451 parts that go into an iPod.[4]

The retail value of the 30-gigabyte video iPod that the authors examined was $299. The most expensive component in it was the hard drive, which was manufactured by Toshiba and costs about $73. The next most costly components were the display module (about $20), the video/multimedia processor chip ($8), and the controller chip ($5). They estimated that the final assembly, done in China, cost only about $4 a unit.

The authors of the report tried to track down where the major parts were manufactured and how much value was added at each stage of the production process. The researchers estimated that $163 of the iPod's $299 retail value in the United States was captured by American companies and workers, breaking it down to $75 for distribution and retail costs, $80 to Apple, and $8 to various domestic component makers. Japan contributed

[4] Greg Linden, Kenneth L. Kraemer, and Jason Dedrick, "Who Captures Value in a Global Innovation Network," *Communications of the ACM*, 52 (3), March 2009, 140–144.

about $26 to the value added (mostly via the Toshiba disk drive), while Korea contributed less than $1.

Ideally, each component was purchased from the lowest-cost provider, and to a large extent these decisions reflected the comparative advantage of the different providers.

Even though the assembly in China only contributed about 1% of the value of the iPod, each imported iPod contributed about $150 of the bilateral trade deficit between China and the United States. What this shows is that the bilateral trade deficit makes no sense. Most of the high-value parts in the iPod were in fact imported into China from other countries in the first place. The highest-value component of the iPod—the design and engineering that went into it—came from the United States.

EXAMPLE: AdWords and AdSense

Two of Google's advertising programs are AdWords, which shows ads targeted to search queries, and AdSense, which shows ads based on the contents of a web page. AdWords shows "search targeted ads" and AdSense shows "contextually targeted ads."

When a user clicks on a contextually targeted ad on a particular site, the advertiser pays a price per click determined by an auction, similar to that described in Chapter 18. The revenue from this ad click is divided between the publisher and Google according to a revenue-sharing formula. Hence the AdSense program provides a simple way for a publisher to generate advertising revenue without having to manage an advertising program on its own.

There is a strong **complementarity** between the AdWords and AdSense programs. By providing a way for publishers to make money from their content, AdSense encourages the production of content. This means that there is more useful information available on the web and therefore content for Google to index and search. By creating a business model for content creation, Google makes its search service more valuable.

36.3 Lock-In

Since IT components often work together as systems, switching any one component often involves switching others as well. This means that the **switching costs** associated with one component in IT industries may be quite substantial. For example, switching from a Macintosh to a Windows-based PC involves not only the hardware costs of the computer itself, but also involves purchasing of a whole new library of software, and, even more importantly, learning how to use a brand new system.

When switching costs are very high, users may find themselves experiencing **lock-in**, a situation where the cost of changing to a different system is so high that switching is virtually inconceivable. This is bad for the consumers, but is, of course, quite attractive for the seller of the components that make up the system in question. Since the locked-in user has a very *inelastic* demand, the seller(s) can jack up the prices of their components to extract consumer surplus from the user.

Of course, wary consumers will try to avoid such lock-in, or, at the very least, bargain hard to be compensated for being locked in. Even if the consumers themselves are poor at bargaining, competition among sellers of systems will force prices down for the *initial* purchase, since the locked-in consumers can provide them with a steady revenue stream afterwords.

Consider, for example, choosing an Internet service provider (ISP). Once you have committed to such a choice, it may be inconvenient to switch due to the cost of notifying all of your correspondents about your new e-mail address, reconfiguring your Internet access programs, and so on. The monopoly power due to these switching costs means that the ISP can charge more than the marginal cost of providing service, once it has acquired you as a customer. But the flip side of this effect is that the stream of profits of the locked-in customers is a valuable asset, and ISPs will compete up front to acquire such customers by offering discounts and other inducements to sign up with them.

A Model of Competition with Switching Costs

Let's examine a model of this phenomenon. We assume that the cost of providing a customer with Internet access is c per month. We also assume a perfectly competitive market, with many identical firms, so that in the absence of any switching costs, the price of Internet service would simply be $p = c$.

But now suppose that there is a cost s of switching ISPs and that ISPs can offer a discount of size d for the first month to attract new customers. At the start of a given month, a consumer contemplates switching to a new ISP. If he does so, he only has to pay the discounted price, $p - d$, but he also has to endure the switching costs s. If he stays with his old provider, he has to pay the price p forever. After the first month, we assume that both providers continue to charge the same price p forever.

The consumer will switch if the present value of the payments to the new provider plus the switching cost is less than the present value of the payments to the original ISP. Letting r be the (monthly) interest rate, the consumer will switch if

$$(p - d) + \frac{p}{r} + s < p + \frac{p}{r}.$$

Competition between providers ensures that the consumer is indifferent between switching or not switching, which implies

$$(p - d) + s = p.$$

It follows that $d = s$, which means the discount offered just covers the switching cost of the consumer.

On the producer side, we suppose that competition forces the present value of profits to be zero. The present value of profit associated with a single customer is the price minus the initial discount, plus the present value of the profits in future months. Letting r be the (monthly) interest rate, and using the fact that $d = s$, the zero-profit condition can be written as

$$(p - s) - c + \frac{p - c}{r} = 0. \tag{36.1}$$

Rearranging this equation gives us two equivalent ways to describe the equilibrium price:

$$p - c + \frac{p - c}{r} = s, \tag{36.2}$$

or

$$p = c + \frac{r}{1 + r} s. \tag{36.3}$$

Equation (36.2) says that the present value of the future profits from the consumer must just equal the consumer's switching cost. Equation (36.3) says that the price of service is a markup on marginal cost, where the amount of the markup is proportional to the switching costs.

Adding switching costs to the model raises the *monthly* price of service above cost, but competition for this profit flow forces the *initial* price down. Effectively, the producer is investing in the discount $d = s$ in order to acquire the flow of markups in the future.

In reality many ISPs have other sources of revenue than just the monthly income from their customers. America Online, for example, derives a substantial part of its operating revenue from advertising. It makes sense for them to offer large up-front discounts, in order to capture advertising revenue, even if they have to provide Internet connections at rates at or below cost.

We can easily add this effect to the model. If a is the advertising revenue generated by the consumer each month, the zero-profit condition requires

$$(p - s) + a - c + \frac{p + a - c}{r} = 0. \tag{36.4}$$

Solving for p we have

$$p = c - a + \frac{r}{1 + r} s.$$

This equation shows that what is relevant is the *net* cost of servicing the customer, $c - a$, which involves both the service cost and the advertising revenues.

EXAMPLE: Online Bill Payment

Many banks offer low-cost or even free bill payment services. Some banks will even pay customers who start using their online bill payment services.

Why the big rush to pay bills online? The answer is that banks have found that once a customer goes to the trouble of setting up the bill-paying service, he or she is much less likely to switch banks. According to a Bank of America study, the frequency of switching goes down by 80 percent for such customers.[5]

It's true that once you get online bill payment up and running, it's hard to give it up. Switching to another bank to get an extra tenth of a percent of interest on your checking account doesn't seem very attractive. As in the analysis of lock-in presented above, investing in services that create switching costs can be very profitable for businesses.

EXAMPLE: Number Portability on Cell Phones

At one time, cell phone providers prevented individuals from transferring their phone numbers when they switched carriers. This prohibition increases individual switching costs significantly, since anyone who switched would have to notify all of his or her friends about the new number.

As the model presented in this chapter describes, the fact that customers could be charged more when they faced high switching costs meant that the phone providers would compete even more aggressively to sign up such highly profitable customers. This competition took the form of providing low-cost or even free phones, along with offers of "free minutes," "rollover plans," "cell-to-cell discounts," and other marketing gimmicks.

The cell phone industry was united in its efforts to block number portability and lobbied regulatory agencies and Congress to maintain the status quo.

Slowly but surely, the tide started to turn against the cell phone industry as consumers demanded number portability. The Federal Communications Commission, which regulates the telephone business, started dropping hints that cell phone providers should consider ways in which they could implement number portability.

In June 2003, Verizon Wireless said it would drop opposition to number portability. Their decision appeared to rest on two considerations. First, it was becoming clear that they were fighting a losing battle: eventually cell-number portability would win out. Perhaps more significantly, several recent consumer surveys showed that Verizon led the industry in terms of

[5] Michelle Higgins, "Banks Use Online Bill Payment In Effort to Lock In Customers," *Wall Street Journal*, September 4, 2002.

customer satisfaction. It appeared quite possible that Verizon would gain more customers than it lost if switching costs were reduced. Indeed, it appears that ultimately Verizon benefited from number portability.

This episode provides a good lesson in business strategy: tactics to increase customer switching costs may be valuable for a while. But ultimately service quality plays a decisive role in attracting and retaining customers.

36.4 Network Externalities

We have already examined the idea of **externalities** in Chapter 35. Recall that economists use this term to describe situations in which one person's consumption directly influences another person's utility. **Network externalities** are a special kind of externalities in which one person's utility for a good depends on the *number* of other people who consume this good.[6]

Take for example a consumer's demand for a fax machine. People want fax machines so they can communicate with each other. If no one else has a fax machine, it certainly isn't worthwhile for you to buy one. Modems have a similar property: a modem is only useful if there is another modem somewhere that you can communicate with.

Another more indirect effect for network externalities arises with complementary goods. There is no reason for a video store to locate in a community where no one owns a video player; but then again, there is little reason to buy a video player unless you have access to pre-recorded video tapes to play in the machine. In this case the demand for video tapes depends on the number of VCRs, and the demand for VCRs depends on the number of video tapes available, resulting in a slightly more general form of network externalities.

36.5 Markets with Network Externalities

Let us try to model network externalities using a simple demand and supply model. Suppose that there are 1000 people in a market for some good and we index the people by $v = 1, \ldots, 1000$. Think of v as measuring the **reservation price** for the good by person v. Then if the price of the good is p, the number of people who think that the good is worth at least p is $1000 - p$. For example, if the price of the good is \$200, then there are 800 people who are willing to pay at least \$200 for the good, so the total number of units sold would be 800. This structure generates a standard, downward-sloping demand curve.

[6] More generally, a person's utility could depend on the *identity* of other users; it is easy to add this to the analysis.

But now let's add a twist to the model. Suppose that the good we are examining exhibits network externalities, like a fax machine or a telephone. For simplicity, let us suppose that the value of the good to person v is vn, where n is the number of people who consume the good—the number of people who are connected to the network. The more people there are who consume the good, the more *each* person is willing to pay to acquire it.[7] What does the demand function look like for this model?

If the price is p, there is someone who is just indifferent between buying the good and not buying it. Let $\hat{v}$ denote the index of this marginal individual. By definition, he is just indifferent to purchasing the good, so his willingness to pay for the good equals its price:

$$p = \hat{v}n. \tag{36.5}$$

Since this "marginal person" is indifferent, everyone with a *higher* value of v than $\hat{v}$ must definitely want to buy. This means that the number of people who want to buy the good is

$$n = 1000 - \hat{v}. \tag{36.6}$$

Putting equations (36.5) and (36.6) together, we have a condition that characterizes equilibrium in this market:

$$p = n(1000 - n).$$

This equation gives us a relationship between the price of the good and the number of users. In this sense, it is a kind of demand curve; if there are n people who purchase the good, then the willingness to pay of the marginal individual is given by the height of the curve.

However, if we look at the plot of this curve in Figure 36.1, we see that it has quite a different shape than a standard demand curve! If the number of people who connect is low, then the willingness to pay of the marginal individual is low, because there aren't many other people out there that he can communicate with. If there are a large number of people connected, then the willingness to pay of the marginal individual is low, because everyone else who valued it more highly has already connected. These two forces lead to the humped shape depicted in Figure 36.1.

Now that we understand the demand side of the market, let's look at the supply side. To keep things simple, let us suppose that the good can be provided by a constant returns to scale technology. As we've seen, this means that the supply curve is a flat line at price equals average cost.

Note that there are three possible intersections of the demand and supply curve. There is a low-level equilibrium where $n^* = 0$. This is where no one

[7] We should really interpret n as the number of people who are *expected* to consume the good, but this distinction won't be very important for what follows.

Network externalities. The demand is given by the curved hump, the supply by the horizontal line. Note that there are three intersections where demand equals supply.

Figure 36.1

consumes the good (connects to the network), so no one is willing to pay anything to consume the good. This might be referred to as a "pessimistic expectations" equilibrium.

The middle equilibrium with a positive but small number of consumers is one where people don't think the network will be very big, so they aren't willing to pay that much to connect to it—and therefore the network isn't very big.

Finally the last equilibrium has a large number of people, n_H. Here the price is small because the marginal person who purchases the good doesn't value it very highly, even though the market is very large.

36.6 Market Dynamics

Which of the three equilibria will we see occur? So far the model gives us no reason to choose among them. At each of these equilibria, demand equals supply. However, we can add a dynamic adjustment process to help us decide which equilibrium is more likely to occur.

It is plausible to assume that when people are willing to pay more than the cost of the good, the size of the market expands and, when they are willing to pay less, the market contracts. Geometrically this is saying that when the demand curve is above the supply curve, the quantity goes up and, when it is beneath the supply curve, the quantity goes down. The arrows in Figure 36.1 illustrate this adjustment process.

These dynamics give us a little more information. It is now evident that the low-level equilibrium, where no one connects, and the high-level equilibrium, where many people connect, are stable whereas the middle equilibrium is unstable. Hence it is unlikely that the final resting point of the system will be the middle equilibrium.

We are now left with two possible stable equilibria; how can we tell which is likely to occur? One idea is to think about how costs might change over time. For the kinds of examples we have discussed—faxes, VCRs, computer networks, and so on—it is natural to suppose that the cost of the good starts out high and then decreases over time due to technological progress. This process is illustrated in Figure 36.2. At a high unit cost there is only one stable equilibrium—where demand equals zero. When the cost decreases sufficiently, there are two stable equilibria.

Figure 36.2

Cost adjustment and network externalities. When the cost is high, the only equilibrium implies a market of size zero. As the cost goes down, other equilibria become possible.

Now add some noise to the system. Think of perturbing the number of people connected to the network around the equilibrium point of $n^* = 0$. These perturbations could be random, or they could be part of business strategies such as initial discounts or other promotions. As the cost gets smaller and smaller, it becomes increasingly likely that one of these perturbations will kick the system up *past* the unstable equilibrium. When this happens, the dynamic adjustment will push the system up to the high-level equilibrium.

A possible path for the number of consumers of the good is depicted in Figure 36.3.

It starts out at essentially zero, with a few small perturbations over time. The cost decreases, and at some point we reach a critical mass that kicks us up past the low-level equilibrium and the system then zooms up to the high-level equilibrium.

Possible adjustment to equilibrium. The number of users connected to the network is initially small, and increases only gradually as costs fall. When a critical mass is reached, the network growth takes off dramatically.

Figure 36.3

A real-life example of this kind of adjustment is the market for fax machines. Figure 36.4 illustrates the price and number of fax machines shipped over a period of 12 years.[8]

EXAMPLE: Network Externalities in Computer Software

Network externalities arise naturally in the provision of computer software. It is very convenient to be able to exchange data files and tips with other

[8] This diagram is taken from "Critical Mass and Network Size with Applications to the US Fax Market," by Nicholas Economides and Charles Himmelberg (Discussion Paper no. EC-95-11, Stern School of Business, N.Y.U., 1995). See also Michael L. Katz and Carl Shapiro, "Systems Competition and Network Effects," *Journal of Economic Perspectives*, 8 (1994), 93–116, for a nice overview of network externalities and their implications.

Figure 36.4 **Fax market**. The demand for fax machines was small for a long time since so few people used them. During the mid-eighties the price fell significantly and the demand suddenly exploded.

users of the same software. This gives a significant advantage to the largest seller in a given market and leads software producers to invest heavily in acquiring market share.

Examples of this abound. Adobe Systems for example, invested heavily in developing a "page description language" called PostScript for desktop publishing. Adobe realized clearly that no one would invest the time and resources necessary to learn PostScript unless it was the clear "industry standard." So the firm deliberately allowed competitors to "clone" its language in order to create a competitive market in PostScript interpreters. Adobe's strategy paid off: several competitors emerged (including one that gave its product away) and PostScript became a widely used standard

for desktop publishing. Adobe kept a few things proprietary—for instance, techniques for displaying fonts at low resolution—and managed to dominate the high end of the market. Ironically, Adobe's market success was due to its ability to encourage entry by its competitors!

In recent years, many software producers have followed this model. Adobe itself gives away several software products, such as the Adobe Acrobat reader. One of the hot new stock issues of 1995, Netscape Communications Corporation, acquired the lion's share of the Web browser market by giving away its main product, making it a prime example of a company that "lost money on every sale, but made up for it in volume."

36.7 Implications of Network Externalities

The model described above, simple though it is, still yields a number of insights. For example, the critical mass issue is very important: if one user's demand depends on how many other users there are, it is very important to try to stimulate growth early in the life cycle of a product. Nowadays it is quite common to see producers offering very cheap access to a piece of software or a communications service in order to "create a market" where none existed before.

Of course, the critical question is how big does the market have to be before it can take off on its own? Theory can provide little guidance here; everything depends on the nature of the good and the costs and benefits the users face in adopting it.

Another important implication of network externalities is the role played by governmental policy. The Internet is a prime example. The Internet was originally used only by a few small research labs to exchange data files. In the mid-eighties the National Science Foundation used the Internet technology to connect several large universities to 12 supercomputers deployed at various locations. The original vision was that researchers at the universities would send data back and forth to the supercomputers. But a fundamental property of communications networks is that if you are all connected to the same thing, you are all connected to each other. This allowed researchers to send email to each other that had nothing to do with the supercomputers. Once a critical mass of users had been connected to the Internet, its value to new users increased dramatically. Most of these new users had no interest in the supercomputer centers, even though this was the original motivation for providing the network.

EXAMPLE: The Yellow Pages

The familiar local yellow pages phone directories are a $14 billion business.

Ten years ago, it was dominated by telephone companies, who had about 95 percent of the market. Nowadays, they have only 85 percent.

The difference is due to competition. Several small upstarts entered the market in recent years, taking business away from the local phone companies. This is no easy task, as the local business directories exhibit a classic form of network effects: it used to be that consumers all used the yellow page directory provided by their local phone companies, so local merchants were forced to advertise in them.

One upstart, Yellow Book, managed to overcome the network effects by using clever business strategies, such as dramatically undercutting the phone companies' ad rates and distributing its directory just before the local phone company's directory came out. The incumbent providers, thinking that their market was secure, dismissed the threat of aggressive newcomers until it was nearly too late. In the last few years, competition has heated up in this industry. This example goes to show that even industries with strong network effects aren't immune to competitive forces, particularly when the incumbents become overconfident.

EXAMPLE: Radio Ads

The "killer app" for radio in 1910 was ship-to-shore communication. Unfortunately, radio conversations were not private since they were broadcast to anyone who tuned into the right frequency. At some point David Sarnoff recognized that this bug might be a feature and offered a "radio music box" that sent music over the airwaves. His colleagues were skeptical, saying, "The wireless music box has no imaginable commercial value. Who would pay for a message sent to nobody in particular?"

They had a point. Even though people found broadcast radio attractive, the industry did not have a business model. How would they make money?

The magazine *Wireless World* held a contest in which it proposed 5 business models for broadcast radio and people voted for their favorite. The business models were:

- support from general taxation;
- donations from the public;
- radio hardware makers subsidized the production of radio content;
- advertising-supported radio;
- vacuum tube tax used to support content production.

The winner was the last model: a tax on vacuum tubes. Some of the other models are still used today. BBC Radio and TV is supported by a tax

on TVs, and National Public Radio in the United States is supported by donations from the public at large. However, in most countries advertising has become the most popular business model.

In 1922, 30 radio stations were in operation in the United States, and a hundred thousand radios were sold. By the next year there were 556 stations with half-a-million receivers being sold. Radio was on its way.

36.8 Two-sided Markets

A **two-sided market** is a special kind of network effect. Think about the case of a new technology, like Blu-ray DVDs. I don't really care what sort of DVD player other people have, so there is no direct network effect. But there is a kind of *indirect* network effect: the more Blu-ray players that are sold, the more disks that will become available, and the more disks there are available, the more attractive it will be to buy a Blu-ray player.

One can think of many other examples. Consider a new credit card: the more merchants accept the credit card, the more attractive the credit card will be to consumers. But the more consumers who adopt the card, the more attractive it will be to merchants.

Or think about Adobe's PDF platform. The more users who have the PDF-viewing software (Acrobat Reader) the more graphics designers will want to distribute content in this format, and the more demand there will be for the Acrobat Distiller, the software used to create PDF files.

This last example illustrates an important point: it may pay Adobe to give away one product (Reader) in order to encourage demand for another product (Distiller). This is as old as "giving away the razor to sell the blades," but since the combination of digital goods and the Internet has made distribution so cheap, the strategy has become very common.

Apple, for example, sells the popular iPod music player. They also distribute music for the iPod on their iTunes store. According to industry reports, Apple makes very little money on the music—most of the profits go to the music studios. However, from Apple's point of view it makes sense to give away the blades (songs) to sell the razors (iPods).

A Model of Two-sided Markets

Let us generalize the model used in section 34.5 to apply to two-sided markets.

Suppose now that there are two goods. The reservation price for good 1 is v_1 and it takes on values $v_1 = 1, \ldots, 1000$. Similarly the reservation price for good 2 takes on values $v_2 = 1, \ldots, 1000$.

The total value for good 1 depends on how many people adopt good 2, and the total value for good 2 depends on how many people adopt

good 1, so we write $U_1 = v_1 n_2$ and $U_2 = v_2 n_1$. Finally, there are some exogenous prices for supplying good 1 and good 2, which we denote by p_1 and p_2. (You can think of these as costs from a constant-returns to scale production process.)

The marginal adopters for goods 1 and 2 are determined by $\hat{v}_1 n_2 = p_1$ and $\hat{v}_2 n_1 = p_2$. Everybody who has a value higher than $\hat{v}_1$ will purchase good 1, so $n_1 = 1000 - \hat{v}_1$. Similarly, $n_2 = 1000 - \hat{v}_2$.

Putting all these equations together we have

$$\begin{aligned} \hat{v}_1 n_2 &= p_1 \\ \hat{v}_2 n_1 &= p_2 \\ n_1 &= 1000 - \hat{v}_1 \\ n_2 &= 1000 - \hat{v}_2 \end{aligned}$$

Substituting from equations (3) and (4) into (1) and (2) we find

$$\begin{aligned} (1000 - n_1) n_2 &= p_1 \\ (1000 - n_2) n_1 &= p_2 \end{aligned}$$

The first thing we observe is that there is always an equilibrium at $n_1 = n_2 = 0$. If no one purchases good 1, the value of good 2 will be zero and vice versa. To find the other solutions, we plot the two functions. As you might guess, generally there will be two solutions as depicted in the example in Figure 36.5. There is a low-level equilibrium where little is sold of either good and a high-level equilibrium where there are substantial sales of both goods.

The challenge facing a supplier is how to get to the high-level equilibrium. One strategy, mentioned above, is to subsidize the production of one of the goods. Selling one good below cost can make sense if it leads to a larger market and more profits for other goods you sell.

36.9 Rights Management

There is much interest these days in new business models for intellectual property (IP). IP transactions take a variety of forms: books are sold outright and also borrowed from libraries. Videos can either be sold or rented. Some software is licensed for particular uses; other software is sold outright. Shareware is a form of software in which payment is voluntary.

Choosing the terms and conditions under which a piece of intellectual property is offered is a critical business decision. Should you use copy protection? Should you encourage users to share a news item with a friend? Should you sell to individuals or use site license?

Some simple economics helps to understand the relevant issues. Let's consider a purely digital good, such as an online newspaper, so we don't

Equilibrium in a two-sided market. In general there will be three equilibria in a two-sided market.

Figure 36.5

have to worry about marginal cost of production. First let us consider behavior under some default set of terms and conditions. The owner of the digital good will choose a price and, implicitly, a quantity to sell so as to maximize profit:

$$\max_y \; p(y)y \tag{36.7}$$

This yields some optimal (p^*, y^*).

Now the seller of the good contemplates liberalizing terms and conditions: let's say extending a trial period of free use from 1 week to 1 month. This has two effects on the demand curve. First, it increases the value of the product to each of the potential users, shifting the demand curve up. But it also may easily result in less of the item being sold, since some users will find the longer trial period enough to meet their needs.

Let us model this by defining the new amount consumed by $Y = by$, where $b > 1$, and the new demand curve by $P(Y) = ap(Y)$, where $a > 1$.

The new profit-maximization problem now becomes

$$\max_Y \; P(Y)y.$$

Note that we multiply price times the amount sold, y, not the amount consumed, Y.

Applying the definitions $Y = by$ and $P(Y) = ap(Y)$, we can write this as

$$\max_Y \; ap(Y)\frac{Y}{b} = \max_Y \; \frac{a}{b}p(Y)Y.$$

This maximization problem looks like problem (36.7) except for the constant a/b in front of the max. This will not affect the optimal choice, so we can conclude that $Y^* = y^*$.

This simple analysis allows us to make several conclusions:

- The amount of the good consumed, Y^*, is independent of the terms and conditions.
- The amount of the good produced is y^*/b which is less than y^*.
- The profits could go up or down depending on whether a/b is greater or less than 1. Profits go up if the increase in value to the consumers who buy the product compensates for the reduced number of buyers.

EXAMPLE: Video Rental

Video stores can choose the terms and conditions under which they rent videos. The longer you can keep the video, the more valuable it is to you, since you have a longer period of time during which you can watch it. But the longer you keep the video, the less profit the store makes from it, since it is unable to rent it to someone else. The optimal choice for the rental period involves trading off these two effects.

In practice, this has tended to lead to a form of product differentiation. New releases are rented for short periods, since the the profits from other renters being excluded are very substantial. Older videos are rented for longer periods, since there is less cost to the store from the video being unavailable.

36.10 Sharing Intellectual Property

Intellectual property is often shared. Libraries, for example, facilitate the sharing of books. Video stores help people to "share" videos—and charge a price for doing so. Interlibrary loan helps libraries share books among

themselves. Even textbooks—such as the one you are holding—are shared among students from one term to the next via the resale market.

There is considerable debate in the publishing and library communities about the proper role of sharing. Librarians have established an informal "rule of five" for interlibrary loan: an item may be loaned out up to five times before additional royalty payments should be made to the publisher. Publishers and authors have traditionally been unenthusiastic about the resale market for books.

The advent of digital information has made this situation even more acute. Digital information can be perfectly reproduced, and "sharing" can be taken to new extremes. Recently, a well-known country music singer engaged in a vociferous public relations campaign against stores selling used CDs. The problem was that CDs do not deteriorate with replay and it is possible to buy a CD, tape it, and then sell the CD to the used-CD store.

Let us try to construct a model of this sort of sharing phenomenon. We begin with the baseline case in which there is no sharing. In this case a video maker chooses to produce y copies of a video to maximize profit:

$$\max_y \; p(y)y - cy - F. \tag{36.8}$$

As usual, $p(y)$ is the inverse demand function, c is the (constant) marginal cost, and F is the fixed cost. Let the profit maximizing output be denoted by y_n, where the n stands for "no sharing."

Now suppose that a video rental market is allowed. In this case the number of videos *viewed* will be distinct from the number of copies produced. If y is the number of videos produced and each video is shared among k viewers, then the number of viewings will be $x = ky$. (For simplicity we are assuming that *all* copies of the video are rented in this case.)

We need to specify how the consumers sort themselves into the "clubs" that share the videos. The simplest assumption is that the consumers with high values associate with each other, and the consumers with low-values associate with each other. That is, one club consists of consumers with the k highest values, another club consists of the consumers with the next k highest values, and so on. (Other assumptions could be used, but this one gives a very simple analysis.)

If y copies of the video are produced, $x = ky$ copies will be viewed, so the willingness to pay of the marginal individual will be $p(x) = p(ky)$. However, it is clearly the case that there is some inconvenience cost to renting a video rather than owning it yourself. Let us denote this "transactions cost" by t, so that the willingness to pay of the marginal individual becomes $p(x) - t$.

Recall that we have assumed that all copies of the video are shared among k users. Therefore the willingness to pay of a *video store* will just be k times the willingness to pay of the marginal individual. That is, if y

copies are produced, the willingness to pay of the video store will be

$$P(y) = k[p(ky) - t]. \tag{36.9}$$

Equation (36.9) contains the two key effects that arise from sharing: the willingness to pay goes *down* since more videos are viewed than are produced; but the willingness to pay also goes *up* since the cost of a single video is shared among several individuals.

The profit maximization problem of the producer now becomes

$$\max_y \; P(y)y - cy - F,$$

which can be written as

$$\max_y \; k[p(ky) - t]y - cy - F,$$

or

$$\max_y \; p(ky)ky - \left(\frac{c}{k} + t\right) ky - F.$$

Recalling that the number of viewings, x, is related to the number produced, y, via $x = ky$, we can also write the maximization problem as

$$\max_x \; p(x)x - \left(\frac{c}{k} + t\right) x - F.$$

Note that this problem is identical to problem (36.8), with the exception that the marginal cost is now $(c/k + t)$ rather than c.

The close relationship between the two problems is very useful since it allows us to make the following observation: *profits will be larger when rental is possible than when it is not if and only if*

$$\frac{c}{k} + t < c.$$

Rearranging this condition, we have

$$\left(\frac{k}{k+1}\right) t < c.$$

For large k, the fraction on the left is about 1. Hence the critical issue is the relationship between the marginal cost of production, c, and the transactions cost of renting, t.

If the cost of production is large and the cost of renting is small, then the most profitable thing for a producer to do is to produce a few copies, sell them at a high price, and let the consumers rent. On the other hand, if the transactions cost of renting is larger than the cost of production, it is more profitable for a producer to have renting prohibited: since renting is so inconvenient for the consumers, video stores aren't willing to pay much more for the "shared" videos, and so the producer is better off selling.

EXAMPLE: Online Two-sided Markets

There are several examples of two-sided markets on the Internet. For example, eBay serves as a meeting place for those who wish to buy and sell collectibles. If you are selling rare coins, for example, you want to offer them on a market where there are many potential buyers. Similarly, if you are a buyer, you want to go to a market where there are several competing sellers. This two-sided network effect tends to lead to a single meeting place. In the last several years, eBay has expanded beyond collectibles and now sells a variety of mechandise.

Another set of interesting examples are social networking sites, such as Facebook, MySpace, LinkedIn, and others. Participants want to register on sites where their friends have registered. This again leads to a network effect—the largest network attracts the most new participants.

Facebook has seen particularly rapid growth. It was launched in February of 2004 and by December of that year had 1 million active users. In September of 2009, Facebook had over 300 million active users worldwide, according to statistics from its website.

Summary

1. Because information technology works together in systems, it is costly to consumers to switch any one component.

2. If two monopoly providers of complementary products coordinate their price setting, then they will both set their prices lower than they would than if they set them independently.

3. This will increase profit for the two monopolists *and* make consumers better off.

4. There are many ways to achieve this coordination, including merger, negotiation, revenue sharing, and commoditization.

5. In a lock-in equilibrium the discount offered first period is paid for by increased prices in future periods.

6. Network externalities arise when one person's willingness to pay for a good depends on the number of other users of that good.

7. Models with network externalities typically exhibit multiple equilibria. The ultimate outcome often depends on the history of the industry.

8. Rights management involves a tradeoff between increased value and prices versus reduced sales.

9. Information goods like books and videos are often rented or shared as well as purchased. Rental or purchase can be more profitable depending on how transactions costs compare with production costs.

REVIEW QUESTIONS

1. If the cost to a customer from switching long-distance carriers is on the order of $50, how much should a long-distance carrier be willing to pay to acquire a new customer?

2. Describe how the demand for a word processing package might exhibit network externalities.

3. Suppose that the marginal cost of producing an extra video is zero and the transactions cost of renting a video is zero. Does a producer make more money by selling the video or by renting it?

CHAPTER 37

PUBLIC GOODS

In Chapter 35 we argued that for certain kinds of externalities, it was not difficult to eliminate the inefficiencies. In the case of a consumption externality between two people, for example, all one had to do was to ensure that initial property rights were clearly specified. People could then trade the right to generate the externality in the normal way. In the case of production externalities, the market itself provided profit signals to sort out the property rights in the most efficient way. In the case of common property, assigning property rights to someone would eliminate the inefficiency.

Unfortunately, not all externalities can be handled in that manner. As soon as there are more than two economic agents involved things become much more difficult. Suppose, for example, that instead of the two roommates examined in the last chapter, we had *three* roommates—one smoker and two nonsmokers. Then the amount of smoke would be a negative externality for both of the nonsmokers.

Let's suppose that property rights are well defined—say the nonsmokers have the right to demand clean air. Just as before, although they have the *right* to clean air, they also have the right to trade some of that clean air away in return for appropriate compensation. But now there is a problem involved—the nonsmokers have to agree among themselves how much smoke should be allowed and what the compensation should be.

Perhaps one of the nonsmokers is much more sensitive than the other, or one of them is much richer than the other. They may have very different preferences and resources, and yet they both have to reach some kind of agreement to allow for an efficient allocation of smoke.

Instead of roommates, we can think of inhabitants of a whole country. How much pollution should be allowed in the country? If you think that reaching an agreement is difficult with only three roommates, imagine what it is like with millions of people!

The smoke externality with three people is an example of a **public good**—a good that must be provided in the same amount to all the affected consumers. In this case the amount of smoke generated will be the same for all consumers—each person may value it differently, but they all have to face the same amount.

Many public goods are provided by the government. For example, streets and sidewalks are provided by local municipalities. There are a certain number and quality of streets in a town, and everyone has that number available to use. National defense is another good example; there is one level of national defense provided for all the inhabitants of a country. Each citizen may value it differently—some may want more, some may want less—but they are all provided with the same amount.

Public goods are an example of a particular kind of consumption externality: everyone must consume the same amount of the good. They are a particularly troublesome kind of externality, for the decentralized market solutions that economists are fond of don't work very well in allocating public goods. People can't purchase different amounts of public defense; somehow they have to decide on a common amount.

The first issue to examine is what the ideal amount of the public good should be. Then we'll discuss some ways that might be used to make social decisions about public goods.

37.1 When to Provide a Public Good?

Let us start with a simple example. Suppose that there are two roommates, 1 and 2. They are trying to decide whether or not to purchase a TV. Given the size of their apartment, the TV will necessarily go in the living room, and both roommates will be able to watch it. Thus it will be a public good, rather than a private good. The question is, is it worth it for them to acquire the TV?

Let's use w_1 and w_2 to denote each person's initial wealth, g_1 and g_2 to denote each person's contribution to the TV, and x_1 and x_2 to denote each person's money left over to spend on private consumption. The budget constraints are given by

$$x_1 + g_1 = w_1$$
$$x_2 + g_2 = w_2.$$

We also suppose that the TV costs c dollars, so that in order to purchase it, the sum of the two contributions must be at least c:

$$g_1 + g_2 \geq c.$$

This equation summarizes the technology available to provide the public good: the roommates can acquire one TV if together they pay the cost c.

The utility function of person 1 will depend on his or her private consumption, x_1, and the availability of the TV—the public good. We'll write person 1's utility function as $u_1(x_1, G)$, where G will either be 0, indicating no TV, or 1, indicating that a TV is present. Person 2 will have utility function $u_2(x_2, G)$. Each person's private consumption has a subscript to indicate that the good is consumed by person 1 or person 2, but the public good has no subscript. It is "consumed" by both people. Of course, it isn't really consumed in the sense of being "used up"; rather, it is the *services* of the TV that are consumed by the two roommates.

The roommates may value the services of the TV quite differently. We can measure the value that each person places on the TV by asking how much each person would be willing to pay to have the TV available. To do this, we'll use the concept of the **reservation price**, introduced in Chapter 15.

The reservation price of person 1 is the maximum amount that person 1 would be willing to pay to have the TV present. That is, it is that price, r_1, such that person 1 is just indifferent between paying r_1 and having the TV available, and not having the TV at all. If person 1 pays the reservation price and gets the TV, he will have $w_1 - r_1$ available for private consumption. If he doesn't get the TV, he will have w_1 available for private consumption. If he is to be just indifferent between these two alternatives, we must have

$$u_1(w_1 - r_1, 1) = u_1(w_1, 0).$$

This equation defines the reservation price for person 1—the maximum amount that he would be willing to pay to have the TV present. A similar equation defines the reservation price for person 2. Note that in general the reservation price of each person will depend on that person's wealth: the maximum amount that an individual will be *willing* to pay will depend to some degree on how much that individual is *able* to pay.

Recall that an allocation is Pareto efficient if there is no way to make both people better off. An allocation is Pareto *inefficient* if there *is* some way to make both people better off; in this case, we say that a **Pareto improvement** is possible. In the TV problem there are only two sorts of allocations that are of interest. One is an allocation where the TV is not provided. This allocation takes the simple form $(w_1, w_2, 0)$; that is, each person spends his wealth only on his private consumption.

The other kind of allocation is the one where the public good is provided. This will be an allocation of the form $(x_1, x_2, 1)$, where

$$x_1 = w_1 - g_1$$

$$x_2 = w_2 - g_2.$$

These two equations come from rewriting the budget constraints. They say that each individual's private consumption is determined by the wealth that he has left over after making his contribution to the public good.

Under what conditions should the TV be provided? That is, when is there a payment scheme (g_1, g_2) such that both people will be better off having the TV and paying their share than not having the TV? In the language of economics, when will it be a Pareto improvement to provide the TV?

It will be a Pareto improvement to provide the allocation $(x_1, x_2, 1)$ if both people would be better off having the TV provided than not having it provided. This means

$$\begin{aligned} u_1(w_1, 0) &< u_1(x_1, 1) \\ u_2(w_2, 0) &< u_2(x_2, 1). \end{aligned}$$

Now use the definition of the reservation prices r_1 and r_2 and the budget constraint to write

$$\begin{aligned} u_1(w_1 - r_1, 1) = u_1(w_1, 0) &< u_1(x_1, 1) = u_1(w_1 - g_1, 1) \\ u_2(w_2 - r_2, 1) = u_2(w_2, 0) &< u_2(x_2, 1) = u_2(w_2 - g_2, 1). \end{aligned}$$

Looking at the left- and the right-hand sides of these inequalities, and remembering that more private consumption must increase utility, we can conclude that

$$\begin{aligned} w_1 - r_1 &< w_1 - g_1 \\ w_2 - r_2 &< w_2 - g_2, \end{aligned}$$

which in turn implies

$$\begin{aligned} r_1 &> g_1 \\ r_2 &> g_2. \end{aligned}$$

This is a condition that must be satisfied if an allocation $(w_1, w_2, 0)$ is Pareto inefficient: it must be that the contribution that each person is making to the TV is less than his willingness to pay for the TV. If a consumer can acquire the good for less than the maximum that he would be willing to pay, then the acquisition would be to his benefit. Thus the condition that the reservation price exceeds the cost share simply says that a Pareto improvement will result when each roommate can acquire the services of the TV for less than the maximum that he would be willing to pay for it. This is clearly a *necessary* condition for purchase of the TV to be a Pareto improvement.

If each roommate's willingness to pay exceeds his cost share, then the *sum* of the willingnesses to pay must be greater than the cost of the TV:

$$r_1 + r_2 > g_1 + g_2 = c. \tag{37.1}$$

This condition is a *sufficient* condition for it to be a Pareto improvement to provide the TV. If the condition is satisfied, then there will be some payment plan such that both people will be made better off by providing the public good. If $r_1 + r_2 \geq c$, then the total amount that the roommates will be willing to pay is at least as large as the cost of purchase, so they can easily find a payment plan (g_1, g_2) such that $r_1 \geq g_1$, $r_2 \geq g_2$, and $g_1 + g_2 = c$. This condition is so simple that you might wonder why we went through all the detail in deriving it. Well, there are a few subtleties involved.

First, it is important to note that the condition describing when provision of the public good will be a Pareto improvement only depends on each agent's *willingness* to pay and on the total cost. If the sum of the reservation prices exceeds the cost of the TV, then there will always *exist* a payment scheme such that both people will be better off having the public good than not having it.

Second, whether or not it is Pareto efficient to provide the public good will, in general, depend on the initial distribution of wealth (w_1, w_2). This is true because, in general, the reservation prices r_1 and r_2 will depend on the distribution of wealth. It perfectly possible that for some distributions of wealth $r_1 + r_2 > c$, and for other distributions of wealth $r_1 + r_2 < c$.

To see how this can be, imagine a situation where one roommate really loves the TV and the other roommate is nearly indifferent about acquiring it. Then if the TV-loving roommate had all of the wealth, he would be willing to pay more than the cost of the TV all by himself. Thus it would be a Pareto improvement to provide the TV. But if the indifferent roommate had all of the wealth, then the TV lover wouldn't have much money to contribute toward the TV, and it would be Pareto efficient *not* to provide the TV.

Thus, in general, whether or not the public good should be provided will depend on the distribution of wealth. But in specific cases the provision of the public good may be independent of the distribution of wealth. For example, suppose that the preferences of the two roommates were quasilinear. This means that the utility functions take the form

$$u_1(x_1, G) = x_1 + v_1(G)$$
$$u_2(x_2, G) = x_2 + v_2(G),$$

where G will be 0 or 1, depending on whether or not the public good is available. For simplicity, suppose that $v_1(0) = v_2(0) = 0$. This says that no TV provides zero utility from watching TV.[1]

In this case the definitions of the reservation prices become

$$u_1(w_1 - r_1, 1) = w_1 - r_1 + v_1(1) = u_1(w_1, 0) = w_1$$
$$u_2(w_2 - r_2, 1) = w_2 - r_2 + v_2(1) = u_2(w_2, 0) = w_2,$$

[1] Perhaps watching TV should be assigned a negative utility.

which implies that the reservation prices are given by

$$r_1 = v_1(1)$$
$$r_2 = v_2(1).$$

Thus the reservation prices are independent of the amount of wealth, and hence the optimal provision of the public good will be independent of wealth, at least over some range of wealths.[2]

37.2 Private Provision of the Public Good

We have seen above that acquiring the TV will be Pareto efficient for the two roommates if the sum of their willingnesses to pay exceeds the cost of providing the public good. This answers the question about efficient allocation of the good, but it does not necessarily follow that they will actually decide to acquire the TV. Whether they actually decide to acquire the TV depends on the particular method they adopt to make joint decisions.

If the two roommates cooperate and truthfully reveal how much they value the TV, then it should not be difficult for them to agree on whether or not they should buy the TV. But under some circumstances, they may not have incentives to tell the truth about their values.

For example, suppose that each person valued the TV equally, and that each person's reservation price was greater than the cost, so that $r_1 > c$ and $r_2 > c$. Then person 1 might think that if he said he had 0 value for the TV, the other person would acquire it anyway. But person 2 could reason the same way! One can imagine other situations where both people would refuse to contribute in the hopes that the other person would go out and unilaterally purchase the TV.

In this kind of situation, economists say that the people are attempting to **free ride** on each other: each person hopes that the other person will purchase the public good on his own. Since each person will have full use of the services of the TV if it is acquired, each person has an incentive to try to pay as little as possible toward the provision of the TV.

37.3 Free Riding

Free riding is similar, but not identical, to the prisoner's dilemma that we examined in Chapter 29. To see this, let us construct a numerical example of the TV problem described above. Suppose that each person has a wealth of \$500, that each person values the TV at \$100, and that the cost of the

[2] Even this will only be true for some ranges of wealth, since we must always require that $r_1 \leq w_1$ and $r_2 \leq w_2$—i.e., the willingness to pay is less than the ability to pay.

TV is $150. Since the sum of the reservation prices exceeds the cost, it is Pareto efficient to buy the TV.

Let us suppose that there is no way for one of the roommates to exclude the other one from watching the TV and that each roommate will decide independently whether or not to buy the TV. Consider the decision of one of the roommates, Player A. If he buys the TV, he gets benefits of $100 and pays a cost of $150, leaving him with net benefits of −50. However, if Player A buys the TV, Player B gets to watch it for free, which gives B a benefit of $100. The payoffs to the game are depicted in Table 37.1.

Free riding game matrix. **Table 37.1**

		Player B	
		Buy	Don't buy
Player A	Buy	–50, –50	–50, 100
	Don't buy	100, –50	0, 0

The dominant strategy equilibrium for this game is for neither player to buy the TV. If player A decides to buy the TV, then it is in player B's interest to free ride: to watch the TV but not contribute anything to paying for it. If player A decides not to buy, then it is in player B's interest not to buy the TV either. This is similar to the prisoners' dilemma, but not exactly the same. In the prisoners' dilemma, the strategy that maximizes the sum of the players' utilities is for each player to make the *same* choice. Here the strategy that maximizes the sum of the utilities is for just one of the players to buy the TV (and both players to watch it).

If Player A buys the TV and both players watch it, we can construct a Pareto improvement simply by having Player B make a "sidepayment" to Player A. For example, if Player B gives Player A $51, then both players will be made better off when Player A buys the TV. More generally, any payment between $50 and $100 will result in a Pareto improvement for this example.

In fact, this is probably what would happen in practice: each player would contribute some fraction of the cost of the TV. This public goods problem is relatively easy to solve, but more difficult free riding problems can arise in the sharing of other household public goods. For example, what about cleaning the living room? Each person may prefer to see the living room clean and is willing to do his part. But each may also be tempted to free ride on the other—so that neither one ends up cleaning the room, with the usual untidy results.

The situation becomes even worse if there are more than just two people involved—since there are more people on whom to free ride! Letting the other guy do it may be optimal from an *individual* point of view, but it is Pareto inefficient from the viewpoint of society as a whole.

37.4 Different Levels of the Public Good

In the above example, we had an either/or decision: either provide the TV or not. But the same kind of phenomena occurs when there is a choice of *how much* of the public good to provide. Suppose, for example, that the two roommates have to decide how much money to spend on the TV. The more money they decide to spend, the better the TV they can get.

As before we'll let x_1 and x_2 measure the private consumption of each person and g_1 and g_2 be their contributions to the TV. Let G now measure the "quality" of the TV they buy, and let the cost function for quality be given by $c(G)$. This means that if the two roommates want to purchase a TV of quality G, they have to spend $c(G)$ dollars to do so.

The constraint facing the roommates is that the total amount that they spend on their public and private consumption has to add up to how much money they have:

$$x_1 + x_2 + c(G) = w_1 + w_2.$$

A Pareto efficient allocation is one where consumer 1 is as well-off as possible given consumer 2's level of utility. If we fix the utility of consumer 2 at $\overline{u}_2$, we can write this problem as

$$\max_{x_1, x_2, G} u_1(x_1, G)$$

$$\text{such that } u_2(x_2, G) = \overline{u}_2$$

$$x_1 + x_2 + c(G) = w_1 + w_2.$$

It turns out that the appropriate optimality condition for this problem is that the *sum* of the absolute values of the marginal rates of substitution between the private good and the public good for the two consumers equals the marginal cost of providing an extra unit of the public good:

$$|MRS_1| + |MRS_2| = MC(G)$$

or, spelling out the definitions of the marginal rates of substitution,

$$\left|\frac{\Delta x_1}{\Delta G}\right| + \left|\frac{\Delta x_2}{\Delta G}\right| = \frac{MU_G}{MU_{x_1}} + \frac{MU_G}{MU_{x_2}} = MC(G).$$

In order to see why this must be the right efficiency condition, let us apply the usual trick and think about what would be the case if it were

violated. Suppose, for example, that the sum of the marginal rates of substitution were less than the marginal cost: say $MC = 1$, $|MRS_1| = 1/4$, and $|MRS_2| = 1/2$. We need to show that there is some way to make both people better off.

Given his marginal rate of substitution, we know that person 1 would be willing to accept 1/4 more dollars of the private good for the loss of 1 dollar of the public good (since both goods cost \$1 per unit). Similarly, person 2 would accept 1/2 more dollars of the private good for a 1-dollar decrease in the public good. Suppose we reduce the amount of the public good and offer to compensate both individuals. When we reduce the public good by one unit we save a dollar. After we pay each individual the amount he requires to allow this change ($3/4 = 1/4 + 1/2$), we find that we still have 1/4 of a dollar left over. This remaining money could be shared between the two individuals, thereby making them both better off.

Similarly, if the sum of the marginal rates of substitution were greater than 1, we could increase the amount of the public good to make them both better off. If $|MRS_1| = 2/3$ and $|MRS_2| = 1/2$, say, this means that person 1 would give up 2/3 of a dollar of private consumption to get 1 unit more of the public good and person 2 would give up 1/2 of a dollar of private consumption to get 1 unit more of the public good. But if person 1 gave up his 2/3 units, and person 2 gave up his 1/2 unit, we would have more than enough to produce the extra unit of the public good, since the marginal cost of providing the public good is 1. Thus we could give the left-over amount back to both people, thereby making them both better off.

What does the condition for Pareto efficiency mean? One way to interpret it is to think of the marginal rate of substitution as measuring the *marginal* willingness to pay for an extra unit of the public good. Then the efficiency condition just says that the *sum* of the marginal willingnesses to pay must equal the marginal cost of providing an extra unit of the public good.

In the case of a discrete good that was either provided or not provided, we said that the efficiency condition was that the sum of the willingnesses to pay should be at least as large as the cost. In the case we're considering here, where the public good can be provided at different levels, the efficiency condition is that the sum of the *marginal* willingnesses to pay should *equal* the marginal cost at the optimal amount of the public good. For whenever the sum of the marginal willingnesses to pay for the public good exceeds the marginal cost, it is appropriate to provide more of the public good.

It is worthwhile comparing the efficiency condition for a public good to the efficiency condition for a private good. For a private good, each person's marginal rate of substitution must equal the marginal cost; for a public good, the *sum* of the marginal rates of substitution must equal the marginal cost. In the case of a private good, each person can consume a different amount of the private good, but they all must value it the same at the margin—otherwise they would want to trade. In the case of a public good, each person must consume the same amount of the public good, but

they can all value it differently at the margin.

We can illustrate the public good efficiency condition in Figure 37.1. We simply draw each person's MRS curve and then add them vertically to get the sum of the MRS curves. The efficient allocation of the public good will occur where the sum of the MRSs equals the marginal cost, as illustrated in Figure 37.1.

Figure 37.1 **Determining the efficient amount of a public good**. The sum of the marginal rates of substitution must equal the marginal cost.

37.5 Quasilinear Preferences and Public Goods

In general, the optimal amount of the public good will be different at different allocations of the private good. But if the consumers have quasilinear preferences it turns out that there will be a unique amount of the public good supplied at every efficient allocation. The easiest way to see this is to think about the kind of utility function that represents quasilinear preferences.

As we saw in Chapter 4, quasilinear preferences have a utility representation of the form: $u_i(x_i, G) = x_i + v_i(G)$. This means that the marginal

utility of the private good is always 1, and thus the marginal rate of substitution between the private and the public good—the ratio of the marginal utilities—will depend only on G. In particular:

$$|MRS_1| = \frac{\Delta u_1(x_1, G)/\Delta G}{\Delta u_1/\Delta x_1} = \frac{\Delta v_1(G)}{\Delta G}$$
$$|MRS_2| = \frac{\Delta u_2(x_2, G)/\Delta G}{\Delta u_2/\Delta x_2} = \frac{\Delta v_2(G)}{\Delta G}.$$

We already know that a Pareto efficient level of the public good must satisfy the condition

$$|MRS_1| + |MRS_2| = MC(G).$$

Using the special form of the MRSs in the case of quasilinear utility, we can write this condition as

$$\frac{\Delta v_1(G)}{\Delta G} + \frac{\Delta v_2(G)}{\Delta G} = MC(G).$$

Note that this equation determines G without any reference to x_1 or x_2. Thus there is a unique efficient level of provision of the public good.

Another way to see this is to think about the behavior of the indifference curves. In the case of quasilinear preferences, all of the indifference curves are just shifted versions of each other. This means, in particular, that the slope of the indifference curves—the marginal rate of substitution—doesn't change as we change the amount of the private good. Suppose that we find one efficient allocation of the public and private goods, where the sum of the absolute value of the MRSs equals $MC(G)$. Now if we take some amount of the private good away from one person and give it to another, the slopes of both indifference curves stay the same, so the sum of the absolute value of the MRSs is still equal to $MC(G)$ and we have another Pareto efficient allocation.

In the case of quasilinear preferences, all Pareto efficient allocations are found by just redistributing the private good. The amount of the public good stays fixed at the efficient level.

EXAMPLE: Pollution Revisited

Recall the model of the steel firm and the fishery described in Chapter 35. There we argued that the efficient provision of pollution was one which internalized the pollution costs borne by the steel firm and the fishery. Suppose now that there are two fisheries, and that the amount of pollution produced by the steel firm is a public good. (Or, perhaps more appropriately, is a public bad!)

Then the efficient provision of pollution will involve maximizing the sum of the profits of all three firms—that is, minimizing the total social cost of the pollution. Formally, let $c_s(s, x)$ be the cost to the steel firm of producing s units of steel and x units of pollution, and write $c_f^1(f_1, x)$ for the costs for firm 1 to catch f_1 fish when the pollution level is x, and $c_f^2(f_2, x)$ as the analogous expression for firm 2. Then to compute the Pareto efficient amount of pollution, we maximize the sum of the three firms' profits:

$$\max_{s, f_1, f_2, x} \; p_s s + p_f f_1 + p_f f_2 - c_s(s, x) - c_f^1(f_1, x) - c_f^2(f_2, x).$$

The interesting effect for our purposes is the effect on aggregate profits of increasing pollution. Increasing pollution lowers the cost of producing steel but raises the costs of producing fish for each of the fisheries. The appropriate optimality condition from the profit-maximization problem is

$$\frac{\Delta c_s(\hat{s}, \hat{x})}{\Delta x} + \frac{\Delta c_f^1(\hat{f}_1, \hat{x})}{\Delta x} + \frac{\Delta c_f^2(\hat{f}_2, \hat{x})}{\Delta x} = 0,$$

which simply says that the *sum* of the marginal costs of pollution over the three firms should equal zero. Just as in the case of a public consumption good, it is the *sum* of the marginal benefits or costs over the economic agents that is relevant for determining the Pareto efficient provision of a public good.

37.6 The Free Rider Problem

Now that we know what the Pareto efficient allocations of public goods are, we can turn our attention to asking how to get there. In the case of private goods with no externalities we saw that the market mechanism will generate an efficient allocation. Will the market work in the case of public goods?

We can think of each person as having some endowment of a private good, w_i. Each person can spend some fraction of this private good on his own private consumption, or he or she can contribute some of it to purchase the public good. Let's use x_1 for 1's private consumption, and let g_1 denote the amount of the public good he buys, and similarly for person 2. Suppose for simplicity that $c(G) \equiv G$, which implies that the marginal cost of providing a unit of the public good is constant at 1. The total amount of the public good provided will be $G = g_1 + g_2$. Since each person cares about the *total* amount of the public good provided, the utility function of person i will have the form $u_i(x_i, g_1 + g_2) = u_i(x_i, G)$.

In order for person 1 to decide how much he should contribute to the public good, he has to have some forecast of how much person 2 will contribute. The simplest thing to do here is to adopt the Nash equilibrium

model described in Chapter 29, and suppose that person 2 will make some contribution $\overline{g}_2$. We assume that person 2 also makes a guess about person 1's contribution, and we look for an equilibrium where each person is making an optimal contribution given the other person's behavior.

Thus person 1's maximization problem takes the form

$$\max_{x_1, g_1} u_1(x_1, g_1 + \overline{g}_2)$$

$$\text{such that } x_1 + g_1 = w_1.$$

This is just like an ordinary consumer maximization problem. The optimization condition is therefore the same: if both people purchase both goods the marginal rate of substitution between the public and the private goods should be 1 for each consumer:

$$|MRS_1| = 1$$

$$|MRS_2| = 1.$$

However, we have to be careful here. It is true that if person 2 purchases any amount of the public good at all, he will purchase it until the marginal rate of substitution equals one. But it can easily happen that person 2 decides that the amount already contributed by person 1 is sufficient and that it would therefore be unnecessary for him to contribute anything toward the public good at all.

Formally, we are assuming that the individuals can only make positive contributions to the public good—they can put money into the collection plate, but they can't take money out. Thus there is an extra constraint on each person's contributions, namely, that $g_1 \geq 0$ and $g_2 \geq 0$. Each person can only decide whether or not he wants to *increase* the amount of the public good. But then it may well be that one person decides that the amount provided by the other is just fine and would prefer to make no contribution at all.

A case like this is depicted in Figure 37.2. Here we have illustrated each person's private consumption on the horizontal axis and his or her public consumption on the vertical axis. The "endowment" of each person consists of his or her wealth, w_i, along with the amount of the public good contribution of the *other* person—since this is how much of the public good will be available if the person in question decides not to contribute. Figure 37.2A shows a case where person 1 is the only contributor to the public good, so that $g_1 = G$. If person 1 contributes G units to the public good, then person 2's endowment will consist of her private wealth, w_2, and the amount of the public good G—since person 2 gets to consume the public good whether or not she contributes to it. Since person 2 cannot reduce the amount of the public good, but can only increase it, her budget constraint is the bold line in Figure 37.2B. Given the shape of 2's indifference curve,

Figure 37.2 **The free rider problem**. Person 1 contributes while person 2 free rides.

it is optimal from her point of view to free ride on 1's contribution and simply consume her endowment, as depicted.

This is an example where person 2 is free riding on person 1's contribution to the public good. Since a public good is a good that everyone must consume in the same amount, the provision of a public good by any one person will tend to reduce the other peoples' provision. Thus in general there will be too little of the public good supplied in a voluntary equilibrium, relative to an efficient provision of the public good.

37.7 Comparison to Private Goods

In our discussion of private goods, we were able to show that a particular social institution—the competitive market—was capable of achieving a Pareto efficient allocation of private goods. Each consumer deciding for himself or herself how much to purchase of various goods would result in a pattern of consumption that was Pareto efficient. A major assumption in this analysis was that an individual's consumption did not affect other people's utility—that is, that there were no consumption externalities. Thus each person optimizing with respect to his or her own consumption was sufficient to achieve a kind of social optimum.

The situation is radically different with respect to public goods. In this case, the utilities of the individuals are inexorably linked since everyone is required to consume the same amount of the public good. In this case

the market provision of *public* goods would be very unlikely to result in a Pareto efficient provision.

Indeed, for the most part we use *different* social institutions to determine the provision of public goods. Sometimes people use a **command mechanism**, where one person or small group of people determines the amount of various public goods that will be provided by the populace. Other times people use a **voting system** where individuals vote on the provision of public goods. One can well ask the same sorts of questions about voting, or other social mechanisms for decision making, that we asked about the private market: are they capable of achieving a Pareto efficient allocation of public goods? Can any Pareto efficient allocation of public goods be achieved by such mechanisms? A complete analysis of these questions is beyond the scope of this book, but we will be able to shed a little light on how some methods work below.

37.8 Voting

Private provision of a public good doesn't work very well, but there are several other mechanisms for social choice. One of the most common mechanisms in democratic countries is **voting**. Let's examine how well it works for the provision of public goods.

Voting isn't very interesting in the case of two consumers, so we will suppose that we have n consumers. Furthermore, so as not to worry about ties, we'll suppose that n is an odd number. Let's imagine that the consumers are voting about the size of some public good—say the magnitude of expenditures on public defense. Each consumer has a most-preferred level of expenditure, and his valuation of other levels of expenditure depends on how close they are to his preferred level of expenditure.

The first problem with voting as a way of determining social outcomes has already been examined in Chapter 34. Suppose that we are considering three levels of expenditure, A, B, and C. It is perfectly possible that there is a majority of the consumers who prefer A to B, a majority who prefer B to C . . . and a majority who prefer C to A!

Using the terminology of Chapter 34, the social preferences generated by these consumers are not transitive. This means that the outcome of voting on the level of public good may not be well defined—there is always a level of expenditure that beats every expenditure. If a society is allowed to vote many times on an issue, this means that it may "cycle" around various choices. Or if a society votes only once on an issue, the outcome depends on the order in which the choices are presented.

If first you vote on A versus B and then on A versus C, C will be the outcome. But if you vote on C versus A and then C versus B, B will be the outcome. You can get any of the three outcomes by choosing how the alternatives are presented!

The "paradox of voting" described above is disturbing. One natural thing to do is to ask what restrictions on preferences will allow us to rule it out; that is, what form must preferences have so as to ensure that the kinds of cycles described above cannot happen?

Let us depict the preferences of consumer i by a graph like those in Figure 37.3, where the height of the graph illustrates the value or the net utility for different levels of the expenditure on the public good. The term "net utility" is appropriate since each person cares both about the level of the public good, and the amount that he has to contribute to it. Higher levels of expenditure mean more public goods but also higher taxes in order to pay for those public goods. Thus it is reasonable to assume that the net utility of expenditure on the public good rises at first due to the benefits of the public good but then eventually falls, due to the costs of providing it.

One restriction on preferences of this sort is that they be **single-peaked**. This means that preferences must have the shape depicted in Figure 37.3A rather than that depicted in Figure 37.3B. With single-peaked preferences, the net utility of different levels of expenditure rises until the most-preferred point and then falls, as it does in Figure 37.3A; it never goes up, down, and then up again, as it does in Figure 37.3B.

Figure 37.3 **Shapes of preferences**. Single-peaked preferences are shown in panel A and multiple peaked preferences in panel B.

If each individual has single-peaked preferences, then it can be shown that the social preferences revealed by majority vote will never exhibit the

kind of intransitivity we described above. Accepting this result for the moment, we can ask which level of expenditure will be chosen if everyone has single-peaked preferences. The answer turns out to be the **median expenditure**—that expenditure such that one-half of the population wants to spend more, and one-half wants to spend less. This result is reasonably intuitive: if more than one-half wanted more expenditure on the public good, they would vote for more, so the only possible equilibrium voting outcome is when the votes for increasing and decreasing expenditure on the public good are just balanced.

Will this be an efficient level of the public good? In general, the answer is no. The median outcome just means that half the population wants more and half wants less; it doesn't say anything about *how much more* they want of the public good. Since efficiency takes this kind of information into account, voting will not in general lead to an efficient outcome.

Furthermore, even if peoples' true preferences are single-peaked, so that voting may lead to a reasonable outcome, individuals may choose to misrepresent their true preferences when they vote. Thus people will have an incentive to vote differently than their true preferences would indicate in order to manipulate the final outcome.

EXAMPLE: Agenda Manipulation

We have seen that the outcome of a sequence of votes may depend on the order in which the votes are taken. Experienced politicians are well aware of this possibility. In the U.S. Congress, amendments to a bill must be voted on before the bill itself, and this provides a commonly used way to influence the legislative process.

In 1956 the House of Representatives considered a bill calling for Federal aid to school construction. One representative offered an amendment requiring that the bill would only provide Federal aid to states with integrated schools. There were three more-or-less equally sized groups of representatives with strongly held views on this issue.

- Republicans. They were opposed to Federal aid to education, but preferred the amended bill to the original. Their ranking of the alternatives was no bill, amended bill, original bill.

- Northern Democrats. They wanted Federal aid to education and supported integrated schools, so they ranked the alternatives amended bill, original bill, no bill.

- Southern Democrats. This group wanted Federal aid to education, but would not get any aid under the amended bill due to the segregated schools in the South. Their ranking was original bill, no bill, amended bill.

In the vote on the amendment, the Republicans and the Northern Democrats were in the majority, thereby substituting the amended bill for the original. In the vote on the amended bill, the Republicans and the Southern Democrats were in the majority, and the amended bill was defeated. However, before being amended the original bill had a majority of the votes!

37.9 The Vickrey-Clarke-Groves Mechanism

Let us think about the public good problem in a very general framework. The goal is to choose some outcome (for example, whether or not to provide a streetlight) so as to maximize the sum of utilities of the agents involved. The challenge is to determine just what those individual utility functions are, since consumers may not have good incentives to report true values.

In the simplest case the choice might be a zero-one decision: if $x = 1$ the streetlight is built, if $x = 0$ it is not. In a more general case, the choice might be how much of something to provide—how many streetlights, or how bright they are, or where they are located. We will use x to represent the possible choices, whatever they may be. We suppose that there are n agents, and let $u_i(x)$ be the utility of agent i. The goal is to choose x to maximize the sum of the agents' utility, $\sum_i u_i(x)$.

This would be easy if the decision maker knew the utility functions. Unfortunately, in any realistic situation, the decision maker won't know this. And, as we've seen, the agents may well have an incentive to misrepresent their true utility functions.

Somewhat surprisingly, there is a clever way to get the agents to tell the truth and achieve an efficient outcome. This **economic mechanism** is known as the **Vickrey-Clarke-Groves mechanism**, or **VCG mechanism**.

Groves Mechanism

We will describe the VCG mechanism in two stages. First, we describe what is known as a **Groves mechanism**.

1. The center asks each agent i to report how much he is willing to pay to have x units of the public good provided. We denote this reported utility for x units of the public good by $r_i(x)$.

2. The center chooses the level of the public good x^* that maximizes the sum of the reported utilities, $R = \sum_{i=1}^{n} r_i(x)$.

3. Each agent i receives a sidepayment which is the sum of the reported utilities of everyone else, evaluated at the level of x determined in step 2. Denote this sidepayment by $R_i = \sum_{j \neq i} r_j(x^*)$.

It turns out that in this mechanism it is a **dominant strategy** for each agent to report his true utility function. To see why, consider the total payoff to agent i, which is his utility plus his sidepayment

$$u_i(x) + \sum_{j \neq i} r_j(x).$$

Note that agent i cares about his *true* utility function but his sidepayment depends on the sum of the others' *reported* utility functions.

Agent i recognizes that the decision maker will maximize the sum of utilities using his *reported* utility,

$$r_i(x) + \sum_{j \neq i} r_j(x).$$

However, agent i wants the decision maker to maximize his own true utility plus sidepayment,

$$u_i(x) + \sum_{j \neq i} r_j(x).$$

Agent i can ensure that the decision maker makes a choice that will maximize this expression by reporting his true utility; that is, by setting $r_i(x) = u_i(x)$.

The Groves mechanism essentially "internalizes the externality" among the agents. It makes each agent face the costs and benefits that his report imposes on the other agents. Each agent wants to report his true utility, since that is what he wants to be maximized.

The VCG Mechanism

The trouble with the Groves mechanism alone is it is potentially very costly: the center has to pay every agent an amount equal to the sum of the others' reported utilities. How can the magnitude of the sidepayments be reduced?

One important observation is that we can impose a "tax" on each agent as long as this tax is independent of the agent's choice. If the tax is independent of i's choice, then it can't affect his decision.[3] We will choose the tax in a way that guarantees that the net payments the center will receive are either positive or zero. Thus the center will always have at least as much money as necessary to pay for the public good.

A particularly convenient tax is to charge agent i an amount equal to the *maximum* sum of reported utilities excluding agent i. That is, we charge

[3] This is where the quasilinear assumption about utility is important.

each agent the sum of the reported utilities that would occur if he were not present. The net tax imposed on agent i is then

$$W_i - R_i = \sum_{j \neq i} r_j(x) - \max_z \sum_{j \neq i} r_j(z).$$

Note that this number is either positive or zero. Why? Because the *maximum* sum of the $n-1$ reported utilities has to be larger than any other value for that sum.

What we are computing here is the difference between what would happen with agent i present, and what would happen with him absent. Thus it measures the net cost that agent i imposes on the other agents. As long as i faces the cost that he imposes on the other agents, he will have appropriate incentives to report his true utility.

Now we can complete the description of the VCG mechanism. We use steps 1 and 2 above, but then substitute the following steps for step 3 above.

3. The center also calculates the outcome that maximizes the sum of the $n-1$ reported utilities if agent $1, 2, \ldots, n$ were not present. Let W_i be the maximum sum of reported utilities that results without agent i.

4. Each agent i pays a tax equal to $W_i - R_i$.

37.10 Examples of VCG

The discussion in the last section was admittedly abstract, so it is helpful to examine some specific cases.

Vickrey Auction

The first case we look at is the **Vickrey auction**, as described in Chapter 18. Here the outcome is simple: which person should get the item being auctioned. Let $v_1 > v_2$ be the true values of two bidders and $r_1 > r_2$ be the reported values.

If agent 1 is present, he gets a utility of v_1. If he is absent, the item is awarded to the other agent so agent 1's total payoff is $v_1 - r_2$. Agent 2 gets a payoff of zero no matter what. Each agent has an incentive to report its true value, so we end up with the optimal outcome.

Clarke-Groves Mechanism

The next example is a public goods problem along the lines of the TV-buying game described in Table 37.1. As in that example, suppose that

there are two roommates who are trying to decide whether they will buy a TV. Let c_i be how much agent i will pay if the TV is purchased. Since the total cost of the TV is \$150, we must have $c_1 + c_2 = 150$.

According to the VCG mechanism, each agent reports a value for the TV, denoted by r_i. If $r_1 + r_2 > 150$ the TV will be purchased and the agents will make the payments according to the mechanism. Let $x = 1$ if the TV is purchased and $x = 0$ if it is not.

Before we look at the VCG mechanism, let us think about what would happen if we followed a naive mechanism: ask each agent to report his value and then acquire the TV if the sum of the reported values exceeds the cost of the TV.

Suppose person 1's value exceeds his cost share, so that $v_1 - c_1 > 0$. Then person 1 may as well report a million dollars; this will ensure the TV get purchased, which is what he wants to see. On the other hand if $v_1 < c_1$ person 1 may as well report a negative million dollars.

The problem is that each agent, acting independently, has no reason to take into account the other agent's values. The agents have a strong incentive to exaggerate their reported values one way or the other.

Let's see how the VCG mechanism solves this problem. The payoff to agent 1 is

$$(v_1 - c_1)x + (r_2 - c_2)x - max_y(r_2 - c_2)y.$$

The first term is his net utility from the TV: the value to him minus the cost he has to pay. The second term is the reported net utility to his roommate. The last term is the maximum utility his roommate would get if agent 1 were not present. Since agent 1 can't influence this, we can just ignore it for now.

Rearranging the first 2 terms we have agent's payoff as

$$[(v_1 + r_2) - (c_1 + c_2)]x.$$

If this is positive then he can ensure the TV is purchased if he reports $r_1 = v_1$, since then the sum of the *reported* values will exceed the total cost. If this is negative he can ensure that the TV is not purchased by reporting $r_1 = v_1$. Either way, it is optimal to report the true value. The same thing is true for agent 2. If both report the truth, the TV will be purchased only when $v_1 + v_2 > 150$, which is the optimal thing to do.

Note that agent i will have to make a payment only if he changes the social decision. In this case we say agent i is **pivotal**. The amount of the payment a pivotal agent makes is simply the cost that he imposes on the other agents.

37.11 Problems with the VCG

The VCG mechanism leads to truthtelling and leads to the optimal level of the public good. However, it is not without problems.

The first problem is that it only works with quasilinear preferences. This is because we can't have the amount that you have to pay influence your demand for the public good. It is important that there is a unique optimal level of the public good.

The second problem is that the VCG mechanism doesn't really generate a Pareto efficient outcome. The level of the public good will be optimal, but the private consumption could be greater. This is because of the tax collection. Remember that in order to have the correct incentives, the pivotal people must actually pay some taxes that reflect the harm that they do to the other people. And these taxes cannot go to anybody else involved in the decision process, since that might affect their decisions. The taxes have to disappear from the system. And that's the problem—if the taxes actually have to be paid, the private consumption will end up being lower than it could be otherwise, and therefore be Pareto inefficient.

However, the taxes only have to be paid if someone is pivotal. If there are many people involved in the decision, the probability that any one person is pivotal may not be very large; thus the tax collections might typically be expected to be rather small.

A third problem with VCG is that it is susceptible to collusion. Consider, for example, the public goods problem described above. Suppose that there are 3 roommates participating in the TV auction, but two of them collude. The colluders agree to each state $1 million as their net benefit from the TV. This ensures that the TV will be purchased but since neither of the agents is pivotal (i.e., neither of the colluding agents changed the decision) then neither one has to pay the tax.

The final problem concerns the equity and efficiency tradeoff inherent in the VCG mechanism. Since the payment scheme must be fixed in advance, there will generally be situations where some people will be made worse off by providing the public good, even though the Pareto efficient *amount* of the public good will be provided. To say that it is Pareto efficient to provide the public good is to say that there is *some* payment scheme under which everyone is better off having the public good provided than not having it. But this doesn't mean that for an *arbitrary* payment scheme everyone will be better off. The VCG mechanism ensures that if everyone *could* be better off having the good provided, then it will be provided. But that doesn't imply that everyone will actually be better off.

It would be nice if there were a scheme that determined not only whether or not to provide the public good, but also a Pareto efficient way to pay for it—that is, a payment plan that makes everyone better off. However, it does not appear that such a general plan is available.

Summary

1. Public goods are goods for which everyone must "consume" the same

amount, such as national defense, air pollution, and so on.

2. If a public good is to be provided in some fixed amount or not provided at all, then a necessary and sufficient condition for provision to be Pareto efficient is that the sum of the willingnesses to pay (the reservation prices) exceeds the cost of the public good.

3. If a public good can be provided in a variable amount, then the necessary condition for a given amount to be Pareto efficient is that the sum of the marginal willingnesses to pay (the marginal rates of substitution) should equal the marginal cost.

4. The free rider problem refers to the temptation of individuals to let others provide the public goods. In general, purely individualistic mechanisms will not generate the optimal amount of a public good because of the free rider problem.

5. Various collective decision methods have been proposed to determine the supply of a public good. Such methods include the command mechanism, voting, and the VCG mechanism.

REVIEW QUESTIONS

1. Suppose that 10 people live on a street and that each of them is willing to pay \$2 for each extra streetlight, regardless of the number of streetlights provided. If the cost of providing x streetlights is given by $c(x) = x^2$, what is the Pareto efficient number of streetlights to provide?

APPENDIX

Let's solve the maximization problem that determines the Pareto efficient allocations of the public good:

$$\max_{x_1, x_2, G} u_1(x_1, G)$$
$$\text{such that } u_2(x_2, G) = \overline{u}_2$$
$$x_1 + x_2 + c(G) = w_1 + w_2.$$

We set up the Lagrangian:

$$L = u_1(x_1, G) - \lambda[u_2(x_2, G) - \overline{u}_2] - \mu[x_1 + x_2 + c(G) - w_1 - w_2]$$

and differentiate with respect to x_1, x_2, and G to get

$$\frac{\partial L}{\partial x_1} = \frac{\partial u_1(x_1, G)}{\partial x_1} - \mu = 0$$
$$\frac{\partial L}{\partial x_2} = -\lambda \frac{\partial u_2(x_2, G)}{\partial x_2} - \mu = 0$$
$$\frac{\partial L}{\partial G} = \frac{\partial u_1(x_1, G)}{\partial G} - \lambda \frac{\partial u_2(x_2, G)}{\partial G} - \mu \frac{\partial c(G)}{\partial G} = 0.$$

If we divide the third equation by μ and rearrange, we get

$$\frac{1}{\mu}\frac{\partial u_1(x_1, G)}{\partial G} - \frac{\lambda}{\mu}\frac{\partial u_2(x_2, G)}{\partial G} = \frac{\partial c(G)}{\partial G}. \tag{37.2}$$

Now solve the first equation for μ to get

$$\mu = \frac{\partial u_1(x_1, G)}{\partial x_1},$$

and solve the second equation for μ/λ to get

$$\frac{\mu}{\lambda} = -\frac{\partial u_2(x_2, G)}{\partial x_2}.$$

Substitute these two equations into equation (37.2) to find

$$\frac{\partial u_1(x_1, G)/\partial G}{\partial u_1(x_1, G)/\partial x_1} + \frac{\partial u_2(x_2, G)/\partial G}{\partial u_2(x_2, G)/\partial x_2} = \frac{\partial c(G)}{\partial G},$$

which is just

$$MRS_1 + MRS_2 = MC(G)$$

as given in the text.

CHAPTER 38

ASYMMETRIC INFORMATION

So far in our study of markets we have not examined the problems raised by differences in information: by assumption buyers and sellers were both perfectly informed about the quality of the goods being sold in the market. This assumption can be defended if it is easy to verify the quality of an item. If it is not costly to tell which goods are high-quality goods and which are low-quality goods, then the prices of the goods will simply adjust to reflect the quality differences.

But if information about quality is costly to obtain, then it is no longer plausible that buyers and sellers have the same information about goods involved in transactions. There are certainly many markets in the real world in which it may be very costly or even impossible to gain accurate information about the quality of the goods being sold.

One obvious example is the labor market. In the simple models described earlier, labor was a homogeneous product—everyone had the same "kind" of labor and supplied the same amount of effort per hour worked. This is clearly a drastic simplification! In reality, it may be very difficult for a firm to determine how productive its employees are.

Costly information is not just a problem with labor markets. Similar problems arise in markets for consumer products. When a consumer buys

a used car it may be very difficult for him to determine whether or not it is a good car or a lemon. By contrast, the seller of the used car probably has a pretty good idea of the quality of the car. We will see that this **asymmetric information** may cause significant problems with the efficient functioning of a market.

38.1 The Market for Lemons

Let us look at a model of a market where the demanders and suppliers have different information about the qualities of the goods being sold.[1]

Consider a market with 100 people who want to sell their used cars and 100 people who want to buy a used car. Everyone knows that 50 of the cars are "plums" and 50 are "lemons."[2] The current owner of each car knows its quality, but the prospective purchasers don't know whether any given car is a plum or a lemon.

The owner of a lemon is willing to part with it for \$1000 and the owner of a plum is willing to part with it for \$2000. The buyers of the car are willing to pay \$2400 for a plum and \$1200 for a lemon.

If it is easy to verify the quality of the cars there will be no problems in this market. The lemons will sell at some price between \$1000 and \$1200 and the plums will sell at some price between \$2000 and \$2400. But what happens to the market if the buyers *can't* observe the quality of the car?

In this case the buyers have to guess about how much each car is worth. We'll make a simple assumption about the form that this guess takes: we assume that if a car is equally likely to be a plum as a lemon, then a typical buyer would be willing to pay the expected value of the car. Using the numbers described above this means that the buyer would be willing to pay $\frac{1}{2}1200 + \frac{1}{2}2400 = \$1800$.

But who would be willing to sell their car at that price? The owners of the lemons certainly would, but the owners of the plums wouldn't be willing to sell their cars—by assumption they need at least \$2000 to part with their cars. The price that the buyers are willing to pay for an "average" car is less than the price that the sellers of the plums want in order to part with their cars. At a price of \$1800 only lemons would be offered for sale.

But if the buyer was certain that he would get a lemon, then he wouldn't be willing to pay \$1800 for it! In fact, the equilibrium price in this market would have to be somewhere between \$1000 and \$1200. For a price in this range only owners of lemons would offer their cars for sale, and buyers

[1] The first paper to point out some of the difficulties in markets of this sort was George Akerlof, "The Market for Lemons: Quality Uncertainty and the Market Mechanism," *The Quarterly Journal of Economics*, 84, 1970, pp. 488-500. He was awarded the 2001 Nobel Prize in economics for this work.

[2] A "plum" is slang for a good car; a "lemon" is slang for a bad car.

would therefore (correctly) expect to get a lemon. In this market, none of the plums ever get sold! Even though the price at which buyers are willing to buy plums exceeds the price at which sellers are willing to sell them, no such transactions will take place.

It is worth contemplating the source of this market failure. The problem is that there is an externality between the sellers of good cars and bad cars; when an individual decides to try to sell a bad car, he affects the purchasers' perceptions of the quality of the average car on the market. This lowers the price that they are willing to pay for the average car, and thus hurts the people who are trying to sell good cars. It is this externality that creates the market failure.

The cars that are most likely to be offered for sale are the ones that people want most to get rid of. The very act of offering to sell something sends a signal to the prospective buyer about its quality. If too many low-quality items are offered for sale it makes it difficult for the owners of high-quality items to sell their products.

38.2 Quality Choice

In the lemons model there were a fixed number of cars of each quality. Here we consider a variation on that model where quality may be determined by the producers. We will show how the equilibrium quality is determined in this simple market.

Suppose that each consumer wants to buy a single umbrella and that there are two different qualities available. Consumers value high-quality umbrellas at \$14 and low-quality umbrellas at \$8. It is impossible to tell the quality of the umbrellas in the store; this can only be determined after a few rainstorms.

Suppose that some manufacturers produce high-quality umbrellas and some produce low-quality umbrellas. Suppose further that both high-quality and low-quality umbrellas cost \$11.50 to manufacture and that the industry is perfectly competitive. What would we expect to be the equilibrium quality of umbrellas produced?

We suppose that consumers judge the quality of the umbrellas available in the market by the *average* quality sold, just as in the case of the lemons market. If the fraction of high-quality umbrellas is q, then the consumer would be willing to pay $p = 14q + 8(1 - q)$ for an umbrella.

There are three cases to consider.

Only low-quality manufacturers produce. In this case then the consumers would be willing to pay only \$8 for an average umbrella. But it costs \$11.50 to produce an umbrella, so none would be sold.

Only high-quality manufacturers produce. In this case the producers would compete the price of an umbrella down to marginal cost, \$11.50. The

consumers are willing to pay \$14 for an umbrella, so they would get some consumers' surplus.

Both qualities are produced. In this case competition ensures that the price will be \$11.50. The average quality available must therefore have a value to the consumer of at least \$11.50. This means that we must have

$$14q + 8(1 - q) \geq 11.50.$$

The lowest value of q that satisfies this inequality is $q = 7/12$. This means that if 7/12 of the suppliers are high-quality the consumers are just willing to pay \$11.50 for an umbrella.

The determination of the equilibrium ratio of high-quality producers is depicted in Figure 38.1. The horizontal axis measures q, the fraction of high-quality producers. The vertical axis measures the consumers' willingness to pay for an umbrella if the fraction of high-quality umbrellas offered is q. Producers are willing to supply either quality of umbrella at a price of \$11.50, so the supply conditions are summarized by the colored horizontal line at \$11.50.

Consumers are willing to purchase umbrellas only if $14q+8(1-q) \geq 11.50$; the boundary of this region is illustrated by the dashed line. Any value of q between 7/12 and 1 is an equilibrium.

In this market the equilibrium price is \$11.50, but the value of the average umbrella to a consumer can be anywhere between \$11.50 and \$14, depending on the fraction of high-quality producers. Any value of q between 1 and 7/12 is an equilibrium.

However, all of these equilibria are not equivalent from the social point of view. The producers get zero producer surplus in all the equilibria, due to the assumption of pure competition and constant marginal cost, so we only have to examine the consumers' surplus. Here it is easy to see that the higher the average quality, the better off the consumers are. The best equilibrium from the viewpoint of the consumers is the one in which only the high-quality goods are produced.

Choosing the Quality

Now let us change the model a bit. Suppose that each producer can choose the quality of umbrella that he produces and that it costs \$11.50 to produce a high-quality umbrella and \$11 to produce a low-quality umbrella. What will happen in this case?

Suppose that the fraction of producers who choose high-quality umbrellas is q, where $0 < q < 1$. Consider one of these producers. If it behaves competitively and believes that it has only a negligible effect on the market

Figure 38.1

Equilibrium quality. The horizontal line represents the supply conditions: the market is willing to supply any quality of umbrella for \$11.50. The slanted line represents the demand conditions: consumers are willing to pay more if the average quality is higher. The market is in equilibrium if the fraction of high-quality producers is at least 7/12.

price *and* quality, then it would always want to produce only low-quality umbrellas. Since this producer is by assumption only a small part of the market, it neglects its influence on the market price and therefore chooses to produce the more profitable product.

But every producer will reason the same way and only low-quality umbrellas will be produced. But consumers are only willing to pay \$8 for a low-quality umbrella, so there is no equilibrium. Or, if you will, the only equilibrium involves zero production of *either* quality of umbrella! The possiblity of low-quality production has destroyed the market for *both* qualities of the good!

38.3 Adverse Selection

The phenomenon described in the last section is an example of **adverse selection**. In the model we just examined the low-quality items crowded out the high-quality items because of the high cost of acquiring information. As we just saw, this adverse selection problem may be so severe that it can

completely destroy the market. Let's consider a few other examples of adverse selection.

Consider first an example from the insurance industry. Suppose that an insurance company wants to offer insurance for bicycle theft. They do a careful market survey and find that the incident of theft varies widely across communities. In some areas there is a high probability that a bicycle will be stolen, and in other areas thefts are quite rare. Suppose that the insurance company decides to offer the insurance based on the *average* theft rate. What do you think will happen?

Answer: the insurance company is likely to go broke quickly! Think about it. Who is going to buy the insurance at the average rate? Not the people in the safe communities—they don't need much insurance anyway. Instead the people in the communities with a high incidence of theft will want the insurance—they're the ones who need it.

But this means that the insurance claims will mostly be made by the consumers who live in the high-risk areas. Rates based on the *average* probability of theft will be a misleading indication of the actual experience of claims filed with the insurance company. The insurance company will not get an unbiased selection of customers; rather they will get an *adverse* selection. In fact the term "adverse selection" was first used in the insurance industry to describe just this sort of problem.

It follows that in order to break even the insurance company must base their rates on the "worst-case" forecasts and that consumers with a low, but not negligible, risk of bicycle theft will be unwilling to purchase the resulting high-priced insurance.

A similar problem arises with health insurance—insurance companies can't base their rates on the *average* incidence of health problems in the population. They can only base their rates on the average incidence of health problems in the group of potential purchasers. But the people who want to purchase health insurance the most are the ones who are likely to need it the most and thus the rates must reflect this disparity.

In such a situation it is possible that everyone can be made better off by *requiring* the purchase of insurance that reflects the average risk in the population. The high-risk people are better off because they can purchase insurance at rates that are lower than the actual risk they face and the low-risk people can purchase insurance that is more favorable to them than the insurance offered if *only* high-risk people purchased it.

A situation like this, where the market equilibrium is dominated by a compulsary purchase plan, is quite surprising to most economists. We usually think that "more choice is better," so it is peculiar that restricting choice can result in a Pareto improvement. But it should be emphasized that this paradoxical result is due to the externality between the low-risk and high-risk people.

In fact there are social institutions that help to solve this market inefficiency. It is commonly the case that employers offer health plans to their

employees as part of the package of fringe benefits. The insurance company can base its rates on the averages over the set of employees and is assured that *all* employees must participate in the program, thus eliminating the adverse selection.

38.4 Moral Hazard

Another interesting problem that arises in the insurance industry is known as the **moral hazard** problem. The term is somewhat peculiar, but the phenomenon is not hard to describe. Consider the bicycle-theft insurance market again and suppose for simplicity that all of the consumers live in areas with identical probabilities of theft, so that there is no problem of adverse selection. On the other hand, the probability of theft may be affected by the *actions* taken by the bicycle owners.

For example, if the bicycle owners don't bother to lock their bikes or use only a flimsy lock, the bicycle is much more likely to be stolen than if they use a secure lock. Similar examples arise in other sorts of insurance. In the case of health insurance, for example, the consumers are less likely to need the insurance if they take actions associated with a healthy lifestyle. We will refer to actions that affect the probability that some event occurs as *taking care*.

When it sets its rates the insurance company has to take into account the incentives that the consumers have to take an appropriate amount of care. If no insurance is available consumers have an incentive to take the maximum possible amount of care. If it is impossible to buy bicycle-theft insurance, then all bicyclists would use large expensive locks. In this case the individual bears the full cost of his actions and accordingly he wants to "invest" in taking care until the marginal benefit from more care just equals the marginal cost of doing so.

But if a consumer can purchase bicycle insurance, then the cost inflicted on the individual of having his bicycle stolen is much less. After all, if the bicycle is stolen then the person simply has to report it to the insurance company and he will get insurance money to replace it. In the extreme case, where the insurance company completely reimburses the individual for the theft of his bicycle, the individual has no incentive to take care at all. This lack of incentive to take care is called **moral hazard**.

Note the tradeoff involved: too little insurance means that people bear a lot of risk, too much insurance means that people will take inadequate care.

If the amount of care is observable, then there is no problem. The insurance company can base its rates on the amount of care taken. In real life it is common for insurance companies to give different rates to businesses that have a fire sprinkler system in their building, or to charge smokers different rates than nonsmokers for health insurance. In these cases the insurance

firm attempts to discriminate among users depending on the choices they have made that influence the probability of damage.

But insurance companies can't observe all the relevant actions of those they insure. Therefore we will have the tradeoff described above: full insurance means too little care will be undertaken because the individuals don't face the full costs of their actions.

What does this imply about the types of insurance contracts that will be offered? In general, the insurance companies will not want to offer the consumers "complete" insurance. They will always want the consumer to face some part of the risk. This is why most insurance policies include a "deductible," an amount that the insured party has to pay in any claim. By making the consumers pay part of a claim, the insurance companies can make sure that the consumer always has an incentive to take *some* amount of care. Even though the insurance company would be willing to insure a consumer completely if they could verify the amount of care taken, the fact that the consumer can *choose* the amount of care he takes implies that the insurance company will not allow the consumer to purchase as much insurance as he wants if the company cannot observe the level of care.

This is also a paradoxical result when compared with the standard market analysis. Typically the amount of a good traded in a competitive market is determined by the condition that demand equals supply—the marginal willingness to pay equals the marginal willingness to sell. In the case of moral hazard, a market equilibrium has the property that each consumer would like to buy more insurance, and the insurance companies would be willing to provide more insurance if the consumers continued to take the same amount of care . . . but this trade won't occur because if the consumers were able to purchase more insurance they would rationally choose to take less care!

38.5 Moral Hazard and Adverse Selection

Moral hazard refers to situations where one side of the market can't observe the actions of the other. For this reason it is sometimes called a **hidden action** problem.

Adverse selection refers to situations where one side of the market can't observe the "type" or quality of the goods on other side of the market. For this reason it is sometimes called a **hidden information** problem.

Equilibrium in a market involving hidden action typically involves some form of rationing—firms would like to provide more than they do, but they are unwilling to do so since it will change the incentives of their customers. Equilibrium in a market involving hidden information will typically involve too little trade taking place because of the externality between the "good" and "bad" types.

Equilibrium outcomes in this market appear to be inefficient, but one has to be careful in making such a claim. The question to ask is "inefficient relative to what?" The equilibrium will always be inefficient relative to the equilibrium with full information. But this is of little help in making policy decisions: if the firms in the industry find it too costly to collect more information the government would probably find it too costly as well.

The real question to ask is whether some sort of governmental intervention in the market could improve efficiency even if the government had the same information problems as the firms.

In the case of hidden action considered above, the answer is usually "no." If the government can't observe the care taken by the consumers, then it can do no better than the insurance companies. Of course the government might have other tools at its disposal that are not available to the insurance company—it could compel a particular level of care, and it could set criminal punishments for those who did not take due care. But if the government can only set prices and quantities, then it can do no better than the private market can do.

Similar issues arise in the case of hidden information. We have already seen that if the government can *compel* people of all risk classes to purchase insurance, it is possible for everyone to be made better off. This is, on the face of it, a good case for intervention. On the other hand, there are costs to government intervention as well; economic decisions made by governmental decree may not be as cost-effective as those made by private firms. Just because there are governmental actions that *can* improve social welfare doesn't mean that these actions will be taken!

Furthermore, there may be purely private solutions to the adverse selection problems. For example, we have already seen how providing health insurance as a fringe benefit can help to eliminate the adverse selection problem.

38.6 Signaling

Recall our model of the used-car market: the owners of the used cars knew the quality, but the purchasers had to guess at the quality. We saw that this asymmetric information could cause problems in the market; in some cases, the adverse selection problem would result in too few transactions being made.

However, the story doesn't end there. The owners of the good used cars have an incentive to try to convey the fact that they have a good car to the potential purchasers. They would like to choose actions that **signal** the quality of their car to those who might buy it.

One sensible signal in this context would be for the owner of a good used car to offer a **warranty**. This would be a promise to pay the purchaser some agreed upon amount if the car turned out to be a lemon. Owners of

the good used cars can afford to offer such a warranty while the owners of the lemons can't afford this. This is a way for the owners of the good used cars to signal that they have good cars.

In this case signaling helps to make the market perform better. By offering the warranty—the signal—the sellers of the good cars can distinguish themselves from the sellers of the bad used cars. But there are other cases where signaling can make a market perform less well.

Let's consider a very simplified model of the education market first examined by Michael Spence.[3] Suppose that we have two types of workers, able and unable. The able workers have a marginal product of a_2, and the unable workers have a marginal product of a_1, where $a_2 > a_1$. Suppose that a fraction b of the workers are able and $1 - b$ of them are unable.

For simplicity we assume a linear production function so that the total output produced by L_2 able workers and L_1 unable workers is $a_1L_1 + a_2L_2$. We also assume a competitive labor market.

If worker quality is easily observable, then firms would just offer a wage of $w_2 = a_2$ to the able workers and of $w_1 = a_1$ to the unable workers. That is, each worker would be paid his marginal product and we would have an efficient equilibrium.

But what if the firm can't observe the marginal products? If a firm can't distinguish the types of workers, then the best that it can do is to offer the average wage, which is $w = (1-b)a_1 + ba_2$. As long as the good and the bad workers both agree to work at this wage there is no problem with adverse selection. And, given our assumption about the production function, the firm produces just as much output and makes just as much profit as it would if it could perfectly observe the type of the worker.

However, suppose now that there is some signal that the workers can acquire that will distinguish the two types. For example, suppose that the workers can acquire education. Let e_1 be the amount of education attained by the type 1 workers and e_2 the amount attained by the type 2 workers. Suppose that the workers have different costs of acquiring education, so that the total cost of education for the able workers is c_2e_2 and the total cost of education for the unable workers is c_1e_1. These costs are meant to include not only the dollar costs of attending school, but also includes the opportunity costs, the costs of the effort required, and so on.

Now we have two decisions to consider. The workers have to decide how much education to acquire and the firms have to decide how much to pay workers with different amounts of education. Let us make the extreme assumption that the education doesn't affect worker productivity at all. Of course this isn't true in real life—especially for economics courses—but it helps to keep the model simple.

[3] Michael Spence, *Market Signaling* (Cambridge, Mass: Harvard University Press, 1974).

It turns out that the nature of the equilibrium in this model depends crucially on the cost of acquiring education. Suppose that $c_2 < c_1$. This says that the marginal cost of acquiring education is less for the able workers than the unable workers. Let e^* be an education level that satisfies the following inequalities:

$$\frac{a_2 - a_1}{c_1} < e^* < \frac{a_2 - a_1}{c_2}.$$

Given our assumption that $a_2 > a_1$ and that $c_2 < c_1$ there must be such an e^*.

Now consider the following set of choices: the able workers all acquire education level e^* and the unable workers all acquire education level 0, and the firm pays workers with education level e^* a wage of a_2 and workers with less education than this a wage of a_1. Note that the choice of the education level of a worker perfectly signals his type.

But is this an equilibrium? Does anyone have an incentive to change his or her behavior? Each firm is paying each worker his or her marginal product, so the firms have no incentive to do anything differently. The only question is whether the workers are behaving rationally given the wage schedule they face.

Would it be in the interest of an unable worker to purchase education level e^*? The benefit to the worker would be the increase in wages $a_2 - a_1$. The cost to the unable worker would be $c_1 e^*$. The benefits are less than the costs if

$$a_2 - a_1 < c_1 e^*.$$

But we are guaranteed that this condition holds by the choice of e^*. Hence the unable workers find it optimal to choose a zero educational level.

Is it actually in the interest of the able workers to acquire the level of education e^*? The condition for the benefits to exceed the costs is

$$a_2 - a_1 > c_2 e^*,$$

and this condition also holds due to the choice of e^*.

Hence this pattern of wages is indeed an equilibrium: if each able worker chooses education level e^* and each unable worker chooses a zero educational level, then no worker has any reason to change his or her behavior. Due to our assumption about the cost differences, the education level of a worker can, in equilibrium, serve as a signal of the different productivities. This type of signaling equilibrium is sometimes called a **separating equilibrium** since the equilibrium involves each type of worker making a choice that allows him to separate himself from the other type.

Another possibility is a **pooling equilibrium**, in which each type of worker makes the *same* choice. For example, suppose that $c_2 > c_1$, so that the able workers have a higher cost of acquiring education than the unable

workers. In this case it can be shown that the only equilibrium involves the workers all getting paid a wage based on their average ability, and so no signaling occurs.

The separating equilibrium is especially interesting since it is inefficient from a social point of view. Each able worker finds it in his interest to pay for acquiring the signal, even though it doesn't change his productivity at all. The able workers want to acquire the signal not because it makes them any more productive, but just because it distinguishes them from the unable workers. Exactly the same amount of output is produced in the (separating) signaling equilibrium as would be if there were no signaling at all. In this model the acquisition of the signal is a total waste from the social point of view.

It is worth thinking about the nature of this inefficiency. As before, it arises because of an externality. If both able and unable workers were paid their *average* product, the wage of the able workers would be depressed because of the presence of the unable workers. Thus they would have an incentive to invest in signals that will distinguish them from the less able. This investment offers a private benefit but no social benefit.

Of course signaling doesn't always lead to inefficiencies. Some types of signals, such as the used-car warranties described above, help to facilitate trade. In that case the equilibrium with signals is preferred to the equilibrium without signals. So signaling can make things better or worse; each case has to be examined on its own merits.

EXAMPLE: The Sheepskin Effect

In the extreme form of the educational signaling model described above education has no effect on productivity: the years spent in school serve only to signal the fixed ability of an individual. This is obviously an exaggeration: a student with 11 years of schooling almost certainly is more productive than one with 10 years of schooling due to the fact that he has acquired more useful skills during the additional year. Presumably part of the returns to schooling are due to signaling, and part are due to the acquisition of useful skills while in school. How can we separate these two factors?

Labor economists who have studied the returns to education have observed the following suggestive fact: the earnings of people who have graduated from high school are much higher than the incomes of people who have only completed 3 years of high school. One study found that graduating from high school increases earnings by 5 to 6 times as much as does completing a year in high school that does not result in graduation. The same discontinuous jump occurs for people who graduate from college. According to one estimate, the economic return to the 16th year of schooling

are about three times as high as the return to the 15th year of schooling.[4]

If education imparts productive skills, we might well expect that people with 11 years of education are paid more than people with 10 years of education. What is surprising is that there is a huge jump in earnings associated with high school graduation. Economists have termed this the **sheepskin effect**, in reference to the fact that diplomas were often written on sheepskins. Presumably, graduation from high school is some kind of signal. But what is it a signal of? In the educational signaling model described earlier, educational attainment was a signal of ability. Is that what high school graduation signals? Or is it something else?

Andrew Weiss, a Boston University economist, attempted to answer these questions.[5] He looked at a set of data describing how workers assembled equipment and was able to obtain a measure of how much output they produced in their first month on the job. He found that there was a very small effect of education on output: each year of secondary education increased a worker's output by about 1.3 percent. Furthermore, high school graduates produced essentially the same amount of output as nongraduates. Apparently education contributed only a small amount to the initial productivity of these workers.

Weiss then looked at another data set that described various characteristics of workers in a variety of occupations. He found that high school graduates had significantly lower quit and absentee rates than nongraduates. It seems that high school graduates receive higher wages because they are more productive—but the reason that they are more productive is because they stay with the firm longer and have fewer absences. This suggests that the signaling model does give us insight into real-world labor markets. However, the actual signal sent by educational attainment is considerably more complex than the simplest version of the signaling model suggests.

38.7 Incentives

We turn now to a slightly different topic, the study of **incentive systems**. As it turns out, our investigation of this topic will naturally involve asymmetric information. But it is useful to start with the case of full information.

The central question in the design of incentive systems is "How can I get someone to do something for me?" Let's pose this question in a specific

[4] See Thomas Hungerford and Gary Solon, "Sheepskin Effects in the Returns to Education," *Review of Economics and Statistics,* 69, 1987, 175–77.

[5] "High School Graduation, Performance and Wages," *Journal of Political Economy,* 96, 4, 1988, 785–820.

context. Suppose that you own a plot of land but you are unable to work on the land yourself. So you try to hire someone to do the farming for you. What sort of compensation system should you set up?

One plan might involve paying the worker a lump-sum fee independent of how much he produces. But then he would have little incentive to work. In general a good incentive plan will make the payment of the worker depend in some way on the output he produces. The problem of incentive design is to determine exactly how sensitive the payment should be to the produced output.

Let x be the amount of "effort" that the worker expends, and let $y = f(x)$ be the amount of output produced; for simplicity we suppose that the price of output is 1 so that y also measures the value of the output. Let $s(y)$ be the amount that you pay the worker if he produces y dollars worth of output. Presumably you would like to choose the function $s(y)$ to maximize your profits $y - s(y)$.

What are the constraints that you face? In order to answer this question we have to look at things from the worker's perspective.

We assume that the worker finds effort costly, and write $c(x)$ for the cost of effort x. We assume that this cost function has the usual shape: both total and marginal costs increase as effort increases. The utility of the worker who chooses effort level x is then simply $s(y)-c(x) = s(f(x))-c(x)$. The worker may have other alternatives available that give him some utility $\overline{u}$. This could come from working at other jobs or from not working at all. All that is relevant for the design of the incentive scheme is that the utility that the worker gets from this job must be at least as great as the utility he could get elsewhere. This gives us the **participation constraint**:

$$s(f(x)) - c(x) \geq \overline{u}.$$

Given this constraint we can determine how much output we can get from the worker. You want to induce the worker to choose an effort level x that yields you the greatest surplus given the constraint that the worker is willing to work for you:

$$\max_x \; f(x) - s(f(x))$$
$$\text{such that } s(f(x)) - c(x) \geq \overline{u}.$$

In general, you will want the worker to choose x to just satisfy the constraint so that $s(f(x)) - c(x) = \overline{u}$. Substituting this into the objective function we have the unconstrained maximization problem

$$\max_x \; f(x) - c(x) - \overline{u}.$$

But it is easy to solve this problem! Just choose x^* so that the marginal product equals the marginal cost:

$$MP(x^*) = MC(x^*).$$

Any choice of x^* where the marginal benefit is not equal to the marginal cost cannot maximize profits.

This tells us what level of effort the owner wants to achieve; now we have to ask what he has to pay the worker to achieve that effort. That is, what does the function $s(y)$ have to look like to induce the worker to choose to make x^* the optimal choice?

Suppose that you decide that you want to induce the worker to put in x^* amount of effort. Then you must make it in his interest to do so; that is, you must design your incentive scheme $s(y)$ so that the utility from choosing to work x^* is larger than the utility of worker any other amount x. This gives us the constraint

$$s(f(x^*)) - c(x^*) \geq s(f(x)) - c(x) \quad \text{for all } x.$$

This constraint is called the **incentive compatibility constraint**. It simply says that the utility to the worker from choosing x^* must be greater than the utility of any other choice of effort.

So we have two conditions that the incentive scheme must satisfy: first, it must give total utility to the worker of $\overline{u}$, and second, it must make the marginal product of effort equal to the marginal cost of effort at the effort level x^*. There are several ways to do this.

Rent. The landowner could simply rent the land to the worker for some price R, so that the worker gets all the output he produces after he pays the owner R. For this scheme

$$s(f(x)) = f(x) - R.$$

If the worker maximizes $s(f(x)) - c(x) = f(x) - R - c(x)$, he will choose the effort level where $MP(x^*) = MC(x^*)$, which is exactly what the owner wants. The rental rate R is determined from the participation condition. Since the total utility to the worker must be $\overline{u}$ we have

$$f(x^*) - c(x^*) - R = \overline{u},$$

which says $R = f(x^*) - c(x^*) - \overline{u}$.

Wage labor. In this scheme the landowner pays the worker a constant wage per unit of effort along with a lump sum K. This means that the incentive payment takes the form

$$s(x) = wx + K.$$

The wage rate w is equal to the marginal product of the worker at the optimal choice x^*, $MP(x^*)$. The constant K is chosen to just make the worker indifferent between working for the landowner and working elsewhere; that is, it is chosen to satisfy the participation constraint.

The problem of maximizing $s(f(x)) - c(x)$ then becomes

$$\max_x \; wx + K - c(x),$$

which means that the worker will choose x so as to set his marginal cost equal to the wage: $w = MC(x)$. Since the wage is $MP(x^*)$, this means that the *optimal* choice of the worker will be x^* such that $MP(x^*) = MC(x^*)$ which is just what the firm wants.

Take-it-or-leave-it. In this scheme the landowner pays the worker B^* if he works x^* and zero otherwise. The amount B^* is determined by the participation constraint $B^* - c(x^*) = \overline{u}$, so $B^* = \overline{u} + c(x^*)$. If the worker chooses any level of effort $x \neq x^*$, he gets a utility of $-c(x)$. If he chooses x^*, he gets a utility of $\overline{u}$. Hence the optimal choice for the worker is to set $x = x^*$.

Each of these schemes is equivalent as far as the analysis goes: each one gives the worker a utility of $\overline{u}$, and each one gives the worker an incentive to work the optimal amount x^*. At this level of generality there is no reason to choose between them.

If all of these schemes are optimal, what could a nonoptimal scheme look like? Here is an example.

Sharecropping. In sharecropping the worker and the landowner each get some fixed percentage of the output. Suppose that the worker's share takes the form $s(x) = \alpha f(x) + F$, where F is some constant and $\alpha < 1$. This is *not* an efficient scheme for the problem under consideration. It is easy to see why. The worker's maximization problem is

$$\max_x \; \alpha f(x) + F - c(x),$$

which means that he would choose a level of effort $\hat{x}$ where

$$\alpha MP(\hat{x}) = MC(\hat{x}).$$

Such an effort level clearly cannot satisfy the efficiency condition that $MP(x) = MC(x)$.

Here is a way to summarize this analysis. In order to design an efficient incentive scheme it is necessary to ensure that the person who makes the effort decision is the **residual claimant** to the output. The way the owner can make himself as well off as possible is to make sure that he gets the worker to produce the optimal amount of output. This is the output level where the marginal product of the worker's extra effort equals the marginal cost of putting forth that effort. It follows that the incentive scheme must provide a marginal benefit to the worker equal to his marginal product.

EXAMPLE: Voting Rights in the Corporation

Normally shareholders in a corporation have the right to vote on various issues related to the management of the corporation while bondholders do not. Why is this? The answer comes from looking at the structure of payoffs to stockholders and bondholders. If a corporation produces X dollars of profit in a given years, the bondholders have first claim on these profits, while the amount that is left over goes to the stockholders. If the total claim by the bondholders is B, then the amount that goes to the stockholders is $X - B$. This makes the stockholders the residual claimants—so they have an incentive to make X as large as possible. The bondholders on the other hand only have an incentive to make sure that X is at least B, since that is the most that they are entitled to. Hence giving the stockholders the right to make decisions will generally result in larger profits.

EXAMPLE: Chinese Economic Reforms

Prior to 1979 Chinese rural communes were organized along orthodox Marxist lines. Workers were paid according to a rough estimate of how much they contributed to the commune income. Five percent of the commune's land was set aside for private plots, but peasants were not allowed to travel to cities to sell the output from their private farms. All trade had to take place through a highly regulated government market.

At the end of 1978 the Chinese central government instituted a major reform in the structure of agriculture, known as the "responsibility system." In the responsibility system, any production in excess of a fixed quota was kept by the household and could be sold on private markets. The government removed restrictions on private plots and increased the amount of land devoted to private farming. By the end of 1984, 97 percent of the farmers operated under this responsibility system.

Note that the structure of the system is very much like the optimal incentive mechanism described above: each household makes a lump-sum payment to the commune but can keep anything in excess of this quota. Hence the *marginal* incentives for household production are the economically appropriate ones.

The effect of this new system on agricultural output was phenomenal: between 1978 and 1984, the output of Chinese agriculture increased by over 61 percent! However, not all of this increase is due to better incentives; at the same time these reforms were going on, the Chinese government also changed the controlled prices of agricultural goods, and even allowed some of these prices to be determined on private markets.

Three economists attempted to divide the increase in output into the part

due to better incentives and the part due to the change in prices.[6] They found that over three-fourths of the increase was due to the improvement in incentives, and only one-fourth was due to the price reforms.

38.8 Asymmetric Information

The above analysis provides some insights about the use of different sorts of incentive schemes. For example, it shows that renting the land to a worker is better than sharecropping. But this really proves too much. If our analysis is a good description of the world, then we would expect to see rental or wage labor used in agriculture and never see sharecropping used, except by mistake.

Clearly this isn't right. Sharecropping has been used for thousands of years in some parts of the world, so it is likely that it fulfills some kind of need. What have we left out of our model?

Given the title of this section it is not hard to guess the answer: we've left out problems involving imperfect information. We assumed that the owner of the firm could perfectly observe the effort of the worker. In many situations of interest it may be impossible to observe the effort. At best the owner may observe some *signal* of the effort such as the resulting production of output. The amount of output produced by a farmer may depend in part on his effort, but it may also depend on the weather, the quality of the inputs, and many other factors. Because of this kind of "noise," a payment from the owner to the worker based on output will not in general be equivalent to a payment based on effort alone.

This is essentially a problem of asymmetric information: the worker can choose his effort level, but the owner cannot perfectly observe it. The owner has to guess the effort from the observed output, and the design of the optimal incentive scheme has to reflect this inference problem.

Consider the four incentive schemes described above. What goes wrong if effort is not perfectly correlated with output?

Rent. If the firm rents the technology to the worker, then the worker can get all of the output that remains after paying the fixed rental fee. If output has a random component, this means that the worker will have to bear all the risk from the random factors. If the worker is more risk averse than the owner—which is the likely case—this will be inefficient. In general, the worker would be willing to give up some of the residual profits in order to have a less risky income stream.

[6] J. McMillan, J. Whalley, and L. Zhu, "The Impact of China's Economic Reforms on Agricultural Productivity Growth," *Journal of Political Economy,* 97, 4, 1989, 781–807.

Wage labor. The problem with wage labor is that it requires observation of the *amount* of labor input. The wage has to be based on the effort put in to production, not just the hours spent in the firm. If the owner can't observe the amount of labor input, then it will be impossible to implement this kind of incentive scheme.

Take-it-or-leave-it. If the incentive payment is based on the labor input, then we have the same problem with this scheme as with wage labor. If the payment is based on *output,* then the scheme involves the worker bearing all the risk. Even missing the "target output" by a small amount results in a zero payment.

Sharecropping. This is something of a happy medium. The payment to the worker depends in part on observed output, but the worker and the owner share the risk of output fluctuations. This gives the worker an incentive to produce output but it doesn't leave him bearing all the risk.

The introduction of asymmetric information has made a drastic change in our evaluation of the incentive methods. If the owner can't observe effort, then wage labor is infeasible. Rent and the take-it-or-leave-it scheme leave the worker bearing too much risk. Sharecropping is a compromise between the two extremes: it gives the worker some incentive to produce, but it doesn't leave him with all the risk.

EXAMPLE: Monitoring Costs

It is not always easy to observe the amount of effort an employee puts into his or her job. Consider, for example, a job as a clerk in a 24-hour convenience store. How can the manager observe the employees' performance when the manager isn't around? Even if there are ways to observe the physical output of the employee (shelves stocked, sales rung up) it is much harder to observe things like politeness to customers.

There is little doubt that some of the worst service in the world was provided in the formerly Communist countries in Eastern Europe: once you managed to attract the attention of a clerk, you were more likely to be greeted by a scowl than a smile. Nevertheless, a Hungarian entrepreneur, Gabor Varszegi, has made millions by providing high-quality service in his photo developing shops in Budapest.[7]

Varszegi says that he got his start as a businessman in the mid-sixties by playing bass guitar and managing a rock group. "Back then," he says, "the only private businessmen in Eastern Europe were rock musicians."

[7] See Steven Greenhouse, "A New Formula in Hungary: Speed Service and Grow Rich," *New York Times,* June 5, 1990, A1.

He introduced one-hour film developing to Hungary in 1985; the next best alternative to his one-hour developing shops was the state-run agency that took one month.

Varszegi follows two rules in labor relations: he never hires anyone who worked under Communism, and he pays his workers four times the market wage. This makes perfect sense in light of the above remarks about monitoring costs: there are very few employees per store and monitoring their behavior is very costly. If there were only a small penalty to being fired, there would be great temptation to slack off. By paying the workers much more than they could get elsewhere, Varszegi makes it very costly for them to be fired—and reduces his monitoring costs significantly.

EXAMPLE: The Grameen Bank

A village moneylender in Bangladesh charges over 150 percent interest a year. Any American banker would love a return of that size: why isn't Citibank installing money machines in Bangladesh? To ask the question is to answer it: Citibank would probably not do as well as the moneylender. The village moneylender has a comparative advantage in these small-scale loans for several reasons.

- The village moneylender can deal more effectively with the small scale of lending involved;
- The moneylender has better access to information about who are good and bad credit risks than an outsider does.
- The moneylender is in a better position to monitor the progress of the loan payments to insure repayment.

These three problems—returns to scale, adverse selection and moral hazard—allow the village moneylender to maintain a local monopoly in the credit market.

Such a local monopoly is especially pernicious in an underdeveloped country such as Bangladesh. At an interest rate of 150 percent there are many profitable projects that are not being undertaken by the peasants. Improved access to credit could lead to a major increase in investment, and a corresponding increase in the standards of living.

Muhammad Yunus, an American-trained economist from Bangladesh, has developed an ingenious institution known as the Grameen Bank (village bank) to address some of these problems. In the Grameen plan, entrepreneurs with separate projects get together and apply for a loan as a group. If the loan is approved, two members of the group get their loan and commence their investment activity. If they are successful in meeting

the repayment schedule, two more members get loans. If they are also successful the last member, the group leader, will get a loan.

The Grameen bank addresses each of the three problems described above. Since the quality of the group influences whether or not individual members will get loans, potential members are highly selective about who they will join with. Since members of the group can only get loans if other members succeed in their investments, there are strong incentives to help each other out and share expertise. Finally, these activities of choosing candidates for loans and monitoring the progress of the repayments are all done by the peasants themselves, not directly by the loan officers at the bank.

The Grameen bank has been very successful. It makes about 475,000 loans a month with an average size of $70. Their loan-recovery rate is about 98 percent, while conventional lenders in Bangladesh achieve a loan-recovery rate of about 30 to 40 percent. The success of the group responsibility program in encouraging investment has led to its adoption in a number of other poverty-stricken areas in North and South America.

Summary

1. Imperfect and asymmetric information can lead to drastic differences in the nature of market equilbrium.

2. Adverse selection refers to situations where the type of the agents is not observable so that one side of the market has to guess the type or quality of a product based on the behavior of the other side of the market.

3. In markets involving adverse selection too little trade may take place. In this case it is possible that everyone can be made better off by forcing them to transact.

4. Moral hazard refers to a situation where one side of the market can't observe the actions of the other side.

5. Signaling refers to the fact that when adverse selection or moral hazard are present some agents will want to invest in signals that will differentiate them from other agents.

6. Investment in signals may be privately beneficial but publically wasteful. On the other hand, investment in signals may help to solve problems due to asymmetric information.

7. Efficient incentive schemes (with perfect observability of effort) leave the worker as the residual claimant. This means that the worker will equate marginal benefits and marginal costs.

8. But if information is imperfect this is no longer true. In general, an incentive scheme that shares risks as well as providing incentives will be appropriate.

REVIEW QUESTIONS

1. Consider the model of the used-car market presented in this chapter. What is the maximum amount of consumers' surplus that is created by trade in the market equilibrium?

2. In the same model, how much consumers' surplus would be created by *randomly* assigning buyers to sellers? Which method gives the larger surplus?

3. A worker can produce x units of output at a cost of $c(x) = x^2/2$. He can achieve a utility level of $\overline{u} = 0$ working elsewhere. What is the optimal wage-labor incentive scheme $s(x)$ for this worker?

4. Given the setup of the previous problem, what would the worker be willing to pay to rent the production technology?

5. How would your answer to the last problem change if the worker's alternative employment gave him $\overline{u} = 1$?

MATHEMATICAL APPENDIX

In this Appendix we will provide a brief review of some of the mathematical concepts that are used in the text. This material is meant to serve as a reminder of the definitions of various terms used in the text. It is emphatically not a tutorial in mathematics. The definitions given will generally be the simplest, not the most rigorous.

A.1 Functions

A **function** is a rule that describes a relationship between numbers. For each number x, a function assigns a *unique* number y according to some rule. Thus a function can be indicated by describing the rule, as "take a number and square it," or "take a number and multiply it by 2," and so on. We write these particular functions as $y = x^2$, $y = 2x$. Functions are sometimes referred to as **transformations**.

Often we want to indicate that some variable y depends on some other variable x, but we don't know the specific algebraic relationship between the two variables. In this case we write $y = f(x)$, which should be interpreted as saying that the variable y depends on x according to the rule f.

Given a function $y = f(x)$, the number x is often called the **independent variable**, and the number y is often called the **dependent variable**.

The idea is that x varies independently, but the value of y *depends* on the value of x.

Often some variable y depends on several other variables x_1, x_2, and so on, so we write $y = f(x_1, x_2)$ to indicate that both variables together determine the value of y.

A.2 Graphs

A **graph** of a function depicts the behavior of a function pictorially. Figure A.1 shows two graphs of functions. In mathematics the independent variable is usually depicted on the horizontal axis, and the dependent variable is depicted on the vertical axis. The graph then indicates the relationship between the independent and the dependent variables.

However, in economics it is common to graph functions with the independent variable on the vertical axis and the dependent variable on the horizontal axis. Demand functions, for example, are usually depicted with the price on the vertical axis and the amount demanded on the horizontal axis.

Figure A.1 **Graphs of functions**. Panel A denotes the graph of $y = 2x$, and panel B denotes the graph of $y = x^2$.

A.3 Properties of Functions

A **continuous function** is one that can be drawn without lifting a pencil from the paper: there are no jumps in a continuous function. A **smooth**

function is one that has no "kinks" or corners. A **monotonic** function is one that always increases or always decreases; a **positive monotonic function** always increases as x increases, while a **negative monotonic function** always decreases as x increases.

A.4 Inverse Functions

Recall that a function has the property that for each value of x there is a unique value of y associated with it and that a monotonic function is one that is always increasing or always decreasing. This implies that for a monotonic function there will be a unique value of x associated with each value of y.

We call the function that relates x to y in this way an **inverse function**. If you are given y as a function of x, you can calculate the inverse function just by solving for x as a function of y. If $y = 2x$, then the inverse function is $x = y/2$. If $y = x^2$, then there is no inverse function; given any y, both $x = +\sqrt{y}$ and $x = -\sqrt{y}$ have the property that their square is equal to y. Thus there is not a *unique* value of x associated with each value of y, as is required by the definition of a function.

A.5 Equations and Identities

An **equation** asks when a function is equal to some particular number. Examples of equations are

$$2x = 8$$
$$x^2 = 9$$
$$f(x) = 0.$$

The **solution** to an equation is a value of x that satisfies the equation. The first equation has a solution of $x = 4$. The second equation has two solutions, $x = 3$ and $x = -3$. The third equation is just a general equation. We don't know its solution until we know the actual rule that f stands for, but we can denote its solution by x^*. This simply means that x^* is a number such that $f(x^*) = 0$. We say that x^* **satisfies** the equation $f(x) = 0$.

An **identity** is a relationship between variables that holds for *all* values of the variables. Here are some examples of identities:

$$(x + y)^2 \equiv x^2 + 2xy + y^2$$
$$2(x + 1) \equiv 2x + 2.$$

The special symbol $\equiv$ means that the left-hand side and the right-hand side are equal for *all* values of the variables. An equation only holds for some values of the variables, whereas an identity is true for all values of the variables. Often an identity is true by the definition of the terms involved.

A.6 Linear Functions

A **linear function** is a function of the form

$$y = ax + b,$$

where a and b are constants. Examples of linear functions are

$$y = 2x + 3$$
$$y = x - 99.$$

Strictly speaking, a function of the form $y = ax + b$ should be called an **affine function**, and only functions of the form $y = ax$ should be called linear functions. However, we will not insist on this distinction.

Linear functions can also be expressed implicitly in forms like $ax+by = c$. In such a case, we often like to solve for y as a function of x to convert this to the "standard" form:

$$y = \frac{c}{b} - \frac{a}{b}x.$$

A.7 Changes and Rates of Change

The notation Δx is read as "the change in x." It does *not* mean Δ times x. If x changes from x^* to x^{**}, then the change in x is just

$$\Delta x = x^{**} - x^*.$$

We can also write

$$x^{**} = x^* + \Delta x$$

to indicate that x^{**} is x^* plus a change in x.

Typically Δx will refer to a *small* change in x. We sometimes express this by saying that Δx represents a **marginal change**.

A **rate of change** is the ratio of two changes. If y is a function of x given by $y = f(x)$, then the rate of change of y with respect to x is denoted by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The rate of change measures how y changes as x changes.

A linear function has the property that the rate of change of y with respect to x is constant. To prove this, note that if $y = a + bx$, then

$$\frac{\Delta y}{\Delta x} = \frac{a + b(x + \Delta x) - a - bx}{\Delta x} = \frac{b\Delta x}{\Delta x} = b.$$

For nonlinear functions, the rate of change of the function will depend on the value of x. Consider, for example, the function $y = x^2$. For this function

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = 2x + \Delta x.$$

Here the rate of change from x to $x + \Delta x$ depends on the value of x and on the size of the change, Δx. But if we consider very small changes in x, Δx will be nearly zero, so the rate of change of y with respect to x will be approximately $2x$.

A.8 Slopes and Intercepts

The rate of change of a function can be interpreted graphically as the **slope** of the function. In Figure A.2A we have depicted a linear function $y = -2x + 4$. The **vertical intercept** of this function is the value of y when $x = 0$, which is $y = 4$. The **horizontal intercept** is the value of x when $y = 0$, which is $x = 2$. The slope of the function is the rate of change of y as x changes. In this case, the slope of the function is -2.

Slopes and intercepts. Panel A depicts the function $y = -2x + 4$, and panel B depicts the function $y = x^2$.

Figure A.2

In general, if a linear function has the form $y = ax + b$, the vertical intercept will be $y^* = b$ and the horizontal intercept will be $x^* = -b/a$. If a linear function is expressed in the form

$$a_1x_1 + a_2x_2 = c,$$

then the horizontal intercept will be the value of x_1 when $x_2 = 0$, which is $x_1^* = c/a_1$, and the vertical intercept will occur when $x_1 = 0$, which means $x_2^* = c/a_2$. The slope of this function is $-a_1/a_2$.

A nonlinear function has the property that its slope changes as x changes. A **tangent** to a function at some point x is a linear function that has the same slope. In Figure A.2B we have depicted the function x^2 and the tangent line at $x = 1$.

If y increases whenever x increases, then Δy will always have the same sign as Δx, so that the slope of the function will be positive. If on the other hand y decreases when x increases, or y increases when x decreases, Δy and Δx will have opposite signs, so that the slope of the function will be negative.

A.9 Absolute Values and Logarithms

The **absolute value** of a number is a function $f(x)$ defined by the following rule:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Thus the absolute value of a number can be found by dropping the sign of the number. The absolute value function is usually written as $|x|$.

The (natural) **logarithm** or **log** of x describes a particular function of x, which we write as $y = \ln x$ or $y = \ln(x)$. The logarithm function is the unique function that has the properties

$$\ln(xy) = \ln(x) + \ln(y)$$

for all positive numbers x and y and

$$\ln(e) = 1.$$

(In this last equation, e is the base of natural logarithms which is equal to $2.7183\ldots$) In words, the log of the product of two numbers is the sum of the individual logs. This property implies another important property of logarithms:

$$\ln(x^y) = y\ln(x),$$

which says that the log of x raised to the power y is equal to y times the log of x.

A.10 Derivatives

The **derivative** of a function $y = f(x)$ is defined to be

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

In words, the derivative is the limit of the rate of change of y with respect to x as the change in x goes to zero. The derivative gives precise meaning to the phrase "the rate of change of y with respect to x for small changes in x." The derivative of $f(x)$ with respect to x is also denoted by $f'(x)$.

We have already seen that the rate of change of a linear function $y = ax + b$ is constant. Thus for this linear function

$$\frac{df(x)}{dx} = a.$$

For a nonlinear function the rate of change of y with respect to x will usually depend on x. We saw that in the case of $f(x) = x^2$, we had $\Delta y/\Delta x = 2x + \Delta x$. Applying the definition of the derivative

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x.$$

Thus the derivative of x^2 with respect to x is $2x$.

It can be shown by more advanced methods that if $y = \ln x$, then

$$\frac{df(x)}{dx} = \frac{1}{x}.$$

A.11 Second Derivatives

The **second derivative** of a function is the derivative of the derivative of that function. If $y = f(x)$, the second derivative of $f(x)$ with respect to x is written as $d^2f(x)/dx^2$ or $f''(x)$. We know that

$$\frac{d(2x)}{dx} = 2$$
$$\frac{d(x^2)}{dx} = 2x.$$

Thus

$$\frac{d^2(2x)}{dx^2} = \frac{d(2)}{dx} = 0$$
$$\frac{d^2(x^2)}{dx^2} = \frac{d(2x)}{dx} = 2.$$

The second derivative measures the curvature of a function. A function with a negative second derivative at some point is concave near that point; its slope is decreasing. A function with a positive second derivative at a point is convex near that point; its slope is increasing. A function with a zero second derivative at a point is flat near that point.

A.12 The Product Rule and the Chain Rule

Suppose that $g(x)$ and $h(x)$ are both functions of x. We can define the function $f(x)$ that represents their product by $f(x) = g(x)h(x)$. Then the derivative of $f(x)$ is given by

$$\frac{df(x)}{dx} = g(x)\frac{dh(x)}{dx} + h(x)\frac{dg(x)}{dx}.$$

Given two functions $y = g(x)$ and $z = h(y)$, the **composite function** is

$$f(x) = h(g(x)).$$

For example, if $g(x) = x^2$ and $h(y) = 2y + 3$, then the composite function is

$$f(x) = 2x^2 + 3.$$

The **chain rule** says that the derivative of a composite function, $f(x)$, with respect to x is given by

$$\frac{df(x)}{dx} = \frac{dh(y)}{dy}\,\frac{dg(x)}{dx}.$$

In our example, $dh(y)/dy = 2$, and $dg(x)/dx = 2x$, so the chain rule says that $df(x)/dx = 2 \times 2x = 4x$. Direct calculation verifies that this is the derivative of the function $f(x) = 2x^2 + 3$.

A.13 Partial Derivatives

Suppose that y depends on both x_1 and x_2, so that $y = f(x_1, x_2)$. Then the **partial derivative** of $f(x_1, x_2)$ with respect to x_1 is defined by

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1}.$$

The partial derivative of $f(x_1, x_2)$ with respect to x_1 is just the derivative of the function with respect to x_1, *holding x_2 fixed*. Similarly, the partial derivative with respect to x_2 is

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2) - f(x_1, x_2)}{\Delta x_2}.$$

Partial derivatives have exactly the same properties as ordinary derivatives; only the name has been changed to protect the innocent (that is, people who haven't seen the ∂ symbol).

In particular, partial derivatives obey the chain rule, but with an extra twist. Suppose that x_1 and x_2 both depend on some variable t and that we define the function $g(t)$ by

$$g(t) = f(x_1(t), x_2(t)).$$

Then the derivative of $g(t)$ with respect to t is given by

$$\frac{dg(t)}{dt} = \frac{\partial f(x_1, x_2)}{\partial x_1}\frac{dx_1(t)}{dt} + \frac{\partial f(x_1, x_2)}{\partial x_2}\frac{dx_2(t)}{dt}.$$

When t changes, it affects both $x_1(t)$ and $x_2(t)$. Therefore, we need to calculate the derivative of $f(x_1, x_2)$ with respect to each of those changes.

A.14 Optimization

If $y = f(x)$, then $f(x)$ achieves a **maximum** at x^* if $f(x^*) \geq f(x)$ for all x. It can be shown that if $f(x)$ is a smooth function that achieves its maximum value at x^*, then

$$\frac{df(x^*)}{dx} = 0$$
$$\frac{d^2 f(x^*)}{dx^2} \leq 0.$$

These expressions are referred to as the **first-order condition** and the **second-order condition** for a maximum. The first-order condition says that the function is flat at x^*, while the second-order condition says that the function is concave near x^*. Clearly both of these properties have to hold if x^* is indeed a maximum.

We say that $f(x)$ achieves its **minimum** value at x^* if $f(x^*) \leq f(x)$ for all x. If $f(x)$ is a smooth function that achieves its minimum at x^*, then

$$\frac{df(x^*)}{dx} = 0$$
$$\frac{d^2 f(x^*)}{dx^2} \geq 0.$$

The first-order condition again says that the function is flat at x^*, while the second-order condition now says that the function is convex near x^*.

If $y = f(x_1, x_2)$ is a smooth function that achieves its maximum or minimum at some point (x_1^*, x_2^*), then we must satisfy

$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_1} = 0$$
$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_2} = 0.$$

These are referred to as the **first-order conditions**. There are also second-order conditions for this problem, but they are more difficult to describe.

A.15 Constrained Optimization

Often we want to consider the maximum or minimum of some function over some restricted values of (x_1, x_2). The notation

$$\max_{x_1, x_2} f(x_1, x_2)$$

$$\text{such that } g(x_1, x_2) = c.$$

means

find x_1^* and x_2^* such that $f(x_1^*, x_2^*) \geq f(x_1, x_2)$ for all values of x_1 and x_2 that satisfy the equation $g(x_1, x_2) = c$.

The function $f(x_1, x_2)$ is called the **objective function**, and the equation $g(x_1, x_2) = c$ is called the **constraint**. Methods for solving this kind of constrained maximization problem are described in the Appendix to Chapter 5.

ANSWERS

1 The Market

1.1. It would be constant at \$500 for 25 apartments and then drop to \$200.

1.2. In the first case, \$500, and in the second case, \$200. In the third case, the equilibrium price would be any price between \$200 and \$500.

1.3. Because if we want to rent one more apartment, we have to offer a lower price. The number of people who have reservation prices greater than p must always increase as p decreases.

1.4. The price of apartments in the inner ring would go up since demand for apartments would not change but supply would decrease.

1.5. The price of apartments in the inner ring would rise.

1.6. A tax would undoubtedly reduce the number of apartments supplied in the long run.

1.7. He would set a price of 25 and rent 50 apartments. In the second case he would rent all 40 apartments at the maximum price the market would bear. This would be given by the solution to $D(p) = 100 - 2p = 40$, which is $p^* = 30$.

1.8. Everyone who had a reservation price higher than the equilibrium price in the competitive market, so that the final outcome would be Pareto efficient. (Of course in the long run there would probably be fewer new apartments built, which would lead to another kind of inefficiency.)

2 Budget Constraint

2.1. The new budget line is given by $2p_1x_1 + 8p_2x_2 = 4m$.

2.2. The vertical intercept (x_2 axis) decreases and the horizontal intercept (x_1 axis) stays the same. Thus the budget line becomes flatter.

2.3. Flatter. The slope is $-2p_1/3p_2$.

2.4. A good whose price has been set to 1; all other goods' prices are measured relative to the numeraire good's price.

2.5. A tax of 8 cents a gallon.

2.6. $(p_1 + t)x_1 + (p_2 - s)x_2 = m - u$.

2.7. Yes, since all of the bundles the consumer could afford before are affordable at the new prices and income.

3 Preferences

3.1. No. It might be that the consumer was indifferent between the two bundles. All we are justified in concluding is that $(x_1, x_2) \succeq (y_1, y_2)$.

3.2. Yes to both.

3.3. It is transitive, but it is not complete—two people might be the same height. It is not reflexive since it is false that a person is strictly taller than himself.

3.4. It is transitive, but not complete. What if A were bigger but slower than B? Which one would he prefer?

3.5. Yes. An indifference curve can cross itself, it just can't cross another distinct indifference curve.

3.6. No, because there are bundles on the indifference curve that have strictly more of both goods than other bundles on the (alleged) indifference curve.

3.7. A negative slope. If you give the consumer more anchovies, you've made him worse off, so you have to take away some pepperoni to get him back on his indifference curve. In this case the direction of increasing utility is *toward* the origin.

3.8. Because the consumer weakly prefers the weighted average of two bundles to either bundle.

3.9. If you give up one \$5 bill, how many \$1 bills do you need to compensate you? Five \$1 bills will do nicely. Hence the answer is -5 or $-1/5$, depending on which good you put on the horizontal axis.

3.10. Zero—if you take away some of good 1, the consumer needs zero units of good 2 to compensate him for his loss.

3.11. Anchovies and peanut butter, scotch and Kool Aid, and other similar repulsive combinations.

4 Utility

4.1. The function $f(u) = u^2$ is a monotonic transformation for positive u, but not for negative u.

4.2. (1) Yes. (2) No (works for v positive). (3) No (works for v negative). (4) Yes (only defined for v positive). (5) Yes. (6) No. (7) Yes. (8) No.

4.3. Suppose that the diagonal intersected a given indifference curve at two points, say (x, x) and (y, y). Then either $x > y$ or $y > x$, which means that one of the bundles has more of *both* goods. But if preferences are monotonic, then one of the bundles would have to be preferred to the other.

4.4. Both represent perfect substitutes.

4.5. Quasilinear preferences. Yes.

4.6. The utility function represents Cobb-Douglas preferences. No. Yes.

4.7. Because the MRS is measured *along* an indifference curve, and utility remains constant along an indifference curve.

5 Choice

5.1. $x_2 = 0$ when $p_2 > p_1$, $x_2 = m/p_2$ when $p_2 < p_1$, and anything between 0 and m/p_2 when $p_1 = p_2$.

5.2. The optimal choices will be $x_1 = m/p_1$ and $x_2 = 0$ if $p_1/p_2 < b$, $x_1 = 0$ and $x_2 = m/p_2$ if $p_1/p_2 > b$, and any amount on the budget line if $p_1/p_2 = b$.

5.3. Let z be the number of cups of coffee the consumer buys. Then we know that $2z$ is the number of teaspoons of sugar he or she buys. We must satisfy the budget constraint

$$2p_1 z + p_2 z = m.$$

Solving for z we have

$$z = \frac{m}{2p_1 + p_2}.$$

5.4. We know that you'll either consume all ice cream or all olives. Thus the two choices for the optimal consumption bundles will be $x_1 = m/p_1$, $x_2 = 0$, or $x_1 = 0$, $x_2 = m/p_2$.

5.5. This is a Cobb-Douglas utility function, so she will spend $4/(1+4) = 4/5$ of her income on good 2.

5.6. For kinked preferences, such as perfect complements, where the change in price doesn't induce any change in demand.

6 Demand

6.1. No. If her income increases, and she spends it all, she must be purchasing more of at least one good.

6.2. The utility function for perfect substitutes is $u(x_1, x_2) = x_1 + x_2$. Thus if $u(x_1, x_2) > u(y_1, y_2)$, we have $x_1 + x_2 > y_1 + y_2$. It follows that $tx_1 + tx_2 > ty_1 + ty_2$, so that $u(tx_1, tx_2) > u(ty_1, ty_2)$.

6.3. The Cobb-Douglas utility function has the property that

$$u(tx_1, tx_2) = (tx_1)^a (tx_2)^{1-a} = t^a t^{1-a} x_1^a x_2^{1-a} = tx_1^a x_2^{1-a} = tu(x_1, x_2).$$

Thus if $u(x_1, x_2) > u(y_1, y_2)$, we know that $u(tx_1, tx_2) > u(ty_1, ty_2)$, so that Cobb-Douglas preferences are indeed homothetic.

6.4. The demand curve.

6.5. No. Concave preferences can only give rise to optimal consumption bundles that involve zero consumption of one of the goods.

6.6. Normally they would be complements, at least for non-vegetarians.

6.7. We know that $x_1 = m/(p_1 + p_2)$. Solving for p_1 as a function of the other variables, we have

$$p_1 = \frac{m}{x_1} - p_2.$$

6.8. False.

7 Revealed Preference

7.1. No. This consumer violates the Weak Axiom of Revealed Preference since when he bought (x_1, x_2) he could have bought (y_1, y_2) and vice versa. In symbols:

$$p_1 x_1 + p_2 x_2 = 1 \times 1 + 2 \times 2 = 5 > 4 = 1 \times 2 + 2 \times 1 = p_1 y_1 + p_2 y_2$$

and

$$q_1y_1 + q_2y_2 = 2 \times 2 + 1 \times 1 = 5 > 4 = 2 \times 1 + 1 \times 2 = q_1x_1 + q_2x_2.$$

7.2. Yes. No violations of WARP are present, since the y-bundle is not affordable when the x-bundle was purchased and vice versa.

7.3. Since the y-bundle was more expensive than the x-bundle when the x-bundle was purchased and vice versa, there is no way to tell which bundle is preferred.

7.4. If both prices changed by the same amount. Then the base-year bundle would still be optimal.

7.5. Perfect complements.

8 Slutsky Equation

8.1. Yes. To see this, use our favorite example of red pencils and blue pencils. Suppose red pencils cost 10 cents a piece, and blue pencils cost 5 cents a piece, and the consumer spends \$1 on pencils. She would then consume 20 blue pencils. If the price of blue pencils falls to 4 cents a piece, she would consume 25 blue pencils, a change which is entirely due to the income effect.

8.2. Yes.

8.3. Then the income effect would cancel out. All that would be left would be the pure substitution effect, which would automatically be negative.

8.4. They are receiving tx' in revenues and paying out tx, so they are losing money.

8.5. Since their old consumption is affordable, the consumers would have to be at least as well-off. This happens because the government is giving them back *more* money than they are losing due to the higher price of gasoline.

9 Buying and Selling

9.1. Her gross demands are $(9, 1)$.

9.2. The bundle $(y_1, y_2) = (3, 5)$ costs more than the bundle $(4, 4)$ at the current prices. The consumer will not necessarily prefer consuming this

bundle, but would certainly prefer to own it, since she could sell it and purchase a bundle that she would prefer.

9.3. Sure. It depends on whether she was a net buyer or a net seller of the good that became more expensive.

9.4. Yes, but only if the U.S. switched to being a net exporter of oil.

9.5. The new budget line would shift outward and remain parallel to the old one, since the increase in the number of hours in the day is a pure endowment effect.

9.6. The slope will be positive.

10 Intertemporal Choice

10.1. According to Table 10.1, \$1 20 years from now is worth 3 cents today at a 20 percent interest rate. Thus \$1 million is worth $.03 \times 1,000,000 =$ \$30,000 today.

10.2. The slope of the intertemporal budget constraint is equal to $-(1+r)$. Thus as r increases the slope becomes more negative (steeper).

10.3. If goods are perfect substitutes, then consumers will only purchase the cheaper good. In the case of intertemporal food purchases, this implies that consumers only buy food in one period, which may not be very realistic.

10.4. In order to remain a lender after the change in interest rates, the consumer must be choosing a point that he could have chosen under the old interest rates, but decided not to. Thus the consumer must be worse off. If the consumer becomes a borrower after the change, then he is choosing a previously unavailable point that cannot be compared to the initial point (since the initial point is no longer available under the new budget constraint), and therefore the change in the consumer's welfare is unknown.

10.5. At an interest rate of 10%, the present value of \$100 is \$90.91. At a rate of 5% the present value is \$95.24.

11 Asset Markets

11.1. Asset A must be selling for $11/(1 + .10) =$ \$10.

11.2. The rate of return is equal to $(10,000 + 10,000)/100,000 = 20\%$.

11.3. We know that the rate of return on the nontaxable bonds, r, must be such that $(1-t)r_t = r$, therefore $(1-.40).10 = .06 = r$.

11.4. The price today must be $40/(1+.10)^{10} = \$15.42$.

12 Uncertainty

12.1. We need a way to reduce consumption in the bad state and increase consumption in the good state. To do this you would have to *sell* insurance against the loss rather than buy it.

12.2. Functions (a) and (c) have the expected utility property (they are affine transformations of the functions discussed in the chapter), while (b) does not.

12.3. Since he is risk-averse, he prefers the expected value of the gamble, \$325, to the gamble itself, and therefore he would take the payment.

12.4. If the payment is \$320 the decision will depend on the form of the utility function; we can't say anything in general.

12.5. Your picture should show a function that is initially convex, but then becomes concave.

12.6. In order to self-insure, the risks must be independent. However, this does not hold in the case of flood damage. If one house in the neighborhood is damaged by a flood it is likely that all of the houses will be damaged.

13 Risky Assets

13.1. To achieve a standard deviation of 2% you will need to invest $x = \sigma_x/\sigma_m = 2/3$ of your wealth in the risky asset. This will result in a rate of return equal to $(2/3).09 + (1-2/3).06 = 8\%$.

13.2. The price of risk is equal to $(r_m - r_f)/\sigma_m = (9-6)/3 = 1$. That is, for every additional percent of standard deviation you can gain 1% of return.

13.3. According to the CAPM pricing equation, the stock should offer an expected rate of return of $r_f + \beta(r_m - r_f) = .05 + 1.5(.10 - .05) = .125$ or 12.5%. The stock should be selling for its expected present value, which is equal to $100/1.125 = \$88.89$.

14 Consumer's Surplus

14.1. The equilibrium price is \$10 and the quantity sold is 100 units. If the tax is imposed, the price rises to \$11, but 100 units of the good will still be sold, so there is no deadweight loss.

14.2. We want to compute the area under the demand curve to the left of the quantity 6. Break this up into the area of a triangle with a base of 6 and a height of 6 and a rectangle with base 6 and height 4. Applying the formulas from high school geometry, the triangle has area 18 and the rectangle has area 24. Thus gross benefit is 42.

14.3. When the price is 4, the consumer's surplus is given by the area of a triangle with a base of 6 and a height of 6; i.e., the consumer's surplus is 18. When the price is 6, the triangle has a base of 4 and a height of 4, giving an area of 8. Thus the price change has reduced consumer's surplus by \$10.

14.4. Ten dollars. Since the demand for the discrete good hasn't changed, all that has happend is that the consumer has had to reduce his expenditure on other goods by ten dollars.

15 Market Demand

15.1. The inverse demand curve is $P(q) = 200 - 2q$.

15.2. The decision about whether to consume the drug at all could well be price sensitive, so the adjustment of market demand on the extensive margin would contribute to the elasticity of the market demand.

15.3. Revenue is $R(p) = 12p - 2p^2$, which is maximized at $p = 3$.

15.4. Revenue is $pD(p) = 100$, regardless of the price, so all prices maximize revenue.

15.5. True. The weighted average of the income elasticities must be 1, so if one good has a *negative* income elasticity, the other good must have an elasticity *greater* than 1 to get the average to be 1.

16 Equilibrium

16.1. The entire subsidy gets passed along to the consumers if the supply curve is flat, but the subsidy is totally received by the producers when the supply curve is vertical.

16.2. The consumer.

16.3. In this case the demand curve for red pencils is horizontal at the price p_b, since that is the most that they would be willing to pay for a red pencil. Thus, if a tax is imposed on red pencils, consumers will end up paying p_b for them, so the entire amount of the tax will end up being borne by the producers (if any red pencils are sold at all—it could be that the tax would induce the producer to get out of the red pencil business).

16.4. Here the supply curve of foreign oil is flat at \$25. Thus the price to the consumers must rise by the \$5 amount of the tax, so that the net price to the consumers becomes \$30. Since foreign oil and domestic oil are perfect substitutes as far as the consumers are concerned, the domestic producers will sell their oil for \$30 as well and get a windfall gain of \$5 per barrel.

16.5. Zero. The deadweight loss measures the value of lost output. Since the same amount is supplied before and after the tax, there is no deadweight loss. Put another way: the suppliers are paying the entire amount of the tax, and everything they pay goes to the government. The amount that the suppliers would pay to avoid the tax is simply the tax revenue the government receives, so there is no excess burden of the tax.

16.6. Zero revenue.

16.7. It raises negative revenue, since in this case we have a net subsidy of borrowing.

17 Measurement

17.1. This is an example of Simpson's paradox.

17.2. If the coin is fair, then it has a probability 1/2 of coming up heads the first time, 1/2 of coming up heads the second time, and so on. The probability of coming up heads 5 times in a row would be $1/2^5 = 1/32 \sim .03$.

17.3. If you take the natural log of each side, you see that $\log(x) = c + bp$ which is a semi-log demand function.

18 Auctions

18.1. Since the collectors likely have their own values for the quilts, and don't particularly care about the other bidders' values, it is a private-value auction.

18.2. Following the analysis in the text, there are four equally likely configurations of bidders: (8,8), (8,10), (10,8), and (10,10). With zero reservation price, the optimal bids will be (8,9,9,10), resulting in expected profit of \$9. The only candidate for a reservation price is \$10, which yields expected profit of $30/4 = \$7.50$. Hence zero is a profit-maximizing reservation price in this auction.

18.3. Have each person write down a value, then award the two books to the students with the two highest values, but just charge them the bid of the third highest student.

18.4. It was efficient in the sense that it awarded the license to the firm that valued it most highly. But it took a year for this to happen, which is inefficient. A Vickrey auction or an English auction would have achieved the same result more quickly.

18.5. This is a common-value auction since the value of the prize is the same to all bidders. Normally, the winning bidder overestimates the number of pennies in the jar, illustrating the winner's curse.

19 Technology

19.1. Increasing returns to scale.

19.2. Decreasing returns to scale.

19.3. If $a + b = 1$, we have constant returns to scale, $a + b < 1$ gives decreasing returns to scale, and $a + b > 1$ gives increasing returns to scale.

19.4. $4 \times 3 = 12$ units.

19.5. True.

19.6. Yes.

20 Profit Maximization

20.1. Profits will decrease.

20.2. Profit would increase, since output would go up more than the cost of the inputs.

20.3. If the firm really had decreasing returns to scale, dividing the scale of all inputs by 2 would produce more than half as much output. Thus the

subdivided firm would make more profits than the big firm. This is one argument why having everywhere decreasing returns to scale is implausible.

20.4. The gardener has ignored opportunity costs. In order to accurately account for the true costs, the gardener must include the cost of her own time used in the production of the crop, even if no explicit wage was paid.

20.5. Not in general. For example, consider the case of uncertainty.

20.6. Increase.

20.7. The use of x_1 does not change, and profits will increase.

20.8. May not.

21 Cost Minimization

21.1. Since profit is equal to total revenue minus total costs, if a firm is not minimizing costs then there exists a way for the firm to increase profits; however, this contradicts the fact that the firm is a profit maximizer.

21.2. Increase the use of factor 1 and decrease the use of factor 2.

21.3. Since the inputs are identically priced perfect substitutes, the firm will be indifferent between which of the inputs it uses. Thus the firm will use any amounts of the two inputs such that $x_1 + x_2 = y$.

21.4. The demand for paper either goes down or stays constant.

21.5. It implies that $\sum_{i=1}^{n} \Delta w_i \Delta x_i \leq 0$, where $\Delta w_i = w_i^t - w_i^s$ and $\Delta x_i = x_i^t - x_i^s$.

22 Cost Curves

22.1. True, true, false.

22.2. By simultaneously producing more output at the second plant and reducing production at the first plant, the firm can reduce costs.

22.3. False.

23 Firm Supply

23.1. The inverse supply curve is $p = 20y$, so the supply curve is $y = p/20$.

23.2. Set $AC = MC$ to find $10y + 1000/y = 20y$. Solve to get $y^* = 10$.

23.3. Solve for p to get $P_s(y) = (y - 100)/20$.

23.4. At 10 the supply is 40 and at 20 the supply is 80. The producer's surplus is composed of a rectangle of area 10×40 plus a triangle of area $\frac{1}{2} \times 10 \times 40$, which gives a total change in producer's surplus of 600. This is the same as the change in profits, since the fixed costs don't change.

23.5. The supply curve is given by $y = p/2$ for all $p \geq 2$, and $y = 0$ for all $p \leq 2$. At $p = 2$ the firm is indifferent between supplying 1 unit of output or not supplying it.

23.6. Mostly technical (in more advanced models this could be market), market, could be either market or technical, technical.

23.7. That all firms in the industry take the market price as given.

23.8. The market price. A profit-maximizing firm will set its output such that the marginal cost of producing the last unit of output is equal to its marginal revenue, which in the case of pure competition is equal to the market price.

23.9. The firm should produce zero output (with or without fixed costs).

23.10. In the short run, if the market price is greater than the average variable cost, a firm should produce some output even though it is losing money. This is true because the firm would have lost more had it not produced since it must still pay fixed costs. However, in the long run there are no fixed costs, and therefore any firm that is losing money can produce zero output and lose a maximum of zero dollars.

23.11. The market price must be equal to the marginal cost of production for all firms in the industry.

24 Industry Supply

24.1. The inverse supply curves are $P_1(y_1) = 10 + y_1$ and $P_2(y_2) = 15 + y_2$. When the price is below 10 neither firm supplies output. When the price is 15 firm 2 will enter the market, and at any price above 15, both firms are in the market. Thus the kink occurs at a price of 15.

24.2. In the short run, the consumers pay the entire amount of the tax. In the long run it is paid by the producers.

24.3. False. A better statement would be: convenience stores can charge high prices because they are near the campus. Because of the high prices the stores are able to charge, the landowners can in turn charge high rents for the use of the convenient location.

24.4. True.

24.5. The profits or losses of the firms that are currently operating in the industry.

24.6. Flatter.

24.7. No, it does not violate the model. In accounting for the costs we failed to value the rent on the license.

25 Monopoly

25.1. No. A profit-maximizing monopolist would never operate where the demand for its product was inelastic.

25.2. First solve for the inverse demand curve to get $p(y) = 50 - y/2$. Thus the marginal revenue is given by $MR(y) = 50 - y$. Set this equal to marginal cost of 2, and solve to get $y = 48$. To determine the price, substitute into the inverse demand function, $p(48) = 50 - 48/2 = 26$.

25.3. The demand curve has a constant elasticity of -3. Using the formula $p[1 + 1/\epsilon] = MC$, we substitute to get $p[1 - 1/3] = 2$. Solving, we get $p = 3$. Substitute back into the demand function to get the quantity produced: $D(3) = 10 \times 3^{-3}$.

25.4. The demand curve has a constant elasticity of -1. Thus marginal revenue is zero for all levels of output. Hence it can never be equal to marginal cost.

25.5. For a linear demand curve the price rises by half the change in cost. In this case, the answer is \$3.

25.6. In this case $p = kMC$, where $k = 1/(1 - 1/3) = 3/2$. Thus the price rises by \$9.

25.7. Price will be two times marginal cost.

25.8. A subsidy of 50 percent, so the marginal costs facing the monopolist are half the actual marginal costs. This will ensure that price equals marginal cost at the monopolist's choice of output.

25.9. A monopolist operates where $p(y)+y\Delta p/\Delta y = MC(y)$. Rearranging, we have $p(y) = MC(y) - y\Delta p/\Delta y$. Since demand curves have a negative slope, we know that $\Delta p/\Delta y < 0$, which proves that $p(y) > MC(y)$.

25.10. False. Imposing a tax on a monopolist may cause the market price to rise more than, the same as, or less than the amount of the tax.

25.11. A number of problems arise, including: determining the true marginal costs for the firm, making sure that all customers will be served, and ensuring that the monopolist will not make a loss at the new price and output level.

25.12. Some appropriate conditions are: large fixed costs and small marginal costs, large minimum efficient scale relative to the market, ease of collusion, etc.

26 Monopoly Behavior

26.1. Yes, if it can perfectly price discriminate.

26.2. $p_i = \epsilon_i c/(1+\epsilon_i)$ for $i = 1, 2$.

26.3. If he can perfectly price discriminate, he can extract the entire consumers' surplus; if he can charge for admission, he can do the same. Hence, the monopolist does equally well under either pricing policy. (In practice, it is much easier to charge for admission than to charge a different price for every ride.)

26.4. This is third-degree price discrimination. Apparently the Disneyland administrators believe that residents of Southern California have more elastic demands than other visitors to their park.

27 Factor Markets

27.1. Sure. A monopsonist can produce at any level of supply elasticity.

27.2. Since the supply of labor would exceed the demand for labor at such a wage, we would presumably see unemployment.

27.3. We find the equilibrium prices by substituting into the demand functions. Since $p = a - by$, we can use the solution for y to find

$$p = \frac{3a + c}{4}.$$

Since $k = a - 2bx$, we can use the solution for x to find

$$k = \frac{a + c}{2}.$$

28 Oligopoly

28.1. In equilibrium each firm will produce $(a - c)/3b$, so the total industry output is $2(a - c)/3b$.

28.2. Nothing. Since all firms have the same marginal cost, it doesn't matter which of them produces the output.

28.3. No, because one of the choices open to the Stackelberg leader is to choose the level of output it would have in the Cournot equilibrium. So it always has to be able to do at least this well.

28.4. We know from the text that we must have $p[1 - 1/n|\epsilon|] = MC$. Since $MC > 0$, and $p > 0$, we must have $1 - 1/n|\epsilon| > 0$. Rearranging this inequality gives the result.

28.5. Make $f_2(y_1)$ steeper than $f_1(y_2)$.

28.6. In general, no. Only in the case of the Bertrand solution does price equal the marginal cost.

29 Game Theory

29.1. The second player will defect in response to the first player's (mistaken) defection. But then the first player will defect in response to that, and each player will continue to defect in response to the other's defection! This example shows that tit-for-tat may not be a very good strategy when players can make mistakes in either their actions or their perceptions of the other player's actions.

29.2. Yes and no. A player prefers to play a dominant strategy regardless of the strategy of the opponent (even if the opponent plays her own dominant

strategy). Thus, if all of the players are using dominant strategies then it is the case that they are all playing a strategy that is optimal given the strategy of their opponents, and therefore a Nash equilibrium exists. However, not all Nash equilibria are dominant strategy equilibria; for example, see Table 29.2.

29.3. Not necessarily. We know that your Nash equilibrium strategy is the best thing for you to do as long as your opponent is playing her Nash equilibrium strategy, but if she is not then perhaps there is a better strategy for you to pursue.

29.4. Formally, if the prisoners are allowed to retaliate the payoffs in the game may change. This could result in a Pareto efficient outcome for the game (for example, think of the case where the prisoners both agree that they will kill anyone who confesses, and assume death has a very low utility).

29.5. The dominant Nash equilibrium strategy is to defect in every round. This strategy is derived via the same backward induction process that was used to derive the finite 10-round case. The experimental evidence using much smaller time periods seems to indicate that players rarely use this strategy.

29.6. The equilibrium has player B choosing left and player A choosing top. Player B prefers to move first since that results in a payoff of 9 versus a payoff of 1. (Note, however, that moving first is not always advantageous in a sequential game. Can you think of an example?)

30 Game Applications

30.1. In a Nash equilibrium, each player is making a best response to the other player's best response. In a dominant strategy equilibrium, each player's choice is a best response to any choice the other player makes.

30.2. No, because when $r = 1/3$ there is an infinity of best responses, not a single one, as is required for the mathematical definition of a function.

30.3. Not necessarily; it depends on the payoffs of the game. In chicken if both choose to drive straight they receive the worst payoff.

30.4. It is row's expected payoff in the equilibrium strategy of kicking to the left with probability .7, while column jumps to the left with probability .6. We have to sum the payoffs to row over four events: the probability row kicks left and column defends left $\times$ row's payoff in this case + probability

row kicks right and column defends left $\times$ row's payoff in this case, and so on. The numbers are $(.7)(.6)50 + (.7)(.4)80 + (.3)(.6)90 + (.3)(.4)20 = 62$.

30.5. He means that he will bid low in order to get the contract, but then charge high prices subsequently for any changes. The client has to go along, since it is costly for him to switch in the middle of a job.

31 Behavioral Economics

31.1. The first group is more likely to buy, due to the "framing effect."

31.2. The "bracketing effect" makes it likely that the meals chosen by Mary will have more variety.

31.3. From the viewpoint of classical consumer theory, more choice is better. But it is certainly possible that too much choice could confuse the employees, so 10 might be a safer choice. If you did decide to offer 50 mutual funds, it would be a good idea to group them into a relatively small number of categories.

31.4. The probability of heads coming up 3 times in a row is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = .125$. The probability of tails coming up in a row is also .125, so the probability of a run of 3 heads or tails is .25.

31.5. It is called "time inconsistency."

32 Exchange

32.1. Yes. For example, consider the allocation where one person has everything. Then the other person is worse off at this allocation than he would be at an allocation where he had something.

32.2. No. For this would mean that at the allegedly Pareto efficient allocation there is some way to make everyone better off, contradicting the assumption of Pareto efficiency.

32.3. If we know the contract curve, then any trading should end up somewhere on the curve; however, we don't know where.

32.4. Yes, but not without making someone else worse off.

32.5. The value of excess demand in the remaining two markets must sum to zero.

33 Production

33.1. Giving up 1 coconut frees up \$6 worth of resources that could be used to produce 2 pounds (equals \$6 worth) of fish.

33.2. A higher wage would produce a steeper isoprofit line, implying that the profit maximizing level for the firm would occur at a point to the left of the current equilibrium, entailing a lower level of labor demand. However, under this new budget constraint Robinson will want to supply more than the required level of labor (why?) and therefore the labor market will not be in equilibrium.

33.3. Given a few assumptions, an economy that is in competitive equilibrium is Pareto efficient. It is generally recognized that this is a good thing for a society since it implies that there are no opportunities to make any individual in the economy better off without hurting someone else. However, it may be that the society would prefer a different distribution of welfare; that is, it may be that society prefers making one group better off at the expense of another group.

33.4. He should produce more fish. His marginal rate of substitution indicates that he is willing to give up two coconuts for an additional fish. The marginal rate of transformation implies that he only has to give up one coconut to get an additional fish. Therefore, by giving up a single coconut (even though he would have been willing to give up two) he can have an additional fish.

33.5. Both would have to work 9 hours per day. If they both work for 6 hours per day (Robinson producing coconuts, and Friday catching fish) and give half of their total production to the other, they can produce the same output. The reduction in the hours of work from 9 to 6 hours per day is due to rearranging production based on each individual's comparative advantage.

34 Welfare

34.1. The major shortcoming is that there are many allocations that cannot be compared—there is no way to decide between any two Pareto efficient allocations.

34.2. It would have the form: $W(u_1, \ldots, u_n) = \max\{u_1, \ldots, u_n\}$.

34.3. Since the Nietzschean welfare function cares only about the best off individual, welfare maxima for this allocation would typically involve one person getting everything.

34.4. Suppose that this is not the case. Then each individual envies someone else. Let's construct a list of who envies whom. Person A envies someone—call him person B. Person B in turn envies someone—say person C. And so on. But eventually we will find someone who envies someone who came earlier in the list. Suppose the cycle is "C envies D envies E envies C." Then consider the following swap: C gets what D has, D gets what E has, and E gets what C has. Each person in the cycle gets a bundle that he prefers, and thus each person is made better off. But then the original allocation couldn't have been Pareto efficient!

34.5. First vote between $\mathbf{x}$ and $\mathbf{z}$, and then vote between the winner ($\mathbf{z}$) and $\mathbf{y}$. First pair $\mathbf{x}$ and $\mathbf{y}$, and then vote between the winner ($\mathbf{x}$) and $\mathbf{z}$. The fact that the social preferences are intransitive is responsible for this agenda-setting power.

35 Externalities

35.1. True. Usually, efficiency problems can be eliminated by the delineation of property rights. However, when we impose property rights we are also imposing an endowment, which may have important distributional consequences.

35.2. False.

35.3. Come on, your roommates aren't all bad ...

35.4. The government could just give away the optimal number of grazing rights. Another alternative would be to sell the grazing rights. (Question: how much would these rights sell for? Hint: think about rents.) The government could also impose a tax, t per cow, such that $f(c^*)/c^* + t = a$.

36 Information Technology

36.1. They should be willing to pay up to $50, since this is the present value of the profit they can hope to get from that customer in the long run.

36.2. Users would gravitate toward packages with the most users, since that would make it more convenient for them to exchange files and information about how to use the program.

36.3. In this case the profit maximization conditions are identical. If two people share a video, the producer would just double the price and make exactly the same profits.

37 Public Goods

37.1. We want the sum of the marginal rates of substitution to equal the marginal cost of providing the public good. The sum of the MRSs is 20 ($= 10 \times 2$), and the marginal cost is $2x$. Thus we have the equation $2x = 20$, which implies that $x = 10$. So the Pareto efficient number of streetlights is 10.

38 Asymmetric Information

38.1. Since only the low-quality cars get exchanged in equilibrium and there is a surplus of \$200 per transaction, the total surplus created is $50 \times 200 = \$10,000$.

38.2. If the cars were assigned randomly, the average surplus per transaction would be the average willingness to pay, \$1800, minus the average willingness to sell, \$1500. This gives an average surplus of \$300 per transaction and there are 100 transactions, so we get a total surplus of \$30,000, which is much better than the market solution.

38.3. We know from the text that the optimal incentive plan takes the form $s(x) = wx + K$. The wage w must equal the marginal product of the worker, which in this case is 1. The constant K is chosen so that the worker's utility at the optimal choice is $\overline{u} = 0$. The optimal choice of x occurs where price, 1, equals marginal cost, x, so $x^* = 1$. At this point the worker gets a utility of $x^* + K - c(x^*) = 1 + K - 1/2 = 1/2 + K$. Since the worker's utility must equal 0, it follows that $K = -1/2$.

38.4. We saw in the last answer that the profits at the optimal level of production are 1/2. Since $\overline{u} = 0$, the worker would be willing to pay 1/2 to rent the technology.

38.5. If the worker is to achieve a utility level of 1, the firm would have to give the worker a lump-sum payment of 1/2.

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Instructor's Manual

Intermediate Microeconomics

Ninth Edition

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Instructor's Manual
by
Hal R. Varian

Answers to Workouts
by
Hal R. Varian and Theodore C. Bergstrom

W. W. Norton & Company • New York • London

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Printed in the United States of America

Typeset in TEX by Hal R. Varian

NINTH EDITION

ISBN 978-0-393-93676-6

W. W. Norton & Company, Inc., 500 Fifth Avenue, New York, N.Y. 10110
W. W. Norton & Company, Ltd., Castle House, 75/76 Wells Street, London W1T 3QT

www.wwnorton.com

1 2 3 4 5 6 7 8 9 0

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Chapter 1

The Market

This chapter was written so I would have something to talk about on the first day of class. I wanted to give students an idea of what economics was all about, and what my lectures would be like, and yet not have anything that was really critical for the course. (At Michigan, students are still shopping around on the first day, and a good number of them won't necessarily be at the lecture.)

I chose to discuss a housing market since it gives a way to describe a number of economic ideas in very simple language and gives a good guide to what lies ahead. In this chapter I was deliberately looking for *surprising results*—analytic insights that wouldn't arise from "just thinking" about a problem. The two most surprising results that I presented are the condominium example and the tax example in Section 1.6. It is worth emphasizing in class just why these results are true, and how they illustrate the power of economic modeling.

It also makes sense to describe their limitations. Suppose that every condominium conversion involved knocking out the walls and creating two apartments. Then what would happen to the price of apartments? Suppose that the condominiums attracted suburbanites who wouldn't otherwise consider renting an apartment. In each of these cases, the price of remaining apartments would rise when condominium conversion took place.

The point of a simple economic model of the sort considered here is to focus our thoughts on what the relevant effects are, not to come to a once-and-for-all conclusion about the urban housing market. The real insight that is offered by these examples is that you have to consider both the supply *and* the demand side of the apartment market when you analyze the impact of this particular policy.

The only concept that the students seem to have trouble with in this chapter is the idea of Pareto efficiency. I usually talk about the idea a little more than is in the book and rephrase it a few times. But then I tell them not to worry about it too much, since we'll look at it in great detail later in the course.

The workbook problems here are pretty straightforward. The biggest problem is getting the students to draw the true (discontinuous) demand curve, as in Figure 1.1, rather than just to sketch in a downward-sloping curve as in Figure 1.2. This is a good time to emphasize to the students that when they are given numbers describing a curve, they have to use the numbers—they can't just sketch in any old shape.

The Market

A. Example of an economic model — the market for apartments
   1. models are simplifications of reality
   2. for example, assume all apartments are identical
   3. some are close to the university, others are far away
   4. price of outer-ring apartments is **exogenous** — determined outside the model
   5. price of inner-ring apartments is **endogenous** — determined within the model

B. Two principles of economics
   1. **optimization principle** — people choose actions that are in their interest
   2. **equilibrium principle** — people's actions must eventually be consistent with each other

C. Constructing the demand curve
   1. line up the people by willingness-to-pay. See Figure 1.1.
   2. for large numbers of people, this is essentially a smooth curve as in Figure 1.2.

D. Supply curve
   1. depends on time frame
   2. but we'll look at the **short run** — when supply of apartments is fixed.

E. Equilibrium
   1. when demand equals supply
   2. price that clears the market

F. Comparative statics
   1. how does equilibrium adjust when economic conditions change?
   2. "comparative" — compare two equilibria
   3. "statics" — only look at equilibria, not at adjustment
   4. example — increase in supply lowers price; see Figure 1.5.
   5. example — create condos which are purchased by renters; no effect on price; see Figure 1.6.

G. Other ways to allocate apartments
   1. discriminating monopolist
   2. ordinary monopolist
   3. rent control

H. Comparing different institutions
   1. need a criterion to compare how efficient these different allocation methods are.
   2. an allocation is **Pareto efficient** if there is no way to make some group of people better off without making someone else worse off.
   3. if something is *not* Pareto efficient, then there *is* some way to make some people better off without making someone else worse off.
   4. if something is not Pareto efficient, then there is some kind of "waste" in the system.

I. Checking efficiency of different methods
   1. free market — efficient
   2. discriminating monopolist — efficient
   3. ordinary monopolist — not efficient
   4. rent control — not efficient

J. Equilibrium in long run
   1. supply will change
   2. can examine efficiency in this context as well

Chapter 2

Budget Constraint

Most of the material here is pretty straightforward. Drive home the formula for the slope of the budget line, emphasizing the derivation on page 23. Try some different notation to make sure that they see the *idea* of the budget line, and don't just memorize the formulas. In the workbook, we use a number of different choices of notation for precisely this reason. It is also worth pointing out that the slope of a line depends on the (arbitrary) choice of which variable is plotted on the vertical axis. It is surprising how often confusion arises on this point.

Students sometimes have problems with the idea of a numeraire good. They understand the algebra, but they don't understand when it would be used. One nice example is in foreign currency exchange. If you have English pounds and American dollars, then you can measure the total wealth that you have in either dollars or pounds by choosing one or the other of the two goods as numeraire.

In the workbook, students sometimes get thrown in exercises where one of the goods has a negative price, so the budget line has a positive slope. This comes from trying to memorize formulas and figures rather than thinking about the problem. This is a good exercise to go over in order to warn students about the dangers of rote learning!

Budget Constraint

A. Consumer theory: consumers choose the best bundles of goods they can afford.
   1. this is virtually the entire theory in a nutshell
   2. but this theory has many surprising consequences

B. Two parts to theory
   1. "can afford" — **budget constraint**
   2. "best" — according to consumers' **preferences**

C. What do we want to do with the theory?
   1. test it — see if it is adequate to describe consumer behavior
   2. predict how behavior changes as economic environment changes
   3. use observed behavior to estimate underlying values
      a) cost-benefit analysis
      b) predicting impact of some policy

D. Consumption bundle
   1. (x_1, x_2) — how much of each good is consumed
   2. (p_1, p_2) — prices of the two goods
   3. m — money the consumer has to spend
   4. budget constraint: $p_1x_1 + p_2x_2 \leq m$
   5. all (x_1, x_2) that satisfy this constraint make up the **budget set** of the consumer. See Figure 2.1.

E. Two goods
   1. theory works with more than two goods, but can't draw pictures.
   2. often think of good 2 (say) as a composite good, representing money to spend on other goods.
   3. budget constraint becomes $p_1x_1 + x_2 \leq m$.
   4. money spent on good 1 (p_1x_1) plus the money spent on good 2 (x_2) has to be less than or equal to the amount available (m).

F. Budget line
   1. $p_1x_1 + p_2x_2 = m$
   2. also written as $x_2 = m/p_2 - (p_1/p_2)x_1$.
   3. budget line has slope of $-p_1/p_2$ and vertical intercept of m/p_2.
   4. set $x_1 = 0$ to find vertical intercept (m/p_2); set $x_2 = 0$ to find horizontal intercept (m/p_1).
   5. slope of budget line measures opportunity cost of good 1 — how much of good 2 you must give up in order to consume more of good 1.

G. Changes in budget line
   1. increasing m makes parallel shift out. See Figure 2.2.
   2. increasing p_1 makes budget line steeper. See Figure 2.3.
   3. increasing p_2 makes budget line flatter
   4. just see how intercepts change
   5. multiplying all prices by t is just like dividing income by t
   6. multiplying all prices and income by t doesn't change budget line
      a) "a perfectly balanced inflation doesn't change consumption possibilities"

H. The numeraire
   1. can arbitrarily assign one price a value of 1 and measure other price relative to that
   2. useful when measuring relative prices; e.g., English pounds per dollar, 1987 dollars versus 1974 dollars, etc.

I. Taxes, subsidies, and rationing
   1. quantity tax — tax levied on units bought: $p_1 + t$
   2. value tax — tax levied on dollars spent: $p_1 + \tau p_1$. Also known as *ad valorem* tax
   3. subsidies — opposite of a tax
      a) $p_1 - s$
      b) $(1 - \sigma)p_1$

4. lump sum tax or subsidy — amount of tax or subsidy is independent of the consumer's choices. Also called a head tax or a poll tax
5. rationing — can't consume more than a certain amount of some good

J. Example — food stamps
1. before 1979 was an *ad valorem* subsidy on food
   a) paid a certain amount of money to get food stamps which were worth more than they cost
   b) some rationing component — could only buy a maximum amount of food stamps
2. after 1979 got a straight lump-sum grant of food coupons. Not the same as a pure lump-sum grant since could only spend the coupons on food.

Chapter 3

Preferences

This chapter is more abstract and therefore needs somewhat more motivation than the previous chapters. It might be a good idea to talk about relations in general before introducing the particular idea of preference relations. Try the relations of "taller," and "heavier," and "taller and heavier." Point out that "taller and heavier" isn't a complete relation, while the other two are. This general discussion can motivate the general idea of preference relations.

Make sure that the students learn the specific examples of preferences such as perfect substitutes, perfect complements, etc. They will use these examples many, many times in the next few weeks!

When describing the ideas of perfect substitutes, emphasize that the defining characteristic is that the slope of the indifference curves is constant, not that it is -1. In the text, I always stick with the case where the slope is -1, but in the workbook, we often treat the general case. The same warning goes with the perfect complements case. I work out the symmetric case in the text and try to get the students to do the asymmetric case in the workbook.

The definition of the marginal rate of substitution is fraught with "sign confusion." Should the MRS be defined as a negative or a positive number? I've chosen to give the MRS its natural sign in the book, but I warn the students that many economists tend to speak of the MRS in terms of absolute value. Example: diminishing marginal rate of substitution refers to a situation where the *absolute value* of the MRS decreases as we move along an indifference curve. The actual value of the MRS (a negative number) is *increasing* in this movement!

Students often begin to have problems with the workbook exercises here. The first confusion they have is that they get mixed up about the idea that indifference curves measure the directions where preferences are constant, and instead draw lines that indicate the directions that preferences are increasing. The second problem that they have is in knowing when to draw just arbitrary curves that qualitatively depict some behavior or other, and when to draw exact shapes.

Try asking your students to draw their indifference curves between five dollar bills and one dollar bills. Offer to trade with them based on what they draw. In addition to getting them to think, this is a good way to supplement your faculty salary.

Preferences

A. Preferences are relationships between bundles.
   1. if a consumer would choose bundle (x_1, x_2) when (y_1, y_2) is available, then it is natural to say that bundle (x_1, x_2) is preferred to (y_1, y_2) by this consumer.
   2. preferences have to do with the entire *bundle* of goods, not with individual goods.

B. Notation
   1. $(x_1, x_2) \succ (y_1, y_2)$ means the x-bundle is **strictly preferred** to the y-bundle
   2. $(x_1, x_2) \sim (y_1, y_2)$ means that the x-bundle is regarded as **indifferent** to the y-bundle
   3. $(x_1, x_2) \succeq (y_1, y_2)$ means the x-bundle is **at least as good as** (preferred to or indifferent to) the y-bundle

C. Assumptions about preferences
   1. complete — any two bundles can be compared
   2. reflexive — any bundle is at least as good as itself
   3. transitive — if $X \succeq Y$ and $Y \succeq Z$, then $X \succeq Z$
      a) transitivity necessary for theory of *optimal* choice

D. Indifference curves
   1. graph the set of bundles that are indifferent to some bundle. See Figure 3.1.
   2. indifference curves are like contour lines on a map
   3. note that indifference curves describing two distinct levels of preference cannot cross. See Figure 3.2.
      a) proof — use transitivity

E. Examples of preferences
   1. perfect substitutes. Figure 3.3.
      a) red pencils and blue pencils; pints and quarts
      b) constant rate of trade-off between the two goods
   2. perfect complements. Figure 3.4.
      a) always consumed together
      b) right shoes and left shoes; coffee and cream
   3. bads. Figure 3.5.
   4. neutrals. Figure 3.6.
   5. satiation or bliss point Figure 3.7.

F. Well-behaved preferences
   1. monotonicity — more of either good is better
      a) implies indifference curves have negative slope. Figure 3.9.
   2. convexity — averages are preferred to extremes. Figure 3.10.
      a) slope gets flatter as you move further to right
      b) example of non-convex preferences

G. Marginal rate of substitution
   1. slope of the indifference curve
   2. $MRS = \Delta x_2 / \Delta x_1$ along an indifference curve. Figure 3.11.
   3. sign problem — natural sign is negative, since indifference curves will generally have negative slope
   4. measures how the consumer is willing to trade off consumption of good 1 for consumption of good 2. Figure 3.12.

5. measures marginal willingness to pay (give up)
   a) not the same as how much you have to pay
   b) but how much you would be *willing* to pay

Chapter 4

Utility

In this chapter, the level of abstraction kicks up another notch. Students often have trouble with the idea of utility. It is sometimes hard for trained economists to sympathize with them sufficiently, since it seems like such an obvious notion to us.

Here is a way to approach the subject. Suppose that we return to the idea of the "heavier than" relation discussed in the last chapter. Think of having a big balance scale with two trays. You can put someone on each side of the balance scale and see which person is heavier, but you don't have any standardized weights. Nevertheless you have a way to determine whether x is heavier than y.

Now suppose that you decide to establish a scale. You get a bunch of stones, check that they are all the same weight, and then measure the weight of individuals in stones. It is clear that x is heavier than y if x's weight in stones is heavier than y's weight in stones.

Somebody else might use different units of measurements—kilograms, pounds, or whatever. It doesn't make any difference in terms of deciding who is heavier. At this point it is easy to draw the analogy with utility—just as pounds give a way to represent the "heavier than" order numerically, utility gives a way to represent the preference order numerically. Just as the units of weight are arbitrary, so are the units of utility.

This analogy can also be used to explore the concept of a positive monotonic transformation, a concept that students have great trouble with. Tell them that a monotonic transformation is just like changing units of measurement in the weight example.

However, it is also important for students to understand that nonlinear changes of units are possible. Here is a nice example to illustrate this. Suppose that wood is always sold in piles shaped like cubes. Think of the relation "one pile has more wood than another." Then you can represent this relation by looking at the measure of the sides of the piles, the surface area of the piles, or the volume of the piles. That is, x, x^2, or x^3 gives exactly the same comparison between the piles. Each of these numbers is a different representation of the utility of a cube of wood.

Be sure to go over carefully the examples here. The Cobb-Douglas example is an important one, since we use it so much in the workbook. Emphasize that it is just a nice functional form that gives convenient expressions. Be sure to

elaborate on the idea that $x_1^a x_2^b$ is the general form for Cobb-Douglas preferences, but various monotonic transformations (e.g., the log) can make it look quite different. It's a good idea to calculate the MRS for a few representations of the Cobb-Douglas utility function in class so that people can see how to do them and, more importantly, that the MRS doesn't change as you change the representation of utility.

The example at the end of the chapter, on commuting behavior, is a very nice one. If you present it right, it will convince your students that utility is an operational concept. Talk about how the same methods can be used in marketing surveys, surveys of college admissions, etc.

The exercises in the workbook for this chapter are very important since they drive home the ideas. A lot of times, students *think* that they understand some point, but they don't, and these exercises will point that out to them. It is a good idea to let the students discover for themselves that a sure-fire way to tell whether one utility function represents the same preferences as another is to compute the two marginal rate of substitution functions. If they don't get this idea on their own, you can pose it as a question and lead them to the answer.

Utility

A. Two ways of viewing utility
   1. old way
      a) measures how "satisfied" you are
         1) not operational
         2) many other problems
   2. new way
      a) summarizes preferences
      b) a utility function assigns a number to each bundle of goods so that more preferred bundles get higher numbers
      c) that is, $u(x_1, x_2) > u(y_1, y_2)$ if and only if $(x_1, x_2) \succ (y_1, y_2)$
      d) only the ordering of bundles counts, so this is a theory of **ordinal utility**
      e) advantages
         1) operational
         2) gives a complete theory of demand

B. Utility functions are not unique
   1. if $u(x_1, x_2)$ is a utility function that represents some preferences, and $f(\cdot)$ is any increasing function, then $f(u(x_1, x_2))$ represents the same preferences
   2. why? Because $u(x_1, x_2) > u(y_1, y_2)$ only if $f(u(x_1, x_2)) > f(u(y_1, y_2))$
   3. so if $u(x_1, x_2)$ is a utility function then any positive monotonic transformation of it is also a utility function that represents the same preferences

C. Constructing a utility function
   1. can do it mechanically using the indifference curves. Figure 4.2.
   2. can do it using the "meaning" of the preferences

D. Examples
   1. utility to indifference curves
      a) easy — just plot all points where the utility is constant
   2. indifference curves to utility
   3. examples
      a) perfect substitutes — all that matters is total number of pencils, so $u(x_1, x_2) = x_1 + x_2$ does the trick

1) can use any monotonic transformation of this as well, such as $\log(x_1 + x_2)$

b) perfect complements — what matters is the minimum of the left and right shoes you have, so $u(x_1, x_2) = \min\{x_1, x_2\}$ works

c) quasilinear preferences — indifference curves are vertically parallel. Figure 4.4.

1) utility function has form $u(x_1, x_2) = v(x_1) + x_2$

d) Cobb-Douglas preferences. Figure 4.5.

1) utility has form $u(x_1, x_2) = x_1^b x_2^c$

2) convenient to take transformation $f(u) = u^{\frac{1}{b+c}}$ and write $x_1^{\frac{b}{b+c}} x_2^{\frac{c}{b+c}}$

3) or $x_1^a x_2^{1-a}$, where $a = b/(b+c)$

E. Marginal utility

1. extra utility from some extra consumption of one of the goods, holding the other good fixed
2. this is a derivative, but a special kind of derivative — a *partial* derivative
3. this just means that you look at the derivative of $u(x_1, x_2)$ keeping x_2 fixed — treating it like a constant
4. examples
   a) if $u(x_1, x_2) = x_1 + x_2$, then $MU_1 = \partial u/\partial x_1 = 1$
   b) if $u(x_1, x_2) = x_1^a x_2^{1-a}$, then $MU_1 = \partial u/\partial x_1 = a x_1^{a-1} x_2^{1-a}$
5. note that marginal utility depends on which utility function you choose to represent preferences
   a) if you multiply utility times 2, you multiply marginal utility times 2
   b) thus it is not an operational concept
   c) however, MU is closely related to MRS, which is an operational concept
6. relationship between MU and MRS
   a) $u(x_1, x_2) = k$, where k is a constant, describes an indifference curve
   b) we want to measure slope of indifference curve, the MRS
   c) so consider a change (dx_1, dx_2) that keeps utility constant. Then

$$MU_1 dx_1 + MU_2 dx_2 = 0$$
$$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0$$

   d) hence

$$\frac{dx_2}{dx_1} = -\frac{MU_1}{MU_2}$$

   e) so we can compute MRS from knowing the utility function

F. Example

1. take a bus or take a car to work?
2. let x_1 be the time of taking a car, y_1 be the time of taking a bus. Let x_2 be cost of car, etc.
3. suppose utility function takes linear form $U(x_1, \ldots, x_n) = \beta_1 x_1 + \ldots + \beta_n x_n$
4. we can observe a number of choices and use statistical techniques to estimate the parameters β_i that best describe choices
5. one study that did this could forecast the actual choice over 93% of the time
6. once we have the utility function we can do many things with it:
   a) calculate the marginal rate of substitution between two characteristics
      1) how much money would the average consumer give up in order to get a shorter travel time?
   b) forecast consumer response to proposed changes
   c) estimate whether proposed change is worthwhile in a benefit-cost sense

Chapter 5

Choice

This is the chapter where we bring it all together. Make sure that students understand the *method* of maximization and don't just memorize the various special cases. The problems in the workbook are designed to show the futility of memorizing special cases, but often students try it anyway.

The material in Section 5.4 is very important—I introduce it by saying "Why should you *care* that the MRS equals the price ratio?" The answer is that this allows economists to determine something about peoples' trade-offs by observing market prices. Thus it allows for the possibility of benefit-cost analysis.

The material in Section 5.5 on choosing taxes is the first big non-obvious result from using consumer theory ideas. I go over it very carefully, to make sure that students understand the result, and emphasize how this analysis uses the techniques that we've developed. Pound home the idea that the analytic techniques of microeconomics have a big payoff—they allow us to answer questions that we wouldn't have been able to answer without these techniques.

If you are doing a calculus-based course, be sure to spend some time on the appendix to this chapter. Emphasize that to solve a constrained maximization problem, you must have two equations. One equation is the constraint, and one equation is the optimization condition. I usually work a Cobb-Douglas and a perfect complements problem to illustrate this. In the Cobb-Douglas case, the optimization condition is that the MRS equals the price ratio. In the perfect complements case, the optimization condition is that the consumer chooses a bundle at the corner.

Choice

A. Optimal choice
   1. move along the budget line until preferred set doesn't cross the budget set. Figure 5.1.
   2. note that tangency occurs at optimal point — necessary condition for optimum. In symbols: $MRS = -\text{price ratio} = -p_1/p_2$.
      a) exception — kinky tastes. Figure 5.2.
      b) exception — boundary optimum. Figure 5.3.
   3. tangency is not sufficient. Figure 5.4.
      a) unless indifference curves are convex.

   b) unless optimum is interior.
4. optimal choice is demanded bundle
   a) as we vary prices and income, we get demand functions.
   b) want to study how optimal choice — the demanded bundle – changes as price and income change

B. Examples
1. perfect substitutes: $x_1 = m/p_1$ if $p_1 < p_2$; 0 otherwise. Figure 5.5.
2. perfect complements: $x_1 = m/(p_1 + p_2)$. Figure 5.6.
3. neutrals and bads: $x_1 = m/p_1$.
4. discrete goods. Figure 5.7.
   a) suppose good is either consumed or not
   b) then compare $(1, m - p_1)$ with $(0, m)$ and see which is better.
5. concave preferences: similar to perfect substitutes. Note that tangency doesn't work. Figure 5.8.
6. Cobb-Douglas preferences: $x_1 = am/p_1$. Note constant budget shares, $a =$ budget share of good 1.

C. Estimating utility function
1. examine consumption data
2. see if you can "fit" a utility function to it
3. e.g., if income shares are more or less constant, Cobb-Douglas does a good job
4. can use the fitted utility function as guide to policy decisions
5. in real life more complicated forms are used, but basic idea is the same

D. Implications of MRS condition
1. why do we care that $MRS = -$price ratio?
2. if everyone faces the same prices, then everyone has the same local trade-off between the two goods. This is independent of income and tastes.
3. since everyone locally values the trade-off the same, we can make policy judgments. Is it worth sacrificing one good to get more of the other? Prices serve as a guide to relative marginal valuations.

E. Application — choosing a tax. Which is better, a commodity tax or an income tax?
1. can show an income tax is always better in the sense that given any commodity tax, there is an income tax that makes the consumer better off. Figure 5.9.
2. outline of argument:
   a) original budget constraint: $p_1x_1 + p_2x_2 = m$
   b) budget constraint with tax: $(p_1 + t)x_1 + p_2x_2 = m$
   c) optimal choice with tax: $(p_1 + t)x_1^* + p_2x_2^* = m$
   d) revenue raised is tx_1^*
   e) income tax that raises same amount of revenue leads to budget constraint: $p_1x_1 + p_2x_2 = m - tx_1^*$
      1) this line has same slope as original budget line
      2) also passes through (x_1^*, x_2^*)
      3) proof: $p_1x_1^* + p_2x_2^* = m - tx_1^*$
      4) this means that (x_1^*, x_2^*) is affordable under the income tax, so the optimal choice under the income tax must be even better than (x_1^*, x_2^*)
3. caveats
   a) only applies for one consumer — for each consumer there is an income tax that is better

b) income is exogenous — if income responds to tax, problems
c) no supply response — only looked at demand side

F. Appendix — solving for the optimal choice
1. calculus problem — constrained maximization
2. max $u(x_1, x_2)$ s.t. $p_1x_1 + p_2x_2 = m$
3. method 1: write down $MRS = p_1/p_2$ and budget constraint and solve.
4. method 2: substitute from constraint into objective function and solve.
5. method 3: Lagrange's method
   a) write Lagrangian: $L = u(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - m)$.
   b) differentiate with respect to x_1, x_2, λ.
   c) solve equations.
6. example 1: Cobb-Douglas problem in book
7. example 2: quasilinear preferences
   a) max $u(x_1) + x_2$ s.t. $p_1x_1 + x_2 = m$
   b) easiest to substitute, but works each way

Chapter 6

Demand

This is a very important chapter, since it unifies all the material in the previous chapter. It is also the chapter that separates the sheep from the goats. If the student has been paying attention for the previous 5 chapters and has been religiously doing the homework, then it is fairly easy to handle this chapter. Alas, I have often found that students have developed a false sense of confidence after seeing budget constraints, drift through the discussions of preference and utility, and come crashing down to earth at Chapter 6.

So, the first thing to do is to get them to review the previous chapters. Emphasize how each chapter builds on the previous chapters, and how Chapter 6 represents a culmination of this building. In turn Chapter 6 is a foundation for further analysis, and must be mastered in order to continue.

Part of the problem is that there is a large number of new concepts in this chapter: offer curves, demand curves, Engel curves, inferior goods, Giffen goods, etc. A list of these ideas along with their definitions and page references is often helpful just for getting the concepts down pat.

If you are doing a calculus-based course, the material in the appendix on quasilinear preferences is quite important. We will refer to this treatment later on when we discuss consumer's surplus, so it is a good idea to go through it carefully now.

Students usually have a rough time with the workbook problems. In part, I think that this is due to the fact that we have now got a critical mass of ideas, and that it has to percolate a bit before they can start brewing some new ideas. A few words of encouragement help a lot here, as well as drawing links with the earlier chapters. Most students will go back on their own and see what they missed on first reading, if you indicate that is a good thing to do. Remember: the point of the workbook problems is to show the students what they *don't* understand, not to give them a pat on the back. The role of the professor is to give them a pat on the back, or a nudge in the behind, whichever seems more appropriate.

Demand

A. Demand functions — relate prices and income to choices

B. How do choices change as economic environment changes?
   1. changes in income
      a) this is a parallel shift out of the budget line
      b) increase in income increases demand — **normal** good. Figure 6.1.
      c) increase in income decreases demand — **inferior** good. Figure 6.2.
      d) as income changes, the optimal choice moves along the **income expansion path**
      e) the relationship between the optimal choice and income, with prices fixed, is called the **Engel curve**. Figure 6.3.
   2. changes in price
      a) this is a tilt or pivot of the budget line
      b) decrease in price increases demand — **ordinary** good. Figure 6.9.
      c) decrease in price decreases demand — **Giffen** good. Figure 6.10.
      d) as price changes the optimal choice moves along the **offer curve**
      e) the relationship between the optimal choice and a price, with income and the other price fixed, is called the **demand curve**

C. Examples
   1. perfect substitutes. Figure 6.12.
   2. perfect complements. Figure 6.13.
   3. discrete good. Figure 6.14.
      a) reservation price — price where consumer is just indifferent between consuming next unit of good and not consuming it
      b) $u(0, m) = u(1, m - r_1)$
      c) special case: quasilinear preferences
      d) $v(0) + m = v(1) + m - r_1$
      e) assume that $v(0) = 0$
      f) then $r_1 = v(1)$
      g) similarly, $r_2 = v(2) - v(1)$
      h) reservation prices just measure **marginal utilities**

D. Substitutes and complements
   1. increase in p_2 increases demand for x_1 — substitutes
   2. increase in p_2 decreases demand for x_1 — complements

E. Inverse demand curve
   1. usually think of demand curve as measuring quantity as a function of price — but can also think of price as a function of quantity
   2. this is the **inverse demand curve**
   3. same *relationship*, just represented differently

Chapter 7

Revealed Preference

This is a big change of pace, and usually a welcome one. The basic idea of revealed preference, as described in Section 7.1, is a very intuitive one. All I want to do in this chapter is give the students the tools to express that intuition algebraically.

I think that the material in Section 7.3, on recovering preferences, is very exciting. Start out with the idea of indirect revealed preference, as depicted in Figure 7.2. Point out that the optimization model allows us to predict how this person would behave when faced with a choice between (x_1, x_2) and (z_1, z_2), *even though we have never observed the person when faced with this choice!* This is a *big* idea, and a very important one. Again, drive home how the economic model of optimization allows us to make strong predictions about behavior.

Figure 7.3 is the natural extension of this line of reasoning. Given the idea of revealed preference, and more importantly the idea of *indirect* revealed preference, we can determine the shape of underlying indifference curves from looking at choice data. I motivate this in terms of benefit-cost issues, but you could also choose to think about forecasting demand for products in a marketing survey, or similar applications.

Once students understand the idea of revealed preference, they can usually understand the Weak Axiom right away. However, they generally have difficulty in actually checking whether the Weak Axiom is satisfied by some real numbers. I added Section 7.5 for this reason; it just outlines one systematic way to check WARP. The students can omit this in their first reading, but they might want to come back to it when they start to do the exercises. If your students know a little computer programming, you might ask them to think about how to write a computer program to check WARP.

The same comments go for the treatment of the Strong Axiom and checking SARP. This is probably overkill, but I found that students couldn't really handle problem 7.5 in the workbook without some guidance about how to systematically check SARP. Speaking of the workbook, the problems in this section are really fun. I am especially fond of 7.6 and 7.7. Problem 7.9 had some wrong numbers in it in early printings of *Workouts*, so people with old books should be warned.

Finally, the material on index numbers is very worthwhile. Students here about price indices and cost-of-living indices all the time, so it's nice to describe the theory that lies behind these ideas.

Revealed Preference

A. Motivation
   1. up until now we've started with preference and then described behavior
   2. revealed preference is "working backwards" — start with behavior and describe preferences
   3. recovering preferences — how to use observed choices to "estimate" the indifference curves

B. Basic idea
   1. if (x_1, x_2) is chosen when (y_1, y_2) is affordable, then we know that (x_1, x_2) is at least as good as (y_1, y_2)
   2. in equations: if (x_1, x_2) is chosen when prices are (p_1, p_2) and $p_1x_1+p_2x_2 \geq p_1y_1 + p_2y_2$, then $(x_1, x_2) \succeq (y_1, y_2)$
   3. see Figure 7.1.
   4. if $p_1x_1 + p_2x_2 \geq p_1y_1 + p_2y_2$, we say that (x_1, x_2) is **directly revealed preferred** to (y_1, y_2)
   5. if X is directly revealed preferred to Y, and Y is directly revealed preferred to Z (etc.), then we say that X is **indirectly revealed preferred** to Z. See Figure 7.2.
   6. the "chains" of revealed preference can give us a lot of information about the preferences. See Figure 7.3.
   7. the information revealed about tastes by choices can be used in formulating economic policy

C. Weak Axiom of Revealed Preference
   1. recovering preferences makes sense only if consumer is actually maximizing
   2. what if we observed a case like Figure 7.4.
   3. in this case X is revealed preferred to Y and Y is also revealed preferred to X!
   4. in symbols, we have (x_1, x_2) purchased at prices (p_1, p_2) and (y_1, y_2) purchased at prices (q_1, q_2) and $p_1x_1+p_2x_2 > p_1y_1+p_2y_2$ and $q_1y_1+q_2y_2 > q_1x_1 + q_2x_2$
   5. this kind of behavior is inconsistent with the optimizing model of consumer choice
   6. the Weak Axiom of Revealed Preference (WARP) rules out this kind of behavior
   7. WARP: if (x_1, x_2) is directly revealed preferred to (y_1, y_2), then (y_1, y_2) cannot be directly revealed preferred to (x_1, x_2)
   8. WARP: if $p_1x_1+p_2x_2 \geq p_1y_1+p_2y_2$, then it must happen that $q_1y_1+q_2y_2 \leq q_1x_1 + q_2x_2$
   9. this condition can be checked by hand or by computer

D. Strong Axiom of Revealed Preference
   1. WARP is only a necessary condition for behavior to be consistent with utility maximization
   2. Strong Axiom of Revealed Preference (SARP): if (x_1, x_2) is directly or *indirectly* revealed preferred to (y_1, y_2), then (y_1, y_2) cannot be directly or indirectly revealed preferred to (x_1, x_2)
   3. SARP is a *necessary and sufficient* condition for utility maximization
   4. this means that if the consumer is maximizing utility, then his behavior must be consistent with SARP
   5. furthermore if his observed behavior is consistent with SARP, then we can always find a utility function that explains the behavior of the consumer as maximizing behavior.

6. can also be tested by a computer

E. Index numbers
1. given consumption and prices in 2 years, base year b and some other year t
2. how does consumption in year t compare with base year consumption?
3. general form of a consumption index:

$$\frac{w_1 x_1^t + w_2 x_2^t}{w_1 x_1^b + w_2 x_2^b}$$

4. natural to use prices as weights
5. get two indices depending on whether you use period t or period b prices
6. Paasche index uses period t (current period) weights:

$$\frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^t x_1^b + p_2^t x_2^b}$$

7. Laspeyres index uses period b (base period) weights:

$$\frac{p_1^b x_1^t + p_2^b x_2^t}{p_1^b x_1^b + p_2^b x_2^b}$$

8. note connection with revealed preference: if Paasche index is greater than 1, then period t must be better than period b:
   a)
   $$\frac{p_1^t x_1^t + p_2^t x_2^t}{p_1^t x_1^b + p_2^t x_2^b} > 1$$
   b)
   $$p_1^t x_1^t + p_2^t x_2^t > p_1^t x_1^b + p_2^t x_2^b$$
   c) so period t is revealed preferred to period b
9. same sort of thing can be done with Laspeyres index — if Laspeyres index is less than 1, consumer is worse off

Chapter 8

Slutsky Equation

Most books talk about income and substitution effects, but then they don't do anything with the ideas. My view is that you have to give the student enough of an understanding of an idea to be able to compute with it; otherwise, why bother?

The Slutsky decomposition is an analytical tool that allows us to understand how demand changes when a price changes. It does this by breaking the total change in demand up into smaller pieces. The sign of the overall effect depends on the sign of the pieces, but the sign of the pieces is easier to determine.

I have used the Slutsky definition of substitution effect in this chapter. This is because it is much easier to compute examples using this definition. The Hicksian definition is theoretically more elegant, but students can't compute with it until they have more advanced mathematical tools.

A large part of getting this material across is just convincing the students to read the book. The change in income necessary to compensate for a change in price is neither a difficult concept nor a difficult calculation, but it has to be repeated a few times before the students grasp it.

One way to describe income and substitution effects is to give an example based on their own consumption patterns. Talk about a student who spends all of her allowance on food and books. Suppose that the price of books drops in half, but her parents find out about it and cut her allowance. How much do they cut her allowance if they want her to keep her old consumption level affordable?

Once they grasp the idea of the substitution and income effect, it isn't hard to put them together in Section 8.4. The next real hurdle is expressing the Slutsky equation in terms of rates of change, as is done in Section 8.5. This is the way that we usually refer to the Slutsky equation in later chapters, so it is worthwhile going through the algebra so they can see where it comes from. However, if you don't want to go through the algebraic computations, just make sure that they get the basic point: the change in demand can be decomposed into a substitution effect (always negative, i.e., opposite the direction of price change) and an income effect (positive or negative depending on whether we have a normal or inferior good).

I usually skip the Optional sections in this chapter, but they are there for reference if needed. I like the tax rebate section, but it is a little sophisticated. Emphasize the idea that even if you give the money from the tax back to the

consumers, the demand for the good will go down and consumers will be left worse off.

Slutsky Equation

A. We want a way to decompose the effect of a price change into "simpler" pieces.
   1. that's what *analysis* is all about
   2. break up into simple pieces to determine behavior of whole

B. Break up price change into a **pivot** and a **shift** — see Figure 8.2.
   1. these are hypothetical changes
   2. we can examine each change in isolation and look at sum of two changes

C. Change in demand due to pivot is the **substitution effect**.
   1. this measures how demand changes when we change prices, keeping purchasing power fixed
   2. how much would a person demand if he had just enough money to consume the original bundle?
   3. this isolates the pure effect from changing the relative prices
   4. substitution effect *must* be negative due to revealed preference.
      a) "negative" means quantity moves opposite the direction of price

D. Change in demand due to shift is the **income effect**.
   1. increase income, keep prices fixed
   2. income effect can increase or decrease demand depending on whether we have a normal or inferior good

E. Total change in demand is substitution effect plus the income effect.
   1. if good is normal good, the substitution effect and the income effect reinforce each other
   2. if good is inferior good, total effect is ambiguous
   3. see Figure 8.3.

F. Specific examples
   1. perfect complements — Figure 8.4.
   2. perfect substitutes — Figure 8.5.
   3. quasilinear — Figure 8.6.

G. Application — rebating a tax
   1. put a tax on gasoline and return the revenues
   2. original budget constraint: $px^* + y^* = m$
   3. after tax budget constraint: $(p + t)x' + y' = m + tx'$
   4. so consumption after tax satisfies $px' + y' = m$
   5. so (x', y') was affordable originally and rejected in favor of (x^*, y^*)
   6. consumer must be worse off

H. Rates of change
   1. can also express Slutsky effect in terms of *rates of change*
   2. takes the form
   $$\frac{\partial x}{\partial p} = \frac{\partial x^s}{\partial p} - \frac{\partial x}{\partial m} x$$
   3. can interpret each part just as before

Chapter 9

Buying and Selling

The idea of an endowment is an important one, and I wanted to devote a whole chapter to it rather than give it the cursory treatment it gets in most books. It is somewhat unnatural in a two-good context, so it is worth pointing out to students that artificiality and emphasizing that the concept of an endowment does make perfectly good sense in a more general context.

Emphasize the statement in Section 9.3 that an increase in the value of the endowment allows for greater consumption possibilities of both goods. You'll be happy you did this when you discuss present value! Be sure to explain why a consumer would necessarily prefer an endowment with higher value, while she may or may not prefer a consumption bundle with higher value.

The section on price changes is a very nice application of revealed preference arguments. Students often appreciate this idea a lot more after seeing these applications.

The Slutsky equation treatment in this chapter is quite neat, but a trifle involved. Point out that in the original treatment of the Slutsky equation money income didn't change when prices changed—only the purchasing power of the money changed. In this chapter, where consumers get their money from selling their endowments, money income *does* change when purchasing power changes, and this effect has to be accounted for.

I have found that blowing up Figure 9.7 and carefully stepping through the movements is a big help in seeing this point. Point out that if we take away the budget line through point C, we have the standard diagram of the previous chapter. The movement from D to C is the only new thing that has been added in this chapter.

If you've got a group that is pretty comfortable with abstraction, the treatment in the appendix to this chapter will be of interest. It gives an *exact* derivation of the Slutsky equation in this case.

Section 9.7 gives a very short example of the Slutsky equation when an endowment is present. Point out how the result comes solely from the maximization hypothesis, and how hard it would be to figure this out without some analytic tools. That's the point of analytic tools like the Slutsky equation: they make this kind of calculation mechanical so that you don't have to reproduce a complicated path of reasoning in each particular case.

The last topic in the chapter is the analysis of labor supply. The first thing we do is manipulate the budget constraint so it fits into the framework studied earlier. Emphasize that this is a common strategy for analysis: arrange the problem at hand so that it looks like something we've seen before. Also, it is useful to emphasize the interpretation of the endowment in this context: the endowment is what you end up consuming if you don't engage in any market transactions.

Once the labor supply problem has been put in the standard framework, we can apply all the tools that we have at our disposal. The first one is the Slutsky equation. In Section 9.9 I go through a mistaken analysis, and then correct it to give the right analysis. I think that this is appropriate in this case, since so many people get the labor supply analysis wrong. A backward-bending labor supply curve is not a Giffen phenomenon. The supply curve of labor slopes backward because the endowment of leisure is worth more when the wage rate rises, and this can lead to an increased consumption of leisure due to the income effect.

The overtime example is really a dandy illustration of substitution effects. I sometimes introduce the idea by considering the following paradox: if an employer increases a flat wage by some amount, and pays a higher wage for all hours worked, his employees could easily end up choosing to work less. But if the employer pays the *same* increased wage as an overtime wage, the employees will never choose to work less, and will likely choose to work more. Isn't it paradoxical that giving the workers more money (via the flat wage increase) results in less labor forthcoming? Seen in terms of substitution effects and revealed preference, it all makes very good sense, but without those ideas, this common phenomenon can seem very confusing.

Buying and Selling

A. Up until now, people have only had money to exchange for goods. But in reality, people sell things they own (e.g., labor) to acquire goods. Want to model this idea.

B. Net and gross demands
   1. **endowment**: (ω_1, ω_2) — what you have before you enter the market.
   2. **gross demands**: (x_1, x_2) — what you end up consuming.
   3. **net demands**: $(x_1 - \omega_1, x_2 - \omega_2)$ — what you actually buy (positive) and sell (negative).
   4. for economists gross demands are more important; for laypeople net demands are more important.

C. Budget constraint
   1. value of what you consume = value of what you sell.
   2. $p_1 x_1 + p_2 x_2 = p_1 \omega_1 + p_2 \omega_2$
   3. $p_1(x_1 - \omega_1) + p_2(x_2 - \omega_2) = 0$
   4. budget line depicted in Figure 9.1. Note endowment is always affordable.
   5. with two goods, the consumer is always a net demander of one good, a net supplier of the other.

D. Comparative statics
   1. changing the endowment
      a) normal and inferior
      b) increasing the value of the endowment makes the consumer better off. Note that this is different from increasing the value of the consumption bundle. Need access to market.
   2. changing prices

   a) if the price of a good the consumer is selling goes down, and the consumer decides to remain a seller, then welfare goes down. See Figure 9.3.
   b) if the consumer is a net buyer of a good and the price decreases, then the consumer will remain a net buyer. Figure 9.4.
   c) etc.
3. offer curves and demand curves
   a) offer curves — what consumer "offers" to buy or sell
   b) gross demand curve
   c) net demand curves (and net supply curves)

E. Slutsky equation
1. when prices change, we now have three effects
   a) ordinary substitution effect
   b) ordinary income effect
   c) endowment income effect — change in the value of the endowment affects demand.
2. three effects shown in Figure 9.7.
3. the income effect depends on the *net demand*.
4. Slutsky equation now takes the form

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1} + (\omega_1 - x_1)\frac{\partial x_1}{\partial m}$$

5. read through proof in appendix.

F. Labor supply

G. Two goods
1. consumption (C)
2. labor (L) — maximum amount you can work is $\bar{L}$
3. money (M)

H. Budget constraint for labor supply
1. $pC = M + wL$
2. define $\bar{C} = M/p$
3. $pC + w(\bar{L} - L) = p\bar{C} + w\bar{L}$
4. define leisure $R = \bar{L} - L$; note $\bar{R} = \bar{L}$
5. $pC + wR = p\bar{C} + w\bar{R} = p\bar{C} + w\bar{L}$
6. this is just like ordinary budget constraint
7. supply of labor is like demand for leisure
8. w/p is price of leisure

I. Comparative statics
1. apply Slutsky equation to demand for leisure to get

$$\frac{\partial R}{\partial w} = \text{substitution effect} + (\bar{R} - R) \times \text{ income effect}$$

2. increase in the wage rate has an ambiguous effect on supply of labor. Depends on how much labor is supplied already.
3. backward bending labor supply curve

J. Overtime
1. offer workers a higher straight wage, they may work less.
2. offer them a higher overtime wage, they must work at least as much.
3. overtime is a way to get at the substitution effect.

Chapter 10

Intertemporal Choice

This is one of my favorite topics, since it uses consumer theory in such fundamental ways, and yet has many important and practical consequences.

The intertemporal budget constraint is pretty straightforward. I sometimes draw the kinked shape that results from different borrowing and lending rates, just to drive the point home. It is good to spell out the importance of convexity and monotonicity for intertemporal preferences. Ask your students what savings behavior would be exhibited by a person with *convex* intertemporal preferences.

The difference between the present value and the future value formulation of the budget constraint can be seen as a choice of numeraire.

The comparative statics is simply relabeled graphs we've seen before, but it is still worth describing in detail as a concrete example.

I think that it is worth repeating the conclusion of Section 10.6 several times, as students seem to have a hard time absorbing it. An investment that shifts the endowment in a way that increases its present value is an investment that *every* consumer must prefer (as long as they can borrow and lend at the same interest rate). It is a good idea to express this point in several different ways. One especially important way is to talk explicitly about investments as changes in the endowment $(\Delta m_1, \Delta m_2)$, and then point out that any investment with a positive net present value is worthwhile.

Emphasize that present value is really a linear operation, despite appearances. Given a table of present values, as Table 11.1, show how easy it is to calculate present values.

The installment loan example is a very nice one. It is good to motivate it by first considering a person who borrows \$1,000 and then pays back \$1,200 a year later. What rate of interest is he paying? Show that this rate can be found by solving the equation

$$1000(1 + r) = 1200,$$

which can be written as

$$1000 = \frac{1200}{1 + r}.$$

It is then very natural to argue that the *monthly* rate of interest for the installment loan is given by the i that solves the equation

$$1000 = \frac{100}{1+i} + \frac{100}{(1+i)^2} + \ldots + \frac{100}{(1+i)^{12}}$$

There are (at least) two ways to compute the yearly rate. One way is to follow the accountant's convention (and the Truth in Lending Act) and use the formula $r = 12i$. Another, perhaps more sensible, way is to compound the monthly returns and use the formula $1 + r = (1 + i)^{12}$. I followed the accountant's convention in the figures reported in the text.

The workbook problems for this chapter are also quite worthwhile. Problem 11.1 is a nice example of present value analysis, using the perpetuity formulas. Problem 11.6 illustrates the budget constraint with different borrowing and lending rates.

Intertemporal Choice

A. Budget constraint
   1. (m_1, m_2) money in each time period is endowment
   2. allow the consumer to borrow and lend at rate r
   3. $c_2 = m_2 + (1 + r)(m_1 - c_1)$
   4. note that this works for both borrowing and lending, as long as it is at the same interest rate
   5. various forms of the budget constraint
      a) $(1 + r)c_1 + c_2 = (1 + r)m_1 + m_2$ — future value
      b) $c_1 + c_2/(1 + r) = m_1 + m_2/(1 + r)$ — present value
      c) choice of numeraire
      d) see Figure 10.2.
   6. preferences — convexity and monotonicity are very natural

B. Comparative statics
   1. if consumer is initially a lender and interest rate increases, he remains a lender. Figure 10.4.
   2. a borrower is made worse off by an increase in the interest rate. Figure 10.5.
   3. Slutsky allows us to look at the effect of increasing the price of today's consumption (increasing the interest rate)
      a) change in consumption today when interest rate increases = substitution effect + $(m_1 - c_1)$ income effect
      b) assuming normality, an increase in interest rate lowers current consumption for a borrower, and has an ambiguous effect for lender
      c) provide intuition

C. Inflation
   1. put in prices, $p_1 = 1$ and p_2
   2. budget constraint takes the form

$$p_2 c_2 = m_2 + (1 + r)(m_1 - c_1)$$

   3. or

$$c_2 = \frac{m_2}{p_2} + \frac{(1+r)}{p_2}(m_1 - c_1)$$

   4. if π is rate of inflation, then $p_2 = (1 + \pi)p_1$
   5. $1 + \rho = (1 + r)/(1 + \pi)$ is the real interest rate
   6. $\rho = (r - \pi)/(1 + \pi)$ or $\rho \approx r - \pi$

D. Present value — a closer look
   1. future value and present value — what do they mean?
   2. if the consumer can borrow and lend freely, then she would always prefer a consumption pattern with a greater present value.

E. Present value works for any number of periods.

F. Use of present value
   1. the one correct way to rank investment decisions
   2. linear operation, so relatively easy to calculate

G. Bonds
   1. coupon x, maturity date T, face value F
   2. consols
   3. the value of a console is given by $PV = x/r$
      a) proof: $x = r \times PV$

H. Installment loans
   1. borrow some money and pay it back over a period of time
   2. what is the true rate of interest?
   3. example: borrow \$1,000 and pay back 12 equal installments of \$100.
   4. have to value a stream of payments of $1,000, -100, \ldots, -100$.
   5. turns out that the true interest rate is about 35%!

Chapter 11

Asset Markets

This chapter fits in very nicely with the present value calculations in the last chapter. The idea that all riskless assets should earn the same rate of return in equilibrium is a very powerful idea, and generally receives inadequate treatment in intermediate micro texts.

I especially like the arbitrage argument, and showing how it is equivalent to all assets selling for their present values. The applications of the Hotelling oil price model and the forest management model are quite compelling to students.

One interesting twist that you might point out in the forestry problem is that the *market* value of the standing forest will always be its present value, and that present value will grow at the rate of interest—like any other asset. However, the value of the *harvested* forest will grow more rapidly than the interest rate until we reach the optimal harvest time, and then grow less rapidly.

The problems in *Workouts* are quite practical in nature, and it is worth pointing this out to students. Emphasize that present value calculations are the meat and potatoes of investment analysis.

Asset Markets

A. Consider a world of perfect certainty. Then all assets must have the same rate of return.
   1. if one asset had a higher rate of return than another, who would buy the asset with the lower return?
   2. how do asset prices adjust? Answer: **Riskless arbitrage.**
      a) two assets. Bond earns r, other asset costs p_0 now.
      b) invest \$1 in bond, get $1 + r$ dollars tomorrow.
      c) invest $p_0 x = 1$ dollars in other asset, get $p_1 x$ dollars tomorrow.
      d) amounts must be equal, which says that $1 + r = p_1/p_0$.
   3. this is just another way to say present value.
      a) $p_0 = p_1/(1 + r)$.
   4. think about the process of adjustment.

B. Example from stock market
   1. index futures and underlying assets that make up the futures.
   2. no risk in investment, even though asset values are risky, because there is a fixed relationship between the two assets at the time of expiration.

C. Adjustments for differences in characteristics
   1. liquidity and transactions cost
   2. taxes
   3. form of returns — consumption return and financial return

D. Applications
   1. depletable resource — price of oil
      a) let p_t = price of oil at time t
      b) oil in the ground is like money in the bank, so $p_{t+1} = (1 + r)p_t$
      c) demand equals supply over time
      d) let T = time to exhaustion, D = demand per year, and S = available supply. Hence $T = S/D$
      e) let C = cost of next best alternative (e.g., liquified coal)
      f) arbitrage implies $p_0 = C/(1 + r)^T$
   2. harvesting a forest
      a) $F(t)$ = value of forest at time t
      b) natural to think of this increasing rapidly at first and then slowing down
      c) harvest when rate of growth of forest = rate of interest. Figure 11.1.

E. This theory tells you relationships that have to hold between asset prices, given the interest rate.

F. But what determines the interest rate?
   1. answer: aggregate borrowing and lending behavior
   2. or: consumption and investment choices over time

G. What do financial institutions do?
   1. adjust interest rate so that amount people want to borrow equals amount they want to lend
   2. change pattern of consumption possible over time. Example of college student and retiree
   3. example of entrepreneur and investors

Chapter 12

Uncertainty

This chapter begins with the idea of contingent consumption and an insurance market example. Make sure that you define "contingent" since a lot of students don't know the term. (The definition is given in the book.) The emphasis in this first section is on the idea that exactly the same tools that we have used earlier can be used to analyze choice under uncertainty, so it is worth talking about what happens to the budget line when the price of insurance changes, etc.

The expected utility discussion is reasonably elementary. However, it is often hard to motivate the expected utility hypothesis without seeing a lot of applications. I put it in since some schools might want to have an elementary treatment of the subject for use in other courses, such as finance courses.

The easiest application of expected utility theory that I could think of was the result that expected utility maximizers facing actuarially fair insurance would fully insure. In the Information chapter I talk about moral hazard and adverse selection in insurance markets, and those might be fun ideas to touch on in class discussion.

The last three sections on diversification, risk spreading, and the role of the stock market are important economic ideas. I usually discuss these ideas in verbal terms and skip the details of the expected utility material. This seems like a reasonable compromise for a general purpose intermediate micro course.

Uncertainty

A. Contingent consumption
   1. what consumption or wealth you will get in each possible outcome of some random event.
   2. example: rain or shine, car is wrecked or not, etc.
   3. consumer cares about pattern of contingent consumption: $U(c_1, c_2)$.
   4. market allows you to trade patterns of contingent consumption — insurance market. Insurance premium is like a relative price for the different kinds of consumption.
   5. can use standard apparatus to analyze choice of contingent consumption.

B. Utility functions
   1. preferences over the consumption in different events depend on the probabilities that the events will occur.
   2. so $u(c_1, c_2, \pi_1, \pi_2)$ will be the general form of the utility function.
   3. under certain plausible assumptions, utility can be written as being linear in the probabilities, $p_1 u(c_1) + p_2 u(c_2)$. That is, the utility of a pattern of consumption is just the expected utility over the possible outcomes.

C. Risk aversion
   1. shape of expected utility function describes attitudes towards risk.
   2. draw utility of wealth and expected utility of gamble. Note that a person prefers a sure thing to expected value. Figure 12.2.
   3. diversification and risk sharing

D. Role of the stock market
   1. aids in diversification and in risk sharing.
   2. just as entrepreneur can rearrange his consumption patterns through time by going public, he can also rearrange his consumption across states of nature.

Chapter 13

Risky Assets

The first part of this chapter is just notation and review of the concepts of mean and standard deviation. If your students have had some statistics, these ideas should be pretty standard. If they haven't had any statistics, then be sure to get the basics down before proceeding.

The big idea here is in Figure 13.2. In mean-standard deviation space, the "budget constraint" is a straight line. Again, all of the technical apparatus of consumer theory can be brought to bear on analyzing this particular kind of choice problem. Ask what happens to the "price of risk" when the risk-free rate goes down. What do students think this will do to the budget line and the portfolio choice? Don't let them guess—make them give reasons for their statements.

Section 13.2 is a little bit of a fudge. I do give the actual definition of beta in a footnote, but I don't really go through the calculations for the Capital Asset Pricing Model.

The idea of the risk-adjusted interest rate and the story of how returns adjust is a nice one and should be accessible to most students who understood the case of adjustment with certainty.

It might be worth pointing out that participants in the stock market take all this stuff very seriously. There are consulting services that sell their estimates of beta for big bucks and use them as measures of risk all the time.

Risky Assets

A. Utility depends on mean and standard deviation of wealth.
   1. utility $= u(\mu_w, \sigma_w)$
   2. this form of utility function describes tastes.

B. Invest in a risky portfolio (with expected return r_m) and a riskless asset (with return r_f)
   1. suppose you invest a fraction x in the risky asset
   2. expected return $= xr_m + (1-x)r_f$
   3. standard deviation of return $= x\sigma_m$
   4. this relationship gives "budget line" as in Figure 13.2.

C. At optimum we must have the price of risk equal to the slope of the budget line: $MRS = (r_m - r_f)/\sigma_m$
   1. the observable value $(r_m - r_f)/\sigma_m$ is the price of risk
   2. can be used to value other investments, like any other price

D. Measuring the risk of a stock — depends on how it contributes to the risk of the overall portfolio.
   1. β_i = covariance of asset i with the market portfolio/standard deviation of market portfolio
   2. roughly speaking, β_i measures how sensitive a particular asset is to the market as a whole
   3. assets with negative betas are worth a lot, since they reduce risk
   4. how returns adjust — plot the market line

E. Equilibrium
   1. the risk-adjusted rates of return should be equalized
   2. in equations:
   $$r_i - \beta_i(r_m - r_f) = r_j - \beta_j(r_m - r_f)$$
   3. suppose asset j is riskless; then
   $$r_i - \beta_i(r_m - r_f) = r_f$$
   4. this is called the Capital Asset Pricing Model (CAPM)

F. Examples of use of CAPM
   1. how returns adjust — see Figure 13.4.
   2. public utility rate of return choice
   3. ranking mutual funds
   4. investment analysis, public and private

Chapter 14

Consumer's Surplus

This chapter derives consumer's surplus using the demand theory for discrete goods that was developed earlier in Chapters 5 and 6. I review this material in Section 14.1 just to be safe. Given that derivation, it is easy to work backwards to get utility.

Later in the chapter I introduce the idea of compensating and equivalent variation. In my treatment, I use the example of a tax, but another example that is somewhat closer to home is the idea of cost-of-living indexes for various places to live. Take an example of an executive in New York who is offered a job in Tucson. Relative prices differ drastically in these two locations. How much money would the executive need at the Tucson prices to make him as well off as he was in New York? How much money would his New York company have to pay him to make him as well off in New York as he would be if he moved to Tucson?

The example right before Section 14.9 shows that the compensating and the equivalent variation are the same in the case of quasilinear utility. Finally the appendix to this chapter gives a calculus treatment of consumer's surplus, along with some calculations for a few special demand functions and a numerical comparison of consumer's surplus, compensating variation, and equivalent variation.

Consumer's Surplus

A. Basic idea of consumer's surplus
   1. want a measure of how much a person is willing to pay for something. How much a person is willing to sacrifice of one thing to get something else.
   2. price measures marginal willingness to pay, so add up over all different outputs to get total willingness to pay.
   3. total benefit (or gross consumer's surplus), net consumer's surplus, change in consumer's surplus. See Figure 14.1.

B. Discrete demand
   1. remember that the reservation prices measure the "marginal utility"
   2. $r_1 = v(1) - v(0)$, $r_2 = v(2) - v(1)$, $r_3 = v(3) - v(2)$, etc.
   3. hence, $r_1 + r_2 + r_3 = v(3) - v(0) = v(3)$ (since $v(0) = 0$)
   4. this is just the total area under the demand curve.
   5. in general to get the "net" utility, or net consumer's surplus, have to subtract the amount that the consumer has to spend to get these benefits

C. Continuous demand. Figure 14.2.
   1. suppose utility has form $v(x) + y$
   2. then inverse demand curve has form $p(x) = v'(x)$
   3. by fundamental theorem of calculus:

$$v(x) - v(0) = \int_0^x v'(t)\,dt = \int_0^x p(t)\,dt$$

   4. This is the generalization of discrete argument

D. Change in consumer's surplus. Figure 14.3.

E. Producer's surplus — area above supply curve. Change in producer's surplus
   1. see Figure 14.6.
   2. intuitive interpretation: the sum of the marginal willingnesses to supply

F. This all works fine in the case of quasilinear utility, but what do you do in general?

G. Compensating and equivalent variation. See Figure 14.4.
   1. compensating: how much extra money would you need *after* a price change to be as well off as you were before the price change?
   2. equivalent: how much extra money would you need *before* the price change to be just as well off as you would be after the price change?
   3. in the case of quasilinear utility, these two numbers are just equal to the change in consumer's surplus.
   4. in general, they are different ... but the change in consumer's surplus is usually a good approximation to them.

Chapter 15

Market Demand

It would be logical to proceed directly to discussing the theory of the firm, but I wanted to take a break from pure optimization analysis, and discuss instead some ideas from equilibrium analysis. I think that this switch of gears helps students to see where they are going and why all this stuff is useful.

The most important idea in this chapter is elasticity. Elasticity was introduced earlier in Chapter 6, but I never did anything much with it there. Here we can really put it through its paces. The calculations here are all pretty standard, but I'm more careful than usual to distinguish between elasticity and the absolute value of elasticity.

If you use calculus, make sure that you compute elasticities for the linear and log-linear cases.

I love the Laffer curve example in the appendix. Here are some totally trivial elasticity calculations that give a major insight into a big policy issue. I really push on this example in class to show people how what they have learned can really help in making informed judgments about policy.

Market Demand

A. To get market demand, just add up individual demands.
   1. add horizontally
   2. properly account for zero demands; Figure 15.2.

B. Often think of market behaving like a single individual.
   1. **representative consumer** model
   2. not true in general, but reasonable assumption for this course

C. Inverse of aggregate demand curve measures the MRS for each individual.

D. Reservation price model
   1. appropriate when one good comes in large discrete units
   2. reservation price is price that just makes a person indifferent
   3. defined by $u(0, m) = u(1, m - p_1^*)$
   4. see Figure 15.3.
   5. add up demand curves to get aggregate demand curve

E. Elasticity
1. measures responsiveness of demand to price
2. $$\epsilon = \frac{p}{q}\frac{dq}{dp}$$
3. example for linear demand curve
   a) for linear demand, $q = a - bp$, so $\epsilon = -bp/q = -bp/(a - bp)$
   b) note that $\epsilon = -1$ when we are halfway down the demand curve
   c) see Figure 15.4.
4. suppose demand takes form $q = Ap^{-b}$
5. then elasticity is given by
$$\epsilon = -\frac{p}{q}bAp^{-b-1} = \frac{-bAp^{-b}}{Ap^{-b}} = -b$$
6. thus elasticity is constant along this demand curve
7. note that $\log q = \log A - b \log p$
8. what does elasticity depend on? In general how many and how close substitutes a good has.

F. How does revenue change when you change price?
1. $R = pq$, so $\Delta R = (p + dp)(q + dq) - pq = pdq + qdp + dpdq$
2. last term is very small relative to others
3. $dR/dp = q + p\, dq/dp$
4. see Figure 15.5.
5. $dR/dp > 0$ when $|e| < 1$

G. How does revenue change as you change quantity?
1. marginal revenue $= MR = dR/dq = p + q\, dp/dq = p[1 + 1/\epsilon]$.
2. **elastic**: absolute value of elasticity greater than 1
3. **inelastic**: absolute value of elasticity less than 1
4. application: Monopolist never sets a price where $|\epsilon| < 1$ — because it could always make more money by reducing output.

H. Marginal revenue curve
1. always the case that $dR/dq = p + q\, dp/dq$.
2. in case of linear (inverse) demand, $p = a - bq$, $MR = dR/dq = p - bq = (a - bq) - bq = a - 2bq$.

I. Laffer curve
1. how does *tax revenue* respond to changes in tax rates?
2. idea of Laffer curve: Figure 15.8.
3. theory is OK, but what do the magnitudes have to be?
4. model of labor market, Figure 15.9.
5. tax revenue $= T = t\bar{w}S(w(t))$ where $w(t) = (1 - t)\bar{w}$
6. when is $dT/dt < 0$?
7. calculate derivative to find that Laffer curve will have negative slope when
$$\frac{dS}{dw}\frac{w}{S} > \frac{1 - t}{t}$$
8. so if tax rate is .50, would need labor supply elasticity greater than 1 to get Laffer effect
9. very unlikely to see magnitude this large

Chapter 16

Equilibrium

Some people have suggested that it would make more sense to save this chapter until after deriving supply curves, but I still feel that it is in a better position here. After all, the students have seen labor supply curves and net supply curves earlier in the course, and it isn't any shock to see demand and supply treated together now.

The first part of the chapter is pretty standard, although I go to extra pains to be clear to emphasize the idea of the inverse demand and supply curves. I tell the students that the inverse functions describe the same relationship, but just from a different viewpoint.

The treatment of taxes is more thorough than is usually the case. I like the idea of looking at taxation in several different ways. It is a good idea to emphasize that there are really *four* different variables in a taxation problem: the demand price p_d, the supply price p_s, the amount demanded q_d, and the amount supplied q_s. When confronted with a tax problem, the *first* thing you should do is write down the relationships between these four variables.

The most typical set of relationships is

$$p_d = p_s + t$$
$$q_d = q_s$$

But other relationships are possible. For example, if a tax-in-kind is levied, as in the King Kanuta problem in the workbook, then the amount demanded will be *different* than the amount supplied. In fact the only systematic way to work out the King Kanuta problme is to be very careful about writing down the relationships among the four variables.

You should emphasize that the incidence of the tax doesn't depend on the legal requirements of who is responsible for paying the tax. The Social Security tax is a really nice example for this. The Social Security tax is based on 15% of the nominal wage. The employer "pays" half of the tax and the worker "pays" the other half. But of course, this is a fiction. Show the students how we could redefine the nominal wage so that the worker paid all the tax or the employer paid all the tax, and leave the take-home pay of the worker unchanged.

This leads naturally to a discussion of the real incidence of a tax, the ideas of "passing along a tax," and so on.

I like to use the old red pencil/blue pencil example at this point. If red pencils and blue pencils are perfect substitutes in consumption and production, what is the impact of a tax on red pencils? There is a big output effect—consumption and production of red pencils would drop to zero. But what is the effect on consumer utility and producer profits? Zero—consumers and producers just substitute to other activities. This leads naturally to the idea of measuring the impact of a tax via consumer and producer surplus, as is done in Section 16.8.

The two examples that end the chapter, the market for loans and the food subsidies, are really wonderful examples and deserve careful discussion. I like to point out to the students how confused they would be in trying to understand these examples without the analytic methods of economics.

Equilibrium

A. Supply curves — measure amount the supplier wants to supply at each price
   1. review idea of net supply from Chapter 9

B. Equilibrium
   1. competitive market — each agent takes prices as outside his or her control
      a) many small agents
      b) a few agents who think that the others keep fixed prices
   2. equilibrium price — that price where desired demand equals desired supply
      a) $D(p) = S(p)$
   3. special cases — Figure 16.1.
      a) vertical supply — quantity determined by supply, price determined by demand
      b) horizontal supply — quantity determined by demand, price determined by supply
   4. an equivalent definition of equilibrium: where inverse demand curve crosses inverse supply curve
      a) $P_d(q) = P_s(q)$
   5. examples with linear curves

C. Comparative statics
   1. shift each curve separately
   2. shift both curves together

D. Taxes — nice example of comparative statics
   1. demand price and supply price — different in case of taxes
   2. $p_d = p_s + t$
   3. equilibrium happens when $D(p_d) = S(p_s)$
   4. put equations together:
      a) $D(p_s + t) = S(p_s)$
      b) or $D(p_d) = S(p_d - t)$
   5. also can solve using inverse demands:
      a) $P_d(q) = P_s(q) + t$
      b) or $P_d(q) - t = P_s(q)$
   6. see Figure 16.3. and Figure 16.4.

E. Passing along a tax — Figure 16.5.
   1. flat supply curve
   2. vertical supply curve

F. Deadweight loss of a tax — Figure 16.7.
   1. benefits to consumers
   2. benefits to producers
   3. value of lost output

G. Market for loans
   1. tax system subsidizes borrowing, tax lending
   2. with no tax: $D(r^*) = S(r^*)$
   3. with tax: $D((1-t)r') = S((1-t)r')$
   4. hence, $(1-t)r' = r^*$. Quantity transacted is same
   5. see Figure 16.8.

H. Food subsidies
   1. buy up harvest and resell at half price.
   2. before program: $D(p^*) + K = S$
   3. after program: $D(\hat{p}/2) + K = S$
   4. so, $\hat{p} = 2p^*$.
   5. subsidized mortgages — unless the housing stock changes, no effect on cost.

I. Pareto efficiency
   1. efficient output is where demand equals supply
   2. because that is where demand price equals supply price.
   3. that is, the marginal willingness to buy equals the marginal willingness to sell.
   4. deadweight loss measures loss due to inefficiency.

Chapter 17

Measurement

The purpose of this chapter is to "bridge the gap" between the undergraduate theory and econometric classes. It can be covered in a lecture or two and can fit in almost anywhere in the course once you have covered the basics.

I describe the 5 basic things you do with data: summarize, estimate, test, forecast, and predict. Most students will already be familiar with tables and bar charts. Conditional expectation is likely a novel term, but the concept is pretty clear if you use concrete examples.

Simpson's paradox is great fun. Wikipedia has other examples that can be used.

I give a brief introduction to the logic of hypothesis testing, but it is necessarily quite incomplete.

The real meat comes in demand estimation. Here I try to relate the theory of demand estimation to what is done in practice, which is to run an experiment. If we have to use observational data, then we would normally have to worry about identification, which I discuss extensively. I raise the important issue of causal inference, but that is mostly just to make the students aware of the concept, since I can't do a lot with it in this short chapter.

Measurement

A. What do you do with statistics?

B. Example: Coffee consumption
   1. Summarize: how many cups of coffee consumed per person per day?
   2. Estimate: what is the elasticity of demand for coffee?
   3. Test: do men and women drink the same amount of coffee on average?
   4. Forecast: What will the price of coffee be next year?
   5. Predict: What will happen to consumption of coffee if a tax is imposed?

C. Summarizing data with tables and graphs
   1. Summary statistics (mean, median, model)
   2. Conditional mean: an average over those observations that satisfy some other condition

D. Simpson's paradox
   1. Coffee example, admissions example

E. Hypothesis testing
   1. Do men and women drink the same amount of coffee on average?
   2. Sampling variation
   3. Logic of testing: if men and women do drink the same amount, how likely would it be to see a difference as big as we see in the data?
   4. (If the students have taken statistics you might talk about t-tests.)

F. Estimating demand using experimental data
   1. Importance of randomized treatment
   2. Controlling for systematic effects (e.g., weather)
   3. Effect of treatment on population versus effect of treatment on the treated
   4. Experiment should be as close as possible to the proposed policy

G. Estimating demand using observational data

H. How does demand for coffee in the U.S. change as price changes?
   1. Regression analysis: conditional expectation again
   2. How was the data generated?
   3. Did someone choose the price or was it generated by the impersonal forces of supply and demand?

I. Specifying demand function
   1. p_1 = price of coffee, p_2 = price of everything else, m = income.
   2. Demand for single consumer = $D(p_1, p_2, m)$.
   3. Can divide by p_2 to get demand for = $D(p_1/p_2, m/p_2)$
   4. Think of p_2 as being a price index
   5. Demand depends on real price and real income
   6. Add up to get aggregate demand
   7. Functional form: linear, log-linear, semi-log

J. Specifying statistical model
   1. Error term: cumulative effect of other factors
   2. Critical assumption: price and expenditure are not correlated with error
   3. What happens if *only* the price changes, holding everything else constant?
   4. This is the causal effect of price change
   5. Ideally, you run an experiment
   6. In this case, it is reasonable to think that supply-side factors like weather, political events, and transportation costs are what move the price

K. Identification problem
   1. Is demand shifting or supply shifting (or both?)
   2. What can go wrong?
      a) When income is high, consumption is high, price sensitivity is low and firms raise prices
      b) When income is low, consumption is low, price sensitivity is high and firms cut prices
      c) The estimated demand curve slopes the wrong way!
      d) Omitted variable bias
      e) Easy fix in this case, but not so easy in other cases

L. Policy evaluation
   1. Natural experiments
      a) Oregon health care
      b) Police and crime

Chapter 18

Auctions

This is a fun chapter, since it brings in some real-life examples and non-obvious points about a widely-used form of market. The classification is straightforward, but it would be good to ask the students for examples of common-value and private-value auctions. (If there is a resale market, the distinction may be a bit blurred.) You might talk about online auctions like eBay.

The bidding rules are pretty straightforward as well. The interesting part is the section on auction design, which deserves some careful treatment. It is useful to describe the source of inefficiency in the profit-maximization case. Essentially the profit-maximizing monopoly seller restricts expected output, just as the ordinary monopolist restricts actual output.

The Vickrey auction argument is very nice—a little bit abstract, but not too hard to prove.

It is worth going over the bidding ring example in Chapter 24 here, just to show how collusion can work.

Finally the Winner's Curse is a nice story. I know one person who auctions off a jar of pennies, which is a nice common-value auction. He always makes money on the auction, a good example of the winner's curse!

Auctions

A. Auctions are one of the oldest form of markets
   1. 500 BC in Babylon
   2. 1970s offshore oil
   3. 1990s FCC airwave auctions
   4. various privatization projects

B. Classification of auctions
   1. private-value auctions
   2. common-value auctions

C. Bidding rules
   1. English auction, reserve price, bid increment
   2. Dutch auction
   3. sealed-bid auction
   4. Vickrey auction (philatelist auction, second-price auction)

D. Auction design
   1. special case of economic mechanism design
   2. possible goals
      a) Pareto efficiency
      b) profit maximization
   3. Pareto efficiency in private value auction
      a) person who values the good most highly gets it
      b) otherwise would be Pareto improvement possible
   4. Case 1: seller knows values $v_1, \ldots, v_n$
      a) trivial answer: set price at highest value
      b) this is Pareto efficient
   5. Case 2: seller doesn't know value
      a) run English auction
      b) person with highest value gets the good
      c) Pareto efficient
      d) pays price equal to second-highest value
   6. profit maximization in private-value auctions
      a) depends on sellers' beliefs about buyers' values
      b) example: 2 bidders with values of either \$10 or \$100
      c) assume equally likely so possibilities are (10,10), (10,100), (100,10), or (100,100)
      d) minimal bid increment of \$1, flip a coin for ties
      e) revenue will be (10,11,11,100)
      f) expected revenue will be \$33
      g) is this the best the seller can do?
      h) No! If he sets a reserve price of \$100 he gets (0,100,100,100)
      i) expected profit is \$75 which is much better
      j) *not* Pareto efficient
   7. Dutch auction, sealed-bid auction
      a) might not be Pareto efficient
   8. Vickrey auction
      a) if everyone reveals true value will be efficient
      b) but will they want to tell the truth?
      c) Yes! Look at special case of two buyers
      d) payoff = $\text{Prob}(b_1 \geq b_2)[v_1 - b_2]$
      e) if $v_1 > b_2$, want to make probability = 1
      f) if $v_1 < b_2$, want to make probability = 0
      g) it pays to tell the truth (in this case)
      h) note that this is essentially the same outcome as English auction

E. Problems with auctions
   1. susceptible to collusion (bidding rings)
   2. dropping out (Australian satellite-TV licenses)

F. Winner's curse
   1. common value auction
   2. assume that each person bids estimated value
   3. then most optimistic bidder wins
   4. but this is almost certainly an overestimate of value
   5. optimal strategy is to adjust bid downward
   6. amount that you adjust down depends on number of other bidders

Chapter 19

Technology

Here we start our discussion of firm behavior. This chapter discusses the concepts that economists use to describe technologies. Almost all of the material here is quite straightforward, especially given all of the exposure that the students have had to indifference curves, utility functions, etc.

Since students are by now quite familiar with Cobb-Douglas utility functions, you should be sure to emphasize that monotonic transformations are no longer warranted, since now the value of the production function represents some real, physical amount of output. Of course, you could choose to measure the output in different units, in which case the parameters of the production function would change. But given the units of measurement, we don't have any choice about how to measure production.

The new ideas are the ideas of the short and long runs, and the idea of returns to scale. These ideas will show up several times in the next few chapters, so the initial discussion is rather brief. In the workbook we give several examples of technologies and ask about their return-to-scale properties. It's a good idea to work one or two examples to show the students what is going on.

Technology

A. Need a way to describe the technological constraints facing a firm
   1. what patterns of inputs and outputs are feasible?

B. Inputs
   1. factors of production
   2. classifications: labor, land, raw materials, capital
   3. usually try to measure in flows
   4. financial capital vs. physical capital

C. Describing technological constraints
   1. production set — combinations of inputs and outputs that are feasible patterns of production
   2. production function — upper boundary of production set
   3. see Figure 17.1.
   4. isoquants — all combinations of inputs that produce a constant level of output
   5. isoquants (constant output) are just like indifference curves (constant utility)

D. Examples of isoquants
   1. fixed proportions — one man, one shovel
   2. perfect substitutes — pencils
   3. Cobb-Douglas — $y = Ax_1^a x_2^b$
   4. can't take monotonic transformations any more!

E. Well-behaved technologies
   1. monotonic — more inputs produce more output
   2. convex — averages produce more than extremes

F. Marginal product
   1. MP_1 is how much extra output you get from increasing the input of good 1
   2. holding good 2 fixed
   3. $MP_1 = \partial f(x_1, x_2)/\partial x_1$

G. Technical rate of substitution
   1. like the marginal rate of substitution
   2. given by the ratio of marginal products
   3. $$TRS = \frac{dx_2}{dx_1} = -\frac{\partial f/\partial x_1}{\partial f/\partial x_2}$$

H. Diminishing marginal product
   1. more and more of a single input produces more output, but at a decreasing rate. See Figure 17.5.
   2. law of diminishing returns

I. Diminishing technical rate of substitution
   1. equivalent to convexity
   2. note difference between diminishing MP and diminishing TRS

J. Long run and short run
   1. All factors varied — long run
   2. Some factors fixed — short run

K. Returns to scale
   1. constant returns — baseline case
   2. increasing returns
   3. decreasing returns

Chapter 20

Profit Maximization

I start out the chapter with a careful definition of profits: you must value each output and input at its *market* price, whether or not the good is actually sold on a market. This is because the market price measures the price at which you *could* sell the input, and thus measures the true opportunity cost of using the factor in this production process rather than in some other use.

I give some commonplace examples of this idea, but more examples won't hurt. It's good to get this idea across carefully now, since it will make it much easier to discuss the idea of zero long-run profits when it comes up. This idea is usually a stumbling block for students, and a careful examination about just what it is that goes into the definition of economic profits helps a lot in getting the point across.

The material on stock market value is something that is left out of most texts, but since we have had a careful discussion of asset markets, we can draw the link between maximizing profits and maximizing stock market value.

The rest of the material in the chapter is fairly standard. The one novel feature is the revealed profitability approach to firm behavior. This section, Section 18.10, shows how you can use the fact that the firm is maximizing profits to derive comparative statics conclusions. If you have treated revealed preference in consumption carefully, students should have no trouble with this approach.

Profit Maximization

A. Profits defined to be revenues minus costs
   1. value each output and input at its market price — even if it is not sold on a market.
   2. it could be sold, so using it in production rather than somewhere else is an **opportunity cost**.
   3. measure in terms of flows. In general, maximize present value of flow of profits.

B. Stock market value
   1. in world of certainty, stock market value equals present value of stream of profits
   2. so maximizing stock market value is the same as maximizing present value of profits
   3. uncertainty — more complicated, but still works

C. Short-run and long-run maximization
   1. fixed factors — plant and equipment
   2. quasi-fixed factors — can be eliminated if operate at zero output (advertising, lights, heat, etc.)

D. Short-run profit maximization. Figure 18.1.
   1. max $pf(x) - wx$
   2. $pf'(x^*) - w = 0$
   3. in words: the value of the marginal product equals wage rate
   4. comparative statics: change w and p and see how x and $f(x)$ respond

E. Long-run profit maximization
   1. $p\,\partial f/\partial x_1 = w_1$, $p\,\partial f/\partial x_2 = w_2$

F. Profit maximization and returns to scale
   1. constant returns to scale implies profits are zero
      a) note that this doesn't mean that economic factors aren't all appropriately rewarded
      b) use examples
   2. increasing returns to scale implies competitive model doesn't make sense

G. revealed profitability
   1. simple, rigorous way to do comparative statics
   2. observe two choices, at time t and time s
   3. (p^t, w^t, y^t, x^t) and (p^s, w^s, y^s, x^s)
   4. if firm is profit maximizing, then must have

$$p^t y^t - w^t x^t \geq p^t y^s - w^t x^s$$
$$p^s y^s - w^s x^s \geq p^s y^t - w^s x^t$$

   5. write these equations as

$$p^t y^t - w^t x^t \geq p^t y^s - w^t x^s$$
$$-p^s y^t + w^s x^t \geq -p^s y^s + w^s x^s$$

   6. add these two inequalities:

$$(p^t - p^s)y^t - (w^t - w^s)x^t \geq (p^t - p^s)y^s - (w^t - w^s)x^s$$

   7. rearrange:

$$(p^t - p^s)(y^t - y^s) - (w^t - w^s)(x^t - x^s) \geq 0$$

   8. or

$$\Delta p \Delta y - \Delta w \Delta x \geq 0$$

   9. implications for changing output and factor prices

Chapter 21

Cost Minimization

The treatment in this chapter is pretty standard, except for the material on revealed cost minimization. However, by now the students have seen this kind of material three times, so they shouldn't have much difficulty with it.

It is worthwhile emphasizing the difference between the unconditional factor demand functions of Chapter 18 and the conditional factor demand functions of Chapter 19. Here we are looking at the best input choice holding the physical level of output fixed. In Chapter 18 we looked for the best input choice holding the price of output fixed, where the level of output is adjusted to its most profitable level.

The material on returns to scale and the cost function is important to get across, as we will refer in future chapters to cases of increasing average cost, decreasing average cost, etc. It is important to be able to link these ideas to the returns-to-scale ideas discussed in earlier chapters.

The material in Sections 21.4 and 21.5 lays the groundwork for ideas that will be further explored in the next chapter. Both sections are just exploring various definitions. Section 21.4 will be used in discussing the shapes of short-run and long-run cost curves. Section 21.5 will be used to distinguish between two different concepts of fixed costs in the short and long runs.

Cost Minimization

A. Cost minimization problem
   1. minimize cost to produce some given level of output:

$$\min_{x_1,x_2} \; w_1x_1 + w_2x_2$$
$$\text{s.t. } f(x_1,x_2) = y$$

   2. geometric solution: slope of isoquant equals slope of isocost curve. Figure 19.1.
   3. equation is: $w_1/w_2 = MP_1/MP_2$
   4. optimal choices of factors are the **conditional factor demand functions**
   5. optimal cost is the **cost function**
   6. examples
      a) if $f(x_1,x_2) = x_1 + x_2$, then $c(w_1,w_2,y) = \min\{w_1,w_2\}y$
      b) if $f(x_1,x_2) = \min\{x_1,x_2\}$, then $c(w_1,w_2,y) = (w_1+w_2)y$
      c) can calculate other answers using calculus

B. Revealed cost minimization
   1. suppose we hold output fixed and observe choices at different factor prices.
   2. when prices are (w_1^s, w_2^s), choice is (x_1^s, x_2^s), and when prices are (w_1^t, w_2^t), choice is (x_1^t, x_2^t).
   3. if choices minimize cost, then we must have

$$\begin{aligned} w_1^t x_1^t + w_2^t x_2^t &\leq w_1^t x_1^s + w_1^t x_2^s \\ w_1^s x_1^s + w_1^s x_2^s &\leq w_1^s x_1^t + w_2^s x_2^t \end{aligned}$$

   4. this is the **Weak Axiom of Cost Minimization (WACM)**
   5. what does it imply about firm behavior?
   6. multiply the second equation by -1 and get

$$\begin{aligned} w_1^t x_1^t + w_2^t x_2^t &\leq w_1^t x_1^s + w_1^t x_2^s \\ -w_1^s x_1^t - w_1^s x_2^t &\leq -w_1^s x_1^s - w_2^s x_2^s \end{aligned}$$

   7. add these two inequalites:

$$\begin{aligned} (w_1^t - w_1^s)(x_1^t - x_1^s) + (w_2^t - w_2^s)(x_1^t - x_1^s) &\leq 0 \\ \Delta w_1 \Delta x_1 + \Delta w_2 \Delta x_2 &\leq 0 \end{aligned}$$

   8. roughly speaking, "factor demands move opposite to changes in factor prices"
   9. in particular, *factor demand curves must slope downward.*

C. Returns to scale and the cost function
   1. increasing returns to scale implies decreasing AC
   2. constant returns implies constant AC
   3. decreasing returns implies increasing AC

D. Long-run and short-run costs
   1. long run: all inputs variable
   2. short run: some inputs fixed

E. Fixed and quasi-fixed costs
   1. fixed: must be paid, whatever the output level
   2. quasi-fixed: only paid when output is positive (heating, lighting, etc.)

Chapter 22

Cost Curves

Now we get to the standard meat and potatoes of undergraduate microeconomics. The first section lays out the rationale behind U-shaped average cost curves. To me the most natural rationale is constant fixed costs and increasing average variable costs.

The link between marginal costs and variable costs is left out of a lot of books, but is important for understanding producer's surplus.

I am very keen on the cost function $c(y) = y^2 + 1$, and use it in a lot of the examples. Be sure to go over its derivation and show how it gives rise to the various cost curves.

The material on how to get the long-run cost curve from the short-run cost curve is pretty straightforward. It may be a little easier to first do Section 22.5, and then draw in a lot of extra short-run curves to get to the diagram in Figure 22.7.

Cost Curves

A. Family of cost curves
   1. total cost: $c(y) = c_v(y) + F$
   2. $$\frac{c(y)}{y} = \frac{c_v(y)}{y} + \frac{F}{y}$$
      $$AC = AVC + AFC$$
   3. see Figure 20.1.
   4. marginal cost is the change in cost due to change in output $c'(y) = dc(y)/dy = dc_v(y)/dy$
      a) marginal cost equals AVC at zero units of output
      b) goes through minimum point of AC and AVC. Figure 20.2.
         1) $$\frac{d}{dy}\frac{c(y)}{y} = \frac{yc'(y) - c(y)}{y^2}$$
         2) this is negative (for example) when $c'(y) < c(y)/y$

c) fundamental theorem of calculus implies that

$$c_v(y) = \int_0^y c'(t)\,dt$$

d) geometrically: the area under the marginal cost curve gives the total variable costs. Figure 20.3.

e) intuitively: the maginal cost curve measures the cost of each additional unit, so adding up the MCs gives the variable cost

B. Example: $c(y) = y^2 + 1$
   1. $AC = y + 1/y$
   2. $AVC = y$
   3. $MC = 2y$
   4. Figure 20.4.

C. Long-run cost from short-run cost
   1. average costs: Figure 20.8.
   2. marginal costs: Figure 20.9.

Chapter 23

Firm Supply

After all that material on technology and optimization problems, it is fun to get back to the behavior of "real" economic units. I devote a fair amount of time to laying out the idea of a purely competitive market. It is important to distinguish between the definition of a competitive market and the rationale for that definition. The *definition* is that it is a market where firms take the market price as being given, independent of the actions of any particular firm. The usual rationale for this assumption is that each firm is a negligible part of the market.

However, it is also important to emphasize that even markets with a middling number of firms may act in a reasonably competitive fashion. For example, if each firm believes that the other firms will keep their prices fixed no matter what price it charges, we have a model where each firm faces a demand curve for its product that is essentially flat. This idea—the distinction between the market demand curve and the demand curve facing a firm—is an important one to get across. Economists often talk about a quantity-setting firm or a price-setting firm, but these ideas are really rather unnatural. Real firms set both variables. But a firm in a highly competitive market has no real choice about what price to set—it has to meet the price at which everyone else is selling if it wants to make any sales at all. For a competitive firm, the only real choice variable is how much it wants to sell at the going market price.

Firm Supply

A. Firms face two sorts of constraints
   1. technological constraints — summarize in cost function
   2. market constraints — how will consumers and other firms react to a given firm's choice?

B. Pure competition
   1. formally — takes market price as given, outside of any particular firm's control
   2. example: many small price takers
   3. demand curve facing a competitive firm — Figure 21.1.

C. Supply decision of competitive firm
   1. $\max_y py - c(y)$
   2. first-order condition: $p = c'(y)$
   3. price equals marginal cost determines supply as function of price
   4. second-order condition: $-c''(y) \leq 0$, or $c'(y) \geq 0$.
   5. only upward-sloping part of marginal cost curve matters
   6. is it profitable to operate at all?
      a) compare $py - c_v(y) - F$ with $-F$
      b) profits from operating will be greater when $p > c_v(y)/y$
      c) operate when price covers average variable costs

D. So supply curve is the upward-sloping part of MC curve that lies above the AVC curve
   1. see Figure 21.3.

E. Inverse supply curve
   1. $p = c'(y)$ measures the marginal cost curve directly

F. Example: $c(y) = y^2 + 1$
   1. $p = 2y$ gives the (inverse) supply curve
   2. is $p \geq AVC$?
      a) yes, since $2y \geq y$ for all $y \geq 0$
   3. see Figure 21.7.

G. Producer's surplus
   1. producer's surplus is defined to be $py - c_v(y)$
   2. since $c_v(y)$ = area under marginal cost curve
   3. producer's surplus is also the area above the marginal cost curve
   4. we can also use the "rectangle" for part of PS and the "area above MC" for the rest
   5. see Figure 21.5.

H. Long-run supply — use long-run MC. In long run, price must be greater than AC

I. Special case — constant average cost (CRS): flat supply curve
   1. see Figure 21.10.

Chapter 24

Industry Supply

The treatment of industry supply in the case of free entry given in this chapter is more satisfactory than that one usually sees. I simply draw the supply curves for different numbers of firms and look for the lowest intersection that allows for nonnegative profits. After drawing a few examples of this sort, students are quite ready to believe that the equilibrium price can never get very far above minimum average cost. This naturally leads to the standard approximation of taking the supply curve of a competitive industry as being flat at price equals minimum average cost.

The idea that long-run profits are zero in Sections 22.4 and 22.5 is a very important one, and often misunderstood. Be sure to emphasize the exact sense in which it is true.

The other big idea in this chapter is the idea of economic rent. I like to express the relationship between the two ideas this way:

> Long-run profits in competitive industries are always zero. If there are no barriers to entry, then entry competes profits away to zero. If there are specific factors that prevent entry, then competition to acquire those factors forces profits to zero. In a sense, it is always the attempt to enter an industry that forces profits to zero: new firms either enter an industry by adding firms to the industry or by buying out existing firms. The first form of entry increases supply and decreases prices; the second form of entry doesn't affect supply, but simply pushes up the factor prices and costs. But either way, profits get driven to zero.

I like the discussion of economic rent and the politics of rent quite a bit. One great example of rent seeking is to discuss the social costs of theft. It's not the transfer of property that represents a social loss; it's all the expense that one has to go to *to prevent* theft that represents the social loss. The true social cost of theft is not the lost TVs, but the cost of the locks on the doors! If students appreciate the insight in this sentence, they are well on their way to becoming real economists. (If they don't appreciate the insight, they'll just think you're nuts.)

Finally, the treatment of energy policy in Section 24.10 is a lot of fun. The students really begin to appreciate why marginal cost is important after they see this example.

Industry Supply

A. Short-run industry supply
   1. sum of the MC curves
   2. equilibrium in short run
      a) look for point where $D(p) = S(p)$
      b) can then measure profits of firms
      c) see Figure 22.2.

B. Long-run industry supply
   1. change to long-run technology
   2. entry and exit by firms
      a) look at curves with different number of firms
      b) find lowest curve consistent with nonnegative profits
      c) see Figure 22.3.

C. Long-run supply curve
   1. exact — see Figure 22.4.
   2. approximate — flat at $p =$ minimum AC
   3. like replication argument

D. Taxation in long and short runs
   1. see Figure 22.6.
   2. in industry with entry and exit
   3. part of tax is borne by each side
   4. long run — all borne by consumers

E. Meaning of zero profits
   1. pure economic profit means anyone can get it
   2. a mature industry may show accounting profits, but economic profits are probably zero

F. Economic rent
   1. what if some factors are scarce in the long run?
      a) licenses — liquor, taxicab
      b) raw materials, land, etc.
   2. fixed from viewpoint of industry, variable from viewpoint of firm
   3. in this case, industry can only support a certain number of firms
   4. whatever factor is preventing entry earns rents
      a) always the possibility of entry that drives profits to zero
      b) if profits are being made, firms enter industry by
         1) bringing in new resources
         2) bidding up prices of existing resources
   5. see Figure 22.7.
   6. discount flow of rents to get asset value
   7. politics of rent
      a) rents are a pure surplus payment
      b) but people compete for those rents
      c) taxicab licenses — current holders want very much to prevent entry
      d) subsidies and rents — incidence of subsidy falls on the rents
         1) tobacco subsidies
         2) farm policy in general
      e) rent seeking

G. Energy policy
   1. two-tiered oil pricing
   2. price controls
   3. entitlement program

Chapter 25

Monopoly

This chapter discusses the theory of monopoly and compares it with that of competition. The big idea here is the inefficiency of monopoly. The first way to drive it home is to use the fundamental definition of Pareto improvement: whenever price exceeds marginal cost, there must be a whole set of transactions that are Pareto improving. The second way is to add up the consumer and producer surpluses to measure the deadweight loss of monopoly.

The example of an optimal patent life is a nice way to illustrate why society might want to allow certain kinds of monopolies. I often talk about the current hot topic of software manufacturers wanting to protect the "look and feel" of their software.

Monopoly

A. Profit maximization
   1. max $r(y) - c(y)$ implies $r'(y) = c'(y)$
   2. max $p(y)y - c(y)$ implies $p(y) + p'(y)y = c'(y)$
   3. can also write this as

$$p(y)\left[1 + \frac{dp}{dy}\frac{y}{p}\right] = c'(y)$$

   4. or $p(y)[1 + 1/\epsilon] = c'(y)$
   5. linear case
      a) in case of linear demand, $p = a - by$, marginal revenue is given by $MR = a - 2by$
      b) see Figure 23.1.
   6. constant elasticity, $q = Ap^{\epsilon}$
      a) in this case, $MR = p[1 + 1/\epsilon]$
      b) so, optimal condition is $p[1 + 1/\epsilon] = c'(y)$
      c) markup on marginal cost
      d) see Figure 23.2.

B. Taxes
   1. linear case — price goes up by half of tax. Figure 23.3.
   2. log case — price goes up by more than tax, since price is a markup on MC

C. Inefficiency of monopoly
   1. Pareto efficient means no way to make some group better off without hurting some other group
   2. Pareto *in*efficient means that there *is* some way to make some group better off without hurting some other group
   3. monopoly is Pareto inefficient since $P > MC$
   4. measure of the deadweight loss — value of lost output
   5. see Figure 23.5.

D. Patents
   1. sometimes we want to pay this cost of inefficiency
   2. patents: trade-off of innovation against monopoly losses

E. Natural monopoly
   1. public utilities (gas, electricity, telephone) are often thought of as **natural monopolies**
   2. occurs when $p = mc$ is unprofitable — decreasing AC
   3. Figure 23.6.
   4. often occurs when fixed costs are big and marginal costs are small
   5. how to handle
      a) government operates and covers deficit from general revenues
      b) regulates pricing behavior so that price = AC

F. Cause of monopoly
   1. MES large relative to size of market
   2. collusion
   3. law (oranges, sports, etc.)
   4. trademarks, copyrights, brand names, etc.

Chapter 26

Monopoly Behavior

This chapter is concerned with price discrimination, product differentiation, monopolistic competition, and the like.

Price discrimination is a great topic for discussion. It is good to bring up examples of price discrimination from the students' lives. A local movie theater here offers discounts on Tuesday nights. Local bars have happy hours. The electricity company charges nonlinear prices for electricity service. There are many more examples of this sort of thing.

In the fourth edition I have described first-, second- and third-degree price discrimination in a more systematic way. The second-degree analysis is quite interesting, I think, since it uses only consumers' surplus analysis.

The sections on bundling and on two-part tariffs are also quite interesting. There are hundreds of examples of bundling you might discuss. Two-part tariffs are also quite common and have the bonus of serving as a good example of first-degree price discrimination. You might want to talk about the efficiency implications of two-part tariffs.

Finally, the Hotelling boardwalk example is very useful. In the text I emphasize the point that the example can lead to extreme product differentiation, as well as no differentiation, even in the case of two players.

Monopoly Behavior

A. Price discrimination
   1. first degree — perfect price discrimination
      a) gives Pareto efficient output
      b) same as take-it-or-leave-it offer
      c) producer gets all surplus
   2. second degree — nonlinear pricing
      a) two demand curves
      b) would like to charge each full surplus
      c) but have to charge bigger one less to ensure self-selection
      d) but then want to reduce the amount offered to smaller consumer
   3. third degree — most common

a)

$$\max \; p_1(y_1)y_1 + p_2(y_2)y_2 - c(y_1 + y_2)$$

b) gives us the first-order conditions

$$p_1 + p_1'(y_1)y_1 = c'(y_1 + y_2)$$
$$p_2 + p_2'(y_2)y_2 = c'(y_1 + y_2)$$

c) or

$$p_1 \left[1 - \frac{1}{|\epsilon_1|}\right] = MC$$
$$p_2 \left[1 - \frac{1}{|\epsilon_2|}\right] = MC$$

d) result: if $p_1 > p_2$, then $|\epsilon_1| < |\epsilon_2|$
e) more elastic users pay lower prices

B. Two-part tariffs
1. what happens if everyone is the same?
2. entrance fee = full surplus
3. usage fee = marginal cost

C. Bundling
1. type A: wtp \$120 for word processor, \$100 for spreadsheet
2. type B: wtp \$100 for word processor, \$120 for spreadsheet
3. no bundling profits = \$400
4. bundling profits =\$440
5. reduce dispersion of wtp

D. Monopolistic competition
1. rare to see pure monopoly
2. product differentiation – so some market power
3. free entry
4. result — excess capacity theorem
   a) see Figure 24.3.
   b) (but is it really?)
5. location model of product differentiation
   a) ice cream vendors on the boardwalk
   b) socially optimal to locate at 1/4 and 3/4
   c) but this is "unstable"
   d) only stable configuration is for both to locate at middle
   e) is there too much conformity in differentiated markets?

Chapter 27

Factor Markets

I added this chapter in order to discuss some of the problems with imperfect competition in factor markets. There are three topics: a monopolist's demand for a factor, monopsony, and vertically integrated monopolies.

The monopolist's demand is a snap if the students know a little calculus. If they don't, you have to talk your way through it. I'm not sure that this topic really deserves much emphasis if the students don't have the math to handle it.

The section on monopsony is pretty standard; the minimum wage example is useful. You might talk some about the situation in professional sports, to show students that there *are* real-life examples of monopsonies.

I'm very fond of the integrated monopoly example. The fact that the integrated monopoly has a *lower* price is surprising. It shows that in antitrust, sometimes the cure is worse than the disease!

Factor Markets

A. Monopoly in output market
   1. marginal product, MP_x
   2. marginal revenue, MR_y
   3. marginal revenue product, MRP_x
   4. value of the marginal product, pMP_x
      a) $MRP = p\left[1 - \frac{1}{|\epsilon|}\right]$
      b) note that this is less than value of MP

B. Monopoly/monoposony in input market
   1. market power by demander of factor
   2. maximize $pf(x) - w(x)x$
   3. get $MR = MC$, but with particular form
   4. now $MC = w\left[1 + \frac{1}{\epsilon}\right]$
   5. linear example: Figure 25.2.
   6. minimum wage

C. Upstream and downstream monopoly
   1. one monopolist produces a factor that he sells to another monopolist
   2. suppose that one unit of the input produces one unit of output in downstream monopolist
   3. each monopolist wants to mark up its output price over marginal cost
   4. results in a **double markup**
   5. if firms integrated, would only have a single markup
   6. price would go down

Chapter 28

Oligopoly

This chapter is a serious attempt to convey some of the standard models of strategic interaction to intermediate microeconomics students. This is an ambitious goal, but with some motivation it can be done. I have pursued a middle ground in this chapter between the traditional approach to oligopoly and the more modern game theoretic approach.

I've departed from the standard order of discussing things here since I think that it is *much* clearer the way I do it.

I start with a little classification scheme: firms can choose prices or quantities, and they can move simultaneously or sequentially. This gives us four cases to analyse. You might discuss other strategic variables at this point: product differentiation, investment decisions, entry, etc.

I proceed to analyse the case of quantity leadership—the Stackelberg model. Here you should emphasize the importance of thinking strategically: putting yourself in the shoes of the other guy and thinking about how he will react to your choices. Once that insight is there, it is fairly straightforward to do the analysis.

The next case to look at is the case of price leadership. The logic is just the same, and the calculations are even easier.

Then we move to simultaneous quantity setting—the Cournot/Nash model. I have been careful to phrase the concept of a Cournot equilibrium as an equilibrium in beliefs as well as actions—each firm is maximizing given its beliefs about the other firm's choices, and each firm finds that its beliefs are confirmed in equilibrium. I find that it is very useful to calculate out an equilibrium example, so that students can see the richness of the idea involved. The graphical treatment is also very helpful.

Section 28.7, on adjustment to equilibrium, is a little bit of a cheat. This is not really consistent with a thoroughgoing game theoretic analysis, but I put it in anyway since the students seem to like it. It shows in a graphic way how an apparently sensible adjustment process can lead to the Cournot equilibrium.

Section 28.8, on many firms, is a very nice illustration of what the idea of a "demand curve facing a firm" looks like. The idea that a Cournot equilibrium approaches the competitive equilibrium as market shares go to zero is a useful one, and the calculations in this section motivate this idea quite powerfully.

Next, I treat simultaneous price setting—the Bertrand model. I like the interpretation of "bidding" for the consumers. There are a number of real-world examples where forcing firms to make sealed bids results in much lower prices. The logic behind this is essentially that of Bertrand competition. In Ann Arbor, the local coursepack providers quote you a 5-cents-a-page price for copying, but the sealed bids usually end up under 2.5 cents.

Section 28.10 on collusion is also very important. I usually motivate this using OPEC as an example. Each firm negotiates to set a quota that maximizes overall cartel profits ... and then each firm goes home and tries to cheat on the cartel. It is worth pointing out that equation 28.6 implies that the smaller firm 1's output is, relative to firm 2, the more incentive firm 2 has to cheat on the cartel agreement. This is true since

$$\frac{\Delta\pi_1}{\Delta y_1} = -\frac{\Delta p}{\Delta Y} y_2^*.$$

If the output of firm 2 is large, then $\Delta\pi_1/\Delta y_1$ will be large.

In Figure 28.5 it is useful to point out that the reason that we get a whole range of outputs that maximize industry profits is that we have assumed that marginal costs are identical—in fact, we have assumed that they are zero for both firms. If the marginal costs were different, we would most likely get a unique cartel solution.

Oligopoly

A. Oligopoly is the study of the interaction of a small number of firms
   1. duopoly is simplest case
   2. unlikely to have a general solution; depends on market structure and specific details of how firms interact

B. Classification of theories
   1. non-collusive
      a) sequential moves
         1) quantity setting — Stackelberg
         2) price setting — price leader
      b) simultaneous moves
         1) quantity setting — Cournot
         2) price setting — Bertrand
   2. collusive

C. Stackelberg behavior
   1. asymmetry — one firm, quantity leader, gets to set quantity first
   2. maximize profits, given the reaction behavior of the other firm
   3. take into response that the other firm will follow my lead
   4. analyze in reverse
   5. firm 2
      a) $\max_{y_2} P(y_1 + y_2)y_2 - c(y_2)$
      b) FOC: $P(y_1 + y_2) + P'(y_1 + y_2)y_2 = c'(y_2)$
      c) solution gives **reaction function**, $f_2(y_1)$
   6. firm 1
      a) $\max_{y_1} P(y_1 + f_2(y_1))y_1 - c(y_1)$
      b) FOC: $P(y_1 + f_2(y_1)) + P'(y_1 + f_2(y_1))y_1 = c'(y_1)$
      c) see Figure 26.2.
   7. graphical solution in Figure 26.4.

D. Price-setting behavior
   1. leader sets price, follower takes it as given
   2. given p_1, firm 2 supplies $S_2(p_1)$
   3. if demand is $D(p)$, this leaves $D(p_1) - S_2(p_1)$ for leader
   4. hence leader wants to maximize $p_1 y_1 - c(y_1)$ such that $y_1 = D(p_1) - S_2(p_1)$
   5. leader faces "residual demand curve"

E. Cournot equilibrium — simultaneous quantity setting
   1. each firm makes a choice of output, given its forecast of the other firm's output
   2. let y_1 be the output choice of firm 1 and y_2^e be firm 1's beliefs about firm 2's output choice
   3. maximization problem $\max_{y_1} \; p(y_1 + y_2^e)y_1 - c(y_1)$
   4. let $Y = y_1 + y_2^e$
   5. first-order condition is

$$p(Y) + p'(Y)y_1 = c'(y_1)$$

   6. this gives firm 1's reaction curve — how it chooses output given its beliefs about firm 2's output
   7. see Figure 26.1.
   8. look for Cournot equilibrium — where each firm finds its expectations confirmed in equilibrium
   9. so $y_1 = y_1^e$ and $y_2 = y_2^e$

F. Example of Cournot
   1. assume zero costs
   2. linear demand function $p(Y) = a - bY$
   3. profit function: $[a - b(y_1 + y_2)]y_1 = ay_1 - by_1^2 - by_1 y_2$
   4. derive reaction curve
      a) maximize profits
      b) $a - 2by_1 - by_2 = 0$
      c) calculate to get $y_1 = (a - by_2)/2b$
      d) do same sort of thing to get reaction curve for other firm
   5. look for intersection of reaction curves

G. Bertrand – simultaneous price setting
   1. consider case with constant identical marginal cost
   2. if firm 1 thinks that other firm will set p_2, what should it set?
   3. if I think p_2 is greater than my MC, set p_1 slightly smaller than p_2
   4. I get all the customers and make positive profits
   5. only *consistent* (equilibrium) beliefs are $p_1 = p_2 = MC$

H. Collusion
   1. firms get together to maximize joint profits
   2. marginal impact on joint profits from selling output of either firm must be the same
   3. $\max p(y_1 + y_2)[y_1 + y_2] - c(y_1) - c(y_2)$
   4. $P(y_1 + y_2) + P'(y_1 + y_2)[y_1 + y_2] = c'(y_1) = c'(y_2)$
   5. note instability — if firm 1 believes firm 2 will keep its output fixed, it will always pay it to increase its own output
   6. problems with OPEC
   7. if it doesn't believe other firm will keep its output fixed, it will cheat first!

Chapter 29

Game Theory

This is a fun chapter. Students like it a lot, and faculty usually enjoy teaching it. Game theory is hot stuff in economics these days, and this chapter tries to convey some of the reasons why.

The first two equilibrium concepts, that of a dominant strategy equilibrium and that of a Nash equilibrium, are reasonably easy to convey. The idea of a Nash equilibrium in mixed strategies is a little harder. Here's an example that will motivate the idea.

Consider the game of baseball. The pitcher has two strategies: pitch high or pitch low. Likewise, the batter has two strategies, swing high or swing low. If the batter connects, he gets a payoff of 1 and the pitcher gets zero. If the batter misses, the pitcher gets a payoff of 1.

What are the Nash equilibria in this game? If the pitcher always pitches high, the batter will always swing high, and if the pitcher always pitches low, then the batter will always swing low. It is clear from this observation—and from observing baseball games—that the equilibrium strategy must involve a *mixed strategy*. The pitcher will flip a coin and decide whether to pitch high or low, and the batter flips a coin to decide whether to swing high or low. The batter will connect 50% of the time. Here students are very willing to accept that the optimal strategy must involve randomization.

If you really want to get them buzzing, you can talk about the following paradox. If the batter really believes that the pitcher will really randomize 50–50, then he might as well swing high all the time. But of course, once the pitcher detects this departure from randomizing, he will modify his own behavior to exploit the batter's sloppiness. This example drives home the important point that what keeps the players at the Nash equilibrium is the desire to avoid being psyched out by their opponents.

Most students have heard of the prisoners' dilemma by now, but they haven't seen the analysis of the repeated game. The reason why the repeated game is different from the one-shot game is that in the repeated game, the strategy choice at time t can depend on the entire history of the game up until t. Thus choices at time $t-1$ may have some influence on choices at time t. This opens the possibility of tit-for-tat and other strategies that can allow for cooperative solutions.

The analysis of the sequential games, and especially the game of entry deterrence, is a very interesting topic. Students really get excited about this kind of analysis since they think that it will help them be better managers. (Well, who knows, maybe it will!)

Game Theory

A. Game theory studies strategic interaction, developed by von Neumann and Morgenstern around 1950

B. How to depict payoffs of game from different strategies
   1. two players
   2. two strategies
   3. example

		Column	
		Left	Right
Row	Top	$1, 2$	$0, 1$
	Bottom	$2, 1$	$1, 0$

   a) this depicts a **dominant strategy**
   b) each person has a strategy that is best no matter what the other person does
   c) nice when it happens, but doesn't happen that often

C. Nash equilibrium
   1. what if there is no dominant strategy?
   2. in this case, look for strategy that is best if the other player plays his best strategy
   3. note the "circularity" of definition
   4. appropriate when you are playing against a "rational" opponent
   5. each person is playing the best given his expectations about the other person's play and expectations are actually confirmed
   6. example

		Column	
		Left	Right
Row	Top	$2, 1$	$0, 0$
	Bottom	$0, 0$	$1, 2$

   a) note (top, left) is Nash; (bottom, right) is also Nash
   7. Nash equilibrium in pure strategies may not exist.

		Column	
		Left	Right
Row	Top	$0, 0$	$0, -1$
	Bottom	$1, 0$	$-1, 3$

   8. but if allow mixed strategies (and people only care about *expected payoff*), then Nash equilibrium will always exist

D. Prisoner's dilemma
   1. 2 prisoners, each may confess (and implicate other) or deny
   2. gives payoff matrix

		Column	
		Left	Right
Row	Top	$-3, -3$	$0, -6$
	Bottom	$-6, 0$	$-1, -1$

   3. note that (confess, confess) is unique dominant strategy equilibrium, but (deny, deny) is Pareto efficient
   4. example: cheating in a cartel
   5. example: agreeing to get rid of spies
   6. problem — no way to communicate and make binding agreements

E. Repeated games
   1. if game is repeated with same players, then there may be ways to enforce a better solution to prisoner's dilemma
   2. suppose PD is repeated 10 times and people know it
      a) then backward induction says it is a dominant strategy to cheat every round
   3. suppose that PD is repeated an indefinite number of times
      a) then may pay to cooperate
   4. Axelrod's experiment: tit-for-tat

F. Example – enforcing cartel and price wars

G. Sequential game — time of choices matters

H. Example:

		Column	
		Left	Right
Row	Top	$1, 9$	$1, 9$
	Bottom	$0, 0$	$2, 1$

   1. (Top, Left) and (Bottom, Right) are both Nash equilibria
   2. but in extensive form (Top, Left) is not reasonable. Figure 27.6.
   3. to solve game, start at end and work backward
   4. (Top, Left) is not an equilibrium, since the choice of Top is not a credible choice

I. Example: entry deterrence
   1. stay out and fight
   2. excess capacity to prevent entry — change payoffs
   3. see Figure 27.7.
   4. strategic inefficiency

Chapter 30

Game Applications

This is a chapter for those who want more game theory. Basically it is a set of examples of game theory in action. Most of the examples are self-contained so you can pick and choose what you want.

There are 4 major themes: cooperation (games of coordination), competition (zero-sum games), coexistence (evolutionary games), and commitment (subgame perfection).

I start with a description of best response functions, since they are a useful tool for calculating Nash equilibria. This part just formalizes some of the material already introduced in the last chapter.

I then describe the basic coordination games: battle of the sexes, prisoner's dilemma, assurance games, and chicken. The idea of a focal point and of the importance of observability and committment are introduced here.

I then move on to games of competition: zero-sum games, using a soccer free kick as an example. This section illustrates the importance of mixed strategies.

The next section is on coexistence, which is a very brief introduction to evolutionary game theory. I fudge the topic of ESS since I didn't want to introduce all the details.

Finally I look at a number of sequential games where committment is important.

Note that Schelling's game of the "kindly kidnapper" has a plot line similar to that of the movie *Nasty People*. A kidnapper grabs a victim and then gets cold feet. The problem is that if he releases his victim, the rational strategy for the victim is to go to the police and identify the kidnapper. The problem with this sequential game is that the victim has no way to commit to staying away from the police.

Schelling's solution is characteristically inventive: he suggests that the victim allow the kidnapper to photograph him in some unspeakably disgusting act. This gives the kidnapper a threat—if the victim ever exposes the kidnapper, the kidnapper can release the photo. The students think that this game is great fun. You can ask them to suggest various "unspeakably disgusting acts" that the victim might suggest.

Game Applications

A. Last chapter presented the basics, here we apply them
   1. four themes: cooperation, competition, coexistence, and commitment

B. Best response curves
   1. two-person game (for simplicity)
   2. best response curve: choice that maximizes one player's payoff, given the other player's choice
   3.

		Column	
		Left	Right
Row	Top	$2, 1$	$0, 0$
	Bottom	$0, 0$	$1, 2$

      a) if column chooses Left, row should choose Top
      b) if column chooses Right, row's best response is Bottom
   4. note that (Top, Left) is mutually consistent
      a) if row thinks column will play Left, row wants to play Top
      b) if column thinks row will play Top, column wants to play Left
      c) this is a Nash equilibrium

C. Definition of Nash equilibrium
   1. $b_c(r)$ = column's best response to row's choice r
   2. $b_r(c)$ = row's best response to column's choice c
   3. Nash equilibrium

$$c^* = b_c(r^*)$$
$$r^* = b_r(c^*).$$

D. Examples: Cournot equilibrium is Nash in output; Bertrand equilibrium is Nash in prices

E. Mixed strategies
   1. choices are probabilities of playing different pure strategies
   2. best response is then a choice of probabilities

F. Games of coordination
   1. Battle of the sexes
      a) boy and girl want to meet, but each prefers a different place
      b) e.g., an action movie or an art film
      c)

		Boy	
		Action	Art
Girl	Action	$2, 1$	$0, 0$
	Art	$0, 0$	$1, 2$

      d) three equilibria: both Art, both Action, or each chooses their preferred place with probability $2/3$
      e) focal point: something outside the game that helps them pick an equilibrium, such as which theater is closest
   2. Prisoner's dilemma
      a) want to coordinate on deny
      b) dominate strategy to confess
   3. Assurance game
      a) U.S.-U.S.S.R missile deployment

   b) best for both to refrain from deployment
   c) but if other deploys, each wants to as well
   d) solutions: inspections
4. Chicken
   a) two cars race towards each other
   b) choices: swerve or straight
   c) whoever swerves loses face, but if neither swerves, both crash
   d) one solution: commitment (tie steering wheel)
5. Commitment
   a) one player moves first
   b) commits to a course of action
   c) good for assurance, battle of sexes; doesn't help with prisoner's dilemma

G. Competition
1. zero-sum games: what one gets the other loses
2. penalty kick in soccer
   a) kicker: kick left, kick right
   b) goalie: defend left, defend right
   c) obviously want a mixed strategy (why?)
   d) see example in book
   e) each player wants to maximize his minimum payoff (maxmin)

H. Coexistence
1. evolutionary forces adjust population size
2. mixed equilibrium comes from different behaviors in population
3. example: hawk-dove game
   a) wild dogs can either fight or share
   b) fighting works well when other dog is dovish
   c) fighting works poorly when other dog is hawkish
   d) sharing works well when other dog is dovish
   e) sharing works poorly when other dog is hawkish
   f) suppose that type with higher payoff reproduces more rapidly
   g) if payoff to being a hawk is higher than to being a dove, then more hawks, tending to reduce the payoff to being a hawk
   h) thus there will be some equilibrium fraction of hawk players
   i) ESS = evolutionarily stable strategy
   j) ESS equilibrium is also Nash equilibrium

I. Commitment
1. frog and scorpion
   a) scorpion says: carry me across river
   b) frog says: you'll sting me
   c) what are payoffs?
2. kindly kidnapper
   a) wants to let hostage go, but fears being identified
   b) how can hostage commit to not turning the kidnapper in?
3. pigs playing games
   a) dominant pig and subordinate pig in a pen
   b) press lever at one end, food comes at other
   c) who presses the lever, who gets the food?
   d) how can dominant pig commit to sharing?
4. savings and Social Security
   a) oldsters: save or spend
   b) youngsters: support or let starve
   c) subgame perfect equilibrium: spend, support

5. Hold up
   a) contractor: charge high price, charge cost
   b) client: pay the money, look for another painter
   c) contractor charges high price, client pays
   d) solutions: contracts, post bonds, reputation, repetition

J. Bargaining
1. Rubinstein bargaining model
   a) one dollar to divide
   b) Alice makes offer, Bob can make counteroffer, etc.
   c) value of dollar shrinks each day (due to time value of money)
   d) start on last day and work backward
2. ultimatum game
   a) looks at last move of Rubinstein model
   b) how do people actually behave?
   c) tend to reject "unfair" offers
   d) not necessarily rational behavior for chooser, but divider has to recognize the possibility of irrational behavior

Chapter 31

Behavioral Economics

This is a new chapter on one of the hottest fields in economics, behavioral economics. I've tried to provide a survey of some of the most interesting results, but there is lots of new and interesting material available on the web, so be sure to look there as well.

I've found that everybody can identify with this material, as we are all subject to periods of non-rational behavior.

Most of this material can be introduced right after the chapters on consumer theory, but I put it after game theory since some of the material at the end of the chapter is based on strategic issues.

Behavioral Economics

A. Consumer choice theory is elegant but not always accurate
   1. Behavioral economics combines insights of economics and psychology
   2. Often based on psychology experiments

B. Framing effects
   1. Choices may depend on how they are presented or framed
   2. A serious disease threatens 600 people. You are offered a choice between two treatments, A and B, which will yield the following outcomes.
      a) **Treatment A.** Saving 200 lives for sure.
      b) **Treatment B.** A 1/3 chance of saving 600 lives and a 2/3 chance of saving no one.
   3. Which would you choose? Now consider:
      a) **Treatment C.** Having 400 people die for sure.
      b) **Treatment D.** A 2/3 chance of 600 people dying and a 1/3 chance of no one dying.
   4. Which would you choose?
   5. Many people choose A over B, but D over C.
   6. But they are the same!

C. Anchoring effects
   1. Spurious information can influence choices
   2. Spin a wheel of fortune
   3. Is the number of African countries in the UN greater or less than the number shown?
   4. Then asked: how many countries in Africa?
   5. The number shown on wheel of fortune had an influence on answer.
   6. Wine experiment
      a) Would you pay an amount for this bottle of wine equal to the last 2 digits of your Social Security number?
      b) How much are you willing to pay?
      c) Again, the first answer influenced the second.
   7. Pension plans
      a) People's behavior in signing up for plans is highly influence by default choice
      b) Could get more savings by changing default

D. Bracketing
   1. Hard to predict your own behavior
   2. Snack choice
      a) Pick in advance
      b) Pick at each class
      c) Choices made in advance had more variety

E. Too much choice
   1. Jam experiment
      a) One booth had 24 flavors
      b) Other booth had 6 flavors
      c) Stopped more at first, bought more at second
   2. Constructed preferences
      a) According to psychologies we discover our preferences
      b) Watch the process of choice in action
      c) Conventional theory takes preferences as preexisting
      d) Psychological evidence suggests they are constructed
      e) Not necessarily incompatible

F. Uncertainty
   1. Law of small numbers: people unduly influenced by small samples
   2. Large hospital: 45 babies born each day
   3. Small hospital: 15 babies born each day
   4. During 1 year, each hospital recorded the days during which more than 60 percent of the babies born were boys
   5. Which one had more such days?
   6. Survey results
      a) 22% thought the larger hospital had more such days
      b) 56% thought the days would be about the same
      c) Only 22% had correct answer: smaller hospital would have more such days
   7. People find it very difficult to recognize randomness
      a) Write down 150 random coin tosses
      b) 15 percent of these sequences had heads or tails 3 times in a row
      c) Would occur randomly about 25 percent of the time
      d) Important in game theory due to mixed strategies
      e) Tennis example: serving is not quite random, but close

G. Asset integration and loss aversion
   1. Should care in principle about total wealth
   2. But many people care about the gamble level itself
      a) Suppose you make $100,000 per year
      b) Offered $14 if heads comes up, lose $10 if tails comes up
      c) Positive expected value, trivial effect on wealth
      d) But many people won't take bet
      e) In insurance markets, people often overinsure against small events
      f) Risk aversion or loss aversion?

H. Coffee mug experiment
   1. Give half the class coffee mugs
   2. Asked those with mugs: what is lowest price at which you would sell?
   3. Asked those without mugs: what is highest price at which you would buy?
   4. Answer should be about the same, but were quite different
   5. The subjects with mugs were more reluctant to part with them
   6. Similar to sunk cost fallacy

I. Time discounting
   1. Exponential discounting v hyperbolic discounting
   2. Time consistency and self control
   3. Time inconsistency: plans differ from outcomes
   4. How can I commit myself to future choices?
      a) Pills for alcoholics?
      b) Staple stomach?

J. Overconfidence in stock trading
   1. Infrequent traders: 18% return
   2. Frequent traders: 11.3% return
   3. An important factor: gender
   4. Trading can be hazardous to your wealth!

K. Social norms
   1. Ultimatum game
   2. What affects behavior?
      a) Gender
      b) Culture
      c) Specify what you would do ahead of time or in real time
      d) Fairness
      e) Size of pie
   3. Punishment games
      a) Third parties will punish those who make unfair divisions, even if it costs them

L. Assessment
   1. No theory is 100 percent correct (at least in social sciences)
   2. How big a violation do we need to reject theory?
   3. People don't even get simple physics problems right, even though we live in a physical world
   4. Apparently intuitive physics is good enough for everyday life
   5. Markets may help correct serious biases (such as sunk cost)
   6. Some behavioral anomalies appear to be similar to optical illusions
   7. Others are more profound
   8. Best advice: be critical, objective so far as possible

Chapter 32

Exchange

This chapter starts out with a relatively standard treatment of trade in an Edgeworth box. This leads naturally to the idea of Pareto efficient allocations as the outcome of a voluntary trading process. Given the many possibilities that can result from unstructured voluntary trade, I then turn to examining a particular mechanism for trade: the competitive market mechanism.

It is important to emphasize that if there are really only two players, the market mechanism isn't very plausible. We assume that our two players take prices as given; this is sensible only in a model with many players. One way out of this problem is to suppose that there are one hundred A players and one hundred B players, and that the Edgeworth box depicts the bundle that each type has. If there are two hundred small consumers in the Edgeworth box, then there is no problem with them behaving competitively.

The rest of the treatment here is fairly standard. The *reductio ad absurdum* proof in 32.10 throws a few students—they've generally forgotten any logic they've learned by the time they get to college. So if you want to go over this proof, you should remind them of the logic that it uses.

The main problem with presenting the two welfare theorems is that the students don't have any other examples of resource allocation mechanisms with which to compare the Walrasian market. That's why I like the monopoly in the Edgeworth box example. A standard monopolist in the Edgeworth box gives an example of a market-based resource allocation system that results in a Pareto inefficient allocation. A perfectly discriminating monopolist gives an example of a market-based resource allocation scheme other than pure competition that results in Pareto efficiency. These two examples help to illustrate the richness of the idea of Pareto efficiency, as well as some of its limitations.

The implications of the first and second welfare theorems are profound, but it is sometimes hard to convey that profundity. It helps people to see the various aspects of these ideas if they can discuss them a little.

Exchange

A. Partial equilbrium — theory of single market

B. General equilibrium — interactions among many markets
   1. complements and substitutes
   2. prices affect income . . . but income affects prices

C. We do pure exchange first, then production

D. Edgeworth box
   1. Figure 28.1.
   2. allocation
   3. feasible allocation
   4. consumption bundles
   5. initial endowment
   6. final allocation

E. Trade
   1. move to Pareto preferred point
   2. keep going until no more mutally preferred trades

F. Pareto efficient allocations
   1. where trade stops — no mutual improvement possible
   2. Pareto efficient — no way to make both people better off
   3. indifference curves must be tangent
   4. Pareto set, or contract curve — locus of all PE points

G. Market trade
   1. specific way to trade — using price system
   2. gross demands and net demands; Figure 28.3.
   3. market equilibrium — where supply equals demand
   4. see Figure 28.4.

H. Algebra
   1. only one of the markets needs to clear
   2. Walras's law: if each individual satisfies his or her budget constraint, then the market as a whole must satisfy its budget constraint
   3. existence of equilibrium?

I. Efficiency
   1. does the market exhaust all the gains from trade?
   2. is the market outcome efficient?
   3. First Theorem of Welfare Economics — yes
   4. is any efficient allocation a market equilbrium?
   5. Second Theorem of Welfare Economics — yes, if things are appropriately convex
   6. see Figure 28.8.

J. Meaning of First Welfare Theorem
   1. implicit assumptions — no externalities
   2. competitive behavior
   3. existence
   4. shows that there is a general mechanism that will achieve efficient outcomes
   5. can decentralize decisions

K. Meaning of Second Welfare Theorem
   1. prices play allocative and distributive role
   2. use market for allocative role and income redistribution for distributive role
   3. but problems in production economy
      a) how to measure endowments?
      b) how to redistribute endowments?

Chapter 33

Production

In this chapter I describe a general equilibrium model of production, the classical Robinson Crusoe economy. I usually start my lecture by apologizing for the two-good, one-person nature of this example, since this is a context where a two-good treatment seems quite unnatural. On the other hand, there isn't much way to avoid this unnatural discussion and still stick to a graphical treatment.

The fundamental idea is that the price system serves as a way to decentralize resource allocation problems. Robinson, the consumer, only has to know the public prices, his own income, and his own tastes. Robinson, the producer, only has to know the prices. The consumer doesn't have to know anything about what is technologically feasible, and the firm doesn't have to know anything about tastes. All of the relevant information about tastes and technology end up being summarized in the equilibrium prices.

This decentralization role of the price system isn't very interesting in the one- or two-person economy, but if there are thousands of people, it can be extremely important. Thus it is important to understand those cases where the price system works well as a decentralization device and those cases where it works poorly.

In this chapter, the efficacy of the price system depends on the nature of the technology—everything works out just dandy if there are decreasing or constant returns to scale, but if there are increasing returns to scale, it all breaks down. It is a good idea to compare the problems that arise with increasing-returns-to-scale technology discussed here with the problems that arise with the decreasing-average-costs technology discussed in the chapter on monopoly. These are just two different ways of depicting the same phenomenon—marginal cost pricing is not viable since it results in negative profits.

In Section 33.10 I describe the basic idea of comparative advantage. This is a very important idea in economics, but unless students take international trade, they probably won't see it after the standard treatment in their principles course.

Production

A. Want to study production in a general equilibrium context
   1. two-good model is somewhat artificial
   2. but necessary for a graphical treatment

B. Robinson Crusoe economy
   1. Robinson is both consumer and producer
   2. consumes leisure and coconuts
   3. can make leisure-consumption choice directly as in Figure 29.1.
   4. or can make it indirectly via the market

C. Crusoe, Inc. — the firm's choices
   1. firm looks at prices and chooses a profit-maximizing plan
   2. generates some profits π^*. See Figure 29.2.

D. Robinson the consumer
   1. Robinson collects profits as nonlabor income
   2. looks at price and wage and decides how much to work
   3. chooses optimal consumption point. See Figure 29.3.

E. In equilibrium, demand equals supply
   1. demand for labor equals supply of labor
   2. demand for consumption equals supply of consumption

F. Decentralization
   1. each "agent" in the economy only has to look at the prices and make his own decisions
   2. the consumer doesn't have to know anything about the production problem
   3. the producer doesn't have to know anything about the consumer's problem
   4. all information is conveyed in prices
   5. in a one-person economy, this is silly
   6. but in a many-person economy, there can be great savings

G. Different kinds of technologies
   1. constant returns to scale — zero profits
   2. decreasing returns to scale — positive profits
   3. increasing returns to scale — competitive markets don't work. Natural monopoly problem

H. Welfare theorems
   1. First welfare theorem — competitive markets are Pareto efficient
   2. Second welfare theorem — any Pareto efficient outcome can be achieved by competitive markets

I. Production possibilities
   1. if there is more than one good, we can illustrate the production set. Figure 29.7.
   2. if there is more than one way to produce output, producers can exploit comparative advantage. Figure 29.8.
   3. production possibilities and the Edgeworth box. Figure 29.9.

Chapter 34

Welfare

I like to describe the aggregation of preference issues in terms of manipulation. Majority voting is bad because the outcome can depend on the order in which the vote is taken and this can lead to agenda manipulation. Rank-order voting is bad because introducing a new alternative can change the outcome of the process, which creates another way to manipulate the political process. Arrow's theorem can be interpreted to say that there is no way to avoid such manipulation possibilities.

However, that being said, we typically resort to looking at simple ways to aggregate preferences through the use of welfare functions. The essential point to get across here is the connection between Pareto efficiency and welfare maximization: every welfare maximum is efficient. Furthermore, subject to the usual convexity conditions, every efficient allocation is a welfare maximum for *some* welfare function.

The fair allocation stuff is fun. Students like it, since it addresses problems of equity in a nice way. I sometimes talk about other methods of fair division, such as one person cuts and the other chooses, etc.

Welfare

A. Incorporate distributional considerations into the analysis

B. Need some way to compare individual preferences or utilities

C. Aggregation of preferences
   1. majority voting
   2. paradox of voting; see Table 30.1
   3. rank order voting
   4. dependence of irrelevant alternatives; see Table 30.2

D. Arrow's impossibility theorem

E. Social welfare fuctions
   1. add together utilities in some way
   2. classical utilitarian: $\sum_{i=1}^{n} u_i$
   3. weighted sum of utilities: $\sum_{i=1}^{n} a_i u_i$
   4. minimax: $\min\{u_1, \ldots, u_n\}$

F. Maximizing welfare
   1. every welfare maximum is Pareto efficient. Figure 30.1.
   2. every Pareto efficient allocation is welfare maximum (if utility possibilities set is convex)

G. Fair allocations
   1. generalized the idea of symmetric treatment
   2. if $u_i(x_j) > u_i(x_i)$, then we say that i **envies** j
   3. typically will be possible to find allocations that are envy-free and efficient
   4. proof: start out with equal division and let people trade using a competitive market
   5. end up with equal incomes; if someone envies someone else, then they couldn't have purchased the best bundle they could afford

Chapter 35

Externalities

I really like the smokers and nonsmokers example in the Edgeworth box. I think that it gets the main points about externalities across very simply. Students sometimes get confused about the vertical axis. Emphasize that this is the total amount of smoke in the apartment, not how much each person smokes. Only one person generates the smoke—but both people have to consume it.

This presentation shows how special it is when there is a unique optimal level of the externality. Essentially that only occurs when preferences are quasilinear, as shown in Figure 35.2. By the way, Figure 35.2 is a great optical illusion; the Pareto efficient allocations form a horizontal line, although it looks as though the line slants from right to left.

Quasilinear preferences make a lot of sense in the production context; after all, profit functions are quasilinear. I treat the standard Pigouvian tax in Section 35.4, but the deeper idea in that section is the idea that the efficient outcome is independent of the assignment of property rights. Students resist the idea that a polluter could have the right to pollute, and its victim would have to buy back clean water from it. But if they understand that idea, they will understand externalities a lot better.

Section 35.5 is an important one too, since it shows that if there is a productive externality involving only a few firms, then there is a natural market signal to internalize the externality. There was a wonderful example of this on the TV show *L.A. Law* a few weeks ago. A water company was polluting the groundwater of a neighboring trailer park. The nasty executive from the water company said that it was reasonable for the water company to make a million-dollar damage settlement every few years since it would cost $30 million to clean up their technology. This provided a lot of drama on the TV, but it was terrible economics. The sensible thing to do was for the water company to buy up the trailer park—certain to cost a lot less than one million dollars—and evict the tenants. This way they could internalize the externality, and make everyone better off. Unfortunately, the L.A. lawyers didn't suggest this to the waterworks. Perhaps they thought that they would lose their fees.

Externalities

A. Consumption externality occurs when an agent cares directly about another agent's consumption or production of some good
   1. playing loud music
   2. smoking a cheap cigar

B. Production externality occurs when a firm's production function depends on choices of another firm or consumer
   1. apple orchard and honeybees
   2. pollution

C. Example: smokers and nonsmokers
   1. two roommates who consume smoke and money; one likes smoke, the other doesn't
   2. depict preferences
   3. depict endowment
      a) each has $100
      b) but what is initial endowment of smoke?
      c) endowment depends on legal system — just like rights to private property
      d) right to clean air
      e) right to smoke
      f) Pareto efficient amounts of smoke and money
      g) contract curve; how to trade
      h) Figure 31.1.
      i) price mechanism generates a "price of smoke"
      j) problems arise because property rights are poorly determined.
   4. under some conditions, the amount of smoke is independent of the assignment of property rights. Figure 31.2.

D. Production externalities
   1. S, a steel firm and F, a fishery
   2. steel: $\max_s \; p_s s - c_s(s, x)$
   3. fishery: $\max_f \; p_f f - c_f(f, x)$
   4. FOC for steel mill:
   $$p_s = \frac{\partial c_s}{\partial s}$$
   $$0 = \frac{\partial c_s}{\partial x}$$
   5. FOC for fishery:
   $$p_f = \frac{\partial c_f(f, x)}{\partial f}$$

E. Efficient solution
   1. merge and maximize joint profits
   2. **internalize** the externality
   3. $$\max_{s,f} p_s s + p_f f - c_s(s, x) - c_f(f, x)$$
   4. get $p_s = \partial c_s / \partial s$, $p_f = \partial c_f / \partial f$ and
   $$0 = \frac{\partial c_s}{\partial x} + \frac{\partial c_f}{\partial x} = \text{marginal social cost}$$
   5. joint firm takes interaction into account

6. private costs and social costs
7. how to get firms to recognize social cost
   a) Pigouvian tax — set price of pollution to equal social cost
   b) market pollution rights
   c) assign property rights and let firms bargain over amount of pollution
8. market solution to externalities
   a) either firm has incentive to buy out the other and internalize the externality
   b) since profits from coordination are greater than profits without
   c) sometimes works with consumption externalities

Chapter 36

Information Technology

There's so much discussion these days about the Internet, the information economy, and the information society that I thought it would be fun to try to get some of these ideas into the classroom. What's remarkable is how poorly informed most of the commentators are about basic economics—not to mention some of the research work that has been done on network economics, intellectual property, and the like. In this chapter I try to provide some simple illustrations of the way that simple application of intermediate microeconomics can lead to significant insight. Consult Carl Shapiro and Hal R. Varian, *Information Rules: A Strategic Guide to the Network Economy* (Boston: Harvard Business School Press, 1999) for lots of stories to spice up your lecture.

I start with a discussion of *systems competition*. I introduced the notion of "complementors" which is very useful for understanding high-tech industries. This concept is illustrated with the example of a CPU (central processing unit) and an OS (operating system). I describe the Cournot analysis that leads to the surprising result that a merger of complementors reduces the system price. This leads naturally to various ways (beside merger) to solve the "problem of complements."

I then proceed to switching costs and lock-in. The major point here is that in a competitive market, companies invest in getting their customers locked in, but the profit from doing so gets competed down to zero.

Next I describe some ideas in network economics. This analysis (which is more than 25 years old) is a very nice way to look at all sorts of network phenomena. Once your students understand the basic idea, you might have them come up with other examples of networks. An interesting set of examples arises where there are *two* goods involved in the network: video tapes and video players, or computers and software. You don't want a computer unless there's software available, and you don't want the software unless the computer is available. This is a slight generalization of the model presented in the book, and it is fun to work it out.

The next two topics in this chapter have to do with intellectual property. The first is a simple model of rights management that illustrates the trade-off between value and sales: more liberal terms and conditions lead to higher values but lower sales. The trick is to balance these two effects so as to maximize profit.

The final model is a nice little model of sharing. The final result says, basically, that producers make more money by allowing a product to be shared if it is cheaper to share a single copy that it is to produce multiple copies. This is a little surprising at first, but on reflection it makes a lot of sense. Again, it is useful to discuss examples: video rentals, library books, interlibrary loan, rental skis, rental cars, etc.

Information Technology

A. Systems competition
   1. info tech components are *complements*
   2. worry about complementers as much as competitors

B. CPU and OS as complements
   1. price of system is $p_1 + p_2$ so demand is $D(p_1 + p_2)$
   2. each setting price alone ignores spillover effect
   3. merging internalizes this externality
   4. other ways to solve the problem of complements
      a) negotiate
      b) revenue sharing
      c) "commoditize the complement"

C. Lock-in
   1. cost of switching
   2. when very large, we have lock-in
   3. e.g., switching ISPs

D. Example of switching ISPs
   1. $c =$ cost of providing service
   2. $p =$ price of service
   3. if no switching costs, $p = c$
   4. now add switching cost of s
   5. allow seller to discount first period by d
      a) consumer switches if

$$(p - d) + \frac{p}{r} + s > p + \frac{p}{r}$$

      b) implies $d = s$, which means supplier covers switching costs
   6. competition forces profit to zero

$$(p - s) - c + \frac{p - c}{r} = 0$$

      a) implies

$$p = c + \frac{r}{1 + r} s$$

      b) interpretation: ISP invests in discount, earns back premium over cost in subsequent periods

E. Network externalities occur when the value of a good to one consumer depends on how many other consumers purchase it.
   1. examples: fax machines, modems, Internet connections, ...

F. Model: think of 1,000 people who have willingness to pay of $v = 1, 2, 3, \ldots, 1000$..
   1. so number of people with willingness to pay greater or equal to p is $1000 - p$.
   2. this is, in fact, the demand curve for the good

G. But now suppose that the value of a fax machine is vn, where n is the number of people who purchase a fax machine
   1. if the price is p, then the marginal person satisfies

$$p = \hat{v}n$$

   2. everyone with value greater than this person buys the fax machine, so

$$n = 1000 - \hat{v}$$

   3. putting these two equations together gives us

$$p = n(1000 - n)$$

   4. note the peculiar shape of this demand curve!

H. Suppose the fax machines are produced at a constant marginal cost of c
   1. there will then be 3 levels of output where demand equals supply
   2. note that the middle equilibrium is unstable; if costs decrease over time, then system may reach "critical mass"
   3. examples: Adobe, Internet

I. Rights management
   1. offering more liberal terms and conditions increases value, decreases sales
   2. baseline case
      a) $y =$ amount consumed
      b) $p(y) =$ inverse demand
      c) $\max_y \; p(y)y$
   3. more liberal terms and conditions
      a) $Y = \beta y$ with $\beta > 1$
      b) $P(Y) = \alpha p(Y)$ with $\alpha > 1$
      c)

$$\max_Y \alpha p(Y)\frac{Y}{\beta}$$

      d)

$$\max_Y \frac{\alpha}{\beta} p(Y)Y$$

   4. conclusions
      a) same amount consumed
      b) less produced
      c) profits go up if $\alpha > \beta$, down if inequality reversed

J. Sharing intellectual property
   1. examples of sharing
   2. monopoly profit maximization: $p(y)y - cy - F$ gives output $\hat{y}$.
   3. What if good is shared among k users? If y copies produced, $x = kx$ copies used, so marginal wtp is $p(x)$. Inconvenience of sharing gives us marginal wtp of $p(x) - t$.
   4. What about demand by group? It is $k[p(ky) - t]$.
   5. willingness to pay goes up due to k in front, down due to k in argument

K. profit maximization

$$\max_y \; k[p(ky) - t]y - cy - F$$

L. rearrange:

$$\max_x \; p(x)x - \left(\frac{c}{k} + t\right) x - F.$$

M. Marginal cost in this problem is $(c/k+t)$. How does this compare to marginal cost in original problem?

1. Profits will be larger when rental is possible when

$$\frac{c}{k} + t < c$$

or,

$$\left(\frac{k}{k+1}\right) t < c.$$

a) If k is large, this reduces to $t < c$.

2. Interpretation: is it cheaper to produce an extra copy or have an existing copy shared among more consumers?

Chapter 37

Public Goods

I start by introducing the basic idea of a public good—a good that lacks exclusion in consumption. The standard textbook treatment leaps right into the Samuelson conditions, but I think that it makes much more sense to look at the public provision of a discrete good. I derive the optimality condition in this context, namely that the sum of the reservation prices exceeds the cost of the good. Once students are armed with this example, the Samuelson case is a relatively easy extension.

I then turn to a discussion of free riding and relate it to the prisoners' dilemma. The example there is a little forced, but it gets the point across: if each person makes his decision about the public good independently, there may be inadequate provision of the public good. It is fun to talk about other kinds of free riding; e.g., who cleans up the living room?

I next look at the classical Samuelson conditions for efficiency when the public good can be provided at different levels of output. I treat the free rider problem in Section 37.6. Figure 37.2 is really quite a nice diagram and repays careful study.

The next topic for discussion is how to "solve" the public goods problem. Students have been taught democratic ideals in high school civics classes, so it might come as a shock to them that voting isn't that good a mechanism for making decisions about public goods. Here it is worthwhile to give some examples where one person cares a lot about something and would be willing to compensate others, but voting won't be able to reach the Pareto efficient decision. You might talk about ways that real-life political processes get around this problem—e.g., logrolling—but that may tend to confuse them unless they've had some political science.

Finally I discuss the Clarke tax—a way to really "solve" the public goods problem, at least for a special case. The best way to get students to understand the Clarke tax is to actually have them use it. One faculty member I know had his class use a Clarke tax procedure to determine the date of the midterm exam. This is certainly a public goods problem, and the students really understood what was going on when they actually participated. But even if you can't determine the provision of a real public good, like the date of the midterm, it is still of interest to run through a numerical example, such as the one given in the book.

Public Goods

A. Public goods involve a particular kind of externality — where the same amount of the good has to be available to everyone.

B. Examples: national defense, street lights, roads, etc. — same amount must be provided to all.

C. But people can value the public good in different ways.

D. Private goods
   1. each person consumes different amount, but values it the same (at the margin).

E. Public goods
   1. each person consumes the same amount, but values it differently.

F. Two questions about public goods
   1. what is the optimal amount of a public good?
   2. how well do various social institutions work in providing the optimal amount of a public good?

G. Example: a TV for two roommates. Roommate i will contribute $g_i \geq 0$ towards the purchase. TV will be purchased if $g_1 + g_2 \geq C$.
   1. consider the reservation prices r_1 and r_2. These measure maximum willingness-to-pay for TV by each person.
   2. suppose that we can find (g_1, g_2) such that $r_1 \geq g_1$, $r_2 \geq g_2$ and $g_1+g_2 \geq C$.
   3. then clearly it is a good idea to provide the TV.
   4. so, if $r_1 + r_2 \geq C$, then we can find g_1 and g_2 that cover costs and should provide the TV.
   5. if $r_1 + r_2 < C$, then shouldn't provide the TV.
   6. condition for efficiency is that the sum of the willingnesses to pay must exceed the cost of provision.
   7. in case of divisible good (e.g., how much to spend on TV), the optimum occurs when the sum of the marginal willingness-to-pay equals marginal cost.
      a) if sum of MRSs exceeds marginal cost, can make everyone better off by increasing the amount of public good.
      b) if sum of MRSs is less than marginal cost, then should reduce the amount of the public good.

H. Example of divisible good
   1. two people each contributes g_i to a TV. Person i gets utility $u_i(g_1+g_2)-g_i$.
   2. efficient alloction maximizes sum of utilities:

$$\max\ u_1(g_1 + g_2) + u_2(g_1 + g_2) + -g_1 - g_2.$$

   3. FOC:

$$u_1'(g_1^* + g_2^*) + u_2'(g_1^* + g_2^*) = 1.$$

   4. this determines the optimal amount of the public good: $G^* = g_1^* + g_2^*$.
   5. if there were n people, the condition would be

$$\sum_{i=1}^{n} u_i'(G^*) = 1.$$

I. Consider various social institutions to provide the public good.
   1. voluntary contributions
      a) person 1 will contribute until $u_1'(g_1 + g_2) = 1$.
      b) person 2 will contribute until $u_2'(g_1 + g_2) = 1$.
      c) person who has higher willingness to pay will contribute the entire amount.
      d) other person **free rides** — contributes zero.
   2. majority voting
      a) assume that there are $n > 2$ people.
      b) suppose that each person pays $1/n$ of the public good if it is provided.
      c) if G units of the public good are provided, then person i gets benefit

$$u_i(G) - \frac{1}{n}G.$$

      d) person i will vote for an increase in the amount of the public good if

$$u_i'(G) > \frac{1}{n}.$$

      e) if a majority of the people vote for an increase in the public good, then we get a small increase.
      f) so the amount of the public good is determined by the condition that the median voter is happy with the current amount.
         1) median voter means half the voters want more, half the voters want less.
         2) if m is the median voter, then want

$$u_m'(G) = \frac{1}{n}.$$

      g) in general, this won't be the optimal amount of the public good.
      h) think of case where some voters really want a lot more of the public good and would be willing to compensate those who don't want more.
      i) voting doesn't take into account *intensity* of preference.
   3. Clarke-Groves tax
      a) in order to get an efficient amount of the public good, each person must face the social costs of his decision.
      b) there are ways of "bidding" for public good that do this.

Chapter 38

Information

The students really like this material on information economics, but you have to work at it to really get the ideas across.

The first topic is the famous lemon's market. I found it easy to get the idea across, but the logic needs emphasizing. That's why I go through the quality choice model in the next section. The first part of this model is basically the same idea, but in a different context. I then summarize the fundamental idea—the idea of adverse selection. Here it is fun to discuss other examples of adverse selection.

The next topic is that of moral hazard. Again, it is useful to discuss other examples to make sure the students have the idea straight.

The third topic is signaling. The Spence educational signaling model is a wonderful example for college students. In particular, be sure to discuss the "sheepskin effect" example given in the text. It seems that the diploma must carry signaling value, over and above the actual learning that it represents; your class may want to discuss just what a diploma signals.

Finally, I discuss the topic of incentives. The basic thing to get across here is the equivalence of all the compensation schemes in the presence of symmetric information. The idea that sharecropping can be an efficient incentive scheme when information is imperfect is a nice insight that deserves emphasis.

Information

A. Up until now, we have assumed complete information — consumers and firms know the quality of the goods they buy and sell

B. But in real life, information may often be incomplete

C. Then people have to infer quality from price or other signals

D. Firms may supply such signals intentionally or inadvertently
   1. used cars — why are you selling it?
   2. warranties — signal of quality

E. Model of used-car market
   1. 50 lemons for sale, 50 plums
   2. buyers willing to pay \$2,400 for plum, \$1,200 for lemon
   3. sellers will sell plum for \$2,000 and lemon for \$1,000
   4. full information solution
      a) plum sells for price between \$2,400 and \$2,000
      b) lemon sells for price between \$1,200 and \$1,000
   5. incomplete information solution
      a) can't tell if car is a plum or a lemon
      b) estimate quality by looking at average quality of cars on the market
      c) suppose all cars were offered for sale
      d) then the willingness to pay for a car would be

$$\frac{1}{2}2,400 + \frac{1}{2}1,200 = 1,800.$$

      e) at this price, the owners of plums wouldn't sell
      f) only owners of lemons would sell
      g) but then the maximum that buyers would be willing to pay would be \$1,200!
      h) only equilibrium is for lemons to get offered on market, and price to be between \$1,000 and \$1,200
      i) the bad cars have "driven out" the good cars
      j) there is an externality between the good and bad cars

F. Quality choice
   1. in the lemons model, quality is exogenous; what if quality is endogenous?
   2. umbrella market
      a) consumers are willing to pay \$14 for a high-quality umbrella, \$8 for a low-quality umbrella
      b) if a fraction q are high quality, willing to pay

$$p = 14q + 8(1 - q).$$

      c) suppose it costs \$11.50 to produce high quality and \$11 to produce low quality
      d) then if there are many firms, and each thinks that it will have a negligible effect on the price, each will choose to produce a low-quality umbrella
      e) but the amount that people are willing to pay for a low-quality umbrella (\$8) exceeds the cost of production (\$11)
      f) the possibility of production of a low-quality good has destroyed the market!
   3. Adverse selection
      a) consider insurance market
      b) people who need insurance the most are more likely to buy it
      c) rates based on *average* experience over population will not necessarily cover costs
      d) high-risk consumers can drive out low-risk consumers
      e) mandatory insurance can make people better off on average

G. Signaling
1. we have seen that when quality in market is mixed, the bad quality can drive out the good
2. incentive for firms to identify high-quality goods by sending a *signal* to the consumers
3. example: a warranty
4. high-quality producers can afford to offer a warranty, low-quality producers can't
5. in equilibrium a warranty can differentiate the two qualities
6. example—signaling by educational choice
   a) two kinds of workers, able and unable
   b) able have MP of a_2, unable have an MP of a_1, and $a_2 > a_1$.
   c) if firm can observe quality of worker, each type gets paid its MP
   d) if can't observe quality, must pay wage equal to average of MPs
   e) suppose that there is some signal that they can acquire that will indicate which type they are
   f) for example, suppose that workers can choose education level
   g) more able workers have cheaper costs of acquiring education (not necessarily dollar costs)
   h) then can have an equilibrium where able workers acquire education, to distinguish themselves from unable workers
   i) *even though* the education doesn't change their MP
   j) socially wasteful investment — only use is to distinguish one group from another

Chapter 1

NAME_____

The Market

Introduction. The problems in this chapter examine some variations on the apartment market described in the text. In most of the problems we work with the true demand curve constructed from the reservation prices of the consumers rather than the "smoothed" demand curve that we used in the text.

Remember that the reservation price of a consumer is that price where he is just indifferent between renting or not renting the apartment. At any price below the reservation price the consumer will demand one apartment, at any price above the reservation price the consumer will demand zero apartments, and exactly at the reservation price the consumer will be indifferent between having zero or one apartment.

You should also observe that when demand curves have the "staircase" shape used here, there will typically be a *range* of prices where supply equals demand. Thus we will ask for the the highest and lowest price in the range.

1.1 (3) Suppose that we have 8 people who want to rent an apartment. Their reservation prices are given below. (To keep the numbers small, think of these numbers as being daily rent payments.)

Person	=	A	B	C	D	E	F	G	H
Price	=	40	25	30	35	10	18	15	5

(a) Plot the market demand curve in the following graph. (Hint: When the market price is equal to some consumer i's reservation price, there will be two different quantities of apartments demanded, since consumer i will be indifferent between having or not having an apartment.)

(b) Suppose the supply of apartments is fixed at 5 units. In this case there is a whole range of prices that will be equilibrium prices. What is the highest price that would make the demand for apartments equal to 5 units? $18.

(c) What is the lowest price that would make the market demand equal to 5 units? $15.

(d) With a supply of 4 apartments, which of the people A–H end up getting apartments? A, B, C, D.

(e) What if the supply of apartments increases to 6 units. What is the range of equilibrium prices? $10 to $15.

1.2 (3) Suppose that there are originally 5 units in the market and that 1 of them is turned into a condominium.

(a) Suppose that person A decides to buy the condominium. What will be the highest price at which the demand for apartments will equal the supply of apartments? What will be the lowest price? Enter your answers in column A, in the table. Then calculate the equilibrium prices of apartments if B, C, ..., decide to buy the condominium.

Person	A	B	C	D	E	F	G	H
High price	18	18	18	18	25	25	25	25
Low price	15	15	15	15	18	15	18	18

(b) Suppose that there were two people at each reservation price and 10 apartments. What is the highest price at which demand equals supply?

18. Suppose that one of the apartments was turned into a condominium. Is that price still an equilibrium price? Yes.

1.3 (2) Suppose now that a monopolist owns all the apartments and that he is trying to determine which price and quantity maximize his revenues.

(a) Fill in the box with the maximum price and revenue that the monopolist can make if he rents 1, 2, ..., 8 apartments. (Assume that he must charge one price for all apartments.)

Number	1	2	3	4	5	6	7	8
Price	40	35	30	25	18	15	10	5
Revenue	40	70	90	100	90	90	70	40

(b) Which of the people A–F would get apartments? A, B, C, D.

(c) If the monopolist were required by law to rent exactly 5 apartments, what price would he charge to maximize his revenue? $18.

(d) Who would get apartments? A, B, C, D, F.

(e) If this landlord could charge each individual a different price, and he knew the reservation prices of all the individuals, what is the maximum revenue he could make if he rented all 5 apartments? $148.

(f) If 5 apartments were rented, which individuals would get the apartments? A, B, C, D, F.

1.4 (2) Suppose that there are 5 apartments to be rented and that the city rent-control board sets a maximum rent of $9. Further suppose that people A, B, C, D, and E manage to get an apartment, while F, G, and H are frozen out.

(a) If subletting is legal—or, at least, practiced—who will sublet to whom in equilibrium? (Assume that people who sublet can evade the city rent-control restrictions.) E, who is willing to pay only $10 for an apartment would sublet to F, who is willing to pay $18.

(b) What will be the maximum amount that can be charged for the sublet payment? $18.

(c) If you have rent control with unlimited subletting allowed, which of the consumers described above will end up in the 5 apartments? A, B, C, D, F.

(d) How does this compare to the market outcome? It's the same.

1.5 (2) In the text we argued that a tax on landlords would not get passed along to the renters. What would happen if instead the tax was imposed on renters?

(a) To answer this question, consider the group of people in Problem 1.1. What is the maximum that they would be willing to pay to the landlord if they each had to pay a $5 tax on apartments to the city? Fill in the box below with these reservation prices.

Person	A	B	C	D	E	F	G	H
Reservation Price	35	20	25	30	5	13	10	0

(b) Using this information determine the maximum equilibrium price if there are 5 apartments to be rented. $13.

(c) Of course, the total price a renter pays consists of his or her rent plus the tax. This amount is $18.

(d) How does this compare to what happens if the tax is levied on the landlords? It's the same.

Chapter 2

NAME_____

Budget Constraint

Introduction. These workouts are designed to build your skills in describing economic situations with graphs and algebra. Budget sets are a good place to start, because both the algebra and the graphing are very easy. Where there are just two goods, a consumer who consumes x_1 units of good 1 and x_2 units of good 2 is said to consume the *consumption bundle*, (x_1, x_2). Any consumption bundle can be represented by a point on a two-dimensional graph with quantities of good 1 on the horizontal axis and quantities of good 2 on the vertical axis. If the prices are p_1 for good 1 and p_2 for good 2, and if the consumer has income m, then she can afford any consumption bundle, (x_1, x_2), such that $p_1x_1+p_2x_2 \leq m$. On a graph, the *budget line* is just the line segment with equation $p_1x_1 + p_2x_2 = m$ and with x_1 and x_2 both nonnegative. The budget line is the boundary of the *budget set*. All of the points that the consumer can afford lie on one side of the line and all of the points that the consumer cannot afford lie on the other.

If you know prices and income, you can construct a consumer's budget line by finding two commodity bundles that she can "just afford" and drawing the straight line that runs through both points.

Example: Myrtle has 50 dollars to spend. She consumes only apples and bananas. Apples cost 2 dollars each and bananas cost 1 dollar each. You are to graph her budget line, where apples are measured on the horizontal axis and bananas on the vertical axis. Notice that if she spends all of her income on apples, she can afford 25 apples and no bananas. Therefore her budget line goes through the point $(25, 0)$ on the horizontal axis. If she spends all of her income on bananas, she can afford 50 bananas and no apples. Therfore her budget line also passes throught the point $(0, 50)$ on the vertical axis. Mark these two points on your graph. Then draw a straight line between them. This is Myrtle's budget line.

What if you are not told prices or income, but you know two commodity bundles that the consumer can just afford? Then, if there are just two commodities, you know that a unique line can be drawn through two points, so you have enough information to draw the budget line.

Example: Laurel consumes only ale and bread. If she spends all of her income, she can just afford 20 bottles of ale and 5 loaves of bread. Another commodity bundle that she can afford if she spends her entire income is 10 bottles of ale and 10 loaves of bread. If the price of ale is 1 dollar per bottle, how much money does she have to spend? You could solve this problem graphically. Measure ale on the horizontal axis and bread on the vertical axis. Plot the two points, $(20, 5)$ and $(10, 10)$, that you know to be on the budget line. Draw the straight line between these points and extend the line to the horizontal axis. This point denotes the amount of

ale Laurel can afford if she spends all of her money on ale. Since ale costs 1 dollar a bottle, her income in dollars is equal to the largest number of bottles she can afford. Alternatively, you can reason as follows. Since the bundles $(20, 5)$ and $(10, 10)$ cost the same, it must be that giving up 10 bottles of ale makes her able to afford an extra 5 loaves of bread. So bread costs twice as much as ale. The price of ale is 1 dollar, so the price of bread is 2 dollars. The bundle $(20, 5)$ costs as much as her income. Therefore her income must be $20 \times 1 + 5 \times 2 = 30$.

When you have completed this workout, we hope that you will be able to do the following:

- Write an equation for the budget line and draw the budget set on a graph when you are given prices and income or when you are given two points on the budget line.
- Graph the effects of changes in prices and income on budget sets.
- Understand the concept of *numeraire* and know what happens to the budget set when income and all prices are multiplied by the same positive amount.
- Know what the budget set looks like if one or more of the prices is negative.
- See that the idea of a "budget set" can be applied to constrained choices where there are other constraints on what you can have, in addition to a constraint on money expenditure.

2.1 (0) You have an income of \$40 to spend on two commodities. Commodity 1 costs \$10 per unit, and commodity 2 costs \$5 per unit.

(a) Write down your budget equation. $10x_1 + 5x_2 = 40.$

(b) If you spent all your income on commodity 1, how much could you buy? 4.

(c) If you spent all of your income on commodity 2, how much could you buy? 8. Use blue ink to draw your budget line in the graph below.

(d) Suppose that the price of commodity 1 falls to \$5 while everything else stays the same. Write down your new budget equation. $5x_1 + 5x_2 = 40.$ On the graph above, use red ink to draw your new budget line.

(e) Suppose that the amount you are allowed to spend falls to \$30, while the prices of both commodities remain at \$5. Write down your budget equation. $5x_1 + 5x_2 = 30.$ Use black ink to draw this budget line.

(f) On your diagram, use blue ink to shade in the area representing commodity bundles that you can afford with the budget in Part (e) but could not afford to buy with the budget in Part (a). Use black ink or pencil to shade in the area representing commodity bundles that you could afford with the budget in Part (a) but cannot afford with the budget in Part (e).

2.2 (0) On the graph below, draw a budget line for each case.

(a) $p_1 = 1$, $p_2 = 1$, $m = 15$. (Use blue ink.)

(b) $p_1 = 1$, $p_2 = 2$, $m = 20$. (Use red ink.)

(c) $p_1 = 0$, $p_2 = 1$, $m = 10$. (Use black ink.)

(d) $p_1 = p_2$, $m = 15p_1$. (Use pencil or black ink. Hint: How much of good 1 could you afford if you spend your entire budget on good 1?)

2.3 (0) Your budget is such that if you spend your entire income, you can afford either 4 units of good x and 6 units of good y or 12 units of x and 2 units of y.

(a) Mark these two consumption bundles and draw the budget line in the graph below.

(b) What is the ratio of the price of x to the price of y? 1/2.

(c) If you spent all of your income on x, how much x could you buy?

16.

(d) If you spent all of your income on y, how much y could you buy?
8.

(e) Write a budget equation that gives you this budget line, where the price of x is 1. $x + 2y = 16$.

(f) Write another budget equation that gives you the same budget line, but where the price of x is 3. $3x + 6y = 48$.

2.4 (1) Murphy was consuming 100 units of X and 50 units of Y. The price of X rose from 2 to 3. The price of Y remained at 4.

(a) How much would Murphy's income have to rise so that he can still exactly afford 100 units of X and 50 units of Y? $100.

2.5 (1) If Amy spent her entire allowance, she could afford 8 candy bars and 8 comic books a week. She could also just afford 10 candy bars and 4 comic books a week. The price of a candy bar is 50 cents. Draw her budget line in the box below. What is Amy's weekly allowance? $6.

2.6 (0) In a small country near the Baltic Sea, there are only three commodities: potatoes, meatballs, and jam. Prices have been remarkably stable for the last 50 years or so. Potatoes cost 2 crowns per sack, meatballs cost 4 crowns per crock, and jam costs 6 crowns per jar.

(a) Write down a budget equation for a citizen named Gunnar who has an income of 360 crowns per year. Let P stand for the number of sacks of potatoes, M for the number of crocks of meatballs, and J for the number of jars of jam consumed by Gunnar in a year. $2P + 4M + 6J = 360$.

(b) The citizens of this country are in general very clever people, but they are not good at multiplying by 2. This made shopping for potatoes excruciatingly difficult for many citizens. Therefore it was decided to introduce a new unit of currency, such that potatoes would be the *numeraire.* A sack of potatoes costs one unit of the new currency while the same relative prices apply as in the past. In terms of the new currency, what is the price of meatballs? 2 crowns.

(c) In terms of the new currency, what is the price of jam? 3 crowns.

(d) What would Gunnar's income in the new currency have to be for him to be exactly able to afford the same commodity bundles that he could afford before the change? 180 crowns.

(e) Write down Gunnar's new budget equation. $P + 2M + 3J = 180$. Is Gunnar's budget set any different than it was before the change? No.

2.7 (0) Edmund Stench consumes two commodities, namely garbage and punk rock video cassettes. He doesn't actually eat the former but keeps it in his backyard where it is eaten by billy goats and assorted vermin. The reason that he accepts the garbage is that people pay him \$2 per sack for taking it. Edmund can accept as much garbage as he wishes at that price. He has no other source of income. Video cassettes cost him \$6 each.

(a) If Edmund accepts zero sacks of garbage, how many video cassettes can he buy? 0.

(b) If he accepts 15 sacks of garbage, how many video cassettes can he buy? 5.

(c) Write down an equation for his budget line. $6C - 2G = 0$.

(d) Draw Edmund's budget line and shade in his budget set.

2.8 (0) If you think Edmund is odd, consider his brother Emmett. Emmett consumes speeches by politicians and university administrators. He is paid \$1 per hour for listening to politicians and \$2 per hour for listening to university administrators. (Emmett is in great demand to help fill empty chairs at public lectures because of his distinguished appearance and his ability to refrain from making rude noises.) Emmett consumes one good for which he must pay. We have agreed not to disclose what that good is, but we can tell you that it costs \$15 per unit and we shall call it Good X. In addition to what he is paid for consuming speeches, Emmett receives a pension of \$50 per week.

(a) Write down a budget equation stating those combinations of the three commodities, Good X, hours of speeches by politicians (P), and hours of speeches by university administrators (A) that Emmett could afford to consume per week. $15X - 1P - 2A = 50$.

(b) On the graph above, draw a two-dimensional diagram showing the locus of consumptions of the two kinds of speeches that would be possible for Emmett if he consumed 10 units of Good X per week.

2.9 (0) Jonathan Livingstone Yuppie is a prosperous lawyer. He has, in his own words, "outgrown those confining two-commodity limits." Jonathan consumes three goods, unblended Scotch whiskey, designer tennis shoes, and meals in French gourmet restaurants. The price of Jonathan's brand of whiskey is \$20 per bottle, the price of designer tennis shoes is \$80 per pair, and the price of gourmet restaurant meals is \$50 per meal. After he has paid his taxes and alimony, Jonathan has \$400 a week to spend.

(a) Write down a budget equation for Jonathan, where W stands for the number of bottles of whiskey, T stands for the number of pairs of tennis shoes, and M for the number of gourmet restaurant meals that he consumes. $20W + 80T + 50M = 400$.

(b) Draw a three-dimensional diagram to show his budget set. Label the intersections of the budget set with each axis.

(c) Suppose that he determines that he will buy one pair of designer tennis shoes per week. What equation must be satisfied by the combinations of restaurant meals and whiskey that he could afford? $20W + 50M = 320$.

2.10 (0) Martha is preparing for exams in economics and sociology. She has time to read 40 pages of economics and 30 pages of sociology. In the same amount of time she could also read 30 pages of economics and 60 pages of sociology.

(a) Assuming that the number of pages per hour that she can read of either subject does not depend on how she allocates her time, how many pages of sociology could she read if she decided to spend all of her time on sociology and none on economics? 150 pages. (Hint: You have two points on her budget line, so you should be able to determine the entire line.)

(b) How many pages of economics could she read if she decided to spend all of her time reading economics? 50 pages.

2.11 (1) Harry Hype has $5,000 to spend on advertising a new kind of dehydrated sushi. Market research shows that the people most likely to buy this new product are recent recipients of M.B.A. degrees and lawyers who own hot tubs. Harry is considering advertising in two publications, a boring business magazine and a trendy consumer publication for people who wish they lived in California.

Fact 1: Ads in the boring business magazine cost $500 each and ads in the consumer magazine cost $250 each.

Fact 2: Each ad in the business magazine will be read by 1,000 recent M.B.A.'s and 300 lawyers with hot tubs.

Fact 3: Each ad in the consumer publication will be read by 300 recent M.B.A.'s and 250 lawyers who own hot tubs.

Fact 4: Nobody reads more than one ad, and nobody who reads one magazine reads the other.

(a) If Harry spends his entire advertising budget on the business publication, his ad will be read by 10,000 recent M.B.A.'s and by 3,000 lawyers with hot tubs.

(b) If he spends his entire advertising budget on the consumer publication, his ad will be read by 6,000 recent M.B.A.'s and by 5,000 lawyers with hot tubs.

(c) Suppose he spent half of his advertising budget on each publication. His ad would be read by 8,000 recent M.B.A.'s and by 4,000 lawyers with hot tubs.

(d) Draw a "budget line" showing the combinations of number of readings by recent M.B.A.'s and by lawyers with hot tubs that he can obtain if he spends his entire advertising budget. Does this line extend all the way to the axes? No. Sketch, shade in, and label the budget set, which includes all the combinations of MBA's and lawyers he can reach if he spends *no more than* his budget.

(e) Let M stand for the number of instances of an ad being read by an M.B.A. and L stand for the number of instances of an ad being read by a lawyer. This budget line is a line segment that lies on the line with equation $M + 2L = 16$. With a fixed advertising budget, how many readings by M.B.A.'s must he sacrifice to get an additional reading by a lawyer with a hot tub? 2.

2.12 (0) On the planet Mungo, they have two kinds of money, blue money and red money. Every commodity has two prices—a red-money price and a blue-money price. Every Mungoan has two incomes—a red income and a blue income.

In order to buy an object, a Mungoan has to pay that object's red-money price in red money and its blue-money price in blue money. (The shops simply have two cash registers, and you have to pay at both registers to buy an object.) It is forbidden to trade one kind of money for the other, and this prohibition is strictly enforced by Mungo's ruthless and efficient monetary police.

- There are just two consumer goods on Mungo, ambrosia and bubble gum. All Mungoans prefer more of each good to less.
- The blue prices are 1 bcu (bcu stands for blue currency unit) per unit of ambrosia and 1 bcu per unit of bubble gum.
- The red prices are 2 rcus (red currency units) per unit of ambrosia and 6 rcus per unit of bubble gum.

(a) On the graph below, draw the red budget (with red ink) and the blue budget (with blue ink) for a Mungoan named Harold whose blue income is 10 and whose red income is 30. Shade in the "budget set" containing all of the commodity bundles that Harold can afford, given

its* two budget constraints. Remember, Harold has to have enough blue money *and* enough red money to pay both the blue-money cost and the red-money cost of a bundle of goods.

(b) Another Mungoan, Gladys, faces the same prices that Harold faces and has the same red income as Harold, but Gladys has a blue income of 20. Explain how it is that Gladys will not spend its entire blue income no matter what its tastes may be. (Hint: Draw Gladys's budget lines.)

The blue budget line lies strictly outside the red budget line, so to satisfy both budgets, one must be strictly inside the red budget line.

(c) A group of radical economic reformers on Mungo believe that the currency rules are unfair. "Why should everyone have to pay two prices for everything?" they ask. They propose the following scheme. Mungo will continue to have two currencies, every good will have a blue price and a red price, and every Mungoan will have a blue income and a red income. But nobody has to pay both prices. Instead, everyone on Mungo must declare itself to be either a Blue-Money Purchaser (a "Blue") or a Red-Money Purchaser (a "Red") before it buys anything at all. Blues must make all of their purchases in blue money at the blue prices, spending only their blue incomes. Reds must make all of their purchases in red money, spending only their red incomes.

Suppose that Harold has the same income after this reform, and that prices do not change. Before declaring which kind of purchaser it will be,

* We refer to all Mungoans by the gender-neutral pronoun, "it." Although Mungo has two sexes, neither of them is remotely like either of ours.

Harold contemplates the set of commodity bundles that it could afford by making one declaration or the other. Let us call a commodity bundle "attainable" if Harold can afford it by declaring itself to be a "Blue" and buying the bundle with blue money or if Harold can afford the bundle by declaring itself to be a "Red" and buying it with red money. On the diagram below, shade in all of the attainable bundles.

2.13 (0) Are Mungoan budgets really so fanciful? Can you think of situations on earth where people must simultaneously satisfy more than one budget constraint? Is money the only scarce resource that people use up when consuming? Consumption of many commodities takes time as well as money. People have to simultaneously satisfy a time budget and a money budget. Other examples--people may have a calorie budget or a cholesterol budget or an alcohol-intake budget.

Chapter 3

NAME_____

Preferences

Introduction. In the previous section you learned how to use graphs to show the set of commodity bundles that a consumer can afford. In this section, you learn to put information about the consumer's preferences on the same kind of graph. Most of the problems ask you to draw indifference curves.

Sometimes we give you a formula for the indifference curve. Then all you have to do is graph a known equation. But in some problems, we give you only "qualitative" information about the consumer's preferences and ask you to sketch indifference curves that are consistent with this information. This requires a little more thought. Don't be surprised or disappointed if you cannot immediately see the answer when you look at a problem, and don't expect that you will find the answers hiding somewhere in your textbook. The best way we know to find answers is to "think and doodle." Draw some axes on scratch paper and label them, then mark a point on your graph and ask yourself, "What other points on the graph would the consumer find indifferent to this point?" If possible, draw a curve connecting such points, making sure that the shape of the line you draw reflects the features required by the problem. This gives you one indifference curve. Now pick another point that is preferred to the first one you drew and draw an indifference curve through it.

Example: Jocasta loves to dance and hates housecleaning. She has strictly convex preferences. She prefers dancing to any other activity and never gets tired of dancing, but the more time she spends cleaning house, the less happy she is. Let us try to draw an indifference curve that is consistent with her preferences. There is not enough information here to tell us exactly where her indifference curves go, but there is enough information to determine some things about their shape. Take a piece of scratch paper and draw a pair of axes. Label the horizontal axis "Hours per day of housecleaning." Label the vertical axis "Hours per day of dancing." Mark a point a little ways up the vertical axis and write a 4 next to it. At this point, she spends 4 hours a day dancing and no time housecleaning. Other points that would be indifferent to this point would have to be points where she did more dancing *and* more housecleaning. The pain of the extra housekeeping should just compensate for the pleasure of the extra dancing. So an indifference curve for Jocasta must be upward sloping. Because she loves dancing and hates housecleaning, it must be that she prefers all the points above this indifference curve to all of the points on or below it. If Jocasta has strictly convex preferences, then it must be that if you draw a line between any two points on the same indifference curve, all the points on the line (except the endpoints) are preferred to the endpoints. For this to be the case, it must be that the indifference curve slopes upward ever more steeply as you move to the right along it. You should convince yourself of this by making some drawings on scratch

paper. Draw an upward-sloping curve passing through the point $(0, 4)$ and getting steeper as one moves to the right.

When you have completed this workout, we hope that you will be able to do the following:

- Given the formula for an indifference curve, draw this curve, and find its slope at any point on the curve.
- Determine whether a consumer prefers one bundle to another or is indifferent between them, given specific indifference curves.
- Draw indifference curves for the special cases of perfect substitutes and perfect complements.
- Draw indifference curves for someone who dislikes one or both commodities.
- Draw indifference curves for someone who likes goods up to a point but who can get "too much" of one or more goods.
- Identify weakly preferred sets and determine whether these are convex sets and whether preferences are convex.
- Know what the marginal rate of substitution is and be able to determine whether an indifference curve exhibits "diminishing marginal rate of substitution."
- Determine whether a preference relation or any other relation between pairs of things is transitive, whether it is reflexive, and whether it is complete.

3.1 (0) Charlie likes both apples and bananas. He consumes nothing else. The consumption bundle where Charlie consumes x_A bushels of apples per year and x_B bushels of bananas per year is written as (x_A, x_B). Last year, Charlie consumed 20 bushels of apples and 5 bushels of bananas. It happens that the set of consumption bundles (x_A, x_B) such that Charlie is indifferent between (x_A, x_B) and $(20, 5)$ is the set of all bundles such that $x_B = 100/x_A$. The set of bundles (x_A, x_B) such that Charlie is just indifferent between (x_A, x_B) and the bundle $(10, 15)$ is the set of bundles such that $x_B = 150/x_A$.

(a) On the graph below, plot several points that lie on the indifference curve that passes through the point $(20, 5)$, and sketch this curve, using blue ink. Do the same, using red ink, for the indifference curve passing through the point $(10, 15)$.

(b) Use pencil to shade in the set of commodity bundles that Charlie weakly prefers to the bundle $(10, 15)$. Use blue ink to shade in the set of commodity bundles such that Charlie weakly prefers $(20, 5)$ to these bundles.

For each of the following statements about Charlie's preferences, write "true" or "false."

(c) $(30, 5) \sim (10, 15)$. True.

(d) $(10, 15) \succ (20, 5)$. True.

(e) $(20, 5) \succeq (10, 10)$. True.

(f) $(24, 4) \succeq (11, 9.1)$. False.

(g) $(11, 14) \succ (2, 49)$. True.

(h) A set is convex if for any two points in the set, the line segment between them is also in the set. Is the set of bundles that Charlie weakly prefers to $(20, 5)$ a convex set? Yes.

(i) Is the set of bundles that Charlie considers inferior to $(20, 5)$ a convex set? No.

(j) The slope of Charlie's indifference curve through a point, (x_A, x_B), is known as his marginal rate of substitution at that point.

(k) Remember that Charlie's indifference curve through the point $(10, 10)$ has the equation $x_B = 100/x_A$. Those of you who know calculus will remember that the slope of a curve is just its derivative, which in this case is $-100/x_A^2$. (If you don't know calculus, you will have to take our word for this.) Find Charlie's marginal rate of substitution at the point, $(10, 10)$. -1.

(l) What is his marginal rate of substitution at the point $(5, 20)$? -4.

(m) What is his marginal rate of substitution at the point $(20, 5)$? $(-.25)$.

(n) Do the indifference curves you have drawn for Charlie exhibit diminishing marginal rate of substitution? Yes.

3.2 (0) Ambrose consumes only nuts and berries. Fortunately, he likes both goods. The consumption bundle where Ambrose consumes x_1 units of nuts per week and x_2 units of berries per week is written as (x_1, x_2). The set of consumption bundles (x_1, x_2) such that Ambrose is indifferent between (x_1, x_2) and $(1, 16)$ is the set of bundles such that $x_1 \geq 0$, $x_2 \geq 0$, and $x_2 = 20 - 4\sqrt{x_1}$. The set of bundles (x_1, x_2) such that $(x_1, x_2) \sim (36, 0)$ is the set of bundles such that $x_1 \geq 0$, $x_2 \geq 0$ and $x_2 = 24 - 4\sqrt{x_1}$.

(a) On the graph below, plot several points that lie on the indifference curve that passes through the point $(1, 16)$, and sketch this curve, using blue ink. Do the same, using red ink, for the indifference curve passing through the point $(36, 0)$.

(b) Use pencil to shade in the set of commodity bundles that Ambrose weakly prefers to the bundle $(1, 16)$. Use red ink to shade in the set of all commodity bundles (x_1, x_2) such that Ambrose weakly prefers $(36, 0)$ to these bundles. Is the set of bundles that Ambrose prefers to $(1, 16)$ a convex set? Yes.

(c) What is the slope of Ambrose's indifference curve at the point $(9, 8)$? (Hint: Recall from calculus the way to calculate the slope of a curve. If you don't know calculus, you will have to draw your diagram carefully and estimate the slope.) $-2/3$.

(d) What is the slope of his indifference curve at the point $(4, 12)$? -1.

(e) What is the slope of his indifference curve at the point $(9, 12)$? $-2/3$

at the point $(4, 16)$? -1.

(f) Do the indifference curves you have drawn for Ambrose exhibit diminishing marginal rate of substitution? Yes.

(g) Does Ambrose have convex preferences? Yes.

3.3 (0) Shirley Sixpack is in the habit of drinking beer each evening while watching "The Best of Bowlerama" on TV. She has a strong thumb and a big refrigerator, so she doesn't care about the size of the cans that beer comes in, she only cares about how much beer she has.

(a) On the graph below, draw some of Shirley's indifference curves between 16-ounce cans and 8-ounce cans of beer. Use blue ink to draw these indifference curves.

(b) Lorraine Quiche likes to have a beer while she watches "Masterpiece Theatre." She only allows herself an 8-ounce glass of beer at any one time. Since her cat doesn't like beer and she hates stale beer, if there is more than 8 ounces in the can she pours the excess into the sink. (She has no moral scruples about wasting beer.) On the graph above, use red ink to draw some of Lorraine's indifference curves.

3.4 (0) Elmo finds himself at a Coke machine on a hot and dusty Sunday. The Coke machine requires exact change—two quarters and a dime. No other combination of coins will make anything come out of the machine. No stores are open; no one is in sight. Elmo is so thirsty that the only thing he cares about is how many soft drinks he will be able to buy with the change in his pocket; the more he can buy, the better. While Elmo searches his pockets, your task is to draw some indifference curves that describe Elmo's preferences about what he finds.

(a) If Elmo has 2 quarters and a dime in his pockets, he can buy 1 soft drink. How many soft drinks can he buy if he has 4 quarters and 2 dimes?

2.

(b) Use red ink to shade in the area on the graph consisting of all combinations of quarters and dimes that Elmo thinks are just indifferent to having 2 quarters and 1 dime. (Imagine that it is possible for Elmo to have fractions of quarters or of dimes, but, of course, they would be useless in the machine.) Now use blue ink to shade in the area consisting of all combinations that Elmo thinks are just indifferent to having 4 quarters and 2 dimes. Notice that Elmo has indifference "bands," not indifference curves.

(c) Does Elmo have convex preferences between dimes and quarters?

Yes.

(d) Does Elmo always prefer more of both kinds of money to less? No.

(e) Does Elmo have a bliss point? No.

(f) If Elmo had arrived at the Coke machine on a Saturday, the drugstore across the street would have been open. This drugstore has a soda fountain that will sell you as much Coke as you want at a price of 4 cents an ounce. The salesperson will take any combination of dimes and quarters in payment. Suppose that Elmo plans to spend all of the money in his pocket on Coke at the drugstore on Saturday. On the graph above, use pencil or black ink to draw one or two of Elmo's indifference curves between quarters and dimes in his pocket. (For simplicity, draw your graph

as if Elmo's fractional quarters and fractional dimes are accepted at the corresponding fraction of their value.) Describe these new indifference curves in words. Line segments with slope -2.5.

3.5 (0) Randy Ratpack hates studying both economics and history. The more time he spends studying either subject, the less happy he is. But Randy has strictly convex preferences.

(a) Sketch an indifference curve for Randy where the two commodities are hours per week spent studying economics and hours per week spent studying history. Will the slope of an indifference curve be positive or negative? Negative.

(b) Do Randy's indifference curves get steeper or flatter as you move from left to right along one of them? Steeper.

3.6 (0) Flossy Toothsome likes to spend some time studying and some time dating. In fact her indifference curves between hours per week spent studying and hours per week spent dating are concentric circles around her favorite combination, which is 20 hours of studying and 15 hours of dating per week. The closer she is to her favorite combination, the happier she is.

(a) Suppose that Flossy is currently studying 25 hours a week and dating 3 hours a week. Would she prefer to be studying 30 hours a week and dating 8 hours a week? Yes. (Hint: Remember the formula for the distance between two points in the plane?)

(b) On the axes below, draw a few of Flossy's indifference curves and use your diagram to illustrate which of the two time allocations discussed above Flossy would prefer.

3.7 (0) Joan likes chocolate cake and ice cream, but after 10 slices of cake, she gets tired of cake, and eating more cake makes her less happy. Joan always prefers more ice cream to less. Joan's parents require her to eat everything put on her plate. In the axes below, use blue ink to draw a set of indifference curves that depict her preferences between plates with different amounts of cake and ice cream. Be sure to label the axes.

(a) Suppose that Joan's preferences are as before, but that her parents allow her to leave anything on her plate that she doesn't want. On the graph below, use red ink to draw some indifference curves depicting her preferences between plates with different amounts of cake and ice cream.

3.8 (0) Professor Goodheart always gives two midterms in his communications class. He only uses the higher of the two scores that a student gets on the midterms when he calculates the course grade.

(a) Nancy Lerner wants to maximize her grade in this course. Let x_1 be her score on the first midterm and x_2 be her score on the second midterm. Which combination of scores would Nancy prefer, $x_1 = 20$ and $x_2 = 70$ or $x_1 = 60$ and $x_2 = 60$? (20,70).

(b) On the graph below, use red ink to draw an indifference curve showing all of the combinations of scores that Nancy likes exactly as much as $x_1 = 20$ and $x_2 = 70$. Also use red ink to draw an indifference curve showing the combinations that Nancy likes exactly as much as $x_1 = 60$ and $x_2 = 60$.

(c) Does Nancy have convex preferences over these combinations? No.

(d) Nancy is also taking a course in economics from Professor Stern. Professor Stern gives two midterms. Instead of discarding the lower grade, Professor Stern discards the higher one. Let x_1 be her score on the first midterm and x_2 be her score on the second midterm. Which combination of scores would Nancy prefer, $x_1 = 20$ and $x_2 = 70$ or $x_1 = 60$ and $x_2 = 50$? (60,50).

(e) On the graph above, use blue ink to draw an indifference curve showing all of the combinations of scores on her econ exams that Nancy likes exactly as well as $x_1 = 20$ and $x_2 = 70$. Also use blue ink to draw an indifference curve showing the combinations that Nancy likes exactly as well as $x_1 = 60$ and $x_2 = 50$. Does Nancy have convex preferences over these combinations? Yes.

3.9 (0) Mary Granola loves to consume two goods, grapefruits and avocados.

(a) On the graph below, the slope of an indifference curve through any point where she has more grapefruits than avocados is -2. This means that when she has more grapefruits than avocados, she is willing to give up 2 grapefruit(s) to get one avocado.

(b) On the same graph, the slope of an indifference curve at points where she has fewer grapefruits than avocados is $-1/2$. This means that when she has fewer grapefruits than avocados, she is just willing to give up 1/2 grapefruit(s) to get one avocado.

(c) On this graph, draw an indifference curve for Mary through bundle $(10A, 10G)$. Draw another indifference curve through $(20A, 20G)$.

(d) Does Mary have convex preferences? Yes.

3.10 (2) Ralph Rigid likes to eat lunch at 12 noon. However, he also likes to save money so he can buy other consumption goods by attending the "early bird specials" and "late lunchers" promoted by his local diner. Ralph has 15 dollars a day to spend on lunch and other stuff. Lunch at noon costs \$5. If he delays his lunch until t hours after noon, he is able to buy his lunch for a price of $\$5 - t$. Similarly if he eats his lunch t hours before noon, he can buy it for a price of $\$5 - t$. (This is true for fractions of hours as well as integer numbers of hours.)

(a) If Ralph eats lunch at noon, how much money does he have per day to spend on other stuff? \$10.

(b) How much money per day would he have left for other stuff if he ate at 2 P.M.? $12.

(c) On the graph below, use blue ink to draw the broken line that shows combinations of meal time and money for other stuff that Ralph can just afford. On this same graph, draw some indifference curves that would be consistent with Ralph choosing to eat his lunch at 11 A.M.

3.11 (0) Henry Hanover is currently consuming 20 cheeseburgers and 20 Cherry Cokes a week. A typical indifference curve for Henry is depicted below.

(a) If someone offered to trade Henry one extra cheeseburger for every Coke he gave up, would Henry want to do this? No.

(b) What if it were the other way around: for every cheeseburger Henry gave up, he would get an extra Coke. Would he accept this offer? Yes.

(c) At what rate of exchange would Henry be willing to stay put at his current consumption level? 2 cheeseburgers for 1 Coke.

3.12 (1) Tommy Twit is happiest when he has 8 cookies and 4 glasses of milk per day. Whenever he has more than his favorite amount of either food, giving him still more makes him worse off. Whenever he has less than his favorite amount of either food, giving him more makes him better off. His mother makes him drink 7 glasses of milk and only allows him 2 cookies per day. One day when his mother was gone, Tommy's sadistic sister made him eat 13 cookies and only gave him 1 glass of milk, despite the fact that Tommy complained bitterly about the last 5 cookies that she made him eat and begged for more milk. Although Tommy complained later to his mother, he had to admit that he liked the diet that his sister forced on him better than what his mother demanded.

(a) Use black ink to draw some indifference curves for Tommy that are consistent with this story.

(b) Tommy's mother believes that the optimal amount for him to consume is 7 glasses of milk and 2 cookies. She measures deviations by absolute values. If Tommy consumes some other bundle, say, (c, m), she measures his departure from the optimal bundle by $D = |7 - m| + |2 - c|$. The larger D is, the worse off she thinks Tommy is. Use blue ink in the graph above to sketch a few of Mrs. Twit's indifference curves for Tommy's consumption. (Hint: Before you try to draw Mrs. Twit's indifference curves, we suggest that you take a piece of scrap paper and draw a graph of the locus of points (x_1, x_2) such that $|x_1| + |x_2| = 1$.)

3.13 (0) Coach Steroid likes his players to be big, fast, and obedient. If player A is better than player B in two of these three characteristics, then Coach Steroid prefers A to B, but if B is better than A in two of these three characteristics, then Steroid prefers B to A. Otherwise, Steroid is indifferent between them. Wilbur Westinghouse weighs 340 pounds, runs very slowly, and is fairly obedient. Harold Hotpoint weighs 240 pounds, runs very fast, and is very disobedient. Jerry Jacuzzi weighs 150 pounds, runs at average speed, and is extremely obedient.

(a) Does Steroid prefer Westinghouse to Hotpoint or vice versa? He prefers Westinghouse to Hotpoint.

(b) Does Steroid prefer Hotpoint to Jacuzzi or vice versa? He prefers Hotpoint to Jacuzzi.

(c) Does Steroid prefer Westinghouse to Jacuzzi or vice versa? He prefers Jacuzzi to Westinghouse.

(d) Does Coach Steroid have transitive preferences? No.

(e) After several losing seasons, Coach Steroid decides to change his way of judging players. According to his new preferences, Steroid prefers player A to player B if player A is better in all three of the characteristics that Steroid values, and he prefers B to A if player B is better at all three things. He is indifferent between A and B if they weigh the same, are equally fast, and are equally obedient. In all other cases, Coach Steroid simply says "A and B are not comparable."

(f) Are Coach Steroid's new preferences complete? No.

(g) Are Coach Steroid's new preferences transitive? Yes.

(h) Are Coach Steroid's new preferences reflexive? Yes.

3.14 (0) The Bear family is trying to decide what to have for dinner. Baby Bear says that his ranking of the possibilities is (honey, grubs, Goldilocks). Mama Bear ranks the choices (grubs, Goldilocks, honey), while Papa Bear's ranking is (Goldilocks, honey, grubs). They decide to take each pair of alternatives and let a majority vote determine the family rankings.

(a) Papa suggests that they first consider honey vs. grubs, and then the winner of that contest vs. Goldilocks. Which alternative will be chosen?

Goldilocks.

(b) Mama suggests instead that they consider honey vs. Goldilocks and then the winner vs. grubs. Which gets chosen? Grubs.

(c) What order should Baby Bear suggest if he wants to get his favorite food for dinner? Grubs versus Goldilocks, then Honey versus the winner.

(d) Are the Bear family's "collective preferences," as determined by voting, transitive? No.

3.15 (0) Olson likes strong coffee, the stronger the better. But he can't distinguish small differences. Over the years, Mrs. Olson has discovered that if she changes the amount of coffee by more than one teaspoon in her six-cup pot, Olson can tell that she did it. But he cannot distinguish differences smaller than one teaspoon per pot. Where A and B are two different cups of coffee, let us write $A \succ B$ if Olson prefers cup A to cup B. Let us write $A \succeq B$ if Olson either prefers A to B, or can't tell the difference between them. Let us write $A \sim B$ if Olson can't tell the difference between cups A and B. Suppose that Olson is offered cups A, B, and C all brewed in the Olsons' six-cup pot. Cup A was brewed using 14 teaspoons of coffee in the pot. Cup B was brewed using 14.75 teaspoons of coffee in the pot and cup C was brewed using 15.5 teaspoons of coffee in the pot. For each of the following expressions determine whether it is true of false.

(a) $A \sim B$. True.

(b) $B \sim A$. True.

(c) $B \sim C$. `True.`

(d) $A \sim C$. `False.`

(e) $C \sim A$. `False.`

(f) $A \succeq B$. `True.`

(g) $B \succeq A$. `True.`

(h) $B \succeq C$. `True.`

(i) $A \succeq C$. `False.`

(j) $C \succeq A$. `True.`

(k) $A \succ B$. `False.`

(l) $B \succ A$. `False.`

(m) $B \succ C$. `False.`

(n) $A \succ C$. `False.`

(o) $C \succ A$. `True.`

(p) Is Olson's "at-least-as-good-as" relation, $\succeq$, transitive? `No.`

(q) Is Olson's "can't-tell-the-difference" relation, $\sim$, transitive? `No.`

(r) is Olson's "better-than" relation, $\succ$, transitive. `Yes.`

Chapter 4

NAME_____

Utility

Introduction. In the previous chapter, you learned about preferences and indifference curves. Here we study another way of describing preferences, the *utility function.* A utility function that represents a person's preferences is a function that assigns a utility number to each commodity bundle. The numbers are assigned in such a way that commodity bundle (x, y) gets a higher utility number than bundle (x', y') if and only if the consumer prefers (x, y) to (x', y'). If a consumer has the utility function $U(x_1, x_2)$, then she will be indifferent between two bundles if they are assigned the same utility.

If you know a consumer's utility function, then you can find the indifference curve passing through any commodity bundle. Recall from the previous chapter that when good 1 is graphed on the horizontal axis and good 2 on the vertical axis, the slope of the indifference curve passing through a point (x_1, x_2) is known as the *marginal rate of substitution.* An important and convenient fact is that *the slope of an indifference curve is minus the ratio of the marginal utility of good 1 to the marginal utility of good 2.* For those of you who know even a tiny bit of calculus, calculating marginal utilities is easy. To find the marginal utility of either good, you just take the derivative of utility with respect to the amount of that good, treating the amount of the other good as a constant. (If you don't know any calculus at all, you can calculate an approximation to marginal utility by the method described in your textbook. Also, at the beginning of this section of the workbook, we list the marginal utility functions for commonly encountered utility functions. Even if you can't compute these yourself, you can refer to this list when later problems require you to use marginal utilities.)

Example: Arthur's utility function is $U(x_1, x_2) = x_1 x_2$. Let us find the indifference curve for Arthur that passes through the point $(3, 4)$. First, calculate $U(3, 4) = 3 \times 4 = 12$. The indifference curve through this point consists of all (x_1, x_2) such that $x_1 x_2 = 12$. This last equation is equivalent to $x_2 = 12/x_1$. Therefore to draw Arthur's indifference curve through $(3, 4)$, just draw the curve with equation $x_2 = 12/x_1$. At the point (x_1, x_2), the marginal utility of good 1 is x_2 and the marginal utility of good 2 is x_1. Therefore Arthur's marginal rate of substitution at the point $(3, 4)$ is $-x_2/x_1 = -4/3$.

Example: Arthur's uncle, Basil, has the utility function $U^*(x_1, x_2) = 3x_1 x_2 - 10$. Notice that $U^*(x_1, x_2) = 3U(x_1, x_2) - 10$, where $U(x_1, x_2)$ is Arthur's utility function. Since U^* is a positive multiple of U minus a constant, it must be that any change in consumption that increases U will also increase U^* (and vice versa). Therefore we say that Basil's utility function is a *monotonic increasing transformation* of Arthur's utility function. Let

us find Basil's indifference curve through the point $(3, 4)$. First we find that $U^*(3, 4) = 3 \times 3 \times 4 - 10 = 26$. The indifference curve passing through this point consists of all (x_1, x_2) such that $3x_1x_2 - 10 = 26$. Simplify this last expression by adding 10 to both sides of the equation and dividing both sides by 3. You find $x_1x_2 = 12$, or equivalently, $x_2 = 12/x_1$. This is exactly the same curve as Arthur's indifference curve through $(3, 4)$. We could have known in advance that this would happen, because if two consumers' utility functions are monotonic increasing transformations of each other, then these consumers must have the same preference relation between any pair of commodity bundles.

When you have finished this workout, we hope that you will be able to do the following:

- Draw an indifference curve through a specified commodity bundle when you know the utility function.
- Calculate marginal utilities and marginal rates of substitution when you know the utility function.
- Determine whether one utility function is just a "monotonic transformation" of another and know what that implies about preferences.
- Find utility functions that represent preferences when goods are perfect substitutes and when goods are perfect complements.
- Recognize utility functions for commonly studied preferences such as perfect substitutes, perfect complements, and other kinked indifference curves, quasilinear utility, and Cobb-Douglas utility.

4.0 Warm Up Exercise. This is the first of several "warm up exercises" that you will find in *Workouts*. These are here to help you see how to do calculations that are needed in later problems. The answers to *all* warm up exercises are in your answer pages. If you find the warm up exercises easy and boring, go ahead—skip them and get on to the main problems. You can come back and look at them if you get stuck later.

This exercise asks you to calculate marginal utilities and marginal rates of substitution for some common utility functions. These utility functions will reappear in several chapters, so it is a good idea to get to know them now. If you know calculus, you will find this to be a breeze. Even if your calculus is shaky or nonexistent, you can handle the first three utility functions just by using the definitions in the textbook. These three are easy because the utility functions are linear. If you do not know any calculus, fill in the rest of the answers from the back of the workbook and keep a copy of this exercise for reference when you encounter these utility functions in later problems.

$u(x_1, x_2)$	$MU_1(x_1, x_2)$	$MU_2(x_1, x_2)$	$MRS(x_1, x_2)$
$2x_1 + 3x_2$	2	3	$-2/3$
$4x_1 + 6x_2$	4	6	$-2/3$
$ax_1 + bx_2$	a	b	$-a/b$
$2\sqrt{x_1} + x_2$	$\frac{1}{\sqrt{x_1}}$	1	$-\frac{1}{\sqrt{x_1}}$
$\ln x_1 + x_2$	$1/x_1$	1	$-1/x_1$
$v(x_1) + x_2$	$v'(x_1)$	1	$-v'(x_1)$
$x_1 x_2$	x_2	x_1	$-x_2/x_1$
$x_1^a x_2^b$	$ax_1^{a-1}x_2^b$	$bx_1^a x_2^{b-1}$	$-\frac{ax_2}{bx_1}$
$(x_1 + 2)(x_2 + 1)$	$x_2 + 1$	$x_1 + 2$	$-\left(\frac{x_2+1}{x_1+2}\right)$
$(x_1 + a)(x_2 + b)$	$x_2 + b$	$x_1 + a$	$-\left(\frac{x_2+b}{x_1+a}\right)$
$x_1^a + x_2^a$	ax_1^{a-1}	ax_2^{a-1}	$-\left(\frac{x_1}{x_2}\right)^{a-1}$

4.1 (0) Remember Charlie from Chapter 3? Charlie consumes apples and bananas. We had a look at two of his indifference curves. In this problem we give you enough information so you can find *all* of Charlie's indifference curves. We do this by telling you that Charlie's utility function happens to be $U(x_A, x_B) = x_A x_B$.

(a) Charlie has 40 apples and 5 bananas. Charlie's utility for the bundle $(40, 5)$ is $U(40, 5) =$ 200. The indifference curve through $(40, 5)$ includes all commodity bundles (x_A, x_B) such that $x_A x_B =$ 200. So the indifference curve through $(40, 5)$ has the equation $x_B = \frac{200}{x_A}$. On the graph below, draw the indifference curve showing all of the bundles that Charlie likes exactly as well as the bundle $(40, 5)$.

(b) Donna offers to give Charlie 15 bananas if he will give her 25 apples. Would Charlie have a bundle that he likes better than $(40, 5)$ if he makes this trade? Yes. What is the largest number of apples that Donna could demand from Charlie in return for 15 bananas if she expects him to be willing to trade or at least indifferent about trading? 30. (Hint: If Donna gives Charlie 15 bananas, he will have a total of 20 bananas. If he has 20 bananas, how many apples does he need in order to be as well-off as he would be without trade?)

4.2 (0) Ambrose, whom you met in the last chapter, continues to thrive on nuts and berries. You saw two of his indifference curves. One indifference curve had the equation $x_2 = 20 - 4\sqrt{x_1}$, and another indifference curve had the equation $x_2 = 24 - 4\sqrt{x_1}$, where x_1 is his consumption of

nuts and x_2 is his consumption of berries. Now it can be told that Ambrose has quasilinear utility. In fact, his preferences can be represented by the utility function $U(x_1, x_2) = 4\sqrt{x_1} + x_2$.

(a) Ambrose originally consumed 9 units of nuts and 10 units of berries. His consumption of nuts is reduced to 4 units, but he is given enough berries so that he is just as well-off as he was before. After the change, how many units of berries does Ambrose consume? 14.

(b) On the graph below, indicate Ambrose's original consumption and sketch an indifference curve passing through this point. As you can verify, Ambrose is indifferent between the bundle (9,10) and the bundle (25,2). If you doubled the amount of each good in each bundle, you would have bundles (18,20) and (50,4). Are these two bundles on the same indifference curve? No. (Hint: How do you check whether two bundles are indifferent when you know the utility function?)

(c) What is Ambrose's marginal rate of substitution, $MRS(x_1, x_2)$, when he is consuming the bundle $(9, 10)$? (Give a numerical answer.) $-2/3$. What is Ambrose's marginal rate of substitution when he is consuming the bundle $(9, 20)$? $-2/3$.

(d) We can write a general expression for Ambrose's marginal rate of substitution when he is consuming commodity bundle (x_1, x_2). This is $MRS(x_1, x_2) =$ $-2/\sqrt{x_1}$. Although we always write $MRS(x_1, x_2)$ as a function of the two variables, x_1 and x_2, we see that Ambrose's utility function has the special property that his marginal rate of substitution does not change when the variable x_2 changes.

4.3 (0) Burt's utility function is $U(x_1, x_2) = (x_1 + 2)(x_2 + 6)$, where x_1 is the number of cookies and x_2 is the number of glasses of milk that he consumes.

(a) What is the slope of Burt's indifference curve at the point where he is consuming the bundle $(4, 6)$? -2. Use pencil or black ink to draw a line with this slope through the point $(4, 6)$. (Try to make this graph fairly neat and precise, since details will matter.) The line you just drew is the *tangent line* to the consumer's indifference curve at the point $(4, 6)$.

(b) The indifference curve through the point $(4, 6)$ passes through the points (10 ,0), (7, 2), and (2, 12). Use blue ink to sketch in this indifference curve. Incidentally, the equation for Burt's indifference curve through the point $(4, 6)$ is $x_2 =$ $72/(x_1 + 2) - 6$.

(c) Burt currently has the bundle $(4, 6)$. Ernie offers to give Burt 9 glasses of milk if Burt will give Ernie 3 cookies. If Burt makes this trade, he would have the bundle $(1, 15)$. Burt refuses to trade. Was this a wise decision? Yes, $U(1, 15) = 63 < U(4, 6) = 72$. Mark the bundle $(1, 15)$ on your graph.

(d) Ernie says to Burt, "Burt, your marginal rate of substitution is -2. That means that an extra cookie is worth only twice as much to you as an extra glass of milk. I offered to give you 3 glasses of milk for every cookie you give me. If I offer to give you more than your marginal rate of substitution, then you should want to trade with me." Burt replies,

"Ernie, you are right that my marginal rate of substitution is -2. That means that I am willing to make *small* trades where I get more than 2 glasses of milk for every cookie I give you, but 9 glasses of milk for 3 cookies is too big a trade. My indifference curves are not straight lines, you see." Would Burt be willing to give up 1 cookie for 3 glasses of milk? Yes, $U(3,9) = 75 > U(4,6) = 72$. Would Burt object to giving up 2 cookies for 6 glasses of milk? No, $U(2,12) = 72 = U(4,6)$.

(e) On your graph, use red ink to draw a line with slope -3 through the point $(4,6)$. This line shows all of the bundles that Burt can achieve by trading cookies for milk (or milk for cookies) at the rate of 1 cookie for every 3 glasses of milk. Only a segment of this line represents trades that make Burt better off than he was without trade. Label this line segment on your graph AB.

4.4 (0) Phil Rupp's utility function is $U(x,y) = \max\{x, 2y\}$.

(a) On the graph below, use blue ink to draw and label the line whose equation is $x = 10$. Also use blue ink to draw and label the line whose equation is $2y = 10$.

(b) If $x = 10$ and $2y < 10$, then $U(x,y) =$ 10. If $x < 10$ and $2y = 10$, then $U(x,y) =$ 10.

(c) Now use red ink to sketch in the indifference curve along which $U(x,y) = 10$. Does Phil have convex preferences? No.

4.5 (0) As you may recall, Nancy Lerner is taking Professor Stern's economics course. She will take two examinations in the course, and her score for the course is the minimum of the scores that she gets on the two exams. Nancy wants to get the highest possible score for the course.

(a) Write a utility function that represents Nancy's preferences over alternative combinations of test scores x_1 and x_2 on tests 1 and 2 respectively. $U(x_1, x_2) =$ $\min\{x_1, x_2\}$, or any monotonic transformation.

4.6 (0) Remember Shirley Sixpack and Lorraine Quiche from the last chapter? Shirley thinks a 16-ounce can of beer is just as good as two 8-ounce cans. Lorraine only drinks 8 ounces at a time and hates stale beer, so she thinks a 16-ounce can is no better or worse than an 8-ounce can.

(a) Write a utility function that represents Shirley's preferences between commodity bundles comprised of 8-ounce cans and 16-ounce cans of beer. Let X stand for the number of 8-ounce cans and Y stand for the number of 16-ounce cans. $u(X, Y) = X + 2Y$.

(b) Now write a utility function that represents Lorraine's preferences.

$u(X, Y) = X + Y$.

(c) Would the function utility $U(X, Y) = 100X + 200Y$ represent Shirley's preferences? Yes. Would the utility function $U(x, y) = (5X + 10Y)^2$ represent her preferences? Yes. Would the utility function $U(x, y) = X + 3Y$ represent her preferences? No.

(d) Give an example of two commodity bundles such that Shirley likes the first bundle better than the second bundle, while Lorraine likes the second bundle better than the first bundle. Shirley prefers (0,2) to (3,0). Lorraine disagrees.

4.7 (0) Harry Mazzola has the utility function $u(x_1, x_2) = \min\{x_1 + 2x_2, 2x_1 + x_2\}$, where x_1 is his consumption of corn chips and x_2 is his consumption of french fries.

(a) On the graph below, use a pencil to draw the locus of points along which $x_1 + 2x_2 = 2x_1 + x_2$. Use blue ink to show the locus of points for which $x_1 + 2x_2 = 12$, and also use blue ink to draw the locus of points for which $2x_1 + x_2 = 12$.

(b) On the graph you have drawn, shade in the region where *both* of the following inequalities are satisfied: $x_1 + 2x_2 \geq 12$ and $2x_1 + x_2 \geq 12$.

At the bundle $(x_1, x_2) = (8, 2)$, one sees that $2x_1 + x_2 =$ 18 and $x_1 + 2x_2 =$ 12. Therefore $u(8, 2) =$ 12.

(c) Use black ink to sketch in the indifference curve along which Harry's utility is 12. Use red ink to sketch in the indifference curve along which Harry's utility is 6. (Hint: Is there anything about Harry Mazzola that reminds you of Mary Granola?)

(d) At the point where Harry is consuming 5 units of corn chips and 2 units of french fries, how many units of corn chips would he be willing to trade for one unit of french fries? 2.

4.8 (1) Vanna Boogie likes to have large parties. She also has a strong preference for having exactly as many men as women at her parties. In fact, Vanna's preferences among parties can be represented by the utility function $U(x, y) = \min\{2x - y, 2y - x\}$ where x is the number of women and y is the number of men at the party. On the graph below, let us try to draw the indifference curve along which Vanna's utility is 10.

(a) Use pencil to draw the locus of points at which $x = y$. What point on this gives Vanna a utility of 10? $(10, 10)$. Use blue ink to draw the line along which $2y - x = 10$. When $\min\{2x - y, 2y - x\} = 2y - x$,

there are (more men than women, more women than men)? More women. Draw a squiggly red line over the part of the blue line for which $U(x, y) = \min\{2x - y, 2y - x\} = 2y - x$. This shows all the combinations that Vanna thinks are just as good as $(10, 10)$ but where there are (more men than women, more women than men)? More women. Now draw a blue line along which $2x - y = 10$. Draw a squiggly red line over the part of this new blue line for which $\min\{2x - y, 2y - x\} = 2x - y$. Use pencil to shade in the area on the graph that represents all combinations that Vanna likes at least as well as $(10, 10)$.

(b) Suppose that there are 9 men and 10 women at Vanna's party. Would Vanna think it was a better party or a worse party if 5 more men came to her party? Worse.

(c) If Vanna has 16 women at her party and more men than women, and if she thinks the party is exactly as good as having 10 men and 10 women, how many men does she have at the party? 22. If Vanna has 16 women at her party and more women than men, and if she thinks the party is exactly as good as having 10 men and 10 women, how many men does she have at her party? 13.

(d) Vanna's indifference curves are shaped like what letter of the alphabet? V.

4.9 (0) Suppose that the utility functions $u(x, y)$ and $v(x, y)$ are related by $v(x, y) = f(u(x, y))$. In each case below, write "Yes" if the function f is a positive monotonic transformation and "No" if it is not. (Hint for

calculus users: A differentiable function $f(u)$ is an increasing function of u if its derivative is positive.)

(a) $f(u) = 3.141592u$. Yes.

(b) $f(u) = 5,000 - 23u$. No.

(c) $f(u) = u - 100,000$. Yes.

(d) $f(u) = \log_{10} u$. Yes.

(e) $f(u) = -e^{-u}$. Yes.

(f) $f(u) = 1/u$. No.

(g) $f(u) = -1/u$. Yes.

4.10 (0) Martha Modest has preferences represented by the utility function $U(a, b) = ab/100$, where a is the number of ounces of animal crackers that she consumes and b is the number of ounces of beans that she consumes.

(a) On the graph below, sketch the locus of points that Martha finds indifferent to having 8 ounces of animal crackers and 2 ounces of beans. Also sketch the locus of points that she finds indifferent to having 6 ounces of animal crackers and 4 ounces of beans.

(b) Bertha Brassy has preferences represented by the utility function $V(a, b) = 1,000a^2b^2$, where a is the number of ounces of animal crackers that she consumes and b is the number of ounces of beans that she consumes. On the graph below, sketch the locus of points that Bertha finds indifferent to having 8 ounces of animal crackers and 2 ounces of beans. Also sketch the locus of points that she finds indifferent to having 6 ounces of animal crackers and 4 ounces of beans.

(c) Are Martha's preferences convex? Yes. Are Bertha's? Yes.

(d) What can you say about the difference between the indifference curves you drew for Bertha and those you drew for Martha? There is no difference.

(e) How could you tell this was going to happen without having to draw the curves? Their utility functions only differ by a monotonic transformation.

4.11 (0) Willy Wheeler's preferences over bundles that contain nonnegative amounts of x_1 and x_2 are represented by the utility function $U(x_1, x_2) = x_1^2 + x_2^2$.

(a) Draw a few of his indifference curves. What kind of geometric figure are they? Quarter circles centered at the origin. Does Willy have convex preferences? No.

Calculus **4.12 (0)** Joe Bob has a utility function given by $u(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$.

(a) Compute Joe Bob's marginal rate of substitution: $MRS(x_1, x_2) =$ -1.

(b) Joe Bob's straight cousin, Al, has a utility function $v(x_1, x_2) = x_2 + x_1$. Compute Al's marginal rate of substitution. $MRS(x_1, x_2) =$ -1.

(c) Do $u(x_1, x_2)$ and $v(x_1, x_2)$ represent the same preferences? Yes. Can you show that Joe Bob's utility function is a monotonic transformation of Al's? (Hint: Some have said that Joe Bob is square.) Notice that $u(x_1, x_2) = [v(x_1, x_2)]^2$.

4.13 (0) The idea of assigning numerical values to determine a preference ordering over a set of objects is not limited in application to commodity bundles. The *Bill James Baseball Abstract* argues that a baseball player's batting average is not an adequate measure of his offensive productivity. Batting averages treat singles just the same as extra base hits. Furthermore they do not give credit for "walks," although a walk is almost as good as a single. James argues that a double in two at-bats is better than a single, but not as good as two singles. To reflect these considerations, James proposes the following index, which he calls "runs created." Let A be the number of hits plus the number of walks that a batter gets in a season. Let B be the number of total bases that the batter gets in the season. (Thus, if a batter has S singles, W walks, D doubles, T triples, and H

home runs, then $A = S+D+T+H+W$ and $B = S+W+2D+3T+4H$.) Let N be the number of times the batter bats. Then his index of runs created in the season is defined to be AB/N and will be called his RC.

(a) In 1987, George Bell batted 649 times. He had 39 walks, 105 singles, 32 doubles, 4 triples, and 47 home runs. In 1987, Wade Boggs batted 656 times. He had 105 walks, 130 singles, 40 doubles, 6 triples, and 24 home runs. In 1987, Alan Trammell batted 657 times. He had 60 walks, 140 singles, 34 doubles, 3 triples, and 28 home runs. In 1987, Tony Gwynn batted 671 times. He had 82 walks, 162 singles, 36 doubles, 13 triples, and 7 home runs. We can calculate A, the number of hits plus walks, B the number of total bases, and RC, the runs created index for each of these players. For Bell, $A = 227$, $B = 408$, $RC = 143$. For Boggs, $A = 305$, $B = 429$, $RC = 199$. For Trammell, $A = 265$, $B = 389$, $RC = 157$. For Gwynn, $A =$ 300 , $B =$ 383 , $RC =$ 171.

(b) If somebody has a preference ordering among these players, based only on the runs-created index, which player(s) would she prefer to Trammell?

Boggs and Gwynn.

(c) The differences in the number of times at bat for these players are small, and we will ignore them for simplicity of calculation. On the graph below, plot the combinations of A and B achieved by each of the players. Draw four "indifference curves," one through each of the four points you have plotted. These indifference curves should represent combinations of A and B that lead to the same number of runs-created.

4.14 (0) This problem concerns the runs-created index discussed in the preceding problem. Consider a batter who bats 100 times and always either makes an out, hits for a single, or hits a home run.

(a) Let x be the number of singles and y be the number of home runs in 100 at-bats. Suppose that the utility function $U(x, y)$ by which we evaluate alternative combinations of singles and home runs is the runs-created index. Then the formula for the utility function is $U(x, y) =$

$(x + y)(x + 4y)/100$.

(b) Let's try to find out about the shape of an indifference curve between singles and home runs. Hitting 10 home runs and no singles would give him the same runs-created index as hitting 20 singles and no home runs. Mark the points $(0, 10)$ and $(x, 0)$, where $U(x, 0) = U(0, 10)$.

(c) Where x is the number of singles you solved for in the previous part, mark the point $(x/2, 5)$ on your graph. Is $U(x/2, 5)$ greater than or less than or equal to $U(0, 10)$? Greater than. Is this consistent with the batter having convex preferences between singles and home runs?

Yes.

Chapter 5

NAME_____

Choice

Introduction. You have studied budgets, and you have studied preferences. Now is the time to put these two ideas together and do something with them. In this chapter you study the commodity bundle chosen by a utility-maximizing consumer from a given budget.

Given prices and income, you know how to graph a consumer's budget. If you also know the consumer's preferences, you can graph some of his indifference curves. The consumer will choose the "best" indifference curve that he can reach given his budget. But when you try to do this, you have to ask yourself, "How do I find the most desirable indifference curve that the consumer can reach?" The answer to this question is "look in the likely places." Where are the likely places? As your textbook tells you, there are three kinds of likely places. These are: (*i*) a tangency between an indifference curve and the budget line; (*ii*) a kink in an indifference curve; (*iii*) a "corner" where the consumer specializes in consuming just one good.

Here is how you find a point of tangency if we are told the consumer's utility function, the prices of both goods, and the consumer's income. The budget line and an indifference curve are tangent at a point (x_1, x_2) if they have the same slope at that point. Now the slope of an indifference curve at (x_1, x_2) is the ratio $-MU_1(x_1, x_2)/MU_2(x_1, x_2)$. (This slope is also known as the marginal rate of substitution.) The slope of the budget line is $-p_1/p_2$. Therefore an indifference curve is tangent to the budget line at the point (x_1, x_2) when $MU_1(x_1, x_2)/MU_2(x_1, x_2) = p_1/p_2$. This gives us one equation in the two unknowns, x_1 and x_2. If we hope to solve for the x's, we need another equation. That other equation is the budget equation $p_1x_1 + p_2x_2 = m$. With these two equations you can solve for (x_1, x_2).*

Example: A consumer has the utility function $U(x_1, x_2) = x_1^2x_2$. The price of good 1 is $p_1 = 1$, the price of good 2 is $p_2 = 3$, and his income is 180. Then, $MU_1(x_1, x_2) = 2x_1x_2$ and $MU_2(x_1, x_2) = x_1^2$. Therefore his marginal rate of substitution is $-MU_1(x_1, x_2)/MU_2(x_1, x_2) = -2x_1x_2/x_1^2 = -2x_2/x_1$. This implies that his indifference curve will be tangent to his budget line when $-2x_2/x_1 = -p_1/p_2 = -1/3$. Simplifying this expression, we have $6x_2 = x_1$. This is one of the two equations we need to solve for the two unknowns, x_1 and x_2. The other equation is the budget equation. In this case the budget equation is $x_1 + 3x_2 = 180$. Solving these two equations in two unknowns, we find $x_1 = 120$ and

* Some people have trouble remembering whether the marginal rate of substitution is $-MU_1/MU_2$ or $-MU_2/MU_1$. It isn't really crucial to remember which way this goes as long as you remember that a tangency happens when the marginal utilities of any two goods are in the same proportion as their prices.

$x_2 = 20$. Therefore we know that the consumer chooses the bundle $(x_1, x_2) = (120, 20)$.

For equilibrium at kinks or at corners, we don't need the slope of the indifference curves to equal the slope of the budget line. So we don't have the tangency equation to work with. But we still have the budget equation. The second equation that you can use is an equation that tells you that you are at one of the kinky points or at a corner. You will see exactly how this works when you work a few exercises.

Example: A consumer has the utility function $U(x_1, x_2) = \min\{x_1, 3x_2\}$. The price of x_1 is 2, the price of x_2 is 1, and her income is 140. Her indifference curves are L-shaped. The corners of the L's all lie along the line $x_1 = 3x_2$. She will choose a combination at one of the corners, so this gives us one of the two equations we need for finding the unknowns x_1 and x_2. The second equation is her budget equation, which is $2x_1 + x_2 = 140$. Solve these two equations to find that $x_1 = 60$ and $x_2 = 20$. So we know that the consumer chooses the bundle $(x_1, x_2) = (60, 20)$.

When you have finished these exercises, we hope that you will be able to do the following:

- Calculate the best bundle a consumer can afford at given prices and income in the case of simple utility functions where the best affordable bundle happens at a point of tangency.
- Find the best affordable bundle, given prices and income for a consumer with kinked indifference curves.
- Recognize standard examples where the best bundle a consumer can afford happens at a corner of the budget set.
- Draw a diagram illustrating each of the above types of equilibrium.
- Apply the methods you have learned to choices made with some kinds of nonlinear budgets that arise in real-world situations.

5.1 (0) We begin again with Charlie of the apples and bananas. Recall that Charlie's utility function is $U(x_A, x_B) = x_A x_B$. Suppose that the price of apples is 1, the price of bananas is 2, and Charlie's income is 40.

(a) On the graph below, use blue ink to draw Charlie's budget line. (Use a ruler and try to make this line accurate.) Plot a few points on the indifference curve that gives Charlie a utility of 150 and sketch this curve with red ink. Now plot a few points on the indifference curve that gives Charlie a utility of 300 and sketch this curve with black ink or pencil.

(b) Can Charlie afford any bundles that give him a utility of 150? Yes.

(c) Can Charlie afford any bundles that give him a utility of 300? No.

(d) On your graph, mark a point that Charlie can afford and that gives him a higher utility than 150. Label that point A.

(e) Neither of the indifference curves that you drew is tangent to Charlie's budget line. Let's try to find one that is. At any point, (x_A, x_B), Charlie's marginal rate of substitution is a function of x_A and x_B. In fact, if you calculate the ratio of marginal utilities for Charlie's utility function, you will find that Charlie's marginal rate of substitution is $MRS(x_A, x_B) = -x_B/x_A$. This is the slope of his indifference curve at (x_A, x_B). The slope of Charlie's budget line is $-1/2$ (give a numerical answer).

(f) Write an equation that implies that the budget line is tangent to an indifference curve at (x_A, x_B). $-x_B/x_A = -1/2.$ There are many solutions to this equation. Each of these solutions corresponds to a point on a different indifference curve. Use pencil to draw a line that passes through all of these points.

(g) The best bundle that Charlie can afford must lie somewhere on the line you just penciled in. It must also lie on his budget line. If the point is outside of his budget line, he can't afford it. If the point lies inside of his budget line, he can afford to do better by buying more of both goods. On your graph, label this best affordable bundle with an E. This happens where $x_A=$ 20 and $x_B=$ 10. Verify your answer by solving the two simultaneous equations given by his budget equation and the tangency condition.

(h) What is Charlie's utility if he consumes the bundle $(20, 10)$? 200.

(i) On the graph above, use red ink to draw his indifference curve through (20,10). Does this indifference curve cross Charlie's budget line, just touch it, or never touch it? Just touch it.

5.2 (0) Clara's utility function is $U(X, Y) = (X + 2)(Y + 1)$, where X is her consumption of good X and Y is her consumption of good Y.

(a) Write an equation for Clara's indifference curve that goes through the point $(X, Y) = (2, 8)$. $Y = \frac{36}{X+2} - 1$. On the axes below, sketch Clara's indifference curve for $U = 36$.

(b) Suppose that the price of each good is 1 and that Clara has an income of 11. Draw in her budget line. Can Clara achieve a utility of 36 with this budget? Yes.

(c) At the commodity bundle, (X, Y), Clara's marginal rate of substitution is $-\frac{Y+1}{X+2}$.

(d) If we set the absolute value of the MRS equal to the price ratio, we have the equation $\frac{Y+1}{X+2} = 1$.

(e) The budget equation is $X + Y = 11$.

(f) Solving these two equations for the two unknowns, X and Y, we find $X =$ 5 and $Y =$ 6.

5.3 (0) Ambrose, the nut and berry consumer, has a utility function $U(x_1, x_2) = 4\sqrt{x_1} + x_2$, where x_1 is his consumption of nuts and x_2 is his consumption of berries.

(a) The commodity bundle $(25, 0)$ gives Ambrose a utility of 20. Other points that give him the same utility are $(16, 4)$, (9, 8), (4, 12), (1, 16), and (0, 20). Plot these points on the axes below and draw a red indifference curve through them.

(b) Suppose that the price of a unit of nuts is 1, the price of a unit of berries is 2, and Ambrose's income is 24. Draw Ambrose's budget line with blue ink. How many units of nuts does he choose to buy? 16 units.

(c) How many units of berries? 4 units.

(d) Find some points on the indifference curve that gives him a utility of 25 and sketch this indifference curve (in red).

(e) Now suppose that the prices are as before, but Ambrose's income is 34. Draw his new budget line (with pencil). How many units of nuts will he choose? 16 units. How many units of berries? 9 units.

(f) Now let us explore a case where there is a "boundary solution." Suppose that the price of nuts is still 1 and the price of berries is 2, but Ambrose's income is only 9. Draw his budget line (in blue). Sketch the indifference curve that passes through the point $(9, 0)$. What is the slope of his indifference curve at the point $(9, 0)$? $-2/3$.

(g) What is the slope of his budget line at this point? $-1/2$.

(h) Which is steeper at this point, the budget line or the indifference curve? Indifference curve.

(i) Can Ambrose afford any bundles that he likes better than the point $(9, 0)$? No.

5.4 (1) Nancy Lerner is trying to decide how to allocate her time in studying for her economics course. There are two examinations in this course. Her overall score for the course will be the *minimum* of her scores on the two examinations. She has decided to devote a total of 1,200 minutes to studying for these two exams, and she wants to get as high an overall score as possible. She knows that on the first examination if she doesn't study at all, she will get a score of zero on it. For every 10 minutes that she spends studying for the first examination, she will increase her score by one point. If she doesn't study at all for the second examination she will get a zero on it. For every 20 minutes she spends studying for the second examination, she will increase her score by one point.

(a) On the graph below, draw a "budget line" showing the various combinations of scores on the two exams that she can achieve with a total of 1,200 minutes of studying. On the same graph, draw two or three "indifference curves" for Nancy. On your graph, draw a straight line that goes through the kinks in Nancy's indifference curves. Label the point where this line hits Nancy's budget with the letter A. Draw Nancy's indifference curve through this point.

(b) Write an equation for the line passing through the kinks of Nancy's indifference curves. $x_1 = x_2$.

(c) Write an equation for Nancy's budget line. $10x_1 + 20x_2 = 1,200$.

(d) Solve these two equations in two unknowns to determine the intersection of these lines. This happens at the point $(x_1, x_2) =$ $(40, 40)$.

(e) Given that she spends a total of 1,200 minutes studying, Nancy will maximize her overall score by spending 400 minutes studying for the first examination and 800 minutes studying for the second examination.

5.5 (1) In her communications course, Nancy also takes two examinations. Her overall grade for the course will be the *maximum* of her scores on the two examinations. Nancy decides to spend a total of 400 minutes studying for these two examinations. If she spends m_1 minutes studying

for the first examination, her score on this exam will be $x_1 = m_1/5$. If she spends m_2 minutes studying for the second examination, her score on this exam will be $x_2 = m_2/10$.

(a) On the graph below, draw a "budget line" showing the various combinations of scores on the two exams that she can achieve with a total of 400 minutes of studying. On the same graph, draw two or three "indifference curves" for Nancy. On your graph, find the point on Nancy's budget line that gives her the best overall score in the course.

(b) Given that she spends a total of 400 minutes studying, Nancy will maximize her overall score by achieving a score of 80 on the first examination and 0 on the second examination.

(c) Her overall score for the course will then be 80.

5.6 (0) Elmer's utility function is $U(x, y) = \min\{x, y^2\}$.

(a) If Elmer consumes 4 units of x and 3 units of y, his utility is 4.

(b) If Elmer consumes 4 units of x and 2 units of y, his utility is 4.

(c) If Elmer consumes 5 units of x and 2 units of y, his utility is 4.

(d) On the graph below, use blue ink to draw the indifference curve for Elmer that contains the bundles that he likes exactly as well as the bundle $(4, 2)$.

(e) On the same graph, use blue ink to draw the indifference curve for Elmer that contains bundles that he likes exactly as well as the bundle $(1, 1)$ and the indifference curve that passes through the point $(16, 5)$.

(f) On your graph, use black ink to show the locus of points at which Elmer's indifference curves have kinks. What is the equation for this curve? $x = y^2$.

(g) On the same graph, use black ink to draw Elmer's budget line when the price of x is 1, the price of y is 2, and his income is 8. What bundle does Elmer choose in this situation? (4,2).

(h) Suppose that the price of x is 10 and the price of y is 15 and Elmer buys 100 units of x. What is Elmer's income? 1,150. (Hint: At first you might think there is too little information to answer this question. But think about how much y he must be demanding if he chooses 100 units of x.)

5.7 (0) Linus has the utility function $U(x, y) = x + 3y$.

(a) On the graph below, use blue ink to draw the indifference curve passing through the point $(x, y) = (3, 3)$. Use black ink to sketch the indifference curve connecting bundles that give Linus a utility of 6.

(b) On the same graph, use red ink to draw Linus's budget line if the price of x is 1 and the price of y is 2 and his income is 8. What bundle does Linus choose in this situation? (0,4).

(c) What bundle would Linus choose if the price of x is 1, the price of y is 4, and his income is 8? (8,0).

5.8 (2) Remember our friend Ralph Rigid from Chapter 3? His favorite diner, Food for Thought, has adopted the following policy to reduce the crowds at lunch time: if you show up for lunch t hours before or after 12 noon, you get to deduct t dollars from your bill. (This holds for any fraction of an hour as well.)

(a) Use blue ink to show Ralph's budget set. On this graph, the horizontal axis measures the time of day that he eats lunch, and the vertical axis measures the amount of money that he will have to spend on things other than lunch. Assume that he has \$20 total to spend and that lunch at noon costs \$10. (Hint: How much money would he have left if he ate at noon? at 1 P.M.? at 11 A.M.?)

(b) Recall that Ralph's preferred lunch time is 12 noon, but that he is willing to eat at another time if the food is sufficiently cheap. Draw some red indifference curves for Ralph that would be consistent with his choosing to eat at 11 A.M.

5.9 (0) Joe Grad has just arrived at the big U. He has a fellowship that covers his tuition and the rent on an apartment. In order to get by, Joe has become a grader in intermediate price theory, earning \$100 a month. Out of this \$100 he must pay for his food and utilities in his apartment. His utilities expenses consist of heating costs when he heats his apartment and air-conditioning costs when he cools it. To raise the temperature of his apartment by one degree, it costs \$2 per month (or \$20 per month to raise it ten degrees). To use air-conditioning to cool his apartment by a degree, it costs \$3 per month. Whatever is left over after paying the utilities, he uses to buy food at \$1 per unit.

(a) When Joe first arrives in September, the temperature of his apartment is 60 degrees. If he spends nothing on heating or cooling, the temperature in his room will be 60 degrees and he will have $100 left to spend on food.

If he heated the room to 70 degrees, he would have $80 left to spend on food. If he cooled the room to 50 degrees, he would have $70 left to spend on food. On the graph below, show Joe's September budget constraint (with black ink). (Hint: You have just found three points that Joe can afford. Apparently, his budget set is not bounded by a single straight line.)

(b) In December, the outside temperature is 30 degrees and in August poor Joe is trying to understand macroeconomics while the temperature outside is 85 degrees. On the same graph you used above, draw Joe's budget constraints for the months of December (in blue ink) and August (in red ink).

(c) Draw a few smooth (unkinky) indifference curves for Joe in such a way that the following are true. (*i*) His favorite temperature for his apartment would be 65 degrees if it cost him nothing to heat it or cool it. (*ii*) Joe chooses to use the furnace in December, air-conditioning in August, and neither in September. (*iii*) Joe is better off in December than in August.

(d) In what months is the slope of Joe's budget constraint equal to the slope of his indifference curve? August and December.

(e) In December Joe's marginal rate of substitution between food and degrees Fahrenheit is -2. In August, his MRS is 3.

(f) Since Joe neither heats nor cools his apartment in September, we cannot determine his marginal rate of substitution exactly, but we do know that it must be no smaller than -2 and no larger than 3. (Hint: Look carefully at your graph.)

5.10 (0) Central High School has \$60,000 to spend on computers and other stuff, so its budget equation is $C + X = 60,000$, where C is expenditure on computers and X is expenditures on other things. C.H.S. currently plans to spend \$20,000 on computers.

The State Education Commission wants to encourage "computer literacy" in the high schools under its jurisdiction. The following plans have been proposed.

Plan A: This plan would give a grant of \$10,000 to each high school in the state that the school could spend as it wished.

Plan B: This plan would give a \$10,000 grant to any high school, so long as the school spent at least \$10,000 *more* than it currently spends on computers. Any high school can choose not to participate, in which case it does not receive the grant, but it doesn't have to increase its expenditure on computers.

Plan C: Plan C is a "matching grant." For every dollar's worth of computers that a high school orders, the state will give the school 50 cents.

Plan D: This plan is like plan C, except that the maximum amount of matching funds that any high school could get from the state would be limited to \$10,000.

(a) Write an equation for Central High School's budget if plan A is adopted. $C + X = 70,000$. Use black ink to draw the budget line for Central High School if plan A is adopted.

(b) If plan B is adopted, the boundary of Central High School's budget set has two separate downward-sloping line segments. One of these segments describes the cases where C.H.S. spends at least \$30,000 on computers. This line segment runs from the point $(C, X) = (70,000, 0)$ to the point $(C, X) =$ (30,000, 40,000).

(c) Another line segment corresponds to the cases where C.H.S. spends less than \$30,000 on computers. This line segment runs from $(C, X) =$ (30,000,30,000) to the point $(C, X) = (0, 60,000)$. Use red ink to draw these two line segments.

(d) If plan C is adopted and Central High School spends C dollars on computers, then it will have $X = 60,000 - .5C$ dollars left to spend on other things. Therefore its budget line has the equation `.5C+X=60,000.` Use blue ink to draw this budget line.

(e) If plan D is adopted, the school district's budget consists of two line segments that intersect at the point where expenditure on computers is `20,000` and expenditure on other instructional materials is `50,000.`

(f) The slope of the flatter line segment is `−.5.` The slope of the steeper segment is `−1.` Use pencil to draw this budget line.

5.11 (0) Suppose that Central High School has preferences that can be represented by the utility function $U(C, X) = CX^2$. Let us try to determine how the various plans described in the last problem will affect the amount that C.H.S. spends on computers.

(a) If the state adopts none of the new plans, find the expenditure on computers that maximizes the district's utility subject to its budget constraint. 20,000.

(b) If plan A is adopted, find the expenditure on computers that maximizes the district's utility subject to its budget constraint. 23,333.

(c) On your graph, sketch the indifference curve that passes through the point (30,000, 40,000) if plan B is adopted. At this point, which is steeper, the indifference curve or the budget line? The budget line.

(d) If plan B is adopted, find the expenditure on computers that maximizes the district's utility subject to its budget constraint. (Hint: Look at your graph.) 30,000.

(e) If plan C is adopted, find the expenditure on computers that maximizes the district's utility subject to its budget constraint. 40,000.

(f) If plan D is adopted, find the expenditure on computers that maximizes the district's utility subject to its budget constraint. 23,333.

5.12 (0) The telephone company allows one to choose between two different pricing plans. For a fee of \$12 per month you can make as many local phone calls as you want, at no additional charge per call. Alternatively, you can pay \$8 per month and be charged 5 cents for each local phone call that you make. Suppose that you have a total of \$20 per month to spend.

(a) On the graph below, use black ink to sketch a budget line for someone who chooses the first plan. Use red ink to draw a budget line for someone who chooses the second plan. Where do the two budget lines cross?

$(80, 8)$.

(b) On the graph above, use pencil to draw indifference curves for someone who prefers the second plan to the first. Use blue ink to draw an indifference curve for someone who prefers the first plan to the second.

5.13 (1) This is a puzzle—just for fun. Lewis Carroll (1832-1898), author of *Alice in Wonderland* and *Through the Looking Glass,* was a mathematician, logician, and political scientist. Carroll loved careful reasoning about puzzling things. Here Carroll's Alice presents a nice bit of economic analysis. At first glance, it may seem that Alice is talking nonsense, but, indeed, her reasoning is impeccable.

"I should like to buy an egg, please." she said timidly. "How do you sell them?"

"Fivepence farthing for one—twopence for two," the Sheep replied.

"Then two are cheaper than one?" Alice said, taking out her purse.

"Only you must eat them both if you buy two," said the Sheep.

"Then I'll have one please," said Alice, as she put the money down on the counter. For she thought to herself, "They mightn't be at all nice, you know."

(a) Let us try to draw a budget set and indifference curves that are consistent with this story. Suppose that Alice has a total of 8 pence to spend and that she can buy either 0, 1, or 2 eggs from the Sheep, but no fractional eggs. Then her budget set consists of just three points. The point where she buys no eggs is $(0, 8)$. Plot this point and label it A. On your graph, the point where she buys 1 egg is $(1, 2\frac{3}{4})$. (A farthing is 1/4 of a penny.) Plot this point and label it B.

(b) The point where she buys 2 eggs is $(2, 6)$. Plot this point and label it C. If Alice chooses to buy 1 egg, she must like the bundle B better than either the bundle A or the bundle C. Draw indifference curves for Alice that are consistent with this behavior.

5.14 (1) You will remember Harry Mazzola, who consumes only corn chips and french fries. Harry's utility function is $u(x_1, x_2) = \min\{x_1 + 2x_2, 2x_1 + x_2\}$ where x_1 is his consumption of corn chips and x_2 is his consumption of french fries.

(a) On the graph below, use black ink to draw in the indifference curve along which Harry's utility is 6. Use red ink to draw the budget line for Harry if the price of corn chips is $p_1 = 3$, the price of french fries is $p_2 = 2$, and income is $m = 10$. How many units of corn chips and how many units of french fries should he consume to maximize his utility subject to this budget? 2 units of corn chips and 2 units of french fries

(b) On the graph below use blue ink to draw Harry's budget line if the price of corn chips is 1, the price of french fries is 3, and Harry's income is 6. How many units of corn chips and how many units of french fries will Harry consume? 6 units of corn chips and no french fries

(c) At what prices will Harry consume only corn chips and no french fries? Whenever $p_2 > 2p_1$ At what prices will he consume only french fries and no corn chips? Whenever $p_1 > 2p_2$

(d) At what price-income combinations does Harry choose to consume equal quantities of corn chips and french fries? At any income, if $1/2 < p_1/p_2 < 2$, Harry will choose $x_1 = x_2$.

Chapter 6

NAME_____

Demand

Introduction. In the previous chapter, you found the commodity bundle that a consumer with a given utility function would choose in a specific price-income situation. In this chapter, we take this idea a step further. We find demand *functions*, which tell us for *any* prices and income you might want to name, how much of each good a consumer would want. In general, the amount of each good demanded may depend not only on its own price, but also on the price of other goods and on income. Where there are two goods, we write demand functions for Goods 1 and 2 as $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$.*

When the consumer is choosing positive amounts of all commodities and indifference curves have no kinks, the consumer chooses a point of tangency between her budget line and the highest indifference curve that it touches.

Example: Consider a consumer with utility function $U(x_1, x_2) = (x_1 + 2)(x_2 + 10)$. To find $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$, we need to find a commodity bundle (x_1, x_2) on her budget line at which her indifference curve is tangent to her budget line. The budget line will be tangent to the indifference curve at (x_1, x_2) if the price ratio equals the marginal rate of substitution. For this utility function, $MU_1(x_1, x_2) = x_2 + 10$ and $MU_2(x_1, x_2) = x_1 + 2$. Therefore the "tangency equation" is $p_1/p_2 = (x_2 + 10)/(x_1 + 2)$. Cross-multiplying the tangency equation, one finds $p_1x_1 + 2p_1 = p_2x_2 + 10p_2$.

The bundle chosen must also satisfy the budget equation, $p_1x_1 + p_2x_2 = m$. This gives us two linear equations in the two unknowns, x_1 and x_2. You can solve these equations yourself, using high school algebra. You will find that the solution for the two "demand functions" is

$$x_1 = \frac{m - 2p_1 + 10p_2}{2p_1}$$
$$x_2 = \frac{m + 2p_1 - 10p_2}{2p_2}.$$

There is one thing left to worry about with the "demand functions" we just found. Notice that these expressions will be positive only if $m - 2p_1 + 10p_2 > 0$ and $m + 2p_1 - 10p_2 > 0$. If either of these expressions is negative, then it doesn't make sense as a demand function. What happens in this

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* For some utility functions, demand for a good may not be affected by all of these variables. For example, with Cobb-Douglas utility, demand for a good depends on the good's own price and on income but not on the other good's price. Still, there is no harm in writing demand for Good 1 as a function of p_1, p_2, and m. It just happens that the derivative of $x_1(p_1, p_2, m)$ with respect to p_2 is zero.

case is that the consumer will choose a "boundary solution" where she consumes only one good. At this point, her indifference curve will not be tangent to her budget line.

When a consumer has kinks in her indifference curves, she may choose a bundle that is located at a kink. In the problems with kinks, you will be able to solve for the demand functions quite easily by looking at diagrams and doing a little algebra. Typically, instead of finding a tangency equation, you will find an equation that tells you "where the kinks are." With this equation and the budget equation, you can then solve for demand.

You might wonder why we pay so much attention to kinky indifference curves, straight line indifference curves, and other "funny cases." Our reason is this. In the funny cases, computations are usually pretty easy. But often you may have to draw a graph and think about what you are doing. That is what we want you to do. Think and fiddle with graphs. Don't just memorize formulas. Formulas you will forget, but the habit of thinking will stick with you.

When you have finished this workout, we hope that you will be able to do the following:

- Find demand functions for consumers with Cobb-Douglas and other similar utility functions.
- Find demand functions for consumers with quasilinear utility functions.
- Find demand functions for consumers with kinked indifference curves and for consumers with straight-line indifference curves.
- Recognize complements and substitutes from looking at a demand curve.
- Recognize normal goods, inferior goods, luxuries, and necessities from looking at information about demand.
- Calculate the equation of an inverse demand curve, given a simple demand equation.

6.1 (0) Charlie is back—still consuming apples and bananas. His utility function is $U(x_A, x_B) = x_A x_B$. We want to find his demand function for apples, $x_A(p_A, p_B, m)$, and his demand function for bananas, $x_B(p_A, p_B, m)$.

(a) When the prices are p_A and p_B and Charlie's income is m, the equation for Charlie's budget line is $p_A x_A + p_B x_B = m$. The slope of Charlie's indifference curve at the bundle (x_A, x_B) is $-MU_1(x_A, x_B)/MU_2(x_A, x_B) = -x_B/x_A$. The slope of Charlie's budget line is $-p_A/p_B$. Charlie's indifference curve will be tangent to his budget line at the point (x_A, x_B) if the following equation is satisfied: $p_A/p_B = x_B/x_A$.

(b) You now have two equations, the budget equation and the tangency equation, that must be satisfied by the bundle demanded. Solve these two equations for x_A and x_B. Charlie's demand function for apples is $x_A(p_A, p_B, m) = \frac{m}{2p_A}$, and his demand function for bananas is $x_B(p_A, p_B, m) = \frac{m}{2p_B}$.

(c) In general, the demand for both commodities will depend on the price of both commodities and on income. But for Charlie's utility function, the demand function for apples depends only on income and the price of apples. Similarly, the demand for bananas depends only on income and the price of bananas. Charlie always spends the same fraction of his income on bananas. What fraction is this? $1/2$.

6.2 (0) Douglas Cornfield's preferences are represented by the utility function $u(x_1, x_2) = x_1^2 x_2^3$. The prices of x_1 and x_2 are p_1 and p_2.

(a) The slope of Cornfield's indifference curve at the point (x_1, x_2) is $-2x_2/3x_1$.

(b) If Cornfield's budget line is tangent to his indifference curve at (x_1, x_2), then $\frac{p_1 x_1}{p_2 x_2} =$ 2/3. (Hint: Look at the equation that equates the slope of his indifference curve with the slope of his budget line.) When he is consuming the best bundle he can afford, what fraction of his income does Douglas spend on x_1? 2/5.

(c) Other members of Doug's family have similar utility functions, but the exponents may be different, or their utilities may be multiplied by a positive constant. If a family member has a utility function $U(x, y) = cx_1^a x_2^b$ where a, b, and c are positive numbers, what fraction of his or her income will that family member spend on x_1? a/(a+b).

6.3 (0) Our thoughts return to Ambrose and his nuts and berries. Ambrose's utility function is $U(x_1, x_2) = 4\sqrt{x_1} + x_2$, where x_1 is his consumption of nuts and x_2 is his consumption of berries.

(a) Let us find his demand function for nuts. The slope of Ambrose's indifference curve at (x_1, x_2) is $-\frac{2}{\sqrt{x_1}}$. Setting this slope equal to the slope of the budget line, you can solve for x_1 without even using the budget equation. The solution is $x_1 = \left(\frac{2p_2}{p_1}\right)^2$.

(b) Let us find his demand for berries. Now we need the budget equation. In Part (a), you solved for the amount of x_1 that he will demand. The budget equation tells us that $p_1x_1 + p_2x_2 = M$. Plug the solution that you found for x_1 into the budget equation and solve for x_2 as a function of income and prices. The answer is $x_2 = \frac{M}{p_2} - 4\frac{p_2}{p_1}$.

(c) When we visited Ambrose in Chapter 5, we looked at a "boundary solution," where Ambrose consumed only nuts and no berries. In that example, $p_1 = 1$, $p_2 = 2$, and $M = 9$. If you plug these numbers into the formulas we found in Parts (a) and (b), you find $x_1 =$ 16 , and $x_2 =$ -3.5 . Since we get a negative solution for x_2, it must be that the budget line $x_1 + 2x_2 = 9$ is not tangent to an indifference curve when $x_2 \geq 0$. The best that Ambrose can do with this budget is to spend all of his income on nuts. Looking at the formulas, we see that at the prices $p_1 = 1$ and $p_2 = 2$, Ambrose will demand a positive amount of both goods if and only if $M >$ 16.

6.4 (0) Donald Fribble is a stamp collector. The only things other than stamps that Fribble consumes are Hostess Twinkies. It turns out that Fribble's preferences are represented by the utility function $u(s,t) = s + \ln t$ where s is the number of stamps he collects and t is the number of Twinkies he consumes. The price of stamps is p_s and the price of Twinkies is p_t. Donald's income is m.

(a) Write an expression that says that the ratio of Fribble's marginal utility for Twinkies to his marginal utility for stamps is equal to the ratio of the price of Twinkies to the price of stamps. $1/t = p_t/p_s$. (Hint: The derivative of $\ln t$ with respect to t is $1/t$, and the derivative of s with respect to s is 1.)

(b) You can use the equation you found in the last part to show that if he buys both goods, Donald's demand function for Twinkies depends only on the price ratio and not on his income. Donald's demand function for Twinkies is $t(p_s, p_t, m) = p_s/p_t$.

(c) Notice that for this special utility function, if Fribble buys both goods, then the total amount of money that he spends on Twinkies has the peculiar property that it depends on only one of the three variables m, p_t, and p_s, namely the variable p_s. (Hint: The amount of money that he spends on Twinkies is $p_t t(p_s, p_t, m)$.)

(d) Since there are only two goods, any money that is not spent on Twinkies must be spent on stamps. Use the budget equation and Donald's demand function for Twinkies to find an expression for the number of stamps he will buy if his income is m, the price of stamps is p_s and the price of Twinkies is p_t. $s = \frac{m}{p_s} - 1$.

(e) The expression you just wrote down is negative if $m < p_s$. Surely it makes no sense for him to be demanding negative amounts of postage stamps. If $m < p_s$, what would Fribble's demand for postage stamps be? $s = 0$ What would his demand for Twinkies be? $t = m/p_t$. (Hint: Recall the discussion of boundary optimum.)

(f) Donald's wife complains that whenever Donald gets an extra dollar, he always spends it all on stamps. Is she right? (Assume that $m > p_s$.)

Yes.

(g) Suppose that the price of Twinkies is \$2 and the price of stamps is \$1. On the graph below, draw Fribble's Engel curve for Twinkies in red ink and his Engel curve for stamps in blue ink. (Hint: First draw the Engel curves for incomes greater than \$1, then draw them for incomes less than \$1.)

6.5 (0) Shirley Sixpack, as you will recall, thinks that two 8-ounce cans of beer are exactly as good as one 16-ounce can of beer. Suppose that these are the only sizes of beer available to her and that she has \$30 to spend on beer. Suppose that an 8-ounce beer costs \$.75 and a 16-ounce beer costs \$1. On the graph below, draw Shirley's budget line in blue ink, and draw some of her indifference curves in red.

(a) At these prices, which size can will she buy, or will she buy some of each? 16-ounce cans.

(b) Suppose that the price of 16-ounce beers remains \$1 and the price of 8-ounce beers falls to \$.55. Will she buy more 8-ounce beers? No.

(c) What if the price of 8-ounce beers falls to \$.40? How many 8-ounce beers will she buy then? 75 cans.

(d) If the price of 16-ounce beers is \$1 each and if Shirley chooses some 8-ounce beers and some 16-ounce beers, what must be the price of 8-ounce beers? \$.50.

(e) Now let us try to describe Shirley's demand function for 16-ounce beers as a function of general prices and income. Let the prices of 8-ounce and 16-ounce beers be p_8 and p_{16}, and let her income be m. If $p_{16} < 2p_8$, then the number of 16-ounce beers she will demand is m/p_{16}. If $p_{16} > 2p_8$, then the number of 16-ounce beers she will demand is 0. If $p_{16} =$ 2 p_8, she will be indifferent between any affordable combinations.

6.6 (0) Miss Muffet always likes to have things "just so." In fact the only way she will consume her curds and whey is in the ratio of 2 units of whey per unit of curds. She has an income of \$20. Whey costs \$.75 per unit. Curds cost \$1 per unit. On the graph below, draw Miss Muffet's budget line, and plot some of her indifference curves. (Hint: Have you noticed something kinky about Miss Muffet?)

(a) How many units of curds will Miss Muffet demand in this situation? 8 units. How many units of whey? 16 units.

(b) Write down Miss Muffet's demand function for whey as a function of the prices of curds and whey and of her income, where p_c is the price of curds, p_w is the price of whey, and m is her income. $D(p_c, p_w, m) = \frac{m}{p_w + p_c/2}$. (Hint: You can solve for her demands by solving two equations in two unknowns. One equation tells you that she consumes twice as much whey as curds. The second equation is her budget equation.)

6.7 (1) Mary's utility function is $U(b, c) = b + 100c - c^2$, where b is the number of silver bells in her garden and c is the number of cockle shells. She has 500 square feet in her garden to allocate between silver bells and cockle shells. Silver bells each take up 1 square foot and cockle shells each take up 4 square feet. She gets both kinds of seeds for free.

(a) To maximize her utility, given the size of her garden, Mary should plant 308 silver bells and 48 cockle shells. (Hint: Write down her "budget constraint" for space. Solve the problem as if it were an ordinary demand problem.)

(b) If she suddenly acquires an extra 100 square feet for her garden, how much should she increase her planting of silver bells? 100 extra silver bells. How much should she increase her planting of cockle shells? Not at all.

(c) If Mary had only 144 square feet in her garden, how many cockle shells would she grow? 36.

(d) If Mary grows both silver bells and cockle shells, then we know that the number of square feet in her garden must be greater than 192.

6.8 (0) Casper consumes cocoa and cheese. He has an income of \$16. Cocoa is sold in an unusual way. There is only one supplier and the more cocoa one buys from him, the higher the price one has to pay per unit. In fact, x units of cocoa will cost Casper a total of x^2 dollars. Cheese is sold in the usual way at a price of \$2 per unit. Casper's budget equation, therefore, is $x^2 + 2y = 16$ where x is his consumption of cocoa and y is his consumption of cheese. Casper's utility function is $U(x, y) = 3x + y$.

(a) On the graph below, draw the boundary of Casper's budget set in blue ink. Use red ink to sketch two or three of his indifference curves.

(b) Write an equation that says that at the point (x, y), the slope of Casper's budget "line" equals the slope of his indifference "curve." $2x/2 = 3/1$. Casper demands 3 units of cocoa and 3.5 units of cheese.

6.9 (0) Perhaps after all of the problems with imaginary people and places, you would like to try a problem based on actual fact. The U.S. government's Bureau of Labor Statistics periodically makes studies of family budgets and uses the results to compile the consumer price index. These budget studies and a wealth of other interesting economic data can be found in the annually published *Handbook of Labor Statistics.* The

tables below report total current consumption expenditures and expenditures on certain major categories of goods for 5 different income groups in the United States in 1961. People within each of these groups all had similar incomes. Group A is the lowest income group and Group E is the highest.

Table 6.1
Expenditures by Category for Various Income Groups in 1961

Income Group	A	B	C	D	E
Food Prepared at Home	465	783	1078	1382	1848
Food Away from Home	68	171	213	384	872
Housing	626	1090	1508	2043	4205
Clothing	119	328	508	830	1745
Transportation	139	519	826	1222	2048
Other	364	745	1039	1554	3490
Total Expenditures	1781	3636	5172	7415	14208

Table 6.2
Percentage Allocation of Family Budget

Income Group	A	B	C	D	E
Food Prepared at Home	26	22	21	19	13
Food Away from Home	3.8	4.7	4.1	5.2	6.1
Housing	35	30	29	28	30
Clothing	6.7	9.0	9.8	11	12
Transportation	7.8	14	16	17	14

(a) Complete Table 6.2.

(b) Which of these goods are normal goods? All of them.

(c) Which of these goods satisfy your textbook's definition of *luxury goods* at most income levels? Food away from home, clothing, transportation.

(d) Which of these goods satisfy your textbook's definition of *necessity goods* at most income levels? Food prepared at home, housing.

(e) On the graph below, use the information from Table 6.1 to draw "Engel curves." (Use total expenditure on current consumption as income for purposes of drawing this curve.) Use red ink to draw the Engel curve for food prepared at home. Use blue ink to draw an Engel curve for food away from home. Use pencil to draw an Engel curve for clothing. How does the shape of an Engel curve for a luxury differ from the shape of an Engel curve for a necessity? The curve for a luxury gets flatter as income rises, the curve for a necessity gets steeper.

6.10 (0) Percy consumes cakes and ale. His demand function for cakes is $q_c = m - 30p_c + 20p_a$, where m is his income, p_a is the price of ale, p_c is the price of cakes, and q_c is his consumption of cakes. Percy's income is \$100, and the price of ale is \$1 per unit.

(a) Is ale a substitute for cakes or a complement? Explain. A substitute. An increase in the price of ale increases demand for cakes.

(b) Write an equation for Percy's demand function for cakes where income and the price of ale are held fixed at \$100 and \$1. $q_c = 120 - 30p_c$.

(c) Write an equation for Percy's inverse demand function for cakes where income is \$100 and the price of ale remains at \$1. $p_c = 4 - q_c/30$.

At what price would Percy buy 30 cakes? \$3. Use blue ink to draw Percy's inverse demand curve for cakes.

(d) Suppose that the price of ale rises to \$2.50 per unit and remains there. Write an equation for Percy's inverse demand for cakes. $p_c = 5 - q_c/30$. Use red ink to draw in Percy's new inverse demand curve for cakes.

6.11 (0) Richard and Mary Stout have fallen on hard times, but remain rational consumers. They are making do on \$80 a week, spending \$40 on food and \$40 on all other goods. Food costs \$1 per unit. On the graph below, use black ink to draw a budget line. Label their consumption bundle with the letter A.

(a) The Stouts suddenly become eligible for food stamps. This means that they can go to the agency and buy coupons that can be exchanged for \$2 worth of food. Each coupon costs the Stouts \$1. However, the maximum number of coupons they can buy per week is 10. On the graph, draw their new budget line with red ink.

(b) If the Stouts have homothetic preferences, how much more food will they buy once they enter the food stamp program? 5 units.

Calculus **6.12 (2)** As you may remember, Nancy Lerner is taking an economics course in which her overall score is the *minimum* of the number of correct answers she gets on two examinations. For the first exam, each correct answer costs Nancy 10 minutes of study time. For the second exam, each correct answer costs her 20 minutes of study time. In the last chapter, you found the best way for her to allocate 1200 minutes between the two exams. Some people in Nancy's class learn faster and some learn slower than Nancy. Some people will choose to study more than she does, and some will choose to study less than she does. In this section, we will find a general solution for a person's choice of study times and exam scores as a function of the time costs of improving one's score.

(a) Suppose that if a student does not study for an examination, he or she gets no correct answers. Every answer that the student gets right on the first examination costs P_1 minutes of studying for the first exam. Every answer that he or she gets right on the second examination costs P_2 minutes of studying for the second exam. Suppose that this student spends a total of M minutes studying for the two exams and allocates the time between the two exams in the most efficient possible way. Will the student have the same number of correct answers on both exams?

Yes. Write a general formula for this student's overall score for the course as a function of the three variables, P_1, P_2, and M: $S = \frac{M}{P_1+P_2}$. If this student wants to get an overall score of S, with the smallest possible total amount of studying, this student must spend $P_1 S$ minutes studying for the first exam and $P_2 S$ studying for the second exam.

(b) Suppose that a student has the utility function

$$U(S, M) = S - \frac{A}{2}M^2,$$

where S is the student's overall score for the course, M is the number of minutes the student spends studying, and A is a variable that reflects how much the student dislikes studying. In Part (a) of this problem, you found that a student who studies for M minutes and allocates this time wisely between the two exams will get an overall score of $S = \frac{M}{P_1+P_2}$. Substitute $\frac{M}{P_1+P_2}$ for S in the utility function and then differentiate with respect to M to find the amount of study time, M, that maximizes the student's utility. $M = \frac{1}{A(P_1+P_2)}$. Your answer will be a function of the variables P_1, P_2, and A. If the student chooses the utility-maximizing amount of study time and allocates it wisely between the two exams, he or she will have an overall score for the course of $S = \frac{1}{A(P_1+P_2)^2}$.

(c) Nancy Lerner has a utility function like the one presented above. She chose the utility-maximizing amount of study time for herself. For Nancy, $P_1 = 10$ and $P_2 = 20$. She spent a total of $M = 1,200$ minutes studying for the two exams. This gives us enough information to solve for the variable A in Nancy's utility function. In fact, for Nancy, $A = \frac{1}{36{,}000}$.

(d) Ed Fungus is a student in Nancy's class. Ed's utility function is just like Nancy's, with the same value of A. But Ed learns more slowly than Nancy. In fact it takes Ed exactly twice as long to learn anything as it takes Nancy, so that for him, $P_1 = 20$ and $P_2 = 40$. Ed also chooses his amount of study time so as to maximize his utility. Find the ratio of the amount of time Ed spends studying to the amount of time Nancy spends studying. 1/2. Will his score for the course be greater than half, equal to half, or less than half of Nancy's? Less than half.

6.13 (1) Here is a puzzle for you. At first glance, it would appear that there is not nearly enough information to answer this question. But when you graph the indifference curve and think about it a little, you will see that there is a neat, easily calculated solution.

Kinko spends all his money on whips and leather jackets. Kinko's utility function is $U(x, y) = \min\{4x, 2x + y\}$, where x is his consumption of whips and y is his consumption of leather jackets. Kinko is consuming 15 whips and 10 leather jackets. The price of whips is \$10. You are to find Kinko's income.

(a) Graph the indifference curve for Kinko that passes through the point $(15, 10)$. What is the slope of this indifference curve at $(15, 10)$? -2.

What must be the price of leather jackets if Kinko chooses this point? \$5. Now, what is Kinko's income? $15 \times 10 + 10 \times 5 = 200$.

Chapter 7

NAME_____

Revealed Preference

Introduction. In the last section, you were given a consumer's preferences and then you solved for his or her demand behavior. In this chapter we turn this process around: you are given information about a consumer's demand behavior and you must deduce something about the consumer's preferences. The main tool is the *weak axiom of revealed preference*. This axiom says the following. If a consumer chooses commodity bundle A when she can afford bundle B, then she will never choose bundle B from any budget in which she can also afford A. The idea behind this axiom is that if you choose A when you could have had B, you must like A better than B. But if you like A better than B, then you will never choose B when you can have A. If somebody chooses A when she can afford B, we say that for her, A is *directly revealed preferred* to B. The weak axiom says that if A is directly revealed preferred to B, then B is not directly revealed preferred to A.

Example: Let us look at an example of how you check whether one bundle is revealed preferred to another. Suppose that a consumer buys the bundle $(x_1^A, x_2^A) = (2, 3)$ at prices $(p_1^A, p_2^A) = (1, 4)$. The cost of bundle (x_1^A, x_2^A) at these prices is $(2 \times 1) + (3 \times 4) = 14$. Bundle $(2, 3)$ is directly revealed preferred to all the other bundles that she can afford at prices $(1, 4)$, when she has an income of 14. For example, the bundle $(5, 2)$ costs only 13 at prices $(1, 4)$, so we can say that for this consumer $(2, 3)$ is directly revealed preferred to $(1, 4)$.

You will also have some problems about price and quantity indexes. A price index is a comparison of average price levels between two different times or two different places. If there is more than one commodity, it is not necessarily the case that all prices changed in the same proportion. Let us suppose that we want to compare the price level in the "current year" with the price level in some "base year." One way to make this comparison is to compare the costs in the two years of some "reference" commodity bundle. Two reasonable choices for the reference bundle come to mind. One possibility is to use the current year's consumption bundle for the reference bundle. The other possibility is to use the bundle consumed in the base year. Typically these will be different bundles. If the base-year bundle is the reference bundle, the resulting price index is called the *Laspeyres price index*. If the current year's consumption bundle is the reference bundle, then the index is called the *Paasche price index*.

Example: Suppose that there are just two goods. In 1980, the prices were $(1, 3)$ and a consumer consumed the bundle $(4, 2)$. In 1990, the prices were $(2, 4)$ and the consumer consumed the bundle $(3, 3)$. The cost of the 1980 bundle at 1980 prices is $(1 \times 4) + (3 \times 2) = 10$. The cost of this same bundle at 1990 prices is $(2 \times 4) + (4 \times 2) = 16$. If 1980 is treated as the base year and 1990 as the current year, the Laspeyres price ratio

is 16/10. To calculate the Paasche price ratio, you find the ratio of the cost of the 1990 bundle at 1990 prices to the cost of the same bundle at 1980 prices. The 1990 bundle costs $(2 \times 3) + (4 \times 3) = 18$ at 1990 prices. The same bundle cost $(1 \times 3) + (3 \times 3) = 12$ at 1980 prices. Therefore the Paasche price index is 18/12. Notice that both price indexes indicate that prices rose, but because the price changes are weighted differently, the two approaches give different price ratios.

Making an index of the "quantity" of stuff consumed in the two periods presents a similar problem. How do you weight changes in the amount of good 1 relative to changes in the amount of good 2? This time we could compare the cost of the two periods' bundles evaluated at some reference prices. Again there are at least two reasonable possibilities, the *Laspeyres quantity index* and the *Paasche quantity index.* The Laspeyres quantity index uses the base-year prices as the reference prices, and the Paasche quantity index uses current prices as reference prices.

Example: In the example above, the Laspeyres quantity index is the ratio of the cost of the 1990 bundle at 1980 prices to the cost of the 1980 bundle at 1980 prices. The cost of the 1990 bundle at 1980 prices is 12 and the cost of the 1980 bundle at 1980 prices is 10, so the Laspeyres quantity index is 12/10. The cost of the 1990 bundle at 1990 prices is 18 and the cost of the 1980 bundle at 1990 prices is 16. Therefore the Paasche quantity index is 18/16.

When you have completed this section, we hope that you will be able to do the following:

- Decide from given data about prices and consumption whether one commodity bundle is preferred to another.
- Given price and consumption data, calculate Paasche and Laspeyres price and quantity indexes.
- Use the weak axiom of revealed preferences to make logical deductions about behavior.
- Use the idea of revealed preference to make comparisons of well-being across time and across countries.

7.1 (0) When prices are $(4, 6)$, Goldie chooses the bundle $(6, 6)$, and when prices are $(6, 3)$, she chooses the bundle $(10, 0)$.

(a) On the graph below, show Goldie's first budget line in red ink and her second budget line in blue ink. Mark her choice from the first budget with the label A, and her choice from the second budget with the label B.

(b) Is Goldie's behavior consistent with the weak axiom of revealed preference? No.

7.2 (0) Freddy Frolic consumes only asparagus and tomatoes, which are highly seasonal crops in Freddy's part of the world. He sells umbrellas for a living, which provides a fluctuating income depending on the weather. But Freddy doesn't mind; he never thinks of tomorrow, so each week he spends as much as he earns. One week, when the prices of asparagus and tomatoes were each \$1 a pound, Freddy consumed 15 pounds of each. Use blue ink to show the budget line in the diagram below. Label Freddy's consumption bundle with the letter A.

(a) What is Freddy's income? \$30.

(b) The next week the price of tomatoes rose to \$2 a pound, but the price of asparagus remained at \$1 a pound. By chance, Freddy's income had changed so that his old consumption bundle of (15,15) was just affordable at the new prices. Use red ink to draw this new budget line on the graph below. Does your new budget line go through the point A? Yes.

What is the slope of this line? $-1/2$.

(c) How much asparagus can he afford now if he spent all of his income on asparagus? 45 pounds.

(d) What is Freddy's income now? \$45.

(e) Use pencil to shade the bundles of goods on Freddy's new red budget line that he definitely will *not* purchase with this budget. Is it possible that he would increase his consumption of tomatoes when his budget changes from the blue line to the red one? No.

7.3 (0) Pierre consumes bread and wine. For Pierre, the price of bread is 4 francs per loaf, and the price of wine is 4 francs per glass. Pierre has an income of 40 francs per day. Pierre consumes 6 glasses of wine and 4 loaves of bread per day.

Bob also consumes bread and wine. For Bob, the price of bread is 1/2 dollar per loaf and the price of wine is 2 dollars per glass. Bob has an income of $15 per day.

(a) If Bob and Pierre have the same tastes, can you tell whether Bob is better off than Pierre or vice versa? Explain. Bob is better off. He can afford Pierre's bundle and still have income left.

(b) Suppose prices and incomes for Pierre and Bob are as above and that Pierre's consumption is as before. Suppose that Bob spends all of his income. Give an example of a consumption bundle of wine and bread such that, if Bob bought this bundle, we would know that Bob's tastes are not the same as Pierre's tastes. 7.5 wine and 0 bread, for example. If they had the same preferences,

this violates WARP, since each can afford but rejects the other's bundle.

7.4 (0) Here is a table of prices and the demands of a consumer named Ronald whose behavior was observed in 5 different price-income situations.

Situation	p_1	p_2	x_1	x_2
A	1	1	5	35
B	1	2	35	10
C	1	1	10	15
D	3	1	5	15
E	1	2	10	10

(a) Sketch each of his budget lines and label the point chosen in each case by the letters A, B, C, D, and E.

(b) Is Ronald's behavior consistent with the Weak Axiom of Revealed Preference? Yes.

(c) Shade lightly in red ink all of the points that you are certain are worse for Ronald than the bundle C.

(d) Suppose that you are told that Ronald has convex and monotonic preferences and that he obeys the strong axiom of revealed preference. Shade lightly in blue ink all of the points that you are certain are *at least as good as* the bundle C.

7.5 (0) Horst and Nigel live in different countries. Possibly they have different preferences, and certainly they face different prices. They each consume only two goods, x and y. Horst has to pay 14 marks per unit of x and 5 marks per unit of y. Horst spends his entire income of 167 marks on 8 units of x and 11 units of y. Good x costs Nigel 9 quid per unit and good y costs him 7 quid per unit. Nigel buys 10 units of x and 9 units of y.

(a) Which prices and income would Horst prefer, Nigel's income and prices or his own, or is there too little information to tell? Explain your answer.

Horst prefers Nigel's budget to his own. With Nigel's budget, he can afford his own bundle with money left over.

(b) Would Nigel prefer to have Horst's income and prices or his own, or is there too little information to tell? There is too little information to tell.

7.6 (0) Here is a table that illustrates some observed prices and choices for three different goods at three different prices in three different situations.

Situation	p_1	p_2	p_3	x_1	x_2	x_3
A	1	2	8	2	1	3
B	4	1	8	3	4	2
C	3	1	2	2	6	2

(a) We will fill in the table below as follows. Where i and j stand for any of the letters A, B, and C in Row i and Column j of the matrix, write the value of the Situation-j bundle at the Situation-i prices. For example, in Row A and Column A, we put the value of the bundle purchased in Situation A at Situation A prices. From the table above, we see that in Situation A, the consumer bought bundle $(2, 1, 3)$ at prices $(1, 2, 8)$. The cost of this bundle A at prices A is therefore $(1\times 2)+(2\times 1)+(8\times 3) = 28$, so we put 28 in Row A, Column A. In Situation B the consumer bought bundle $(3, 4, 2)$. The value of the Situation-B bundle, evaluated at the situation-A prices is $(1 \times 3) + (2 \times 4) + (8 \times 2) = 27$, so put 27 in Row A, Column B. We have filled in some of the boxes, but we leave a few for you to do.

Prices/Quantities	A	B	C
A	28	27	30
B	33	32	30
C	13	17	16

(b) Fill in the entry in Row i and Column j of the table below with a D if the Situation-i bundle is directly revealed preferred to the Situation-j bundle. For example, in Situation A the consumer's expenditure is \$28. We see that at Situation-A prices, he could also afford the Situation-B bundle, which cost 27. Therefore the Situation-A bundle is directly revealed preferred to the Situation-B bundle, so we put a D in Row A, Column B. Now let us consider Row B, Column A. The cost of the Situation-B bundle at Situation-B prices is 32. The cost of the Situation-A bundle at Situation-B prices is 33. So, in Situation B, the consumer could not afford the Situation-A bundle. Therefore Situation B is *not* directly revealed preferred to Situation A. So we leave the entry in Row B, Column A blank. Generally, there is a D in Row i Column j if the number in the ij entry of the table in part (a) is less than or equal to the entry in Row i, Column i. There will be a violation of WARP if for some i and j, there is a D in Row i Column j and also a D in Row j, Column i. Do these observations violate WARP? No.

Situation	A	B	C
A	—	D	I
B	I	—	D
C	D	I	—

(c) Now fill in Row i, Column j with an I if observation i is *indirectly* revealed preferred to j. Do these observations violate the Strong Axiom of Revealed Preference? Yes.

7.7 (0) It is January, and Joe Grad, whom we met in Chapter 5, is shivering in his apartment when the phone rings. It is Mandy Manana, one of the students whose price theory problems he graded last term. Mandy asks if Joe would be interested in spending the month of February in her apartment. Mandy, who has switched majors from economics to political science, plans to go to Aspen for the month and so her apartment will be empty (alas). All Mandy asks is that Joe pay the monthly service charge of \$40 charged by her landlord and the heating bill for the month of February. Since her apartment is much better insulated than Joe's, it only costs \$1 per month to raise the temperature by 1 degree. Joe

thanks her and says he will let her know tomorrow. Joe puts his earmuffs back on and muses. If he accepts Mandy's offer, he will still have to pay rent on his current apartment but he won't have to heat it. If he moved, heating would be cheaper, but he would have the $40 service charge. The outdoor temperature averages 20 degrees Fahrenheit in February, and it costs him $2 per month to raise his apartment temperature by 1 degree. Joe is still grading homework and has $100 a month left to spend on food and utilities after he has paid the rent on his apartment. The price of food is still $1 per unit.

(a) Draw Joe's budget line for February if he moves to Mandy's apartment and on the same graph, draw his budget line if he doesn't move.

(b) After drawing these lines himself, Joe decides that he would be better off not moving. From this, we can tell, using the principle of revealed preference that Joe must plan to keep his apartment at a temperature of less than 60 degrees.

(c) Joe calls Mandy and tells her his decision. Mandy offers to pay half the service charge. Draw Joe's budget line if he accepts Mandy's new offer. Joe now accepts Mandy's offer. From the fact that Joe accepted this offer we can tell that he plans to keep the temperature in Mandy's apartment above 40 degrees.

7.8 (0) Lord Peter Pommy is a distinguished criminologist, schooled in the latest techniques of forensic revealed preference. Lord Peter is investigating the disappearance of Sir Cedric Pinchbottom who abandoned his aging mother on a street corner in Liverpool and has not been seen

since. Lord Peter has learned that Sir Cedric left England and is living under an assumed name somewhere in the Empire. There are three suspects, R. Preston McAfee of Brass Monkey, Ontario, Canada, Richard Manning of North Shag, New Zealand, and Richard Stevenson of Gooey Shoes, Falkland Islands. Lord Peter has obtained Sir Cedric's diary, which recorded his consumption habits in minute detail. By careful observation, he has also discovered the consumption behavior of McAfee, Manning, and Stevenson. All three of these gentlemen, like Sir Cedric, spend their entire incomes on beer and sausage. Their dossiers reveal the following:

- **Sir Cedric Pinchbottom** — In the year before his departure, Sir Cedric consumed 10 kilograms of sausage and 20 liters of beer per week. At that time, beer cost 1 English pound per liter and sausage cost 1 English pound per kilogram.

- **R. Preston McAfee** — McAfee is known to consume 5 liters of beer and 20 kilograms of sausage. In Brass Monkey, Ontario beer costs 1 Canadian dollar per liter and sausage costs 2 Canadian dollars per kilogram.

- **Richard Manning** — Manning consumes 5 kilograms of sausage and 10 liters of beer per week. In North Shag, a liter of beer costs 2 New Zealand dollars and sausage costs 2 New Zealand dollars per kilogram.

- **Richard Stevenson** — Stevenson consumes 5 kilograms of sausage and 30 liters of beer per week. In Gooey Shoes, a liter of beer costs 10 Falkland Island pounds and sausage costs 20 Falkland Island pounds per kilogram.

(a) Draw the budget line for each of the three fugitives, using a different color of ink for each one. Label the consumption bundle that each chooses. On this graph, superimpose Sir Cedric's budget line and the bundle he chose.

(b) After pondering the dossiers for a few moments, Lord Peter announced. "Unless Sir Cedric has changed his tastes, I can eliminate one of the suspects. Revealed preference tells me that one of the suspects is innocent." Which one? McAfee.

(c) After thinking a bit longer, Lord Peter announced. "If Sir Cedric left voluntarily, then he would have to be better off than he was before. Therefore if Sir Cedric left voluntarily and if he has not changed his tastes, he must be living in Falklands.

7.9 (1) The McCawber family is having a tough time making ends meet. They spend $100 a week on food and $50 on other things. A new welfare program has been introduced that gives them a choice between receiving a grant of $50 per week that they can spend any way they want, and buying any number of $2 food coupons for $1 apiece. (They naturally are not allowed to resell these coupons.) Food is a normal good for the McCawbers. As a family friend, you have been asked to help them decide on which option to choose. Drawing on your growing fund of economic knowledge, you proceed as follows.

(a) On the graph below, draw their old budget line in red ink and label their current choice C. Now use black ink to draw the budget line that they would have with the grant. If they chose the coupon option, how much food could they buy if they spent all their money on food coupons?

$300. How much could they spend on other things if they bought

no food? $150. Use blue ink to draw their budget line if they choose the coupon option.

(b) Using the fact that food is a normal good for the McCawbers, and knowing what they purchased before, darken the portion of the black budget line where their consumption bundle could possibly be if they chose the lump-sum grant option. Label the ends of this line segment A and B.

(c) After studying the graph you have drawn, you report to the McCawbers. "I have enough information to be able to tell you which choice to make. You should choose the coupon because you can get more food even when other expenditure is constant.

(d) Mr. McCawber thanks you for your help and then asks, "Would you have been able to tell me what to do if you hadn't known whether food was a normal good for us?" On the axes below, draw the same budget lines you drew on the diagram above, but draw indifference curves for which food is not a normal good and for which the McCawbers would be better off with the program you advised them not to take.

7.10 (0) In 1933, the Swedish economist Gunnar Myrdal (who later won a Nobel prize in economics) and a group of his associates at Stockholm University collected a fantastically detailed historical series of prices and price indexes in Sweden from 1830 until 1930. This was published in a book called *The Cost of Living in Sweden*. In this book you can find 100 years of prices for goods such as oat groats, hard rye bread, salted codfish, beef, reindeer meat, birchwood, tallow candles, eggs, sugar, and coffee. There are also estimates of the quantities of each good consumed by an average working-class family in 1850 and again in 1890.

The table below gives prices in 1830, 1850, 1890, and 1913, for flour, meat, milk, and potatoes. In this time period, these four staple foods accounted for about 2/3 of the Swedish food budget.

Prices of Staple Foods in Sweden

Prices are in Swedish kronor per kilogram, except for milk, which is in Swedish kronor per liter.

	1830	1850	1890	1913
Grain Flour	.14	.14	.16	.19
Meat	.28	.34	.66	.85
Milk	.07	.08	.10	.13
Potatoes	.032	.044	.051	.064

Based on the tables published in Myrdal's book, typical consumption bundles for a working-class Swedish family in 1850 and 1890 are listed below. (The reader should be warned that we have made some

approximations and simplifications to draw these simple tables from the much more detailed information in the original study.)

Quantities Consumed by a Typical Swedish Family

Quantities are measured in kilograms per year, except for milk, which is measured in liters per year.

	1850	1890
Grain Flour	165	220
Meat	22	42
Milk	120	180
Potatoes	200	200

(a) Complete the table below, which reports the annual cost of the 1850 and 1890 bundles of staple foods at various years' prices.

Cost of 1850 and 1890 Bundles at Various Years' Prices

Cost	1850 bundle	1890 bundle
Cost at 1830 Prices	44.1	61.6
Cost at 1850 Prices	49.0	68.3
Cost at 1890 Prices	63.1	91.1
Cost at 1913 Prices	78.5	113.7

(b) Is the 1890 bundle revealed preferred to the 1850 bundle? Yes.

(c) The Laspeyres quantity index for 1890 with base year 1850 is the ratio of the value of the 1890 bundle at 1850 prices to the value of the 1850 bundle at 1850 prices. Calculate the Laspeyres quantity index of staple food consumption for 1890 with base year 1850. 1.39.

(d) The Paasche quantity index for 1890 with base year 1850 is the ratio of the value of the 1890 bundle at 1890 prices to the value of the 1850 bundle at 1890 prices. Calculate the Paasche quantity index for 1890 with base year 1850. 1.44.

(e) The Laspeyres price index for 1890 with base year 1850 is calculated using 1850 quantities for weights. Calculate the Laspeyres price index for 1890 with base year 1850 for this group of four staple foods. 1.29.

(f) If a Swede were rich enough in 1850 to afford the 1890 bundle of staple foods in 1850, he would have to spend 1.39 times as much on these foods as does the typical Swedish worker of 1850.

(g) If a Swede in 1890 decided to purchase the same bundle of food staples that was consumed by typical 1850 workers, he would spend the fraction .69 of the amount that the typical Swedish worker of 1890 spends on these goods.

7.11 (0) This question draws from the tables in the previous question. Let us try to get an idea of what it would cost an American family at today's prices to purchase the bundle consumed by an average Swedish family in 1850. In the United States today, the price of flour is about $.40 per kilogram, the price of meat is about $3.75 per kilogram, the price of milk is about $.50 per liter, and the price of potatoes is about $1 per kilogram. We can also compute a Laspeyres price index across time and across countries and use it to estimate the value of a current US dollar relative to the value of an 1850 Swedish kronor.

(a) How much would it cost an American at today's prices to buy the bundle of staple food commodities purchased by an average Swedish working-class family in 1850? $408.

(b) Myrdal estimates that in 1850, about 2/3 of the average family's budget was spent on food. In turn, the four staples discussed in the last question constitute about 2/3 of the average family's food budget. If the prices of other goods relative to the price of the food staples are similar in the United States today to what they were in Sweden in 1850, about how much would it cost an American at current prices to consume the same overall consumption bundle consumed by a Swedish working-class family in 1850? $919.

(c) Using the Swedish consumption bundle of staple foods in 1850 as weights, calculate a Laspeyres price index to compare prices in current American dollars relative to prices in 1850 Swedish kronor. 8.35. If we use this to estimate the value of current dollars relative to 1850 Swedish kronor, we would say that a U.S. dollar today is worth about .12 1850 Swedish kronor.

7.12 (0) Suppose that between 1960 and 1985, the price of all goods exactly doubled while every consumer's income tripled.

(a) Would the Laspeyres price index for 1985, with base year 1960 be less than 2, greater than 2, or exactly equal to 2? Exactly 2. What about the Paasche price index? Exactly 2.

(b) If bananas are a normal good, will total banana consumption increase? Yes. If everybody has homothetic preferences, can you determine by what percentage total banana consumption must have increased? Explain. Yes, by 50%. Everybody's budget line shifted out by 50%. With homothetic preferences, the consumption of each good increases in the same proportion.

7.13 (1) Norm and Sheila consume only meat pies and beer. Meat pies used to cost \$2 each and beer was \$1 per can. Their gross income used to be \$60 per week, but they had to pay an income tax of \$10. Use red ink to sketch their old budget line for meat pies and beer.

(a) They used to buy 30 cans of beer per week and spent the rest of their income on meat pies. How many meat pies did they buy? 10.

(b) The government decided to eliminate the income tax and to put a sales tax of $1 per can on beer, raising its price to $2 per can. Assuming that Norm and Sheila's pre-tax income and the price of meat pies did not change, draw their new budget line in blue ink.

(c) The sales tax on beer induced Norm and Sheila to reduce their beer consumption to 20 cans per week. What happened to their consumption of meat pies? Stayed the same--10. How much revenue did this tax raise from Norm and Sheila? $20.

(d) This part of the problem will require some careful thinking. Suppose that instead of just taxing beer, the government decided to tax *both* beer and meat pies at the *same* percentage rate, and suppose that the price of beer and the price of meat pies each went up by the full amount of the tax. The new tax rate for both goods was set high enough to raise exactly the same amount of money from Norm and Sheila as the tax on beer used to raise. This new tax collects $ .50 for every bottle of beer sold and $ 1 for every meat pie sold. (Hint: If both goods are taxed at the same rate, the effect is the same as an income tax.) How large an income tax would it take to raise the same revenue as the $1 tax on beer? $20. Now you can figure out how big a tax on each good is equivalent to an income tax of the amount you just found.

(e) Use black ink to draw the budget line for Norm and Sheila that corresponds to the tax in the last section. Are Norm and Sheila better off having just beer taxed or having both beer and meat pies taxed if both sets of taxes raise the same revenue? Both. (Hint: Try to use the principle of revealed preference.)

Chapter 8

NAME_____

Slutsky Equation

Introduction. It is useful to think of a price change as having two distinct effects, a substitution effect and an income effect. The **substitution effect** of a price change is the change that would have happened *if* income changed at the same time in such a way that the consumer could exactly afford her old consumption bundle. The rest of the change in the consumer's demand is called the **income effect**. Why do we bother with breaking a real change into the sum of two hypothetical changes? Because we know things about the pieces that we wouldn't know about the whole without taking it apart. In particular, we know that the substitution effect of increasing the price of a good *must* reduce the demand for it. We also know that the income effect of an increase in the price of a good is equivalent to the effect of a *loss* of income. Therefore if the good whose price has risen is a normal good, then both the income and substitution effect operate to reduce demand. But if the good is an inferior good, income and substitution effects act in opposite directions.

Example: A consumer has the utility function $U(x_1, x_2) = x_1 x_2$ and an income of \$24. Initially the price of good 1 was \$1 and the price of good 2 was \$2. Then the price of good 2 rose to \$3 and the price of good 1 stayed at \$1. Using the methods you learned in Chapters 5 and 6, you will find that this consumer's demand function for good 1 is $D_1(p_1, p_2, m) = m/2p_1$ and her demand function for good 2 is $D_2(p_1, p_2, m) = m/2p_2$. Therefore initially she will demand 12 units of good 1 and 6 units of good 2. If, when the price of good 2 rose to \$3, her income had changed enough so that she could exactly afford her old bundle, her new income would have to be $(1 \times 12) + (3 \times 6) = \$30$. At an income of \$30, at the new prices, she would demand $D_2(1, 3, 30) = 5$ units of good 2. Before the change she bought 6 units of 2, so the substitution effect of the price change on her demand for good 2 is $5 - 6 = -1$ units. Our consumer's income didn't *really* change. Her income stayed at \$24. Her actual demand for good 2 after the price change was $D_2(1, 3, 24) = 4$. The difference between what she actually demanded after the price change and what she would have demanded if her income had changed to let her just afford the old bundle is the income effect. In this case the income effect is $4 - 5 = -1$ units of good 2. Notice that in this example, both the income effect and the substitution effect of the price increase worked to reduce the demand for good 2.

When you have completed this workout, we hope that you will be able to do the following:

- Find Slutsky income effect and substitution effect of a specific price change if you know the demand function for a good.
- Show the Slutsky income and substitution effects of a price change

on an indifference curve diagram.

- Show the Hicks income and substitution effects of a price change on an indifference curve diagram.
- Find the Slutsky income and substitution effects for special utility functions such as perfect substitutes, perfect complements, and Cobb-Douglas.
- Use an indifference-curve diagram to show how the case of a Giffen good might arise.
- Show that the substitution effect of a price increase unambiguously decreases demand for the good whose price rose.
- Apply income and substitution effects to draw some inferences about behavior.

8.1 (0) Gentle Charlie, vegetarian that he is, continues to consume apples and bananas. His utility function is $U(x_A, x_B) = x_A x_B$. The price of apples is \$1, the price of bananas is \$2, and Charlie's income is \$40 a day. The price of bananas suddenly falls to \$1.

(a) Before the price change, Charlie consumed 20 apples and 10 bananas per day. On the graph below, use black ink to draw Charlie's original budget line and put the label A on his chosen consumption bundle.

(b) If, after the price change, Charlie's income had changed so that he could exactly afford his old consumption bundle, his new income would have been 30. With this income and the new prices, Charlie would consume 15 apples and 15 bananas. Use red ink to draw the budget line corresponding to this income and these prices. Label the bundle that Charlie would choose at this income and the new prices with the letter B.

(c) Does the substitution effect of the fall in the price of bananas make him buy more bananas or fewer bananas? More bananas. How many more or fewer? 5 more.

(d) After the price change, Charlie actually buys 20 apples and 20 bananas. Use blue ink to draw Charlie's actual budget line after the price change. Put the label C on the bundle that he actually chooses after the price change. Draw 3 horizontal lines on your graph, one from A to the vertical axis, one from B to the vertical axis, and one from C to the vertical axis. Along the vertical axis, label the income effect, the substitution effect, and the total effect on the demand for bananas. Is the

blue line parallel to the red line or the black line that you drew before?

Red line.

(e) The income effect of the fall in the price of bananas on Charlie's demand for bananas is the same as the effect of an (increase, decrease) increase in his income of \$ 10 per day. Does the income effect make him consume more bananas or fewer? More. How many more or how many fewer? 5 more.

(f) Does the substitution effect of the fall in the price of bananas make Charlie consume more *apples* or fewer? Fewer. How many more or fewer? 5 fewer. Does the income effect of the fall in the price of bananas make Charlie consume more apples or fewer? More. What is the total effect of the change in the price of bananas on the demand for apples? Zero.

8.2 (0) Neville's passion is fine wine. When the prices of all other goods are fixed at current levels, Neville's demand function for high-quality claret is $q = .02m - 2p$, where m is his income, p is the price of claret (in British pounds), and q is the number of bottles of claret that he demands. Neville's income is 7,500 pounds, and the price of a bottle of suitable claret is 30 pounds.

(a) How many bottles of claret will Neville buy? 90.

(b) If the price of claret rose to 40 pounds, how much income would Neville have to have in order to be exactly able to afford the amount of claret and the amount of other goods that he bought before the price change?

8,400 pounds. At this income, and a price of 40 pounds, how many bottles would Neville buy? 88 bottles.

(c) At his original income of 7,500 and a price of 40, how much claret would Neville demand? 70 bottles.

(d) When the price of claret rose from 30 to 40, the number of bottles that Neville demanded decreased by 20. The substitution effect (increased, reduced) reduced his demand by 2 bottles and the income effect (increased, reduced) reduced his demand by 18 bottles.

8.3 (0) *Note: Do this problem only if you have read the section entitled "Another Substitution Effect" that describes the "Hicks substitution effect".* Consider the figure below, which shows the budget constraint and the indifference curves of good King Zog. Zog is in equilibrium with an income of \$300, facing prices $p_X = \$4$ and $p_Y = \$10$.

(a) How much X does Zog consume? 30.

(b) If the price of X falls to \$2.50, while income and the price of Y stay constant, how much X will Zog consume? 35.

(c) How much income must be taken away from Zog to isolate the Hicksian income and substitution effects (i.e., to make him just able to afford to reach his old indifference curve at the new prices)? \$75.

(d) The total effect of the price change is to change consumption from the point E to the point C.

(e) The income effect corresponds to the movement from the point F to the point C while the substitution effect corresponds to the movement from the point E to the point F.

(f) Is X a normal good or an inferior good? An inferior good.

(g) On the axes below, sketch an Engel curve and a demand curve for Good X that would be reasonable given the information in the graph above. Be sure to label the axes on both your graphs.

8.4 (0) Maude spends *all* of her income on delphiniums and hollyhocks. She thinks that delphiniums and hollyhocks are perfect substitutes; one delphinium is just as good as one hollyhock. Delphiniums cost \$4 a unit and hollyhocks cost \$5 a unit.

(a) If the price of delphiniums decreases to \$3 a unit, will Maude buy more of them? Yes. What part of the change in consumption is due to the income effect and what part is due to the substitution effect?

All due to income effect.

(b) If the prices of delphiniums and hollyhocks are respectively $p_d = \$4$ and $p_h = \$5$ and if Maude has \$120 to spend, draw her budget line in blue ink. Draw the highest indifference curve that she can attain in red ink, and label the point that she chooses as A.

(c) Now let the price of hollyhocks fall to \$3 a unit, while the price of delphiniums does not change. Draw her new budget line in black ink. Draw the highest indifference curve that she can now reach with red ink. Label the point she chooses now as B.

(d) How much would Maude's income have to be after the price of hollyhocks fell, so that she could just exactly afford her old commodity bundle A? \$120.

(e) When the price of hollyhocks fell to \$3, what part of the change in Maude's demand was due to the income effect and what part was due to the substitution effect? All substitution effect.

8.5 (1) Suppose that two goods are perfect complements. If the price of one good changes, what part of the change in demand is due to the substitution effect, and what part is due to the income effect? All income effect.

8.6 (0) Douglas Cornfield's demand function for good x is $x(p_x, p_y, m) = 2m/5p_x$. His income is \$1,000, the price of x is \$5, and the price of y is \$20. If the price of x falls to \$4, then his demand for x will change from

80 to 100.

(a) If his income were to change at the same time so that he could exactly afford his old commodity bundle at $p_x = 4$ and $p_y = 20$, what would his new income be? 920. What would be his demand for x at this new level of income, at prices $p_x = 4$ and $p_y = 20$? 92.

(b) The substitution effect is a change in demand from 80 to 92. The income effect of the price change is a change in demand from 92 to 100.

(c) On the axes below, use blue ink to draw Douglas Cornfield's budget line before the price change. Locate the bundle he chooses at these prices on your graph and label this point A. Use black ink to draw Douglas Cornfield's budget line after the price change. Label his consumption bundle after the change by B.

(d) On the graph above, use black ink to draw a budget line with the new prices but with an income that just allows Douglas to buy his old bundle, A. Find the bundle that he would choose with this budget line and label this bundle C.

8.7 (1) Mr. Consumer allows himself to spend \$100 per month on cigarettes and ice cream. Mr. C's preferences for cigarettes and ice cream are unaffected by the season of the year.

(a) In January, the price of cigarettes was \$1 per pack, while ice cream cost \$2 per pint. Faced with these prices, Mr. C bought 30 pints of ice cream and 40 packs of cigarettes. Draw Mr. C's January budget line with blue ink and label his January consumption bundle with the letter J.

(b) In February, Mr. C again had \$100 to spend and ice cream still cost \$2 per pint, but the price of cigarettes rose to \$1.25 per pack. Mr. C consumed 30 pints of ice cream and 32 packs of cigarettes. Draw Mr. C's February budget line with red ink and mark his February bundle with the letter F. The substitution effect of this price change would make him buy (less, more, the same amount of) `less` cigarettes and (less, more, the same amount of) `more` ice cream. Since this is true and the total change in his ice cream consumption was zero, it must be that the income effect of this price change on his consumption of ice cream makes him buy (more, less, the same amount of) `less` ice cream. The income effect of this price change is like the effect of an (increase, decrease)

decrease in his income. Therefore the information we have suggests that ice cream is a(n) (normal, inferior, neutral) normal good.

(c) In March, Mr. C again had $100 to spend. Ice cream was on sale for $1 per pint. Cigarette prices, meanwhile, increased to $1.50 per pack. Draw his March budget line with black ink. Is he better off than in January, worse off, or can you not make such a comparison? Better off. How does your answer to the last question change if the price of cigarettes had increased to $2 per pack? Now you can't tell.

8.8 (1) This problem continues with the adventures of Mr. Consumer from the previous problem.

(a) In April, cigarette prices rose to $2 per pack and ice cream was still on sale for $1 per pint. Mr. Consumer bought 34 packs of cigarettes and 32 pints of ice cream. Draw his April budget line with pencil and label his April bundle with the letter A. Was he better off or worse off than in January? Worse off. Was he better off or worse off than in February, or can't one tell? Better off.

(b) In May, cigarettes stayed at $2 per pack and as the sale on ice cream ended, the price returned to $2 per pint. On the way to the store, however, Mr. C found $30 lying in the street. He then had $130 to spend on cigarettes and ice cream. Draw his May budget with a dashed line. Without knowing what he purchased, one can determine whether he is better off than he was in at least one previous month. Which month or months?

He is better off in May than in February.

(c) In fact, Mr. C buys 40 packs of cigarettes and 25 pints of ice cream in May. Does he satisfy WARP? No.

8.9 (2) In the last chapter, we studied a problem involving food prices and consumption in Sweden in 1850 and 1890.

(a) Potato consumption was the same in both years. Real income must have gone up between 1850 and 1890, since the amount of food staples purchased, as measured by either the Laspeyres or the Paasche quantity index, rose. The price of potatoes rose less rapidly than the price of either meat or milk, and at about the same rate as the price of grain flour. So real income went up and the price of potatoes went down relative to other goods. From this information, determine whether potatoes were

most likely a normal or an inferior good. Explain your answer.

If potatoes were a normal good, both the fall in potato price and the rise in income would increase the demand for potatoes. But potato consumption did not increase. So potatoes must be an inferior good.

(b) Can one also tell from these data whether it is likely that potatoes were a Giffen good? If potatoes were a Giffen good, then the fall in the price of potatoes would decrease demand and the rise in income would also decrease demand for potatoes. But potato demand stayed constant. So potatoes were probably not a Giffen good.

8.10 (1) Agatha must travel on the Orient Express from Istanbul to Paris. The distance is 1,500 miles. A traveler can choose to make any fraction of the journey in a first-class carriage and travel the rest of the way in a second-class carriage. The price is 10 cents a mile for a second-class carriage and 20 cents a mile for a first-class carriage. Agatha much prefers first-class to second-class travel, but because of a misadventure in an Istanbul bazaar, she has only $200 left with which to buy her tickets. Luckily, she still has her toothbrush and a suitcase full of cucumber sandwiches to eat on the way. Agatha plans to spend her entire $200 on her tickets for her trip. She will travel first class as much as she can afford to, but she must get all the way to Paris, and $200 is not enough money to get her all the way to Paris in first class.

(a) On the graph below, use red ink to show the locus of combinations of first- and second-class tickets that Agatha can just afford to purchase with her $200. Use blue ink to show the locus of combinations of first- and second-class tickets that are sufficient to carry her the entire distance from Istanbul to Paris. Locate the combination of first- and second-class miles that Agatha will choose on your graph and label it A.

(b) Let m_1 be the number of miles she travels by first-class coach and m_2 be the number of miles she travels by second-class coach. Write down two equations that you can solve to find the number of miles she chooses to travel by first-class coach and the number of miles she chooses to travel by second-class coach. $.2m_1 + .1m_2 = 200$, $m_1 + m_2 = 1,500$.

(c) The number of miles that she travels by second-class coach is 1,000.

(d) Just before she was ready to buy her tickets, the price of second-class tickets fell to \$.05 while the price of first-class tickets remained at \$.20. On the graph that you drew above, use pencil to show the combinations of first-class and second-class tickets that she can afford with her \$200 at these prices. On your graph, locate the combination of first-class and second-class tickets that she would now choose. (Remember, she is going to travel as much first-class as she can afford to and still make the 1,500 mile trip on \$200.) Label this point B. How many miles does she travel by second class now? 666.66. (Hint: For an exact solution you will have to solve two linear equations in two unknowns.) Is second-class travel a normal good for Agatha? No. Is it a Giffen good for her? Yes.

8.11 (0) We continue with the adventures of Agatha, from the previous problem. Just after the price change from \$.10 per mile to \$.05 per mile for second-class travel, and just before she had bought any tickets, Agatha misplaced her handbag. Although she kept most of her money in her sock, the money she lost was just enough so that at the new prices, she could exactly afford the combination of first- and second-class tickets that she would have purchased at the old prices. How much money did she lose?

`$50.` On the graph you started in the previous problem, use black ink to draw the locus of combinations of first- and second-class tickets that she can just afford after discovering her loss. Label the point that she chooses with a C. How many miles will she travel by second class now?

`1,000.`

(a) Finally, poor Agatha finds her handbag again. How many miles will she travel by second class now (assuming she didn't buy any tickets before she found her lost handbag)? `666.66.` When the price of second-class tickets fell from \$.10 to \$.05, how much of a change in Agatha's demand for second-class tickets was due to a substitution effect? `None.` How much of a change was due to an income effect? -333.33.

Chapter 9

NAME_____

Buying and Selling

Introduction. In previous chapters, we studied the behavior of consumers who start out without owning any goods, but who had some money with which to buy goods. In this chapter, the consumer has an *initial endowment*, which is the bundle of goods the consumer owns before any trades are made. A consumer can trade away from his initial endowment by selling one good and buying the other.

The techniques that you have already learned will serve you well here. To find out how much a consumer demands at given prices, you find his budget line and then find a point of tangency between his budget line and an indifference curve. To determine a budget line for a consumer who is trading from an initial endowment and who has no source of income other than his initial endowment, notice two things. First, *the initial endowment must lie on the consumer's budget line.* This is true because, no matter what the prices are, the consumer can always afford his initial endowment. Second, *if the prices are p_1 and p_2, the slope of the budget line must be $-p_1/p_2$.* This is true, since for every unit of good 1 the consumer gives up, he can get exactly p_1/p_2 units of good 2. Therefore if you know the prices and you know the consumer's initial endowment, then you can always write an equation for the consumer's budget line. After all, if you know one point on a line and you know its slope, you can either draw the line or write down its equation. Once you have the budget equation, you can find the bundle the consumer chooses, using the same methods you learned in Chapter 5.

Example: A peasant consumes only rice and fish. He grows some rice and some fish, but not necessarily in the same proportion in which he wants to consume them. Suppose that if he makes no trades, he will have 20 units of rice and 5 units of fish. The price of rice is 1 yuan per unit, and the price of fish is 2 yuan per unit. The value of the peasant's endowment is $(1 \times 20) + (2 \times 5) = 30$. Therefore the peasant can consume any bundle (R, F) such that $(1 \times R) + (2 \times F) = 30$.

Perhaps the most interesting application of trading from an initial endowment is the theory of labor supply. To study labor supply, we consider the behavior of a consumer who is choosing between leisure and other goods. The only thing that is at all new or "tricky" is finding the appropriate budget constraint for the problem at hand. To study labor supply, we think of the consumer as having an initial endowment of leisure, some of which he may trade away for goods.

In most applications we set the price of "other goods" at 1. The wage rate is the price of leisure. The role that is played by income in the ordinary consumer-good model is now played by "full income." A worker's full income is the income she would have if she chose to take no leisure.

Example: Sherwin has 18 hours a day which he divides between labor and leisure. He can work as many hours a day as he wishes for a wage of \$5 per hour. He also receives a pension that gives him \$10 a day whether he works or not. The price of other goods is \$1 per unit. If Sherwin makes no trades at all, he will have 18 hours of leisure and 10 units of other goods. Therefore Sherwin's initial endowment is 18 hours of leisure a day and \$10 a day for other goods. Let R be the amount of leisure that he has per day, and let C be the number of dollars he has to spend per day on other goods. If his wage is \$5 an hour, he can afford to consume bundle (R, C) if it costs no more per day than the value of his initial endowment. The value of his initial endowment (his full income) is $\$10 + (\$5 \times 18) = \$100$ per day. Therefore Sherwin's budget equation is $5R + C = 100$.

9.1 (0) Abishag Appleby owns 20 quinces and 5 kumquats. She has no income from any other source, but she can buy or sell either quinces or kumquats at their market prices. The price of kumquats is four times the price of quinces. There are no other commodities of interest.

(a) How many quinces could she have if she was willing to do without kumquats? 40. How many kumquats could she have if she was willing to do without quinces? 10.

(b) Draw Abishag's budget set, using blue ink, and label the endowment bundle with the letter E. If the price of quinces is 1 and the price of kumquats is 4, write Abishag's budget equation. $Q + 4K = 40$. If the price of quinces is 2 and the price of kumquats is 8, write Abishag's budget equation. $2Q + 8K = 80$. What effect does doubling both

prices have on the set of commodity bundles that Abishag can afford?

No effect.

(c) Suppose that Abishag decides to sell 10 quinces. Label her final consumption bundle in your graph with the letter C.

(d) Now, after she has sold 10 quinces and owns the bundle labeled C, suppose that the price of kumquats falls so that kumquats cost the same as quinces. On the diagram above, draw Abishag's new budget line, using red ink.

(e) If Abishag obeys the weak axiom of revealed preference, then there are some points on her red budget line that we can be sure Abishag will not choose. On the graph, make a squiggly line over the portion of Abishag's red budget line that we can be sure she will not choose.

9.2 (0) Mario has a small garden where he raises eggplant and tomatoes. He consumes some of these vegetables, and he sells some in the market. Eggplants and tomatoes are perfect complements for Mario, since the only recipes he knows use them together in a 1:1 ratio. One week his garden yielded 30 pounds of eggplant and 10 pounds of tomatoes. At that time the price of each vegetable was \$5 per pound.

(a) What is the monetary value of Mario's endowment of vegetables?

\$200.

(b) On the graph below, use blue ink to draw Mario's budget line. Mario ends up consuming 20 pounds of tomatoes and 20 pounds of eggplant. Draw the indifference curve through the consumption bundle that Mario chooses and label this bundle A.

(c) Suppose that before Mario makes any trades, the price of tomatoes rises to \$15 a pound, while the price of eggplant stays at \$5 a pound. What is the value of Mario's endowment now? \$300. Draw his new budget line, using red ink. He will now choose a consumption bundle consisting of 15 tomatoes and 15 eggplants.

(d) Suppose that Mario had sold his entire crop at the market for a total of \$200, intending to buy back some tomatoes and eggplant for his own consumption. Before he had a chance to buy anything back, the price of tomatoes rose to \$15, while the price of eggplant stayed at \$5. Draw his budget line, using pencil or black ink. Mario will now consume 10 pounds of tomatoes and 10 pounds of eggplant.

(e) Assuming that the price of tomatoes rose to \$15 from \$5 before Mario made any transactions, the change in the demand for tomatoes due to the substitution effect was 0. The change in the demand for tomatoes due to the ordinary income effect was -10. The change in the demand for tomatoes due to the endowment income effect was $+5$. The total change in the demand for tomatoes was -5.

9.3 (0) Lucetta consumes only two goods, A and B. Her only source of income is gifts of these commodities from her many admirers. She doesn't always get these goods in the proportions in which she wants to consume them, but she can always buy or sell A at the price $p_A = 1$ and B at the price $p_B = 2$. Lucetta's utility function is $U(a, b) = ab$, where a is the amount of A she consumes and b is the amount of B she consumes.

(a) Suppose that Lucetta's admirers give her 100 units of A and 200 units of B. In the graph below, use red ink to draw her budget line. Label her initial endowment E.

(b) What are Lucetta's gross demands for A? 250 units. And for B? 125 units.

(c) What are Lucetta's net demands? 150 of A and -75 of B.

(d) Suppose that before Lucetta has made any trades, the price of good B falls to 1, and the price of good A stays at 1. Draw Lucetta's budget line at these prices on your graph, using blue ink.

(e) Does Lucetta's consumption of good B rise or fall? It rises. By how much? 25 units. What happens to Lucetta's consumption of good A? It decreases by 100 units.

(f) Suppose that before the price of good B fell, Lucetta had exchanged all of her gifts for money, planning to use the money to buy her consumption bundle later. How much of good B will she choose to consume? 250 units. How much of good A? 250 units.

(g) Explain why her consumption is different depending on whether she was holding goods or money at the time of the price change. In the former case, the fall in p_B makes her poorer because she is a net seller of good B. In the latter case, her income doesn't

change.

9.4 (0) Priscilla finds it optimal not to engage in trade at the going prices and just consumes her endowment. Priscilla has no kinks in her indifference curves, and she is endowed with positive amounts of both goods. Use pencil or black ink to draw a budget line and an indifference curve for Priscilla that would be consistent with these facts. Suppose that the price of good 2 stays the same, but the price of good 1 falls. Use blue ink to show her new budget line. Priscilla satisfies the weak axiom of revealed preference. Could it happen that Priscilla will consume less of good 1 than before? Explain. No. If p_1 falls, then with the new budget, she can still afford her old bundle. She could afford the bundles with less of good 1 than her endowment at the old prices. By WARP, she won't choose them now.

9.5 (0) Potatoes are a Giffen good for Paddy, who has a small potato farm. The price of potatoes fell, but Paddy increased his potato consumption. At first this astonished the village economist, who thought that a decrease in the price of a Giffen good was supposed to reduce demand. But then he remembered that Paddy was a net supplier of potatoes. With the help of a graph, he was able to explain Paddy's behavior. In the axes below, show how this could have happened. Put "potatoes" on the horizontal axis and "all other goods" on the vertical axis. Label the old equilibrium A and the new equilibrium B. Draw a point C so that the

Slutsky substitution effect is the movement from A to C and the Slutsky income effect is the movement from C to B. On this same graph, you are also going to have to show that potatoes are a Giffen good. To do this, draw a budget line showing the effect of a fall in the price of potatoes if Paddy didn't own any potatoes, and only had money income. Label the new consumption point under these circumstances by D. (Warning: You probably will need to make a few dry runs on some scratch paper to get the whole story straight.)

9.6 (0) Recall the travails of Agatha, from the previous chapter. She had to travel 1,500 miles from Istanbul to Paris. She had only \$200 with which to buy first-class and second-class tickets on the Orient Express when the price of first-class tickets was \$.20 a mile and the price of second-class tickets was \$.10 a mile. She bought tickets that enabled her to travel all the way to Paris, with as many miles of first class as she could afford. After she boarded the train, she discovered to her amazement that the price of second-class tickets had fallen to \$.05 a mile while the price of first-class tickets remained at \$.20 a mile. She also discovered that on the train it was possible to buy or sell first-class tickets for \$.20 a mile and to buy or sell second-class tickets for \$.05 a mile. Agatha had no money left to buy either kind of ticket, but she did have the tickets that she had already bought.

(a) On the graph below, use pencil to show the combinations of tickets that she could afford at the old prices. Use blue ink to show the combinations of tickets that would take her exactly 1,500 miles. Mark the point that she chooses with the letter A.

(b) Use red ink to draw a line showing all of the combinations of first-class and second-class travel that she can afford when she is on the train, by trading her endowment of tickets at the new prices that apply on board the train.

(c) On your graph, show the point that she chooses after finding out about the price change. Does she choose more, less, or the same amount of second-class tickets? The same.

9.7 (0) Mr. Cog works in a machine factory. He can work as many hours per day as he wishes at a wage rate of w. Let C be the number of dollars he spends on consumer goods and let R be the number of hours of leisure that he chooses.

(a) Mr. Cog earns \$8 an hour and has 18 hours per day to devote to labor or leisure, and he has \$16 of nonlabor income per day. Write an equation for his budget between consumption and leisure. $C + 8R = 160$. Use blue ink to draw his budget line in the graph below. His initial endowment is the point where he does no work and enjoys 18 hours of leisure per day. Mark this point on the graph below with the letter A. (Remember that although Cog can choose to work and thereby "sell" some of his endowment of leisure, he cannot "buy leisure" by paying somebody else to loaf for him.) If Mr. Cog has the utility function $U(R, C) = CR$, how many hours of leisure per day will he choose? 10. How many hours per day will he work? 8.

(b) Suppose that Mr. Cog's wage rate rises to \$12 an hour. Use red ink to draw his new budget line. (He still has \$16 a day in nonlabor income.) If he continued to work exactly as many hours as he did before the wage increase, how much more money would he have each day to spend on consumption? \$32. But with his new budget line, he chooses to work $8\frac{1}{3}$ hours, and so his consumption increases by \$36.

(c) Suppose that Mr. Cog still receives \$8 an hour but that his nonlabor income rises to \$48 per day. Use black ink to draw his budget line. How many hours does he choose to work? 6.

(d) Suppose that Mr. Cog has a wage of $\$w$ per hour, a nonlabor income of $\$m$, and that he has 18 hours a day to divide between labor and leisure. Cog's budget line has the equation $C + wR = m + 18w$. Using the same methods you used in the chapter on demand functions, find the amount of leisure that Cog will demand as a function of wages and of nonlabor income. (Hint: Notice that this is the same as finding the demand for R when the price of R is w, the price of C is 1, and income is $m + 18w$.)

Mr. Cog's demand function for leisure is $R(w, m) =$ $9 + (m/2w)$.

Mr. Cog's supply function for labor is therefore $18 - R(w, m) =$ $9 - m/2w$.

9.8 (0) Fred has just arrived at college and is trying to figure out how to supplement the meager checks that he gets from home. "How can anyone live on \$50 a week for spending money?" he asks. But he asks to no avail. "If you want more money, get a job," say his parents. So Fred glumly investigates the possibilities. The amount of leisure time that he has left after allowing for necessary activities like sleeping, brushing teeth, and studying for economics classes is 50 hours a week. He can work as many hours per week at a nearby Taco Bell for \$5 an hour. Fred's utility function for leisure and money to spend on consumption is $U(C, L) = CL$.

(a) Fred has an endowment that consists of \$50 of money to spend on consumption and 50 hours of leisure, some of which he might "sell" for money. The money value of Fred's endowment bundle, including both his money allowance and the market value of his leisure time is therefore \$300. Fred's "budget line" for leisure and consumption is like a budget line for someone who can buy these two goods at a price of \$1 per unit of consumption and a price of \$5 per unit of leisure. The only difference is that this budget line doesn't run all the way to the horizontal axis.

(b) On the graph below, use black ink to show Fred's budget line. (Hint: Find the combination of leisure and consumption expenditures that he could have if he didn't work at all. Find the combination he would have if he chose to have no leisure at all. What other points are on your graph?) On the same graph, use blue ink to sketch the indifference curves that give Fred utility levels of 3,000, 4,500, and 7,500.

(c) If you maximized Fred's utility subject to the above budget, how much consumption would he choose? \$150. (Hint: Remember how to solve for the demand function of someone with a Cobb-Douglas utility function?)

(d) The amount of leisure that Fred will choose to consume is 30 hours. This means that his optimal labor supply will be 20 hours.

9.9 (0) George Johnson earns \$5 per hour in his job as a truffle sniffer. After allowing time for all of the activities necessary for bodily upkeep, George has 80 hours per week to allocate between leisure and labor. Sketch the budget constraints for George resulting from the following government programs.

(a) There is no government subsidy or taxation of labor income. (Use blue ink on the graph below.)

(b) All individuals receive a lump sum payment of \$100 per week from the government. There is no tax on the first \$100 per week of labor income. But all labor income above \$100 per week is subject to a 50% income tax. (Use red ink on the graph above.)

(c) If an individual is not working, he receives a payment of \$100. If he works he does not receive the \$100, and all wages are subject to a 50% income tax. (Use blue ink on the graph below.)

(d) The same conditions as in Part (c) apply, except that the first 20 hours of labor are exempt from the tax. (Use red ink on the graph above.)

(e) All wages are taxed at 50%, but as an incentive to encourage work, the government gives a payment of $100 to anyone who works more than 20 hours a week. (Use blue ink on the graph below.)

9.10 (0) In the United States, real wage rates in manufacturing have risen steadily from 1890 to the present. In the period from 1890 to 1930, the length of the workweek was reduced dramatically. But after 1930, despite the continuing growth of real wage rates, the length of the work week has stayed remarkably constant at about 40 hours per week.

Hourly Wages and Length of Work Week in U.S. Manufacturing, 1890–1983

Sources: *Handbook of Labor Statistics, 1983* and *U.S. Economic History*, by Albert Niemi (p. 274). Wages are in 1983 dollars.

Year	Wage	Hours Worked
1890	1.89	59.0
1909	2.63	51.0
1920	3.11	47.4
1930	3.69	42.1
1940	5.27	38.1
1950	6.86	40.5
1960	8.56	39.7
1970	9.66	39.8
1983	10.74	40.1

(a) Use these data to plot a "labor supply curve" on the graph below.

(b) At wage rates below $4 an hour, does the workweek get longer or shorter as the wage rate rises? Shorter.

(c) The data in this table could be consistent with workers choosing various hours a week to work, given the wage rate. An increase in wages has both an endowment income effect and a substitution effect. The substitution effect alone would make for a (longer, shorter) longer workweek. If leisure is a normal good, the endowment income effect tends to make people choose (more, less) more leisure and a (longer, shorter) shorter workweek. At wage rates below $4 an hour, the (substitution, endowment income) endowment income effect appears to dominate. How would you explain what happens at wages above $4 an hour? Substitution and endowment income effects cancel each other out, so the workweek stays roughly constant.

(d) Between 1890 and 1909, wage rates rose by 39 percent, but weekly earnings rose by only 20 percent. For this period, the

gain in earnings (overstates, understates) understates the gain in workers' wealth, since they chose to take (more, less) more leisure in 1909 than they took in 1890.

9.11 (0) Professor Mohamed El Hodiri of the University of Kansas, in a classic tongue-in-cheek article "The Economics of Sleeping," *Manifold*, 17 (1975), offered the following analysis. "Assume there are 24 hours in a day. Daily consumption being x and hours of sleep s, the consumer maximizes a utility function of the form $u = x^2 s$, where $x = w(24 - s)$, with w being the wage rate."

(a) In El Hodiri's model, does the optimal amount of sleeping increase, decrease, or stay the same as wages increase? Stays the same.

(b) How many hours of sleep per day is best in El Hodiri's model? 8.

9.12 (0) Wendy and Mac work in fast food restaurants. Wendy gets \$4 an hour for the first 40 hours that she works and \$6 an hour for every hour beyond 40 hours a week. Mac gets \$5 an hour no matter how many hours he works. Each has 80 hours a week to allocate between work and leisure and neither has any income from sources other than labor. Each has a utility function $U = cr$, where c is consumption and r is leisure. Each can choose the number of hours to work.

(a) How many hours will Mac choose to work? 40.

(b) Wendy's budget "line" has a kink in it at the point where $r =$ 40 and $c =$ 160. Use blue ink for the part of her budget line where she would be if she does not work overtime. Use red ink for the part where she would be if she worked overtime.

(c) The blue line segment that you drew lies on a line with equation $c + 4r = 320$. The red line that you drew lies on a line with equation $c + 6r = 400$. (Hint: For the red line, you know one point on the line and you know its slope.)

(d) If Wendy was paid \$4 an hour no matter how many hours she worked, she would work 40 hours and earn a total of \$160 a week. On your graph, use black ink to draw her indifference curve through this point.

(e) Will Wendy choose to work overtime? Yes. What is the best choice for Wendy from the red budget line? $(c, r) = (200, 33.3)$. How many hours a week will she work? 46.6.

(f) Suppose that the jobs are equally agreeable in all other respects. Since Wendy and Mac have the same preferences, they will be able to agree about who has the better job. Who has the better job? Mac. (Hint: Calculate Wendy's utility when she makes her best choice. Calculate what her utility would be if she had Mac's job and chose the best amount of time to work.)

9.13 (1) Wally Piper is a plumber. He charges \$10 per hour for his work and he can work as many hours as he likes. Wally has no source of income other than his labor. He has 168 hours per week to allocate between labor and leisure. On the graph below, draw Wally's budget set, showing the various combinations of weekly leisure and income that Wally can afford.

(a) Write down Wally's budget equation. $I + 10R = 1,680.$

(b) While self-employed, Wally chose to work 40 hours per week. The construction firm, Glitz and Drywall, had a rush job to complete. They offered Wally \$20 an hour and said that he could work as many hours as he liked. Wally still chose to work only 40 hours per week. On the graph you drew above, draw in Wally's new budget line.

(c) Wally has convex preferences and no kinks in his indifference curves. On the graph, draw indifference curves that are consistent with his choice of working hours when he was self-employed and when he worked for Glitz and Drywall.

(d) Glitz and Drywall were in a great hurry to complete their project and wanted Wally to work more than 40 hours. They decided that instead of paying him \$20 per hour, they would pay him only \$10 an hour for the first 40 hours that he worked per week and \$20 an hour for every hour of

"overtime" that he worked beyond 40 hours per week. On the graph that you drew above, use red ink to sketch in Wally's budget line with this pay schedule. Draw the indifference curve through the point that Wally chooses with this pay schedule. Will Wally work more than 40 hours or less than 40 hours per week with this pay schedule? More.

9.14 (1) Felicity loves her job. She is paid \$10 an hour and can work as many hours a day as she wishes. She chooses to work only 5 hours a day. She says the job is so interesting that she is happier working at this job than she would be if she made the same income without working at all. A skeptic asks, "If you like the job better than not working at all, why don't you work more hours and earn more money?" Felicity, who is entirely rational, patiently explains that work may be desirable on average but undesirable on the margin. The skeptic insists that she show him her indifference curves and her budget line.

(a) On the axes below, draw a budget line and indifference curves that are consistent with Felicity's behavior and her remarks. Put leisure on the horizontal axis and income on the vertical axis. (Hint: Where does the indifference curve through her actual choice hit the vertical line $l = 24$?)

9.15 (2) Dudley's utility function is $U(C, R) = C - (12 - R)^2$, where R is the amount of leisure he has per day. He has 16 hours a day to divide between work and leisure. He has an income of \$20 a day from nonlabor sources. The price of consumption goods is \$1 per unit.

(a) If Dudley can work as many hours a day as he likes but gets zero wages for his labor, how many hours of leisure will he choose? 12.

(b) If Dudley can work as many hours a day as he wishes for a wage rate of $10 an hour, how many hours of leisure will he choose? 7.

How many hours will he work? 9. (Hint: Write down Dudley's budget constraint. Solve for the amount of leisure that maximizes his utility subject to this constraint. Remember that the amount of labor he wishes to supply is 16 minus his demand for leisure.)

(c) If Dudley's nonlabor income decreased to $5 a day, while his wage rate remained at $10, how many hours would he choose to work? 9.

(d) Suppose that Dudley has to pay an income tax of 20 percent on all of his income, and suppose that his before-tax wage remained at $10 an hour and his before-tax nonlabor income was $20 per day. How many hours would he choose to work? 8.

Chapter 10

NAME_____

Intertemporal Choice

Introduction. The theory of consumer saving uses techniques that you have already learned. In order to focus attention on consumption over time, we will usually consider examples where there is only one consumer good, but this good can be consumed in either of two time periods. We will be using two "tricks." One trick is to treat *consumption in period 1* and *consumption in period 2* as two distinct commodities. If you make period-1 consumption the **numeraire**, then the "price" of period-2 consumption is the amount of period-1 consumption that you have to give up to get an extra unit of period-2 consumption. This price turns out to be $1/(1+r)$, where r is the interest rate.

The second trick is in the way you treat income in the two different periods. Suppose that a consumer has an income of m_1 in period 1 and m_2 in period 2 and that there is no inflation. The total amount of period-1 consumption that this consumer could buy, if he borrowed as much money as he could possibly repay in period 2, is $m_1 + \frac{m_2}{1+r}$. As you work the exercises and study the text, it should become clear that the consumer's budget equation for choosing consumption in the two periods is always

$$c_1 + \frac{c_2}{1+r} = m_1 + \frac{m_2}{1+r}.$$

This budget constraint looks just like the standard budget constraint that you studied in previous chapters, where the price of "good 1" is 1, the price of "good 2" is $1/(1+r)$, and "income" is $m_1 + \frac{m_2}{(1+r)}$. Therefore if you are given a consumer's utility function, the interest rate, and the consumer's income in each period, you can find his demand for consumption in periods 1 and 2 using the methods you already know. Having solved for consumption in each period, you can also find saving, since the consumer's saving is just the difference between his period-1 income and his period-1 consumption.

Example: A consumer has the utility function $U(c_1, c_2) = c_1 c_2$. There is no inflation, the interest rate is 10%, and the consumer has income 100 in period 1 and 121 in period 2. Then the consumer's budget constraint $c_1 + c_2/1.1 = 100 + 121/1.1 = 210$. The ratio of the price of good 1 to the price of good 2 is $1 + r = 1.1$. The consumer will choose a consumption bundle so that $MU_1/MU_2 = 1.1$. But $MU_1 = c_2$ and $MU_2 = c_1$, so the consumer must choose a bundle such that $c_2/c_1 = 1.1$. Take this equation together with the budget equation to solve for c_1 and c_2. The solution is $c_1 = 105$ and $c_2 = 115.50$. Since the consumer's period-1 income is only 100, he must borrow 5 in order to consume 105 in period 1. To pay back principal and interest in period 2, he must pay 5.50 out of his period-2 income of 121. This leaves him with 115.50 to consume.

You will also be asked to determine the effects of inflation on consumer behavior. The key to understanding the effects of inflation is to see what happens to the budget constraint.

Example: Suppose that in the previous example, there happened to be an inflation rate of 6%, and suppose that the price of period-1 goods is 1. Then if you save \$1 in period 1 and get it back with 10% interest, you will get back \$1.10 in period 2. But because of the inflation, goods in period 2 cost 1.06 dollars per unit. Therefore the amount of period-1 consumption that you have to give up to get a unit of period-2 consumption is $1.06/1.10 = .964$ units of period-2 consumption. If the consumer's money income in each period is unchanged, then his budget equation is $c_1 + .964c_2 = 210$. This budget constraint is the same as the budget constraint would be if there were no inflation and the interest rate were r, where $.964 = 1/(1+r)$. The value of r that solves this equation is known as the **real rate of interest**. In this case the real rate of interest is about .038. When the interest rate and inflation rate are both small, the real rate of interest is closely approximated by the difference between the **nominal interest rate**, (10% in this case) and the inflation rate (6% in this case), that is, $.038 \sim .10 - .06$. As you will see, this is not such a good approximation if inflation rates and interest rates are large.

10.1 (0) Peregrine Pickle consumes (c_1, c_2) and earns (m_1, m_2) in periods 1 and 2 respectively. Suppose the interest rate is r.

(a) Write down Peregrine's intertemporal budget constraint in present value terms. $c_1 + \frac{c_2}{(1+r)} = m_1 + \frac{m_2}{(1+r)}$.

(b) If Peregrine does not consume anything in period 1, what is the most he can consume in period 2? $m_1(1+r) + m_2$. This is the (future value, present value) of his endowment. Future value.

(c) If Peregrine does not consume anything in period 2, what is the most he can consume in period 1? $m_1 + \frac{m_2}{(1+r)}$. This is the (future value, present value) of his endowment. Present value. What is the slope of Peregrine's budget line? $-(1+r)$.

10.2 (0) Molly has a Cobb-Douglas utility function $U(c_1, c_2) = c_1^a c_2^{1-a}$, where $0 < a < 1$ and where c_1 and c_2 are her consumptions in periods 1 and 2 respectively. We saw earlier that if utility has the form $u(x_1, x_2) = x_1^a x_2^{1-a}$ and the budget constraint is of the "standard" form $p_1x_1 + p_2x_2 = m$, then the demand functions for the goods are $x_1 = am/p_1$ and $x_2 = (1-a)m/p_2$.

(a) Suppose that Molly's income is m_1 in period 1 and m_2 in period 2. Write down her budget constraint in terms of present values. $c_1 + c_2/(1+r) = m_1 + m_2/(1+r)$.

(b) We want to compare this budget constraint to one of the standard form. In terms of Molly's budget constraint, what is p_1? 1. What is p_2? $1/(1+r)$. What is m? $m_1 + m_2/(1+r)$.

(c) If $a = .2$, solve for Molly's demand functions for consumption in each period as a function of m_1, m_2, and r. Her demand function for consumption in period 1 is $c_1 =$ $.2m_1 + .2m_2/(1+r)$. Her demand function for consumption in period 2 is $c_2 =$ $.8(1+r)m_1 + .8m_2$.

(d) An increase in the interest rate will decrease her period-1 consumption. It will increase her period-2 consumption and increase her savings in period 1.

10.3 (0) Nickleby has an income of \$2,000 this year, and he expects an income of \$1,100 next year. He can borrow and lend money at an interest rate of 10%. Consumption goods cost \$1 per unit this year and there is no inflation.

(a) What is the present value of Nickleby's endowment? $3,000.

What is the future value of his endowment? $3,300. With blue ink, show the combinations of consumption this year and consumption next year that he can afford. Label Nickelby's endowment with the letter E.

(b) Suppose that Nickleby has the utility function $U(C_1, C_2) = C_1C_2$. Write an expression for Nickleby's marginal rate of substitution between consumption this year and consumption next year. (Your answer will be a function of the variables C_1, C_2.) MRS $= -C_2/C_1$.

(c) What is the slope of Nickleby's budget line? -1.1. Write an equation that states that the slope of Nickleby's indifference curve is equal to the slope of his budget line when the interest rate is 10%. $1.1 = C_2/C_1$. Also write down Nickleby's budget equation. $C_1 + C_2/1.1 = 3,000$.

(d) Solve these two equations. Nickleby will consume 1,500 units in period 1 and 1,650 units in period 2. Label this point A on your diagram.

(e) Will he borrow or save in the first period? Save. How much?

500.

(f) On your graph use red ink to show what Nickleby's budget line would be if the interest rate rose to 20%. Knowing that Nickleby chose the point A at a 10% interest rate, even without knowing his utility function, you can determine that his new choice cannot be on certain parts of his new budget line. Draw a squiggly mark over the part of his new budget line where that choice can *not* be. (Hint: Close your eyes and think of WARP.)

(g) Solve for Nickleby's optimal choice when the interest rate is 20%.

Nickleby will consume 1,458.3 units in period 1 and 1,750 units in period 2.

(h) Will he borrow or save in the first period? Save. How much?

541.7.

10.4 (0) Decide whether each of the following statements is true or false. Then explain why your answer is correct, based on the Slutsky decomposition into income and substitution effects.

(a) "If both current and future consumption are normal goods, an increase in the interest rate will necessarily make a saver save more." False. Substitution effect makes him consume less in period 1 and save more. For a saver, income effect works in opposite direction. Either effect could dominate.

(b) "If both current and future consumption are normal goods, an increase in the interest rate will necessarily make a saver choose more consumption in the second period." True. The income and substitution effects both lead to more consumption in the second period.

10.5 (1) Laertes has an endowment of \$20 each period. He can borrow money at an interest rate of 200%, and he can lend money at a rate of 0%. (Note: If the interest rate is 0%, for every dollar that you save, you get back \$1 in the next period. If the interest rate is 200%, then for every dollar you borrow, you have to pay back \$3 in the next period.)

(a) Use blue ink to illustrate his budget set in the graph below. (Hint: The boundary of the budget set is not a single straight line.)

(b) Laertes could invest in a project that would leave him with $m_1 = 30$ and $m_2 = 15$. Besides investing in the project, he can still borrow at 200% interest or lend at 0% interest. Use red ink to draw the new budget set in the graph above. Would Laertes be better off or worse off by investing in this project given his possibilities for borrowing or lending? Or can't one tell without knowing something about his preferences? Explain.

Better off. If he invests in the project, he can borrow or lend to get any bundle he could afford without investing.

(c) Consider an alternative project that would leave Laertes with the endowment $m_1 = 15$, $m_2 = 30$. Again suppose he can borrow and lend as above. But if he chooses this project, he can't do the first project. Use pencil or black ink to draw the budget set available to Laertes if he chooses this project. Is Laertes better off or worse off by choosing this project than if he didn't choose either project? Or can't one tell without knowing more about his preferences? Explain. Can't tell. He can afford some things he couldn't afford originally. But some things he could afford before, he can't afford if he invests in this project.

10.6 (0) The table below reports the inflation rate and the annual rate of return on treasury bills in several countries for the years 1984 and 1985.

Inflation Rate and Interest Rate for Selected Countries

Country	% Inflation Rate, 1984	% Inflation Rate, 1985	% Interest Rate, 1984	% Interest Rate, 1985
United States	3.6	1.9	9.6	7.5
Israel	304.6	48.1	217.3	210.1
Switzerland	3.1	0.8	3.6	4.1
W. Germany	2.2	−0.2	5.3	4.2
Italy	9.2	5.8	15.3	13.9
Argentina	90.0	672.2	NA	NA
Japan	0.6	2.0	NA	NA

(a) In the table below, use the formula that your textbook gives for the exact real rate of interest to compute the exact real rates of interest.

(b) What would the nominal rate of return on a bond in Argentina have to be to give a real rate of return of 5% in 1985? 710.8%. What would the nominal rate of return on a bond in Japan have to be to give a real rate of return of 5% in 1985? 7.1%.

(c) Subtracting the inflation rate from the nominal rate of return gives a good approximation to the real rate for countries with a low rate of inflation. For the United States in 1984, the approximation gives you 6% while the more exact method suggested by the text gives you 5.79%. But for countries with very high inflation this is a poor approximation. The approximation gives you −87.3% for Israel in 1984, while the more exact formula gives you −21.57%. For Argentina in 1985, the approximation would tell us that a bond yielding a nominal rate of 677.7% would yield a real interest rate of 5%. This contrasts with the answer 710.8% that you found above.

Real Rates of Interest in 1984 and 1985

Country	1984	1985
United States	5.7	5.5
Israel	−21.57	109.4
Switzerland	0.5	3.33
W. Germany	3.0	4.4
Italy	5.6	7.6

10.7 (0) We return to the planet Mungo. On Mungo, macroeconomists and bankers are jolly, clever creatures, and there are two kinds of money, red money and blue money. Recall that to buy something in Mungo you have to pay for it twice, once with blue money and once with red money. Everything has a blue-money price and a red-money price, and nobody is ever allowed to trade one kind of money for the other. There is a blue-money bank where you can borrow and lend blue money at a 50% annual interest rate. There is a red-money bank where you can borrow and lend red money at a 25% annual interest rate.

A Mungoan named Jane consumes only one commodity, ambrosia, but it must decide how to allocate its consumption between this year and next year. Jane's income this year is 100 blue currency units and no red currency units. Next year, its income will be 100 red currency units and no blue currency units. The blue currency price of ambrosia is one b.c.u. per flagon this year and will be two b.c.u.'s per flagon next year. The red currency price of ambrosia is one r.c.u. per flagon this year and will be the same next year.

(a) If Jane spent all of its blue income in the first period, it would be enough to pay the blue price for 100 flagons of ambrosia. If Jane saved all of this year's blue income at the blue-money bank, it would have 150 b.c.u.'s next year. This would give it enough blue currency to pay the blue price for 75 flagons of ambrosia. On the graph below, draw Jane's blue budget line, depicting all of those combinations of current and next period's consumption that it has enough blue income to buy.

(b) If Jane planned to spend no red income in the next period and to borrow as much red currency as it can pay back with interest with next period's red income, how much red currency could it borrow? 80.

(c) The (exact) real rate of interest on blue money is -25%. The real rate of interest on red money is 25%.

(d) On the axes below, draw Jane's blue budget line and its red budget line. Shade in all of those combinations of current and future ambrosia consumption that Jane can afford given that it has to pay with both currencies.

(e) It turns out that Jane finds it optimal to operate *on* its blue budget line and *beneath* its red budget line. Find such a point on your graph and mark it with a C.

(f) On the following graph, show what happens to Jane's original budget set if the blue interest rate rises and the red interest rate does not change. On your graph, shade in the part of the new budget line where Jane's new demand could possibly be. (Hint: Apply the principle of revealed preference. Think about what bundles were available but rejected when Jane chose to consume at C before the change in blue interest rates.)

10.8 (0) Mr. O. B. Kandle will only live for two periods. In the first period he will earn \$50,000. In the second period he will retire and live on his savings. His utility function is $U(c_1, c_2) = c_1 c_2$, where c_1 is consumption in period 1 and c_2 is consumption in period 2. He can borrow and lend at the interest rate $r = .10$.

(a) If the interest rate rises, will his period-1 consumption increase, decrease, or stay the same? Stay the same. His demand for c_1 is $.5(m_1 + m_2/(1+r))$ and $m_2 = 0$.

(b) Would an increase in the interest rate make him consume more or less in the second period? More. He saves the same amount, but with higher interest rates, he gets more back next period.

(c) If Mr. Kandle's income is zero in period 1, and \$ 55,000 in period 2, would an increase in the interest rate make him consume more, less, or the same amount in period 1? Less.

10.9 (1) Harvey Habit's utility function is $U(c_1, c_2) = \min\{c_1, c_2\}$, where c_1 is his consumption of bread in period 1 and c_2 is his consumption of bread in period 2. The price of bread is \$1 per loaf in period 1. The interest rate is 21%. Harvey earns \$2,000 in period 1 and he will earn \$1,100 in period 2.

(a) Write Harvey's budget constraint in terms of future value, assuming no inflation. $1.21c_1 + c_2 = 3,520$.

(b) How much bread does Harvey consume in the first period and how much money does he save? (The answer is not necessarily an integer.)

He picks $c_1 = c_2$. Substitute into the budget to find $c_1 = 3,520/2.21 = 1,592.8$. He saves $2,000 - 3,520/2.21 = 407.2$.

(c) Suppose that Harvey's money income in both periods is the same as before, the interest rate is still 21%, but there is a 10% inflation rate. Then in period 2, a loaf of bread will cost \$ 1.10. Write down Harvey's budget equation for period-1 and period-2 bread, given this new information. $1.21c_1 + 1.1c_2 = 3,520$.

10.10 (2) In an isolated mountain village, the only crop is corn. Good harvests alternate with bad harvests. This year the harvest will be 1,000 bushels. Next year it will be 150 bushels. There is no trade with the outside world. Corn can be stored from one year to the next, but rats will eat 25% of what is stored in a year. The villagers have Cobb-Douglas utility functions, $U(c_1, c_2) = c_1c_2$ where c_1 is consumption this year, and c_2 is consumption next year.

(a) Use red ink to draw a "budget line," showing consumption possibilities for the village, with this year's consumption on the horizontal axis and next year's consumption on the vertical axis. Put numbers on your graph to show where the budget line hits the axes.

(b) How much corn will the villagers consume this year? 600 bushels. How much will the rats eat? 100 bushels. How much corn will the villagers consume next year? 450 bushels.

(c) Suppose that a road is built to the village so that now the village is able to trade with the rest of the world. Now the villagers are able to buy and sell corn at the world price, which is $1 per bushel. They are also able to borrow and lend money at an interest rate of 10%. On your graph, use blue ink to draw the new budget line for the villagers. Solve for the amount they would now consume in the first period 568 bushels and the second period 624 bushels.

(d) Suppose that all is as in the last part of the question except that there is a transportation cost of $.10 per bushel for every bushel of grain hauled into or out of the village. On your graph, use black ink or pencil to draw the budget line for the village under these circumstances.

10.11 (0) The table below records percentage interest rates and inflation rates for the United States in some recent years. Complete this table.

Inflation and Interest in the United States, 1965-1985

Year	1965	1970	1975	1978	1980	1985
CPI, Start of Year	38.3	47.1	66.3	79.2	100.0	130.0
CPI, End of Year	39.4	49.2	69.1	88.1	110.4	133
% Inflation Rate	2.9	4.3	4.2	11.3	10.4	2.3
Nominal Int. Rate	4.0	6.4	5.8	7.2	11.6	7.5
Real Int. Rate	1.1	2.1	1.6	−3.7	1.09	5.07

(a) People complained a great deal about the high interest rates in the late 70s. In fact, interest rates had never reached such heights in modern times. Explain why such complaints are misleading.

Nominal interest rates were high, but so was inflation. Real interest rates were not high. (They were negative in 1978.)

(b) If you gave up a unit of consumption goods at the beginning of 1985 and saved your money at interest, you could use the proceeds of your saving to buy 1.05 units of consumption goods at the beginning of 1986. If you gave up a unit of consumption goods at the beginning of 1978 and saved your money at interest, you would be able to use the proceeds of your saving to buy .96 units of consumption goods at the beginning of 1979.

10.12 (1) Marsha Mellow doesn't care whether she consumes in period 1 or in period 2. Her utility function is simply $U(c_1, c_2) = c_1 + c_2$. Her initial endowment is \$20 in period 1 and \$40 in period 2. In an antique shop, she discovers a cookie jar that is for sale for \$12 in period 1 and that she is certain she can sell for \$20 in period 2. She derives no consumption benefits from the cookie jar, and it costs her nothing to store it for one period.

(a) On the graph below, label her initial endowment, E, and use blue ink to draw the budget line showing combinations of period-1 and period-2 consumption that she can afford if she doesn't buy the cookie jar. On the same graph, label the consumption bundle, A, that she would have if she did not borrow or lend any money but bought the cookie jar in period 1, sold it in period 2, and used the proceeds to buy period-2 consumption.

If she cannot borrow or lend, should Marsha invest in the cookie jar?

Yes.

(b) Suppose that Marsha can borrow and lend at an interest rate of 50%. On the graph where you labelled her initial endowment, draw the budget line showing all of the bundles she can afford if she invests in the cookie jar and borrows or lends at the interest rate of 50%. On the same graph use red ink to draw one or two of Marsha's indifference curves.

(c) Suppose that instead of consumption in the two periods being perfect substitutes, they are perfect complements, so that Marsha's utility function is $\min\{c_1, c_2\}$. If she cannot borrow or lend, should she buy the cookie jar? No. If she can borrow and lend at an interest rate of 50%, should she invest in the cookie jar? Yes. If she can borrow or lend as much at an interest rate of 100%, should she invest in the cookie jar?

No.

Chapter 11

NAME_____

Asset Markets

Introduction. The fundamental equilibrium condition for asset markets is that in equilibrium the rate of return on all assets must be the same. Thus if you know the rate of interest and the cash flow generated by an asset, you can predict what its market equilibrium price will be. This condition has many interesting implications for the pricing of durable assets. Here you will explore several of these implications.

Example: A drug manufacturing firm owns the patent for a new medicine. The patent will expire on January 1, 1996, at which time anyone can produce the drug. Whoever owns the patent will make a profit of $1,000,000 per year until the patent expires. For simplicity, let us suppose that profits for any year are all collected on December 31. The interest rate is 5%. Let us figure out what the selling price of the patent rights will be on January 1, 1993. On January 1, 1993, potential buyers realize that owning the patent will give them $1,000,000 every year starting 1 year from now and continuing for 3 years. The present value of this cash flow is

$$\$\frac{1,000,000}{(1.05)} + \frac{1,000,000}{(1.05)^2} + \frac{1,000,000}{(1.05)^3} \sim \$2,723,248.$$

Nobody would pay more than this amount for the patent since if you put $2,723,248 at 5% interest, you could collect $1,000,000 a year from the bank for 3 years, starting 1 year from now. The patent wouldn't sell for less than $2,723,248, since if it sold for less, one would get a higher rate of return by investing in this patent than one could get from investing in anything else. What will the price of the patent be on January 1, 1994? At that time, the patent is equivalent to a cash flow of $1,000,000 in 1 year and another $1,000,000 in 2 years. The present value of this flow, viewed from the standpoint of January 1, 1994, will be

$$\$\frac{1,000,000}{(1.05)} + \frac{1,000,000}{(1.05)^2} \sim \$1,859,310.$$

A slightly more difficult problem is one where the cash flow from an asset depends on how the asset is used. To find the price of such an asset, one must ask what will be the present value of the cash flow that the asset yields if it is managed in such a way as to maximize its present value.

Example: People will be willing to pay $15 a bottle to drink a certain wine this year. Next year they would be willing to pay $25, and the year after that they would be willing to pay $26. After that, it starts to deteriorate and the amount people are willing to pay to drink it falls. The interest rate is 5%. We can determine not only what the wine will sell for but also when it will be drunk. If the wine is drunk in the first year, it would have to sell for $15. But no rational investor is going to sell the wine for

\$15 in the first year, because it will sell for \$25 one year later. This is a 66.66% rate of return, which is better than the rate of interest. When the interest rate is 5%, investors are willing to pay at least $\$25/1.05 = \$23.81$ for the wine. So investors must outbid drinkers, and none will be drunk this year. Will investors want to hold onto the wine for 2 years? In 2 years, the wine will be worth \$26, so the present value of buying the wine and storing it for 2 years is $\$26/(1.05)^2 = \$23.58$. This is less than the present value of holding the wine for 1 year and selling it for \$25. So, we conclude that the wine will be drunk after 1 year. Its current selling price will be \$23.81, and 1 year from now, it will sell for \$25.

11.0 Warm Up Exercise. Here are a few problems on present values. In all of the following examples, assume that you can both borrow and lend at an annual interest rate of r and that the interest rate will remain the same forever.

(a) You would be indifferent between getting \$1 now and `1+r` dollars, one year from now, because if you put the dollar in the bank, then one year from now you could get back `1+r` dollars from the bank.

(b) You would be indifferent between getting 1 dollar(s) one year from now and getting $\$1/(1+r)$ dollars now, because $1/(1+r)$ deposited in the bank right now would enable you to withdraw principal and interest worth \$1.

(c) For any $X > 0$, you would be indifferent between getting $X/(1+r)$ dollars right now and $\$X$ one year from now. The present value of $\$X$ received one year from now is $X/(1+r)$ dollars.

(d) The present value of an obligation to pay $\$X$ one year from now is $-X/(1+r)$ dollars.

(e) The present value of $\$X$, to be received 2 years from now, is $X/(1+r)^2$ dollars.

(f) The present value of an asset that pays X_t dollars t years from now is $X_t/(1+r)^t$ dollars.

(g) The present value of an asset that pays $\$X_1$ one year from now, $\$X_2$ in two years, and $\$X_{10}$ ten years from now is $X_1/(1+r)+X_2/(1+r)^2+X_{10}/(1+r)^{10}$ dollars.

(h) The present value of an asset that pays a constant amount, $\$X$ per year forever can be computed in two different ways. One way is to figure out the amount of money you need in the bank so that the bank would give you $\$X$ per year, forever, without ever exhausting your principal. The annual interest received on a bank account of X/r dollars will be $\$X$. Therefore having X/r dollars right now is just as good as getting $\$X$ a year forever.

(i) Another way to calculate the present value of $\$X$ a year forever is to evaluate the infinite series $\sum_{i=1}^{\infty} X/(1+r)^i$. This series is known as a `geometric` series. Whenever $r > 0$, this sum is well defined and is equal to X/r.

(j) If the interest rate is 10%, the present value of receiving $1,000 one year from now will be, to the nearest dollar, `$909.` The present value of receiving $1,000 a year forever, will be, to the nearest dollar, `$10,000.`

(k) If the interest rate is 10%, what is the present value of an asset that requires you to pay out $550 one year from now and will pay you back $1,210 two years from now? $-550/(1.1) + 1,210/(1.1)^2 = 500$ dollars.

11.1 (0) An area of land has been planted with Christmas trees. On December 1, ten years from now, the trees will be ready for harvest. At that time, the standing Christmas trees can be sold for $1,000 per acre. The land, after the trees have been removed, will be worth $200 per acre. There are no taxes or operating expenses, but also no revenue from this land until the trees are harvested. The interest rate is 10%.

(a) What can we expect the market price of the land to be?

$\$1,200/(1.1)^{10} \sim \$463$ `per acre.`

(b) Suppose that the Christmas trees do not have to be sold after 10 years, but could be sold in any year. Their value if they are cut before they are 10 years old is zero. After the trees are 10 years old, an acre of trees is worth $1,000 and its value will increase by $100 per year for the next 20 years. After the trees are cut, the land on which the trees stood can always be sold for $200 an acre. When should the trees be cut to maximize the present value of the payments received for trees and land?

After 10 years. What will be the market price of an acre of land? Still $463.

11.2 (0) Publicity agents for the Detroit Felines announce the signing of a phenomenal new quarterback, Archie Parabola. They say that the contract is worth $1,000,000 and will be paid in 20 installments of $50,000 per year starting one year from now and with one new installment each year for next 20 years. The contract contains a clause that guarantees he will get all of the money even if he is injured and cannot play a single game. Sports writers declare that Archie has become an "instant millionaire."

(a) Archie's brother, Fenwick, who majored in economics, explains to Archie that he is not a millionaire. In fact, his contract is worth less than half a million dollars. Explain in words why this is so.

The present value of $50,000 a year for 20 years is less than $1,000,000, since the present value of a dollar received in the future is less than $1.

Archie's college course on "Sports Management" didn't cover present values. So his brother tried to reason out the calculation for him. Here is how it goes:

(b) Suppose that the interest rate is 10% and is expected to remain at 10% forever. How much would it cost the team to buy Archie a perpetuity that would pay him and his heirs $1 per year *forever*, starting in 1 year?

$10.

(c) How much would it cost to buy a perpetuity that paid $50,000 a year forever, starting in one year? $500,000.

In the last part, you found the present value of Archie's contract if he were going to get $50,000 a year forever. But Archie is not going to get $50,000 a year forever. The payments stop after 20 years. The present value of Archie's actual contract is the same as the present value of a contract that pays him $50,000 a year forever, but makes him pay back $50,000 each year, forever, starting 21 years from now. Therefore you can find the present value of Archie's contract by subtracting the

present value of \$50,000 a year forever, starting 21 years from now from the present value of \$50,000 a year forever.

(d) If the interest rate is and will remain at 10%, a stream of payments of \$50,000 a year, starting 21 years from now has the same present value as a lump sum of \$ 500,000 to be received all at once, exactly 20 years from now.

(e) If the interest rate is and will remain at 10%, what is the present value of \$50,000 per year forever, starting 21 years from now? \$75,000. (Hint: The present value of \$1 to be paid in 20 years is $1/(1+r)^{20} = .15$.)

(f) Now calculate the present value of Archie's contract. $8.50 \times 50,000 = \$425,000$.

11.3 (0) Professor Thesis is puzzling over the formula for the present value of a stream of payments of \$1 a year, starting 1 year from now and continuing forever. He knows that the value of this stream is expressed by the infinite series

$$S = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots,$$

but he can't remember the simplified formula for this sum. All he knows is that if the first payment were to arrive today, rather than a year from now, the present value of the sum would be \$1 higher. So he knows that

$$S + 1 = 1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots.$$

Professor Antithesis suffers from a similar memory lapse. He can't remember the formula for S either. But, he knows that the present value of \$1 a year forever, starting right now has to be $1 + r$ times as large as the present value of \$1 a year, starting a year from now. (This is true because if you advance any income stream by a year, you multiply its present value by 1+r.) That is,

$$1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots = (1+r)S.$$

(a) If Professor Thesis and Professor Antithesis put their knowledge together, they can express a simple equation involving only the variable S. This equation is $S + 1 =$ $(1+r)S$. Solving this equation, they find that $S =$ $1/r$.

(b) The two professors have also forgotten the formula for the present value of a stream of \$1 per year starting next year and continuing for K years. They agree to call this number $S(K)$ and they see that

$$S(K) = \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^K}.$$

Professor Thesis notices that if each of the payments came 1 year earlier, the present value of the resulting stream of payments would be

$$1 + \frac{1}{(1+r)} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^{K-1}} = S(K) + 1 - \frac{1}{(1+r)^K}.$$

Professor Antithesis points out that speeding up any stream of payments by a year is also equivalent to multiplying its present value by $(1+r)$. Putting their two observations together, the two professors noticed an equation that could be solved for $S(K)$. This equation is $S(K) + 1 - \frac{1}{(1+r)^K} =$ $S(K)(1+r)$. Solving this equation for $S(K)$, they find that the formula for $S(K)$ is $S(K) = (1 - \frac{1}{(1+r)^K})/r$.

Calculus **11.4 (0)** You are the business manager of P. Bunyan Forests, Inc., and are trying to decide when you should cut your trees. The market value of the lumber that you will get if you let your trees reach the age of t years is given by the function $W(t) = e^{.20t - .001t^2}$. Mr. Bunyan can earn an interest rate of 5% per year on money in the bank.

The rate of growth of the market value of the trees will be greater than 5% until the trees reach 75 years of age. (Hint: It follows from elementary calculus that if $F(t) = e^{g(t)}$, then $F'(t)/F(t) = g'(t)$.)

(a) If he is only interested in the trees as an investment, how old should Mr. Bunyan let the trees get? 75 years.

(b) At what age do the trees have the greatest market value? 100 years.

11.5 (0) You expect the price of a certain painting to rise by 8% per year forever. The market interest rate for borrowing and lending is 10%. Assume there are no brokerage costs in purchasing or selling.

(a) If you pay a price of \$$x$ for the painting now and sell it in a year, how much has it cost you to hold the painting rather than to have loaned the \$$x$ at the market interest rate? It has cost $.02x$.

(b) You would be willing to pay \$100 a year to have the painting on your walls. Write an equation that you can solve for the price x at which you would be just willing to buy the painting $.02x = 100.$

(c) How much should you be willing to pay to buy the painting?

\$5,000.

11.6 (2) Ashley is thinking of buying a truckload of wine for investment purposes. He can borrow and lend as much as he likes at an annual interest rate of 10%. He is looking at three kinds of wine. To keep our calculations simple, let us assume that handling and storage costs are negligible.

• Wine drinkers would pay exactly \$175 a case to drink Wine A today. But if Wine A is allowed to mature for one year, it will improve. In fact wine drinkers will be willing to pay \$220 a case to drink this wine one year from today. After that, the wine gradually deteriorates and becomes less valuable every year.

• From now until one year from now, Wine B is indistinguishable from Wine A. But instead of deteriorating after one year, Wine B will improve. In fact the amount that wine drinkers would be willing to pay to drink Wine B will be \$220 a case in one year and will rise by \$10 per case per year for the next 30 years.

• Wine drinkers would be willing to pay \$100 per case to drink Wine C right now. But one year from now, they will be willing to pay \$250 per case to drink it and the amount they will be willing to pay to drink it will rise by \$50 per case per year for the next 20 years.

(a) What is the most Ashley would be willing to pay per case for Wine A? \$200.

(b) What is the most Ashley would be willing to pay per case for Wine B? \$200. (Hint: When will Wine B be drunk?)

(c) How old will Wine C be when it first becomes worthwhile for investors to sell off their holdings and for drinkers to drink it? 6 years. (Hint: When does the rate of return on holding wine get to 10%?)

(d) What will the price of Wine C be at the time it is first drunk?

\$500 per case.

(e) What is the most that Ashley would be willing to pay today for a case of Wine C? (Hint: What is the present value of his investment if he sells it to a drinker at the optimal time?) Express your answer in exponential notation without calculating it out. $\$500/1.1^6$.

11.7 (0) Fisher Brown is taxed at 40% on his income from ordinary bonds. Ordinary bonds pay 10% interest. Interest on municipal bonds is not taxed at all.

(a) If the interest rate on municipal bonds is 7%, should he buy municipal bonds or ordinary bonds? Brown should buy municipal bonds.

(b) Hunter Black makes less money than Fisher Brown and is taxed at only 25% on his income from ordinary bonds. Which kind of bonds should he buy? Black should buy ordinary bonds.

(c) If Fisher has \$1,000,000 in bonds and Hunter has \$10,000 in bonds, how much tax does Fisher pay on his interest from bonds? 0.
How much tax does Hunter pay on his interest from bonds? \$250.

(d) The government is considering a new tax plan under which no interest income will be taxed. If the interest rates on the two types of bonds do not change, and Fisher and Hunter are allowed to adjust their portfolios, how much will Fisher's after-tax income be increased? \$30,000.
How much will Hunter's after-tax income be increased? \$250.

(e) What would the change in the tax law do to the demand for municipal bonds if the interest rates did not change? It would reduce it to zero.

(f) What interest rate will new issues of municipal bonds have to pay in order to attract purchasers? They will have to pay 10%.

(g) What do you think will happen to the market price of the old municipal bonds, which had a 7% yield originally? The price of the old bonds will fall until their yield equals 10%.

11.8 (0) In the text we discussed the market for oil assuming zero production costs, but now suppose that it is costly to get the oil out of the ground. Suppose that it costs \$5 dollars per barrel to extract oil from the ground. Let the price in period t be denoted by p_t and let r be the interest rate.

(a) If a firm extracts a barrel of oil in period t, how much profit does it make in period t? $p_t - 5$.

(b) If a firm extracts a barrel of oil in period $t+1$, how much profit does it make in period $t+1$? $p_{t+1} - 5$.

(c) What is the present value of the profits from extracting a barrel of oil in period $t+1$? $(p_{t+1}-5)/(1+r)^{t+1}$. What is the present value of profit from extracting a barrel of oil in period t? $(p_t-5)/(1+r)^t$.

(d) If the firm is willing to supply oil in each of the two periods, what must be true about the relation between the present value of profits from sale of a barrel of oil in the two periods? The present values must be equal. Express this relation as an equation.

$$\frac{p_{t+1}-5}{(1+r)^{t+1}} = \frac{p_t-5}{(1+r)^t}.$$

(e) Solve the equation in the above part for p_{t+1} as a function of p_t and r. $p_{t+1} = (1+r)p_t - 5r$.

(f) Is the percentage rate of price increase between periods larger or smaller than the interest rate? The percent change in price is smaller.

11.9 (0) Dr. No owns a bond, serial number 007, issued by the James Company. The bond pays \$200 for each of the next three years, at which time the bond is retired and pays its face value of \$2,000.

(a) How much is the James bond 007 worth to Dr. No at an interest rate of 10%? $200/1.1+200/1.1^2+200/1.1^3+2,000/1.1^3 = 2,000.$

(b) How valuable is James bond 007 at an interest rate of 5%? $200/1.05 + 200/1.05^2 + 200/1.05^3 + 2,000/1.05^3 = 2,272.32.$

(c) Ms. Yes offers Dr. No $2,200 for the James bond 007. Should Dr. No say yes or no to Ms. Yes if the interest rate is 10%? Yes. What if the interest rate is 5%? No.

(d) In order to destroy the world, Dr. No hires Professor Know to develop a nasty zap beam. In order to lure Professor Know from his university position, Dr. No will have to pay the professor $200 a year. The nasty zap beam will take three years to develop, at the end of which it can be built for $2,000. If the interest rate is 5%, how much money will Dr. No need today to finance this dastardly program? $2,272.32, which is the present value calculated in the first part of the problem. If the interest rate were 10%, would the world be in more or less danger from Dr. No? More danger, since the dastardly plan is now cheaper.

11.10 (0) Chillingsworth owns a large, poorly insulated home. His annual fuel bill for home heating averages $300 per year. An insulation contractor suggests to him the following options.

Plan A. Insulate just the attic. If he does this, he will permanently reduce his fuel consumption by 15%. Total cost of insulating the attic is $300.

Plan B. Insulate the attic and the walls. If he does this, he will permanently reduce his fuel consumption by 20%. Total cost of insulating the attic and the walls is $500.

Plan C. Insulate the attic and the walls, and install a solar heating unit. If he does this, he will permanently reduce his fuel costs to zero. Total cost of this option is $7,000 for the solar heater and $500 for the insulating.

(a) Assume for simplicity of calculations that the house and the insulation will last forever. Calculate the present value of the dollars saved on fuel from each of the three options if the interest rate is 10%. The present values are: Plan A? $450. Plan B? $600. Plan C? $3,000.

(b) Each plan requires an expenditure of money to undertake. The difference between the present value and the present cost of each plan is: Plan A? $450 - 300 = 150$. Plan B? $600 - 500 = 100$. Plan C? $3,000 - 7,500 = -4,500$.

(c) If the price of fuel is expected to remain constant, which option should he choose if he can borrow and lend at an annual interest rate of 10%? A.

(d) Which option should he choose if he can borrow and lend at an annual rate of 5%? B.

(e) Suppose that the government offers to pay half of the cost of any insulation or solar heating device. Which option would he now choose at interest rates 10%? B. 5%? C.

(f) Suppose that there is no government subsidy but that fuel prices are expected to rise by 5% per year. What is the present value of fuel savings from each of the three proposals if interest rates are 10%? (Hint: If a stream of income is growing at x% and being discounted at y%, its present value should be the same as that of a constant stream of income discounted at $(y - x)$%.) Plan A? $900. Plan B? $1,200. Plan C? $6,000. Which proposal should Chillingsworth choose if interest rates are 10%? B. 5%? C.

11.11 (1) Have you ever wondered if a college education is financially worthwhile? The U.S. Census Bureau collects data on income and education that throws some light on this question. A recent census publication (Current Population Reports, Series P-70, No. 11) reports the average annual wage income in 1984 of persons aged 35–44 by the level of schooling achieved. The average wage income of high school graduates was $13,000 per year. The average wage income of persons with bachelor's degrees was $24,000 per year. The average wage income of persons with master's degrees was $28,000 per year. The average wage income of persons with Ph.D.'s was $40,000 per year. These income differences probably overstate the return to education itself, because it is likely that those people who get more education tend to be more able than those who get less.

Some of the income difference is, therefore, a return to ability rather than to education. But just to get a rough idea of returns to education, let us see what would be the return if the reported wage differences are all due to education.

(a) Suppose that you have just graduated from high school at age 18. You want to estimate the present value of your lifetime earnings if you do not go to college but take a job immediately. To do this, you have to make some assumptions. Assume that you would work for 47 years, until you are 65 and then retire. Assume also that you would make $13,000 a year for the rest of your life. (If you were going to do this more carefully, you would want to take into account that people's wages vary with their age, but let's keep things simple for this problem.) Assume that the interest rate is 5%. Find the present value of your lifetime earnings. (Hint: First find out the present value of $13,000 a year forever. Subtract from this the present value of $13,000 a year forever, starting 47 years from now.)

$233,753.

(b) Again, supposing you have just graduated from high school at age 18, and you want to estimate the present value of your life time earnings if you go to college for 4 years and do not earn any wages until you graduate from college. Assume that after graduating from college, you would work for 43 years at $24,000 per year. What would be the present value of your lifetime earnings? $346,441.

(c) Now calculate the present value of your lifetime earnings if you get a master's degree. Assume that if you get a master's, you have no earnings for 6 years and then you work for 41 years at $28,000 per year. What would be the present value of your lifetime income? $361,349.

(d) Finally calculate the present value of your lifetime earnings if you get a Ph.D. Assume that if you get a Ph.D., you will have no earnings for 8 years and then you work for 39 years at $40,000 per year. What would be the present value of your lifetime income? $460,712.

(e) Consider the case of someone who married right after finishing high school and stopped her education at that point. Suppose that she is now 45 years old. Her children are nearly adults, and she is thinking about going back to work or going to college. Assuming she would earn the average wage for her educational level and would retire at age 65, what would be the present value of her lifetime earnings if she does not go to college? $162,000.

(f) What would be the present value of her lifetime earnings if she goes to college for 4 years and then takes a job until she is 65? $213,990.

(g) If college tuition is $5,000 per year, is it financially worthwhile for her to go to college? Explain. Yes, the gain in the present value of her income exceeds the present value of tuition.

11.12 (0) As you may have noticed, economics is a difficult major. Are their any rewards for all this effort? The U.S. census publication discussed in the last problem suggests that there might be. There are tables reporting wage income by the field in which one gets a degree. For bachelor's degrees, the most lucrative majors are economics and engineering. The average wage incomes for economists are about $28,000 per year and for engineers are about $27,000. Psychology majors average about $15,000 a year and English majors about $14,000 per year.

(a) Can you think of any explanation for these differences? Some might say that economics, like accounting or mortuary science, is so boring the pay has to be high to get you to do it. Others would suggest that the ability to do well in economics is scarce and is valued by the marketplace. Perhaps the English majors and psychology majors include disproportionately many persons who are not full-time participants in the labor force. No doubt there are several other good partial explanations.

(b) The same table shows that the average person with an advanced degree in business earns $38,000 per year and the average person with a degree in medicine earns $45,000 per year. Suppose that an advanced degree in business takes 2 years after one spends 4 years getting a bachelor's degree and that a medical degree takes 4 years after getting a bachelor's

degree. Suppose that you are 22 years old and have just finished college. If $r = .05$, find the present value of lifetime earnings for a graduating senior who will get an advanced degree in business and earn the average wage rate for someone with this degree until retiring at 65. $596,000.

Make a similar calculation for medicine. $630,000.

11.13 (0) On the planet Stinko, the principal industry is turnip growing. For centuries the turnip fields have been fertilized by guano which was deposited by the now-extinct giant scissor-billed kiki-bird. It costs \$5 per ton to mine kiki-bird guano and deliver it to the fields. Unfortunately, the country's stock of kiki-bird guano is about to be exhausted. Fortunately the scientists on Stinko have devised a way of synthesizing kiki-guano from political science textbooks and swamp water. This method of production makes it possible to produce a product indistinguishable from kiki-guano and to deliver it to the turnip fields at a cost of \$30 per ton. The interest rate on Stinko is 10%. There are perfectly competitive markets for all commodities.

(a) Given the current price and the demand function for kiki-guano, the last of the deposits on Stinko will be exhausted exactly one year from now. Next year, the price of kiki-guano delivered to the fields will have to be \$30, so that the synthetic kiki-guano industry will just break even. The owners of the guano deposits know that next year, they would get a net return of \$25 a ton for any guano they have left to sell. In equilibrium, what must be the current price of kiki-guano delivered to the turnip fields?

The price of guano delivered to the field must be the \$5 + the present value of \$25. This is $5 + 25/1.1 = 27.73$. (Hint: In equilibrium, sellers must be indifferent between selling their kiki-guano right now or at any other time before the total supply is exhausted. But we know that they must be willing to sell it right up until the day, one year from now, when the supply will be exhausted and the price will be \$30, the cost of synthetic guano.)

(b) Suppose that everything is as we have said previously except that the deposits of kiki-guano will be exhausted 10 years from now. What must be the current price of kiki-guano? (Hint: $1.1^{10} = 2.59$.)

$5 + 25/(1.1)^{10} = 14.65$.

Chapter 12

NAME_____

Uncertainty

Introduction. In Chapter 11, you learned some tricks that allow you to use techniques you already know for studying intertemporal choice. Here you will learn some similar tricks, so that you can use the same methods to study risk taking, insurance, and gambling.

One of these new tricks is similar to the trick of treating commodities at different dates as different commodities. This time, we invent new commodities, which we call *contingent commodities.* If either of two events A or B could happen, then we define one contingent commodity as *consumption if A happens* and another contingent commodity as *consumption if B happens.* The second trick is to find a budget constraint that correctly specifies the set of contingent commodity bundles that a consumer can afford.

This chapter presents one other new idea, and that is the notion of von Neumann-Morgenstern utility. A consumer's willingness to take various gambles and his willingness to buy insurance will be determined by how he feels about various combinations of contingent commodities. Often it is reasonable to assume that these preferences can be expressed by a utility function that takes the special form known as *von Neumann-Morgenstern utility.* The assumption that utility takes this form is called the *expected utility hypothesis.* If there are two events, 1 and 2 with probabilities π_1 and π_2, and if the contingent consumptions are c_1 and c_2, then the von Neumann-Morgenstern utility function has the special functional form, $U(c_1, c_2) = \pi_1 u(c_1) + \pi_2 u(c_2)$. The consumer's behavior is determined by maximizing this utility function subject to his budget constraint.

Example: You are thinking of betting on whether the Cincinnati Reds will make it to the World Series this year. A local gambler will bet with you at odds of 10 to 1 against the Reds. You think the probability that the Reds will make it to the World Series is $\pi = .2$. If you don't bet, you are certain to have \$1,000 to spend on consumption goods. Your behavior satisfies the expected utility hypothesis and your von Neumann-Morgenstern utility function is $\pi_1\sqrt{c_1} + \pi_2\sqrt{c_2}$.

The contingent commodities are *dollars if the Reds make the World Series* and *dollars if the Reds don't make the World Series.* Let c_W be your consumption contingent on the Reds making the World Series and c_{NW} be your consumption contingent on their not making the Series. Betting on the Reds at odds of 10 to 1 means that if you bet $\$x$ on the Reds, then if the Reds make it to the Series, you make a net gain of $\$10x$, but if they don't, you have a net loss of $\$x$. Since you had \$1,000 before betting, if you bet $\$x$ on the Reds and they made it to the Series, you would have $c_W = 1,000 + 10x$ to spend on consumption. If you bet $\$x$ on the Reds and they didn't make it to the Series, you would lose $\$x$,

and you would have $c_{NW} = 1,000 - x$. By increasing the amount $\$x$ that you bet, you can make c_W larger and c_{NW} smaller. (You could also bet against the Reds at the same odds. If you bet $\$x$ against the Reds and they fail to make it to the Series, you make a net gain of $.1x$ and if they make it to the Series, you lose $\$x$. If you work through the rest of this discussion for the case where you bet against the Reds, you will see that the same equations apply, with x being a negative number.) We can use the above two equations to solve for a budget equation. From the second equation, we have $x = 1,000 - c_{NW}$. Substitute this expression for x into the first equation and rearrange terms to find $c_W + 10c_{NW} = 11,000$, or equivalently, $.1c_W + c_{NW} = 1,100$. (The same budget equation can be written in many equivalent ways by multiplying both sides by a positive constant.)

Then you will choose your contingent consumption bundle (c_W, c_{NW}) to maximize $U(c_W, c_{NW}) = .2\sqrt{c_W} + .8\sqrt{c_{NW}}$ subject to the budget constraint, $.1c_W + c_{NW} = 1,100$. Using techniques that are now familiar, you can solve this consumer problem. From the budget constraint, you see that consumption contingent on the Reds making the World Series costs 1/10 as much as consumption contingent on their not making it. If you set the marginal rate of substitution between c_W and c_{NW} equal to the price ratio and simplify the resulting expression, you will find that $c_{NW} = .16c_W$. This equation, together with the budget equation implies that $c_W = \$4,230.77$ and $c_{NW} = \$676.92$. You achieve this bundle by betting \$323.08 on the Reds. If the Reds make it to the Series, you will have $\$1,000 + 10 \times 323.08 = \$4,230.80$. If not, you will have \$676.92. (We rounded the solutions to the nearest penny.)

12.1 (0) In the next few weeks, Congress is going to decide whether or not to develop an expensive new weapons system. If the system is approved, it will be very profitable for the defense contractor, *General Statics*. Indeed, if the new system is approved, the value of stock in *General Statics* will rise from \$10 per share to \$15 a share, and if the project is not approved, the value of the stock will fall to \$5 a share. In his capacity as a messenger for Congressman Kickback, Buzz Condor has discovered that the weapons system is much more likely to be approved than is generally thought. On the basis of what he knows, Condor has decided that the probability that the system will be approved is 3/4 and the probability that it will not be approved is 1/4. Let c_A be Condor's consumption if the system is approved and c_{NA} be his consumption if the system is not approved. Condor's von Neumann-Morgenstern utility function is $U(c_A, c_{NA}) = .75 \ln c_A + .25 \ln c_{NA}$. Condor's total wealth is \$50,000, all of which is invested in perfectly safe assets. Condor is about to buy stock in *General Statics*.

(a) If Condor buys x shares of stock, and if the weapons system is approved, he will make a profit of \$5 per share. Thus the amount he can consume, contingent on the system being approved, is $c_A = \$50,000 + 5x$. If Condor buys x shares of stock, and if the weapons system is not approved, then he will make a loss of \$ 5 per share. Thus the

amount he can consume, contingent on the system not being approved, is $c_{NA} =$ $50,000 - 5x$.

(b) You can solve for Condor's budget constraint on contingent commodity bundles (c_A, c_{NA}) by eliminating x from these two equations. His budget constraint can be written as .5 $c_A +$ .5 $c_{NA} = 50,000$.

(c) Buzz Condor has no moral qualms about trading on inside information, nor does he have any concern that he will be caught and punished. To decide how much stock to buy, he simply maximizes his von Neumann-Morgenstern utility function subject to his budget. If he sets his marginal rate of substitution between the two contingent commodities equal to their relative prices and simplifies the equation, he finds that $c_A/c_{NA} =$ 3. (Reminder: Where a is any constant, the derivative of $a \ln x$ with respect to x is a/x.)

(d) Condor finds that his optimal contingent commodity bundle is $(c_A, c_{NA}) =$ (75,000,25,000). To acquire this contingent commodity bundle, he must buy 5,000 shares of stock in *General Statics*.

12.2 (0) Willy owns a small chocolate factory, located close to a river that occasionally floods in the spring, with disastrous consequences. Next summer, Willy plans to sell the factory and retire. The only income he will have is the proceeds of the sale of his factory. If there is no flood, the factory will be worth \$500,000. If there is a flood, then what is left of the factory will be worth only \$50,000. Willy can buy flood insurance at a cost of \$.10 for each \$1 worth of coverage. Willy thinks that the probability that there will be a flood this spring is 1/10. Let c_F denote the contingent commodity *dollars if there is a flood* and c_{NF} denote *dollars if there is no flood*. Willy's von Neumann-Morgenstern utility function is $U(c_F, c_{NF}) = .1\sqrt{c_F} + .9\sqrt{c_{NF}}$.

(a) If he buys no insurance, then in each contingency, Willy's consumption will equal the value of his factory, so Willy's contingent commodity bundle will be $(c_F, c_{NF}) =$ $(50,000, 500,000)$.

(b) To buy insurance that pays him \$$x$ in case of a flood, Willy must pay an insurance premium of $.1x$. (The insurance premium must be paid whether or not there is a flood.) If Willy insures for \$$x$, then if there is a flood, he gets \$$x$ in insurance benefits. Suppose that Willy has contracted for insurance that pays him \$$x$ in the event of a flood. Then after paying his insurance premium, he will be able to consume $c_F =$ $50,000 + .9x$. If Willy has this amount of insurance and there is no flood, then he will be able to consume $c_{NF} =$ $500,000 - .1x$.

(c) You can eliminate x from the two equations for c_F and c_{NF} that you found above. This gives you a budget equation for Willy. Of course there are many equivalent ways of writing the same budget equation, since multiplying both sides of a budget equation by a positive constant yields an equivalent budget equation. The form of the budget equation in which the "price" of c_{NF} is 1 can be written as $.9c_{NF} + .1 c_F =$ 455,000.

(d) Willy's marginal rate of substitution between the two contingent commodities, *dollars if there is no flood* and *dollars if there is a flood*, is $MRS(c_F, c_N F) = -\frac{.1\sqrt{c_{NF}}}{.9\sqrt{c_F}}$. To find his optimal bundle of contingent commodities, you must set this marginal rate of substitution equal to the number $= -9$. Solving this equation, you find that Willy will choose to consume the two contingent commodities in the ratio $c_{NF}/c_F = 1$.

(e) Since you know the ratio in which he will consume c_F and c_{NF}, and you know his budget equation, you can solve for his optimal consumption bundle, which is $(c_F, c_{NF}) = (455,000, 455,000)$. Willy will buy an insurance policy that will pay him \$450,000 if there is a flood. The amount of insurance premium that he will have to pay is \$45,000.

12.3 (0) Clarence Bunsen is an expected utility maximizer. His preferences among contingent commodity bundles are represented by the expected utility function

$$u(c_1, c_2, \pi_1, \pi_2) = \pi_1\sqrt{c_1} + \pi_2\sqrt{c_2}.$$

Clarence's friend, Hjalmer Ingqvist, has offered to bet him \$1,000 on the outcome of the toss of a coin. That is, if the coin comes up heads, Clarence must pay Hjalmer \$1,000 and if the coin comes up tails, Hjalmer must pay Clarence \$1,000. The coin is a fair coin, so that the probability of heads and the probability of tails are both 1/2. If he doesn't accept the bet, Clarence will have \$10,000 with certainty. In the privacy of his car dealership office over at Bunsen Motors, Clarence is making his decision. (Clarence uses the pocket calculator that his son, Elmer, gave him last Christmas. You will find that it will be helpful for you to use a calculator too.) Let Event 1 be "coin comes up heads" and let Event 2 be "coin comes up tails."

(a) If Clarence accepts the bet, then in Event 1, he will have 9,000 dollars and in Event 2, he will have 11,000 dollars.

(b) Since the probability of each event is 1/2, Clarence's expected utility for a gamble in which he gets c_1 in Event 1 and c_2 in Event 2 can be described by the formula $\frac{1}{2}\sqrt{c_1} + \frac{1}{2}\sqrt{c_2}$. Therefore Clarence's expected utility if he accepts the bet with Hjalmer will be 99.8746. (Use that calculator.)

(c) If Clarence decides not to bet, then in Event 1, he will have 10,000 dollars and in Event 2, he will have 10,000 dollars. Therefore if he doesn't bet, his expected utility will be 100.

(d) Having calculated his expected utility if he bets and if he does not bet, Clarence determines which is higher and makes his decision accordingly. Does Clarence take the bet? No.

12.4 (0) It is a slow day at Bunsen Motors, so since he has his calculator warmed up, Clarence Bunsen (whose preferences toward risk were described in the last problem) decides to study his expected utility function more closely.

(a) Clarence first thinks about really *big* gambles. What if he bet his entire $10,000 on the toss of a coin, where he loses if heads and wins if tails? Then if the coin came up heads, he would have 0 dollars and if it came up tails, he would have $20,000. His expected utility if he took the bet would be 70.71 , while his expected utility if he didn't take the bet would be 100. Therefore he concludes that he would not take such a bet.

(b) Clarence then thinks, "Well, of course, I wouldn't want to take a chance on losing all of my money on just an ordinary bet. But, what if somebody offered me a really good deal. Suppose I had a chance to bet where if a fair coin came up heads, I lost my $10,000, but if it came up tails, I would win $50,000. Would I take the bet? If I took the bet, my expected utility would be 122.5. If I didn't take the bet, my expected utility would be 100. Therefore I should take the bet."

(c) Clarence later asks himself, "If I make a bet where I lose my \$10,000 if the coin comes up heads, what is the smallest amount that I would have to win in the event of tails in order to make the bet a good one for me to take?" After some trial and error, Clarence found the answer. You, too, might want to find the answer by trial and error, but it is easier to find the answer by solving an equation. On the left side of your equation, you would write down Clarence's utility if he doesn't bet. On the right side of the equation, you write down an expression for Clarence's utility if he makes a bet such that he is left with zero consumption in Event 1 and x in Event 2. Solve this equation for x. The answer to Clarence's question is where $x = 10,000$. The equation that you should write is $100 = \frac{1}{2}\sqrt{x}$. The solution is $x =$ 40,000.

(d) Your answer to the last part gives you two points on Clarence's indifference curve between the contingent commodities, money in Event 1 and money in Event 2. (Poor Clarence has never heard of indifference curves or contingent commodities, so you will have to work this part for him, while he heads over to the Chatterbox Cafe for morning coffee.) One of these points is where money in both events is \$10,000. On the graph below, label this point A. The other is where money in Event 1 is zero and money in Event 2 is 40,000. On the graph below, label this point B.

(e) You can quickly find a third point on this indifference curve. The coin is a fair coin, and Clarence cares whether heads or tails turn up only because that determines his prize. Therefore Clarence will be indifferent between two gambles that are the same except that the assignment of prizes to outcomes are reversed. In this example, Clarence will be indifferent between point B on the graph and a point in which he gets zero if Event 2 happens and 40,000 if Event 1 happens. Find this point on the Figure above and label it C.

(f) Another gamble that is on the same indifference curve for Clarence as not gambling at all is the gamble where he loses \$5,000 if heads turn up and where he wins 6,715.73 dollars if tails turn up. (Hint: To solve this problem, put the utility of not betting on the left side of an equation and on the right side of the equation, put the utility of having $\$10,000 - \$5,000$ in Event 1 and $\$10,000 + x$ in Event 2. Then solve the resulting equation for x.) On the axes above, plot this point and label it D. Now sketch in the entire indifference curve through the points that you have labeled.

12.5 (0) Hjalmer Ingqvist's son-in-law, Earl, has not worked out very well. It turns out that Earl likes to gamble. His preferences over contingent commodity bundles are represented by the expected utility function

$$u(c_1, c_2, \pi_1, \pi_2) = \pi_1 c_1^2 + \pi_2 c_2^2.$$

(a) Just the other day, some of the boys were down at Skoog's tavern when Earl stopped in. They got to talking about just how bad a bet they could get him to take. At the time, Earl had \$100. Kenny Olson shuffled a deck of cards and offered to bet Earl \$20 that Earl would not cut a spade from the deck. Assuming that Earl believed that Kenny wouldn't cheat, the probability that Earl would win the bet was 1/4 and the probability that Earl would lose the bet was 3/4. If he won the bet, Earl would have 120 dollars and if he lost the bet, he would have 80 dollars. Earl's expected utility if he took the bet would be 8,400, and his expected utility if he did not take the bet would be 10,000. Therefore he refused the bet.

(b) Just when they started to think Earl might have changed his ways, Kenny offered to make the same bet with Earl except that they would bet \$100 instead of \$20. What is Earl's expected utility if he takes that bet? 10,000. Would Earl be willing to take this bet? He is just indifferent about taking it or not.

(c) Let Event 1 be the event that a card drawn from a fair deck of cards is a spade. Let Event 2 be the event that the card is not a spade. Earl's preferences between income contingent on Event 1, c_1, and income contingent on Event 2, c_2, can be represented by the equation $u = \frac{1}{4}c_1^2 + \frac{3}{4}c_2^2$. Use blue ink on the graph below to sketch Earl's indifference curve passing through the point $(100, 100)$.

(d) On the same graph, let us draw Hjalmer's son-in-law Earl's indifference curves between contingent commodities where the probabilities are different. Suppose that a card is drawn from a fair deck of cards. Let Event 1 be the event that the card is black. Let event 2 be the event that the card drawn is red. Suppose each event has probability 1/2. Then Earl's preferences between income contingent on Event 1 and income contingent on Event 2 are represented by the formula $u = \frac{1}{2}c_1^2 + \frac{1}{2}c_2^2$. On the graph, use red ink to show two of Earl's indifference curves, including the one that passes through $(100, 100)$.

12.6 (1) Sidewalk Sam makes his living selling sunglasses at the boardwalk in Atlantic City. If the sun shines Sam makes \$30, and if it rains Sam only makes \$10. For simplicity, we will suppose that there are only two kinds of days, sunny ones and rainy ones.

(a) One of the casinos in Atlantic City has a new gimmick. It is accepting bets on whether it will be sunny or rainy the next day. The casino sells dated "rain coupons" for \$1 each. If it rains the next day, the casino will give you \$2 for every rain coupon you bought on the previous day. If it doesn't rain, your rain coupon is worthless. In the graph below, mark Sam's "endowment" of contingent consumption if he makes no bets with the casino, and label it E.

(b) On the same graph, mark the combination of consumption contingent on rain and consumption contingent on sun that he could achieve by buying 10 rain coupons from the casino. Label it A.

(c) On the same graph, use blue ink to draw the budget line representing all of the other patterns of consumption that Sam can achieve by buying rain coupons. (Assume that he can buy fractional coupons, but not negative amounts of them.) What is the slope of Sam's budget line at points above and to the left of his initial endowment? The slope is -1.

(d) Suppose that the casino also sells sunshine coupons. These tickets also cost \$1. With these tickets, the casino gives you \$2 if it doesn't rain and nothing if it does. On the graph above, use red ink to sketch in the budget line of contingent consumption bundles that Sam can achieve by buying sunshine tickets.

(e) If the price of a dollar's worth of consumption when it rains is set equal to 1, what is the price of a dollar's worth of consumption if it shines?

The price is 1.

12.7 (0) Sidewalk Sam, from the previous problem, has the utility function for consumption in the two states of nature

$$u(c_s, c_r, \pi) = c_s^{1-\pi} c_r^{\pi},$$

where c_s is the dollar value of his consumption if it shines, c_r is the dollar value of his consumption if it rains, and π is the probability that it will rain. The probability that it will rain is $\pi = .5$.

(a) How many units of consumption is it optimal for Sam to consume conditional on rain? 20 units.

(b) How many rain coupons is it optimal for Sam to buy? 10.

12.8 (0) Sidewalk Sam's brother Morgan von Neumanstern is an expected utility maximizer. His von Neumann-Morgenstern utility function for wealth is $u(c) = \ln c$. Sam's brother also sells sunglasses on another beach in Atlantic City and makes exactly the same income as Sam does. He can make exactly the same deal with the casino as Sam can.

(a) If Morgan believes that there is a 50% chance of rain and a 50% chance of sun every day, what would his expected utility of consuming (c_s, c_r) be? $u = \frac{1}{2}\ln c_s + \frac{1}{2}\ln c_r$.

(b) How does Morgan's utility function compare to Sam's? Is one a monotonic transformation of the other? Morgan's utility function is just the natural log of Sam's, so the answer is yes.

(c) What will Morgan's optimal pattern of consumption be? Answer: Morgan will consume 20 on the sunny days and 20 on the rainy days. How does this compare to Sam's consumption? This is the same as Sam's consumption.

12.9 (0) Billy John Pigskin of Mule Shoe, Texas, has a von Neumann-Morgenstern utility function of the form $u(c) = \sqrt{c}$. Billy John also weighs about 300 pounds and can outrun jackrabbits and pizza delivery trucks. Billy John is beginning his senior year of college football. If he is not seriously injured, he will receive a \$1,000,000 contract for playing professional football. If an injury ends his football career, he will receive a \$10,000 contract as a refuse removal facilitator in his home town. There is a 10% chance that Billy John will be injured badly enough to end his career.

(a) What is Billy John's expected utility? We calculate $.1\sqrt{10,000} + .9\sqrt{1,000,000} = 910$.

(b) If Billy John pays \$$p$ for an insurance policy that would give him \$1,000,000 if he suffered a career-ending injury while in college, then he would be sure to have an income of $\$1,000,000 - p$ no matter what happened to him. Write an equation that can be solved to find the largest price that Billy John would be willing to pay for such an insurance policy.

The equation is $910 = \sqrt{1,000,000 - p}$.

(c) Solve this equation for p. $p = 171,900$.

12.10 (1) You have \$200 and are thinking about betting on the Big Game next Saturday. Your team, the Golden Boars, are scheduled to play their traditional rivals the Robber Barons. It appears that the going odds are 2 to 1 against the Golden Boars. That is to say if you want to bet \$10 on the Boars, you can find someone who will agree to pay you \$20 if the Boars win in return for your promise to pay him \$10 if the Robber Barons win. Similarly if you want to bet \$10 on the Robber Barons, you can find someone who will pay you \$10 if the Robber Barons win, in return for your promise to pay him \$20 if the Robber Barons lose. Suppose that you are able to make as large a bet as you like, either on the Boars or on the Robber Barons so long as your gambling losses do not exceed \$200. (To avoid tedium, let us ignore the possibility of ties.)

(a) If you do not bet at all, you will have \$200 whether or not the Boars win. If you bet \$50 on the Boars, then after all gambling obligations are settled, you will have a total of 300 dollars if the Boars win and 150 dollars if they lose. On the graph below, use blue ink to draw a line that represents all of the combinations of "money if the Boars win" and "money if the Robber Barons win" that you could have by betting from your initial \$200 at these odds.

(b) Label the point on this graph where you would be if you did not bet at all with an E.

(c) After careful thought you decide to bet \$50 on the Boars. Label the point you have chosen on the graph with a C. Suppose that after you have made this bet, it is announced that the star Robber Baron quarterback suffered a sprained thumb during a tough economics midterm examination and will miss the game. The market odds shift from 2 to 1 against the Boars to "even money" or 1 to 1. That is, you can now bet on either team and the amount you would win if you bet on the winning team is the same as the amount that you would lose if you bet on the losing team. You cannot cancel your original bet, but you can make new bets at the new odds. Suppose that you keep your first bet, but you now also bet \$50 on the Robber Barons at the new odds. If the Boars win, then after you collect your winnings from one bet and your losses from the other, how much money will you have left? \$250. If the Robber Barons win, how much money will you have left after collecting your winnings and paying off your losses? \$200.

(d) Use red ink to draw a line on the diagram you made above, showing the combinations of "money if the Boars win" and "money if the Robber Barons win" that you could arrange for yourself by adding possible bets at the new odds to the bet you made before the news of the quarterback's misfortune. On this graph, label the point D that you reached by making the two bets discussed above.

12.11 (2) The *certainty equivalent* of a lottery is the amount of money you would have to be given with certainty to be just as well-off with that lottery. Suppose that your von Neumann-Morgenstern utility function

over lotteries that give you an amount x if Event 1 happens and y if Event 1 does not happen is $U(x, y, \pi) = \pi\sqrt{x} + (1-\pi)\sqrt{y}$, where π is the probability that Event 1 happens and $1-\pi$ is the probability that Event 1 does not happen.

(a) If $\pi = .5$, calculate the utility of a lottery that gives you \$10,000 if Event 1 happens and \$100 if Event 1 does not happen. $55 = .5 \times 100 + .5 \times 10.$

(b) If you were sure to receive \$4,900, what would your utility be? 70. (Hint: If you receive \$4,900 with certainty, then you receive \$4,900 in both events.)

(c) Given this utility function and $\pi = .5$, write a general formula for the certainty equivalent of a lottery that gives you \$$x$ if Event 1 happens and \$$y$ if Event 1 does not happen. $(.5x^{1/2} + .5y^{1/2})^2$.

(d) Calculate the certainty equivalent of receiving \$10,000 if Event 1 happens and \$100 if Event 1 does not happen. \$3,025.

12.12 (0) Dan Partridge is a risk averter who tries to maximize the expected value of $\sqrt{c}$, where c is his wealth. Dan has \$50,000 in safe assets and he also owns a house that is located in an area where there are lots of forest fires. If his house burns down, the remains of his house and the lot it is built on would be worth only \$40,000, giving him a total wealth of \$90,000. If his home doesn't burn, it will be worth \$200,000 and his total wealth will be \$250,000. The probability that his home will burn down is .01.

(a) Calculate his expected utility if he doesn't buy fire insurance.

\$498.

(b) Calculate the certainty equivalent of the lottery he faces if he doesn't buy fire insurance. \$248,004.

(c) Suppose that he can buy insurance at a price of \$1 per \$100 of insurance. For example if he buys \$100,000 worth of insurance, he will pay \$1,000 to the company no matter what happens, but if his house burns, he will also receive \$100,000 from the company. If Dan buys \$160,000 worth of insurance, he will be fully insured in the sense that no matter what happens his after-tax wealth will be \$248,400.

(d) Therefore if he buys full insurance, the certainty equivalent of his wealth is $248,400 , and his expected utility is $\sqrt{248,800}$.

12.13 (1) Portia has been waiting a long time for her ship to come in and has concluded that there is a 25% chance that it will arrive today. If it does come in today, she will receive $1,600. If it does not come in today, it will never come and her wealth will be zero. Portia has a von Neumann-Morgenstern utility such that she wants to maximize the expected value of $\sqrt{c}$, where c is total wealth. What is the minimum price at which she will sell the rights to her ship? $100.

Chapter 13

NAME_____

Risky Assets

Introduction. Here you will solve the problems of consumers who wish to divide their wealth optimally between a risky asset and a safe asset. The expected rate of return on a portfolio is just a weighted average of the rate of return on the safe asset and the expected rate of return on the risky asset, where the weights are the fractions of the consumer's wealth held in each. The standard deviation of the portfolio return is just the standard deviation of the return on the risky asset times the fraction of the consumer's wealth held in the risky asset. Sometimes you will look at the problem of a consumer who has preferences over the expected return and the risk of her portfolio and who faces a budget constraint. Since a consumer can always put all of her wealth in the safe asset, one point on this budget constraint will be the combination of the safe rate of return and no risk (zero standard deviation). Now as the consumer puts x percent of her wealth into the risky asset, she gains on that amount the difference between the expected rate of return for the risky asset and the rate of return on the safe asset. But she also absorbs some risk. So the slope of the budget line will be the difference between the two returns divided by the standard deviation of the portfolio that has x percent of the consumer's wealth invested in the risky asset. You can then apply the usual indifference curve–budget line analysis to find the consumer's optimal choice of risk and expected return given her preferences. (Remember that if the standard deviation is plotted on the horizontal axis and if less risk is preferred to more, the better bundles will lie to the northwest.) You will also be asked to apply the result from the Capital Asset Pricing Model that the expected rate of return on any asset is equal to the sum of the risk-free rate of return plus the risk adjustment. Remember too that the expected rate of return on an asset is its expected change in price divided by its current price.

13.1 (3) Ms. Lynch has a choice of two assets: The first is a risk-free asset that offers a rate of return of r_f, and the second is a risky asset (a china shop that caters to large mammals) that has an expected rate of return of r_m and a standard deviation of σ_m.

(a) If x is the percent of wealth Ms. Lynch invests in the risky asset, what is the equation for the expected rate of return on the portfolio? $r_x = xr_m + (1-x)r_f$. What is the equation for the standard deviation of the portfolio? $\sigma_x = x\sigma_m$.

(b) By solving the second equation above for x and substituting the result into the first equation, derive an expression for the rate of return on the portfolio in terms of the portfolio's riskiness. $r_x = \frac{r_m - r_f}{\sigma_m}\sigma_x + r_f$.

(c) Suppose that Ms. Lynch can borrow money at the interest rate r_f and invest it in the risky asset. If $r_m = 20$, $r_f = 10$, and $\sigma_m = 10$, what will be Ms. Lynch's expected return if she borrows an amount equal to 100% of her initial wealth and invests it in the risky asset? (Hint: This is just like investing 200% of her wealth in the risky asset.) Apply the formula $r_x = xr_m + (1-x)r_f$ with $x = 2$ to get $r_x = 2 \times 20 - 1 \times 10 = 30$.

(d) Suppose that Ms. Lynch can borrow or lend at the risk-free rate. If r_f is 10%, r_m is 20%, and σ_m is 10%, what is the formula for the "budget line" Ms. Lynch faces? $r_x = \sigma_x + 10$. Plot this budget line in the graph below.

(e) Which of the following risky assets would Ms. Lynch prefer to her present risky asset, assuming she can only invest in one risky asset at a time and that she can invest a fraction of her wealth in whichever risky asset she chooses? Write the word "better," "worse," or "same" after each of the assets.

Asset A with r_a =17% and σ_a = 5%. Better.

Asset B with r_b =30% and σ_b = 25%. Worse.

Asset C with r_c =11% and $\sigma_c = 1\%$. Same.

Asset D with r_d =25% and $\sigma_d = 14\%$. Better.

(f) Suppose Ms. Lynch's utility function has the form $u(r_x, \sigma_x) = r_x - 2\sigma_x$. How much of her portfolio will she invest in the original risky asset? (You might want to graph a few of Ms. Lynch's indifference curves before answering; e.g., graph the combinations of r_x and σ_x that imply $u(r_x, \sigma_x) = 0, 1, \ldots$). She will not invest anything in the risky asset.

13.2 (3) Fenner Smith is contemplating dividing his portfolio between two assets, a risky asset that has an expected return of 30% and a standard deviation of 10%, and a safe asset that has an expected return of 10% and a standard deviation of 0%.

(a) If Mr. Smith invests x percent of his wealth in the risky asset, what will be his expected return? $r_x = 30x + 10(1 - x)$.

(b) If Mr. Smith invests x percent of his wealth in the risky asset, what will be the standard deviation of his wealth? $\sigma_x = 10x$.

(c) Solve the above two equations for the expected return on Mr. Smith's wealth as a function of the standard deviation he accepts. The budget line is $r_x = 2\sigma_x + 10$.

(d) Plot this "budget line" on the graph below.

(e) If Mr. Smith's utility function is $u(r_x, \sigma_x) = \min\{r_x, 30 - 2\sigma_x\}$, then Mr. Smith's optimal value of r_x is 20, and his optimal value of σ_x is 5. (Hint: You will need to solve two equations in two unknowns. One of the equations is the budget constraint.)

(f) Plot Mr. Smith's optimal choice and an indifference curve through it in the graph.

(g) What fraction of his wealth should Mr. Smith invest in the risky asset?

Using the answer to Part (a), we find an x that solves $20 = r_x = 30x + 10(1 - x)$. The answer is $x = .5$.

13.3 (2) Assuming that the Capital Asset Pricing Model is valid, complete the following table. In this table p_0 is the current price of asset i and Ep_1 is the expected price of asset i in the next period.

r_f	r_m	r_i	β_i	p_0	Ep_1
10	20	10	0	100	110
10	20	25	1.5	100	125
10	15	20	2	200	240
0	30	20	2/3	40	48
10	22	10	0	80	88

13.4 (2) Farmer Alf Alpha has a pasture located on a sandy hill. The return to him from this pasture is a random variable depending on how much rain there is. In rainy years the yield is good; in dry years the yield is poor. The market value of this pasture is \$5,000. The expected return from this pasture is \$500 with a standard deviation of \$100. Every inch of rain above average means an extra \$100 in profit and every inch of rain below average means another \$100 less profit than average. Farmer Alf has another \$5,000 that he wants to invest in a second pasture. There are two possible pastures that he can buy.

(a) One is located on low land that never floods. This pasture yields an expected return of \$500 per year no matter what the weather is like. What is Alf Alpha's expected rate of return on his *total* investment if he buys this pasture for his second pasture? 10%. What is the standard deviation of his rate of return in this case? 10%.

(b) Another pasture that he could buy is located on the very edge of the river. This gives very good yields in dry years but in wet years it floods. This pasture also costs $5,000. The expected return from this pasture is $500 and the standard deviation is $100. Every inch of rain *below* average means an extra $100 in profit and every inch of rain above average means another $100 less profit than average. If Alf buys this pasture and keeps his original pasture on the sandy hill, what is his expected rate of return on his total investment? 10%. What is the standard deviation of the rate of return on his total investment in this case? 0%.

(c) If Alf is a risk averter, which of these two pastures should he buy and why? He should choose the second pasture since it has the same expected return and lower risk.

Chapter 14

NAME_____

Consumer's Surplus

Introduction. In this chapter you will study ways to measure a consumer's valuation of a good given the consumer's demand curve for it. The basic logic is as follows: The height of the demand curve measures how much the consumer is willing to pay for the last unit of the good purchased—the willingness to pay for the marginal unit. Therefore the sum of the willingnesses-to-pay for each unit gives us the total willingness to pay for the consumption of the good.

In geometric terms, the total willingness to pay to consume some amount of the good is just the area under the demand curve up to that amount. This area is called **gross consumer's surplus** or **total benefit** of the consumption of the good. If the consumer has to pay some amount in order to purchase the good, then we must subtract this expenditure in order to calculate the **(net) consumer's surplus**.

When the utility function takes the quasilinear form, $u(x) + m$, the area under the demand curve measures $u(x)$, and the area under the demand curve minus the expenditure on the other good measures $u(x) + m$. Thus in this case, consumer's surplus serves as an exact measure of utility, and the change in consumer's surplus is a monetary measure of a change in utility.

If the utility function has a different form, consumer's surplus will not be an exact measure of utility, but it will often be a good approximation. However, if we want more exact measures, we can use the ideas of the **compensating variation** and the **equivalent variation.**

Recall that the compensating variation is the amount of extra income that the consumer would need at the *new* prices to be as well off as she was facing the old prices; the equivalent variation is the amount of money that it would be necessary to take away from the consumer at the old prices to make her as well off as she would be, facing the new prices. Although different in general, the change in consumer's surplus and the compensating and equivalent variations will be the same if preferences are quasilinear.

In this chapter you will practice:

- Calculating consumer's surplus and the change in consumer's surplus
- Calculating compensating and equivalent variations

Example: Suppose that the inverse demand curve is given by $P(q) = 100 - 10q$ and that the consumer currently has 5 units of the good. How much money would you have to pay him to compensate him for reducing his consumption of the good to zero?

Answer: The inverse demand curve has a height of 100 when $q = 0$ and a height of 50 when $q = 5$. The area under the demand curve is a trapezoid with a base of 5 and heights of 100 and 50. We can calculate

the area of this trapezoid by applying the formula

$$\text{Area of a trapezoid} \quad = \quad \text{base} \quad \times \frac{1}{2}(\text{height}_1 + \text{height}_2).$$

In this case we have $A = 5 \times \frac{1}{2}(100 + 50) = \$375$.

Example: Suppose now that the consumer is purchasing the 5 units at a price of \$50 per unit. If you require him to reduce his purchases to zero, how much money would be necessary to compensate him?

In this case, we saw above that his gross benefits decline by \$375. On the other hand, he has to spend $5 \times 50 = \$250$ less. The decline in *net* surplus is therefore \$125.

Example: Suppose that a consumer has a utility function $u(x_1, x_2) = x_1 + x_2$. Initially the consumer faces prices $(1, 2)$ and has income 10. If the prices change to $(4, 2)$, calculate the compensating and equivalent variations.

Answer: Since the two goods are perfect substitutes, the consumer will initially consume the bundle $(10, 0)$ and get a utility of 10. After the prices change, she will consume the bundle $(0, 5)$ and get a utility of 5. After the price change she would need \$20 to get a utility of 10; therefore the compensating variation is $20 - 10 = 10$. Before the price change, she would need an income of 5 to get a utility of 5. Therefore the equivalent variation is $10 - 5 = 5$.

14.1 (0) Sir Plus consumes mead, and his demand function for tankards of mead is given by $D(p) = 100 - p$, where p is the price of mead in shillings.

(a) If the price of mead is 50 shillings per tankard, how many tankards of mead will he consume? 50.

(b) How much gross consumer's surplus does he get from this consumption? 3,750.

(c) How much money does he spend on mead? 2,500.

(d) What is his net consumer's surplus from mead consumption?

1,250.

14.2 (0) Here is the table of reservation prices for apartments taken from Chapter 1:

Person	=	A	B	C	D	E	F	G	H
Price	=	40	25	30	35	10	18	15	5

(a) If the equilibrium rent for an apartment turns out to be \$20, which consumers will get apartments? A,B,C,D.

(b) If the equilibrium rent for an apartment turns out to be \$20, what is the consumer's (net) surplus generated in this market for person A?

20. For person B? 5.

(c) If the equilibrium rent is \$20, what is the total net consumers' surplus generated in the market? 50.

(d) If the equilibrium rent is \$20, what is the total gross consumers' surplus in the market? 130.

(e) If the rent declines to \$19, how much does the gross surplus increase?

0.

(f) If the rent declines to \$19, how much does the net surplus increase?

4.

Calculus **14.3 (0)** Quasimodo consumes earplugs and other things. His utility function for earplugs x and money to spend on other goods y is given by

$$u(x, y) = 100x - \frac{x^2}{2} + y.$$

(a) What kind of utility function does Quasimodo have? Quasilinear.

(b) What is his inverse demand curve for earplugs? $p = 100 - x$.

(c) If the price of earplugs is \$50, how many earplugs will he consume?

50.

(d) If the price of earplugs is \$80, how many earplugs will he consume?

20.

(e) Suppose that Quasimodo has \$4,000 in total to spend a month. What is his total utility for earplugs and money to spend on other things if the price of earplugs is \$50? \$5,250.

(f) What is his total utility for earplugs and other things if the price of earplugs is \$80? $4,200.

(g) Utility decreases by 1,050 when the price changes from \$50 to \$80.

(h) What is the change in (net) consumer's surplus when the price changes from \$50 to \$80? 1,050.

14.4 (2) In the graph below, you see a representation of Sarah Gamp's indifference curves between cucumbers and other goods. Suppose that the reference price of cucumbers and the reference price of "other goods" are both 1.

(a) What is the minimum amount of money that Sarah would need in order to purchase a bundle that is indifferent to A? 20.

(b) What is the minimum amount of money that Sarah would need in order to purchase a bundle that is indifferent to B? 30.

(c) Suppose that the reference price for cucumbers is 2 and the reference price for other goods is 1. How much money does she need in order to purchase a bundle that is indifferent to bundle A? 30.

(d) What is the minimum amount of money that Sarah would need to purchase a bundle that is indifferent to B using these new prices? 40.

(e) No matter what prices Sarah faces, the amount of money she needs to purchase a bundle indifferent to A must be (higher, lower) than the amount she needs to purchase a bundle indifferent to B. lower.

14.5 (2) Bernice's preferences can be represented by $u(x, y) = \min\{x, y\}$, where x is pairs of earrings and y is dollars to spend on other things. She faces prices $(p_x, p_y) = (2, 1)$ and her income is 12.

(a) Draw in pencil on the graph below some of Bernice's indifference curves and her budget constraint. Her optimal bundle is 4 pairs of earrings and 4 dollars to spend on other things.

(b) The price of a pair of earrings rises to \$3 and Bernice's income stays the same. Using blue ink, draw her new budget constraint on the graph above. Her new optimal bundle is 3 pairs of earrings and 3 dollars to spend on other things.

(c) What bundle would Bernice choose if she faced the original prices and had just enough income to reach the new indifference curve? $(3, 3)$. Draw with red ink the budget line that passes through this bundle at the original prices. How much income would Bernice need at the original prices to have this (red) budget line? \$9.

(d) The maximum amount that Bernice would pay to avoid the price increase is $3. This is the (compensating, equivalent) variation in income. Equivalent.

(e) What bundle would Bernice choose if she faced the new prices and had just enough income to reach her original indifference curve? $(4, 4)$. Draw with black ink the budget line that passes through this bundle at the new prices. How much income would Bernice have with this budget? $16.

(f) In order to be as well-off as she was with her original bundle, Bernice's original income would have to rise by $4. This is the (compensating, equivalent) variation in income. Compensating.

Calculus **14.6 (0)** Ulrich likes video games and sausages. In fact, his preferences can be represented by $u(x, y) = \ln(x + 1) + y$ where x is the number of video games he plays and y is the number of dollars that he spends on sausages. Let p_x be the price of a video game and m be his income.

(a) Write an expression that says that Ulrich's marginal rate of substitution equals the price ratio. (Hint: Remember Donald Fribble from Chapter 6?) $1/(x + 1) = p_x$.

(b) Since Ulrich has quasilinear preferences, you can solve this equation alone to get his demand function for video games, which is $x = 1/p_x - 1$. His demand function for the dollars to spend on sausages is $y = m - 1 + p$.

(c) Video games cost \$.25 and Ulrich's income is \$10. Then Ulrich demands 3 video games and 9.25 dollars' worth of sausages. His utility from this bundle is 10.64. (Round off to two decimal places.)

(d) If we took away all of Ulrich's video games, how much money would he need to have to spend on sausages to be just as well-off as before?

$10.64.

(e) Now an amusement tax of \$.25 is put on video games and is passed on in full to consumers. With the tax in place, Ulrich demands 1 video game and 9.5 dollars' worth of sausages. His utility from this bundle is 10.19. (Round off to two decimal places.)

(f) Now if we took away all of Ulrich's video games, how much money would he have to have to spend on sausages to be just as well-off as with the bundle he purchased after the tax was in place? \$10.19.

(g) What is the change in Ulrich's consumer surplus due to the tax? $-.45$ How much money did the government collect from Ulrich by means of the tax? \$.25.

Calculus **14.7 (1)** Lolita, an intelligent and charming Holstein cow, consumes only two goods, cow feed (made of ground corn and oats) and hay. Her preferences are represented by the utility function $U(x, y) = x - x^2/2 + y$, where x is her consumption of cow feed and y is her consumption of hay. Lolita has been instructed in the mysteries of budgets and optimization and always maximizes her utility subject to her budget constraint. Lolita has an income of $\$m$ that she is allowed to spend as she wishes on cow feed and hay. The price of hay is always \$1, and the price of cow feed will be denoted by p, where $0 < p \leq 1$.

(a) Write Lolita's inverse demand function for cow feed. (Hint: Lolita's utility function is quasilinear. When y is the numeraire and the price of x is p, the inverse demand function for someone with quasilinear utility $f(x) + y$ is found by simply setting $p = f'(x)$.) $p = 1 - x$.

(b) If the price of cow feed is p and her income is m, how much hay does Lolita choose? (Hint: The money that she doesn't spend on feed is used to buy hay.) $m - p(1 - p)$.

(c) Plug these numbers into her utility function to find out the utility level that she enjoys at this price and this income. $u = m + (1-p)^2/2$.

(d) Suppose that Lolita's daily income is \$3 and that the price of feed is \$.50. What bundle does she buy? $(1/2, 11/4)$. What bundle would she buy if the price of cow feed rose to \$1? $(0, 3)$.

(e) How much money would Lolita be willing to pay to avoid having the price of cow feed rise to \$1? $1/8$. This amount is known as the **equivalent** variation.

(f) Suppose that the price of cow feed rose to \$1. How much extra money would you have to pay Lolita to make her as well-off as she was at the old prices? $1/8$. This amount is known as the **compensating** variation. Which is bigger, the compensating or the equivalent variation, or are they the same? **Same.**

(g) At the price \$.50 and income \$3, how much (net) consumer's surplus is Lolita getting? $1/8$.

14.8 (2) F. Flintstone has quasilinear preferences and his inverse demand function for Brontosaurus Burgers is $P(b) = 30 - 2b$. Mr. Flintstone is currently consuming 10 burgers at a price of 10 dollars.

(a) How much money would he be willing to pay to have this amount rather than no burgers at all? **\$200.** What is his level of (net) consumer's surplus? **\$100.**

(b) The town of Bedrock, the only supplier of Brontosaurus Burgers, decides to raise the price from \$10 a burger to \$14 a burger. What is Mr. Flintstone's change in consumer's surplus? **At price \$10, consumer's surplus is \$100. At \$14, he demands 8 burgers, for net consumer's surplus of** $\frac{1}{2}(16 \times 8) = 64$. **The change in consumer's surplus is** $-\$36$.

14.9 (1) Karl Kapitalist is willing to produce $p/2 - 20$ chairs at every price, $p > 40$. At prices below 40, he will produce nothing. If the price of chairs is \$100, Karl will produce **30** chairs. At this price, how much is his producer's surplus? $\frac{1}{2}(60 \times 30) = 900$.

14.10 (2) Ms. Q. Moto loves to ring the church bells for up to 10 hours a day. Where m is expenditure on other goods, and x is hours of bell ringing, her utility is $u(m, x) = m + 3x$ for $x \leq 10$. If $x > 10$, she develops painful blisters and is worse off than if she didn't ring the bells.

Her income is equal to $100 and the sexton allows her to ring the bell for 10 hours.

(a) Due to complaints from the villagers, the sexton has decided to restrict Ms. Moto to 5 hours of bell ringing per day. This is bad news for Ms. Moto. In fact she regards it as just as bad as losing $15 dollars of income.

(b) The sexton relents and offers to let her ring the bells as much as she likes so long as she pays $2 per hour for the privilege. How much ringing does she do now? 10 hours. This tax on her activities is as bad as a loss of how much income? $20.

(c) The villagers continue to complain. The sexton raises the price of bell ringing to $4 an hour. How much ringing does she do now? 0 hours. This tax, as compared to the situation in which she could ring the bells for free, is as bad as a loss of how much income? $30.

Chapter 15

NAME_____

Market Demand

Introduction. Some problems in this chapter will ask you to construct the market demand curve from individual demand curves. The market demand at any given price is simply the sum of the individual demands at that price. The key thing to remember in going from individual demands to the market demand is to *add quantities*. Graphically, you sum the individual demands horizontally to get the market demand. The market demand curve will have a kink in it whenever the market price is high enough that some individual demand becomes zero.

Sometimes you will need to find a consumer's reservation price for a good. Recall that the reservation price is the price that makes the consumer indifferent between having the good at that price and not having the good at all. Mathematically, the reservation price p^* satisfies $u(0, m) = u(1, m - p^*)$, where m is income and the quantity of the other good is measured in dollars.

Finally, some of the problems ask you to calculate price and/or income elasticities of demand. These problems are especially easy if you know a little calculus. If the demand function is $D(p)$, and you want to calculate the price elasticity of demand when the price is p, you only need to calculate $dD(p)/dp$ and multiply it by p/q.

15.0 Warm Up Exercise. (Calculating elasticities.) Here are some drills on price elasticities. For each demand function, find an expression for the price elasticity of demand. The answer will typically be a function of the price, p. As an example, consider the linear demand curve, $D(p) = 30 - 6p$. Then $dD(p)/dp = -6$ and $p/q = p/(30 - 6p)$, so the price elasticity of demand is $-6p/(30 - 6p)$.

(a) $D(p) = 60 - p$. $-p/(60 - p)$.

(b) $D(p) = a - bp$. $-bp/(a - bp)$.

(c) $D(p) = 40p^{-2}$. -2.

(d) $D(p) = Ap^{-b}$. $-b$.

(e) $D(p) = (p + 3)^{-2}$. $-2p/(p + 3)$.

(f) $D(p) = (p + a)^{-b}$. $-bp/(p + a)$.

15.1 (0) In Gas Pump, South Dakota, there are two kinds of consumers, Buick owners and Dodge owners. Every Buick owner has a demand function for gasoline $D_B(p) = 20 - 5p$ for $p \leq 4$ and $D_B(p) = 0$ if $p > 4$. Every Dodge owner has a demand function $D_D(p) = 15 - 3p$ for $p \leq 5$ and $D_D(p) = 0$ for $p > 5$. (Quantities are measured in gallons per week and price is measured in dollars.) Suppose that Gas Pump has 150 consumers, 100 Buick owners, and 50 Dodge owners.

(a) If the price is \$3, what is the total amount demanded by each individual Buick Owner? 5. And by each individual Dodge owner? 6.

(b) What is the total amount demanded by all Buick owners? 500. What is the total amount demanded by all Dodge owners? 300.

(c) What is the total amount demanded by all consumers in Gas Pump at a price of 3? 800.

(d) On the graph below, use blue ink to draw the demand curve representing the total demand by Buick owners. Use black ink to draw the demand curve representing total demand by Dodge owners. Use red ink to draw the market demand curve for the whole town.

(e) At what prices does the market demand curve have kinks? At $p = 4$ and $p = 5$.

(f) When the price of gasoline is \$1 per gallon, how much does weekly demand fall when price rises by 10 cents? 65 gallons.

(g) When the price of gasoline is \$4.50 per gallon, how much does weekly demand fall when price rises by 10 cents? 15 gallons.

(h) When the price of gasoline is \$10 per gallon, how much does weekly demand fall when price rises by 10 cents? Remains at zero.

15.2 (0) For each of the following demand curves, compute the inverse demand curve.

(a) $D(p) = \max\{10 - 2p, 0\}$. $p(q) = 5 - q/2$ `if` $q < 10$.

(b) $D(p) = 100/\sqrt{p}$. $p(q) = 10,000/q^2$.

(c) $\ln D(p) = 10 - 4p$. $p(q) = (10 - \ln q)/4$.

(d) $\ln D(p) = \ln 20 - 2\ln p$. $p(q) = \sqrt{20/q}$.

15.3 (0) The demand function of dog breeders for electric dog polishers is $q_b = \max\{200-p, 0\}$, and the demand function of pet owners for electric dog polishers is $q_o = \max\{90 - 4p, 0\}$.

(a) At price p, what is the price elasticity of dog breeders' demand for electric dog polishers? $-p/(200 - p)$. What is the price elasticity of pet owners' demand? $-4p/(90 - 4p)$.

(b) At what price is the dog breeders' elasticity equal to -1? $100.

At what price is the pet owners' elasticity equal to -1? $11.25.

(c) On the graph below, draw the dog breeders' demand curve in blue ink, the pet owners' demand curve in red ink, and the market demand curve in pencil.

(d) Find a nonzero price at which there is positive total demand for dog polishers and at which there is a kink in the demand curve. $22.50.

What is the market demand function for prices below the kink? $290 - 5p$. What is the market demand function for prices above the kink? $200 - p$.

(e) Where on the market demand curve is the price elasticity equal to -1? $100. At what price will the revenue from the sale of electric dog polishers be maximized? $100. If the goal of the sellers is to maximize revenue, will electric dog polishers be sold to breeders only, to pet owners only, or to both? Breeders only.

Calculus **15.4 (0)** The demand for kitty litter, in pounds, is $\ln D(p) = 1,000 - p + \ln m$, where p is the price of kitty litter and m is income.

(a) What is the price elasticity of demand for kitty litter when $p = 2$ and $m = 500$? -2. When $p = 3$ and $m = 500$? -3. When $p = 4$ and $m = 1,500$? -4.

(b) What is the income elasticity of demand for kitty litter when $p = 2$ and $m = 500$? 1. When $p = 2$ and $m = 1,000$? 1. When $p = 3$ and $m = 1,500$? 1.

(c) What is the price elasticity of demand when price is p and income is m? $-p$. The income elasticity of demand? 1.

Calculus **15.5 (0)** The demand function for drangles is $q(p) = (p+1)^{-2}$.

(a) What is the price elasticity of demand at price p? $-2p/(p+1)$.

(b) At what price is the price elasticity of demand for drangles equal to -1? When the price equals 1.

(c) Write an expression for total revenue from the sale of drangles as a function of their price. $R(p) = pq = p/(p+1)^2$. Use calculus to find the revenue-maximizing price. Don't forget to check the second-order condition. Differentiating and solving gives $p = 1$.

(d) Suppose that the demand function for drangles takes the more general form $q(p) = (p+a)^{-b}$ where $a > 0$ and $b > 1$. Calculate an expression for the price elasticity of demand at price p. $-bp/(p+a)$. At what price is the price elasticity of demand equal to -1? $p = a/(b-1)$.

15.6 (0) Ken's utility function is $u_K(x_1, x_2) = x_1 + x_2$ and Barbie's utility function is $u_B(x_1, x_2) = (x_1 + 1)(x_2 + 1)$. A person can buy 1 unit of good 1 or 0 units of good 1. It is impossible for anybody to buy fractional units or to buy more than 1 unit. Either person can buy any quantity of good 2 that he or she can afford at a price of \$1 per unit.

(a) Where m is Barbie's wealth and p_1 is the price of good 1, write an equation that can be solved to find Barbie's reservation price for good 1. $(m - p_1 + 1)2 = m + 1$. What is Barbie's reservation price for good 1? $p = (m+1)/2$. What is Ken's reservation price for good 1? \$1.

(b) If Ken and Barbie each have a wealth of 3, plot the market demand curve for good 1.

15.7 (0) The demand function for yo-yos is $D(p, M) = 4 - 2p + \frac{1}{100}M$, where p is the price of yo-yos and M is income. If M is 100 and p is 1,

(a) What is the income elasticity of demand for yo-yos? $1/3$.

(b) What is the price elasticity of demand for yo-yos? $-2/3$.

15.8 (0) If the demand function for zarfs is $P = 10 - Q$,

(a) At what price will total revenue realized from their sale be at a maximum? $P = 5$.

(b) How many zarfs will be sold at that price? $Q = 5$.

15.9 (0) The demand function for football tickets for a typical game at a large midwestern university is $D(p) = 200,000 - 10,000p$. The university has a clever and avaricious athletic director who sets his ticket prices so as to maximize revenue. The university's football stadium holds 100,000 spectators.

(a) Write down the inverse demand function. $p(q) = 20 - q/10,000$.

(b) Write expressions for total revenue $R(q) = 20q - q^2/10,000$ and marginal revenue $\texttt{MR} = 20 - q/5,000$ as a function of the number of tickets sold.

(c) On the graph below, use blue ink to draw the inverse demand function and use red ink to draw the marginal revenue function. On your graph, also draw a vertical blue line representing the capacity of the stadium.

(d) What price will generate the maximum revenue? $\texttt{\$10.}$ What quantity will be sold at this price? $\texttt{100,000.}$

(e) At this quantity, what is marginal revenue? $\texttt{0.}$ At this quantity, what is the price elasticity of demand? $-1.$ Will the stadium be full? $\texttt{Yes.}$

(f) A series of winning seasons caused the demand curve for football tickets to shift upward. The new demand function is $q(p) = 300,000 - 10,000p$. What is the new inverse demand function? $p(q) = 30 - q/10,000.$

(g) Write an expression for marginal revenue as a function of output. $MR(q) =$ $30 - q/5,000$. Use red ink to draw the new demand function and use black ink to draw the new marginal revenue function.

(h) Ignoring stadium capacity, what price would generate maximum revenue? $15. What quantity would be sold at this price? 150,000.

(i) As you noticed above, the quantity that would maximize total revenue given the new higher demand curve is greater than the capacity of the stadium. Clever though the athletic director is, he cannot sell seats he hasn't got. He notices that his marginal revenue is positive for any number of seats that he sells up to the capacity of the stadium. Therefore, in order to maximize his revenue, he should sell 100,000 tickets at a price of $20.

(j) When he does this, his marginal revenue from selling an extra seat is 10. The elasticity of demand for tickets at this price quantity combination is $\epsilon = -2$.

15.10 (0) The athletic director discussed in the last problem is considering the extra revenue he would gain from three proposals to expand the size of the football stadium. Recall that the demand function he is now facing is given by $q(p) = 300,000 - 10,000p$.

(a) How much could the athletic director increase the total revenue per game from ticket sales if he added 1,000 new seats to the stadium's capacity and adjusted the ticket price to maximize his revenue? 9,900.

(b) How much could he increase the revenue per game by adding 50,000 new seats? $250,000. 60,000 new seats? (Hint: The athletic director still wants to maximize revenue.) $250,000.

(c) A zealous alumnus offers to build as large a stadium as the athletic director would like and donate it to the university. There is only one hitch. The athletic director must price his tickets so as to keep the stadium full. If the athletic director wants to maximize his revenue from ticket sales, how large a stadium should he choose? 150,000 seats.

Chapter 16

NAME_____

Equilibrium

Introduction. Supply and demand problems are bread and butter for economists. In the problems below, you will typically want to solve for equilibrium prices and quantities by writing an equation that sets supply equal to demand. Where the price received by suppliers is the same as the price paid by demanders, one writes supply and demand as functions of the same price variable, p, and solves for the price that equalizes supply and demand. But if, as happens with taxes and subsidies, suppliers face different prices from demanders, it is a good idea to denote these two prices by separate variables, p_s and p_d. Then one can solve for equilibrium by solving a system of two equations in the two unknowns p_s and p_d. The two equations are the equation that sets supply equal to demand and the equation that relates the price paid by demanders to the net price received by suppliers.

Example: The demand function for commodity x is $q = 1,000 - 10p_d$, where p_d is the price paid by consumers. The supply function for x is $q = 100 + 20p_s$, where p_s is the price received by suppliers. For each unit sold, the government collects a tax equal to half of the price paid by consumers. Let us find the equilibrium prices and quantities. In equilibrium, supply must equal demand, so that $1,000 - 10p_d = 100 + 20p_s$. Since the government collects a tax equal to half of the price paid by consumers, it must be that the sellers only get half of the price paid by consumers, so it must be that $p_s = p_d/2$. Now we have two equations in the two unknowns, p_s and p_d. Substitute the expression $p_d/2$ for p_s in the first equation, and you have $1,000 - 10p_d = 100 + 10p_d$. Solve this equation to find $p_d = 45$. Then $p_s = 22.5$ and $q = 550$.

16.1 (0) The demand for yak butter is given by $120 - 4p_d$ and the supply is $2p_s - 30$, where p_d is the price paid by demanders and p_s is the price received by suppliers, measured in dollars per hundred pounds. Quantities demanded and supplied are measured in hundred-pound units.

(a) On the axes below, draw the demand curve (with blue ink) and the supply curve (with red ink) for yak butter.

(b) Write down the equation that you would solve to find the equilibrium price. Solve $120 - 4p = 2p - 30$.

(c) What is the equilibrium price of yak butter? \$25. What is the equilibrium quantity? 20. Locate the equilibrium price and quantity on the graph, and label them p_1 and q_1.

(d) A terrible drought strikes the central Ohio steppes, traditional homeland of the yaks. The supply schedule shifts to $2p_s - 60$. The demand schedule remains as before. Draw the new supply schedule. Write down the equation that you would solve to find the new equilibrium price of yak butter. $120 - 4p = 2p - 60$.

(e) The new equilibrium price is 30 and the quantity is 0. Locate the new equilibrium price and quantity on the graph and label them p_2 and q_2.

(f) The government decides to relieve stricken yak butter consumers and producers by paying a subsidy of \$5 per hundred pounds of yak butter to producers. If p_d is the price paid by demanders for yak butter, what is the total amount received by producers for each unit they produce? $p_d + 5$. When the price paid by consumers is p_d, how much yak butter is produced? $2p_d - 50$.

(g) Write down an equation that can be solved for the equilibrium price paid by consumers, given the subsidy program. $2p_d - 50 = 120 - 4p_d$. What are the equilibrium price paid by consumers and the equilibrium quantity of yak butter now? $p_d = 170/6$, $q = 170/3 - 50 = 20/3$.

(h) Suppose the government had paid the subsidy to consumers rather than producers. What would be the equilibrium net price paid by consumers? $170/6$. The equilibrium quantity would be $20/3$.

16.2 (0) Here are the supply and demand equations for throstles, where p is the price in dollars:

$$D(p) = 40 - p$$
$$S(p) = 10 + p.$$

On the axes below, draw the demand and supply curves for throstles, using blue ink.

(a) The equilibrium price of throstles is 15 and the equilibrium quantity is 25.

(b) Suppose that the government decides to restrict the industry to selling only 20 throstles. At what price would 20 throstles be demanded? 20. How many throstles would suppliers supply at that price? 30. At what price would the suppliers supply only 20 units? \$10.

(c) The government wants to make sure that only 20 throstles are bought, but it doesn't want the firms in the industry to receive more than the minimum price that it would take to have them supply 20 throstles. One way to do this is for the government to issue 20 ration coupons. Then in order to buy a throstle, a consumer would need to present a ration coupon along with the necessary amount of money to pay for the good. If the ration coupons were freely bought and sold on the open market, what would be the equilibrium price of these coupons? $10.

(d) On the graph above, shade in the area that represents the deadweight loss from restricting the supply of throstles to 20. How much is this expressed in dollars? (Hint: What is the formula for the area of a triangle?)

$25.

16.3 (0) The demand curve for ski lessons is given by $D(p_D) = 100 - 2p_D$ and the supply curve is given by $S(p_S) = 3p_S$.

(a) What is the equilibrium price? $20. What is the equilibrium quantity? 60.

(b) A tax of \$10 per ski lesson is imposed on consumers. Write an equation that relates the price paid by demanders to the price received by suppliers.

$p_D = p_S + 10$. Write an equation that states that supply equals demand. $100 - 2p_D = 3p_S$.

(c) Solve these two equations for the two unknowns p_S and p_D. With the \$10 tax, the equilibrium price p_D paid by consumers would be $26 per lesson. The total number of lessons given would be 48.

(d) A senator from a mountainous state suggests that although ski lesson consumers are rich and deserve to be taxed, ski instructors are poor and deserve a subsidy. He proposes a \$6 subsidy on production while maintaining the \$10 tax on consumption of ski lessons. Would this policy have any different effects for suppliers or for demanders than a tax of \$4 per lesson? No.

16.4 (0) The demand curve for salted codfish is $D(P) = 200 - 5P$ and the supply curve $S(P) = 5P$.

(a) On the graph below, use blue ink to draw the demand curve and the supply curve. The equilibrium market price is $20 and the equilibrium quantity sold is 100.

(b) A quantity tax of $2 per unit sold is placed on salted codfish. Use red ink to draw the new supply curve, where the price on the vertical axis remains the price per unit paid by demanders. The new equilibrium price paid by the demanders will be $21 and the new price received by the suppliers will be $19. The equilibrium quantity sold will be 95.

(c) The deadweight loss due to this tax will be $5 = 2 \times 5/2$. On your graph, shade in the area that represents the deadweight loss.

16.5 (0) The demand function for merino ewes is $D(P) = 100/P$, and the supply function is $S(P) = P$.

(a) What is the equilibrium price? $10.

(b) What is the equilibrium quantity? 10.

(c) An ad valorem tax of 300% is imposed on merino ewes so that the price paid by demanders is four times the price received by suppliers. What is the equilibrium price paid by the demanders for merino ewes now? $20. What is the equilibrium price received by the suppliers for merino ewes? $5. What is the equilibrium quantity? 5.

16.6 (0) Schrecklich and LaMerde are two justifiably obscure nineteenth-century impressionist painters. The world's total stock of paintings by Schrecklich is 100, and the world's stock of paintings by LaMerde is 150. The two painters are regarded by connoisseurs as being very similar in style. Therefore the demand for either painter's work depends both on its own price and the price of the other painter's work. The demand function for Schrecklichs is $D_S(P) = 200 - 4P_S - 2P_L$, and the demand function for LaMerdes is $D_L(P) = 200 - 3P_L - P_S$, where P_S and P_L are respectively the price in dollars of a Schrecklich painting and a LaMerde painting.

(a) Write down two simultaneous equations that state the equilibrium condition that the demand for each painter's work equals supply.

The equations are $200 - 4P_S - 2P_L = 100$ and $200 - 3P_L - P_S = 150$.

(b) Solving these two equations, one finds that the equilibrium price of Schrecklichs is 20 and the equilibrium price of LaMerdes is 10.

(c) On the diagram below, draw a line that represents all combinations of prices for Schrecklichs and LaMerdes such that the supply of Schrecklichs equals the demand for Schrecklichs. Draw a second line that represents those price combinations at which the demand for LaMerdes equals the supply of LaMerdes. Label the unique price combination at which both markets clear with the letter E.

(d) A fire in a bowling alley in Hamtramck, Michigan, destroyed one of the world's largest collections of works by Schrecklich. The fire destroyed a total of 10 Schrecklichs. After the fire, the equilibrium price of Schrecklichs was 23 and the equilibrium price of LaMerdes was 9.

(e) On the diagram you drew above, use red ink to draw a line that shows the locus of price combinations at which the demand for Schrecklichs equals the supply of Schrecklichs after the fire. On your diagram, label the new equilibrium combination of prices E'.

16.7 (0) The price elasticity of demand for oatmeal is constant and equal to -1. When the price of oatmeal is \$10 per unit, the total amount demanded is 6,000 units.

(a) Write an equation for the demand function. $q = 60,000/p$.
Graph this demand function below with blue ink. (Hint: If the demand curve has a constant price elasticity equal to ϵ, then $D(p) = ap^{\epsilon}$ for some constant a. You have to use the data of the problem to solve for the constants a and ϵ that apply in this particular case.)

(b) If the supply is perfectly inelastic at 5,000 units, what is the equilibrium price? $12. Show the supply curve on your graph and label the equilibrium with an E.

(c) Suppose that the demand curve shifts outward by 10%. Write down the new equation for the demand function. $q = 66,000/p$. Suppose that the supply curve remains vertical but shifts to the right by 5%. Solve for the new equilibrium price 12.51 and quantity $5,250$.

(d) By what percentage approximately did the equilibrium price rise?

It rose by about 5 percent. Use red ink to draw the new demand curve and the new supply curve on your graph.

(e) Suppose that in the above problem the demand curve shifts outward by x% and the supply curve shifts right by y%. By approximately what percentage will the equilibrium price rise? By about $(x - y)$ percent.

16.8 (0) An economic historian* reports that econometric studies indicate for the pre–Civil War period, 1820–1860, the price elasticity of demand for cotton from the American South was approximately -1. Due to the rapid expansion of the British textile industry, the demand curve for American cotton is estimated to have shifted outward by about 5% per year during this entire period.

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* Gavin Wright, *The Political Economy of the Cotton South*, W. W. Norton, 1978.

(a) If during this period, cotton production in the United States grew by 3% per year, what (approximately) must be the rate of change of the price of cotton during this period? It would rise by about 2% a year.

(b) Assuming a constant price elasticity of -1, and assuming that when the price is \$20, the quantity is also 20, graph the demand curve for cotton. What is the total revenue when the price is \$20? 400. What is the total revenue when the price is \$10? 400.

(c) If the change in the quantity of cotton supplied by the United States is to be interpreted as a movement along an upward-sloping long-run supply curve, what would the elasticity of supply have to be? (Hint: From 1820 to 1860 quantity rose by about 3% per year and price rose by 2 % per year. [See your earlier answer.] If the quantity change is a movement along the long-run supply curve, then the long-run price elasticity must be what?) 1.5 %.

(d) The American Civil War, beginning in 1861, had a devastating effect on cotton production in the South. Production fell by about 50% and remained at that level throughout the war. What would you predict would be the effect on the price of cotton? It would double if demand didn't change.

(e) What would be the effect on total revenue of cotton farmers in the South? Since the demand has elasticity of -1, the revenue would stay the same.

(f) The expansion of the British textile industry ended in the 1860s, and for the remainder of the nineteenth century, the demand curve for American cotton remained approximately unchanged. By about 1900, the South approximately regained its prewar output level. What do you think happened to cotton prices then? They would recover to their old levels.

16.9 (0) The number of bottles of chardonnay demanded per year is $\$1,000,000 - 60,000P$, where P is the price per bottle (in U.S. dollars). The number of bottles supplied is $40,000P$.

(a) What is the equilibrium price? \$10. What is the equilibrium quantity? $400,000$.

(b) Suppose that the government introduces a new tax such that the wine maker must pay a tax of \$5 per bottle for every bottle that he produces. What is the new equilibrium price paid by consumers? \$12. What is the new price received by suppliers? \$7. What is the new equilibrium quantity? 280,000.

16.10 (0) The inverse demand function for bananas is $P_d = 18 - 3Q_d$ and the inverse supply function is $P_s = 6 + Q_s$, where prices are measured in cents.

(a) If there are no taxes or subsidies, what is the equilibrium quantity? 3. What is the equilibrium market price? 9 cents.

(b) If a subsidy of 2 cents per pound is paid to banana growers, then in equilibrium it still must be that the quantity demanded equals the quantity supplied, but now the price received by sellers is 2 cents higher than the price paid by consumers. What is the new equilibrium quantity? 3.5. What is the new equilibrium price received by suppliers? 9.5 cents. What is the new equilibrium price paid by demanders? 7.5 cents.

(c) Express the change in price as a percentage of the original price. -16.66%. If the cross-elasticity of demand between bananas and apples is +.5, what will happen to the quantity of apples demanded as a consequence of the banana subsidy, if the price of apples stays constant? (State your answer in terms of percentage change.) -8.33%.

16.11 (1) King Kanuta rules a small tropical island, Nutting Atoll, whose primary crop is coconuts. If the price of coconuts is P, then King Kanuta's subjects will demand $D(P) = 1,200 - 100P$ coconuts per week for their own use. The number of coconuts that will be supplied per week by the island's coconut growers is $S(p) = 100P$.

(a) The equilibrium price of coconuts will be 6 and the equilibrium quantity supplied will be 600.

(b) One day, King Kanuta decided to tax his subjects in order to collect coconuts for the Royal Larder. The king required that every subject who consumed a coconut would have to pay a coconut to the king as a tax. Thus, if a subject wanted 5 coconuts for himself, he would have to purchase 10 coconuts and give 5 to the king. When the price that is received by the sellers is p_S, how much does it cost one of the king's subjects to get an extra coconut for himself? $2p_S$.

(c) When the price paid to suppliers is p_S, how many coconuts will the king's subjects demand for their own consumption? (Hint: Express p_D in terms of p_S and substitute into the demand function.) Since $p_D = 2p_S$, they consume $1,200 - 200p_S$.

(d) Since the king consumes a coconut for every coconut consumed by the subjects, the total amount demanded by the king and his subjects is twice the amount demanded by the subjects. Therefore, when the price received by suppliers is p_S, the total number of coconuts demanded per week by Kanuta and his subjects is $2,400 - 400p_S$.

(e) Solve for the equilibrium value of p_S 24/5, the equilibrium total number of coconuts produced 480, and the equilibrium total number of coconuts consumed by Kanuta's subjects. 240.

(f) King Kanuta's subjects resented paying the extra coconuts to the king, and whispers of revolution spread through the palace. Worried by the hostile atmosphere, the king changed the coconut tax. Now, the shopkeepers who sold the coconuts would be responsible for paying the tax. For every coconut sold to a consumer, the shopkeeper would have to pay one coconut to the king. This plan resulted in $480/2 = 240$ coconuts being sold to the consumers. The shopkeepers got $24/5$ per coconut after paying their tax to the king, and the consumers paid a price of $48/5$ per coconut.

16.12 (1) On August 29, 2005, Hurricane Katrina caused severe damage to oil installations in the Gulf of Mexico. Although this damage could eventually be repaired, it resulted in a substantial reduction in the short run supply of gasoline in the United States. In many areas, retail gasoline prices quickly rose by about 30% to an average of $3.06 per gallon.

Georgia governor Sonny Perdue suspended his state's 7.5 cents-a-gallon gas tax and 4% sales tax on gasoline purchases until Oct. 1. Governor Perdue explained that, "I believe it is absolutely wrong for the state to reap a tax windfall in this time of urgency and tragedy." Lawmakers in several other states were considering similar actions.

Let us apply supply and demand analysis to this problem. Before the hurricane, the United States consumed about 180 million gallons of gasoline per day, of which about 30 million gallons came from the Gulf of Mexico. In the short run, the supply of gasoline is extremely inelastic and is limited by refinery and transport capacity. Let us assume that the daily short run supply of gasoline was perfectly inelastic at 180 million gallons before the storm and perfectly inelastic at 150 million gallons after the storm. Suppose that the demand function, measured in millions of gallons per day, is given by $Q = 240 - 30P$ where P is the dollar price, including tax, that consumers pay for gasoline.

(a) What was the market equilibrium price for gasoline before the hurricane? $2.00 After the hurricane? $3.00

(b) Suppose that both before and after the hurricane, a government tax of 10 cents is charged for every gallon of gasoline sold in the United States. How much money would *suppliers* receive *per gallon* of gasoline before the hurricane? $1.90 After the hurricane? $2.90

(c) Suppose that after the hurricane, the federal government removed the gas tax. What would then be the equilibrium price paid by consumers? $3.00 How much money would suppliers receive per gallon of gasoline? $3.00 How much revenue would the government lose per day by

removing the tax? $15 million What is the net effect of removing the tax on gasoline prices? There is no effect. Who are the gainers and who are the losers from removing the tax? Gasoline suppliers gain and government loses $15 million per day.

(d) Suppose that after the hurricane, the ten-cent tax is removed in some states but not in others. The states where the tax is removed constitute just half of the demand in the United States. Thus the demand schedule in each half of the country is $Q = 120 - 15P$ where P is the price paid by consumers in that part of the country. Let P^* be the equilibrium price for consumers in the part of the country where the tax is removed. In equilibrium, suppliers must receive the same price per gallon in all parts of the country. Therefore the equilibrium price for consumers in states that keep the tax must be $\$P^* + \$0.10$. In equilibrium it must be that the total amount of gasoline demanded in the two parts of the country equals the total supply. Write an equation for total demand as a function of P^*.

Q=$120 - 15P^* + 120 - 15(P^* + \$0.10)$. Set demand equal to supply and solve for the price paid by consumers in the states that remove the tax $2.95 and for the price paid by consumers in states that do not remove the tax. $P^* = \$3.05$ How much money do suppliers receive per gallon of gasoline sold in every state? $2.95 How does the tax removal affect daily gasoline consumption in each group of states? Increases consumption by .75 million gallons in states that remove tax and decreases consumption by .75 million gallons in states that do not.

(e) If half of the states remove the gasoline tax, as described above, some groups will be better off and some worse off than they would be if the tax were left in place. Describe the gains or losses for each of the following groups.

Consumers in the states that remove the tax Gain from 5 cent price reduction.

Consumers in other states Lose from 5 cent price

increase.

Gasoline suppliers Gain from 5 cent increase in revenue per gallon.

Governments of the states that remove the tax Lose $15 million per day in revenue.

Governments of states that do not remove the tax Because consumption falls by 0.75 million gallons per day, they lose $75,000 per day.

16.13 (2) Citizens of Zephyronia consume only two goods and do not trade with the outside world. There is a fixed supply of labor in Zephyronia, which can be applied to production of either good. Good 1 is produced at constant returns to scale, using one unit of labor for each unit of output. Good 1 generates no externalities. Good 2 is also produced at constant returns to scale, with one unit of labor needed for each unit of output, but every unit of Good 2 that is produced also generates one unit of pollution. Both industries are perfectly competitive. Zephyronia has 1,000 citizens, and each citizen of this economy supplies 10 units of labor and each has a quasi-linear utility function

$$U(x_1, x_2, s) = x_1 + 4x_2 - \frac{1}{2}x_2^2 - \frac{1}{1000}s,$$

where x_1 and x_2 are her consumptions of goods 1 and 2, and where s is the total amount of pollution generated by the production of good 2. The currency of Zephyronia is known as the Puff. We will treat labor as the *numeraire* so that each consumer has an income of 10 Puffs.

(a) If there are no controls on pollution, then in competitive equilibrium, the price of each good must be equal to the marginal cost of producing the good. The marginal cost of good 1 is 1 Puff and the marginal cost of good 2 is 1 Puff . Suppose that individuals assume that the effect of their own purchases on pollution levels is negligible. Then each citizen chooses x_1 and x_2 to maximize utility subject to the budget constraint $x_1 + x_2 = 10$. Each citizen will choose to consume 3 units of x_2. Total production of good 2 in Zephyronia will be 3,000 units, and the total amount of pollution generated will be 3,000 units. Total utility of each citizen of Zephyronia will be $u(7, 3, 3000) = 11.5$.

(b) Suppose that the government imposes a pollution tax of 1 Puff for each unit of pollution produced by any firm and divides the revenue collected from the tax equally among all citizens. Then the marginal cost of producing a unit of good 2 will rise to 2 Puffs and since the industry is competitive, the price of good 2 will be 2 Puffs. Suppose that individuals ignore the tiny effect of their own purchases on the level of pollution and on the amount of their own rebate. Then each citizen will choose to consume 2 units of good 2 and the total amount of pollution will be $2,000$. The total revenue of the government from the pollution tax will be 2,000 Puffs. Each citizen will therefore receive a rebate of 2 Puffs and hence will have a total income of 12 including wages and rebate. Therefore each consumer will consume 8 units of good 1, 2 units of good 2, and will experience 2,000 units of pollution. The utility of each consumer will now be $u(8, 2, 2000) = 12$

(c) If you did the previous section correctly, you will have found that with a pollution tax of 1 Puff per unit of pollution produced, the total level of pollution produced will be 2,000 units. Suppose that instead of taxing pollution, the government required that firms must have a pollution permit for each unit of pollution produced and the government issued exactly 2,000 pollution permits. Suppose that these permits are purchased and used by producers of good 2. How many units of good 2 will be produced? 2,000 At what price will the total demand for good 2 equal 2,000? 2 (Hint: Each individual's demand for good 2 is given by the equation $x_1 = 4 - p_2$. Market demand is the total demand of 1,000 individuals like this, so market demand is $D(p_2) = 4,000 - 1,000p_2$. In equilibrium, the total amount of good 2 demanded must equal the supply.)

(d) (Extra Credit) Show that a tax of $t = 1$ Puffs per unit of pollution is the tax rate that results in the highest total utility for consumers.

With a tax of t, the price of good 2 is $1 + t$ and each citizen consumes $3 - t$ units of good 2. Total pollution is $s = 1000(3 -$

$t)$, government revenue is $1000t(3-t)$, each citizen receives a rebate of $t(3-t)$. Each consumer consumes $10+t(3t-t)-(1+t)(3-t)$ units of good 1, $3-t$ units of good 2, and experiences $1000(3-t)$ units of smoke. Calculating utility and simplifying, we find each has utility $(3/2)(3-t)(1+t)$ which is maximized when $t=1$.

16.14 (2) Suppose that in Zephyronia all is as in the previous problem except that there are alternative ways of producing good 2, some of which produce less pollution than others. A method of production that emits z units of pollution per unit of output requires $c(z)$ units of labor per unit of output, where

$$c(z) = 2 + z^2 - 2z.$$

(a) If there is no tax on pollution, firms will choose the technology that minimizes labor cost per unit. To do this, they choose $z =$ 1 and then the marginal cost of a unit of good 2 will be 1 Puff. In competitive equilibrium, the price of good 1 will be 1 Puff and the price of good 2 will be 1 Puff. The total amount of good 2 demanded in Zephyronia will be 3,000 units and the amount of pollution generated will be 3,000 units.

(b) Suppose that producers must pay a tax of 1 Puff per unit of pollution generated. Again they will choose a technology that minimizes their per-unit cost. When they take the tax into account, their production costs per unit will be

$$c(z) - tz = 2 + z^2 - 2z + z = 2 + z^2 - z$$

where z is the amount of pollution generated per unit of output. What level of z will they choose? $z = 1/2$ With this choice, what is the unit cost for producers of good 2 including the tax? $7/4$ Puffs What is the equilibrium price of good 2? $7/4$ Puffs At this price, how

much of good 2 will be produced? 2,250 units What is the total amount of pollution that will be generated? $2,250 \times 1/2 = 1,125$

(c) Suppose alternatively that the government requires pollution permits and issues 1,125 of these permits. With 1,125 permits, how many units of good 2 can be produced? 2,250 Find the price of good 2 at which supply equals demand when this number of units is produced? $7/4$

Puffs

Chapter 17

NAME_____

Measurement

Introduction. There are currently no exercises for this chapter

Chapter 18

NAME_____

Auctions

Introduction. An auction is described by a set of rules. The rules specify bidding procedures for participants and the way in which the array of bids made determines who gets the object being sold and how much each bidder pays. Those who are trying to sell an object by auction typically do not know the willingness to pay of potential buyers but have some probabilistic expectations. Sellers are interested in finding rules that maximize their expected revenue from selling the object.

Social planners are often interested not only in the revenue generated from an auction method, but also in its *efficiency*. In the absence of externalities, an auction for a single object will be efficient only if the object is sold to the buyer who values it most highly.

18.1 (1) At Toivo's auction house in Ishpemming, Michigan, a beautiful stuffed moosehead is being sold by auction. There are 5 bidders in attendance: Aino, Erkki, Hannu, Juha, and Matti. The moosehead is worth \$100 to Aino, \$20 to Erkki, and \$5 to each of the others. The bidders do not collude and they don't know each others' valuations.

(a) If the auctioneer sells it in an English auction, who would get the moosehead and approximately how much would the buyer pay? Aino would get it for \$20.

(b) If the auctioneer sells it in a sealed-bid, second-price auction and if no bidder knows the others' values for the moosehead, how much should Aino bid in order to maximize his expected gain? \$100. How much should Erkki bid? \$20. How much would each of the others bid? \$5. Who would get the moosehead and how much would he pay? Aino would get it for \$20.

18.2 (2) Charlie Plopp sells used construction equipment in a quiet Oklahoma town. He has run short of cash and needs to raise money quickly by selling an old bulldozer. If he doesn't sell his bulldozer to a customer today, he will have to sell it to a wholesaler for \$1,000.

Two kinds of people are interested in buying bulldozers. These are professional bulldozer operators and people who use bulldozers only for recreational purposes on weekends. Charlie knows that a professional bulldozer operator would be willing to pay \$6,000 for his bulldozer but no

more, while a weekend recreational user would be willing to pay $\$4,500$ but no more. Charlie puts a sign in his window. "Bulldozer Sale Today."

Charlie is disappointed to discover that only two potential buyers have come to his auction. These two buyers evidently don't know each other. Charlie believes that the probability that either is a professional bulldozer operator is independent of the other's type and he believes that each of them has a probability of 1/2 of being a professional bulldozer operator and a probability of 1/2 of being a recreational user.

Charlie considers the following three ways of selling the bulldozer:

- Method 1. Post a price of $6,000, and if nobody takes the bulldozer at that price, sell it to the wholesaler.
- Method 2. Post a price equal to a recreational bulldozer user's buyer value and sell it to anyone who offers that price.
- Method 3. Run a sealed-bid auction and sell the bulldozer to the high bidder at the second highest bid (if there is a tie, choose one of the high bidders at random and sell the bulldozer to this bidder at the price bid by both bidders.)

(a) What is the probability that both potential buyers are professional bulldozer operators? 1/4. What is the probability that both are recreational bulldozer users? 1/4. What is the probability that one of them is of each type? 1/2.

(b) If Charlie sells by method 1, what is the probability that he will be able to sell the bulldozer to one of the two buyers? 3/4. What is the probability that he will have to sell the bulldozer to the wholesaler? 1/4. What is his expected revenue? $(3/4)\times\$6,000+(1/4)\times\$1,000=\$4,750.$

(c) If Charlie sells by method 2, how much will he receive for his bulldozer? $4,500.

(d) Suppose that Charlie sells by method 3 and that both potential buyers bid rationally. If both bidders are professional bulldozer operators, how much will each bid? $6,000. How much will Charlie receive for his bulldozer? $6,000. If one bidder is a professional bulldozer operator and one is a recreational user, what bids will Charlie receive? Professional bids $6,000. Recreational user bids $4,500. Who will get

the bulldozer? The professional. How much money will Charlie get for his bulldozer? $4,500. If both bidders are recreational bulldozer users, how much will each bid? $4,500. How much will Charlie receive for his bulldozer? $4,500. What will be Charlie's expected revenue from selling the bulldozer by method 3?

$$(1/4) \times \$6,000 + (3/4) \times \$4,500 = \$4,875.$$

(e) Which of the three methods will give Charlie the highest expected revenue? Method 3.

18.3 (2) We revisit our financially afflicted friend, Charlie Plopp. This time we will look at a slightly generalized version of the same problem. All else is as before, but the willingness to pay of recreational bulldozers is an amount $C < \$6,000$ which is known to Charlie. In the previous problem we dealt with the special case where $C = \$4,500$. Now we want to explore the way in which the sales method that gives Charlie the highest expected revenue depends on the size of C.

(a) What will Charlie's expected revenue be if he posts a price equal to the reservation price of professional bulldozer operators? $(3/4) \times \$6,000 + (1/4) \times \$1,000 = \$4,750.$

(b) If Charlie posts a price equal to the reservation price C of recreational bulldozer operators, what is his expected revenue? $\$C$.

(c) If Charlie sells his bulldozer by method 3, the second-price sealed-bid auction, what is his expected revenue? (The answer is a function of C.)

$$(1/4) \times \$6,000 + (3/4) \times \$C = \$1,500 + \$(3/4)C.$$

(d) Show that selling by method 3 will give Charlie a higher expected payoff than selling by method 2 if $C < \$6,000$. With method 3, each bidder will bid his true valuation. If both bidders have valuations of $6,000, he will get $6,000. Otherwise, he will get $\$C$. His expected payoff is then

$\$1,500+(3/4)C$. With method 2 he gets $C. But $\$1,500+(3/4)C > C$ whenever $C < \$6,000$.

(e) For what values of C is Charlie better off selling by method 2 than by method 1? $C > \$(3/4)6,000 + (1/4)1,000 = 4,750$.

(f) For what values of C is Charlie better off selling by method 1 than by method 3? This happens when $4,750 > 1,500 + 3/4C$, which is the case whenever $C < 4,333.33$.

18.4 (3) Yet again we tread the dusty streets of Charlie Plopp's home town. Everything is as in the previous problem. Professional bulldozer operators are willing to pay \$6,000 for a bulldozer and recreational users are willing to pay $\$C$. Charlie is just about to sell his bulldozer when a third potential buyer appears. Charlie believes that this buyer, like the other two, is equally likely to be a professional bulldozer operator as a recreational bulldozer operator and that this probability is independent of the types of the other two.

(a) With three buyers, Charlie's expected revenue from using method 1 is 5375, his expected revenue from using method 2 is C, and his expected revenue from using method 3 is $\$3,000 + (C/2)$.

(b) At which values of C would method 1 give Charlie a higher expected revenue than either of the other two methods of selling proposed above? $C < \$4,750$.

(c) At which values of C (if any) would method 2 give Charlie a higher expected revenue than either of the other two methods of selling proposed above? None.

(d) At which values of C would method 3 give Charlie a higher expected revenue than either of the other two methods of selling proposed above? $C > \$4,750$.

18.5 (2) General Scooters has decided to replace its old assembly line with a new one that makes extensive use of robots. There are two contractors who would be able to build the new assembly line. General Scooters does not know exactly what it would cost either of the contractors to do

this job. However its engineers and investigators have discovered that for either contractor, this cost will take one of three possible values: H, M, and L, where $H > M > L$. The best information that General Scooters's investigators have been able to give it is that for each contractor the probability is 1/3 that the cost is H, 1/3 that the cost is M, and 1/3 that the cost is L and that the probability distribution of costs is independent between the two contractors. Each contractor knows its own costs but thinks that the other's costs are equally likely to be H, M, or L. General Scooters is confident that the contractors will not collude.

Accountants at General Scooters suggested that General Scoooters accept sealed bids from the two contractors for constructing the assembly line and that it announce that it will award the contract to the low bidder but will pay the low bidder the amount bid by the other contractor. (If there is a tie for low bidder, one of the bidders will be selected at random to get the contract.) In this case, as your textbook shows for the Vickrey auction, each contractor would find it in his own interest to bid his true valuation.

(a) Suppose that General Scooters uses the bidding mechanism suggested by the accountants. What is the probability that it will have to pay L? 1/9 (Hint: The only case where it pays L is when both contractors have a cost of L.) What is the probability that it will have to pay H to get the job done? 5/9 (Hint: Notice that it has to pay H if at least one of the two contractors has costs of H.) What is the probability that it will have to pay M? 1/3. Write an expression in terms of the variables H, M, and L for the expected cost of the project to General Scooters.

$5H/9 + 3M/9 + L/9$

(b) When the distinguished-looking, silver-haired chairman of General Scooters was told of the accountants' suggested bidding scheme, he was outraged. "What a stupid bidding system! Any fool can see that it is more profitable for us to pay the lower of the two bids. Why on earth would you ever want to pay the higher bid rather than the lower one?" he roared.

A timid-looking accountant summoned up his courage and answered the chairman's question. What answer would you suggest that he make? The amount that contractors will bid depends on the rules of the auction. If you contract to the low bidder at the low bidder's bid, then all bidders will bid

a higher amount than they would if you contract at the second-lowest bid.

(c) The chairman ignored the accountants and proposed the following plan. "Let us award the contract by means of sealed bids, but let us do it wisely. Since we know that the contractors' costs are either H, M, or L, we will accept only bids of H, M, or L, and we will award the contract to the low bidder at the price he himself bids. (If there is a tie, we will randomly select one of the bidders and award it to him at his bid.)"

If the chairman's scheme is adopted, would it ever be worthwhile for a contractor with costs of L to bid L? (Hint: What are the contractor's profits if he bids L and costs are L. Does he have a chance of a positive profit if he bids M?) No. If he bids L, he is sure to make zero profits; if he bids higher than that, he might get the contract and make a profit.

(d) Suppose that the chairman's bidding scheme is adopted and that both contractors use the strategy of padding their bids in the following way. A contractor will bid M if her costs are L, and she will bid H if her costs are H or M. If contractors use this strategy, what is the expected cost of the project to General Scooters? $4H/9 + 5M/9$. Which of the two schemes will result in a lower expected cost for General Scooters, the accountants' scheme or the chairman's scheme?* The accountants' scheme.

(e) We have not yet demonstrated that the bid-padding strategies proposed above are equilibrium strategies for bidders. Here we will show that this is the case for some (but not all) values of H, M, and L. Suppose that you are one of the two contractors. You believe that the other contractor is equally likely to have costs of H, M, or L and that he will bid H when his costs are M or H and he will bid M when his costs are L. Obviously if your costs are H, you can do no better than to bid H. If your costs are M, your expected profits will be positive if you bid H and

* The chairman's scheme might not have worked out so badly for General Scooters if he had not insisted that the only acceptable bids are H, M, and L. If bidders had been allowed to bid any number between L and H, then the only equilibrium in bidding strategies would involve the use of mixed strategies, and if the contractors used these strategies, the expected cost of the project to General Scooters would be the same as it is with the second-bidder auction proposed by the accountants.

negative or zero if you bid L or M. What if your costs are L? For what values of H, M, and L will the best strategy available to you be to bid M? $3L/5 + 2H/5 < M$.

18.6 (3) Late in the day at an antique rug auction there are only two bidders left, April and Bart. The last rug is brought out and each bidder takes a look at it. The seller says that she will accept sealed bids from each bidder and will sell the rug to the highest bidder at the highest bidder's bid.

Each bidder believes that the other is equally likely to value the rug at any amount between 0 and \$1,000. Therefore for any number X between 0 and 1,000, each bidder believes that the probability that the other bidder values the rug at less than X is $X/1,000$. The rug is actually worth \$800 to April. If she gets the rug, her profit will be the difference between \$800 and what she pays for it, and if she doesn't get the rug, her profit will be zero. She wants to make her bid in such a way as to maximize her expected profit.

(a) Suppose that April thinks that Bart will bid exactly what the rug is worth to him. If she bids \$700 for the rug, what is the probability that she will get the rug? 7/10. If she gets the rug for \$700, what is her profit? \$100. What is her expected profit if she bids \$700? \$70.

(b) Suppose that Bart will pay exactly what the rug is worth to him. If April bids \$600 for the rug, what is the probability that she will get the rug? 6/10. What is her profit if she gets the rug for \$600? \$200. What is her expected profit if she bids \$600? \$120.

(c) Again suppose that Bart will bid exactly what the rug is worth to him. If April bids \$$x$ for the rug (where x is a number between 0 and 1,000) what is the probability that she will get the rug? $x/1,000$ What is her profit if she gets the rug? \$800-x Write a formula for her expected profit if she bids \$$x$. $\$(800 - x)(x/1,000)$. Find the bid x that maximizes her expected profit. (Hint: Take a derivative.) $x = 400$.

(d) Now let us go a little further toward finding a general answer. Suppose that the value of the rug to April is \$$V$ and she believes that Bart will bid exactly what the rug is worth to him. Write a formula that expresses her expected profit in terms of the variables V and x if she bids \$$x$.

$\$(V-x)(x/1,000)$. Now calculate the bid $\$x$ that will maximize her expected profit. (Same hint: Take a derivative.) $x = V/2$.

18.7 (3) If you did the previous problem correctly, you found that if April believes that Bart will bid exactly as much as the rug is worth to him, then she will bid only half as much as the rug is worth to her. If this is the case, it doesn't seem reasonable for April to believe that Bart will bid his full value. Let's see what would the best thing for April to do if she believed that Bart would bid only half as much as the rug is worth to him.

(a) If Bart always bids half of what the rug is worth to him, what is the highest amount that Bart would ever bid? $500. Why would it never pay for April to bid more than $500.01? She can get it for sure by bidding just over $500, since Bart will never bid more than $500.

(b) Suppose that the the rug is worth $800 to April and she bids $300 for it. April will only get the rug if the value of the rug to Bart is less than $600. What is the probability that she will get the rug if she bids $300 for it? 6/10. What is her profit if she bids $300 and gets the rug? $500. What is her expected profit if she bids $300? $300.

(c) Suppose that the rug is worth $800 to April. What is the probability that she will get it if she bids $\$x$ where $\$x < \$500$? $2x/1,000$. Write a formula for her expected profit as a function of her bid $\$x$ when the rug is worth $800 to her. $\$(800-x)2x/1,000$. What bid maximizes her expected profit in this case? $400.

(d) Suppose that April values the rug at $\$V$ and she believes that Bart will bid half of his true value. Show that the best thing for April is to bid half of her own true value. Maximize $(V-x)2x/1,000$. The derivative with respect to x is 0 when $x = V/2$.

(e) Suppose that April believes that Bart will bid half of his actual value and Bart believes that April will bid half of her actual value. Suppose also that they both act to maximize their expected profit given these beliefs. Will these beliefs be self-confirming in the sense that given these beliefs, each will take the action that the other expects? Yes.

18.8 (2) Rod's Auction House in Bent Crankshaft, Oregon, holds sealed-bid used-car auctions every Tuesday. Each used car is sold to the highest bidder at the second-highest bidder's bid. On average, half of the cars sold at Rod's Auction House are lemons and half are good used cars. A good used car is worth $1,000 to any buyer and a lemon is worth only $100. Buyers are allowed to look over the used cars for a few minutes before they are auctioned. Almost all of the buyers who attend the auctions can do no better than random choice at picking good cars from among the lemons. The only exception is Al Crankcase. Al can sometimes, but not always, detect a lemon by licking the oil off of the dipstick. To Al, the oil from a good car's dipstick invariably has a sweet, lingering taste. On the other hand, the oil from the dipsticks of 1/3 of the lemons has a sour, acidic taste, while the oil from the dipsticks of the remaining 2/3 of the lemons has the same sweet taste as the oil from the good cars. Al attends every auction, licks every dipstick, and taking into account the results of his taste test, bids his expected value for every car.

(a) This auction environment is an example of a (common, private) common value auction.

(b) Suppose that Al licks the dipsticks of 900 cars, half of which are good cars and half of which are lemons. Suppose that the dipstick oil from all of the good cars and from 2/3 of the lemons tastes sweet and the oil from 1/3 of the lemons tastes sour to Al. How many good cars will there be whose oil tastes sweet to him? 450. How many lemons will there be whose oil tastes sweet to Al? 300. How many of the 900 cars will have oil that tastes good to Al? 750. What fraction of the cars whose oil tastes sweet to Al will be good cars? 450/750=3/5.

(c) If Al finds that the oil on a car's dipstick tastes sweet, what is the probability that it is a good used car? 3/5.

(d) If Al finds that the oil on a car's dipstick tastes sour, what is the probability that it is a good used car? 0.

(e) Assuming that Al always bids his expected value for a car, given the result of his taste test, how much will Al bid for a car that tastes sweet? $3/5 \times 1,000 + 2/5 \times 100 = \$640.$ How much will he bid for a car that tastes sour? $100.

(f) Consider a naive bidder at Rod's Auction House, who knows that half of the cars are good and half are lemons, but has no clue at all about which ones are good. If this individual bids his expected value for a randomly selected car, how much would he bid? $550.

(g) Given that Al bids his expected value for every used car and the naive bidders bid the expected value of a randomly selected car, will a naive bidder ever get a car whose oil tasted sweet to Al? No.

(h) What is the expected value of cars that naive bidders get if they always bid the expected value of a randomly selected car? $100. Will naive bidders make money, lose money, or break even if they follow this policy? Lose money.

(i) Suppose that bidders other than Al realize that they will get only the cars that taste sour to Al. If they bid the expected value of such a car, how much will they bid? $100.

(j) Suppose that bidders other than Al believe that they will only get cars whose oil tastes sour to Al and suppose that they bid their expected value of such cars. Suppose also that for every car, Al bids his expected value, given the results of his taste test. Who will get the good cars and at what price? (Recall that cars are sold to the highest bidder at the second-highest bid.) Al will get all of the good cars and he will pay $100 for them.

(k) What will Al's expected profit be on a car that passes his test?

$540.

18.9 (3) Steve and Leroy buy antique paintings at an art gallery in Fresno, California. Eighty percent of the paintings that are sold at the gallery are fakes, and the rest are genuine. After a painting is purchased, it will be carefully analyzed, and then everybody will know for certain whether it is genuine or a fake. A genuine antique is worth $1,000. A fake is worthless. Before they place their bids, buyers are allowed to

inspect the paintings briefly and then must place their bids. Because they are allowed only a brief inspection, Steve and Leroy each try to guess whether the paintings are fakes by smelling them. Steve finds that if a painting fails his sniff test, then it is certainly a fake. However, he cannot detect all fakes. In fact the probability that a fake passes Steve's sniff test is 1/2. Leroy detects fakes in the same way as Steve. Half of the fakes fail his sniff test and half of them pass his sniff test. Genuine paintings are sure to pass Leroy's sniff test. For any fake, the probability that Steve recognizes it as a fake is independent of the probability that Leroy recognizes it as a fake.

The auction house posts a price for each painting. Potential buyers can submit a written offer to buy at the posted price on the day of the sale. If more than one person offers to buy the painting, the auction house will select one of them at random and sell it to that person at the posted price.

(a) One day, as the auction house is about to close, Steve arrives and discovers that neither Leroy nor any other bidders have appeared. He sniffs a painting, and it passes his test. Given that it has passed his test, what is the probability that it is a good painting? (Hint: Since fakes are much more common than good paintings, the number of fakes that pass Steve's test will exceed the number of genuine antiques that pass his test.)

1/3. Steve realizes that he can buy the painting for the posted price if he wants it. What is the highest posted price at which he would be willing to buy the painting? $333.33.

(b) On another day, Steve and Leroy see each other at the auction, sniffing all of the paintings. No other customers have appeared at the auction house. In deciding how much to bid for a painting that passes his sniff test, Steve considers the following: If a painting is selected at random and sniffed by both Steve and Leroy, there are five possible outcomes. Fill in the blanks for the probability of each.

- Genuine and passes both dealers' tests. Probability: .2.
- Fake and passes both dealers' tests. Probability: .2.
- Fake and passes Steve's test but fails Leroy's. Probability: .2.
- Fake and passes Leroy's test but fails Steve's. Probability: .2.
- Fake and fails both dealers' tests. Probability: .2.

(c) On the day when Steve and Leroy are the only customers, the auction house sets a reserve price of $300. Suppose that Steve believes that Leroy will offer to buy any painting that passes his sniff test. Recall that if Steve and Leroy both bid on a painting, the probability that Steve gets it is only 1/2. If Steve decides to bid on every painting that passes his

own sniff test, what is the probability that a randomly selected painting is genuine and that Steve is able to buy it? .1. What is the probability that a randomly selected painting is a fake and that Steve will bid on it and get it? .3. If Steve offers to pay \$300 for every painting that passes his sniff test, will his expected profit be positive or negative?

Negative. Suppose that Steve knows that Leroy is willing to pay the reserve price for any painting that passes Leroy's sniff test. What is the highest reserve price that Steve should be willing to pay for a painting that passes his own sniff test? \$250.

18.10 (2) Every day the Repo finance company holds a sealed-bid, second-price auction in which it sells a repossessed automobile. There are only three bidders who bid on these cars, Arnie, Barney, and Carny. Each of these bidders is a used-car dealer whose willingness to pay for another used car fluctuates randomly from day to day in response to the variation in demand at his car lot. The value of one of these used cars to any dealer, on any given day is a random variable which takes a high value $\$H$ with probability 1/2 and a low value $\$L$ with probability 1/2. The value that each dealer places on a car on a given day is independent of the values placed by the other dealers.

Each day the used-car dealers submit written bids for the used car being auctioned. The Repo finance company will sell the car to the dealer with the highest bid at the price bid by the second-highest bidder. If there is a tie for the highest bid, then the second-highest bid is equal to the highest bid and so that day's car will be sold to a randomly selected top bidder at the price bid by all top bidders.

(a) How much should a dealer bid for a used car on a day when he places a value of $\$H$ on a used car? $\$H$. How much should a dealer bid for a used car on a day when he places a value of $\$L$ on a used car? $\$L$.

(b) If the dealers do not collude, how much will Repo get for a used car on days when two or three dealers value the car at $\$H$? $\$H$. How much will Repo get for a used car on days when fewer than two dealers value the car at $\$H$? $\$L$.

(c) On any given day, what is the probability that Repo receives $\$H$ for that day's used car? 1/2. What is the probability that Repo receives $\$L$ for that day's used car? 1/2. What is Repo's expected revenue from the sale? $\$(H+L)/2$.

(d) If there is no collusion and every dealer bids his actual valuation for every used car, what is the probability on any given day that Arnie gets a car for a lower price than the value he places on it? (Hint: This will happen only if the car is worth $\$H$ to Arnie and $\$L$ to the other dealers.)

1/8. Suppose that we measure a car dealer's profit by the difference between what a car is worth to him and what he pays for it. On a randomly selected day, what is Arnie's expected profit? $(H-L)/8$.

(e) The expected total profit of all participants in the market is the sum of the expected profits of the three car dealers and the expected revenue realized by Repo. Used cars are sold by a sealed-bid, second-price auction and the dealers do not collude. What is the sum of the expected profits of all participants in the market? $(3(H-L)/8)+((H+L)/2) = (7/8)H + (1/8)L$.

18.11 (3) This problem (and the two that follow) concerns collusion among bidders in sealed-bid auctions. Many writers have found evidence that collusive bidding occurs. The common name for a group that practices collusive bidding is a "bidding ring."*

Arnie, Barney, and Carny of the previous problem happened to meet at a church social and got to talking about the high prices they were paying for used cars and the low profits they were making. Carny complained, "About half the time the used cars go for $\$H$, and when that happens, none of us makes any money." Arnie got a thoughtful look and then whispered, "Why don't we agree to always bid $\$L$ in Repo's used-car auctions?" Barney said, "I'm not so sure that's a good idea. If we all bid $\$L$, then we will save some money, but the trouble is, when we all bid the same, we are just as likely to get the car if we have a low value as we are to get it if we have a high value. When we bid what we think its worth, then it always goes to one of the people who value it most."

(a) If Arnie, Barney, and Carny agree to always bid $\$L$, then on any given day, what is the probability that Barney gets the car for $\$L$ when it is actually worth $\$H$ to him? 1/6. What is Barney's expected profit per day? $\$(H-L)/6$.

(b) Do the three dealers make higher expected profits with this collusive agreement than they would if they did not collude? Explain. Yes. $(H-L)/6 > (H-L)/8$.

* Our discussion draws extensively on the paper, "Collusive Bidder Behavior at Single-Object, Second-Price, and English Auctions" by Daniel Graham and Robert Marshall in the *Journal of Political Economy*, 1987.

(c) Calculate the expected total profits of all participants in the market (including Repo as well as the three dealers) in the case where the dealers collude. $3(H-L)/6+L=(H+L)/2$. Are these expected total profits larger or smaller than they are when the dealers do not collude? Smaller.

(d) The cars are said to be allocated efficiently if a car never winds up in the hands of a dealer who values it less than some other dealer values it. With a sealed-bid, second-price auction, if there is no collusion, are the cars allocated efficiently? Yes. If the dealers collude as in this problem, are the cars allocated efficiently? No.

18.12 (2) Arnie, Barney, and Carny happily practiced the strategy of "always bid low" for several weeks, until one day Arnie had another idea. Arnie proposed to the others: "When we all bid $\$L$, it sometimes happens that the one who gets the week's car values it at only $\$L$ although it is worth $\$H$ to somebody else. I've thought of a scheme that will increase profits for all of us." Here is Arnie's scheme. Every day, before Repo holds its auction, Arnie, Barney and Carny will hold a sealed-bid, second-price preauction auction among themselves in which they bid for the right to be the only high bidder in that day's auction. The dealer who wins this preauction bidding can bid anything he likes, while the other two bidders must bid $\$L$. A preauction auction like this is known is a "knockout." The revenue that is collected from the "knockout" auction is divided equally among Arnie, Barney, and Carny. For this problem, assume that in the knockout auction, each bidder bids his actual value of winning the knockout auction.*

(a) If the winner of the knockout auction values the day's used car at $\$H$, then he knows that he can bid $\$H$ for this car in Repo's second-price sealed-bid auction and he will get it for a price of $\$L$. Therefore the value of winning the knockout auction to someone who values a used car at $\$H$ must be $\$H-L$. The value of winning the knockout auction to someone who values a used car at $\$L$ is 0

(b) On a day when one dealer values the used car at $\$H$ and the other two value it at $\$L$, the dealer with value $\$H$ will bid $\$H-L$ in

* It is not necessarily the case that this is the best strategy in the knockout auction, since one's bids affect the revenue redistributed from the auction as well as who gets the right to bid. Graham and Marshall present a variation on this mechanism that ensures "honest" bidding in the knockout auction.

the knockout auction and the other two dealers will bid 0 In this case, in the knockout auction, the dealer pays 0 for the right to be the only high bidder in Repo's auction. In this case, the day's used car will go to the only dealer with value $\$H$ and he pays Repo $\$L$ for it. On this day, the dealer with the high buyer value makes a total profit of $\$H - L$

(c) We continue to assume that in the knockout auction, dealers bid their actual values of winning the knockout. On days when two or more buyers value the used car at $\$H$, the winner of the knockout auction pays $H - L$ for the right to be the only high bidder in Repo's auction.

(d) If Arnie's scheme is adopted, what is the expected total profit of each of the three car dealers? (Remember to include each dealer's share of the revenue from the knockout auction.) 7(H-L)/8

18.13 (2) After the passage of several weeks during which Repo never got more than one high bid for a car, the Repo folks guessed that something was amiss. Some members of the board of directors proposed hiring a hit man to punish Arnie, Barney, and Carny, but cooler heads prevailed and they decided instead to hire an economist who had studied *Intermediate Microeconomics*. The economist suggested: "Why don't you set a reserve price $\$R$ which is just a little bit lower than $\$H$ (but of course much larger than $\$L$)? If you get at least one bid of $\$R$, sell it for $\$R$ to one of these bidders, and if you don't get a bid as large as $\$R$, then just dump that day's car into the river. (Sadly, the environmental protection authorities in Repo's hometown are less than vigilant.) "But what a waste," said a Repo official. "Just do the math," replied the economist.

(a) The economist continued. "If Repo sticks to its guns and refuses to sell at any price below $\$R$, then even if Arnie, Barney, and Carny collude, the best they can do is for each to bid $\$R$ when they value a car at $\$H$ and to bid nothing when they value it at $\$L$." If they follow this strategy, the probability that Repo can sell a given car for $\$R$ is 7/8 , so Repo's expected profit will be $\$(7/8)R$.

(b) Setting a reserve price that is just slightly below $\$H$ and destroying cars for which it gets no bid will be more profitable for Repo than setting no reservation price if the ratio H/L is greater than 7/8. and less profitable if H/L is less than 7/8.

18.14 (1) Three boys, Alan, Bill, and Charlie, and three girls, Alice, Betsy, and Clara, are planning dates for the senior prom. Their preferences over possible partners are given in this table:

Person	First Choice	Second Choice	Third Choice
Alan	Alice	Betsy	Clara
Bill	Betsy	Alice	Clara
Charlie	Alice	Clara	Betsy
Alice	Bill	Alan	Charlie
Betsy	Alan	Bill	Charlie
Clara	Alan	Charlie	Bill

For example, Alan likes Alice best, Betsy second, and Clara third, while Alice likes Bill best, Alan second, and Charlie third.

(a) Suppose that partners are assigned by the deferred acceptance algorithm, where the boys ask the girls. On the first round, Alice will be asked by `Alan` and `Charlie`. Betsy will be asked by `Bill`. Clara will be asked by `Nobody`. Who does Alice refuse on the first round? `Charlie` What happens on the second round? `Charlie proposes to Clara` What is the final assignment of partners? Alice is matched with `Alan`, Betsy with `Bill`, and Clara with `Charlie`. Verify that this assignment is stable.

(b) Suppose that instead they assign partners by the deferred acceptance algorithm, with the girls asking the boys. What is the final assignment of partners? Alice is matched with `Bill`, Betsy is matched with `Alan`, and Clara is matched with `Charlie`. Verify that this assignment is stable.

(c) In this example, which group is in general better off, the proposing sex or the proposed-to sex? `The proposers` (It turns out that this result is true in general. See David Gale and Lloyd Shapley: "College Admissions and the Stability of Marriage", *American Mathematical Monthly* volume 69, pages 9-14, 1962.)

(d) Suppose that partners are assigned by the deferred acceptance algorithm, with the boys proposing. Alice knows everybody's preferences and believes that everybody else will act according to their true preferences. Can she get a better outcome by a bit of deception? Suppose that Alice chooses Charlie over Alan, what will happen in the second round?

Alan will propose to Betsy What will happen next? Betsy will choose Alan and tell Bill to go away and Bill will propose to Alice. What will be the final outcome. Alice matches with Bill, Betsy matches with Alan, Clara matches with Charlie. Will Alice be better off than if she had played "honestly" and picked Alan over Charlie? yes

18.15 (1) Four students, Art, Bob, Cal, and Dan, must be assigned rooms in a college dorm. The dean of housing would like to assign them in stable partnerships, so that no two students would prefer each other to their assigned partners. The students' preferences are as follows:

Person	First Choice	Second Choice	Third Choice
Art	Bob	Cal	Dan
Bob	Cal	Art	Dan
Cal	Art	Bob	Dan
Dan	Art	Bob	Cal

(a) Is there any stable assignment? Explain your answer. (Hint: Note that every student except Dan is the first choice of somebody other than Dan.) No. Suppose there is a stable assignment. Somebody would have to be Dan's partner. Dan's partner would prefer anyone to Dan. But Dan's partner is the first choice of someone other than Dan.

(b) The humorist James Thurber once wrote a book called *Is Sex Necessary?* In what sense does this example offer an answer to Thurber's question? The deferred acceptance algorithm guarantees that there exists a stable match when there are two ''sexes'' such that pairs include one member of each sex. But when anyone can match with anyone else, there need not be a stable match.

Chapter 19

NAME_____

Technology

Introduction. In this chapter you work with production functions, relating output of a firm to the inputs it uses. This theory will look familiar to you, because it closely parallels the theory of utility functions. In utility theory, an *indifference curve* is a locus of commodity bundles, all of which give a consumer the same utility. In production theory, an *isoquant* is a locus of input combinations, all of which give the same output. In consumer theory, you found that the slope of an indifference curve at the bundle (x_1, x_2) is the ratio of marginal utilities, $MU_1(x_1, x_2)/MU_2(x_1, x_2)$. In production theory, the slope of an isoquant at the input combination (x_1, x_2) is the ratio of the marginal products, $MP_1(x_1, x_2)/MP_2(x_1, x_2)$. Most of the functions that we gave as examples of utility functions can also be used as examples of production functions.

There is one important difference between production functions and utility functions. Remember that utility functions were only "unique up to monotonic transformations." In contrast, two different production functions that are monotonic transformations of each other describe different technologies.

Example: If the utility function $U(x_1, x_2) = x_1 + x_2$ represents a person's preferences, then so would the utility function $U^*(x_1, x_2) = (x_1 + x_2)^2$. A person who had the utility function $U^*(x_1, x_2)$ would have the same indifference curves as a person with the utility function $U(x_1, x_2)$ and would make the same choices from every budget. But suppose that one firm has the production function $f(x_1, x_2) = x_1 + x_2$, and another has the production function $f^*(x_1, x_2) = (x_1 + x_2)^2$. It is true that the two firms will have the same isoquants, but they certainly do not have the same technology. If both firms have the input combination $(x_1, x_2) = (1, 1)$, then the first firm will have an output of 2 and the second firm will have an output of 4.

Now we investigate "returns to scale." Here we are concerned with the change in output if the amount of every input is multiplied by a number $t > 1$. If multiplying inputs by t multiplies output by more than t, then there are increasing returns to scale. If output is multiplied by exactly t, there are constant returns to scale. If output is multiplied by less than t, then there are decreasing returns to scale.

Example: Consider the production function $f(x_1, x_2) = x_1^{1/2} x_2^{3/4}$. If we multiply the amount of each input by t, then output will be $f(tx_1, tx_2) = (tx_1)^{1/2}(tx_2)^{3/4}$. To compare $f(tx_1, tx_2)$ to $f(x_1, x_2)$, factor out the expressions involving t from the last equation. You get $f(tx_1, tx_2) = t^{5/4} x_1^{1/2} x_2^{3/4} = t^{5/4} f(x_1, x_2)$. Therefore when you multiply the amounts of all inputs by t, you multiply the amount of output by $t^{5/4}$. This means there are *increasing* returns to scale.

Example: Let the production function be $f(x_1, x_2) = \min\{x_1, x_2\}$. Then

$$f(tx_1, tx_2) = \min\{tx_1, tx_2\} = \min t\{x_1, x_2\} = t \min\{x_1, x_2\} = tf(x_1, x_2).$$

Therefore when all inputs are multiplied by t, output is also multiplied by t. It follows that this production function has *constant* returns to scale.

You will also be asked to determine whether the marginal product of each single factor of production increases or decreases as you increase the amount of that factor without changing the amount of other factors. Those of you who know calculus will recognize that the marginal product of a factor is the first derivative of output with respect to the amount of that factor. Therefore the marginal product of a factor will decrease, increase, or stay constant as the amount of the factor increases depending on whether the *second* derivative of the production function with respect to the amount of that factor is negative, positive, or zero.

Example: Consider the production function $f(x_1, x_2) = x_1^{1/2} x_2^{3/4}$. The marginal product of factor 1 is $\frac{1}{2} x_1^{-1/2} x_2^{3/4}$. This is a decreasing function of x_1, as you can verify by taking the derivative of the marginal product with respect to x_1. Similarly, you can show that the marginal product of x_2 decreases as x_2 increases.

19.0 Warm Up Exercise. The first part of this exercise is to calculate marginal products and technical rates of substitution for several frequently encountered production functions. As an example, consider the production function $f(x_1, x_2) = 2x_1 + \sqrt{x_2}$. The marginal product of x_1 is the derivative of $f(x_1, x_2)$ with respect to x_1, holding x_2 fixed. This is just 2. The marginal product of x_2 is the derivative of $f(x_1, x_2)$ with respect to x_2, holding x_1 fixed, which in this case is $\frac{1}{2\sqrt{x_2}}$. The TRS is $-MP_1/MP_2 = -4\sqrt{x_2}$. Those of you who do not know calculus should fill in this table from the answers in the back. The table will be a useful reference for later problems.

Marginal Products and Technical Rates of Substitution

$f(x_1, x_2)$	$MP_1(x_1, x_2)$	$MP_2(x_1, x_2)$	$TRS(x_1, x_2)$
$x_1 + 2x_2$	1	2	$-1/2$
$ax_1 + bx_2$	a	b	$-a/b$
$50x_1x_2$	$50x_2$	$50x_1$	$-\frac{x_2}{x_1}$
$x_1^{1/4}x_2^{3/4}$	$\frac{1}{4}x_1^{-3/4}x_2^{3/4}$	$\frac{3}{4}x_1^{1/4}x_2^{-1/4}$	$-\frac{x_2}{3x_1}$
$Cx_1^a x_2^b$	$Cax_1^{a-1}x_2^b$	$Cbx_1^a x_2^{b-1}$	$-\frac{ax_2}{bx_1}$
$(x_1+2)(x_2+1)$	x_2+1	x_1+2	$-\left(\frac{x_2+1}{x_1+2}\right)$
$(x_1+a)(x_2+b)$	x_2+b	x_1+a	$-\left(\frac{x_2+b}{x_1+a}\right)$
$ax_1 + b\sqrt{x_2}$	a	$\frac{b}{2\sqrt{x_2}}$	$-\frac{2a\sqrt{x_2}}{b}$
$x_1^a + x_2^a$	ax_1^{a-1}	ax_2^{a-1}	$-\left(\frac{x_1}{x_2}\right)^{a-1}$
$(x_1^a + x_2^a)^b$	$bax_1^{a-1}\left(x_1^a + x_2^a\right)^{b-1}$	$bax_2^{a-1}\left(x_1^a + x_2^a\right)^{b-1}$	$-\left(\frac{x_1}{x_2}\right)^{a-1}$

Returns to Scale and Changes in Marginal Products

For each production function in the table below, put an I, C, or D in the first column if the production function has increasing, constant, or decreasing returns to scale. Put an I, C, or D in the second (third) column, depending on whether the marginal product of factor 1 (factor 2) is increasing, constant, or decreasing, as the amount of that factor alone is varied.

$f(x_1, x_2)$	Scale	MP_1	MP_2
$x_1 + 2x_2$	C	C	C
$\sqrt{x_1 + 2x_2}$	D	D	D
$.2x_1x_2^2$	I	C	I
$x_1^{1/4}x_2^{3/4}$	C	D	D
$x_1 + \sqrt{x_2}$	D	C	D
$(x_1 + 1)^{.5}(x_2)^{.5}$	D	D	D
$\left(x_1^{1/3} + x_2^{1/3}\right)^3$	C	D	D

19.1 (0) Prunella raises peaches. Where L is the number of units of labor she uses and T is the number of units of land she uses, her output is $f(L, T) = L^{\frac{1}{2}}T^{\frac{1}{2}}$ bushels of peaches.

(a) On the graph below, plot some input combinations that give her an output of 4 bushels. Sketch a production isoquant that runs through these points. The points on the isoquant that gives her an output of 4 bushels all satisfy the equation $T =$ $16/L$.

(b) This production function exhibits (constant, increasing, decreasing) returns to scale. Constant returns to scale.

(c) In the short run, Prunella cannot vary the amount of land she uses. On the graph below, use blue ink to draw a curve showing Prunella's output as a function of labor input if she has 1 unit of land. Locate the points on your graph at which the amount of labor is 0, 1, 4, 9, and 16 and label them. The slope of this curve is known as the marginal product of labor. Is this curve getting steeper or flatter as the amount of labor increase? Flatter.

(d) Assuming she has 1 unit of land, how much extra output does she get from adding an extra unit of labor when she previously used 1 unit of labor? $\sqrt{2} - 1 \approx .41$. 4 units of labor? $\sqrt{5} - 2 \approx .24$. If you know calculus, compute the marginal product of labor at the input combination $(1, 1)$ and compare it with the result from the unit increase in labor output found above. Derivative is $1/2\sqrt{L}$, so the MP is .5 when $L = 1$ and .25 when $L = 4$.

(e) In the long run, Prunella can change her input of land as well as of labor. Suppose that she increases the size of her orchard to 4 units of land. Use red ink to draw a new curve on the graph above showing output as a function of labor input. Also use red ink to draw a curve showing marginal product of labor as a function of labor input when the amount of land is fixed at 4.

19.2 (0) Suppose x_1 and x_2 are used in fixed proportions and $f(x_1, x_2) = \min\{x_1, x_2\}$.

(a) Suppose that $x_1 < x_2$. The marginal product for x_1 is 1 and (increases, remains constant, decreases) remains constant for small increases in x_1. For x_2 the marginal product is 0, and (increases, remains constant, decreases) remains constant for small increases in x_2. The technical rate of substitution between x_2 and x_1 is infinity. This technology demonstrates (increasing, constant, decreasing) constant returns to scale.

(b) Suppose that $f(x_1, x_2) = \min\{x_1, x_2\}$ and $x_1 = x_2 = 20$. What is the marginal product of a small increase in x_1? 0. What is the marginal product of a small increase in x_2? 0. The marginal product of x_1 will (increase, decrease, stay constant) increase if the amount of x_2 is increased by a little bit.

Calculus **19.3 (0)** Suppose the production function is Cobb-Douglas and $f(x_1, x_2) = x_1^{1/2} x_2^{3/2}$.

(a) Write an expression for the marginal product of x_1 at the point (x_1, x_2). $\frac{1}{2} x_1^{-1/2} x_2^{3/2}$.

(b) The marginal product of x_1 (increases, decreases, remains constant) decreases for small increases in x_1, holding x_2 fixed.

(c) The marginal product of factor 2 is $3/2x_1^{1/2}x_2^{1/2}$, and it (increases, remains constant, decreases) increases for small increases in x_2.

(d) An increase in the amount of x_2 (increases, leaves unchanged, decreases) increases the marginal product of x_1.

(e) The technical rate of substitution between x_2 and x_1 is $-x_2/3x_1$.

(f) Does this technology have diminishing technical rate of substitution? Yes.

(g) This technology demonstrates (increasing, constant, decreasing) increasing returns to scale.

19.4 (0) The production function for fragles is $f(K, L) = L/2 + \sqrt{K}$, where L is the amount of labor used and K the amount of capital used.

(a) There are (constant, increasing, decreasing) decreasing returns to scale. The marginal product of labor is constant (constant, increasing, decreasing).

(b) In the short run, capital is fixed at 4 units. Labor is variable. On the graph below, use blue ink to draw output as a function of labor input in the short run. Use red ink to draw the marginal product of labor as a function of labor input in the short run. The average product of labor is defined as total output divided by the amount of labor input. Use black ink to draw the average product of labor as a function of labor input in the short run.

19.5 (0) General Monsters Corporation has two plants for producing juggernauts, one in Flint and one in Inkster. The Flint plant produces according to $f_F(x_1, x_2) = \min\{x_1, 2x_2\}$ and the Inkster plant produces according to $f_I(x_1, x_2) = \min\{2x_1, x_2\}$, where x_1 and x_2 are the inputs.

(a) On the graph below, use blue ink to draw the isoquant for 40 juggernauts at the Flint plant. Use red ink to draw the isoquant for producing 40 juggernauts at the Inkster plant.

(b) Suppose that the firm wishes to produce 20 juggernauts at each plant. How much of each input will the firm need to produce 20 juggernauts at the Flint plant? $x_1 = 20, x_2 = 10.$ How much of each input will the firm need to produce 20 juggernauts at the Inkster plant? $x_1 = 10, x_2 = 20.$ Label with an a on the graph, the point representing the total amount of each of the two inputs that the firm needs to produce a total of 40 juggernauts, 20 at the Flint plant and 20 at the Inkster plant.

(c) Label with a b on your graph the point that shows how much of each of the two inputs is needed in toto if the firm is to produce 10 juggernauts in the Flint plant and 30 juggernauts in the Inkster plant. Label with a c the point that shows how much of each of the two inputs that the firm needs in toto if it is to produce 30 juggernauts in the Flint plant and 10 juggernauts in the Inkster plant. Use a black pen to draw the firm's isoquant for producing 40 units of output if it can split production in any manner between the two plants. Is the technology available to this firm convex? Yes.

19.6 (0) You manage a crew of 160 workers who could be assigned to make either of two products. Product A requires 2 workers per unit of output. Product B requires 4 workers per unit of output.

(a) Write an equation to express the combinations of products A and B that could be produced using exactly 160 workers. $2A + 4B = 160.$ On the diagram below, use blue ink to shade in the area depicting

the combinations of A and B that could be produced with 160 workers. (Assume that it is also possible for some workers to do nothing at all.)

(b) Suppose now that every unit of product A that is produced requires the use of 4 shovels as well as 2 workers and that every unit of product B produced requires 2 shovels and 4 workers. On the graph you have just drawn, use red ink to shade in the area depicting combinations of A and B that could be produced with 180 shovels if there were no worries about the labor supply. Write down an equation for the set of combinations of A and B that require exactly 180 shovels. $4A + 2B = 180$.

(c) On the same diagram, use black ink to shade the area that represents possible output combinations when one takes into account both the limited supply of labor and the limited supply of shovels.

(d) On your diagram locate the feasible combination of inputs that use up all of the labor and all of the shovels. If you didn't have the graph, what equations would you solve to determine this point? $2A + 4B = 160$ and $4A + 2B = 180$.

(e) If you have 160 workers and 180 shovels, what is the largest amount of product A that you could produce? 45 units. If you produce this amount, you will not use your entire supply of one of the inputs. Which one? Workers. How many will be left unused? 70.

19.7 (0) A firm has the production function $f(x, y) = \min\{2x, x + y\}$. On the graph below, use red ink to sketch a couple of production isoquants for this firm. A second firm has the production function $f(x, y) = x + \min\{x, y\}$. Do either or both of these firms have constant returns to scale?

Both do. On the same graph, use black ink to draw a couple of isoquants for the second firm.

19.8 (0) Suppose the production function has the form

$$f(x_1, x_2, x_3) = Ax_1^a x_2^b x_3^c,$$

where $a + b + c > 1$. Prove that there are increasing returns to scale.

For any $t > 1$**,** $f(tx_1, tx_2, tx_3) = A(tx_1)^a(tx_2)^b(tx_3)^c = t^{a+b+c}f(x_1, x_2, x_3) > tf(x_1, x_2, x_3)$**.**

19.9 (0) Suppose that the production function is $f(x_1, x_2) = Cx_1^a x_2^b$, where a, b, and C are positive constants.

(a) For what positive values of a, b, and C are there decreasing returns to scale? **All** $C > 0$ **and** $a + b < 1$**.** constant returns to scale? **All** $C > 0$ **and** $a + b = 1$**.** increasing returns to scale? **All** $C > 0$ **and** $a + b > 1$**.**

(b) For what positive values of a, b, and C is there decreasing marginal product for factor 1? **All** $C > 0$ **and** $b > 0$ **and** $a < 1$**.**

(c) For what positive values of a, b, and C is there diminishing technical rate of substitution? For all positive values.

19.10 (0) Suppose that the production function is $f(x_1, x_2) = (x_1^a + x_2^a)^b$, where a and b are positive constants.

(a) For what positive values of a and b are there decreasing returns to scale? $ab < 1$. Constant returns to scale? $ab = 1$. Increasing returns to scale? $ab > 1$.

19.11 (0) Suppose that a firm has the production function $f(x_1, x_2) = \sqrt{x_1} + x_2^2$.

(a) The marginal product of factor 1 (increases, decreases, stays constant) decreases as the amount of factor 1 increases. The marginal product of factor 2 (increases, decreases, stays constant) increases as the amount of factor 2 increases.

(b) This production function does not satisfy the definition of increasing returns to scale, constant returns to scale, or decreasing returns to scale. How can this be? Returns to scale are different depending on the ratio in which the factors are used. Find a combination of inputs such that doubling the amount of both inputs will more than double the amount of output. $x_1 = 1$, $x_2 = 4$, for example. Find a combination of inputs such that doubling the amount of both inputs will less than double output. $x_1 = 4$, $x_2 = 0$, for example.

Chapter 20

NAME_____

Profit Maximization

Introduction. A firm in a competitive industry cannot charge more than the market price for its output. If it also must compete for its inputs, then it has to pay the market price for inputs as well. Suppose that a profit-maximizing competitive firm can vary the amount of only one factor and that the marginal product of this factor decreases as its quantity increases. Then the firm will maximize its profits by hiring enough of the variable factor so that the value of its marginal product is equal to the wage. Even if a firm uses several factors, only some of them may be variable in the short run.

Example: A firm has the production function $f(x_1, x_2) = x_1^{1/2} x_2^{1/2}$. Suppose that this firm is using 16 units of factor 2 and is unable to vary this quantity in the short run. In the short run, the only thing that is left for the firm to choose is the amount of factor 1. Let the price of the firm's output be p, and let the price it pays per unit of factor 1 be w_1. We want to find the amount of x_1 that the firm will use and the amount of output it will produce. Since the amount of factor 2 used in the short run must be 16, we have output equal to $f(x_1, 16) = 4x_1^{1/2}$. The marginal product of x_1 is calculated by taking the derivative of output with respect to x_1. This marginal product is equal to $2x_1^{-1/2}$. Setting the value of the marginal product of factor 1 equal to its wage, we have $p2x_1^{-1/2} = w_1$. Now we can solve this for x_1. We find $x_1 = (2p/w_1)^2$. Plugging this into the production function, we see that the firm will choose to produce $4x_1^{1/2} = 8p/w_1$ units of output.

In the long run, a firm is able to vary all of its inputs. Consider the case of a competitive firm that uses two inputs. Then if the firm is maximizing its profits, it must be that the value of the marginal product of each of the two factors is equal to its wage. This gives two equations in the two unknown factor quantities. If there are decreasing returns to scale, these two equations are enough to determine the two factor quantities. If there are constant returns to scale, it turns out that these two equations are only sufficient to determine the *ratio* in which the factors are used.

In the problems on the weak axiom of profit maximization, you are asked to determine whether the observed behavior of firms is consistent with profit-maximizing behavior. To do this you will need to plot some of the firm's isoprofit lines. An isoprofit line relates all of the input-output combinations that yield the same amount of profit for some given input and output prices. To get the equation for an isoprofit line, just write down an equation for the firm's profits at the given input and output prices. Then solve it for the amount of output produced as a function of the amount of the input chosen. Graphically, you know that a firm's behavior is consistent with profit maximization if its input-output choice in each period lies below the isoprofit lines of the other periods.

20.1 (0) The short-run production function of a competitive firm is given by $f(L) = 6L^{2/3}$, where L is the amount of labor it uses. (For those who do not know calculus—if total output is aL^b, where a and b are constants, and where L is the amount of some factor of production, then the marginal product of L is given by the formula abL^{b-1}.) The cost per unit of labor is $w = 6$ and the price per unit of output is $p = 3$.

(a) Plot a few points on the graph of this firm's production function and sketch the graph of the production function, using blue ink. Use black ink to draw the isoprofit line that passes through the point $(0, 12)$, the isoprofit line that passes through $(0, 8)$, and the isoprofit line that passes through the point $(0, 4)$. What is the slope of each of the isoprofit lines? They all have slope 2. How many points on the isoprofit line through $(0, 12)$ consist of input-output points that are actually possible? None. Make a squiggly line over the part of the isoprofit line through $(0, 4)$ that consists of outputs that are actually possible.

(b) How many units of labor will the firm hire? 8. How much output will it produce? 24. If the firm has no other costs, how much will its total profits be? 24.

(c) Suppose that the wage of labor falls to 4, and the price of output remains at p. On the graph, use red ink to draw the new isoprofit line for the firm that passes through its old choice of input and output. Will the firm increase its output at the new price? Yes. Explain why, referring to your diagram. As the diagram shows, the firm can reach a higher isoprofit line by increasing output.

Calculus **20.2 (0)** A Los Angeles firm uses a single input to produce a recreational commodity according to a production function $f(x) = 4\sqrt{x}$, where x is the number of units of input. The commodity sells for \$100 per unit. The input costs \$50 per unit.

(a) Write down a function that states the firm's profit as a function of the amount of input. $\pi = 400\sqrt{x} - 50x$.

(b) What is the profit-maximizing amount of input? 16. of output? 16. How much profits does it make when it maximizes profits? \$800.

(c) Suppose that the firm is taxed \$20 per unit of its output and the price of its input is subsidized by \$10. What is its new input level? 16. What is its new output level? 16. How much profit does it make now? \$640. (Hint: A good way to solve this is to write an expression for the firm's profit as a function of its input and solve for the profit-maximizing amount of input.)

(d) Suppose that instead of these taxes and subsidies, the firm is taxed at 50% of its profits. Write down its after-tax profits as a function of the amount of input. $\pi = .50 \times (400\sqrt{x} - 50x)$. What is the profit-maximizing amount of output? 16. How much profit does it make after taxes? \$400.

20.3 (0) Brother Jed takes heathens and reforms them into righteous individuals. There are two inputs needed in this process: heathens (who are widely available) and preaching. The production function has the following form: $r_p = \min\{h, p\}$, where r_p is the number of righteous

persons produced, h is the number of heathens who attend Jed's sermons, and p is the number of hours of preaching. For every person converted, Jed receives a payment of s from the grateful convert. Sad to say, heathens do not flock to Jed's sermons of their own accord. Jed must offer heathens a payment of w to attract them to his sermons. Suppose the amount of preaching is fixed at $\bar{p}$ and that Jed is a profit-maximizing prophet.

(a) If $h < \bar{p}$, what is the marginal product of heathens? 1. What is the value of the marginal product of an additional heathen? s.

(b) If $h > \bar{p}$, what is the marginal product of heathens? 0. What is the value of the marginal product of an additional heathen in this case?

0.

(c) Sketch the shape of this production function in the graph below. Label the axes, and indicate the amount of the input where $h = \bar{p}$.

(d) If $w < s$, how many heathens will be converted? $\bar{p}$. If $w > s$, how many heathens will be converted? 0.

20.4 (0) Allie's Apples, Inc. purchases apples in bulk and sells two products, boxes of apples and jugs of cider. Allie's has capacity limitations of three kinds: warehouse space, crating facilities, and pressing facilities. A box of apples requires 6 units of warehouse space, 2 units of crating facilities, and no pressing facilities. A jug of cider requires 3 units of warehouse space, 2 units of crating facilities, and 1 unit of pressing facilities. The total amounts available each day are: 1,200 units of warehouse space, 600 units of crating facilities, and 250 units of pressing facilities.

(a) If the only capacity limitations were on warehouse facilities, and if all warehouse space were used for the production of apples, how many boxes of apples could be produced in one day? 200. How many jugs of cider could be produced each day if, instead, all warehouse space were used in

the production of cider and there were no other capacity constraints?

400. Draw a blue line in the following graph to represent the warehouse space constraint on production combinations.

(b) Following the same reasoning, draw a red line to represent the constraints on output to limitations on crating capacity. How many boxes of apples could Allie produce if he only had to worry about crating capacity?

300. How many jugs of cider? 300.

(c) Finally draw a black line to represent constraints on output combinations due to limitations on pressing facilities. How many boxes of apples could Allie produce if he only had to worry about the pressing capacity and no other constraints? An infinite number. How many jugs of cider? 250.

(d) Now shade the area that represents feasible combinations of daily production of apples and cider for Allie's Apples.

(e) Allie's can sell apples for \$5 per box of apples and cider for \$2 per jug. Draw a black line to show the combinations of sales of apples and cider that would generate a revenue of \$1,000 per day. At the profit-maximizing production plan, Allie's is producing 200 boxes of apples and 0 jugs of cider. Total revenues are \$1,000.

20.5 (0) A profit-maximizing firm produces one output, y, and uses one input, x, to produce it. The price per unit of the factor is denoted by w and the price of the output is denoted by p. You observe the firm's behavior over three periods and find the following:

Period	y	x	w	p
1	1	1	1	1
2	2.5	3	.5	1
3	4	8	.25	1

(a) Write an equation that gives the firm's profits, π, as a function of the amount of input x it uses, the amount of output y it produces, the per-unit cost of the input w, and the price of output p. $\pi = py - wx$.

(b) In the diagram below, draw an isoprofit line for each of the three periods, showing combinations of input and output that would yield the same profits that period as the combination actually chosen. What are the equations for these three lines? $y = x$, $y = 1 + .5x$, $y = 2 + .25x$. Using the theory of revealed profitability, shade in the region on the graph that represents input-output combinations that could be feasible as far as one can tell from the evidence that is available. How would you describe this region in words? `The region that is below all 3 isoprofit lines.`

20.6 (0) T-bone Pickens is a corporate raider. This means that he looks for companies that are not maximizing profits, buys them, and then tries to operate them at higher profits. T-bone is examining the financial records of two refineries that he might buy, the Shill Oil Company and the Golf Oil Company. Each of these companies buys oil and produces gasoline. During the time period covered by these records, the price of gasoline fluctuated significantly, while the cost of oil remained constant at $10 a barrel. For simplicity, we assume that oil is the only input to gasoline production.

Shill Oil produced 1 million barrels of gasoline using 1 million barrels of oil when the price of gasoline was $10 a barrel. When the price of gasoline was $20 a barrel, Shill produced 3 million barrels of gasoline using 4 million barrels of oil. Finally, when the price of gasoline was $40 a barrel, Shill used 10 million barrels of oil to produce 5 million barrels of gasoline.

Golf Oil (which is managed by Martin E. Lunch III) did exactly the same when the price of gasoline was $10 and $20, but when the price of gasoline hit $40, Golf produced 3.5 million barrels of gasoline using 8 million barrels of oil.

(a) Using black ink, plot Shill Oil's isoprofit lines and choices for the three different periods. Label them 10, 20, and 40. Using red ink draw Golf Oil's isoprofit line and production choice. Label it with a 40 in red ink.

(b) How much profits could Golf Oil have made when the price of gasoline was \$40 a barrel if it had chosen to produce the same amount that it did when the price was \$20 a barrel? \$80 million. What profits did Golf actually make when the price of gasoline was \$40? \$60 million.

(c) Is there any evidence that Shill Oil is not maximizing profits? Explain.

No. The data satisfy WAPM.

(d) Is there any evidence that Golf Oil is not maximizing profits? Explain.

Yes. When price of gas was \$40, Golf could have made more money by acting as it did when price of gas was \$20.

20.7 (0) After carefully studying Shill Oil, T-bone Pickens decides that it has probably been maximizing its profits. But he still is very interested in buying Shill Oil. He wants to use the gasoline they produce to fuel his delivery fleet for his chicken farms, Capon Truckin'. In order to do this Shill Oil would have to be able to produce 5 million barrels of gasoline from 8 million barrels of oil. Mark this point on your graph. Assuming that Shill always maximizes profits, would it be technologically feasible for it to produce this input-output combination? Why or why not?

No. If it could, then it would have made more profits by choosing this combination than what it chose when price of oil was \$40.

20.8 (0) Suppose that firms operate in a competitive market, attempt to maximize profits, and only use one factor of production. Then we know that for any changes in the input and output price, the input choice and the output choice must obey the Weak Axiom of Profit Maximization, $\Delta p \Delta y - \Delta w \Delta x \geq 0$.

Which of the following propositions can be proven by the Weak Axiom of Profit Maximizing Behavior (WAPM)? Respond yes or no, and give a short argument.

(a) If the price of the input does not change, then a decrease in the price of the output will imply that the firm will produce the same amount or less output. Yes. If price of input doesn't change, $\Delta w = 0$, so WAPM says $\Delta p \Delta y \geq 0$.

(b) If the price of the output remains constant, then a decrease in the input price will imply that the firm will use the same amount or more of the input. Yes. If price of output doesn't change, $\Delta p = 0$, so WAPM says $-\Delta w \Delta x \geq 0$.

(c) If both the price of the output and the input increase and the firm produces less output, then the firm will use more of the input. No. Sign pattern is $(+)(-) - (+)(+) \geq 0$, which cannot happen.

20.9 (1) Farmer Hoglund has discovered that on his farm, he can get 30 bushels of corn per acre if he applies no fertilizer. When he applies N pounds of fertilizer to an acre of land, the *marginal product* of fertilizer is $1 - N/200$ bushels of corn per pound of fertilizer.

(a) If the price of corn is \$3 a bushel and the price of fertilizer is \$$p$ per pound (where $p < 3$), how many pounds of fertilizer should he use per acre in order to maximize profits? $200 - 66.66p$.

(b) (Only for those who remember a bit of easy integral calculus.) Write down a function that states Farmer Hoglund's yield per acre as a function of the amount of fertilizer he uses. $30 + N - N^2/400$.

(c) Hoglund's neighbor, Skoglund, has better land than Hoglund. In fact, for any amount of fertilizer that he applies, he gets exactly twice as much corn per acre as Hoglund would get with the same amount of fertilizer. How much fertilizer will Skoglund use per acre when the price of corn is \$3 a bushel and the price of fertilizer is \$$p$ a pound? $200 - 33.33p$. (Hint: Start by writing down Skoglund's marginal product of fertilizer as a function of N.)

(d) When Hoglund and Skoglund are both maximizing profits, will Skoglund's output be more than twice as much, less than twice as much or exactly twice as much as Hoglund's? Explain. More than twice as much. S. would produce twice as much as H. if they used equal amounts of fertilizer, but S. uses more fertilizer than H. does.

(e) Explain how someone who looked at Hoglund's and Skoglund's corn yields and their fertilizer inputs but couldn't observe the quality of their land, would get a misleading idea of the productivity of fertilizer. Fertilizer did not cause the entire difference in yield. The best land got the most fertilizer.

20.10 (0) A firm has two variable factors and a production function, $f(x_1, x_2) = x_1^{1/2} x_2^{1/4}$. The price of its output is 4. Factor 1 receives a wage of w_1 and factor 2 receives a wage of w_2.

(a) Write an equation that says that the value of the marginal product of factor 1 is equal to the wage of factor 1 $2x_1^{-1/2} x_2^{1/4} = w_1$ and an equation that says that the value of the marginal product of factor 2 is equal to the wage of factor 2. $x_1^{1/2} x_2^{-3/4} = w_2$. Solve two equations in the two unknowns, x_1 and x_2, to give the amounts of factors 1 and 2 that maximize the firm's profits as a function of w_1 and w_2. This gives $x_1 =$ $8/(w_1^3 w_2)$ and $x_2 =$ $4/(w_1^2 w_2^2)$. (Hint: You could use the first equation to solve for x_1 as a function of x_2 and of the factor wages. Then substitute the answer into the second equation and solve for x_2 as a function of the two wage rates. Finally use your solution for x_2 to find the solution for x_1.)

(b) If the wage of factor 1 is 2, and the wage of factor 2 is 1, how many units of factor 1 will the firm demand? 1. How many units of factor 2 will it demand? 1. How much output will it produce? 1. How much profit will it make? 1.

20.11 (0) A firm has two variable factors and a production function $f(x_1, x_2) = x_1^{1/2} x_2^{1/2}$. The price of its output is 4, the price of factor 1 is w_1, and the price of factor 2 is w_2.

(a) Write the two equations that say that the value of the marginal product of each factor is equal to its wage. $2x_1^{-1/2}x_2^{1/2} = w_1$ and $2x_1^{1/2}x_2^{-1/2} = w_2$. If $w_1 = 2w_2$, these two equations imply that $x_1/x_2 =$ 1/2.

(b) For this production function, is it possible to solve the two marginal productivity equations uniquely for x_1 and x_2? No.

20.12 (1) A firm has two variable factors and a production function $f(x_1, x_2) = \sqrt{2x_1 + 4x_2}$. On the graph below, draw production isoquants corresponding to an ouput of 3 and to an output of 4.

(a) If the price of the output good is 4, the price of factor 1 is 2, and the price of factor 2 is 3, find the profit-maximizing amount of factor 1 0, the profit-maximizing amount of factor 2 16/9, and the profit-maximizing output 8/3.

20.13 (0) A profit-maximizing firm produces one output, y, and uses one input, x, to produce it. The price per unit of the factor is denoted by w and the price of the output is denoted by p. You observe the firm's behavior over three periods and find the following:

Period	y	x	w	p
1	1	1	1	1
2	2.5	3	.5	1
3	4	8	.25	1

(a) Write an equation that gives the firm's profits, π, as a function of the amount of input x it uses, the amount of output y it produces, the per-unit cost of the input w, and the price of output p. $\pi = py - wx$.

(b) In the diagram below, draw an isoprofit line for each of the three periods, showing combinations of input and output that would yield the same profits that period as the combination actually chosen. What are the equations for these three lines? $y = x$, $y = 1 + .5x$, $y = 2 + .25x$. Using the theory of revealed profitability, shade in the region on the graph that represents input-output combinations that could be feasible as far as one can tell from the evidence that is available. How would you describe this region in words? `The region that is below all 3 isoprofit lines.`

20.14 (0) T-bone Pickens is a corporate raider. This means that he looks for companies that are not maximizing profits, buys them, and then tries to operate them at higher profits. T-bone is examining the financial records of two refineries that he might buy, the Shill Oil Company and the Golf Oil Company. Each of these companies buys oil and produces

gasoline. During the time period covered by these records, the price of gasoline fluctuated significantly, while the cost of oil remained constant at $10 a barrel. For simplicity, we assume that oil is the only input to gasoline production.

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(a) Using black ink, plot Shill Oil's isoprofit lines and choices for the three different periods. Label them 10, 20, and 40. Using red ink draw Golf Oil's isoprofit line and production choice. Label it with a 40 in red ink.

(b) How much profits could Golf Oil have made when the price of gasoline was $40 a barrel if it had chosen to produce the same amount that it did when the price was $20 a barrel? $80 million. What profits

did Golf actually make when the price of gasoline was \$40? \$60 million.

(c) Is there any evidence that Shill Oil is not maximizing profits? Explain.

No. The data satisfy WAPM.

(d) Is there any evidence that Golf Oil is not maximizing profits? Explain.

Yes. When price of gas was \$40, Golf could have made more money by acting as it did when price of gas was \$20.

20.15 (0) After carefully studying Shill Oil, T-bone Pickens decides that it has probably been maximizing its profits. But he still is very interested in buying Shill Oil. He wants to use the gasoline they produce to fuel his delivery fleet for his chicken farms, Capon Truckin'. In order to do this Shill Oil would have to be able to produce 5 million barrels of gasoline from 8 million barrels of oil. Mark this point on your graph. Assuming that Shill always maximizes profits, would it be technologically feasible for it to produce this input-output combination? Why or why not?

No. If it could, then it would have made more profits by choosing this combination than what it chose when price of oil was \$40.

20.16 (0) Suppose that firms operate in a competitive market, attempt to maximize profits, and only use one factor of production. Then we know that for any changes in the input and output price, the input choice and the output choice must obey the Weak Axiom of Profit Maximization, $\Delta p \Delta y - \Delta w \Delta x \geq 0$.

Which of the following propositions can be proven by the Weak Axiom of Profit Maximizing Behavior (WAPM)? Respond yes or no, and give a short argument.

(a) If the price of the input does not change, then a decrease in the price of the output will imply that the firm will produce the same amount or less output. Yes. If price of input doesn't change, $\Delta w = 0$, so WAPM says $\Delta p \Delta y \geq 0$.

(b) If the price of the output remains constant, then a decrease in the input price will imply that the firm will use the same amount or more of the input. Yes. If price of output doesn't change, $\Delta p = 0$, so WAPM says $-\Delta w \Delta x \geq 0$.

(c) If both the price of the output and the input increase and the firm produces less output, then the firm will use more of the input. No. Sign pattern is $(+)(-) - (+)(+) \geq 0$, which cannot happen.

Chapter 21

NAME_____

Cost Minimization

Introduction. In the chapter on consumer choice, you studied a consumer who tries to maximize his utility subject to the constraint that he has a fixed amount of money to spend. In this chapter you study the behavior of a firm that is trying to produce a fixed amount of output in the cheapest possible way. In both theories, you look for a point of tangency between a curved line and a straight line. In consumer theory, there is an "indifference curve" and a "budget line." In producer theory, there is a "production isoquant" and an "isocost line." As you recall, in consumer theory, finding a tangency gives you only one of the two equations you need to locate the consumer's chosen point. The second equation you used was the budget equation. In cost-minimization theory, again the tangency condition gives you one equation. This time you don't know in advance how much the producer is spending; instead you are told how much output he wants to produce and must find the cheapest way to produce it. So your second equation is the equation that tells you that the desired amount is being produced.

Example. A firm has the production function $f(x_1, x_2) = (\sqrt{x_1} + 3\sqrt{x_2})^2$. The price of factor 1 is $w_1 = 1$ and the price of factor 2 is $w_2 = 1$. Let us find the cheapest way to produce 16 units of output. We will be looking for a point where the technical rate of substitution equals $-w_1/w_2$. If you calculate the technical rate of substitution (or look it up from the warm up exercise in Chapter 18), you find $TRS(x_1, x_2) = -(1/3)(x_2/x_1)^{1/2}$. Therefore we must have $-(1/3)(x_2/x_1)^{1/2} = -w_1/w_2 = -1$. This equation can be simplified to $x_2 = 9x_1$. So we know that the combination of inputs chosen has to lie somewhere on the line $x_2 = 9x_1$. We are looking for the cheapest way to produce 16 units of output. So the point we are looking for must satisfy the equation $(\sqrt{x_1} + 3\sqrt{x_2})^2 = 16$, or equivalently $\sqrt{x_1} + 3\sqrt{x_2} = 4$. Since $x_2 = 9x_1$, we can substitute for x_2 in the previous equation to get $\sqrt{x_1} + 3\sqrt{9x_1} = 4$. This equation simplifies further to $10\sqrt{x_1} = 4$. Solving this for x_1, we have $x_1 = 16/100$. Then $x_2 = 9x_1 = 144/100$.

The amounts x_1 and x_2 that we solved for in the previous paragraph are known as the *conditional factor demands for factors 1 and 2*, conditional on the wages $w_1 = 1$, $w_2 = 1$, and output $y = 16$. We express this by saying $x_1(1, 1, 16) = 16/100$ and $x_2(1, 1, 16) = 144/100$. Since we know the amount of each factor that will be used to produce 16 units of output and since we know the price of each factor, we can now calculate the cost of producing 16 units. This cost is $c(w_1, w_2, 16) = w_1x_1(w_1, w_2, 16) + w_2x_2(w_1, w_2, 16)$. In this instance since $w_1 = w_2 = 1$, we have $c(1, 1, 16) = x_1(1, 1, 16) + x_2(1, 1, 16) = 160/100$.

In consumer theory, you also dealt with cases where the consumer's indifference "curves" were straight lines and with cases where there were

kinks in the indifference curves. Then you found that the consumer's choice might occur at a boundary or at a kink. Usually a careful look at the diagram would tell you what is going on. The story with kinks and boundary solutions is almost exactly the same in the case of cost-minimizing firms. You will find some exercises that show how this works.

21.1 (0) Nadine sells user-friendly software. Her firm's production function is $f(x_1, x_2) = x_1 + 2x_2$, where x_1 is the amount of unskilled labor and x_2 is the amount of skilled labor that she employs.

(a) In the graph below, draw a production isoquant representing input combinations that will produce 20 units of output. Draw another isoquant representing input combinations that will produce 40 units of output.

(b) Does this production function exhibit increasing, decreasing, or constant returns to scale? Constant.

(c) If Nadine uses only unskilled labor, how much unskilled labor would she need in order to produce y units of output? y.

(d) If Nadine uses only skilled labor to produce output, how much skilled labor would she need in order to produce y units of output? $\frac{y}{2}$.

(e) If Nadine faces factor prices $(1, 1)$, what is the cheapest way for her to produce 20 units of output? $x_1 =$ 0 , $x_2 =$ 10.

(f) If Nadine faces factor prices $(1, 3)$, what is the cheapest way for her to produce 20 units of output? $x_1 =$ 20 , $x_2 =$ 0.

(g) If Nadine faces factor prices (w_1, w_2), what will be the minimal cost of producing 20 units of output? $c = \min\{20w_1, 10w_2\} = 10\min\{2w_1, w_2\}$.

(h) If Nadine faces factor prices (w_1, w_2), what will be the minimal cost of producing y units of output? $c(w_1, w_2, y) = \min\{w_1, w_2/2\}y$.

21.2 (0) The Ontario Brassworks produces brazen effronteries. As you know brass is an alloy of copper and zinc, used in fixed proportions. The production function is given by: $f(x_1, x_2) = \min\{x_1, 2x_2\}$, where x_1 is the amount of copper it uses and x_2 is the amount of zinc that it uses in production.

(a) Illustrate a typical isoquant for this production function in the graph below.

(b) Does this production function exhibit increasing, decreasing, or constant returns to scale? Constant.

(c) If the firm wanted to produce 10 effronteries, how much copper would it need? 10 units. How much zinc would it need? 5 units.

(d) If the firm faces factor prices $(1, 1)$, what is the cheapest way for it to produce 10 effronteries? How much will this cost? It can *only* produce 10 units of output by using the bundle $(10, 5)$, so this is the cheapest way. It will cost $15.

(e) If the firm faces factor prices (w_1, w_2), what is the cheapest cost to produce 10 effronteries? $c(w_1, w_2, 10) = 10w_1 + 5w_2$.

(f) If the firm faces factor prices (w_1, w_2), what will be the minimal cost of producing y effronteries? $(w_1 + w_2/2)y$.

Calculus **21.3 (0)** A firm uses labor and machines to produce output according to the production function $f(L, M) = 4L^{1/2}M^{1/2}$, where L is the number of units of labor used and M is the number of machines. The cost of labor is $40 per unit and the cost of using a machine is $10.

(a) On the graph below, draw an isocost line for this firm, showing combinations of machines and labor that cost $400 and another isocost line showing combinations that cost $200. What is the slope of these isocost lines? -4.

(b) Suppose that the firm wants to produce its output in the cheapest possible way. Find the number of machines it would use per worker. (Hint: The firm will produce at a point where the slope of the production isoquant equals the slope of the isocost line.) 4.

(c) On the graph, sketch the production isoquant corresponding to an output of 40. Calculate the amount of labor 5 units and the number of machines 20 that are used to produce 40 units of output in the cheapest possible way, given the above factor prices. Calculate the cost of producing 40 units at these factor prices: $c(40, 10, 40) =$ 400.

(d) How many units of labor y/8 and how many machines y/2 would the firm use to produce y units in the cheapest possible way? How much would this cost? $10y$. (Hint: Notice that there are constant returns to scale.)

21.4 (0) Earl sells lemonade in a competitive market on a busy street corner in Philadelphia. His production function is $f(x_1, x_2) = x_1^{1/3} x_2^{1/3}$, where output is measured in gallons, x_1 is the number of pounds of lemons he uses, and x_2 is the number of labor-hours spent squeezing them.

(a) Does Earl have constant returns to scale, decreasing returns to scale, or increasing returns to scale? Decreasing.

(b) Where w_1 is the cost of a pound of lemons and w_2 is the wage rate for lemon-squeezers, the cheapest way for Earl to produce lemonade is to use w_1/w_2 hours of labor per pound of lemons. (Hint: Set the slope of his isoquant equal to the slope of his isocost line.)

(c) If he is going to produce y units in the cheapest way possible, then the number of pounds of lemons he will use is $x_1(w_1, w_2, y) = w_2^{1/2} y^{3/2} / w_1^{1/2}$ and the number of hours of labor that he will use is $x_2(w_1, w_2, y) = w_1^{1/2} y^{3/2} / w_2^{1/2}$. (Hint: Use the production function and the equation you found in the last part of the answer to solve for the input quantities.)

(d) The cost to Earl of producing y units at factor prices w_1 and w_2 is $c(w_1, w_2, y) = w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y) = 2 w_1^{1/2} w_2^{1/2} y^{3/2}$.

21.5 (0) The prices of inputs (x_1, x_2, x_3, x_4) are $(4, 1, 3, 2)$.

(a) If the production function is given by $f(x_1, x_2) = \min\{x_1, x_2\}$, what is the minimum cost of producing one unit of output? $5.

(b) If the production function is given by $f(x_3, x_4) = x_3 + x_4$, what is the minimum cost of producing one unit of output? $2.

(c) If the production function is given by $f(x_1, x_2, x_3, x_4) = \min\{x_1 + x_2, x_3 + x_4\}$, what is the minimum cost of producing one unit of output?

$3.

(d) If the production function is given by $f(x_1, x_2) = \min\{x_1, x_2\} + \min\{x_3, x_4\}$, what is the minimum cost of producing one unit of output?

$5.

21.6 (0) Joe Grow, an avid indoor gardener, has found that the number of happy plants, h, depends on the amount of light, l, and water, w. In fact, Joe noticed that plants require twice as much light as water, and any more or less is wasted. Thus, Joe's production function is $h = \min\{l, 2w\}$.

(a) Suppose Joe is using 1 unit of light, what is the least amount of water he can use and still produce a happy plant? $1/2$ unit of water.

(b) If Suppose Joe wants to produce 4 happy plants, what are the minimum amounts of light and water required? $(4, 2)$.

(c) Joe's conditional factor demand function for light is $l(w_1, w_2, h) =$ h and his conditional factor demand function for water is $w(w_1, w_2, h) =$ $h/2$.

(d) If each unit of light costs w_1 and each unit of water costs w_2, Joe's cost function is $c(w_1, w_2, h) =$ $w_1 h + \frac{w_2}{2} h$.

21.7 (1) Joe's sister, Flo Grow, is a university administrator. She uses an alternative method of gardening. Flo has found that happy plants only need fertilizer and talk. (*Warning:* Frivolous observations about university administrators' talk being a perfect substitute for fertilizer is in extremely poor taste.) Where f is the number of bags of fertilizer used and t is the number of hours she talks to her plants, the number of happy plants produced is exactly $h = t + 2f$. Suppose fertilizer costs w_f per bag and talk costs w_t per hour.

(a) If Flo uses no fertilizer, how many hours of talk must she devote if she wants one happy plant? 1 hour. If she doesn't talk to her plants at all, how many bags of fertilizer will she need for one happy plant? 1/2 bag.

(b) If $w_t < w_f/2$, would it be cheaper for Flo to use fertilizer or talk to raise one happy plant? It would be cheaper to talk.

(c) Flo's cost function is $c(w_f, w_t, h) =$ $\min\{\frac{w_f}{2}, w_t\}h$.

(d) Her conditional factor demand for talk is $t(w_f, w_t, h) =$ h if $w_t < w_f/2$ and 0 if $w_t > w_f/2$.

21.8 (0) Remember T-bone Pickens, the corporate raider? Now he's concerned about his chicken farms, Pickens's Chickens. He feeds his chickens on a mixture of soybeans and corn, depending on the prices of each. According to the data submitted by his managers, when the price of soybeans was \$10 a bushel and the price of corn was \$10 a bushel, they used 50 bushels of corn and 150 bushels of soybeans for each coop of chickens. When the price of soybeans was \$20 a bushel and the price of corn was \$10 a bushel, they used 300 bushels of corn and no soybeans per coop of chickens. When the price of corn was \$20 a bushel and the price of soybeans was \$10 a bushel, they used 250 bushels of soybeans and no corn for each coop of chickens.

(a) Graph these three input combinations and isocost lines in the following diagram.

(b) How much money did Pickens' managers spend per coop of chickens when the prices were $(10, 10)$? $2,000. When the prices were $(10, 20)$? $2,500. When the prices were $(20, 10)$? $3,000.

(c) Is there any evidence that Pickens's managers were not minimizing costs? Why or why not?

There is no such evidence, since the data satisfy WACM.

(d) Pickens wonders whether there are any prices of corn and soybeans at which his managers will use 150 bushels of corn and 50 bushels of soybeans to produce a coop of chickens. How much would this production method cost per coop of chickens if the prices were $p_s = 10$ and $p_c = 10$? $2,000. if the prices were $p_s = 10$, $p_c = 20$? $3,500. if the prices were $p_s = 20$, $p_c = 10$? $2,500.

(e) If Pickens's managers were always minimizing costs, can it be possible to produce a coop of chickens using 150 bushels and 50 bushels of soybeans? No. At prices $(20, 10)$, this bundle costs less than the bundle actually used at prices $(20, 10)$. If it produced as much as that bundle, the chosen bundle wouldn't have been chosen.

21.9 (0) A genealogical firm called Roots produces its output using only one input. Its production function is $f(x) = \sqrt{x}$.

(a) Does the firm have increasing, constant, or decreasing returns to scale?

Decreasing.

(b) How many units of input does it take to produce 10 units of output? 100 units. If the input costs w per unit, what does it cost to produce 10 units of output? $100w$.

(c) How many units of input does it take to produce y units of output? y^2. If the input costs w per unit, what does it cost to produce y units of output? $y^2 w$.

(d) If the input costs w per unit, what is the average cost of producing y units? $AC(w, y) =$ yw.

21.10 (0) A university cafeteria produces square meals, using only one input and a rather remarkable production process. We are not allowed to say what that ingredient is, but an authoritative kitchen source says that "fungus is involved." The cafeteria's production function is $f(x) = x^2$, where x is the amount of input and $f(x)$ is the number of square meals produced.

(a) Does the cafeteria have increasing, constant, or decreasing returns to scale? Increasing.

(b) How many units of input does it take to produce 144 square meals? 12. If the input costs w per unit, what does it cost to produce 144 square meals? $12w$.

(c) How many units of input does it take to produce y square meals? $\sqrt{y}$. If the input costs w per unit, what does it cost to produce y square meals? $w\sqrt{y}$.

(d) If the input costs w per unit, what is the average cost of producing y square meals? $AC(w, y) =$ $w/\sqrt{y}$.

21.11 (0) Irma's Handicrafts produces plastic deer for lawn ornaments. "It's hard work," says Irma, "but anything to make a buck." Her production function is given by $f(x_1, x_2) = (\min\{x_1, 2x_2\})^{1/2}$, where x_1 is the amount of plastic used, x_2 is the amount of labor used, and $f(x_1, x_2)$ is the number of deer produced.

(a) In the graph below, draw a production isoquant representing input combinations that will produce 4 deer. Draw another production isoquant representing input combinations that will produce 5 deer.

(b) Does this production function exhibit increasing, decreasing, or constant returns to scale? Decreasing returns to scale.

(c) If Irma faces factor prices $(1, 1)$, what is the cheapest way for her to produce 4 deer? Use (16,8). How much does this cost? $24.

(d) At the factor prices $(1, 1)$, what is the cheapest way to produce 5 deer? Use (25,12.5). How much does this cost? $37.50.

(e) At the factor prices $(1, 1)$, the cost of producing y deer with this technology is $c(1, 1, y) =$ $3y^2/2$.

(f) At the factor prices (w_1, w_2), the cost of producing y deer with this technology is $c(w_1, w_2, y) =$ $(w_1 + w_2/2)y^2$.

21.12 (0) Al Deardwarf also makes plastic deer for lawn ornaments. Al has found a way to automate the production process completely. He doesn't use any labor–only wood and plastic. Al says he likes the business "because I need the doe." Al's production function is given by $f(x_1, x_2) = (2x_1 + x_2)^{1/2}$, where x_1 is the amount of plastic used, x_2 is the amount of wood used, and $f(x_1, x_2)$ is the number of deer produced.

(a) In the graph below, draw a production isoquant representing input combinations that will produce 4 deer. Draw another production isoquant representing input combinations that will produce 6 deer.

(b) Does this production function exhibit increasing, decreasing, or constant returns to scale? Decreasing returns to scale.

(c) If Al faces factor prices $(1, 1)$, what is the cheapest way for him to produce 4 deer? $(8, 0)$. How much does this cost? \$8.

(d) At the factor prices $(1, 1)$, what is the cheapest way to produce 6 deer? $(18, 0)$. How much does this cost? \$18.

(e) At the factor prices $(1, 1)$, the cost of producing y deer with this technology is $c(1, 1, y) =$ $y^2/2$.

(f) At the factor prices $(3, 1)$, the cost of producing y deer with this technology is $c(3, 1, y) =$ y^2.

21.13 (0) Suppose that Al Deardwarf from the last problem cannot vary the amount of wood that he uses in the short run and is stuck with using 20 units of wood. Suppose that he can change the amount of plastic that he uses, even in the short run.

(a) How much plastic would Al need in order to make 100 deer? 4,990 units.

(b) If the cost of plastic is \$1 per unit and the cost of wood is \$1 per unit, how much would it cost Al to make 100 deer? \$5,010.

(c) Write down Al's short-run cost function at these factor prices.

$c(1, 1, y) = 20 + (y^2 - 20)/2.$

Chapter 22 NAME_____

Cost Curves

Introduction. Here you continue to work on cost functions. Total cost can be divided into fixed cost, the part that doesn't change as output changes, and variable cost. To get the average (total) cost, average fixed cost, and average variable cost, just divide the appropriate cost function by y, the level of output. The marginal cost function is the derivative of the total cost function with respect to output—or the rate of increase in cost as output increases, if you don't know calculus.

Remember that the marginal cost curve intersects both the average cost curve and the average variable cost curve at their minimum points. So to find the minimum point on the average cost curve, you simply set marginal cost equal to average cost and similarly for the minimum of average variable cost.

Example: A firm has the total cost function $C(y) = 100 + 10y$. Let us find the equations for its various cost curves. Total fixed costs are 100, so the equation of the average fixed cost curve is $100/y$. Total variable costs are $10y$, so average variable costs are $10y/y = 10$ for all y. Marginal cost is 10 for all y. Average total costs are $(100 + 10y)/y = 10 + 10/y$. Notice that for this firm, average total cost decreases as y increases. Notice also that marginal cost is less than average total cost for all y.

22.1 (0) Mr. Otto Carr, owner of Otto's Autos, sells cars. Otto buys autos for \$$c$ each and has no other costs.

(a) What is his total cost if he sells 10 cars? $10c$. What if he sells 20 cars? $20c$. Write down the equation for Otto's total costs assuming he sells y cars: $TC(y) =$ cy.

(b) What is Otto's average cost function? $AC(y) =$ c. For every additional auto Otto sells, by how much do his costs increase? c.

Write down Otto's marginal cost function: $MC(y) =$ c.

(c) In the graph below draw Otto's average and marginal cost curves if $c = 20$.

(d) Suppose Otto has to pay \$$b$ a year to produce obnoxious television commercials. Otto's total cost curve is now $TC(y) =$ $cy + b$, his average cost curve is now $AC(y) =$ $c + b/y$, and his marginal cost curve is $MC(y) =$ c.

(e) If $b = \$100$, use red ink to draw Otto's average cost curve on the graph above.

22.2 (0) Otto's brother, Dent Carr, is in the auto repair business. Dent recently had little else to do and decided to calculate his cost conditions. He found that the total cost of repairing s cars is $TC(s) = 2s^2 + 10$. But Dent's attention was diverted to other things ... and that's where you come in. Please complete the following:

Dent's Total Variable Costs: $2s^2$.

Total Fixed Costs: 10.

Average Variable Costs: $2s$.

Average Fixed Costs: $10/s$.

Average Total Costs: $2s + 10/s$.

Marginal Costs: $4s$.

22.3 (0) A third brother, Rex Carr, owns a junk yard. Rex can use one of two methods to destroy cars. The first involves purchasing a hydraulic car smasher that costs \$200 a year to own and then spending \$1 for every car smashed into oblivion; the second method involves purchasing a shovel that will last one year and costs \$10 and paying the last Carr brother, Scoop, to bury the cars at a cost of \$5 each.

(a) Write down the total cost functions for the two methods, where y is output per year: $TC_1(y) =$ $y + 200$, $TC_2(y) =$ $5y + 10$.

(b) The first method has an average cost function $1 + 200/y$ and a marginal cost function 1. For the second method these costs are $5 + 10/y$ and 5.

(c) If Rex wrecks 40 cars per year, which method should he use? Method 2. If Rex wrecks 50 cars per year, which method should he use? Method 1. What is the smallest number of cars per year for which it would pay him to buy the hydraulic smasher? 48 cars per year.

22.4 (0) Mary Magnolia wants to open a flower shop, the Petal Pusher, in a new mall. She has her choice of three different floor sizes, 200 square feet, 500 square feet, or 1,000 square feet. The monthly rent will be \$1 a square foot. Mary estimates that if she has F square feet of floor space and sells y bouquets a month, her variable costs will be $c_v(y) = y^2/F$ per month.

(a) If she has 200 square feet of floor space, write down her marginal cost function: $MC = \frac{y}{100}$ and her average cost function: $AC = \frac{200}{y} + \frac{y}{200}$. At what amount of output is average cost minimized? 200. At this level of output, how much is average cost? \$2.

(b) If she has 500 square feet, write down her marginal cost function: $MC = y/250$ and her average cost function: $AC = (500/y) + y/500$. At what amount of output is average cost min-

imized? 500. At this level of output, how much is average cost? $2.

(c) If she has 1,000 square feet of floor space, write down her marginal cost function: $MC = y/500$ and her average cost function: $AC = (1,000/y) + y/1,000$. At what amount of output is average cost minimized? 1,000. At this level of output, how much is average cost? $2.

(d) Use red ink to show Mary's average cost curve and her marginal cost curves if she has 200 square feet. Use blue ink to show her average cost curve and her marginal cost curve if she has 500 square feet. Use black ink to show her average cost curve and her marginal cost curve if she has 1,000 square feet. Label the average cost curves AC and the marginal cost curves MC.

(e) Use yellow marker to show Mary's long-run average cost curve and her long-run marginal cost curve in your graph. Label them LRAC and LRMC.

22.5 (0) Touchie MacFeelie publishes comic books. The only inputs he needs are old jokes and cartoonists. His production function is

$$Q = .1J^{\frac{1}{2}}L^{3/4},$$

where J is the number of old jokes used, L the number of hours of cartoonists' labor used as inputs, and Q is the number of comic books produced.

(a) Does this production process exhibit increasing, decreasing, or constant returns to scale? Explain your answer. It exhibits increasing returns to scale since $f(tJ, tL) = t^{5/4} f(J, L) > t f(J, L)$.

(b) If the number of old jokes used is 100, write an expression for the marginal product of cartoonists' labor as a function of L. $MP = \frac{3}{4L^{1/4}}$ Is the marginal product of labor decreasing or increasing as the amount of labor increases? Decreasing.

22.6 (0) Touchie MacFeelie's irascible business manager, Gander MacGrope, announces that old jokes can be purchased for \$1 each and that the wage rate of cartoonists' labor is \$2.

(a) Suppose that in the short run, Touchie is stuck with exactly 100 old jokes (for which he paid \$1 each) but is able to hire as much labor as he wishes. How much labor would he have to hire in order produce Q comic books? $Q^{4/3}$.

(b) Write down Touchie's short-run total cost as a function of his output $2Q^{4/3} + 100$.

(c) His short-run marginal cost function is $8Q^{1/3}/3$.

(d) His short-run average cost function is $2Q^{1/3} + 100/Q$.

Calculus **22.7 (1)** Touchie asks his brother, Sir Francis MacFeelie, to study the long-run picture. Sir Francis, who has carefully studied the appendix to Chapter 19 in your text, prepared the following report.

(a) If all inputs are variable, and if old jokes cost \$1 each and cartoonist labor costs \$2 per hour, the cheapest way to produce exactly one comic book is to use $10^{4/5}(4/3)^{3/5} \approx 7.4$ jokes and $10^{4/5}(3/4)^{2/5} \approx 5.6$ hours of labor. (Fractional jokes are certainly allowable.)

(b) This would cost 18.7 dollars.

(c) Given our production function, the cheapest proportions in which to use jokes and labor are the same no matter how many comic books we print. But when we double the amount of both inputs, the number of comic books produced is multiplied by $2^{5/4}$.

22.8 (0) Consider the cost function $c(y) = 4y^2 + 16$.

(a) The average cost function is $AC = 4y + \frac{16}{y}$.

(b) The marginal cost function is $MC = 8y$.

(c) The level of output that yields the minimum average cost of production is $y = 2$.

(d) The average variable cost function is $AVC = 4y$.

(e) At what level of output does average variable cost equal marginal cost? At $y = 0$.

22.9 (0) A competitive firm has a production function of the form $Y = 2L + 5K$. If $w = \$2$ and $r = \$3$, what will be the minimum cost of producing 10 units of output? \$6.

Chapter 23

NAME_____

Firm Supply

Introduction. The short-run supply curve of a competitive firm is the portion of its short-run marginal cost curve that is upward sloping and lies above its average variable cost curve. The long-run supply curve of a competitive firm is the portion of its short-run marginal cost curve that is upward-sloping and lies above its long-run average cost curve.

Example: A firm has the long-run cost function $c(y) = 2y^2 + 200$ for $y > 0$ and $c(0) = 0$. Let us find its long-run supply curve. The firm's marginal cost when its output is y is $MC(y) = 4y$. If we graph output on the horizontal axis and dollars on the vertical axis, then we find that the long-run marginal cost curve is an upward-sloping straight line through the origin with slope 4. The long-run supply curve is the portion of this curve that lies above the long-run average cost curve. When output is y, long-run average costs of this firm are $AC(y) = 2y + 200/y$. This is a *U-shaped* curve. As y gets close to zero, $AC(y)$ becomes very large because $200/y$ becomes very large. When y is very large, $AC(y)$ becomes very large because $2y$ is very large. When is it true that $AC(y) < MC(y)$? This happens when $2y + 200/y < 4y$. Simplify this inequality to find that $AC(y) < MC(y)$ when $y > 10$. Therefore the long-run supply curve is the piece of the long-run marginal cost curve for which $y > 10$. So the long-run supply curve has the equation $p = 4y$ for $y > 10$. If we want to find quantity supplied as a function of price, we just solve this expression for y as a function of p. Then we have $y = p/4$ whenever $p > 40$.

Suppose that $p < 40$. For example, what if $p = 20$, how much will the firm supply? At a price of 20, if the firm produces where price equals long-run marginal cost, it will produce $5 = 20/4$ units of output. When the firm produces only 5 units, its average costs are $2 \times 5 + 200/5 = 50$. Therefore when the price is 20, the best the firm can do if it produces a positive amount is to produce 5 units. But then it will have total costs of $5 \times 50 = 250$ and total revenue of $5 \times 20 = 100$. It will be losing money. It would be better off producing nothing at all. In fact, for any price $p < 40$, the firm will choose to produce zero output.

23.1 (0) Remember Otto's brother Dent Carr, who is in the auto repair business? Dent found that the total cost of repairing s cars is $c(s) = 2s^2 + 100$.

(a) This implies that Dent's average cost is equal to $2s + 100/s$, his average variable cost is equal to $2s$, and his marginal cost is equal to $4s$. On the graph below, plot the above curves, and also plot Dent's supply curve.

(b) If the market price is \$20, how many cars will Dent be willing to repair? 5. If the market price is \$40, how many cars will Dent repair? 10.

(c) Suppose the market price is \$40 and Dent maximizes his profits. On the above graph, shade in and label the following areas: total costs, total revenue, and total profits.

Calculus **23.2 (0)** A competitive firm has the following short-run cost function: $c(y) = y^3 - 8y^2 + 30y + 5$.

(a) The firm's marginal cost function is $MC(y) =$ $3y^2 - 16y + 30$.

(b) The firm's average variable cost function is $AVC(y) =$ $y^2 - 8y + 30$. (Hint: Notice that total variable costs equal $c(y) - c(0)$.)

(c) On the axes below, sketch and label a graph of the marginal cost function and of the average variable cost function.

(d) Average variable cost is falling as output rises if output is less than 4 and rising as output rises if output is greater than 4.

(e) Marginal cost equals average variable cost when output is 4.

(f) The firm will supply zero output if the price is less than 14.

(g) The smallest positive amount that the firm will ever supply at any price is 4. At what price would the firm supply exactly 6 units of output? 42.

Calculus **23.3 (0)** Mr. McGregor owns a 5-acre cabbage patch. He forces his wife, Flopsy, and his son, Peter, to work in the cabbage patch without wages. Assume for the time being that the land can be used for nothing other than cabbages and that Flopsy and Peter can find no alternative employment. The only input that Mr. McGregor pays for is fertilizer. If he uses x sacks of fertilizer, the amount of cabbages that he gets is $10\sqrt{x}$. Fertilizer costs \$1 per sack.

(a) What is the total cost of the fertilizer needed to produce 100 cabbages? \$100. What is the total cost of the amount of fertilizer needed to produce y cabbages? $y^2/100$.

(b) If the only way that Mr. McGregor can vary his output is by varying the amount of fertilizer applied to his cabbage patch, write an expression for his marginal cost, as a function of y. $MC(y) =$ $y/50$.

(c) If the price of cabbages is \$2 each, how many cabbages will Mr. McGregor produce? 100. How many sacks of fertilizer will he buy? 100. How much profit will he make? \$100.

(d) The price of fertilizer and of cabbages remain as before, but Mr. McGregor learns that he could find summer jobs for Flopsy and Peter in a local sweatshop. Flopsy and Peter would together earn \$300 for the summer, which Mr. McGregor could pocket, but they would have no time to work in the cabbage patch. Without their labor, he would get no cabbages. Now what is Mr. McGregor's total cost of producing y cabbages?

$c(y) = 300 + (y/10)^2$.

(e) Should he continue to grow cabbages or should he put Flopsy and Peter to work in the sweatshop? Sweatshop.

23.4 (0) Severin, the herbalist, is famous for his hepatica. His total cost function is $c(y) = y^2 + 10$ for $y > 0$ and $c(0) = 0$. (That is, his cost of producing zero units of output is zero.)

(a) What is his marginal cost function? $2y$. What is his average cost function? $y + 10/y$.

(b) At what quantity is his marginal cost equal to his average cost? $\sqrt{10}$. At what quantity is his average cost minimized? $\sqrt{10}$.

(c) In a competitive market, what is the lowest price at which he will supply a positive quantity in long-run equilibrium? $2\sqrt{10}$. How much would he supply at that price? $\sqrt{10}$.

23.5 (1) Stanley Ford makes mountains out of molehills. He can do this with almost no effort, so for the purposes of this problem, let us assume that molehills are the only input used in the production of mountains. Suppose mountains are produced at constant returns to scale and that it takes 100 molehills to make 1 mountain. The current market price of molehills is \$20 each. A few years ago, Stan bought an "option" that permits him to buy up to 2,000 molehills at \$10 each. His option contract explicitly says that he can buy fewer than 2,000 molehills if he wishes, but he can not resell the molehills that he buys under this contract. In order to get governmental permission to produce mountains from molehills, Stanley would have to pay \$10,000 for a molehill-masher's license.

(a) The marginal cost of producing a mountain for Stanley is \$1,000 if he produces fewer than 20 mountains. The marginal cost of producing a mountain is \$2,000 if he produces more than 20 mountains.

(b) On the graph below, show Stanley Ford's marginal cost curve (in blue ink) and his average cost curve (in red ink).

(c) If the price of mountains is $1,600, how many mountains will Stanley produce? 20 mountains.

(d) The government is considering raising the price of a molehill-masher's license to $11,000. Stanley claims that if it does so he will have to go out of business. Is Stanley telling the truth? No. What is the highest fee for a license that the government could charge without driving him out of business? The maximum they could charge is the amount of his profits excluding the license fee, $12,000.

(e) Stanley's lawyer, Eliot Sleaze, has discovered a clause in Stanley's option contract that allows him to resell the molehills that he purchased under the option contract at the market price. On the graph above, use a pencil to draw Stanley's new marginal cost curve. If the price of mountains remains $1,600, how many mountains will Stanley produce now? He will sell all of his molehills and produce zero mountains.

23.6 (1) Lady Wellesleigh makes silk purses out of sows' ears. She is the only person in the world who knows how to do so. It takes one sow's ear and 1 hour of her labor to make a silk purse. She can buy as many

sows' ears as she likes for \$1 each. Lady Wellesleigh has no other source of income than her labor. Her utility function is a Cobb-Douglas function $U(c, r) = c^{1/3}r^{2/3}$, where c is the amount of money per day that she has to spend on consumption goods and r is the amount of leisure that she has. Lady Wellesleigh has 24 hours a day that she can devote either to leisure or to working.

(a) Lady Wellesleigh can either make silk purses or she can earn \$5 an hour as a seamstress in a sweatshop. If she worked in the sweat shop, how many hours would she work? 8. (Hint: To solve for this amount, write down Lady Wellesleigh's budget constraint and recall how to find the demand function for someone with a Cobb-Douglas utility function.)

(b) If she could earn a wage of \$$w$ an hour as a seamstress, how much would she work? 8 hours.

(c) If the price of silk purses is \$$p$, how much money will Lady Wellesleigh earn per purse after she pays for the sows' ears that she uses? $p - 1$.

(d) If she can earn \$5 an hour as a seamstress, what is the lowest price at which she will make any silk purses? \$6.

(e) What is the supply function for silk purses? (Hint: The price of silk purses determines the "wage rate" that Lady W. can earn by making silk purses. This determines the number of hours she will choose to work and hence the supply of silk purses.) $S(p) = 8$ for $p > 6$, 0 otherwise.

Calculus **23.7 (0)** Remember Earl, who sells lemonade in Philadelphia? You met him in the chapter on cost functions. Earl's production function is $f(x_1, x_2) = x_1^{1/3}x_2^{1/3}$, where x_1 is the number of pounds of lemons he uses and x_2 is the number of hours he spends squeezing them. As you found out, his cost function is $c(w_1, w_2, y) = 2w_1^{1/2}w_2^{1/2}y^{3/2}$, where y is the number of units of lemonade produced.

(a) If lemons cost \$1 per pound, the wage rate is \$1 per hour, and the price of lemonade is p, Earl's marginal cost function is $MC(y) =$ $3y^{1/2}$ and his supply function is $S(p) =$ $p^2/9$. If lemons cost \$4 per pound and the wage rate is \$9 per hour, his supply function will be $S(p) =$ $p^2/324$.

(b) In general, Earl's marginal cost depends on the price of lemons and the wage rate. At prices w_1 for lemons and w_2 for labor, his marginal cost when he is producing y units of lemonade is $MC(w_1, w_2, y) =$ $3w_1^{1/2}w_2^{1/2}y^{1/2}$. The amount that Earl will supply depends on the three variables, p, w_1, w_2. As a function of these three variables, Earl's supply is $S(p, w_1, w_2) =$ $p^2/9w_1w_2$.

Calculus **23.8 (0)** As you may recall from the chapter on cost functions, Irma's handicrafts has the production function $f(x_1, x_2) = (\min\{x_1, 2x_2\})^{1/2}$, where x_1 is the amount of plastic used, x_2 is the amount of labor used, and $f(x_1, x_2)$ is the number of lawn ornaments produced. Let w_1 be the price per unit of plastic and w_2 be the wage per unit of labor.

(a) Irma's cost function is $c(w_1, w_2, y) =$ $(w_1 + w_2/2)y^2$.

(b) If $w_1 = w_2 = 1$, then Irma's marginal cost of producing y units of output is $MC(y) =$ $3y$. The number of units of output that she would supply at price p is $S(p) =$ $p/3$. At these factor prices, her average cost per unit of output would be $AC(y) =$ $3y/2$.

(c) If the competitive price of the lawn ornaments she sells is $p = 48$, and $w_1 = w_2 = 1$, how many will she produce? 16. How much profit will she make? 384.

(d) More generally, at factor prices w_1 and w_2, her marginal cost is a function $MC(w_1, w_2, y) =$ $(2w_1 + w_2)y$. At these factor prices and an output price of p, the number of units she will choose to supply is $S(p, w_1, w_2) =$ $p/(2w_1 + w_2)$.

23.9 (0) Jack Benny can get blood from a stone. If he has x stones, the number of pints of blood he can extract from them is $f(x) = 2x^{\frac{1}{3}}$. Stones cost Jack \$$w$ each. Jack can sell each pint of blood for \$$p$.

(a) How many stones does Jack need to extract y pints of blood?

$y^3/8$.

(b) What is the cost of extracting y pints of blood? $wy^3/8$.

(c) What is Jack's supply function when stones cost \$8 each? $y = (p/3)^{1/2}$. When stones cost \$$w$ each? $y = (8p/3w)^{1/2}$.

(d) If Jack has 19 relatives who can also get blood from a stone in the same way, what is the aggregate supply function for blood when stones cost \$$w$ each? $Y = 20(8p/3w)^{1/2}$.

23.10 (1) The Miss Manners Refinery in Dry Rock, Oklahoma, converts crude oil into gasoline. It takes 1 barrel of crude oil to produce 1 barrel of gasoline. In addition to the cost of oil there are some other costs involved in refining gasoline. Total costs of producing y barrels of gasoline are described by the cost function $c(y) = y^2/2 + p_o y$, where p_o is the price of a barrel of crude oil.

(a) Express the marginal cost of producing gasoline as a function of p_o and y. $y + p_o$.

(b) Suppose that the refinery can buy 50 barrels of crude oil for \$5 a barrel but must pay \$15 a barrel for any more that it buys beyond 50 barrels. The marginal cost curve for gasoline will be $y + 5$ up to 50 barrels of gasoline and $y + 15$ thereafter.

(c) Plot Miss Manners' supply curve in the diagram below using blue ink.

(d) Suppose that Miss Manners faces a horizontal demand curve for gasoline at a price of \$30 per barrel. Plot this demand curve on the graph above using red ink. How much gasoline will she supply? 25 barrels.

(e) If Miss Manners could no longer get the first 50 barrels of crude for \$5, but had to pay \$15 a barrel for all crude oil, how would her output change? It would decrease to 15 barrels.

(f) Now suppose that an entitlement program is introduced that permits refineries to buy one barrel of oil at \$5 for each barrel of oil that they buy for \$15. What will Miss Manners' supply curve be now? $S(p) = p - 10$. Assume that it can buy fractions of a barrel in the same manner. Plot this supply curve on the graph above using black ink. If the demand curve is horizontal at \$30 a barrel, how much gasoline will Miss Manners supply now? 20 barrels.

Chapter 24 NAME_____

Industry Supply

Introduction. To find the industry supply of output, just add up the supply of output coming from each individual firm. Remember to add quantities, not prices. The industry supply curve will have a kink in it where the market price becomes low enough that some firm reduces its quantity supplied to zero.

The series of problems about the garden gnome industry are designed to help you to understand the distinction between the long run and the short run. To solve these problems, you need to pay careful attention to the timing of decisions. In particular, in this problem, units of capital (gnome molds) can be produced and delivered only one year after they are ordered.

The last three questions of this chapter apply supply and demand analysis to some problems in the economics of illegal activities. In these examples, you will make use of your knowledge of where supply functions come from.

24.0 Warm Up Exercise. Here are some drills for you on finding market supply functions from linear firm supply functions. The trick here is to remember that the market supply function may have kinks in it. For example, if the firm supply functions are $s_1(p) = p$ and $s_2(p) = p - 2$, then the market supply function is $S(p) = p$ for $p \leq 2$ and $S(p) = 2p - 2$ for $p > 2$; that is, only the first firm supplies a positive output at prices below \$2, and both firms supply output at prices above \$2. Now try to construct the market supply function in each of the following cases.

(a) $s_1(p) = p, s_2(p) = 2p, s_3(p) = 3p$. $S(p) = 6p$.

(b) $s_1(p) = 2p, s_2(p) = p - 1$. $S(p) = 2p$ for $p \leq 1, S(p) = 3p - 1$ for $p > 1$.

(c) 200 firms each have a supply function $s_1(p) = 2p - 8$ and 100 firms each have a supply function $s_2(p) = p - 3$. $S(p) = 0$ for $p < 3$, $S(p) = 100p - 300$ for $3 \leq p \leq 4$, $S(p) = 500p - 1,900$ for $p > 4$.

(d) $s_1(p) = 3p - 12, s_2(p) = 2p - 8, s_3(p) = p - 4.$ $S(p) = 6p - 24$ for $p > 4$.

24.1 (1) Al Deardwarf's cousin, Zwerg, makes plaster garden gnomes. The technology in the garden gnome business is as follows. You need a gnome mold, plaster, and labor. A gnome mold is a piece of equipment that costs \$1,000 and will last exactly one year. After a year, a gnome mold is completely worn out and has no scrap value. With a gnome mold, you can make 500 gnomes per year. For every gnome that you make, you also have to use a total of \$7 worth of plaster and labor. The total amounts of plaster and labor used are variable in the short run. If you want to produce only 100 gnomes a year with a gnome mold, you spend only \$700 a year on plaster and labor, and so on. The number of gnome molds in the industry cannot be changed in the short run. To get a newly built one, you have to special-order it from the gnome-mold factory. The gnome-mold factory only takes orders on January 1 of any given year, and it takes one whole year from the time a gnome mold is ordered until it is delivered on the next January 1. When a gnome mold is installed in your plant, it is stuck there. To move it would destroy it. Gnome molds are useless for anything other than making garden gnomes.

For many years, the demand function facing the garden-gnome industry has been $D(p) = 60,000 - 5,000p$, where $D(p)$ is the total number of garden gnomes sold per year and p is the price. Prices of inputs have been constant for many years and the technology has not changed. Nobody expects any changes in the future, and the industry is in long-run equilibrium. The interest rate is 10%. When you buy a new gnome mold, you have to pay for it when it is delivered. For simplicity of calculations, we will assume that all of the gnomes that you build during the one-year life of the gnome mold are sold at Christmas and that the employees and plaster suppliers are paid only at Christmas for the work they have done during the past year. Also for simplicity of calculations, let us approximate the date of Christmas by December 31.

(a) If you invested \$1,000 in the bank on January 1, how much money could you expect to get out of the bank one year later? \$1,100. If you received delivery of a gnome mold on January 1 and paid for it at that time, by how much would your revenue have to exceed the costs of plaster and labor if it is to be worthwhile to buy the machine? (Remember that the machine will be worn out and worthless at the end of the year.)

\$1,100.

(b) Suppose that you have exactly one newly installed gnome mold in your plant; what is your short-run marginal cost of production if you produce up to 500 gnomes? \$7. What is your average *variable* cost

for producing up to 500 gnomes? $7. If you have only one gnome mold, is it possible in the short run to produce more than 500 gnomes? No.

(c) If you have exactly one newly installed gnome mold, you would produce 500 gnomes if the price of gnomes is above 7 dollars. You would produce no gnomes if the price of gnomes is below 7 dollars. You would be indifferent between producing any number of gnomes between 0 and 500 if the price of gnomes is 7 dollars.

(d) If you could sell as many gnomes as you liked for $10 each and none at a higher price, what rate of return would you make on your $1,000 by investing in a gnome mold? 50%. Is this higher than the return from putting your money in the bank? Yes. What is the lowest price for gnomes such that the rate of return you get from ivesting $1000 in a gnome mold is as at least 10%? $9.20. Could the long-run equilibrium price of gnomes be lower than this? No.

(e) At the price you found in the last section, how many gnomes would be demanded each year? 14,000. How many molds would be purchased each year? 28. Is this a long-run equilibrium price? Yes.

24.2 (1) We continue our study of the garden-gnome industry. Suppose that initially everything was as described in the previous problem. To the complete surprise of everyone in the industry, on January 1, 2001, the invention of a new kind of plaster was announced. This new plaster made it possible to produce garden gnomes using the same molds, but it reduced the cost of the plaster and labor needed to produce a gnome from $7 to $5 per gnome. Assume that consumers' demand function for gnomes in 2001 was not changed by this news. The announcement came early enough in the day for everybody to change his order for gnome molds to be delivered on January 1, 2002, but of course, but the total number of molds available to be used in 2001 was just the 28 molds that had been ordered the previous year. The manufacturer of garden gnome molds contracted to sell them for $1,000 when they were ordered, so it can't change the price it charges on delivery.

(a) On the graph below, draw the short run industry supply curve and teh demand curve for garden gnomes that applies in the year 2001, after the discovery of the new plaster is announced.

(b) In 2001, what is the short run equilibrium total output of garden gnomes, 14,000. and what is the short run equilibrium price of garden gnomes? $9.20. (Hint: Look at the intersection of the supply and demand curves you just drew. Cousin Zwerg bought a gnome mold that was delivered on January 1, 2001, and, as had been agreed, he paid $1,000 for it on that day. On January 1, 2002, when he sold the gnomes he had made during the year and when he paid the workers and the suppliers of plaster, he received a net cash flow of $ 2,100. Did he make more than a 10% rate of return on his investment in the gnome mold? Yes. What rate of return did he make? 110%.

(c) Zwerg's neighbor, Munchkin, also makes garden gnomes, and he has a gnome mold that is to be delivered on January 1, 2001. On this day, Zwerg, who is looking for a way to invest some more money, is considering buying Munchkin's new mold from Munchkin and installing it in his own plant. If Munchkin keeps his mold, he will get a net cash flow of $ 2,100 in one year. If the interest rate that Munchkin faces, both for borrowing and lending is 10%, then should he be willing to sell his mold for $1,000? No. What is the lowest price that he would be willing to sell it for? $1,909. If the best rate of return that Zwerg can make on alternative investments of additional funds is 10%, what is the most that Zwerg would be willing to pay for Munchkin's new mold?

$1,909.

(d) What do you think will happen to the number of garden gnomes ordered for delivery on January 1, 2002? Will it be larger, smaller, or the same as the number ordered the previous year? Larger. After the passage of sufficient time, the industry will reach a new long-run equilibrium. What will be the new equilibrium price of gnomes? $7.20.

24.3 (1) In the previous problem, we studied the effects of a cost-saving invention. For this problem, we suppose that there was no such invention, but that a tax is introduced. Suppose that on January 1, 2001, the industry was as described in the previous problem (without the invention of the new kind of plaster). On this day, the government surprised the garden gnome industry by introducing a tax on the production of garden gnomes. For every garden gnome produced, the manufacturer must pay a $1 tax. The announcement came early enough in the day so that there was time for gnome producers to change their orders of gnome molds for 2002. But the gnome molds available to be used in 2001 are those that had been ordered a year previously. Gnome makers had signed contracts promising to pay $1,000 for each gnome mold that they ordered, and they couldn't back out of these promises. Thus in the short run, during the year 2001, the number of gnome molds is stuck at 28.

(a) On the graph below, draw the short run industry supply curve for garden gnomes that applies in the year 2001, after the new tax is introduced. On the same graph, show the demand curve for garden gnomes.

(b) In 2001, after the tax is introduced, what is the short run equilibrium total output of garden gnomes, 14,000. and what is the short run equilibrium price of garden gnomes? $9.20. (Hint: Look at the intersection of the supply and demand curves you just drew.)

(c) If you have a garden gnome mold, the marginal cost of producing a garden gnome, including the tax, is $8. Therefore all gnome molds (will, will not) will be used up to capacity in 2001.

(d) In 2001, what will be the total output of garden gnomes? 14,000. What will be the price of garden gnomes? $9.20. What rate of return will Deardwarf's cousin Zwerg make on his investment in a garden gnome mold that he ordered a year ago and paid $1,000 for at that time? −40%.

(e) Remember that Zwerg's neighbor, Munchkin, also has a gnome mold that is to be delivered on January 1, 2001. Knowing about the tax makes Munchkin's mold a less attractive investment than it was without the tax, but still Zwerg would buy it if he can get it cheap enough so that he makes a 10% rate of return on his investment. How much should he be willing to pay for Munchkin's new mold? $545.45.

(f) What do you think will happen to the number of gnome molds ordered for delivery on January 1, 2002? Will it be larger, smaller, or the same as the number ordered the previous year? Smaller.

(g) The tax on garden gnomes was left in place for many years, and nobody expected any further changes in the tax or in demand or supply conditions. After the passage of sufficient time, the industry reached a new long-run equilibrium. What was the new equilibrium price of gnomes? $10.20.

(h) In the short run, who would end up paying the tax on garden gnomes, the producers or the consumers? Producers. In the long run, did the price of gnomes go up by more, less, or the same amount as the tax per gnome? Same amount.

(i) Suppose that early in the morning of January 1, 2001, the government had announced that there would be a $1 tax on garden gnomes, but that the tax would not go into effect until January 1, 2002. Would the producers of garden gnomes necessarily be worse off than if there were no tax? Why or why not? No. The producers would anticipate the tax increase and restrict supply, thereby raising prices.

(j) Is it reasonable to suppose that the government could introduce "surprise" taxes without making firms suspicious that there would be similar "surprises" in the future? Suppose that the introduction of the tax in January 2001 makes gnome makers suspicious that there will be more taxes introduced in later years. Will this affect equilibrium prices and supplies? How? If a surprise tax makes gnome makers expect similar ''surprises'' in future, it will take a higher current price to get them to enter the industry. This will raise the price paid by consumers.

24.4 (0) Consider a competitive industry with a large number of firms, all of which have identical cost functions $c(y) = y^2 + 1$ for $y > 0$ and $c(0) = 0$. Suppose that initially the demand curve for this industry is given by $D(p) = 52 - p$. (The output of a firm does not have to be an integer number, but the number of firms does have to be an integer.)

(a) What is the supply curve of an individual firm? $S(p) =$ $p/2$. If there are n firms in the industry, what will be the industry supply curve? $Y = np/2$.

(b) What is the smallest price at which the product can be sold? $p^* = 2$.

(c) What will be the equilibrium number of firms in the industry? (Hint: Take a guess at what the industry price will be and see if it works.)

Guess at $p^* = 2$. This gives $D(p) = 52 - 2 = n2/2$, which says $n^* = 50$.

(d) What will be the equilibrium price? $p^* = 2$. What will be the equilibrium output of each firm? $y^* = 1$.

(e) What will be the equilibrium output of the industry? $Y^* = 50$.

(f) Now suppose that the demand curve shifts to $D(p) = 52.5 - p$. What will be the equilibrium number of firms? (Hint: Can a new firm enter the market and make nonnegative profits?) If a new firm entered, there would be 51 firms. The supply-demand equation would be $52.5 - p = 51p/2$. Solve for $p^* = 105/53 < 2$. A new firm would lose money. Therefore in equilibrium there would be 50 firms.

(g) What will be the equilibrium price? Solve $52.5 - p = 50p/2$ to get $p^* = 2.02$. What will be the equilibrium output of each firm? $y^* = 1.01$. What will be the equilibrium profits of each firm? Around .02.

(h) Now suppose that the demand curve shifts to $D(p) = 53 - p$. What will be the equilibrium number of firms? 51. What will be the equilibrium price? 2.

(i) What will be the equilibrium output of each firm? $y = 1$. What will be the equilibrium profits of each firm? Zero.

24.5 (3) In 1990, the town of Ham Harbor had a more-or-less free market in taxi services. Any respectable firm could provide taxi service as long as the drivers and cabs satisfied certain safety standards.

Let us suppose that the constant marginal cost per trip of a taxi ride is \$5, and that the average taxi has a capacity of 20 trips per day. Let the demand function for taxi rides be given by $D(p) = 1,200 - 20p$, where demand is measured in rides per day, and price is measured in dollars. Assume that the industry is perfectly competitive.

(a) What is the competitive equilibrium price per ride? (Hint: In competitive equilibrium, price must equal marginal cost.) 5. What is the equilibrium number of rides per day? 1,100. How many taxicabs will there be in equilibrium? 55.

(b) In 1990 the city council of Ham Harbor created a taxicab licensing board and issued a license to each of the existing cabs. The board stated that it would continue to adjust the taxicab fares so that the demand for rides equals the supply of rides, but no new licenses will be issued in the future. In 1995 costs had not changed, but the demand curve for taxicab rides had become $D(p) = 1,220 - 20p$. What was the equilibrium price of a ride in 1995? $6.

(c) What was the profit per ride in 1995, neglecting any costs associated with acquiring a taxicab license? $1. What was the profit per taxicab license per day? 20. If the taxi operated every day, what was the profit per taxicab license per year? $7,300.

(d) If the interest rate was 10% and costs, demand, and the number of licenses were expected to remain constant forever, what would be the market price of a taxicab license? $73,000.

(e) Suppose that the commission decided in 1995 to issue enough new licenses to reduce the taxicab price per ride to $5. How many more licenses would this take? 1.

(f) Assuming that demand in Ham Harbor is not going to grow any more, how much would a taxicab license be worth at this new fare?

Nothing.

(g) How much money would each current taxicab owner be willing to pay to prevent any new licenses from being issued? $73,000 each. What is the total amount that all taxicab owners together would be willing to pay to prevent any new licences from ever being issued? $4,015,000. The total amount that consumers would be willing to pay to have another taxicab license issued would be (more than, less than, the same as) more than this amount.

24.6 (2) In this problem, we will determine the equilibrium pattern of agricultural land use surrounding a city. Think of the city as being located in the middle of a large featureless plain. The price of wheat at the market at the center of town is $10 a bushel, and it only costs $5 a bushel to grow wheat. However, it costs 10 cents a mile to transport a bushel of wheat to the center of town.

(a) If a farm is located t miles from the center of town, write down a formula for its profit per bushel of wheat transported to market.

Profit per bushel $= 5 - .10t$.

(b) Suppose you can grow 1,000 bushels on an acre of land. How much will an acre of land located t miles from the market rent for? Rent $= 5,000 - 100t$.

(c) How far away from the market do you have to be for land to be worth zero? 50 miles.

24.7 (1) Consider an industry with three firms. Suppose the firms have the following supply functions: $S_1(p) = p$, $S_2(p) = p - 5$, and $S_3(p) = 2p$ respectively. On the graph below plot each of the three supply curves and the resulting industry supply curve.

(a) If the market demand curve has the form $D(p) = 15$, what is the resulting market price? 5. Output? 15. What is the output level for firm 1 at this price? 5. Firm 2? 0. Firm 3? 10.

24.8 (0) Suppose all firms in a given industry have the same supply curve given by $S_i(p) = p/2$. Plot and label the four industry supply curves generated by these firms if there are 1, 2, 3, or 4 firms operating in the industry.

(a) If all of the firms had a cost structure such that if the price was below \$3, they would be losing money, what would be the equilibrium price and output in the industry if the market demand was equal to $D(p) = 3.5$?

Answer: price = \$3.50 , quantity = 3.5. How many firms would exist in such a market? 2.

(b) What if the identical conditions as above held except that the market demand was equal to $D(p) = 8 - p$? Now, what would be the equilibrium price and output? \$3.20 and 4.8. How many firms would operate in such a market? 3.

24.9 (0) There is free entry into the pollicle industry. Anybody can enter this industry and have the same U-shaped average cost curve as all of the other firms in the industry.

(a) On the diagram below, draw a representative firm's average and marginal cost curves using blue ink. Also, indicate the long-run equilibrium level of the market price.

(b) Suppose the government imposes a tax, t, on every unit of output sold by the industry. Use red ink to draw the new conditions on the above graph. After the industry has adjusted to the imposition of the tax, the competitive model would predict the following: the market price would (increase, decrease) increase by amount t, there would be (more, the same, fewer) fewer firms operating in the industry, and the output level for each firm operating in the industry would Stay the same. (increase, stay the same, decrease).

(c) What if the government imposes a tax, l, on every *firm* in the industry. Draw the new cost conditions on the above graph using black ink. After the industry has adjusted to the imposition of the tax the competitive model would predict the following: the market price would (increase, decrease) increase, there would be (more, the same, fewer) fewer firms operating in the industry, and the output level for each firm operating in the industry would increase (increase, stay the same, decrease).

24.10 (0) In many communities, a restaurant that sells alcoholic beverages is required to have a license. Suppose that the number of licenses is limited and that they may be easily transferred to other restaurant owners. Suppose that the conditions of this industry closely approximate perfect competition. If the average restaurant's revenue is $100,000 a year, and if a liquor license can be leased for a year for $85,000 from an existing restaurant, what is the average variable cost in the industry?

$15,000.

24.11 (2) In order to protect the wild populations of cockatoos, the Australian authorities have outlawed the export of these large parrots. An illegal market in cockatoos has developed. The cost of capturing an Australian cockatoo and shipping him to the United States is about \$40 per bird. Smuggled parrots are drugged and shipped in suitcases. This is extremely traumatic for the birds and about 50% of the cockatoos shipped die in transit. Each smuggled cockatoo has a 10% chance of being discovered, in which case the bird is confiscated and a fine of \$500 is charged. Confiscated cockatoos that are alive are returned to the wild. Confiscated cockatoos that are found dead are donated to university cafeterias.*

(a) The probability that a smuggled parrot will reach the buyer alive and unconfiscated is .45. Therefore when the price of smuggled parrots is p, what is the expected gross revenue to a parrot-smuggler from shipping a parrot? $.45p$.

(b) What is the expected cost, including expected fines and the cost of capturing and shipping, per parrot? $\$.10 \times 500 + 40 = \$90$.

(c) The supply schedule for smuggled parrots will be a horizontal line at the market price \$200. (Hint: At what price does a parrot-smuggler just break even?)

(d) The demand function for smuggled cockatoos in the United States is $D(p) = 7,200 - 20p$ per year. How many smuggled cockatoos will be sold in the United States per year at the equilibrium price? 3,200. How many cockatoos must be caught in Australia in order that this number of live birds reaches U.S. buyers? $3,200/.45 = 7,111$.

(e) Suppose that instead of returning live confiscated cockatoos to the wild, the customs authorities sold them in the American market. The profits from smuggling a cockatoo do not change from this policy change. Since the supply curve is horizontal, it must be that the equilibrium price of smuggled cockatoos will have to be the same as the equilibrium price when the confiscated cockatoos were returned to nature. How many live cockatoos will be sold in the United States in equilibrium? 3,200. How many cockatoos will be permanently removed from the Australian wild? 6,400.

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* The story behind this problem is based on actual fact, but the numbers we use are just made up for illustration. It would be very interesting to have some good estimates of the actual demand functions and cost functions.

(f) Suppose that the trade in cockatoos is legalized. Suppose that it costs about $40 to capture and ship a cockatoo to the United States in a comfortable cage and that the number of deaths in transit by this method is negligible. What would be the equilibrium price of cockatoos in the United States? $40. How many cockatoos would be sold in the United States? 6,400. How many cockatoos would have to be caught in Australia for the U.S. market? 6,400.

24.12 (0) The horn of the rhinoceros is prized in Japan and China for its alleged aphrodisiac properties. This has proved to be most unfortunate for the rhinoceroses of East Africa. Although it is illegal to kill rhinoceroses in the game parks of Kenya, the rhinoceros population of these parks has been almost totally depleted by poachers. The price of rhinoceros horns in recent years has risen so high that a poacher can earn half a year's wages by simply killing one rhinoceros. Such high rewards for poaching have made laws against poaching almost impossible to enforce in East Africa. There are also large game parks with rhinoceros populations in South Africa. Game wardens there were able to prevent poaching almost completely and the rhinoceros population of South Africa has prospered. In a recent program from the television series *Nova*, a South African game warden explained that some rhinoceroses even have to be "harvested" in order to prevent overpopulation of rhinoceroses. "What then," asked the interviewer, "do you do with the horns from the animals that are harvested or that die of natural causes?" The South African game warden proudly explained that since international trade in rhinoceros horns was illegal, South Africa did not contribute to international crime by selling these horns. Instead the horns were either destroyed or stored in a warehouse.

(a) Suppose that all of the rhinoceros horns produced in South Africa are destroyed. Label the axes below and draw world supply and demand curves for rhinoceros horns with blue ink. Label the equilibrium price and quantity.

(b) If South Africa were to sell its rhinoceros horns on the world market, which of the curves in your diagram would shift and in what direction? Supply curve to the right. Use red ink to illustrate the shifted curve or curves. If South Africa were to do this, would world consumption of rhinoceros horns be increased or decreased? Increased. Would the world price of rhinoceros horns be increased or decreased? Decreased. Would the amount of rhinoceros poaching be increased or decreased? Decreased.

24.13 (1) The sale of rhinoceros horns is not prohibited because of concern about the wicked pleasures of aphrodisiac imbibers, but because the supply activity is bad for rhinoceroses. Similarly, the Australian reason for restricting the exportation of cockatoos to the United States is not because having a cockatoo is bad for you. Indeed it is legal for Australians to have cockatoos as pets. The motive for the restriction is simply to protect the wild populations from being overexploited. In the case of other commodities, it appears that society has no particular interest in restricting the supply activities but wishes to restrict consumption. A good example is illicit drugs. The growing of marijuana, for example, is a simple pastoral activity, which in itself is no more harmful than growing sweet corn or brussels sprouts. It is the consumption of marijuana to which society objects.

Suppose that there is a constant marginal cost of $5 per ounce for growing marijuana and delivering it to buyers. But whenever the marijuana authorities find marijuana growing or in the hands of dealers, they seize the marijuana and fine the supplier. Suppose that the probability

that marijuana is seized is .3 and that the fine if you are caught is \$10 per ounce.

(a) If the "street price" is \$$p$ per ounce, what is the expected revenue net of fines to a dealer from selling an ounce of marijuana? $.7p - 3.$ What then would be the equilibrium price of marijuana? \$11.4.

(b) Suppose that the demand function for marijuana has the equation $Q = A - Bp$. If all confiscated marijuana is destroyed, what will be the equilibrium consumption of marijuana? $A - 11.4B.$ Suppose that confiscated marijuana is not destroyed but sold on the open market. What will be the equilibrium consumption of marijuana? $A - 11.4B.$

(c) The price of marijuana will (increase, decrease, stay the same) Stay the same.

(d) If there were increasing rather than constant marginal cost in marijuana production, do you think that consumption would be greater if confiscated marijuana were sold than if it were destroyed? Explain. Consumption will increase because the supply curve will shift to the right, lowering the price.

Chapter 25

NAME_____

Monopoly

Introduction. The profit-maximizing output of a monopolist is found by solving for the output at which marginal revenue is equal to marginal cost. Having solved for this output, you find the monopolist's price by plugging the profit-maximizing output into the demand function. In general, the marginal revenue function can be found by taking the derivative of the total revenue function with respect to the quantity. But in the special case of linear demand, it is easy to find the marginal revenue curve graphically. With a linear inverse demand curve, $p(y) = a - by$, the marginal revenue curve always takes the form $MR(y) = a - 2by$.

25.1 (0) Professor Bong has just written the first textbook in Punk Economics. It is called *Up Your Isoquant*. Market research suggests that the demand curve for this book will be $Q = 2,000 - 100P$, where P is its price. It will cost \$1,000 to set the book in type. This setup cost is necessary before any copies can be printed. In addition to the setup cost, there is a marginal cost of \$4 per book for every book printed.

(a) The total revenue function for Professor Bong's book is $R(Q) =$ $20Q - Q^2/100$.

(b) The total cost function for producing Professor Bong's book is $C(Q) =$ $1,000 + 4Q$.

(c) The marginal revenue function is $MR(Q) =$ $20 - Q/50$ and the marginal cost function is $MC(Q) =$ 4. The profit-maximizing quantity of books for professor Bong to sell is $Q^* =$ 800.

25.2 (0) Peter Morgan sells pigeon pies from a pushcart in Central Park. Morgan is the only supplier of this delicacy in Central Park. His costs are zero due to the abundant supplies of raw materials available in the park.

(a) When he first started his business, the inverse demand curve for pigeon pies was $p(y) = 100 - y$, where the price is measured in cents and y measures the number of pies sold. Use black ink to plot this curve in the graph below. On the same graph, use red ink to plot the marginal revenue curve.

(b) What level of output will maximize Peter's profits? 50. What price will Peter charge per pie? 50 cents.

(c) After Peter had been in business for several months, he noticed that the demand curve had shifted to $p(y) = 75 - y/2$. Use blue ink to plot this curve in the graph above. Plot the new marginal revenue curve on the same graph with black ink.

(d) What is his profit-maximizing output at this new price? 75. What is the new profit-maximizing price? 37.5 cents per pie.

25.3 (0) Suppose that the demand function for Japanese cars in the United States is such that annual sales of cars (in thousands of cars) will be $250 - 2P$, where P is the price of Japanese cars in thousands of dollars.

(a) If the supply schedule is horizontal at a price of \$5,000 what will be the equilibrium number of Japanese cars sold in the United States?

240 thousand. How much money will Americans spend in total on Japanese cars? 1.2 billion dollars.

(b) Suppose that in response to pressure from American car manufacturers, the United States imposes an import duty on Japanese cars in such a way that for every car exported to the United States the Japanese manufacturers must pay a tax to the U.S. government of \$2,000. How many Japanese automobiles will now be sold in the United States? 236 thousand. At what price will they be sold? 7 thousand dollars.

(c) How much revenue will the U.S. government collect with this tariff?

472 million dollars.

(d) On the graph below, the price paid by American consumers is measured on the vertical axis. Use blue ink to show the demand and supply schedules before the import duty is imposed. After the import duty is imposed, the supply schedule shifts and the demand schedule stays as before. Use red ink to draw the new supply schedule.

(e) Suppose that instead of imposing an import duty, the U.S. government persuades the Japanese government to impose "voluntary export restrictions" on their exports of cars to the United States. Suppose that the Japanese agree to restrain their exports by requiring that every car exported to the United States must have an export license. Suppose further that the Japanese government agrees to issue only 236,000 export licenses and sells these licenses to the Japanese firms. If the Japanese firms know the American demand curve and if they know that only 236,000 Japanese cars will be sold in America, what price will they be able to charge in America for their cars? 7 thousand dollars.

(f) How much will a Japanese firm be willing to pay the Japanese government for an export license? 2 thousand dollars. (Hint: Think about what it costs to produce a car and how much it can be sold for if you have an export license.)

(g) How much will be the Japanese government's total revenue from the sale of export licenses? 472 million dollars.

(h) How much money will Americans spend on Japanese cars? 1.652 billion dollars.

(i) Why might the Japanese "voluntarily" submit to export controls?

Total revenue of Japanese companies and government is greater with export controls than without them. Since there is less output, costs are lower. Higher revenue, lower costs imply more profit.

25.4 (0) A monopolist has an inverse demand curve given by $p(y) = 12 - y$ and a cost curve given by $c(y) = y^2$.

(a) What will be its profit-maximizing level of output? 3.

(b) Suppose the government decides to put a tax on this monopolist so that for each unit it sells it has to pay the government \$2. What will be its output under this form of taxation? 2.5.

(c) Suppose now that the government puts a lump sum tax of \$10 on the profits of the monopolist. What will be its output? 3.

25.5 (1) In Gomorrah, New Jersey, there is only one newspaper, the *Daily Calumny*. The demand for the paper depends on the price and the amount of scandal reported. The demand function is $Q = 15S^{1/2}P^{-3}$, where Q is the number of issues sold per day, S is the number of column inches of scandal reported in the paper, and P is the price. Scandals are not a scarce commodity in Gomorrah. However, it takes resources to write, edit, and print stories of scandal. The cost of reporting S units of scandal is $\$10S$. These costs are independent of the number of papers sold. In addition it costs money to print and deliver the paper. These cost \$.10 per copy and the cost per unit is independent of the amount of scandal reported in the paper. Therefore the total cost of printing Q copies of the paper with S column inches of scandal is $\$10S + .10Q$.

(a) Calculate the price elasticity of demand for the *Daily Calumny*. -3. Does the price elasticity depend on the amount of scandal reported? No. Is the price elasticity constant over all prices? Yes.

(b) Remember that $MR = P(1 + \frac{1}{\epsilon})$. To maximize profits, the *Daily Calumny* will set marginal revenue equal to marginal cost. Solve for the profit-maximizing price for the *Calumny* to charge per newspaper. $.15. When the newspaper charges this price, the difference between the price and the marginal cost of printing and delivering each newspaper is $.05.

(c) If the *Daily Calumny* charges the profit-maximizing price and prints 100 column inches of scandal, how many copies would it sell? (Round to the nearest integer.) 44,444. Write a general expression for the number of copies sold as a function of S: $Q(S) =$ $Q = 15S^{1/2}(.15)^{-3} = 4,444.44S^{1/2}$.

(d) Assuming that the paper charges the profit-maximizing price, write an expression for profits as a function of Q and S. Profits= $.15Q - .10Q - 10S$. Using the solution for $Q(S)$ that you found in the last section, substitute $Q(S)$ for Q to write an expression for profits as a function of S alone. Profits =$.05(4,444.44S^{1/2}) - 10S = 222.22S^{1/2} - 10S$.

(e) If the *Daily Calumny* charges its profit-maximizing price, and prints the profit-maximizing amount of scandal, how many column inches of scandal should it print? 123.456 inches. How many copies are sold 49,383 and what is the amount of profit for the *Daily Calumny* if it maximizes its profits? 1,234.5.

25.6 (0) In the graph below, use black ink to draw the inverse demand curve, $p_1(y) = 200 - y$.

(a) If the monopolist has zero costs, where on this curve will it choose to operate? At $y = 100$, $p = 100$.

(b) Now draw another demand curve that passes through the profit-maximizing point and is flatter than the original demand curve. Use a red pen to mark the part of this new demand curve on which the monopolist would choose to operate. (Hint: Remember the idea of revealed preference?)

(c) The monopolist would have (larger, smaller) profits at the new demand curve than it had at the original demand curve. `Larger.`

Chapter 26

NAME_____

Monopoly Behavior

Introduction. Problems in this chapter explore the possibilities of price discrimination by monopolists. There are also problems related to spatial markets, where transportation costs are accounted for and we show that lessons learned about spatial models give us a useful way of thinking about competition under product differentiation in economics and in politics.

Remember that a price discriminator wants the *marginal revenue* in each market to be equal to the marginal cost of production. Since he produces all of his output in one place, his marginal cost of production is the same for both markets and depends on his *total* output. The trick for solving these problems is to write marginal revenue in each market as a function of quantity sold in that market and to write marginal cost as a function of the sum of quantities sold in the two markets. The profit-maximizing conditions then become two equations that you can solve for the two unknown quantities sold in the two markets. Of course, if marginal cost is constant, your job is even easier, since all you have to do is find the quantities in each market for which marginal revenue equals the constant marginal cost.

Example: A monopolist sells in two markets. The inverse demand curve in market 1 is $p_1 = 200 - q_1$. The inverse demand curve in market 2 is $p_2 = 300 - q_2$. The firm's total cost function is $C(q_1+q_2) = (q_1+q_2)^2$. The firm is able to price discriminate between the two markets. Let us find the prices that it will charge in each market. In market 1, the firm's marginal revenue is $200 - 2q_1$. In market 2, marginal revenue is $300 - 2q_2$. The firm's marginal costs are $2(q_1 + q_2)$. To maximize its profits, the firm sets marginal revenue in each market equal to marginal cost. This gives us the two equations $200 - 2q_1 = 2(q_1 + q_2)$ and $300 - 2q_2 = 2(q_1 + q_2)$. Solving these two equations in two unknowns for q_1 and q_2, we find $q_1 = 16.67$ and $q_2 = 66.67$. We can find the price charged in each market by plugging these quantities into the demand functions. The price charged in market 1 will be 183.33. The price charged in market 2 will be 233.33.

26.1 (0) Ferdinand Sludge has just written a disgusting new book, *Orgy in the Piggery*. His publisher, Graw McSwill, estimates that the demand for this book in the United States is $Q_1 = 50,000 - 2,000P_1$, where P_1 is the price in the U.S. measured in U.S. dollars. The demand for Sludge's opus in England is $Q_2 = 10,000 - 500P_2$, where P_2 is its price in England measured in U. S. dollars. His publisher has a cost function $C(Q) = \$50,000 + \$2Q$, where Q is the total number of copies of *Orgy* that it produces.

(a) If McSwill must charge the same price in both countries, how many copies should it sell? 27,500. What price should it charge

to maximize its profits $13. How much will those profits be? $252,500.

(b) If McSwill can charge a different price in each country and wants to maximize profits, how many copies should it sell in the United States? 23,000. What price should it charge in the United States? $13.50. How many copies should it sell in England? 4,500. What price should it charge in England? $11. How much will its total profits be? $255,000.

26.2 (0) A monopoly faces an inverse demand curve, $p(y) = 100 - 2y$, and has constant marginal costs of 20.

(a) What is its profit-maximizing level of output? 20.

(b) What is its profit-maximizing price? $60.

(c) What is the socially optimal price for this firm? $20.

(d) What is the socially optimal level of output for this firm? 40.

(e) What is the deadweight loss due to the monopolistic behavior of this firm? 400.

(f) Suppose this monopolist could operate as a perfectly discriminating monopolist and sell each unit of output at the highest price it would fetch. The deadweight loss in this case would be 0.

Calculus **26.3 (1)** Banana Computer Company sells Banana computers in both the domestic and foreign markets. Because of differences in the power supplies, a Banana purchased in one market cannot be used in the other market. The demand and marginal revenue curves associated with the two markets are as follows:

$$P_d = 20,000 - 20Q \quad P_f = 25,000 - 50Q$$
$$MR_d = 20,000 - 40Q \quad MR_f = 25,000 - 100Q.$$

Banana's production process exhibits constant returns to scale and it takes $1,000,000 to produce 100 computers.

(a) Banana's long-run average cost function is $AC(Q) =$ $10,000 and its long-run marginal cost function is $MC(Q) =$ $10,000. (Hint: If there are constant returns to scale, does long-run average cost change as output changes?) Draw the average and marginal cost curves on the graph.

(b) Draw the demand curve for the domestic market in black ink and the marginal revenue curve for the domestic market in pencil. Draw the demand curve for the foreign market in red ink and the marginal revenue curve for the foreign market in blue ink.

(c) If Banana is maximizing its profits, it will sell 250 computers in the domestic market at 15,000 dollars each and 150 computers in the foreign market at 17,500 dollars each. What are Banana's total profits? $2,375,000.

(d) At the profit-maximizing price and quantity, what is the price elasticity of demand in the domestic market? -3. What is the price elasticity of demand in the foreign market? -2.33. Is demand more or less elastic in the market where the higher price is charged? Less elastic.

(e) Suppose that somebody figures out a wiring trick that allows a Banana computer built for either market to be costlessly converted to work in the other. (Ignore transportation costs.) On the graph below, draw the new inverse demand curve (with blue ink) and marginal revenue curve (with black ink) facing Banana.

(f) Given that costs haven't changed, how many Banana computers should Banana sell? 400. What price will it charge? \$15,714. How will Banana's profits change now that it can no longer practice price discrimination? Decrease by \$89,284.

26.4 (0) A monopolist has a cost function given by $c(y) = y^2$ and faces a demand curve given by $P(y) = 120 - y$.

(a) What is its profit-maximizing level of output? 30. What price will the monopolist charge? \$90.

(b) If you put a lump sum tax of \$100 on this monopolist, what would its output be? 30.

(c) If you wanted to choose a price ceiling for this monopolist so as to maximize consumer plus producer surplus, what price ceiling should you choose? \$80.

(d) How much output will the monopolist produce at this price ceiling?

40.

(e) Suppose that you put a specific tax on the monopolist of \$20 per unit output. What would its profit-maximizing level of output be? 25.

26.5 (1) The Grand Theater is a movie house in a medium-sized college town. This theater shows unusual films and treats early-arriving movie goers to live organ music and Bugs Bunny cartoons. If the theater is open, the owners have to pay a fixed nightly amount of \$500 for films, ushers, and so on, regardless of how many people come to the movie. For simplicity, assume that if the theater is closed, its costs are zero. The nightly demand for Grand Theater movies by students is $Q_S = 220 - 40P_S$, where Q_S is the number of movie tickets demanded by students at price P_S. The nightly demand for nonstudent moviegoers is $Q_N = 140 - 20P_N$.

(a) If the Grand Theater charges a single price, P_T, to everybody, then at prices between 0 and \$5.50, the aggregate demand function for movie tickets is $Q_T(P_T) =$ $360 - 60P_T$. Over this range of prices, the inverse demand function is then $P_T(Q_T) =$ $6 - Q_T/60$.

(b) What is the profit-maximizing number of tickets for the Grand Theater to sell if it charges one price to everybody? 180. At what price would this number of tickets be sold? \$3. How much profits would the Grand make? \$40. How many tickets would be sold to students? 100. To nonstudents? 80.

(c) Suppose that the cashier can accurately separate the students from the nonstudents at the door by making students show their school ID cards. Students cannot resell their tickets and nonstudents do not have access to student ID cards. Then the Grand can increase its profits by charging students and nonstudents different prices. What price will be charged to students? \$2.75. How many student tickets will be sold?

110. What price will be charged to nonstudents? $3.50. How many nonstudent tickets will be sold? 70. How much profit will the Grand Theater make? $47.50.

(d) If you know calculus, see if you can do this part. Suppose that the Grand Theater can hold only 150 people and that the manager wants to maximize profits by charging separate prices to students and to nonstudents. If the capacity of the theater is 150 seats and Q_S tickets are sold to students, what is the maximum number of tickets that can be sold to nonstudents? $Q_N = 150 - Q_S$. Write an expression for the price of *nonstudent* tickets as a function of the number of *student* tickets sold. (Hint: First find the inverse nonstudent demand function.) $P_N = -1/2 + Q_S/20$. Write an expression for Grand Theater profits as a function of the number Q_S only. (Hint: Make substitutions using your previous answers.) $Q_S(11/2 - Q_S/40) + (-1/2 + 20/Q_S)(150 - Q_S) - 500 = -3Q_S^2/40 + 27Q_S/2 - 575$. How many student tickets should the Grand sell to maximize profits? 90. What price is charged to students? $3.25. How many nonstudent tickets are sold? 60. What price is charged to nonstudents? $4. How much profit does the Grand make under this arrangement? $32.50.

26.6 (2) The *Mall Street Journal* is considering offering a new service which will send news articles to readers by email. Their market research indicates that there are two types of potential users, impecunious students and high-level executives. Let x be the number of articles that a user requests per year. The executives have an inverse demand function $P_E(x) = 100 - x$ and the students have an inverse demand function $P_U(x) = 80 - x$. (Prices are measured in cents.) The *Journal* has a zero marginal cost of sending articles via email. Draw these demand functions in the graph below and label them.

(a) Suppose that the *Journal* can identify which users are students and which are executives. It offers each type of user a different all or nothing deal. A student can either buy access to 80 articles per year or to none at all. What is the maximum price a student will be willing to pay for access to 80 articles? $32, which equals the area under the student's demand curve. (Hint: Recall the lesson on consumer's surplus and the area under the demand curve.) An executive can either buy access to 100 articles per year or to none at all. What is the maximum price an executive would be willing to pay for access to 100 articles? $50 which is the area under an executive's demand curve.

(b) Suppose that the *Journal* can't tell which users are executives and which are undergraduates. Thus it can't be sure that executives wouldn't buy the student package if they found it to be a better deal for them. In this case, the *Journal* can still offer two packages, but it will have to let the users self-select the one that is optimal for them. Suppose that it offers two packages: one that allows up to 80 articles per year the other that allows up to 100 articles per year. What's the highest price that the undergraduates will pay for the 80-article subscription? $32.

(c) What is the total value to the executives of reading 80 articles per year? $48. (Hint: Look at the area under their demand curve and to the right of a vertical line at 80 articles.)

(d) What is the the maximum price that the *Journal* can charge for 100 articles per year if it wants executives to prefer this deal to buying 80 articles a year at the highest price the undergraduates are willing to pay for 80 articles? Solve $50 - p = 48 - 32$ to find $p = \$34$.

(e) Suppose that the *Mall Street Journal* decides to include only 60 articles in the student package. What is the most it could charge and still get student to buy this package? $30.

(f) If the *Mall Street Journal* offers a "student package" of 60 articles at this price, how much *net* consumer surplus would executives get from buying the student package? $12.

(g) What is the most that the *Mall Street Journal* could charge for a 100-article package and expect executives to buy this package rather than the student package? $38.

(h) If the number of executives in the population equals the number of students, would the *Mall Street Journal* make higher profits by offering a student package of 80 articles or a student package of 60 articles? 60.

26.7 (2) Bill Barriers, CEO of MightySoft software, is contemplating a new marketing strategy: bundling their best-selling wordprocessor and their spreadsheet together and selling the pair of software products for one price.

From the viewpoint of the company, bundling software and selling it at a discounted price has two effects on sales: (1) revenues go up due to to additional sales of the bundle; and (2) revenues go down since there is less of a demand for the individual components of the bundle.

The profitability of bundling depends on which of these two effects dominates. Suppose that MightySoft sells the wordprocessor for $200 and the spreadsheet for $250. A marketing survey of 100 people who purchased either of these packages in the last year turned up the following facts:

1) 20 people bought both.
2) 40 people bought only the wordprocessor. They would be willing to spend up to $120 more for the spreadsheet.
3) 40 people bought only the spreadsheet. They would be willing to spend up to $100 more for the wordprocessor.

In answering the following questions you may assume the following:

1) New purchasers of MightySoft products will have the same characteristics as this group.
2) There is a zero marginal cost to producing extra copies of either software package.
3) There is a zero marginal cost to creating a bundle.

(a) Let us assume that MightySoft also offers the products separately as well as bundled. In order to determine how to price the bundle, Bill Barriers asks himself the following questions. In order to sell the bundle to the wordprocessor purchasers, the price would have to be less than $200 + 120 = 320$.

(b) In order to sell the bundle to the spreadsheet users, the price would have to be less than $250 + 100 = 350$.

(c) What would MightySoft's profits be on a group of 100 users if it priced the bundle at \$320? Everyone buys the bundle so profits are $100 \times 320 = \$32,000$.

(d) What would MightySoft's profits be on a group of 100 users if it priced the bundle at \$350? 20 people would buy both anyway, 40 people would buy the spreadsheet only and would be willing to buy the bundle, 40 people would buy the wordprocessor, but not the spreadsheet. Total profits would be $20 \times 350 + 40 \times 350 + 40 \times 200 = 29,000$.

(e) If MightySoft offers the bundle, what price should it set? \$320 is the more profitable price.

(f) What would profits be without offering the bundle? Without the bundle, profits would be $20 \times (200 + 250) + 40 * 200 + 40 * 250 = 27,000$.

(g) What would be the profits with the bundle? $100 \times 320 = 32,000$

(h) Is it more profitable to bundle or not bundle? Bundle.

(i) Suppose that MightySoft worries about the reliability of their market survey and decides that they believe that without bundling t of the 100 people will buy both products, and $(100-t)/2$ will buy the wordprocessor only and $(100-t)/2$ will buy the spreadsheet only. Calculate profits as a function of t if there is no bundling. $225 \times (100-t) + 450 \times t$.

(j) What are profits with the bundle? $32,000.

(k) At what values of t would it be unprofitable to offer the bundle? Solve for the t that equates the two profits to find $t = 42.22$. So if more than 42 of the 100 new purchasers would buy both products anyway, it is not profitable to bundle them.

(l) This analysis so far has been concerned only with customers who would purchase at least one of the programs at the original set of prices. Is there any additional source of demand for the bundle? What does this say about the calculations we have made about the profitability of bundling? Yes, it may be that there are some consumers who were not willing to pay $200 for the wordprocessor or $250 for the spreadsheet, but would be willing to pay $320 for the bundle. This means that bundling would be even more profitable than the calculations above indicate.

26.8 (0) Colonel Tom Barker is about to open his newest amusement park, Elvis World. Elvis World features a number of exciting attractions: you can ride the rapids in the Blue Suede Chutes, climb the Jailhouse Rock and eat dinner in the Heartburn Hotel. Colonel Tom figures that Elvis World will attract 1,000 people per day, and each person will take $x = 50 - 50p$ rides, where p is the price of a ride. Everyone who visits Elvis World is pretty much the same and negative rides are not allowed. The marginal cost of a ride is essentially zero.

(a) What is each person's inverse demand function for rides? $p(x) = 1 - x/50$.

(b) If Colonel Tom sets the price to maximize profit, how many rides will be taken per day by a typical visitor? 25.

(c) What will the price of a ride be? 50 cents.

(d) What will Colonel Tom's profits be per person? $12.50

(e) What is the Pareto efficient price of a ride? Zero.

(f) If Colonel Tom charged the Pareto efficient price for a ride, how many rides would be purchased? 50.

(g) How much consumers' surplus would be generated at this price and quantity? 25.

(h) If Colonel Tom decided to use a two-part tariff, he would set an admission fee of $25 and charge a price per ride of 0.

26.9 (1) The city of String Valley is squeezed between two mountains and is 36 miles long, running from north to south, and only about 1 block wide. Within the town, the population has a uniform density of 100 people per mile. Because of the rocky terrain, nobody lives outside the city limits on either the north or the south edge of town. Because of strict zoning regulations, the city has only three bowling alleys. One of these is located at the city limits on the north edge of town, one of them is located at the city limits on the south edge of town, and one is located at the exact center of town. Travel costs including time and gasoline are $1 per mile. All of the citizens of the town have the same preferences. They are willing to bowl once a week if the cost of bowling including travel costs and the price charged by the bowling alley does not exceed $15.

(a) Consider one of the bowling alleys at either edge of town. If it charges $10 for a night of bowling, how far will a citizen of String Valley be willing to travel to bowl there? Up to 5 miles. How many customers would this bowling alley have per week if it charged $10 per night of bowling? 500.

(b) Write a formula for the number of customers that a bowling alley at the edge of town will have if it charges $\$p$ per night of bowling.

$100 \times (15 - p)$.

(c) Write a formula for this bowling alley's inverse demand function.

$p = 15 - q/100$.

(d) Suppose that the bowling alleys at the end of town have a marginal cost of \$3 per customer and set their prices to maximize profits. (For the time being assume that these bowling alleys face no competition from the other bowling alleys in town.) How many customers will they have? 600. What price will they charge? \$9. How far away from the edge of town does their most distant customer live? 6 miles.

(e) Now consider the bowling alley in the center of town. If it charges a price of $\$p$, how many customers will it have per week? 2*(15-p).

(f) If the bowling alley in the center of town also has marginal costs of \$3 per customer and maximizes its profits, what price will it charge? \$9. How many customers will it have per week? 1,200. How far away from the center of town will its most distant customers live? 6 miles.

(g) Suppose that the city relaxes its zoning restrictions on where the bowling alleys can locate, but continues to issue operating licenses to only 3 bowling alleys. Both of the bowling alleys at the end of town are about to lose their leases and can locate anywhere in town that they like at about the same cost. The bowling alley in the center of town is committed to stay where it is. Would either of the alleys at the edge of town improve its profits by locating next to the existing bowling alley in the center of town? No. What would be a profit-maximizing location for each of these two bowling alleys? One would be located 12 miles north of the town center and one 12 miles south.

26.10 (1) In a congressional district somewhere in the U.S. West a new representative is being elected. The voters all have one-dimensional political views that can be neatly arrayed on a left-right spectrum. We can define the "location" of a citizen's political views in the following way. The citizen with the most extreme left-wing views is said to be at point 0 and the citizen with the most extreme right-wing views is said to be at point 1. If a citizen has views that are to the right of the views of the fraction x of the state's population, that citizen's views are said to be located at the point x. Candidates for office are forced to publically state their own political position on the zero-one left-right scale. Voters always vote for the candidate whose stated position is nearest to their own views. (If there is a tie for nearest candidate, voters flip a coin to decide which to vote for.)

(a) There are two candidates for the congressional seat. Suppose that each candidate cares only about getting as many votes as possible. Is there an equilibrium in which each candidate chooses the best position given the position of the other candidate? If so, describe this equilibrium.

The only equilibrium is one in which both candidates choose the same position, and that position is at the point 1/2.

26.11 (2) In the congressional district described by the previous problem, let us investigate what will happen if the two candidates do not care about the number of votes that they get but only about the amount of campaign contributions that they receive. Therefore each candidate chooses his ideological location in such a way as to maximize the amount of campaign contributions he receives, given the position of the other.*

Let us define a left-wing extremist as a voter whose political views lie to the left of the leftmost candidate, a right-wing extremist as a voter whose political views lie to the right of the rightmost candidate, and a moderate voter as one whose political views lie between the positions of the two candidates. Assume that each extremist voter contributes to the candidate whose position is closest to his or her own views and that moderate voters make no campaign contributions. The number of dollars that an extremist voter contributes to his or her favorite candidate is proportional to the distance between the two candidates. Specifically, we assume that there is some constant C such that if the left-wing candidate is located at x and the right-wing candidate is located at y, then total campaign contributions received by the left-wing candidate will be $\$Cx(y-x)$ and total campaign contributions received by the right-wing candidate will be $\$C(1-y)(y-x)$.

* This assumption is a bit extreme. Candidates typically spend at least some of their campaign contributions on advertising for votes, and this advertising affects the voting outcomes.

(a) If the right-wing candidate is located at y, the contribution-maximizing position for the left-wing candidate is $x =$ $y/2$ If the left-wing candidate is located at x, the contribution-maximizing position for the right-wing candidate is $y =$ $(1+x)/2$. (Hint: Take a derivative and set it equal to zero.)

(b) Solve for the unique pair of ideological positions for the two candidates such that each takes the position that maximizes his campaign contributions given the position of the other. $x = 1/3,\ y = 2/3$

(c) Suppose that in addition to collecting contributions from extremists on their side, candidates can also collect campaign contributions from moderates whose views are closer to their position than to that of their rivals' position. Suppose that moderates, like extremists, contribute to their preferred candidate and that they contribute in proportion to the difference between their own ideological distance from their less-preferred candidate and their ideological distance from their more-preferred candidate. Show that in this case the unique positions in which the left- and right-wing candidates are each maximizing their campaign contributions, given the position of the other candidate, occurs where $x = 1/4$ and $y = 3/4$. Total contributions received by the left wing candidate will be $C\left(x(y-x)+(y-x)^2/4\right)$. Total contributions received by the right-wing candidate will be $C\left((1-y)(y-x)+(y-x)^2/4\right)$. Differentiating the former expression with respect to x and the latter with respect to y and solving the resulting simultaneous equations yields $x = 1/4$ and $y = 3/4$.

Chapter 27

NAME_____

Factor Markets

Introduction. In this chapter you will examine the factor demand decision of a monopolist. If a firm is a monopolist in some industry, it will produce less output than if the industry were competitively organized. Therefore it will in general want to use less inputs than does a competitive firm. The value marginal product is just the value of the extra output produced by hiring an extra unit of the factor. The ordinary logic of competitive profit maximization implies that a competitive firm will hire a factor up until the point where the **value marginal product** equals the price of the factor.

The **marginal revenue product** is the extra revenue produced by hiring an extra unit of a factor. For a competitive firm, the marginal revenue product is the same as the value of the marginal product, but they differ for monopolist. A monopolist has to take account of the fact that increasing its production will force the price down, so the marginal revenue product of an extra unit of a factor will be *less* than the value marginal product.

Another thing we study in this chapters is **monopsony**, which is the case of a market dominated by a single buyer of some good. The case of monopsony is very similar to the case of a monopoly: The monopsonist hires less of a factor than a similar competitive firm because the monopsony recognizes that the price it has to pay for the factor depends on how much it buys.

Finally, we consider an interesting example of factor supply, in which a monopolist produces a good that is used by another monopolist.

Example: Suppose a monopolist faces a demand curve for output of the form $p(y) = 100 - 2y$. The production function takes the simple form $y = 2x$, and the factor costs \$4 per unit. How much of the factor of production will the monopolist want to employ? How much of the factor would a competitive industry employ if all the firms in the industry had the same production function?

Answer: The monopolist will employ the factor up to the point where the marginal revenue product equals the price of the factor. Revenue as a function of output is $R(y) = p(y)y = (100 - 2y)y$. To find revenue as a function of the input, we substitute $y = 2x$:

$$R(x) = (100 - 4x)2x = (200 - 8x)x.$$

The marginal revenue product function will have the form $MRP_x = 200 - 16x$. Setting marginal revenue product equal to factor price gives us the equation

$$200 - 16x = 4.$$

Solving this equation gives us $x^* = 12.25$.

If the industry were competitive, then the industry would employ the factor up to the point where the value of the marginal product was equal to 4. This gives us the equation

$$p2 = 4,$$

so $p^* = 2$. How much output would be demanded at this price? We plug this into the demand function to get the equation $2 = 100 - 2y$, which implies $y^* = 49$. Since the production function is $y = 2x$, we can solve for $x^* = y^*/2 = 24.5$.

27.1 (0) Gargantuan Enterprises has a monopoly in the production of antimacassars. Its factory is located in the town of Pantagruel. There is no other industry in Pantagruel, and the labor supply equation there is $W = 10 + .1L$, where W is the daily wage and L is the number of person-days of work performed. Antimacassars are produced with a production function, $Q = 10L$, where L is daily labor supply and Q is daily output. The demand curve for antimacassars is $P = 41 - \frac{Q}{1{,}000}$, where P is the price and Q is the number of sales per day.

(a) Find the profit-maximizing output for Gargantuan. (Hint: Use the production function to find the labor input requirements for any level of output. Make substitutions so you can write the firm's total costs as a function of its output and then its profit as a function of output. Solve for the profit–maximizing output.) 10,000.

(b) How much labor does it use? 1,000. What is the wage rate that it pays? $110.

(c) What is the price of antimacassars? $31. How much profit is made? $200,000.

27.2 (0) The residents of Seltzer Springs, Michigan, consume bottles of mineral water according to the demand function $D(p) = 1,000 - p$. Here $D(p)$ is the demand per year for bottles of mineral water if the price per bottle is p.

The sole distributor of mineral water in Seltzer Springs, Bubble Up, purchases mineral water at c per bottle from their supplier Perry Air. Perry Air is the only supplier of mineral water in the area and behaves as a profit-maximizing monopolist. For simplicity we suppose that it has zero costs of production.

(a) What is the equilibrium price charged by the distributor Bubble Up?

$p^* = \frac{1{,}000+c}{2}$.

(b) What is the equilibrium quantity sold by Bubble Up? $D(p^*) = \frac{1{,}000-c}{2}$.

(c) What is the equilibrium price charged by the producer Perry Air? $c^* = 500$.

(d) What is the equilibrium quantity sold by Perry Air? $D(c^*) = 250$.

(e) What are the profits of Bubble Up? $\pi_b = (500-250)(750-500) = 250^2$.

(f) What are the profits of Perry Air? $\pi_p = 500 \cdot 250$.

(g) How much consumer's surplus is generated in this market? $CS_e = 250^2/2$.

(h) Suppose that this situation is expected to persist forever and that the interest rate is expected to be constant at 10% per year. What is the minimum lump sum payment that Perry Air would need to pay to Bubble Up to buy it out? 10×250^2.

(i) Suppose that Perry Air does this. What will be the new price and quantity for mineral water? $p^* = 500$ **and** $D(p^*) = 500$.

(j) What are the profits of the new merged firm? $\pi_p = 500^2$.

(k) What is the total amount of consumers' surplus generated? How does this compare with the previous level of consumers' surplus? $CS_i = 500^2/2 > CS_e$.

Calculus **27.3 (0)** Upper Peninsula Underground Recordings (UPUR) has a monopoly on the recordings of the famous rock group Moosecake. Moosecake's music is only provided on digital tape, and blank digital tapes cost them c per tape. There are no other manufacturing or distribution costs. Let $p(x)$ be the inverse demand function for Moosecake's music as a function of x, the number of tapes sold.

(a) What is the first-order condition for profit maximization? For future reference, let x^* be the profit-maximizing amount produced and p^* be the price at which it sells. (In this part, assume that tapes cannot be copied.)

$$p(x^*) + p'(x^*)x^* = c.$$

Now a new kind of consumer digital tape recorder becomes widely available that allows the user to make 1 and only 1 copy of a prerecorded digital tape. The copies are a perfect substitute in consumption value for the original prerecorded tape, and there are no barriers to their use or sale. However, everyone can see the difference between the copies and the orginals and recognizes that the copies cannot be used to make further copies. Blank tapes cost the consumers c per tape, the same price the monopolist pays.

(b) All Moosecake fans take advantage of the opportunity to make a single copy of the tape and sell it on the secondary market. How is the price of an original tape related to the price of a copy? Derive the inverse demand curve for original tapes facing UPUR. (Hint: There are two sources of demand for a new tape: the pleasure of listening to it, and the profits from selling a copy.) If UPUR produces x tapes, $2x$ tapes reach the market, so UPUR can sell a single tape for $p(2x) + [p(2x) - c]$. The first term is the willingness-to-pay for listening; the second term is profit from selling a copy.

(c) Write an expression for UPUR's profits if it produces x tapes.

$$[p(2x) + p(2x) - c]x - cx = 2p(2x)x - 2cx.$$

(d) Let x^{**} be the profit-maximizing level of production by UPUR. How does it compare to the former profit-maximizing level of production?

From the two profit functions, one sees that $2x^{**} = x^*$, so $x^{**} = x^*/2$.

(e) How does the price of a *copy* of a Moosecake tape compare to the price determined in Part (a)? The prices are the same.

(f) If p^{**} is the price of a copy of a Moosecake tape, how much will a *new* Moosecake tape sell for? $2p^{**} - c$.

Chapter 28

NAME_____

Oligopoly

Introduction. In this chapter you will solve problems for firm and industry outcomes when the firms engage in Cournot competition, Stackelberg competition, and other sorts of oligopoly behavior. In Cournot competition, each firm chooses its own output to maximize its profits given the output that it expects the other firm to produce. The industry price depends on the industry output, say, $q_A + q_B$, where A and B are the firms. To maximize profits, firm A sets its marginal revenue (which depends on the output of firm A and the expected output of firm B since the expected industry price depends on the sum of these outputs) equal to its marginal cost. Solving this equation for firm A's output as a function of firm B's expected output gives you one reaction function; analogous steps give you firm B's reaction function. Solve these two equations simultaneously to get the Cournot equilibrium outputs of the two firms.

Example: In Heifer's Breath, Wisconsin, there are two bakers, Anderson and Carlson. Anderson's bread tastes just like Carlson's—nobody can tell the difference. Anderson has constant marginal costs of \$1 per loaf of bread. Carlson has constant marginal costs of \$2 per loaf. Fixed costs are zero for both of them. The inverse demand function for bread in Heifer's Breath is $p(q) = 6 - .01q$, where q is the total number of loaves sold per day.

Let us find Anderson's Cournot reaction function. If Carlson bakes q_C loaves, then if Anderson bakes q_A loaves, total output will be $q_A + q_C$ and price will be $6 - .01(q_A + q_C)$. For Anderson, the total cost of producing q_A units of bread is just q_A, so his profits are

$$\begin{aligned} pq_A - q_A &= (6 - .01q_A - .01q_C)q_A - q_A \\ &= 6q_A - .01q_A^2 - .01q_Cq_A - q_A. \end{aligned}$$

Therefore if Carlson is going to bake q_C units, then Anderson will choose q_A to maximize $6q_A - .01q_A^2 - .01q_Cq_A - q_A$. This expression is maximized when $6 - .02q_A - .01q_C = 1$. (You can find this out either by setting A's marginal revenue equal to his marginal cost or directly by setting the derivative of profits with respect to q_A equal to zero.) Anderson's reaction function, $R_A(q_C)$ tells us Anderson's best output if he knows that Carlson is going to bake q_C. We solve from the previous equation to find $R_A(q_C) = (5 - .01q_C)/.02 = 250 - .5q_C$.

We can find Carlson's reaction function in the same way. If Carlson knows that Anderson is going to produce q_A units, then Carlson's profits will be $p(q_A + q_C) - 2q_C = (6 - .01q_A - .01q_C)q_C - 2q_C = 6q_C - .01q_Aq_C - .01q_C^2 - 2q_C$. Carlson's profits will be maximized if he chooses q_C to satisfy the equation $6 - .01q_A - .02q_C = 2$. Therefore Carlson's reaction function is $R_C(q_A) = (4 - .01q_A)/.02 = 200 - .5q_A$.

Let us denote the Cournot equilibrium quantities by $\bar{q}_A$ and $\bar{q}_C$. The Cournot equilibrium conditions are that $\bar{q}_A = R_A(\bar{q}_C)$ and $\bar{q}_C = R_C(\bar{q}_A)$. Solving these two equations in two unknowns we find that $\bar{q}_A = 200$ and $\bar{q}_C = 100$. Now we can also solve for the Cournot equilibrium price and for the profits of each baker. The Cournot equilibrium price is $6 - .01(200 + 100) = \$3$. Then in Cournot equilibrium, Anderson makes a profit of \$2 on each of 200 loaves and Carlson makes \$1 on each of 100 loaves.

In Stackelberg competition, the follower's profit-maximizing output choice depends on the amount of output that he expects the leader to produce. His reaction function, $R_F(q_L)$, is constructed in the same way as for a Cournot competitor. The leader knows the reaction function of the follower and gets to choose her own output, q_L, first. So the leader knows that the industry price depends on the *sum* of her own output and the follower's output, that is, on $q_L + R_F(q_L)$. Since the industry price can be expressed as a function of q_L only, so can the leader's marginal revenue. So once you get the follower's reaction function and substitute it into the inverse demand function, you can write down an expression that depends on just q_L and that says marginal revenue equals marginal cost for the leader. You can solve this expression for the leader's Stackelberg output and plug in to the follower's reaction function to get the follower's Stackelberg output.

Example: Suppose that one of the bakers of Heifer's Breath plays the role of Stackelberg leader. Perhaps this is because Carlson always gets up an hour earlier than Anderson and has his bread in the oven before Anderson gets started. If Anderson always finds out how much bread Carlson has in his oven and if Carlson knows that Anderson knows this, then Carlson can act like a Stackelberg leader. Carlson knows that Anderson's reaction function is $R_A(q_C) = 250 - .5q_c$. Therefore Carlson knows that if he bakes q_C loaves of bread, then the total amount of bread that will be baked in Heifer's Breath will be $q_C + R_A(q_C) = q_C + 250 - .5q_C = 250 + .5q_C$. Since Carlson's production decision determines total production and hence the price of bread, we can write Carlson's profit simply as a function of his own output. Carlson will choose the quantity that maximizes this profit. If Carlson bakes q_C loaves, the price will be $p = 6 - .01(250 + .5q_C) = 3.5 - .005q_C$. Then Carlson's profits will be $pq_C - 2q_C = (3.5 - .005q_C)q_C - 2q_C = 1.5q_C - .005q_C^2$. His profits are maximized when $q_C = 150$. (Find this either by setting marginal revenue equal to marginal cost or directly by setting the derivative of profits to zero and solving for q_C.) If Carlson produces 150 loaves, then Anderson will produce $250 - .5 \times 150 = 175$ loaves. The price of bread will be $6 - .01(175 + 150) = 2.75$. Carlson will now make \$.75 per loaf on each of 150 loaves and Anderson will make \$1.75 on each of 175 loaves.

28.1 (0) Carl and Simon are two rival pumpkin growers who sell their pumpkins at the Farmers' Market in Lake Witchisit, Minnesota. They are the only sellers of pumpkins at the market, where the demand function for pumpkins is $q = 3,200 - 1,600p$. The total number of pumpkins sold at the market is $q = q_C + q_S$, where q_C is the number that Carl sells

and q_S is the number that Simon sells. The cost of producing pumpkins for either farmer is \$.50 per pumpkin no matter how many pumpkins he produces.

(a) The inverse demand function for pumpkins at the Farmers' Market is $p = a - b(q_C + q_S)$, where $a =$ 2 and $b =$ $1/1,600$. The marginal cost of producing a pumpkin for either farmer is \$.50.

(b) Every spring, each of the farmers decides how many pumpkins to grow. They both know the local demand function and they each know how many pumpkins were sold by the other farmer last year. In fact, each farmer assumes that the other farmer will sell the same number this year as he sold last year. So, for example, if Simon sold 400 pumpkins last year, Carl believes that Simon will sell 400 pumpkins again this year. If Simon sold 400 pumpkins last year, what does Carl think the price of pumpkins will be if Carl sells 1,200 pumpkins this year? 1. If Simon sold q_S^{t-1} pumpkins in year $t-1$, then in the spring of year t, Carl thinks that if he, Carl, sells q_C^t pumpkins this year, the price of pumpkins this year will be $2 - (q_S^{t-1} + q_C^t)/1,600$.

(c) If Simon sold 400 pumpkins last year, Carl believes that if he sells q_C^t pumpkins this year then the inverse demand function that he faces is $p = 2 - 400/1,600 - q_C^t/1,600 = 1.75 - q_C^t/1,600$. Therefore if Simon sold 400 pumpkins last year, Carl's marginal revenue this year will be $1.75 - q_C^t/800$. More generally, if Simon sold q_S^{t-1} pumpkins last year, then Carl believes that if he, himself, sells q_C^t pumpkins this year, his marginal revenue this year will be $2 - q_S^{t-1}/1,600 - q_C^t/800$.

(d) Carl believes that Simon will never change the amount of pumpkins that he produces from the amount q_S^{t-1} that he sold last year. Therefore Carl plants enough pumpkins this year so that he can sell the amount that maximizes his profits this year. To maximize this profit, he chooses the output this year that sets his marginal revenue this year equal to his marginal cost. This means that to find Carl's output this year when Simon's output last year was q_S^{t-1}, Carl solves the following equation.

$$2 - q_S^{t-1}/1,600 - q_C^t/800 = .5.$$

(e) Carl's Cournot reaction function, $R_C^t(q_S^{t-1})$, is a function that tells us what Carl's profit-maximizing output this year would be as a function of Simon's output last year. Use the equation you wrote in the last answer to find Carl's reaction function, $R_C^t(q_S^{t-1}) =$ $1,200 - q_S^{t-1}/2$. (Hint: This is a linear expression of the form $a - bq_S^{t-1}$. You have to find the constants a and b.)

(f) Suppose that Simon makes his decisions in the same way that Carl does. Notice that the problem is completely symmetric in the roles played by Carl and Simon. Therefore without even calculating it, we can guess that Simon's reaction function is $R_S^t(q_C^{t-1}) =$ $1,200 - q_C^{t-1}/2$. (Of course, if you don't like to guess, you could work this out by following similar steps to the ones you used to find Carl's reaction function.)

(g) Suppose that in year 1, Carl produced 200 pumpkins and Simon produced 1,000 pumpkins. In year 2, how many would Carl produce? 700. How many would Simon produce? 1,100. In year 3, how many would Carl produce? 650. How many would Simon produce? 850. Use a calculator or pen and paper to work out several more terms in this series. To what level of output does Carl's output appear to be converging? 800 How about Simon's? 800.

(h) Write down two simultaneous equations that could be solved to find outputs q_S and q_C such that, if Carl is producing q_C and Simon is producing q_S, then they will both want to produce the same amount in the next period. (Hint: Use the reaction functions.) $q_s = 1,200 - q_C/2$ and $q_C = 1,200 - q_S/2$.

(i) Solve the two equations you wrote down in the last part for an equilibrium output for each farmer. Each farmer, in Cournot equilibrium, produces 800 units of output. The total amount of pumpkins brought to the Farmers' Market in Lake Witchisit is 1,600. The price of pumpkins in that market is \$1. How much profit does each farmer make? \$400.

28.2 (0) Suppose that the pumpkin market in Lake Witchisit is as we described it in the last problem except for one detail. Every spring, the snow thaws off of Carl's pumpkin field a week before it thaws off of Simon's. Therefore Carl can plant his pumpkins one week earlier than Simon can. Now Simon lives just down the road from Carl, and he can tell by looking at Carl's fields how many pumpkins Carl planted and how many Carl will harvest in the fall. (Suppose also that Carl will sell every pumpkin that he produces.) Therefore instead of assuming that Carl will sell the same amount of pumpkins that he did last year, Simon sees how many Carl is actually going to sell this year. Simon has this information before he makes his own decision about how many to plant.

(a) If Carl plants enough pumpkins to yield q_C^t this year, then Simon knows that the profit-maximizing amount to produce this year is $q_S^t =$

Hint: Remember the reaction functions you found in the last problem.

$1,200 - q_C^t/2$.

(b) When Carl plants his pumpkins, he understands how Simon will make his decision. Therefore Carl knows that the amount that Simon will produce this year will be determined by the amount that Carl produces. In particular, if Carl's output is q_C^t, then Simon will produce and sell $1,200 - q_C^t/2$ and the total output of the two producers will be $1,200 + q_C^t/2$. Therefore Carl knows that if his own output is q_C, the price of pumpkins in the market will be $1.25 - q_C^t/3,200$.

(c) In the last part of the problem, you found how the price of pumpkins this year in the Farmers' Market is related to the number of pumpkins that Carl produces this year. Now write an expression for Carl's total revenue in year t as a function of his own output, q_C^t. $1.25q_C^t - (q_C^t)^2/3,200$. Write an expression for Carl's marginal revenue in year t as a function of q_C^t. $1.25 - q_C^t/1,600$.

(d) Find the profit-maximizing output for Carl. 1,200. Find the profit-maximizing output for Simon. 600. Find the equilibrium price of pumpkins in the Lake Witchisit Farmers' Market. $7/8. How much profit does Carl make? $450. How much profit does Simon make? $225. An equilibrium of the type we discuss here is known as a Stackleberg equilibrium.

(e) If he wanted to, it would be possible for Carl to delay his planting until the same time that Simon planted so that neither of them would know the other's plans for this year when he planted. Would it be in Carl's interest to do this? Explain. (Hint: What are Carl's profits in the equilibrium above? How do they compare with his profits in Cournot equilibrium?) No. Carl's profits in Stackleberg equilibrium are larger than in Cournot

equilibrium, so Carl will want Simon to know his output.

28.3 (0) Suppose that Carl and Simon sign a marketing agreement. They decide to determine their total output jointly and to each produce the same number of pumpkins. To maximize their joint profits, how many pumpkins should they produce in toto? 1,200. How much does each one of them produce? 600. How much profit does each one of them make? 450.

28.4 (0) The inverse market demand curve for bean sprouts is given by $P(Y) = 100 - 2Y$, and the total cost function for any firm in the industry is given by $TC(y) = 4y$.

(a) The marginal cost for any firm in the industry is equal to $4. The change in price for a one-unit increase in output is equal to $−2.

(b) If the bean-sprout industry were perfectly competitive, the industry output would be 48, and the industry price would be $4.

(c) Suppose that two Cournot firms operated in the market. The reaction function for Firm 1 would be $y_1 = 24 - y_2/2$. (Reminder: Unlike the example in your textbook, the marginal cost is not zero here.) The reaction function of Firm 2 would be $y_2 = 24 - y_1/2$. If the firms were operating at the Cournot equilibrium point, industry output would be 32, each firm would produce 16, and the market price would be $36.

(d) For the Cournot case, draw the two reaction curves and indicate the equilibrium point on the graph below.

(e) If the two firms decided to collude, industry output would be 24 and the market price would equal $52.

(f) Suppose both of the colluding firms are producing equal amounts of output. If one of the colluding firms assumes that the other firm would not react to a change in industry output, what would happen to a firm's own profits if it increased its output by one unit? Profits would increase by $22.

(g) Suppose one firm acts as a Stackleberg leader and the other firm behaves as a follower. The maximization problem for the leader can be written as $\max_{y_1}[100 - 2(y_1 + 24 - y_1/2)]y_1 - 4y_1$.

Solving this problem results in the leader producing an output of 24 and the follower producing 12. This implies an industry output of 36 and price of $28.

28.5 (0) Grinch is the sole owner of a mineral water spring that costlessly burbles forth as much mineral water as Grinch cares to bottle. It costs Grinch $2 per gallon to bottle this water. The inverse demand curve for Grinch's mineral water is $p = \$20 - .20q$, where p is the price per gallon and q is the number of gallons sold.

(a) Write down an expression for profits as a function of q: $\Pi(q) =$ $(20 - .20q)q - 2q$. Find the profit-maximizing choice of q for Grinch. 45.

(b) What price does Grinch get per gallon of mineral water if he produces the profit-maximizing quantity? \$11. How much profit does he make? \$405.

(c) Suppose, now, that Grinch's neighbor, Grubb finds a mineral spring that produces mineral water that is just as good as Grinch's water, but that it costs Grubb \$6 a bottle to get his water out of the ground and bottle it. Total market demand for mineral water remains as before. Suppose that Grinch and Grubb each believe that the other's quantity decision is independent of his own. What is the Cournot equilibrium output for Grubb? $50/3$. What is the price in the Cournot equilibrium? \$9.33.

28.6 (1) Albatross Airlines has a monopoly on air travel between Peoria and Dubuque. If Albatross makes one trip in each direction per day, the demand schedule for round trips is $q = 160 - 2p$, where q is the number of passengers per day. (Assume that nobody makes one-way trips.) There is an "overhead" fixed cost of \$2,000 per day that is necessary to fly the airplane regardless of the number of passengers. In addition, there is a marginal cost of \$10 per passenger. Thus, total daily costs are $\$2,000 + 10q$ if the plane flies at all.

(a) On the graph below, sketch and label the marginal revenue curve, and the average and marginal cost curves.

(b) Calculate the profit-maximizing price and quantity and total daily profits for Albatross Airlines. $p =$ 45 , $q =$ 70 , $\pi =$ $450 per day.

(c) If the interest rate is 10% per year, how much would someone be willing to pay to own Albatross Airlines's monopoly on the Dubuque-Peoria route. (Assuming that demand and cost conditions remain unchanged forever.) About $1.6 million.

(d) If another firm with the same costs as Albatross Airlines were to enter the Dubuque-Peoria market and if the industry then became a Cournot duopoly, would the new entrant make a profit? No; losses would be about $900 per day.

(e) Suppose that the throbbing night life in Peoria and Dubuque becomes widely known and in consequence the population of both places doubles. As a result, the demand for airplane trips between the two places doubles to become $q = 320 - 4p$. Suppose that the original airplane had a capacity of 80 passengers. If AA must stick with this single plane and if no other airline enters the market, what price should it charge to maximize its output and how much profit would it make? $p =$ $60 , $\pi =$ $2,000.

(f) Let us assume that the overhead costs per plane are constant regardless of the number of planes. If AA added a second plane with the same costs and capacity as the first plane, what price would it charge? $45. How many tickets would it sell? 140. How much would its profits be? $900. If AA could prevent entry by another competitor, would it choose to add a second plane? No.

(g) Suppose that AA stuck with one plane and another firm entered the market with a plane of its own. If the second firm has the same cost function as the first and if the two firms act as Cournot oligopolists, what will be the price, $40 , quantities, 80 , and profits? $400.

28.7 (0) Alex and Anna are the only sellers of kangaroos in Sydney, Australia. Anna chooses her profit-maximizing number of kangaroos to sell, q_1, based on the number of kangaroos that she expects Alex to sell. Alex knows how Anna will react and chooses the number of kangaroos that

she herself will sell, q_2, after taking this information into account. The inverse demand function for kangaroos is $P(q_1 + q_2) = 2,000 - 2(q_1 + q_2)$. It costs \$400 to raise a kangaroo to sell.

(a) Alex and Anna are Stackelberg competitors. Alex is the leader and Anna is the follower.

(b) If Anna expects Alex to sell q_2 kangaroos, what will her own marginal revenue be if she herself sells q_1 kangaroos? $MR(q_1 + q_2) = 2,000 - 4q_1 - 2q_2$.

(c) What is Anna's reaction function, $R(q_2)$? $R(q_2) = 400 - 1/2q_2$.

(d) Now if Alex sells q_2 kangaroos, what is the total number of kangaroos that will be sold? $400 + 1/2q_2$. What will be the market price as a function of q_2 only? $P(q_2) = 1,200 - q_2$.

(e) What is Alex's marginal revenue as a function of q_2 only? $MR(q_2) = 1,200 - 2q_2$. How many kangaroos will Alex sell? 400. How many kangaroos will Anna sell? 200. What will the industry price be? \$800.

28.8 (0) Consider an industry with the following structure. There are 50 firms that behave in a competitive manner and have identical cost functions given by $c(y) = y^2/2$. There is one monopolist that has 0 marginal costs. The demand curve for the product is given by

$$D(p) = 1,000 - 50p.$$

(a) What is the supply curve of one of the competitive firms? $y = p$. The total supply from the competitive sector at price p is $S(p) =$ $50p$.

(b) If the monopolist sets a price p, the amount that it can sell is $D_m(p) =$ $1,000 - 100p$.

(c) The monopolist's profit-maximizing output is $y_m =$ 500. What is the monopolist's profit-maximizing price? $p = 5$.

(d) How much output will the competitive sector provide at this price? $50 \times 5 = 250$. What will be the total amount of output sold in this industry? $y_m + y_c = 750$.

28.9 (0) Consider a market with one large firm and many small firms. The supply curve of the small firms taken together is

$$S(p) = 100 + p.$$

The demand curve for the product is

$$D(p) = 200 - p.$$

The cost function for the one large firm is

$$c(y) = 25y.$$

(a) Suppose that the large firm is forced to operate at a zero level of output. What will be the equilibrium price? 50. What will be the equilibrium quantity? 150.

(b) Suppose now that the large firm attempts to exploit its market power and set a profit-maximizing price. In order to model this we assume that customers always go first to the competitive firms and buy as much as they are able to and then go to the large firm. In this situation, the equilibrium price will be \$37.50. The quantity supplied by the large firm will be 25. and the equilibrium quantity supplied by the competitive firms will be 137.5.

(c) What will be the large firm's profits? \$312.50.

(d) Finally suppose that the large firm could force the competitive firms out of the business and behave as a real monopolist. What will be the equilibrium price? $225/2$. What will be the equilibrium quantity? $175/2$. What will be the large firm's profits? $(175/2)^2$.

Calculus **28.10 (2)** In a remote area of the American Midwest before the railroads arrived, cast iron cookstoves were much desired, but people lived far apart, roads were poor, and heavy stoves were expensive to transport. Stoves could be shipped by river boat to the town of Bouncing Springs, Missouri. Ben Kinmore was the only stove dealer in Bouncing Springs. He could buy as many stoves as he wished for \$20 each, delivered to his store. Ben's only customers were farmers who lived along a road that ran east and west through town. There were no other stove dealers along the road in either direction. No farmers lived in Bouncing Springs, but along the road, in either direction, there was one farm every mile. The cost of hauling a stove was \$1 per mile. The owners of every farm had a reservation price of \$120 for a cast iron cookstove. That is, any of them would be willing to pay up to \$120 to have a stove rather than to not have one. Nobody had use for more than one stove. Ben Kinmore charged a base price of \$$p$ for stoves and added to the price the cost of delivery. For example, if the base price of stoves was \$40 and you lived 45 miles west of Bouncing Springs, you would have to pay \$85 to get a stove, \$40 base price plus a hauling charge of \$45. Since the reservation price of every farmer was \$120, it follows that if the base price were \$40, any farmer who lived within 80 miles of Bouncing Springs would be willing to pay \$40 plus the price of delivery to have a cookstove. Therefore at a base price of \$40, Ben could sell 80 cookstoves to the farmers living west of him. Similarly, if his base price is \$40, he could sell 80 cookstoves to the farmers living within 80 miles to his east, for a total of 160 cookstoves.

(a) If Ben set a base price of \$$p$ for cookstoves where $p < 120$, and if he charged \$1 a mile for delivering them, what would be the total number of cookstoves he could sell? $2(120-p)$. (Remember to count the ones he could sell to his east as well as to his west.) Assume that Ben has no other costs than buying the stoves and delivering them. Then Ben would make a profit of $p - 20$ per stove. Write Ben's total profit as a function of the base price, \$$p$, that he charges: $2(120-p)(p-20) = 2(140p - p^2 - 2,400)$.

(b) Ben's profit-maximizing base price is \$70. (Hint: You just wrote profits as a function of prices. Now differentiate this expression for profits with respect to p.) Ben's most distant customer would be located at a distance of 50 miles from him. Ben would sell 100 cookstoves and make a total profit of \$5,000.

(c) Suppose that instead of setting a single base price and making all buyers pay for the cost of transportation, Ben offers free delivery of cookstoves. He sets a price \$$p$ and promises to deliver for free to any farmer who lives within $p-20$ miles of him. (He won't deliver to anyone who lives

further than that, because it then costs him more than $\$p$ to buy a stove and deliver it.) If he is going to price in this way, how high should he set p? $120. How many cookstoves would Ben deliver? 200. How much would his total revenue be? $24,000 How much would his total costs be, including the cost of deliveries and the cost of buying the stoves? $14,100. (Hint: For any n, the sum of the series $1 + 2 + \ldots + n$ is equal to $n(n+1)/2$.) How much profit would he make? $10,100. Can you explain why it is more profitable for Ben to use this pricing scheme where he pays the cost of delivery himself rather than the scheme where the farmers pay for their own deliveries? If Ben pays for delivery, he can price-discriminate between nearby farmers and faraway ones. He charges a higher price, net of transport cost, to nearby farmers and a lower net price to faraway farmers, who are willing to pay less net of transport cost.

Calculus **28.11 (2)** Perhaps you wondered what Ben Kinmore, who lives off in the woods quietly collecting his monopoly profits, is doing in this chapter on oligopoly. Well, unfortunately for Ben, before he got around to selling any stoves, the railroad built a track to the town of Deep Furrow, just 40 miles down the road, west of Bouncing Springs. The storekeeper in Deep Furrow, Huey Sunshine, was also able to get cookstoves delivered by train to his store for $20 each. Huey and Ben were the only stove dealers on the road. Let us concentrate our attention on how they would compete for the customers who lived between them. We can do this, because Ben can charge different base prices for the cookstoves he ships east and the cookstoves he ships west. So can Huey.

Suppose that Ben sets a base price, p_B, for stoves he sends west and adds a charge of $1 per mile for delivery. Suppose that Huey sets a base price, p_H, for stoves he sends east and adds a charge of $1 per mile for delivery. Farmers who live between Ben and Huey would buy from the seller who is willing to deliver most cheaply to them (so long as the delivered price does not exceed $120). If Ben's base price is p_B and Huey's base price is p_H, somebody who lives x miles west of Ben would have to pay a total of $p_B + x$ to have a stove delivered from Ben and $p_H + (40 - x)$ to have a stove delivered by Huey.

(a) If Ben's base price is p_B and Huey's is p_H, write down an equation that could be solved for the distance x^* to the west of Bouncing Springs that

Ben's market extends. $p_B + x^* = p_H + (40 - x^*)$. If Ben's base price is p_B and Huey's is p_H, then Ben will sell $20 + (p_H - p_B)/2$ cookstoves and Huey will sell $20 + (p_B - p_H)/2$ cookstoves.

(b) Recalling that Ben makes a profit of $p_B - 20$ on every cookstove that he sells, Ben's profits can be expressed as the following function of p_B and p_H. $(20 + (p_H - p_B)/2)(p_B - 20)$.

(c) If Ben thinks that Huey's price will stay at p_H, no matter what price Ben chooses, what choice of p_B will maximize Ben's profits? $p_B = 30 + p_H/2$. (Hint: Set the derivative of Ben's profits with respect to his price equal to zero.) Suppose that Huey thinks that Ben's price will stay at p_B, no matter what price Huey chooses, what choice of p_H will maximize Huey's profits? $p_H = 30 + p_B/2$. (Hint: Use the symmetry of the problem and the answer to the last question.)

(d) Can you find a base price for Ben and a base price for Huey such that each is a profit-maximizing choice given what the other guy is doing? (Hint: Find prices p_B and p_H that simultaneously solve the last two equations.) $p_B = p_H = 60$. How many cookstoves does Ben sell to farmers living west of him? 20. How much profit does he make on these sales? \$800.

(e) Suppose that Ben and Huey decided to compete for the customers who live between them by price discriminating. Suppose that Ben offers to deliver a stove to a farmer who lives x miles west of him for a price equal to the maximum of Ben's total cost of delivering a stove to that farmer and Huey's total cost of delivering to the same farmer less 1 penny. Suppose that Huey offers to deliver a stove to a farmer who lives x miles west of Ben for a price equal to the maximum of Huey's own total cost of delivering to this farmer and Ben's total cost of delivering to him less a penny. For example, if a farmer lives 10 miles west of Ben, Ben's total cost of delivering to him is \$30, \$20 to get the stove and \$10 for hauling it 10 miles west. Huey's total cost of delivering it to him is \$50, \$20 to get the stove and \$30 to haul it 30 miles east. Ben will charge the maximum of his own cost, which is \$30, and Huey's cost less a penny, which is \$49.99. The maximum of these two numbers is \$49.99. Huey will charge the maximum of his own total cost of delivering to this farmer, which is \$50, and Ben's cost less a penny, which is \$29.99. Therefore Huey will charge \$50.00 to deliver to this farmer. This farmer will buy from Ben

whose price to him is cheaper by one penny. When the two merchants have this pricing policy, all farmers who live within 20 miles of Ben will buy from Ben and all farmers who live within 20 miles of Huey will buy from Huey. A farmer who lives x miles west of Ben and buys from Ben must pay $59.99 - x$ dollars to have a cookstove delivered to him. A farmer who lives x miles east of Huey and buys from Huey must pay $59.99 - x$ for delivery of a stove. On the graph below, use blue ink to graph the cost to Ben of delivering to a farmer who lives x miles west of him. Use red ink to graph the total cost to Huey of delivering a cookstove to a farmer who lives x miles west of Ben. Use pencil to mark the lowest price available to a farmer as a function of how far west he lives from Ben.

(f) With the pricing policies you just graphed, which farmers get stoves delivered most cheaply, those who live closest to the merchants or those who live midway between them? Those who live midway between them. On the graph you made, shade in the area representing each merchant's profits. How much profits does each merchant make? $400. If Ben and Huey are pricing in this way, is there any way for either of them to increase his profits by changing the price he charges to some farmers? No.

Chapter 29

NAME_____

Game Theory

Introduction. In this introduction we offer two examples of two-person games. The first game has a dominant strategy equilibrium. The second game is a zero-sum game that has a Nash equilibrium in pure strategies that is not a dominant strategy equilibrium.

Example: Albert and Victoria are roommates. Each of them prefers a clean room to a dirty room, but neither likes housecleaning. If both clean the room, they each get a payoff of 5. If one cleans and the other doesn't clean, the person who does the cleaning has a utility of 2, and the person who doesn't clean has a utility of 6. If neither cleans, the room stays a mess and each has a utility of 3. The payoffs from the strategies "Clean" and "Don't Clean" are shown in the box below.

Clean Room–Dirty Room

		Victoria	
		Clean	Don't Clean
Albert	Clean	$5, 5$	$2, 6$
	Don't Clean	$6, 2$	$3, 3$

In this game, whether or not Victoria chooses to clean, Albert will get a higher payoff if he doesn't clean than if he does clean. Therefore "Don't Clean" is a dominant strategy for Albert. Similar reasoning shows that no matter what Albert chooses to do, Victoria is better off if she chooses "Don't Clean." Therefore the outcome where both roommates choose "Don't Clean" is a dominant strategy equilibrium. This is true despite the fact that both persons would be better off if they both chose to clean the room.

Example: This game is set in the South Pacific in 1943. Admiral Imamura must transport Japanese troops from the port of Rabaul in New Britain, across the Bismarck Sea to New Guinea. The Japanese fleet could either travel north of New Britain, where it is likely to be foggy, or south of New Britain, where the weather is likely to be clear. U.S. Admiral Kenney hopes to bomb the troop ships. Kenney has to choose whether to concentrate his reconnaissance aircraft on the Northern or the Southern route. Once he finds the convoy, he can bomb it until its arrival in New Guinea. Kenney's staff has estimated the number of days of bombing

time for each of the outcomes. The payoffs to Kenney and Imamura from each outcome are shown in the box below. The game is modeled as a "zero-sum game:" for each outcome, Imamura's payoff is the negative of Kenney's payoff.

The Battle of the Bismarck Sea

		Imamura	
		North	South
Kenney	North	$2, -2$	$2, -2$
	South	$1, -1$	$3, -3$

This game does not have a dominant strategy equilibrium, since there is no dominant strategy for Kenney. His best choice depends on what Imamura does. The only Nash equilibrium for this game is where Imamura chooses the northern route and Kenney concentrates his search on the northern route. To check this, notice that if Imamura goes North, then Kenney gets an expected two days of bombing if he (Kenney) chooses North and only one day if he (Kenney) chooses South. Furthermore, if Kenney concentrates on the north, Imamura is indifferent between going north or south, since he can be expected to be bombed for two days either way. Therefore if both choose "North," then neither has an incentive to act differently. You can verify that for any other combination of choices, one admiral or the other would want to change. As things actually worked out, Imamura chose the Northern route and Kenney concentrated his search on the North. After about a day's search the Americans found the Japanese fleet and inflicted heavy damage on it.*

29.1 (0) This problem is designed to give you practice in reading a game matrix and to check that you understand the definition of a dominant strategy. Consider the following game matrix.

A Game Matrix

		Player B	
		Left	Right
Player A	Top	a, b	c, d
	Bottom	e, f	g, h

* This example is discussed in R. Duncan Luce and Howard Raiffa's *Games and Decisions*, John Wiley, 1957, or Dover, 1989. We recommend this book to anyone interested in reading more about game theory.

(a) If (top, left) is a dominant strategy equilibrium, then we know that $a >$ e , $b >$ d , c $> g$, and f $> h$.

(b) If (top, left) is a Nash equilibrium, then which of the above inequalities must be satisfied? $a > e$; $b > d$.

(c) If (top, left) is a dominant strategy equilibrium must it be a Nash equilibrium? Why? Yes. A dominant strategy equilibrium is always a Nash equilibrium.

29.2 (0) In order to learn how people actually play in game situations, economists and other social scientists frequently conduct experiments in which subjects play games for money. One such game is known as the *voluntary public goods game*. This game is chosen to represent situations in which individuals can take actions that are costly to themselves but that are beneficial to an entire community.

In this problem we will deal with a two-player version of the voluntary public goods game. Two players are put in separate rooms. Each player is given \$10. The player can use this money in either of two ways. He can *keep* it or he can *contribute* it to a "public fund." Money that goes into the public fund gets multiplied by 1.6 and then divided equally between the two players. If both contribute their \$10, then each gets back $\$20 \times 1.6/2 = \$16$. If one contributes and the other does not, each gets back $\$10 \times 1.6/2 = \$8$ from the public fund so that the contributor has \$8 at the end of the game and the non-contributor has \$18–his original \$10 plus \$8 back from the public fund. If neither contributes, both have their original \$10. The payoff matrix for this game is:

Voluntary Public Goods Game

		Player B	
		Contribute	Keep
Player A	Contribute	\$16, \$16	\$8, \$18
	Keep	\$18, \$8	\$10, \$10

(a) If the other player keeps, what is your payoff if you keep? \$10.

If the other player keeps, what is your payoff if you contribute? \$8.

(b) If the other player contributes, what is your payoff if you keep? $18. If the other player contributes, what is your payoff if you contribute? $16.

(c) Does this game have a dominant strategy equilibrium? Yes. If so, what is it? Both keep.

29.3 (1) Let us consider a more general version of the voluntary public goods game described in the previous question. This game has N players, each of whom can contribute either \$10 or nothing to the public fund. All money that is contributed to the public fund gets multiplied by some number $B > 1$ and then divided equally among all players in the game (including those who do not contribute.) Thus if all N players contribute \$10 to the fund, the amount of money available to be divided among the N players will be $\$10BN$ and each player will get $\$10BN/N = \$10B$ back from the public fund.

(a) If $B > 1$, which of the following outcomes gives the higher payoff to each player? *a)* All players contribute their \$10 or *b)* all players keep their \$10. All players contribute their $10.

(b) Suppose that exactly K of the other players contribute. If you keep your \$10, you will have this \$10 plus your share of the public fund contributed by others. What will your payoff be in this case? $\$10 + \$10BK/N$. If you contribute your \$10, what will be the total number of contributors? K+1. What will be your payoff? $\$10B(K+1)/N$.

(c) If $B = 3$ and $N = 5$, what is the dominant strategy equilibrium for this game? All keep their $10. Explain your answer. If K other players contribute, a player's payoff will be $\$10 + 30K/5$ if he keeps the $10 and $\$30(K+1)/5$ if he contributes it. Where $B = 3$ and $N = 5$, the difference between these two payoffs is $\$10 - \$6 = \$4 > 0$, so we see that for all possible K, the payoff from keeping the money is always higher.

(d) In general, what relationship between B and N must hold for "Keep" to be a dominant strategy? We ask when the payoff from keeping is larger than the payoff from contributing. Using the formulas above, we want to know when $10(1+BK/N) > 10B(K+1)/N$, which reduces to $B/N < 1$.

(e) Sometimes the action that maximizes a player's *absolute* payoff, does not maximize his *relative* payoff. Consider the example of a voluntary public goods game as described above, where $B = 6$ and $N = 5$. Suppose that four of the five players in the group contribute their \$10, while the fifth player keeps his \$10. What is the payoff of each of the four contributors? $\$60 \times 4/5 = \$48$. What is the payoff of the player who keeps his \$10? $\$10 + \$60 \times 4/ = \$58$. Who has the highest payoff in the group? The player who keeps his \$10. What would be the payoff to the fifth player if instead of keeping his \$10, he contributes, so that all five players contribute. $\$60 \times 5/5 = \$60$. If the other four players contribute, what should the fifth player to maximize his *absolute* payoff? Contribute. What should he do to maximize his payoff *relative* to that of the other players? Keep his \$10.

(f) If $B = 6$ and $N = 5$, what is the dominant strategy equilibrium for this game? All contribute. Explain your answer. Where $B = 6$ and $N = 5$, if K other players contribute, a player's payoff will be $\$10 + 60K/5$ if he keeps the \$10 and $\$60(K+1)/5$ if he contributes it. For every possible K, the difference between these two payoffs is $\$10 - \$12 = -\$2$, so the payoff from contributing the \$10 is

always higher.

29.4 (1) The Stag Hunt game is based on a story told by Jean Jacques Rousseau in his book *Discourses on the Origin and Foundation of Inequality Among Men* (1754). The story goes something like this: "Two hunters set out to kill a stag. One has agreed to drive the stag through the forest, and the other to post at a place where the stag must pass. If both faithfully perform their assigned stag-hunting tasks, they will surely kill the stag and each will get an equal share of this large animal. During the course of the hunt, each hunter has an opportunity to abandon the stag hunt and to pursue a hare. If a hunter pursues the hare instead of the stag he is certain to catch the hare and the stag is certain to escape. Each hunter would rather share half of a stag than have a hare to himself."

The matrix below shows payoffs in a stag hunt game. If both hunters hunt stag, each gets a payoff of 4. If both hunt hare, each gets 3. If one hunts stag and the other hunts hare, the stag hunter gets 0 and the hare hunter gets 3.

The Stag Hunt Game

		Hunter B	
		Hunt Stag	Hunt Hare
Hunter A	Hunt Stag	$4, 4$	$0, 3$
	Hunt Hare	$3, 0$	$3, 3$

(a) If you are sure that the other hunter will hunt stag, what is the best thing for you to do? Hunt stag.

(b) If you are sure that the other hunter will hunt hare, what is the best thing for you to do? Hunt hare.

(c) Does either hunter have a dominant strategy in this game? No. If so, what is it? If not explain why not. The best action to take depends on what the other player does.

(d) This game has two pure strategy Nash equilibria. What are they? Both hunters hunt stag. Both hunters hunt hare.

(e) Is one Nash equilibrium better for both hunters than the other? Yes If so, which is the better equilibrium? Both hunt stag.

(f) If a hunter believes that with probability 1/2 the other hunter will hunt stag and with probability 1/2 he will hunt hare, what should this hunter do to maximize his expected payoff? Hunt hare.

29.5 (1) Evangeline and Gabriel met at a freshman mixer. They want desperately to meet each other again, but they forgot to exchange names or phone numbers when they met the first time. There are two possible strategies available for each of them. These are Go to the Big Party or Stay Home and Study. They will surely meet if they both go to the party, and they will surely not otherwise. The payoff to meeting is 1,000 for each of them. The payoff to not meeting is zero for both of them. The payoffs are described by the matrix below.

Close Encounters of the Second Kind

		Gabriel	
		Go to Party	Stay Home
Evangeline	Go to Party	$1000, 1000$	$0, 0$
	Stay Home	$0, 0$	$0, 0$

(a) A strategy is said to be a weakly dominant strategy for a player if the payoff from using this strategy is *at least as high* as the payoff from using any other strategy. Is there any outcome in this game where both players are using weakly dominant strategies? The only one is (Go to party, Go to party).

(b) Find all of the pure-strategy Nash equilibria for this game. There are two: (Go, Go) and (Stay, Stay).

(c) Do any of the pure Nash equilibria that you found seem more reasonable than others? Why or why not? Although (Stay, Stay) is a Nash equilibrium, it seems a silly one. If either player believes that there is any chance that the other will go to the party, he or she will also go.

(d) Let us change the game a little bit. Evangeline and Gabriel are still desperate to find each other. But now there are two parties that they can go to. There is a little party at which they would be sure to meet if they both went there and a huge party at which they might never see each other. The expected payoff to each of them is 1,000 if they both go to the little party. Since there is only a 50-50 chance that they would find each other at the huge party, the expected payoff to each of them is only 500. If they go to different parties, the payoff to both of them is zero. The payoff matrix for this game is:

More Close Encounters

		Gabriel	
		Little Party	Big Party
Evangeline	Little Party	$1000, 1000$	$0, 0$
	Big Party	$0, 0$	$500, 500$

(e) Does this game have a dominant strategy equilibrium? No. What are the two Nash equilibria in pure strategies? (1) Both go to the little party. (2) Both go to the big party.

(f) One of the Nash equilibria is Pareto superior to the other. Suppose that each person thought that there was some slight chance that the other would go to the little party. Would that be enough to convince them both to attend the little party? No. Can you think of any reason why the

Pareto superior equilibrium might emerge if both players understand the game matrix, if both know that the other understands it, and each knows that the other knows that he or she understands the game matrix?

If both know the game matrix and each knows that the other knows it, then each may predict the other will choose the little party.

29.6 (1) The introduction to this chapter of *Workouts*, recounted the sad tale of roommates Victoria and Albert and their dirty room. The payoff matrix for their relationship was given as follows.

Domestic Life with Victoria and Albert

		Victoria	
		Clean	Don't Clean
Albert	Clean	$5, 5$	$2, 6$
	Don't Clean	$6, 2$	$3, 3$

Suppose that we add a second stage to this game in which Victoria and Albert each have a chance to punish the other. Imagine that at the end of the day, Victoria and Albert are each able to see whether the other has done any housecleaning. After seeing what the other has done, each has the option of starting a quarrel. A quarrel hurts both of them, regardless of who started it. Thus we will assume that if either or both of them starts a quarrel, the day's payoff for each of them is reduced by 2. (For example if Victoria cleans and Albert doesn't clean and if Victoria, on seeing this result, starts a quarrel, Albert's payoff will be $6 - 2 = 4$ and Victoria's will be $2 - 2 = 0$.)

(a) Suppose that it is evening and Victoria sees that Albert has chosen not to clean and she thinks that he will not start a quarrel. Which strategy will give her a higher payoff for the whole day, Quarrel or Not Quarrel?

Not Quarrel.

(b) Suppose that Victoria and Albert each believe that the other will try to take the actions that will maximize his or her total payoff for the day.

Does either believe the other will start a quarrel? No Assuming that each is trying to maximize his or her own payoff, given the actions of the other, what would you expect each of them to do in the first stage of the game, clean or not Clean? Not Clean.

(c) Suppose that Victoria and Albert are governed by emotions that they cannot control. Neither can avoid getting angry if the other does not clean. And if either one is angry, they will quarrel so that the payoff of each is diminished by 2. Given that there is certain to be a quarrel if either does not clean, the payoff matrix for the game between Victoria and Albert becomes:

Vengeful Victoria and Angry Albert

		Victoria	
		Clean	Don't Clean
Albert	Clean	$5, 5$	$0, 4$
	Don't Clean	$4, 0$	$1, 1$

(d) If the other player cleans, is it better to clean or not clean? Clean If the other player does not clean, is it better to clean or not clean. Not Clean Explain If one player does not clean, the other will be angry so there will be a quarrel whether or not the first player cleans. Given that there will be a quarrel anyway, the second player is better off not cleaning than cleaning.

(e) Does this game have a dominant strategy? No. Explain The best response depends on what the other does.

(f) This game has two Nash equilibria. What are they? Both clean the room and both don't clean the room.

(g) Explain how it could happen that Albert and Victoria would both be better off if both are easy to anger than if they are rational about when to get angry, but it might also happen that they would both be worse off. If they both are easy to anger, there are two equilibria, one that is better for both of them and one that is worse for both of them than the equilibrium where neither gets angry and neither cleans.

(h) Suppose that Albert and Victoria are both aware that Albert will get angry and start a quarrel if Victoria does not clean, but that Victoria is level-headed and will not start a quarrel. What would be the equilibrium outcome? Albert does not clean and Victoria does clean. Nobody starts a quarrel.

29.7 (1) Maynard's Cross is a trendy bistro that specializes in carpaccio and other uncooked substances. Most people who come to Maynard's come to see and be seen by other people of the kind who come to Maynard's. There is, however, a hard core of 10 customers per evening who come for the carpaccio and don't care how many other people come. The number of additional customers who appear at Maynard's depends on how many people they expect to see. In particular, if people expect that the number of customers at Maynard's in an evening will be X, then the number of people who actually come to Maynard's is $Y = 10 + .8X$. In equilibrium, it must be true that the number of people who actually attend the restaurant is equal to the number who are expected to attend.

(a) What two simultaneous equations must you solve to find the equilibrium attendance at Maynard's? $Y = 10 + .8X$ and $X = Y$.

(b) What is the equilibrium nightly attendance? 50.

(c) On the following axes, draw the lines that represent each of the two equations you mentioned in Part (a). Label the equilibrium attendance level.

(d) Suppose that one additional carpaccio enthusiast moves to the area. Like the other 10, he eats at Maynard's every night no matter how many others eat there. Write down the new equations determining attendance at Maynard's and solve for the new equilibrium number of customers.

$y = 11 + .8x$ and $y = x$, so $x = y = 55$.

(e) Use a different color ink to draw a new line representing the equation that changed. How many additional customers did the new steady customer attract (besides himself)? 4.

(f) Suppose that everyone bases expectations about tonight's attendance on last night's attendance and that last night's attendance is public knowledge. Then $X_t = Y_{t-1}$, where X_t is expected attendance on day t and Y_{t-1} is actual attendance on day $t-1$. At any time t, $Y_t = 10 + .8X_t$. Suppose that on the first night that Maynard's is open, attendance is 20. What will be attendance on the second night? 26.

(g) What will be the attendance on the third night? 30.8.

(h) Attendance will tend toward some limiting value. What is it? 50.

29.8 (0) Yogi's Bar and Grill is frequented by unsociable types who hate crowds. If Yogi's regular customers expect that the crowd at Yogi's will be X, then the number of people who show up at Yogi's, Y, will be the larger of the two numbers, $120 - 2X$ and 0. Thus $Y = \max\{120 - 2X, 0\}$.

(a) Solve for the equilibrium attendance at Yogi's. Draw a diagram depicting this equilibrium on the axes below.

(b) Suppose that people expect the number of customers on any given night to be the same as the number on the previous night. Suppose that 50 customers show up at Yogi's on the first day of business. How many will show up on the second day? 20. The third day? 80. The fourth day? 0. The fifth day? 120. The sixth day? 0. The ninety-ninth day? 120. The hundredth day? 0.

(c) What would you say is wrong with this model if at least some of Yogi's customers have memory spans of more than a day or two?

They'd notice that last night's attendance is not a good predictor of tonight's. If attendance is low on odd-numbered days and high on even-numbered days, it would be smart to adjust by coming on odd-numbered days.

Chapter 30

NAME_____

Game Applications

Introduction. As we have seen, some games do not have a "Nash equilibrium in pure strategies." But if we allow for the possibility of "Nash equilibrium in mixed strategies," virtually every game of the sort we are interested in will have a Nash equilibrium.

The key to solving for such equilibria is to observe that if a player is indifferent between two strategies, then he is also willing to choose randomly between them. This observation will generally give us an equation that determines the equilibrium.

Example: In the game of baseball, a pitcher throws a ball towards a batter who tries to hit it. In our simplified version of the game, the pitcher can pitch high or pitch low, and the batter can swing high or swing low. The ball moves so fast that the batter has to commit to swinging high or swinging low before the ball is released.

Let us suppose that if the pitcher throws high and batter swings low, or the pitcher throws low and the batter swings high, the batter misses the ball, so the pitcher wins.

If the pitcher throws high and the batter swings high, the batter always connects. If the pitcher throws low and the batter swings low, the batter will connect only half the time.

This story leads us to the following payoff matrix, where if the batter hits the ball he gets a payoff of 1 and the pitcher gets 0, and if the batter misses, the pitcher gets a payoff of 1 and the batter gets 0.

Simplified Baseball

		Batter	
		Swing Low	Swing High
Pitcher	Pitch High	$1, 0$	$0, 1$
	Pitch Low	$.5, .5$	$1, 0$

This game has no Nash equilibrium in pure strategies. There is no combination of actions taken with certainty such that each is making the best response to the other's action. The batter always wants to swing the same place the pitcher throws, and the pitcher always wants to throw to the opposite place. What we can find is a pair of equilibrium mixed strategies.

In a mixed strategy equilibrium each player's strategy is chosen at random. The batter will be willing to choose a random strategy only if the expected payoff to swinging high is the same as the expected payoff to swinging low.

The payoffs from swinging high or swinging low depend on what the pitcher does. Let π_P be the probability that the pitcher throws high and $1 - \pi_P$ be the probability that he throws low. The batter realizes that if he swings high, he will get a payoff of 0 if the pitcher throws low and 1 if the pitcher throws high. The expected payoff to the batter is therefore π_P.

If the pitcher throws low, then the only way the batter can score is if pitcher pitches low, which happens with probability $1 - \pi_P$. Even then the batter only connects half the time. So the expected payoff to the batter from swinging low is $.5(1 - \pi_P)$.

These two expected payoffs are equalized when $\pi_P = .5(1 - \pi_P)$. If we solve this equation, we find $\pi_P = 1/3$. This has to be the probability that the pitcher throws high in a mixed strategy equilibrium.

Now let us find the probability that the batter swings low in a mixed strategy equilibrium. In equilibrium, the batter's probability π_B from swinging low must be such that the pitcher gets the same expected payoff from throwing high as from throwing low The expected payoff to the pitcher is the probability that the batter does *not* score.

If the pitcher throws high, then the batter will not connect if he swings low, but will connect if he swings high, so the expected payoff to the pitcher from pitching high is π_B.

If the pitcher throws low, then with probability $(1 - \pi_B)$, the batter will swing swing high, in which case the pitcher gets a payoff of 1. But when the pitcher throws low, the batter will swing low with probability π_B and connect half the time, giving a payoff to the pitcher of $.5\pi_B$. Therefore the expected payoff to the pitcher from throwing low is $(1-\pi_B)+.5\pi_B = 1 - .5\pi_B$. Equalizing the payoff to the pitcher from throwing high and throwing low requires $\pi_B = 1 - .5\pi_B$. Solving this equation we find that in the equilibrium mixed strategy, $\pi_B = 2/3$.

Summing up, the pitcher should throw low two-thirds of the time, and the batter should swing low two-thirds of the time.

Calculus **30.1 (2)** Two software companies sell competing products. These products are substitutes, so that the number of units that either company sells is a decreasing function of its own price and an increasing function of the other product's price. Let p_1 be the price and x_1 the quantity sold of product 1 and let p_2 and x_2 be the price and quantity sold of product 2. Then $x_1 = 1000\left(90 - \frac{1}{2}p_1 + \frac{1}{4}p_2\right)$ and $x_2 = 1000\left(90 - \frac{1}{2}p_2 + \frac{1}{4}p_1\right)$. Each company has incurred a fixed cost for designing their software and writing the programs, but the cost of selling to an extra user is zero. Therefore each company will maximize its profits by choosing the price that maximizes its total revenue.

(a) Write an expression for the total revenue of company 1, as a function of the its price p_1 and the other company's price p_2. Revenue is $1000\left(90p_1 - \frac{1}{2}p_1^2 + \frac{1}{4}p_1p_2\right)$.

(b) Company 1's *best response function* $BR_1(\cdot)$ is defined so that $BR_1(p_2)$ is the price for product 1 that maximizes company 1's revenue given that the price of product 2 is p_2. With the revenue functions we have specified, the best response function of company 1 is described by the formula $BR_1(p_2) =$ $90 + p_2/4$. (Hint: Take a derivative of revenue with respect to p_1 and solve for the revenue-maximizing p_1 given p_2.)

(c) Use a similar method to solve for company 2's best response function $BR_2(p_1) =$ $90 + p_1/4$

(d) Solve for the Nash equilibrium prices $p_1 =$ 120 and $p_2 =$ 120

(e) Suppose that company 1 sets its price first. Company 2 knows the price p_1 that company 1 has chosen and it knows that company 1 will not change this price. If company 2 sets its price so as to maximize its revenue given that company 1's price is p_1, then what price will company 2 choose? $p_2 =$ $\$90 + p_1/4$ If company 1 is aware of how company 2 will react to its own choice of price, what price will company 1 choose? \$128.55 Given this price for company 1, what price will company 2 choose? \$186.41

30.2 (1) Here is an example of the "Battle of the Sexes" game discussed in the text. Two people, let us call them Michelle and Roger, although they greatly enjoy each other's company, have very different tastes in entertainment. Roger's tastes run to ladies' mud wrestling, while Michelle prefers Italian opera. They are planning their entertainment activities for next Saturday night. For each of them, there are two possible actions, go to the wrestling match or go to the opera. Roger would be happiest if both of them went to see mud wrestling. His second choice would be for both of them to go to the opera. Michelle would prefer if both went to the opera. Her second choice would be that they both went to see the mud wrestling. They both think that the worst outcome would be that they didn't agree on where to go. If this happened, they would both stay home and sulk.

Battle of the Sexes

		Michelle	
		Wrestling	Opera
Roger	Wrestling	$2, 1$	$0, 0$
	Opera	$0, 0$	$1, 2$

(a) Is the sum of the payoffs to Michelle and Roger constant over all outcomes? No. Does this game have a dominant strategy equilibrium? No.

(b) Find two Nash equilibria in pure strategies for this game. Both go to opera. Both go to mud wrestling.

(c) Find a Nash equilibrium in mixed strategies. Michele chooses opera with probability $2/3$ and wrestling with probability $1/3$. Roger chooses opera with probability $1/3$ and mud wrestling with probability $2/3$.

30.3 (1) This is an example of the game of "Chicken." Two teenagers in souped-up cars drive toward each other at great speed. The first one to swerve out of the road is "chicken." The best thing that can happen to you is that the other guy swerves and you don't. Then you are the hero and the other guy is the chicken. If you both swerve, you are both chickens. If neither swerves, you both end up in the hospital. A payoff matrix for a chicken-type game is the following.

Chicken

		Leroy	
		Swerve	Don't Swerve
Joe Bob	Swerve	$1, 1$	$1, 2$
	Don't Swerve	$2, 1$	$0, 0$

(a) Does this game have a dominant strategy? No. What are the two Nash equilibria in pure strategies? The two outcomes where one teenager swerves and the does not.

(b) Find a Nash equilibrium in mixed strategies for this game. Play each strategy with probability 1/2.

30.4 (0) I propose the following game: I flip a coin, and while it is in the air, you call either heads or tails. If you call the coin correctly, you get to keep the coin. Suppose that you know that the coin always comes up heads. What is the best strategy for you to pursue? Always call heads.

(a) Suppose that the coin is unbalanced and comes up heads 80% of the time and tails 20% of the time. Now what is your best strategy? Always call heads.

(b) What if the coin comes up heads 50% of the time and tails 50% of the time? What is your best strategy? It doesn't matter. You can call heads always, tails always, or randomize your calls.

(c) Now, suppose that I am able to choose the type of coin that I will toss (where a coin's type is the probability that it comes up heads), and that you will know my choice. What type of coin should I choose to minimize my losses? A fair coin.

(d) What is the Nash mixed strategy equilibrium for this game? (It may help to recognize that a lot of symmetry exists in the game.) I choose a fair coin, and you randomize with 50% heads and 50% tails.

30.5 (0) Ned and Ruth love to play "Hide and Seek." It is a simple game, but it continues to amuse. It goes like this. Ruth hides upstairs or downstairs. Ned can look upstairs or downstairs but not in both places. If he finds Ruth, Ned gets one scoop of ice cream and Ruth gets none. If

he does not find Ruth, Ruth gets one scoop of ice cream and Ned gets none. Fill in the payoffs in the matrix below.

Hide and Seek

		Ruth	
		Upstairs	Downstairs
Ned	Upstairs	$1, 0$	$0, 1$
	Downstairs	$0, 1$	$1, 0$

(a) Is this a zero-sum game? Yes. What are the Nash equilibria in pure strategies? There are none.

(b) Find a Nash equilibrium in mixed strategies for this game. Ruth hides upstairs and Ned searches upstairs with probability 1/2; Ruth hides downstairs and Ned searches downstairs with probability 1/2.

(c) After years of playing this game, Ned and Ruth think of a way to liven it up a little. Now if Ned finds Ruth upstairs, he gets two scoops of ice cream, but if he finds her downstairs, he gets one scoop. If Ned finds Ruth, she gets no ice cream, but if he doesn't find her she gets one scoop. Fill in the payoffs in the graph below.

Advanced Hide and Seek

		Ruth	
		Upstairs	Downstairs
Ned	Upstairs	$2, 0$	$0, 1$
	Downstairs	$0, 1$	$1, 0$

(d) Are there any Nash equilibria in pure strategies? No. What mixed strategy equilibrium can you find? Ruth hides downstairs 2/3 of the time. Ned looks downstairs 1/2 the time. If both use equilibrium strategies, what fraction of the time will Ned find Ruth? 1/2.

30.6 (0) Perhaps you have wondered what it could mean that "the meek shall inherit the earth." Here is an example, based on the discussion in the book. In a famous experiment, two psychologists* put two pigs—a little one and a big one—into a pen that had a lever at one end and a trough at the other end. When the lever was pressed, a serving of pigfeed would appear in a trough at the other end of the pen. If the little pig would press the lever, then the big pig would eat all of the pigfeed and keep the little pig from getting any. If the big pig pressed the lever, there would be time for the little pig to get some of the pigfeed before the big pig was able to run to the trough and push him away.

Let us represent this situation by a game, in which each pig has two possible strategies. One strategy is Press the Lever. The other strategy is Wait at the Trough. If both pigs wait at the trough, neither gets any feed. If both pigs press the lever, the big pig gets all of the feed and the little pig gets a poke in the ribs. If the little pig presses the lever and the big pig waits at the trough, the big pig gets all of the feed and the little pig has to watch in frustration. If the big pig presses the lever and the little pig waits at the trough, then the little pig is able to eat 2/3 of the feed before the big pig is able to push him away. The payoffs are as follows. (These numbers are just made up, but their relative sizes are consistent with the payoffs in the Baldwin-Meese experiment.)

Big Pig–Little Pig

		Big Pig	
		Press	Wait
Little Pig	Press	$-1, 9$	$-1, 10$
	Wait	$6, 4$	$0, 0$

* Baldwin and Meese (1979), "Social Behavior in Pigs Studied by Means of Operant Conditioning," *Animal Behavior*

(a) Is there a dominant strategy for the little pig? Yes, Wait. Is there a dominant strategy for the big pig? No.

(b) Find a Nash equilibrium for this game. Does the game have more than one Nash equilibrium? The only Nash equilibrium is where little pig waits and big pig presses. (Incidentally, while Baldwin and Meese did not interpret this experiment as a game, the result they observed was the result that would be predicted by Nash equilibrium.)

(c) Which pig gets more feed in Nash equilibrium? Little pig.

30.7 (1) Let's have another look at the soccer example that was discussed in the text. But this time, we will generalize the payoff matrix just a little bit. Suppose the payoff matrix is as follows.

The Free Kick

		Kicker	
		Kick Left	Kick Right
Goalie	Jump Left	$1, 0$	$0, 1$
	Jump Right	1-p,p	$1, 0$

Now the probability that the kicker will score if he kicks to the left and the goalie jumps to the right is p. We will want to see how the equilibrium probabilities change as p changes.

(a) If the goalie jumps left with probability π_G, then if the kicker kicks right, his probability of scoring is π_G.

(b) If the goalie jumps left with probability π_G, then if the kicker kicks left, his probability of scoring is $p(1 - \pi_G)$.

(c) Find the probability π_G that makes kicking left and kicking right lead to the same probability of scoring for the kicker. (Your answer will be a function of p.) $\pi_G = \frac{p}{1+p}$.

(d) If the kicker kicks left with probability π_K, then if the goalie jumps left, the probability that the kicker will *not* score is π_K.

(e) If the kicker kicks left with probability π_K, then if the goalie jumps right, the probability that the kicker will *not* score is $(1-p)\pi_K + (1-\pi_K)$.

(f) Find the probability π_K that makes the payoff to the goalie equal from jumping left or jumping right. $\frac{1}{1+p}$.

(g) The variable p tells us how good the kicker is at kicking the ball into the left side of the goal when it is undefended. As p increases, does the equilibrium probability that the kicker kicks to the left increase or decrease? Decreases. Explain why this happens in a way that even a TV sports announcer might understand. The better the kicker's weak side gets, the less often the goalie defends the kicker's good side. So kicker can kick to good side more often.

30.8 (1) This problem is an illustration of the Hawk-Dove game described in the text. The game was first used by biologist John Maynard Smith to illustrate the uses of game theory in the theory of evolution. Males of a certain species frequently come into conflict with other males over the opportunity to mate with females. If a male runs into a situation of conflict, he has two alternative strategies. If he plays "Hawk," he will fight the other male until he either wins or is badly hurt. If he plays "Dove," he makes a bold display but retreats if his opponent starts to fight. If two Hawk players meet, they are both seriously injured in battle. If a Hawk meets a Dove, the Hawk gets to mate with the female and the Dove slinks off to celibate contemplation. If a Dove meets another Dove, they both strut their stuff but neither chases the other away. Eventually the female may select one of them at random or may get bored and wander off. The expected payoffs to each male are shown in the box below.

The Hawk-Dove Game

		Animal B	
		Hawk	Dove
Animal A	Hawk	$-5, -5$	$10, 0$
	Dove	$0, 10$	$4, 4$

(a) Now while wandering through the forest, a male will encounter many conflict situations of this type. Suppose that he cannot tell in advance whether another animal that he meets is a Hawk or a Dove. The payoff to adopting either strategy for oneself depends on the proportions of Hawks and Doves in the population at large. For example, that there is one Hawk in the forest and all of the other males are Doves. The Hawk would find that his rival always retreated and would therefore enjoy a payoff of 10 on every encounter. Given that all other males are Doves, if the remaining male is a Dove, his payoff on each encounter would be 4.

(b) If strategies that are more profitable tend to be chosen over strategies that are less profitable, explain why there cannot be an equilibrium in which all males act like Doves. If you know that you are meeting a Dove, it pays to be a Hawk.

(c) If all the other males are Hawks, then a male who adopts the Hawk strategy is sure to encounter another Hawk and would get a payoff of -5. If instead, this male adoptled the Dove strategy, he would again be sure to encounter a Hawk, but his payoff would be 0.

(d) Explain why there could not be an equilibrium where all of the animals acted like Hawks. If everyone plays Hawk, it would be profitable to play Dove.

(e) Since there is not an equilibrium in which everybody chooses the same strategy, we look for an equilibrium in which some fraction of the males are Hawks and the rest are Doves. Suppose that there is a large male population and the fraction p are Hawks. Then the fraction of any player's encounters that are with Hawks is about p and the fraction that are with Doves is about $1 - p$. Therefore with probability p a Hawk meets another Hawk and gets a payoff of -5 and with probability $1 - p$ he meets a Dove and gets 10. It follows that the payoff to a Hawk when the fraction of Hawks in the population is p, is $p \times (-5) + (1 - p) \times 10 = 10 - 15p$. Similar calculations show that the average payoff to being a Dove when the proportion of Hawks in the population is p will be

$$p \times 0 + (1 - p) \times 4.$$

(f) Write an equation that states that when the proportion of Hawks in the population is p, the payoff to Hawks is the same as the payoffs to Doves. $4 - 4p = 10 - 15p$.

(g) Solve this equation for the value of p such that at this value Hawks do exactly as well as Doves. This requires that $p =$ $6/11$.

(h) On the axes below, use blue ink to graph the average payoff to the strategy Dove when the proportion of Hawks in the male population who is p. Use red ink to graph the average payoff to the strategy, Hawk, when the proportion of the male population who are Hawks is p. Label the equilibrium proportion in your diagram by E.

(i) If the proportion of Hawks is slightly greater than E, which strategy does better? `Dove.` If the proportion of Hawks is slightly less than E, which strategy does better? `Hawk.` If the more profitable strategy tends to be adopted more frequently in future plays, then if the strategy proportions are out of equilibrium, will changes tend to move the proportions back toward equilibrium or further away from equilibrium?

`Closer.`

30.9 (2) Economic ideas and equilibrium analysis have many fascinating applications in biology. Popular discussions of natural selection and biological fitness often take it for granted that animal traits are selected for the benefit of the species. Modern thinking in biology emphasizes that individuals (or strictly speaking, genes) are the unit of selection. A mutant gene that induces an animal to behave in such a way as to help the species at the expense of the individuals that carry that gene will soon be eliminated, no matter how beneficial that behavior is to the species.

A good illustration is a paper in the *Journal of Theoretical Biology*, 1979, by H. J. Brockmann, A. Grafen, and R. Dawkins, called "Evolutionarily Stable Nesting Strategy in a Digger Wasp." They maintain that natural selection results in behavioral strategies that maximize an individual animal's expected rate of reproduction over the course of its lifetime. According to the authors, "Time is the currency which an animal spends."

Females of the digger wasp *Sphex ichneumoneus* nest in underground burrows. Some of these wasps dig their own burrows. After she has dug her burrow, a wasp goes out to the fields and hunts katydids. These she stores in her burrow to be used as food for her offspring when they hatch. When she has accumulated several katydids, she lays a single egg in the burrow, closes off the food chamber, and starts the process over again. But digging burrows and catching katydids is time-consuming. An alternative strategy for a female wasp is to sneak into somebody else's burrow while she is out hunting katydids. This happens frequently in digger wasp colonies. A wasp will enter a burrow that has been dug by another wasp and partially stocked with katydids. The invader will start catching katydids, herself, to add to the stock. When the founder and the invader finally meet, they fight. The loser of the fight goes away and never comes back. The winner gets to lay her egg in the nest.

Since some wasps dig their own burrows and some invade burrows begun by others, it is likely that we are observing a biological equilibrium in which each strategy is as effective a way for a wasp to use its time for producing offspring as the other. If one strategy were more effective than the other, then we would expect that a gene that led wasps to behave in the more effective way would prosper at the expense of genes that led them to behave in a less effective way.

Suppose the average nesting episode takes 5 days for a wasp that digs its own burrow and tries to stock it with katydids. Suppose that the

average nesting episode takes only 4 days for invaders. Suppose that when they meet, half the time the founder of the nest wins the fight and half the time the invader wins. Let D be the number of wasps that dig their own burrows and let I be the number of wasps that invade the burrows of others. The fraction of the digging wasps that are invaded will be about $\frac{5}{4}\frac{I}{D}$. (Assume for the time being that $\frac{5}{4}\frac{I}{D} < 1$.) Half of the diggers who are invaded will win their fight and get to keep their burrows. The fraction of digging wasps who lose their burrows to other wasps is then $\frac{1}{2}\frac{5}{4}\frac{I}{D} = \frac{5}{8}\frac{I}{D}$. Assume also that all the wasps who are not invaded by other wasps will successfully stock their burrows and lay their eggs.

(a) Then the fraction of the digging wasps who do not lose their burrows is just $1 - \frac{5}{8}\frac{I}{D}$.

Therefore over a period of 40 days, a wasp who dug her own burrow every time would have 8 nesting episodes. Her expected number of successes would be $8 - 5\frac{I}{D}$.

(b) In 40 days, a wasp who chose to invade every time she had a chance would have time for 10 invasions. Assuming that she is successful half the time on average, her expected number of successes would be 5. Write an equation that expresses the condition that wasps who always dig their own burrows do exactly as well as wasps who always invade burrows dug by others. $8 - 5\frac{I}{D} = 5$.

(c) The equation you have just written should contain the expression $\frac{I}{D}$. Solve for the numerical value of $\frac{I}{D}$ that just equates the expected number of successes for diggers and invaders. The answer is $\frac{3}{5}$.

(d) But there is a problem here: the equilibrium we found doesn't appear to be stable. On the axes below, use blue ink to graph the expected number of successes in a 40-day period for wasps that dig their own burrows every time where the number of successes is a function of $\frac{I}{D}$. Use black ink to graph the expected number of successes in a 40-day period for invaders. Notice that this number is the same for all values of $\frac{I}{D}$. Label the point where these two lines cross and notice that this is equilibrium. Just to the right of the crossing, where $\frac{I}{D}$ is just a little bit bigger than the equilibrium value, which line is higher, the blue or the black? Black. At this level of $\frac{I}{D}$, which is the more effective strategy for any individual wasp? Invade. Suppose that if one strategy is more effective than the other, the proportion of wasps adopting the more effective one increases. If, after being in equilibrium, the population got joggled just a little to the right of equilibrium, would the proportions of diggers and

invaders return toward equilibrium or move further away? Further away.

(e) The authors noticed this likely instability and cast around for possible changes in the model that would lead to stability. They observed that an invading wasp does help to stock the burrow with katydids. This may save the founder some time. If founders win their battles often enough and get enough help with katydids from invaders, it might be that the expected number of eggs that a founder gets to lay is an increasing rather than a decreasing function of the number of invaders. On the axes below, show an equilibrium in which digging one's own burrow is an increasingly effective strategy as $\frac{I}{D}$ increases and in which the payoff to invading is constant over all ratios of $\frac{I}{D}$. Is this equilibrium stable? Yes.

30.10 (1) The Iron Chicken restaurant is located on a busy interstate

highway. Most of its customers are just passing through and will never return to the Iron Chicken. But some are truck drivers whose routes take them past the Iron Chicken on a regular basis. Sybil, the nearsighted waitress at the Iron Chicken is unable to distinguish regular customers from one-time customers. Sybil can either give a customer good service or bad service. She knows that if she gives bad service to any customer, then she will get a small tip. If she gives good service to a truck driver, he will give her a large tip in the (futile) hope that she will recognize him the next time he comes, but if she gives good service to a one-time customer, he will still leave a small tip. Suppose that the cost to Sybil of giving a customer good service rather than bad service is \$1. The tips given by dissatisfied customers and customers just passing through average \$0.50 per customer. A truck driver who has received good service and plans to come back will leave a tip of \$2. Sybil believes that the fraction of her customers who are truck drivers who plan to come back is x. In equilibrium we would expect Sybil to give good service if x is greater than 2/3 and bad service if x is smaller.

30.11 (2) Mona is going to be out of town for two days and will not be needing her car during this time. Lisa is visiting relatives and is interested in renting her car. The value to Lisa of having the car during this time is \$50 per day. Mona figures that the total cost to her of letting Lisa use the car is \$20, regardless of how many days Lisa uses it. On the evening before the first of these two days, Mona can send a message to Lisa, offering to rent the car to her for two days for a specified price. Lisa can either accept the offer or reject the offer and make a counteroffer. The only problem is that it takes a full day for a counteroffer to be made and accepted.

Let us consider the Rubinstein bargaining solution to this problem. We start by working back from end. If Lisa rejects the original offer, then the car can only be rented for one day and there will be no time for Mona to make a counteroffer. So if Lisa rejects the original offer, she can offer Mona slightly more than \$20 to rent the car for the last day and Mona will accept. In this case, Lisa will get a profit of slightly less than $\$50 - \$20 = \$30$. Mona understands that this is the case. Therefore when Mona makes her original offer, she is aware that Lisa will reject the offer unless it gives Lisa a profit of slightly more than \$30 . Mona is aware that the value to Lisa of renting the car for two days is \$100 . Therefore the highest price for two days car rental that Lisa will accept is slightly less than \$70 Since Mona's total costs for renting the car are \$20, Mona would make a net profit of slightly less than \$50 and Lisa would make a net profit of slightly more than \$30 .

(a) Suppose that the story is as before except that Mona will be out of town for *three* days. The value to Lisa of having the car is again \$50 per day and the total cost to Mona of letting Lisa use the car is \$20, regardless of how many days Lisa uses it. This time, let us suppose that Lisa makes the first offer. Mona can either accept the offer or refuse it and make a counteroffer. Lisa, in turn, can either accept Mona's counteroffer or refuse it and make another counteroffer. Each time an offer is rejected and a new offer is made, a day passes and so there is one less day in which the car can be rented. On the evening before the first of these three days, Lisa reasons as follows. "If Mona rejects the offer that I make tonight, then there will be two days left and it will be Mona's turn to make an offer. If this happens, Mona will get a profit of slightly less than \$50 (Hint: We found this answer above, for the two-day case.) Since her total costs are \$20, Mona will make a profit of \$50 if I offer her a price of \$70 for the three days rental." Since three days of car rental is worth \$150 to Lisa, Lisa would make a profit of \$80 and Mona would make a profit of \$50.

(b) Now suppose that the story is as before except that Mona will be out of town for four days and suppose that Mona makes the first offer. Mona knows that if Lisa rejects her first offer, there will be three days left, it will be Lisa's turn to make an offer and so Lisa can make a profit of \$80 (See the previous answer.) Having the car for four days is worth \$200 to Lisa, so to the highest price that Mona can expect Lisa to accept for the four days is slightly less than \$120 In this case, Mona makes a profit of slightly less than \$100 and Lisa makes a profit of slightly more than \$80

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Chapter 31

NAME_____

Behavioral Economics

Introduction. In this section we present some problems designed to help you think about the nature of rational and not-so-rational choice. You will meet a hyperbolic procrastinator and an exponential procrastinator. Do these people remind you of anyone you know? You will meet Jake, who is aware that he has a self-control problem with beer-drinking. For those who have not experienced Jake's problem, have you ever avoided putting a full plate of chocolate chip cookies in front of you, because you know what will happen if you start eating them? Have you ever had trouble making a choice because there are too many options available? How would you react if Harriet Hardnose had you over a barrel? How rationally do you think the other people that you deal with are likely to behave?

31.1 (2) It is early Monday morning and Darryl Dawdle must write a term paper. Darryl's instructor does not accept late papers and it is crucial for Darryl to meet the deadline. The paper is due on Thursday morning, so Darryl has three days to work on it. He knows that it will take him 12 hours to do the research and write the paper. Darryl hates working on papers and likes to postpone unpleasant tasks. But he also knows that it is less painful to spread the work over all three days rather than doing it all on the last day. For any day, t, let x_t be the number of hours that he spends on the paper on day t, and x_{t+1}, and x_{t+2} the number of hours he spends on the paper the next day and the day after that. At the beginning of day t, Darryl's preferences about writing time over the next 3 days are described by the utility function

$$U(x_t, x_{t+1}, x_{t+2}) = -x_t^2 - \frac{1}{2}x_{t+1}^2 - \frac{1}{3}x_{t+2}^2.$$

(a) Suppose that on Monday morning, Darryl makes a plan by choosing x_M, x_T, and x_W to maximize his utility function

$$U(x_M, x_T, x_W) = -x_M^2 - \frac{1}{2}x_T^2 - \frac{1}{3}x_W^2$$

subject to the constraint that he puts in a total of 12 hours work on the paper. This constraint can be written as $x_M + x_T + x_W = 12$. How many hours will he plan to work on Monday? $x_M =$ 2 hours. Tuesday? $x_T =$ 4 hours. Wednesday? $x_W =$ 6 hours. (Hint: If he is maximizing his utility subject to this constraint, his marginal disutility for working must be the same on each day. Write two equations, one that sets his marginal disutility for working on Tuesday equal to that of working on Monday and one that sets his marginal disutility for

working on Wednesday equal to that of working on Monday. Use these two equations plus the budget constraint $x_M + x_T + x_W = 24$ to solve for x_M, x_T, and x_W.)

(b) On Monday, Darryl spent 2 hours working on his term paper. On Tuesday morning, when Darryl got up, he knew that he had 10 hours of work left to do. Before deciding how much work to do on Tuesday, Darryl consulted his utility function. Since it is now Tuesday, Darryl's utility function is

$$U(x_T, x_W, x_{Th}) = -x_T^2 - \frac{1}{2}x_W^2 - \frac{1}{3}x_{Th}^2,$$

where x_T, x_W, and x_{Th} are hours spent working on Tuesday, Wednesday and Thursday. Of course work done on Thursday won't be of any use. To meet the deadline, Darryl has to complete the remaining work on Tuesday and Wednesday. Therefore the least painful way to complete his assignment on time is to choose x_T and x_W to maximize

$$U(x_T, x_W, 0) = -x_T^2 - \frac{1}{2}x_W^2$$

subject to $x_T + x_W = 10$. To do this, he sets his marginal disutility for working on Tuesday equal to that for working on Wednesday. This gives the equation $2x_T = x_W$. Use this equation and the budget equation $x_T + x_W = 10$ to determine the number of hours that Darryl will work on Tuesday $31/3$ and on Wednesday $62/3$. On Monday, when Darryl made his initial plan, how much did he plan to work on Tuesday? 4 hours. On Wednesday? 6 hours. Does Darryl have time-consistent preferences? No.

(c) Suppose that on Monday morning Darryl realizes that when Tuesday comes, he will not follow the plan that maximizes his Monday preferences, but will choose to allocate the remainder of the task so as to maximize

$$U(x_T, x_W, 0) = -x_T^2 - \frac{1}{2}x_W^2$$

subject to the constraint that $x_T + x_W = 12 - x_M$. Taking this into account, Darryl makes a new calculation of how much work to do on Monday. He reasons as follows. On Tuesday, he will choose x_T and x_W so that his marginal disutility of working on Tuesday equals that on Monday. To do this he will choose $x_W/x_T =$ 2. Darryl uses this equation, along with the constraint equation to $x_T + x_W = 12 - x_M$ to solve for the amounts of work he will actually do on Tuesday and Wednesday if he does x_M hours on Monday. When he does this, he finds that if he works x_M hours on Monday, he will work $x_T(x_M) = \frac{1}{3}(12 - x_M)$ and $x_W(x_M) = \frac{2}{3}(12 - x_M)$ hours on Wednesday. Now, for each possible choice of x_M, Darryl knows how much work he will do on Tuesday and

Wednesday. Therefore, on Monday, he can calculate his utility as the following function of x_M

$$U_M(x_M) = -x_M^2 - \frac{1}{2}x_T(x_M)^2 - \frac{1}{3}x_W(x_M)^2.$$

Set the derivative of this expression with respect to x_M to find that Darryl maximizes his utility by working 2.03 hours on Monday.

(d) Does Darryl's three-period utility function have exponential discounting or hyperbolic discounting? Hyperbolic. If exponential, what is the discount rate δ; if hyperbolic, what is the parameter k? k=1.

31.2 (2) On Monday morning, Polly Putitov faces the same assignment as Darryl Dawdle. It will also take her 12 hours to finish the term paper. However on day t, her preferences about time spent writing over the next 3 days are represented by

$$U(x_t, x_{t+1}, x_{t+2}) = -x_t^2 - \frac{1}{2}x_{t+1}^2 - \frac{1}{4}x_{t+2}^2.$$

(a) Does Polly's three-period utility function have exponential or hyperbolic discounting? Exponential. If exponential, what is the discount rate δ; if hyperbolic, what is the parameter k? $\delta = 1/2$.

(b) On Monday morning Polly makes a plan for finishing the term paper that maximizes her Monday utility function

$$U(x_M, x_T, x_W) = -x_M^2 - \frac{1}{2}x_T^2 - \frac{1}{4}x_W^2$$

subject to $x_M + x_T + x_W = 12$. How many hours does Polly plan to work on Monday? 12/7=1.71. On Tuesday? $12 \times 2/7 = 3.43$. On Wednesday? $12 \times 4/7 = 6.86$.

(c) Polly spent 12/7 hours working on the project on Monday and completed the amount of work she planned to do on Monday. On Tuesday morning, her utility function is

$$U(x_T, x_W, x_{Th}) = -x_T^2 - \frac{1}{2}x_W^2 - \frac{1}{4}x_{Th}^2.$$

Since work on Thursday won't help to get the paper done before the deadline, she will set $x_{Th} = 0$, and she will choose x_T and x_W to maximize

$$-x_T^2 - \frac{1}{2}x_W^2$$

subject to the constraint that $x_T + x_W$ is equal to $12 - (12/7)$. How much will she work on Tuesday? $12 \times 2/7 = 3.43$. On Wednesday? $12 \times 4/7 = 6.86$. Do these quantities agree with the plans she made on Monday? Yes. Does Polly have time-consistent preferences? Yes.

31.3 (2) Jake likes parties and he likes to drink beer. He knows that if he drinks too much beer he will not feel well the next day and won't be able to get any work done. When Jake is at home, soberly thinking about the day-after effects of drinking, his preferences for drinking x glasses of beer at a party are represented by the utility function $U_0(x) = 10x - x^2$. Jake is invited to a party on Saturday night, and he knows that there will be free beer. His alternative is to spend a quiet evening with a teetotaling friend. Spending the quiet evening with the friend would give him a utility of 20.

(a) If he goes to the party and drinks the amount of beer that maximizes $U_0(x)$, how many glasses will he drink? 5. What will his utility be? 25. Is this higher or lower than the utility he would have by staying home? Higher.

(b) Jake has noticed that beer has a strange effect on him. It changes his utility function. When he drinks more beer he seems to get thirstier and forget the morning-after costs. In fact, for any number of beers t, after he has drunk t beers, his utility function for drinking a *total* of x beers becomes $U_t(x) = (10 + t)x - x^2$. For example, after he has drunk 5 beers, his utility for drinking a total of x beers will be $15x - x^2$, and his marginal utility for drinking more beer will be $15 - 2x$. Since this marginal utility is positive when $x = 5$, he will choose to drink more than 5 beers. How many beers must he drink so that his marginal utility of drinking more beer will be zero? 10. Suppose that before going to the party, Jake knows that the number of beers he would drink is not the number that his sober self tells him is optimal, but that he would drink until his beer-altered preferences tell him to quit. Using his sober preferences, what utility does he expect to get if he goes to the party? 0. Would he be better off spending the quiet evening at home? Yes.

31.4 (1) A rare, but fatal, disease afflicts 1 person in 100,000. Researchers have developed a powerful diagnostic test for this disease. Everyone who has the disease will test positive. Ninety-nine percent of those

who do not have the disease will test negative and one percent will test positive. Harold Dilemma was given this test during a routine physical examination and he tested positive. Harold was horrified.

(a) Harold has read about an available surgical procedure. His insurance will cover the financial cost of this surgery. The surgery would certainly eliminate the disease if he has it, but whether or not he has the disease, there is a probability of 1/200 that he would not survive the surgery. Before doing detailed calculations, do you think that undergoing the surgery would increase his overall probability of survival? Your answer

(b) To find the probability that Harold actually has the disease, given that he tests positive for it, let us reason as follows. The disease afflicts 1 person in 100,000, so in a population of 1,000,000 people, the number of people who have the disease can be expected to be about 10 . Suppose that the test is administered to all 1,000,000 people. Given that one percent of those who do not have the disease will test positive, the total number of people who test positive for the disease can be expected to be about 10,000 . Therefore, of all those who test positive for the disease, the fraction who actually have it is 1/1,000 . So, given that he tests positive, what is the probability that Harold has the disease? 1/1,000 . Would Harold improve his survival probability by undergoing surgery? Explain. No, If he has the surgery he dies with probability 1/200. If he does not have the surgery, he dies with probability 1/1,000.

(c) Suppose that this disease afflicted one person in 10,000 rather than one in 100,000. Then if Harold tested positive, what would be the probability that he has the disease? 1/100. Would he then improve his survival probability by undergoing the surgery? Yes.

31.5 (2) Some economists find experimental evidence of systematic differences between the amounts that people are willing to pay for an object and the amounts that they would have to be paid to give it up, if it is theirs. This is known as the "endowment effect." Professor Daniel McFadden of the University of California devised a classroom experiment to test for an endowment effect. He randomly sorted students in a large class into two groups of equal size. Students in one group were given a pencil, embossed with the class name. McFadden then organized a pencil market. Each student who got a pencil was asked to write down the lowest

price at which she would sell her pencil. Students without a pencil were asked to write down the highest price that they would be willing to pay for one. Students were told that the instructor would construct a "supply curve" by arraying the offers from low to high and a "demand curve" by arraying the bids from high to low. The equilibrium price is the price at which the supply curve meets the demand curve. Buyers who bid at least the equilibrium price would get a pen at the equilibrium price and sellers who offered to sell at prices at or below equilibrium would receive the equilibrium price for their pens. With these rules, it is in the interest of every student to bid his true valuation.

(a) McFadden noted that since students who received pencils were randomly selected, the distribution of willingness to pay for a pencil can be expected to be similar for those who were and those who were not given pencils. If there is no endowment effect, the lowest price at which a pencil owner is willing to sell her pencil is equal to the highest price that she would pay for a pencil. Since preferences in the two groups are approximately the same, we would expect that in equilibrium after the pencils are bought and sold, the number of pencils held by those who were not given pencils would be about equal to the number held by those who were given pencils. If this is the case, what fraction of the non-pencil-owners buy pencils? `Half` What fraction of the pencil-owners would sell pencils? `Half` What fraction of the total number of pencils handed out would be traded? `Half`

(b) In Professor McFadden's classroom, the number of pencils traded turned out to be much smaller than the number that would be expected without an endowment effect. An example will show how an endowment effect might explain this difference. Consider a classroom with 200 students randomly split into two groups of 100. Before pencils are handed out, the distribution of students' willingness to pay for pencils is the same within each group. In particular, for any price P (measured in pennies) between 0 and 100, the number of students in each group who are willing

to pay P or more for a pencil is $100 - P$. Suppose that there is no endowment effect. The demand curve of those without pencils is given by the equation $D(p) =$ $100 - P$. In the figure below, use black ink to draw the demand curve. The pencil-owners have 100 pencils. If they have the same preferences as the non-pencil-owners, then at price p, they will want to keep $D(P)$ pencils for themselves. The number that they will supply is therefore $S(P) = 100 - D(P)$. Substituting from the equation that we found for $D(P)$, this expression simplifies to $S(P) =$ P. In the grid you find here, use blue ink to draw the supply curve. Supply equals demand when the price P is 50 and the number of pencils traded is 50.

(c) Suppose that there is an endowment effect that works as follows: Pencil-owners develop an attachment to their pencils once they get them. The lowest price that any pencil-owner will accept for her pencil is three times the amount that she was willing to pay before she got the pencil. On the figure, use red ink to show the supply curve for pencils, given this endowment effect. What is the equation for this red supply curve? $S(p) =$ $p/3$ Assuming that the demand curve of non-pencil-owners does not change, what is the competitive equilibrium price for pencils? 75 What is the equilibrium number of pencils traded? 25

31.6 (2) Mark Ting works for Sucker's Jams and Jellies and is preparing a supermarket display to highlight the company's many offerings. Mark's previous exhibit was criticized by his boss, Bianca Regina who said "Jam to the left of me, jam to the right of me, but never jam in front of me!" So now Mark is under strong pressure to do well. He hired an expensive economic consultant who discovered that if x types of jam are exhibited, the probability that a given passerby will stop and look is given by $\max\{x/20, 1\}$. Furthermore, if someone stops and looks, the probability that he will buy is given by $\max\{1 - (x/20), 0\}$.

(a) If x types of jam are exhibited and $1,000$ people walk by, how many will stop to look on average? $50x$ `if` $x \leq 20$, `1000 if` $x > 20$.

(b) If x types of jam are exhibited, how many of the people who pass by will actually purchase jam? $(1-(x/20))50x = 50x - 5x^2/2$

(c) In the graph below, plots sales as a function of x.

(d) If Mark chooses x to maximize sales, what value should he choose?

$x = 10$.

(e) How many jars of jam does he sell? $50 \times 10 - 5 \times 100/2 =$ 250 jars.

31.7 (2) Charley Citrus has to leave town quickly and cannot take his car with him. Since he does not have time to seek another buyer, he must either sell it to his neighbor Harriet Hardnose or simply destroy it. Charley and Harriet both know that the car is worth \$500 to Harriet. There is no time for negotiation. Harriet must make one offer for the car and Charley will decide either to sell it to her at that price or destroy it. Harriet has no sympathy for Charley and his dilemma, but she does know that Charley has a temper and she fears that if she offers too little, Charley might get angry and destroy the car. In particular, she believes that the probability that Charley will sell her the car is $x/500$ for any $x \leq \$500$. Harriet realizes that if she can buy the car for $\$x$, her profit will be $\$500 - x$, but if he destroys the car, her profit is zero.

(a) Write an expression for Harriet's expected profit as a function of x.

$(500 - x)x/500$ for $x \leq 500$. What price should she offer Charley in order to maximize her expected profit? \$250

(b) Suppose that Harriet thinks that Charley is sure to sell the car if she offers more than \$300 and that the probability that Charley will sell her the car is $x/300$ for $x \leq 300$.

(c) Write an expression for Harriet's expected profit as a function of x. $(500 - x)x/300$ `for` $x \leq 300$. What price should she offer in order to maximize her expected profit? `$250.`

(d) Suppose that Harriet thinks that Charley is sure to sell the car if she offers more than \$200 and that the probability that Charley will sell her the car is $x/200$ for $x \leq 200$. What price should she offer in order to maximize her expected profit? `$200.`

31.8 (1) A group of people are asked to play the following game. Each person in the group picks a number between 0 and 100. There is a large reward for getting an answer that is closest to 2/3 of the average answer given by the other members of the group.

(a) Sam Smartypants thinks that the other players in the game are really stupid. He believes that they will choose numbers between 0 and 100 at random, with an expected value of 50. Sam will maximize his expected winnings by choosing a number close to $33.3 = 50 \times 2/3$.

(b) Oona Onestep realizes that the average cannot exceed 100 and so half the average cannot exceed 50. She thinks that the other people in the group are about as smart as Sam Smartypants. Given that these people will choose numbers close to `33.3`, Oona will choose a number close to `22.2`.

(c) Talullah Twostep thinks that nobody could be as stupid as Sam Smartypants, but that everyone else will reason just like Oona Onestep. Talullah notices however that if everybody else acts like Oona, then she should not do the same thing that Oona does, but should instead choose a number close to `14.8`.

(d) Raquel Rational notes that Sam, Oona, and Talullah all think they are smarter than everybody else. She asks herself, "What number would I choose if I thought everyone reasons in the same way that I do?" For what value of X is it true that $X = 2X/3$? $X = 0$

(e) This game is known as the "Beauty Contest Game." The Beauty Contest game has been played in many economics laboratory experiments. It has also been played by several thousand people who submitted reader entries in three separate contests conducted by business newspapers in the UK and Spain and by *Spektrum der Wissenschaft*, the German edition of *Scientific American*. Readers were asked to submit numbers between 0

and 100 with a prize awarded to the number closest to 2/3 of the average. The game has also been played in laboratory experiments by groups such as undergraduates at Cal Tech and UCLA, students in Germany and Singapore, high school students, the Cal Tech board of trustees, and portfolio managers. In these experiments, typically only a small proportion of participants respond with numbers of 50 or more. In graphs showing frequencies of the possible responses, there are spikes at 33.3, at 22.2, and a modest spike at zero. The mean number selected by readers of *Spektrum der Wissenschaft* was 22.08. Mean scores among the various laboratory groups varied considerably from a low of 21.9 to a high of 46.1. Which laboratory group do you think chose the lowest number? `Cal Tech undergraduates` (To find out more about this game, where it got its name, and how each of the groups did, look at "Progress in Behavioral Game Theory," a paper in the *Journal of Economic Perspectives*, Autumn 1997, by Professor Colin Camerer of Cal Tech.)

Chapter 32

NAME_____

Exchange

Introduction. The *Edgeworth box* is a thing of beauty. An amazing amount of information is displayed with a few lines, points and curves. In fact one can use an Edgeworth box to show just about everything there is to say about the case of two traders dealing in two commodities. Economists know that the real world has more than two people and more than two commodities. But it turns out that the insights gained from this model extend nicely to the case of many traders and many commodities. So for the purpose of introducing the subject of exchange equilibrium, the Edgeworth box is exactly the right tool. We will start you out with an example of two gardeners engaged in trade. You will get most out of this example if you fill in the box as you read along.

Example: Alice and Byron consume two goods, camelias and dahlias. Alice has 16 camelias and 4 dahlias. Byron has 8 camelias and 8 dahlias. They consume no other goods, and they trade only with each other. To describe the possible allocations of flowers, we first draw a box whose width is the total number of camelias and whose height is the total number of dahlias that Alice and Byron have between them. The width of the box is therefore $16 + 8 = 24$ and the height of the box is $4 + 8 = 12$.

Any feasible allocation of flowers between Alice and Byron is fully described by a single point in the box. Consider, for example, the allocation where Alice gets the bundle $(15, 9)$ and Byron gets the bundle $(9, 3)$. This allocation is represented by the point $A = (15, 9)$ in the Edgeworth box, which you should draw in. The distance 15 from A to the left side of the box is the number of camelias for Alice and the distance 9 from A to the bottom of the box is the number of dahlias for Alice. This point also determines Byron's consumption of camelias and dahlias. The distance 9 from A to the right side of the box is the total number of camelias consumed by Byron, and the distance from A to the top of the box is the number of dahlias consumed by Byron. Since the width of the box is the total supply of camelias and the height of the box is the total supply of

dahlias, these conventions ensure that any point in the box represents a feasible allocation of the total supply of camelias and dahlias.

It is also useful to mark the initial allocation in the Edgeworth box, which, in this case, is the point $E = (16, 4)$. Now suppose that Alice's utility function is $U(c, d) = c + 2d$ and Byron's utility funtion is $U(c, d) = cd$. Alice's indifference curves will be straight lines with slope $-1/2$. The indifference curve that passes through her initial endowment, for example, will be a line that runs from the point $(24, 0)$ to the point $(0, 12)$. Since Byron has Cobb-Douglas utility, his indifference curves will be rectangular hyperbolas, but since quantities for Byron are measured from the upper right corner of the box, these indifference curves will be flipped over as in the Edgeworth box diagrams in your textbook.

The *Pareto set* or *contract curve* is the set of points where Alice's indifference curves are tangent to Byron's. There will be tangency if the slopes are the same. The slope of Alice's indifference curve at any point is $-1/2$. The slope of Byron's indifference curve depends on his consumption of the two goods. When Byron is consuming the bundle (c_B, d_B), the slope of his indifference curve is equal to his marginal rate of substitution, which is $-d_B/c_B$. Therefore Alice's and Byron's indifference curves will nuzzle up in a nice tangency whenever $-d_B/c_B = -1/2$. So the Pareto set in this example is just the diagonal of the Edgeworth box.

Some problems ask you to find a competitive equilibrium. For an economy with two goods, the following procedure is often a good way to calculate equilibrium prices and quantities.

- Since demand for either good depends only on the ratio of prices of good 1 to good 2, it is convenient to set the price of good 1 equal to 1 and let p_2 be the price of good 2.
- With the price of good 1 held at 1, calculate each consumer's demand for good 2 as a function of p_2.
- Write an equation that sets the total amount of good 2 demanded by all consumers equal to the total of all participants' initial endowments of good 2.
- Solve this equation for the value of p_2 that makes the demand for good 2 equal to the supply of good 2. (When the supply of good 2 equals the demand of good 2, it must also be true that the supply of good 1 equals the demand for good 1.)
- Plug this price into the demand functions to determine quantities.

Example: Frank's utility function is $U(x_1, x_2) = x_1 x_2$ and Maggie's is $U(x_1, x_2) = \min\{x_1, x_2\}$. Frank's initial endowment is 0 units of good 1 and 10 units of good 2. Maggie's initial endowment is 20 units of good 1 and 5 units of good 2. Let us find a competitive equilibrium for Maggie and Frank.

Set $p_1 = 1$ and find Frank's and Maggie's demand functions for good 2 as a function of p_2. Using the techniques learned in Chapter 6, we find that Frank's demand function for good 2 is $m/2p_2$, where m is his income. Since Frank's initial endowment is 0 units of good 1 and 10 units of good 2, his income is $10p_2$. Therefore Frank's demand for good 2 is $10p_2/2p_2 = 5$. Since goods 1 and 2 are perfect complements for Maggie, she will choose to consume where $x_1 = x_2$. This fact, together with her

budget constraint implies that Maggie's demand function for good 2 is $m/(1+p_2)$. Since her endowment is 20 units of good 1 and 5 units of good 2, her income is $20+5p_2$. Therefore at price p_2, Maggie's demand is $(20+5p_2)/(1+p_2)$. Frank's demand plus Maggie's demand for good 2 adds up to $5+(20+5p_2)/(1+p_2)$. The total supply of good 2 is Frank's 10 unit endowment plus Maggie's 5 unit endowment, which adds to 15 units. Therefore demand equals supply when

$$5+\frac{(20+5p_2)}{(1+p_2)}=15.$$

Solving this equation, one finds that the equilibrium price is $p_2=2$. At the equilibrium price, Frank will demand 5 units of good 2 and Maggie will demand 10 units of good 2.

32.1 (0) Morris Zapp and Philip Swallow consume wine and books. Morris has an initial endowment of 60 books and 10 bottles of wine. Philip has an initial endowment of 20 books and 30 bottles of wine. They have no other assets and make no trades with anyone other than each other. For Morris, a book and a bottle of wine are perfect substitutes. His utility function is $U(b,w)=b+w$, where b is the number of books he consumes and w is the number of bottles of wine he consumes. Philip's preferences are more subtle and convex. He has a Cobb-Douglas utility function, $U(b,w)=bw$. In the Edgeworth box below, Morris's consumption is measured from the lower left, and Philip's is measured from the upper right corner of the box.

(a) On this diagram, mark the initial endowment and label it E. Use red ink to draw Morris Zapp's indifference curve that passes through his initial endowment. Use blue ink to draw in Philip Swallow's indifference curve that passes through his initial endowment. (Remember that quantities for Philip are measured from the upper right corner, so his indifference curves are "Phlipped over.")

(b) At any Pareto optimum, where both people consume some of each good, it must be that their marginal rates of substitution are equal. No matter what he consumes, Morris's marginal rate of substitution is equal to -1. When Philip consumes the bundle, (b_P, w_P), his MRS is $-w_P/b_P$. Therefore every Pareto optimal allocation where both consume positive amounts of both goods satisfies the equation $w_P = b_P$. Use black ink on the diagram above to draw the locus of Pareto optimal allocations.

(c) At a competitive equilibrium, it will have to be that Morris consumes some books and some wine. But in order for him to do so, it must be that the ratio of the price of wine to the price of books is 1. Therefore we know that if we make books the *numeraire*, then the price of wine in competitive equilibrium must be 1.

(d) At the equilibrium prices you found in the last part of the question, what is the value of Philip Swallow's initial endowment? 50. At these prices, Philip will choose to consume 25 books and 25 bottles of wine. If Morris Zapp consumes all of the books and all of the wine that Philip doesn't consume, he will consume 55 books and 15 bottles of wine.

(e) At the competitive equilibrium prices that you found above, Morris's income is 70. Therefore at these prices, the cost to Morris of consuming all of the books and all of the wine that Philip doesn't consume is (the same as, more than, less than) the same as his income. At these prices, can Morris afford a bundle that he likes better than the bundle $(55, 15)$? No.

(f) Suppose that an economy consisted of 1,000 people just like Morris and 1,000 people just like Philip. Each of the Morris types had the same endowment and the same tastes as Morris. Each of the Philip types had the same endowment and tastes as Philip. Would the prices that you found to be equilibrium prices for Morris and Philip still be competitive equilibrium prices? Yes. If each of the Morris types and each of the Philip types behaved in the same way as Morris and Philip did above, would supply equal demand for both wine and books? Yes.

32.2 (0) Consider a small exchange economy with two consumers, Astrid and Birger, and two commodities, herring and cheese. Astrid's initial endowment is 4 units of herring and 1 unit of cheese. Birger's initial endowment has no herring and 7 units of cheese. Astrid's utility function is $U(H_A, C_A) = H_A C_A$. Birger is a more inflexible person. His utility function is $U(H_B, C_B) = \min\{H_B, C_B\}$. (Here H_A and C_A are the amounts of herring and cheese for Astrid, and H_B and C_B are amounts of herring and cheese for Birger.)

(a) Draw an Edgeworth box, showing the initial allocation and sketching in a few indifference curves. Measure Astrid's consumption from the lower left and Birger's from the upper right. In your Edgeworth box, draw two different indifference curves for each person, using blue ink for Astrid's and red ink for Birger's.

(b) Use black ink to show the locus of Pareto optimal allocations. (Hint: Since Birger is kinky, calculus won't help much here. But notice that because of the rigidity of the proportions in which he demands the two goods, it would be inefficient to give Birger a positive amount of either good if he had less than that amount of the other good. What does that tell you about where the Pareto efficient locus has to be?) Pareto efficient allocations lie on the line with slope 1 extending from Birger's corner of the box.

(c) Let cheese be the numeraire (with price 1) and let p denote the price of herring. Write an expression for the amount of herring that Birger will demand at these prices. $7/(p+1)$ (Hint: Since Birger initially owns 7 units of cheese and no herring and since cheese is the numeraire, the value of his initial endowment is 7. If the price of herring is p, how many units of herring will he choose to maximize his utility subject to his budget constraint?)

(d) Where the price of cheese is 1 and p is the price of herring, what is the value of Astrid's initial endoment? $1+4p$. How much herring will Astrid demand at price p? $2 + (1/2p)$

32.3 (0) Dean Foster Z. Interface and Professor J. Fetid Nightsoil exchange platitudes and bromides. When Dean Interface consumes T_I platitudes and B_I bromides, his utility is given by

$$U_I(B_I, T_I) = B_I + 2\sqrt{T_I}.$$

When Professor Nightsoil consumes T_N platitudes and B_N bromide, his utility is given by

$$U_N(B_N, T_N) = B_N + 4\sqrt{T_N}.$$

Dean Interface's initial endowment is 12 platitudes and 8 bromides. Professor Nightsoil's initial endowment is 4 platitudes and 8 bromides.

(a) If Dean Interface consumes T_I platitudes and B_I bromides, his marginal rate of substitution will be $-T_I^{-1/2}$. If Professor Nightsoil consumes T_N platitudes and B_N bromides, his marginal rate of substitution will be $-2T_N^{-1/2}$.

(b) On the contract curve, Dean Interface's marginal rate of substitution equals Professor Nightsoil's. Write an equation that states this condition. $\sqrt{T_I} = \sqrt{T_N}/2$. This equation is especially simple because each person's marginal rate of substitution depends only on his consumption of platitudes and not on his consumption of bromides.

(c) From this equation we see that $T_I/T_N =$ $1/4$ at all points on the contract curve. This gives us one equation in the two unknowns T_I and T_N.

(d) But we also know that along the contract curve it must be that $T_I + T_N =$ 16 , since the total consumption of platitudes must equal the total endowment of platitudes.

(e) Solving these two equations in two unknowns, we find that everywhere on the contract curve, T_I and T_N are constant and equal to $T_I = 3.2$ and $T_N = 12.8$.

(f) In the Edgeworth box, label the initial endowment with the letter E. Dean Interface has thick gray penciled indifference curves. Professor Nightsoil has red indifference curves. Draw a few of these in the Edgeworth box you made. Use blue ink to show the locus of Pareto optimal points. The contract curve is a (vertical, horizontal, diagonal) horizontal line in the Edgeworth box.

(g) Find the competitive equilibrium prices and quantities. You know what the prices have to be at competitive equilibrium because you know what the marginal rates of substitution have to be at every Pareto optimum. $T_I = 3.2$, $T_N = 12.8$, platitude price/bromide price $= \frac{1}{\sqrt{3.2}}$.

32.4 (0) A little exchange economy has just two consumers, named Ken and Barbie, and two commodities, quiche and wine. Ken's initial endowment is 3 units of quiche and 2 units of wine. Barbie's initial endowment is 1 unit of quiche and 6 units of wine. Ken and Barbie have identical utility functions. We write Ken's utility function as, $U(Q_K, W_K) = Q_K W_K$ and Barbie's utility function as $U(Q_B, W_B) = Q_B W_B$, where Q_K and W_K are the amounts of quiche and wine for Ken and Q_B and W_B are amounts of quiche and wine for Barbie.

(a) Draw an Edgeworth box below, to illustrate this situation. Put quiche on the horizontal axis and wine on the vertical axis. Measure goods for Ken from the lower left corner of the box and goods for Barbie from the upper right corner of the box. (Be sure that you make the length of the box equal to the total supply of quiche and the height equal to the total supply of wine.) Locate the initial allocation in your box, and label it W. On the sides of the box, label the quantities of quiche and wine for each of the two consumers in the initial endowment.

(b) Use blue ink to draw an indifference curve for Ken that shows allocations in which his utility is 6. Use red ink to draw an indifference curve for Barbie that shows allocations in which her utility is 6.

(c) At any Pareto optimal allocation where both consume some of each good, Ken's marginal rate of substitution between quiche and wine must equal Barbie's. Write an equation that states this condition in terms of the consumptions of each good by each person. $W_B/Q_B = W_K/Q_K$.

(d) On your graph, show the locus of points that are Pareto efficient. (Hint: If two people must each consume two goods in the same proportions as each other, and if together they must consume twice as much wine as quiche, what must those proportions be?)

(e) In this example, at any Pareto efficient allocation, where both persons consume both goods, the slope of Ken's indifference curve will be -2. Therefore, since we know that competitive equilibrium must be Pareto efficient, we know that at a competitive equilibrium, $p_Q/p_W =$ 2.

(f) In competitive equilibrium, Ken's consumption bundle must be 2 quiche, 4 wine. How about Barbie's consumption bundle? 2 quiche, 4 wine. (Hint: You found competitive equilibrium prices above. You know Ken's initial endowment and you know the

equilibrium prices. In equilibrium Ken's income will be the value of his endowment at competitive prices. Knowing his income and the prices, you can compute his demand in competitive equilibrium. Having solved for Ken's consumption and knowing that total consumption by Ken and Barbie equals the sum of their endowments, it should be easy to find Barbie's consumption.)

(g) On the Edgeworth box for Ken and Barbie, draw in the competitive equilibrium allocation and draw Ken's competitive budget line (with black ink).

32.5 (0) Linus Straight's utility function is $U(a, b) = a + 2b$, where a is his consumption of apples and b is his consumption of bananas. Lucy Kink's utility function is $U(a, b) = \min\{a, 2b\}$. Lucy initially has 12 apples and no bananas. Linus initially has 12 bananas and no apples. In the Edgeworth box below, goods for Lucy are measured from the upper right corner of the box and goods for Linus are measured from the lower left corner. Label the initial endowment point on the graph with the letter E. Draw two of Lucy's indifference curves in red ink and two of Linus's indifference curves in blue ink. Use black ink to draw a line through all of the Pareto optimal allocations.

(a) In this economy, in competitive equilibrium, the ratio of the price of apples to the price of bananas must be 1/2.

(b) Let a_S be Linus's consumption of apples and let b_S be his consumption of bananas. At competititive equilibrium, Linus's consumption will have to satisfy the budget constraint, $a_s +$ 2 $b_S =$ 24. This gives us one equation in two unknowns. To find a second equation, consider Lucy's

consumption. In competitive equilibrium, total consumption of apples equals the total supply of apples and total consumption of bananas equals the total supply of bananas. Therefore Lucy will consume $12 - a_s$ apples and 12 $-b_s$ bananas. At a competitive equilibrium, Lucy will be consuming at one of her kink points. The kinks occur at bundles where Lucy consumes 2 apples for every banana that she consumes. Therefore we know that $\frac{12-a_s}{12-b_s} =$ 2.

(c) You can solve the two equations that you found above to find the quantities of apples and bananas consumed in competitive equilibrium by Linus and Lucy. Linus will consume 6 units of apples and 9 units of bananas. Lucy will consume 6 units of apples and 3 units of bananas.

32.6 (0) Consider a pure exchange economy with two consumers and two goods. At some given Pareto efficient allocation it is known that both consumers are consuming both goods and that consumer A has a marginal rate of substitution between the two goods of -2. What is consumer B's marginal rate of substitution between these two goods? -2.

32.7 (0) Charlotte loves apples and hates bananas. Her utility function is $U(a, b) = a - \frac{1}{4}b^2$, where a is the number of apples she consumes and b is the number of bananas she consumes. Wilbur likes both apples and bananas. His utility function is $U(a, b) = a + 2\sqrt{b}$. Charlotte has an initial endowment of no apples and 8 bananas. Wilbur has an initial endowment of 16 apples and 8 bananas.

(a) On the graph below, mark the initial endowment and label it E. Use red ink to draw the indifference curve for Charlotte that passes through this point. Use blue ink to draw the indifference curve for Wilbur that passes through this point.

(b) If Charlotte hates bananas and Wilbur likes them, how many bananas can Charlotte be consuming at a Pareto optimal allocation? 0. On the graph above, use black ink to mark the locus of Pareto optimal allocations of apples and bananas between Charlotte and Wilbur.

(c) We know that a competitive equilibrium allocation must be Pareto optimal and the total consumption of each good must equal the total supply, so we know that at a competitive equilibrium, Wilbur must be consuming 16 bananas. If Wilbur is consuming this number of bananas, his marginal utility for bananas will be 1/4 and his marginal utility of apples will be 1. If apples are the *numeraire*, then the only price of bananas at which he will want to consume exactly 16 bananas is 1/4. In competitive equilibrium, for the Charlotte-Wilbur economy, Wilbur will consume 16 bananas and 14 apples and Charlotte will consume 0 bananas and 2 apples.

32.8 (0) Mutt and Jeff have 8 cups of milk and 8 cups of juice to divide between themselves. Each has the same utility function given by $u(m, j) = \max\{m, j\}$, where m is the amount of milk and j is the amount of juice that each has. That is, each of them cares only about the larger of the two amounts of liquid that he has and is indifferent to the liquid of which he has the smaller amount.

(a) Sketch an Edgeworth box for Mutt and Jeff. Use blue ink to show a couple of indifference curves for each. Use red ink to show the locus of Pareto optimal allocations. (Hint: Look for boundary solutions.)

32.9 (1) Remember Tommy Twit from Chapter 3. Tommy is happiest when he has 8 cookies and 4 glasses of milk per day and his indifference curves are concentric circles centered around (8,4). Tommy's mother, Mrs. Twit, has strong views on nutrition. She believes that too much of anything is as bad as too little. She believes that the perfect diet for Tommy would be 7 glasses of milk and 2 cookies per day. In her view, a diet is healthier the smaller is the sum of the absolute values of the differences between the amounts of each food consumed and the ideal amounts. For example, if Tommy eats 6 cookies and drinks 6 glasses of milk, Mrs. Twit believes that he has 4 too many cookies and 1 too few glasses of milk, so the sum of the absolute values of the differences from her ideal amounts is 5. On the axes below, use blue ink to draw the locus of combinations that Mrs. Twit thinks are exactly as good for Tommy as $(6, 6)$. Also, use red ink to draw the locus of combinations that she thinks is just as good as $(8, 4)$. On the same graph, use red ink to draw an indifference "curve" representing the locus of combinations that Tommy likes just as well as 7 cookies and 8 glasses of milk.

(a) On the graph, shade in the area consisting of combinations of cookies and milk that both Tommy and his mother agree are better than 7 cookies and 8 glasses of milk, where "better" for Mrs. Twit means she thinks it is healthier, and where "better" for Tommy means he likes it better.

(b) Use black ink to sketch the locus of "Pareto optimal" bundles of cookies and milk for Tommy. In this situation, a bundle is Pareto optimal if any bundle that Tommy prefers to this bundle is a bundle that Mrs. Twit thinks is worse for him. The locus of Pareto optimal points that you just drew should consist of two line segments. These run from the point (8,4) to the point 5,7 and from that point to the point 2,7.

32.10 (2) This problem combines equilibrium analysis with some of the things you learned in the chapter on intertemporal choice. It concerns the economics of saving and the life cycle on an imaginary planet where life is short and simple. In advanced courses in macroeconomics, you would study more-complicated versions of this model that build in more earthly realism. For the present, this simple model gives you a good idea of how the analysis must go.

On the planet Drongo there is just one commodity, cake, and two time periods. There are two kinds of creatures, "old" and "young." Old creatures have an income of I units of cake in period 1 and no income in period 2. Young creatures have no income in period 1 and an income of I^* units of cake in period 2. There are N_1 old creatures and N_2 young creatures. The consumption bundles of interest to creatures are pairs (c_1, c_2), where c_1 is cake in period 1 and c_2 is cake in period 2. All creatures, old and young, have identical utility functions, representing preferences over cake in the two periods. This utility function is $U(c_1, c_2) = c_1^a c_2^{1-a}$, where a is a number such that $0 \leq a \leq 1$.

(a) If current cake is taken to be the *numeraire*, (that is, its price is set at 1), write an expression for the present value of a consumption bundle (c_1, c_2). $c_1 + c_2/(1+r)$. Write down the present value of income for old creatures I and for young creatures $I^*/(1+r)$. The budget line for any creature is determined by the condition that the present value of its consumption bundle equals the present value of its income. Write down this budget equation for old creatures: $c_1 + c_2/(1+r) = I$ and for young creatures: $c_1 + c_2/(1+r) = I^*/(1+r)$.

(b) If the interest rate is r, write down an expression for an old creature's demand for cake in period 1 $c_1 = aI$ and in period 2 $c_2 = (1-a)I(1+r)$. Write an expression for a young creature's demand for cake in period 1 $c_1 = aI^*/(1+r)$ and in period 2 $c_2 = (1-a)I^*$. (Hint: If its budget line is $p_1c_1 + p_2c_2 = W$ and its utility function is of the form proposed above, then a creature's demand function for good 1 is $c_1 = aW/p$ and demand for good 2 is $c_2 = (1-a)W/p$.) If the interest rate is zero, how much cake would a young creature choose in period 1? aI^*. For what value of a would it choose the same amount in each period if the interest rate is zero? $a = 1/2$. If $a = .55$, what would r have to be in order that young creatures would want to consume the same amount in each period? $.22$.

(c) The total supply of cake in period 1 equals the total cake earnings of all old creatures, since young creatures earn no cake in this period. There are N_1 old creatures and each earns I units of cake, so this total is N_1I. Similarly, the total supply of cake in period 2 equals the total amount earned by young creatures. This amount is N_2I^*.

(d) At the equilibrium interest rate, the total demand of creatures for period-1 cake must equal total supply of period-1 cake, and similarly the demand for period-2 cake must equal supply. If the interest rate is r, then the demand for period-1 cake by each old creature is aI and the demand for period-1 cake by each young creature is $aI^*/(1+r)$. Since there are N_1 old creatures and N_2 young creatures, the total demand for period-1 cake at interest rate r is $N_1aI + N_2aI^*/(1+r)$.

(e) Using the results of the last section, write an equation that sets the demand for period-1 cake equal to the supply. $N_1aI+N_2aI^*/(1+r) = N_1I$. Write a general expression for the equilibrium value of r, given N_1, N_2, I, and I^*. $r = \frac{N_2I^*a}{N_1I(1-a)} - 1$. Solve this equation for the special case when $N_1 = N_2$ and $I = I^*$ and $a = 11/21$.

$r = 10\%$.

(f) In the special case at the end of the last section, show that the interest rate that equalizes supply and demand for period-1 cake will also equalize supply and demand for period-2 cake. (This illustrates Walras's law.)

`Supply = demand for period 2 if` $N_1(1-a)I(1+r) + N_2(1-a)I^* = N_2I^*$. `If` $N_1 = N_2$ `and` $I = I^*$, `then` $(1-a)(1+r) + (1-a) = 1$. `If` $a = 11/21$, `then` $r = 10\%$.

Chapter 33 NAME_____

Production

Introduction. In this section we explore economywide production possibility sets. We pay special attention to the principle of comparative advantage. The principle is simply that efficiency suggests that people should specialize according to their *relative* abilities in different activities rather than absolute abilities.

Example: For simplicity, let us imagine an island with only two people on it, both of them farmers. They do not trade with the outside world. Farmer A has 100 acres and is able to grow two crops, wheat and hay. Each acre of his land that he plants to wheat will give him 50 bushels of wheat. Each acre of his land that he plants to hay will give him 2 tons of hay. Farmer B also has 100 acres, but his land is not so good. Each acre of his land yields only 20 bushels of wheat and only 1 ton of hay. Notice that, although Farmer A's land is better for *both* wheat and hay, Farmer B's land has *comparative* advantage in the production of hay. This is true because the ratio of tons of hay to bushels of wheat per acre $2/50 = .04$ for Farmer A and $1/20 = .05$ for Farmer B. Farmer A, on the other hand, has comparative advantage in the production of wheat, since the ratio of bushels of wheat to tons of hay is $50/2 = 25$ for Farmer A and $20/1 = 20$ for Farmer B. The efficient way to arrange production is to have Farmer A "specialize" in wheat and farmer B "specialize" in hay. If Farmer A devotes all of his land to wheat and Farmer B devotes all of his land to hay, then total wheat production will be 5,000 bushels and total hay production will be 100 tons. Suppose that they decide to produce only 4,000 bushels of wheat. Given that they are going to produce 4,000 bushels of wheat, the most hay they can possibly produce together will be obtained if Farmer A devotes 80 acres to wheat and 20 acres to hay while Farmer B devotes all of his land to hay. Suppose that they decide to produce 6,000 bushels of wheat. Then they will get the most hay possible given that they are producing 6,000 bushels of wheat if Farmer A puts all of his land into wheat and Farmer B puts 50 acres into wheat and the remaining 50 acres into hay.

33.1 (0) Tip and Spot finally got into college. Tip can write term papers at the rate of 10 pages per hour and solve workbook problems at the rate of 3 per hour. Spot can write term papers at the rate of 6 pages per hour and solve workbook problems at the rate of 2 per hour. Which of these two has comparative advantage in solving workbook problems?

Spot.

(a) Tip and Spot each work 6 hours a day. They decide to work together and to produce a combination of term papers and workbook problems that lies on their joint production possibility frontier. On the above graph plot their joint production possibility frontier. If they produce less than 60 pages of term papers, then Tip will write all of the term papers. If they produce more than 60 pages of term papers, then Tip will continue to specialize in writing term papers and Spot will also write some term papers.

33.2 (0) Robinson Crusoe has decided that he will spend exactly 8 hours a day gathering food. He can either spend this time gathering coconuts or catching fish. He can catch 1 fish per hour and he can gather 2 coconuts per hour. On the graph below, show Robinson's production possibility frontier between fish and coconuts per day. Write an equation for the line segment that is Robinson's production possibility frontier.

$F + C/2 = 8$.

(a) Robinson's utility function is $U(F, C) = FC$, where F is his daily fish consumption and C is his daily coconut consumption. On the graph above, sketch the indifference curve that gives Robinson a utility of 4, and also sketch the indifference curve that gives him a utility of 8. How many fish will Robinson choose to catch per day? 4. How many coconuts will he collect? 8. (Hint: Robinson will choose a bundle that maximizes his utility subject to the constraint that the bundle lies in his production possibility set. But for this technology, his production possibility set looks just like a budget set.)

(b) Suppose Robinson is not isolated on an island in the Pacific, but is retired and lives next to a grocery store where he can buy either fish or coconuts. If fish cost \$1 per fish, how much would coconuts have to cost in order that he would choose to consume twice as many coconuts as fish?

\$.50. Suppose that a social planner decided that he wanted Robinson to consume 4 fish and 8 coconuts per day. He could do this by setting the price of fish equal to \$1, the price of coconuts equal to \$.50 and giving Robinson a daily income of \$ 8 .

(c) Back on his island, Robinson has little else to do, so he pretends that he is running a competitive firm that produces fish and coconuts. He wonders, "What would the price have to be to make me do just what I am actually doing? Let's assume that fish are the *numeraire* and have a price of \$1. And let's pretend that I have access to a competitive labor market where I can hire as much labor as I want at some given wage. There is a constant returns to scale technology. An hour's labor produces

one fish or 2 coconuts. At wages above $ 1 per hour, I wouldn't produce any fish at all, because it would cost me more than $1 to produce a fish. At wages below $ 1 per hour, I would want to produce infinitely many fish since I would make a profit on every one. So the only possible wage rate that would make me choose to produce a positive finite amount of fish is $ 1 per hour. Now what would the price of coconuts have to be to induce me to produce a positive number of coconuts. At the wage rate I just found, the cost of producing a coconut is $.50. At this price and only at this price, would I be willing to produce a finite positive number of coconuts."

33.3 (0) We continue the story of Robinson Crusoe from the previous problem. One day, while walking along the beach, Robinson Crusoe saw a canoe in the water. In the canoe was a native of a nearby island. The native told Robinson that on his island there were 100 people and that they all lived on fish and coconuts. The native said that on his island, it takes 2 hours to catch a fish and 1 hour to find a coconut. The native said that there was a competitive economy on his island and that fish were the *numeraire.* The price of coconuts on the neighboring island must have been $.50. The native offered to trade with Crusoe at these prices. "I will trade you either fish for coconuts or coconuts for fish at the exchange rate of 2 coconuts for a fish," said he. "But you will have to give me 1 fish as payment for rowing over to your island." Would Robinson gain by trading with him? No. If so, would he buy fish and sell coconuts or vice versa? Neither. Since their prices are the same as the rate at which he can transform the two goods, he can gain nothing by trading.

(a) Several days later, Robinson saw another canoe in the water on the other side of his island. In this canoe was a native who came from a different island. The native reported that on his island, one could catch only 1 fish for every 4 hours of fishing and that it takes 1 hour to find a coconut. This island also had a competitive economy. The native offered to trade with Robinson at the same exchange rate that prevailed on his own island, but said that he would have to have 2 fish in return for rowing between the islands. If Robinson decides to trade with this island, he chooses to produce only fish and will get his coconuts from the other island. On the graph above, use black ink to draw Robinson's production possibility frontier if he doesn't trade and use blue ink to

show the bundles he can afford if he chooses to trade and specializes appropriately. Remember to take away 2 fish to pay the trader.

(b) Write an equation for Crusoe's "budget line" if he specializes appropriately and trades with the second trader. If he does this, what bundle will he choose to consume? 3 fish, 12 coconuts. Does he like this bundle better than the bundle he would have if he didn't trade? Yes.

33.4 (0) The Isle of Veritas has made it illegal to trade with the outside world. Only two commodities are consumed on this island, milk and wheat. On the north side of the island are 40 farms. Each of these farms can produce any combination of non-negative amounts of milk and wheat that satisfies the equation $m = 60 - 6w$. On the south side of the island are 60 farms. Each of these farms can produce any combination of non-negative amounts of milk and wheat that satisfies the equation $m = 40 - 2w$. The economy is in competitive equilibrium and 1 unit of wheat exchanges for 4 units of milk.

(a) On the diagram below, use black ink to draw the production possibility set for a typical farmer from the *north* side of the island. Given the equilibrium prices, will this farmer specialize in milk, specialize in wheat, or produce both goods? Specialize in milk. Use blue ink to draw the budget that he faces in his role as a consumer if he makes the optimal choice of what to produce.

(b) On the diagram below, use black ink to draw the production possibility set for a typical farmer from the *south* side of the island. Given the equilibrium prices, will this farmer specialize in milk, specialize in wheat, or produce both goods? Specialize in wheat. Use blue ink to draw the budget that he faces in his role as a consumer if he makes the optimal choice of what to produce.

(c) Suppose that peaceful Viking traders discover Veritas and offer to exchange either wheat for milk or milk for wheat at an exchange rate of

1 unit of wheat for 3 units of milk. If the Isle of Veritas allows free trade with the Vikings, then this will be the new price ratio on the island. At this price ratio, would either type of farmer change his output? No.

(d) On the first of the two graphs above, use red ink to draw the budget for northern farmers if free trade is allowed and the farmers make the right choice of what to produce. On the second of the two graphs, use red ink to draw the budget for southern farmers if free trade is allowed and the farmers make the right choice of what to produce.

(e) The council of elders of Veritas will meet to vote on whether to accept the Viking offer. The elders from the north end of the island get 40 votes and the elders from the south end get 60 votes. Assuming that everyone votes in the selfish interest of his end of the island, how will the northerners vote? In favor. How will the southerners vote? Against. How is it that you can make a definite answer to the last two questions without knowing anything about the farmers's consumption preferences? The change strictly enlarges the budget set for northerners and strictly shrinks it for southerners.

(f) Suppose that instead of offering to make exchanges at the rate of 1 unit of wheat for 3 units of milk, the Vikings had offered to trade at the price of 1 unit of wheat for 1 unit of milk and vice versa. Would either type of farmer change his output? Yes. Southerners would now switch to specializing in milk. Use pencil to sketch the budget line for each kind of farmer at these prices if he makes the right production decision. How will the northerners vote now? In favor. How will the southerners vote now? Depends on their preferences about consumption. Explain why it is that your answer to one of the last two questions has to be "it depends." The two alternative budget lines for southerners are not nested.

33.5 (0) Recall our friends the Mungoans of Chapter 2. They have a strange two-currency system consisting of Blue Money and Red Money. Originally, there were two prices for everything, a blue-money price and a red-money price. The blue-money prices are 1 bcu per unit of ambrosia and 1 bcu per unit of bubble gum. The red-money prices are 2 rcu's per unit of ambrosia and 4 rcu's per unit of bubble gum.

(a) Harold has a blue income of 9 and a red income of 24. If it has to pay in *both* currencies for any purchase, draw its budget set in the graph below. (Hint: You answered this question a few months ago.)

(b) The Free Choice party campaigns on a platform that Mungoans should be allowed to purchase goods at *either* the blue-money price or the red-money price, whichever they prefer. We want to construct Harold's budget set if this reform is instituted. To begin with, how much bubble gum could Harold consume if it spent all of its blue money and its red money on bubble gum? 15 units of bubble gum.

(c) How much ambrosia could it consume if it spent all of its blue money and all of its red money on ambrosia? 21 units of ambrosia.

(d) If Harold were spending all of its money of both colors on bubble gum and it decided to purchase a little bit of ambrosia, which currency would it use? The red currency.

(e) How much ambrosia could it buy before it ran out of that color money?

12 units of ambrosia.

(f) What would be the slope of this budget line before it ran out of that kind of money? The slope would be $-\frac{1}{2}$.

(g) If Harold were spending all of its money of both colors on ambrosia and it decided to purchase a little bit of bubble gum, which currency would it use? The blue currency.

(h) How much bubble gum could it buy before it ran out of that color money? He could buy 9 units of bubble gum.

(i) What would be the slope of this budget line before it ran out of that kind of money? The slope would be -1.

(j) Use your answers to the above questions to draw Harold's budget set in the above graph if it could purchase bubble gum and ambrosia using either currency.

Chapter 34

NAME_____

Welfare

Introduction. Here you will look at various ways of determining social preferences. You will check to see which of the Arrow axioms for aggregating individual preferences are satisfied by these welfare relations. You will also try to find optimal allocations for some given social welfare functions. The method for solving these last problems is analogous to solving for a consumer's optimal bundle given preferences and a budget constraint. Two hints. Remember that for a Pareto optimal allocation inside the Edgeworth box, the consumers' marginal rates of substitution will be equal. Also, in a "fair allocation," neither consumer prefers the other consumer's bundle to his own.

Example: A social planner has decided that she wants to allocate income between 2 people so as to maximize $\sqrt{Y_1}+\sqrt{Y_2}$ where Y_i is the amount of income that person i gets. Suppose that the planner has a fixed amount of money to allocate and that she can enforce any income distribution such that $Y_1 + Y_2 = W$, where W is some fixed amount. This planner would have ordinary convex indifference curves between Y_1 and Y_2 and a "budget constraint" where the "price" of income for each person is 1. Therefore the planner would set her marginal rate of substitution between income for the two people equal to the relative price which is 1. When you solve this, you will find that she sets $Y_1 = Y_2 = W/2$. Suppose instead that it is "more expensive" for the planner to give money to person 1 than to person 2. (Perhaps person 1 is forgetful and loses money, or perhaps person 1 is frequently robbed.) For example, suppose that the planner's budget is $2Y_1+Y_2 = W$. Then the planner maximizes $\sqrt{Y_1}+\sqrt{Y_2}$ subject to $2Y_1 + Y_2 = W$. Setting her MRS equal to the price ratio, we find that $\frac{\sqrt{Y_2}}{\sqrt{Y_1}} = 2$. So $Y_2 = 4Y_1$. Therefore the planner makes $Y_1 = W/5$ and $Y_2 = 4W/5$.

34.1 (2) One possible method of determining a social preference relation is the *Borda count*, also known as rank-order voting. Each voter is asked to rank all of the alternatives. If there are 10 alternatives, you give your first choice a 1, your second choice a 2, and so on. The voters' scores for each alternative are then added over all voters. The total score for an alternative is called its Borda count. For any two alternatives, x and y, if the Borda count of x is smaller than or the same as the Borda count for y, then x is "socially at least as good as" y. Suppose that there are a finite number of alternatives to choose from and that every individual has complete, reflexive, and transitive preferences. For the time being, let us also suppose that individuals are never indifferent between any two different alternatives but always prefer one to the other.

(a) Is the social preference ordering defined in this way complete? Yes. Reflexive? Yes. Transitive? Yes.

(b) If everyone prefers x to y, will the Borda count rank x as socially preferred to y? Explain your answer. Yes. If everybody ranks x ahead of y, then everyone must give x a higher rank than y. Then the sum of the ranks of x must be larger than the sum of the ranks of y.

(c) Suppose that there are two voters and three candidates, x, y, and z. Suppose that Voter 1 ranks the candidates, x first, z second, and y third. Suppose that Voter 2 ranks the candidates, y first, x second, and z third. What is the Borda count for x? 3. For y? 4. For z? 5. Now suppose that it is discovered that candidate z once lifted a beagle by the ears. Voter 1, who has rather large ears himself, is appalled and changes his ranking to x first, y second, z third. Voter 2, who picks up his own children by the ears, is favorably impressed and changes his ranking to y first, z second, x third. Now what is the Borda count for x? 4. For y? 3. For z? 5.

(d) Does the social preference relation defined by the Borda count have the property that social preferences between x and y depend only on how people rank x versus y and not on how they rank other alternatives? Explain. No. In the above example, the ranking of z changed, but nobody changed his mind about whether x was better than y or *vice versa*. Before the change x beat y, and after the change y beat x.

34.2 (2) Suppose the utility possibility frontier for two individuals is given by $U_A + 2U_B = 200$. On the graph below, plot the utility frontier.

(a) In order to maximize a "Nietzschean social welfare function," $W(U_A, U_B) = \max\{U_A, U_B\}$, on the utility possibility frontier shown above, one would set U_A equal to 200 and U_B equal to 0.

(b) If instead we use a Rawlsian criterion, $W(U_A, U_B) = \min\{U_A, U_B\}$, then the social welfare function is maximized on the above utility possibility frontier where U_A equals 66.66 and U_B equals 66.66.

(c) Suppose that social welfare is given by $W(U_A, U_B) = U_A^{1/2} U_B^{1/2}$. In this case, with the above utility possibility frontier, social welfare is maximized where U_A equals 100 and U_B is 50. (Hint: You might want to think about the similarities between this maximization problem and the consumer's maximization problem with a Cobb-Douglas utility function.)

(d) Show the three social maxima on the above graph. Use black ink to draw a Nietzschean isowelfare line through the Nietzschean maximum. Use red ink to draw a Rawlsian isowelfare line through the Rawlsian maximum. Use blue ink to draw a Cobb-Douglas isowelfare line through the Cobb-Douglas maximum.

34.3 (2) A parent has two children named A and B and she loves both of them equally. She has a total of $1,000 to give to them.

(a) The parent's utility function is $U(a,b) = \sqrt{a} + \sqrt{b}$, where a is the amount of money she gives to A and b is the amount of money she gives to B. How will she choose to divide the money? $a = b = \$500$.

(b) Suppose that her utility function is $U(a,b) = -\frac{1}{a} - \frac{1}{b}$. How will she choose to divide the money? $a = b = \$500$.

(c) Suppose that her utility function is $U(a,b) = \log a + \log b$. How will she choose to divide the money? $a = b = \$500$.

(d) Suppose that her utility function is $U(a,b) = \min\{a,b\}$. How will she choose to divide the money? $a = b = \$500$.

(e) Suppose that her utility function is $U(a,b) = \max\{a,b\}$. How will she choose to divide the money? $a = \$1,000$, $b = 0$, or vice versa.

(Hint: In each of the above cases, we notice that the parent's problem is to maximize $U(a,b)$ subject to the constraint that $a + b = 1,000$. This is just like the consumer problems we studied earlier. It must be that the parent sets her marginal rate of substitution between a and b equal to 1 since it costs the same to give money to each child.)

(f) Suppose that her utility function is $U(a,b) = a^2 + b^2$. How will she choose to divide the money between her children? Explain why she doesn't set her marginal rate of substitution equal to 1 in this case.

She gives everything to one child. Her preferences are not convex, indifference curves are quarter circles.

34.4 (2) In the previous problem, suppose that A is a much more efficient shopper than B so that A is able to get twice as much consumption goods as B can for every dollar that he spends. Let a be the amount of consumption goods that A gets and b the amount that B gets. We will measure consumption goods so that one unit of consumption goods costs \$1 for A and \$2 for B. Thus the parent's budget constraint is $a + 2b = 1,000$.

(a) If the mother's utility function is $U(a,b) = a + b$, which child will get more money? A. Which child will consume more goods? A.

(b) If the mother's utility function is $U(a, b) = a \times b$, which child will get more money? They get the same amount of money. Which child will get to consume more? A consumes more.

(c) If the mother's utility function is $U(a, b) = -\frac{1}{a} - \frac{1}{b}$, which child will get more money? B gets more money. Which child will get to consume more? They consume the same amount.

(d) If the mother's utility function is $U(a, b) = \max\{a, b\}$, which child will get more money? A. Which child will get to consume more?

A.

(e) If the mother's utility function is $U(a, b) = \min\{a, b\}$, which child will get more money? B. Which child will get to consume more?

They consume the same amount.

Calculus **34.5 (1)** Norton and Ralph have a utility possibility frontier that is given by the following equation, $U_R + U_N^2 = 100$ (where R and N signify Ralph and Norton respectively).

(a) If we set Norton's utility to zero, what is the highest possible utility Ralph can achieve? 100. If we set Ralph's utility to zero, what is the best Norton can do? 10.

(b) Plot the utility possibility frontier on the graph below.

(c) Derive an equation for the slope of the above utility possibility curve.

$$\frac{dU_R}{dU_N} = -2U_N\,.$$

(d) Both Ralph and Norton believe that the ideal allocation is given by maximizing an appropriate social welfare function. Ralph thinks that $U_R = 75$, $U_N = 5$ is the best distribution of welfare, and presents the maximization solution to a weighted-sum-of-the-utilities social welfare function that confirms this observation. What was Ralph's social welfare function? (Hint: What is the slope of Ralph's social welfare function?)

$$W = U_R + 10U_N\,.$$

(e) Norton, on the other hand, believes that $U_R = 19$, $U_N = 9$ is the best distribution. What is the social welfare function Norton presents?

$$W = U_R + 18U_N\,.$$

34.6 (2) Roger and Gordon have identical utility functions, $U(x, y) = x^2 + y^2$. There are 10 units of x and 10 units of y to be divided between them. Roger has blue indifference curves. Gordon has red ones.

(a) Draw an Edgeworth box showing some of their indifference curves and mark the Pareto optimal allocations with black ink. (Hint: Notice that the indifference curves are nonconvex.)

(b) What are the fair allocations in this case? See diagram.

34.7 (2) Paul and David consume apples and oranges. Paul's utility function is $U_P(A_P, O_P) = 2A_P + O_P$ and David's utility function is

$U_D(A_D, O_D) = A_D + 2O_D$, where A_P and A_D are apple consumptions for Paul and David, and O_P and O_D are orange consumptions for Paul and David. There are a total of 12 apples and 12 oranges to divide between Paul and David. Paul has blue indifference curves. David has red ones. Draw an Edgeworth box showing some of their indifference curves. Mark the Pareto optimal allocations on your graph.

(a) Write one inequality that says that Paul likes his own bundle as well as he likes David's and write another inequality that says that David likes his own bundle as well as he likes Paul's. $2A_P + O_P \geq 2A_D + O_D$ and $A_D + 2O_D \geq A_P + 2O_P$.

(b) Use the fact that at feasible allocations, $A_P + A_D = 12$ and $O_P + O_D = 12$ to eliminate A_D and O_D from the first of these equations. Write the resulting inequality involving only the variables A_P and O_P. Now in your Edgeworth box, use blue ink to shade in all of the allocations such that Paul prefers his own allocation to David's. $2A_P + O_P \geq 18$.

(c) Use a procedure similar to that you used above to find the allocations where David prefers his own bundle to Paul's. Describe these points with an inequality and shade them in on your diagram with red ink.

$A_D + 2O_D \geq 18$.

(d) On your Edgeworth box, mark the fair allocations.

34.8 (3) Romeo loves Juliet and Juliet loves Romeo. Besides love, they consume only one good, spaghetti. Romeo likes spaghetti, but he

also likes Juliet to be happy and he knows that spaghetti makes her happy. Juliet likes spaghetti, but she also likes Romeo to be happy and she knows that spaghetti makes Romeo happy. Romeo's utility function is $U_R(S_R, S_J) = S_R^a S_J^{1-a}$ and Juliet's utility function is $U_J(S_J, S_R) = S_J^a S_R^{1-a}$, where S_J and S_R are the amount of spaghetti for Romeo and the amount of spaghetti for Juliet respectively. There is a total of 24 units of spaghetti to be divided between Romeo and Juliet.

(a) Suppose that $a = 2/3$. If Romeo got to allocate the 24 units of spaghetti exactly as he wanted to, how much would he give himself? 16. How much would he give Juliet? 8. (Hint: Notice that this problem is formally just like the choice problem for a consumer with a Cobb-Douglas utility function choosing between two goods with a budget constraint. What is the budget constraint?)

(b) If Juliet got to allocate the spaghetti exactly as she wanted to, how much would she take for herself? 16. How much would she give Romeo? 8.

(c) What are the Pareto optimal allocations? (Hint: An allocation will not be Pareto optimal if both persons' utility will be increased by a gift from one to the other.) The Pareto optimal allocations are all of the allocations in which each person gets at least 8 units of spaghetti.

(d) When we had to allocate two goods between two people, we drew an Edgeworth box with indifference curves in it. When we have just one good to allocate between two people, all we need is an "Edgeworth line" and instead of indifference curves, we will just have indifference dots. Consider the Edgeworth line below. Let the distance from left to right denote spaghetti for Romeo and the distance from right to left denote spaghetti for Juliet.

(e) On the Edgeworth line you drew above, show Romeo's favorite point and Juliet's favorite point.

(f) Suppose that $a = 1/3$. If Romeo got to allocate the spaghetti, how much would he choose for himself? 8. If Juliet got to allocate the spaghetti, how much would she choose for herself? 8. Label the Edgeworth line below, showing the two people's favorite points and the locus of Pareto optimal points.

(g) When $a = 1/3$, at the Pareto optimal allocations what do Romeo and Juliet disagree about? Romeo wants to give spaghetti to Juliet, but she doesn't want to take it. Juliet wants to give spaghetti to Romeo, but he doesn't want to take it. Both like spaghetti for themselves, but would rather the other had it.

34.9 (2) Hatfield and McCoy hate each other but love corn whiskey. Because they hate for each other to be happy, each wants the other to have less whiskey. Hatfield's utility function is $U_H(W_H, W_M) = W_H - W_M^2$ and McCoy's utility function is $U_M(W_M, W_H) = W_M - W_H^2$, where W_M is McCoy's daily whiskey consumption and W_H is Hatfield's daily whiskey consumption (both measured in quarts). There are 4 quarts of whiskey to be allocated.

(a) If McCoy got to allocate all of the whiskey, how would he allocate it? All for himself. If Hatfield got to allocate all of the whiskey, how would he allocate it? All for himself.

(b) If each of them gets 2 quarts of whiskey, what will the utility of each of them be? -2. If a bear spilled 2 quarts of their whiskey and they divided the remaining 2 quarts equally between them, what would the utility of each of them be? 0. If it is possible to throw away some of the whiskey, is it Pareto optimal for them each to consume 2 quarts of whiskey? No.

(c) If it is possible to throw away some whiskey and they must consume equal amounts of whiskey, how much should they throw away? 3 quarts.

Chapter 35

NAME_____

Externalities

Introduction. When there are externalities, the outcome from independently chosen actions is typically not Pareto efficient. In these exercises, you explore the consequences of alternative mechanisms and institutional arrangements for dealing with externalities.

Example: A large factory pumps its waste into a nearby lake. The lake is also used for recreation by 1,000 people. Let X be the amount of waste that the firm pumps into the lake. Let Y_i be the number of hours per day that person i spends swimming and boating in the lake, and let C_i be the number of dollars that person i spends on consumption goods. If the firm pumps X units of waste into the lake, its profits will be $1,200X - 100X^2$. Consumers have identical utility functions, $U(Y_i, C_i, X) = C_i + 9Y_i - Y_i^2 - XY_i$, and identical incomes. Suppose that there are no restrictions on pumping waste into the lake and there is no charge to consumers for using the lake. Also, suppose that the factory and the consumers make their decisions independently. The factory will maximize its profits by choosing $X = 6$. (Set the derivative of profits with respect to X equal to zero.) When $X = 6$, each consumer maximizes utility by choosing $Y_i = 1.5$. (Set the derivative of utility with respect to Y_i equal to zero.) Notice from the utility functions that when each person is spending 1.5 hours a day in the lake, she will be willing to pay 1.5 dollars to reduce X by 1 unit. Since there are 1,000 people, the total amount that people will be willing to pay to reduce the amount of waste by 1 unit is \$1,500. If the amount of waste is reduced from 6 to 5 units, the factory's profits will fall from \$3,600 to \$3,500. Evidently the consumers could afford to bribe the factory to reduce its waste production by 1 unit.

35.1 (2) The picturesque village of Horsehead, Massachusetts, lies on a bay that is inhabited by the delectable crustacean, *homarus americanus*, also known as the lobster. The town council of Horsehead issues permits to trap lobsters and is trying to determine how many permits to issue. The economics of the situation is this:

1. It costs \$2,000 dollars a month to operate a lobster boat.
2. If there are x boats operating in Horsehead Bay, the total revenue from the lobster catch per month will be $f(x) = \$1,000(10x - x^2)$.

(a) In the graph below, plot the curves for the average product, $AP(x) = f(x)/x$, and the marginal product, $MP(x) = 10,000 - 2,000x$. In the same graph, plot the line indicating the cost of operating a boat.

(b) If the permits are free of charge, how many boats will trap lobsters in Horsehead, Massachusetts? (Hint: How many boats must enter before there are zero profits?) 8 boats.

(c) What number of boats maximizes total profits? Set MP equal to cost to give $10 - 2x = 2$, or $x = 4$ boats.

(d) If Horsehead, Massachusetts, wants to restrict the number of boats to the number that maximizes total profits, how much should it charge per month for a lobstering permit? (Hint: With a license fee of F thousand dollars per month, the marginal cost of operating a boat for a month would be $(2 + F)$ thousand dollars per month.) $4,000 per month.

35.2 (2) Suppose that a honey farm is located next to an apple orchard and each acts as a competitive firm. Let the amount of apples produced be measured by A and the amount of honey produced be measured by H. The cost functions of the two firms are $c_H(H) = H^2/100$ and $c_A(A) = A^2/100 - H$. The price of honey is \$2 and the price of apples is \$3.

(a) If the firms each operate independently, the equilibrium amount of honey produced will be 100 and the equilibrium amount of apples produced will be 150.

(b) Suppose that the honey and apple firms merged. What would be the profit-maximizing output of honey for the combined firm? 150. What would be the profit-maximizing amount of apples? 150.

(c) What is the socially efficient output of honey? 150. If the firms stayed separate, how much would honey production have to be subsidized to induce an efficient supply? $1 per unit.

35.3 (2) In El Carburetor, California, population 1,001, there is not much to do except to drive your car around town. Everybody in town is just like everybody else. While everybody likes to drive, everybody complains about the congestion, noise, and pollution caused by traffic. A typical resident's utility function is $U(m, d, h) = m + 16d - d^2 - 6h/1,000$, where m is the resident's daily consumption of Big Macs, d is the number of hours per day that he, himself, drives, and h is the total amount of driving (measured in person-hours per day) done by all other residents of El Carburetor. The price of Big Macs is $1 each. Every person in El Carburetor has an income of $40 per day. To keep calculations simple, suppose it costs nothing to drive a car.

(a) If an individual believes that the amount of driving he does won't affect the amount that others drive, how many hours per day will he choose to drive? 8. (Hint: What value of d maximizes $U(m, d, h)$?)

(b) If everybody chooses his best d, then what is the total amount h of driving by other persons? 8,000.

(c) What will be the utility of each resident? 56.

(d) If everybody drives 6 hours a day, what will be the utility level of a typical resident of El Carburetor? 64.

(e) Suppose that the residents decided to pass a law restricting the total number of hours that anyone is allowed to drive. How much driving should everyone be allowed if the objective is to maximize the utility of the typical resident? (Hint: Rewrite the utility function, substituting $1,000d$ for h, and maximize with respect to d.) 5 hours per day.

(f) The same objective could be achieved with a tax on driving. How much would the tax have to be per hour of driving? (Hint: This price would have to equal an individual's marginal rate of substitution between driving and Big Macs when he is driving the "right" amount.) $6.

35.4 (3) Tom and Jerry are roommates. They spend a total of 80 hours a week together in their room. Tom likes loud music, even when he sleeps. His utility function is $U_T(C_T, M) = C_T + M$, where C_T is the number of cookies he eats per week and M is the number of hours of loud music per week that is played while he is in their room. Jerry hates all kinds of music. His utility function is $U(C_J, M) = C_J - M^2/12$. Every week, Tom and Jerry *each* get two dozen chocolate chip cookies sent from home. They have no other source of cookies. We can describe this situation with a box that looks like an Edgeworth box. The box has cookies on the horizontal axis and hours of music on the vertical axis. Since cookies are private goods, the number of cookies that Tom consumes per week plus the number that Jerry consumes per week must equal 48. But music in their room is a public good. Each must consume the same number of hours of music, whether he likes it or not. In the box, let the height of a point represent the total number of hours of music played in their room per week. Let the distance of the point from the left side of the box be "cookies for Tom" and the distance of the point from the right side of the box be "cookies for Jerry."

(a) Suppose the dorm's policy is that you must have your roommate's permission to play music. The *initial endowment* in this case denotes the situation if Tom and Jerry make no deals. There would be no music, and each person would consume 2 dozen cookies a week. Mark this initial endowment on the box above with the label A. Use red ink to sketch the indifference curve for Tom that passes through this point, and use

blue ink to sketch the indifference curve for Jerry that passes through this point. [Hint: When you draw Jerry's indifference curve, remember two things: (1) He hates music, so he prefers lower points on the graph to higher ones. (2) Cookies for Jerry are measured from the right side of the box, so he prefers points that are toward the left side of the box to points that are toward the right.] Use blue ink to shade in the points representing situations that would make both roommates better off than they are at point A.

(b) Suppose, alternatively, that the dorm's policy is "rock-n-roll is good for the soul." You don't need your roommate's permission to play music. Then the *initial endowment* is one in which Tom plays music for all of the 80 hours per week that they are in the room together and where each consumes 2 dozen cookies per week. Mark this endowment point in the box above and label it B. Use red ink to sketch the indifference curve for Tom that passes through this point, and use blue ink to sketch the indifference curve for Jerry that passes through this point. Given the available resources, can *both* Tom and Jerry be made better off than they are at point B? Yes.

Calculus **35.5 (0)** A clothing store and a jewelry store are located side by side in a small shopping mall. The number of customers who come to the shopping mall intending to shop at either store depends on the amount of money that the store spends on advertising per day. Each store also attracts some customers who came to shop at the neighboring store. If the clothing store spends $\$x_C$ per day on advertising, and the jeweller spends $\$x_J$ on advertising per day, then the total profits per day of the clothing store are $\Pi_C(x_C, x_J) = (60 + x_J)x_C - 2x_C^2$, and the total profits per day of the jewelry store are $\Pi_J(x_C, x_J) = (105 + x_C)x_J - 2x_J^2$. (In each case, these are profits net of *all* costs, including advertising.)

(a) If each store believes that the other store's amount of advertising is independent of its own advertising expenditure, then we can find the equilibrium amount of advertising for each store by solving two equations in two unknowns. One of these equations says that the derivative of the clothing store's profits with respect to its own advertising is zero. The other equation requires that the derivative of the jeweller's profits with respect to its own advertising is zero. These two equations are written as

$60 + x_J - 4x_C = 0$ and $105 + x_C - 4x_J = 0$. The equilibrium amounts of advertising are $x_C = 23$ and $x_J = 32$. Profits of the clothing store are \$1,058 and profits of the jeweller are 2,048.

(b) The extra profit that the jeweller would get from an extra dollar's worth of advertising by the clothing store is approximately equal to the derivative of the jeweller's profits with respect to the clothing store's advertising expenditure. When the two stores are doing the equilibrium amount of advertising that you calculated above, a dollar's worth of advertising by the clothing store would give the jeweller an extra profit of about $32 and an extra dollar's worth of advertising by the jeweller would give the clothing store an extra profit of about $23.

(c) Suppose that the owner of the clothing store knows the profit functions of both stores. She reasons to herself as follows. Suppose that I can decide how much advertising I will do before the jeweller decides what he is going to do. When I tell him what I am doing, he will have to adjust his behavior accordingly. I can calculate his reaction function to my choice of x_C, by setting the derivative of his profits with respect to his own advertising equal to zero and solving for his amount of advertising as a function of my own advertising. When I do this, I find that $x_J =$ $105/4 + x_C/4$. If I substitute this value of x_J into my profit function and then choose x_C to maximize my own profits, I will choose $x_C =$ 24.64 and he will choose $x_J =$ 32.41 . In this case my profits will be $1,062.72 and his profits will be $2,100.82.

(d) Suppose that the clothing store and the jewelry store have the same profit functions as before but are owned by a single firm that chooses the amounts of advertising so as to maximize the sum of the two stores' profits. The single firm would choose $x_C =$ $37.50 and $x_J =$ $45. Without calculating actual profits, can you determine whether total profits will be higher, lower, or the same as total profits would be when they made their decisions independently? Yes, they would be higher. How much would the total profits be? $3,487.50.

35.6 (2) The cottagers on the shores of Lake Invidious are an unsavory bunch. There are 100 of them, and they live in a circle around the lake. Each cottager has two neighbors, one on his right and one on his left. There is only one commodity, and they all consume it on their front lawns in full view of their two neighbors. Each cottager likes to consume the commodity but is very envious of consumption by the neighbor on his left. Curiously, nobody cares what the neighbor on his right is doing. In fact every consumer has a utility function $U(c, l) = c - l^2$, where c is his own consumption and l is consumption by his neighbor on the left.

Suppose that each consumer owns 1 unit of the consumption good and consumes it.

(a) Calculate his utility level. 0.

(b) Suppose that each consumer consumes only 3/4 of a unit. Will all individuals be better off or worse off? Better off.

(c) What is the best possible consumption if all are to consume the same amount? 1/2.

(d) Suppose that everybody around the lake is consuming 1 unit. Can any two people make themselves both better off either by redistributing consumption between them or by throwing something away? No.

(e) How about a group of three people? No.

(f) How large is the smallest group that could cooperate to benefit all its members? 100.

35.7 (0) Jim and Tammy are partners in Business and in Life. As is all too common in this imperfect world, each has a little habit that annoys the other. Jim's habit, we will call activity X, and Tammy's habit, activity Y. Let x be the amount of activity X that Jim pursues and y be the amount of activity Y that Tammy pursues. Due to a series of unfortunate reverses, Jim and Tammy have a total of only \$1,000,000 a year to spend. Jim's utility function is $U_J = c_J + 500 \ln x - 10y$, where c_J is the money he spends per year on goods other than his habit, x is the number of units of activity X that he consumes per year, and y is the number of units of activity Y that Tammy consumes per year. Tammy's utility function is $U_T = c_T + 500 \ln y - 10x$, where c_T is the amount of money she spends on goods other than activity Y, y is the number of units of activity Y that she consumes, and x is the number of units of activity X that Jim consumes. Activity X costs \$20 per unit. Activity Y costs \$100 per unit.

(a) Suppose that Jim has a right to half their joint income and Tammy has a right to the other half. Suppose further that they make no bargains with each other about how much activity X and Y they will consume.

How much of activity X will Jim choose to consume? 25 units.

How much of activity Y will Tammy consume? 5 units.

(b) Because Jim and Tammy have quasilinear utility functions, their utility possibility frontier includes a straight line segment. Furthermore, this segment can be found by maximizing the sum of their utilities. Notice that

$$\begin{aligned} U_J(c_J, x, y) + U_T(c_T, x, y) \\ &= c_J + 500 \ln x - 20y + c_T + 500 \ln y - 10x \\ &= c_J + c_T + 500 \ln x - 10x + 500 \ln y - 10y. \end{aligned}$$

But we know from the family budget constraint that $c_J + c_T = 1,000,000 - 20x - 100y$. Therefore we can write

$$\begin{aligned} U_J(c_J, x, y) + U_T(c_T, x, y) &= 1,000,000 - 20x - 100y + 500 \ln x - 10x \\ &\quad + 500 \ln y - 10y \\ &= 1,000,000 + 500 \ln x + 500 \ln y - 30x - 110y. \end{aligned}$$

Let us now choose x and y so as to maximize $U_J(c_J, x, y) + U_T(c_T, x, y)$. Setting the partial derivatives with respect to x and y equal to zero, we find the maximum where $x =$ 16.67 and $y =$ 4.54 . If we plug these numbers into the equation $U_J(c_J, x, y) + U_T(c_T, x, y) = 1,000,000 + 500 \ln x + 500 \ln y - 30x - 110y$, we find that the utility possibility frontier is described by the equation $U_J + U_C =$ 1,001,163.86. (You need a calculator or a log table to find this answer.) Along this frontier, the total expenditure on the annoying habits X and Y by Jim and Tammy is 787.34 . The rest of the \$1,000,000 is spent on c_J and c_T. Each possible way of dividing this expenditure corresponds to a different point on the utility possibility frontier. The slope of the utility possibility frontier constructed in this way is -1.

35.8 (0) An airport is located next to a large tract of land owned by a housing developer. The developer would like to build houses on this land, but noise from the airport reduces the value of the land. The more planes that fly, the lower is the amount of profits that the developer makes. Let X be the number of planes that fly per day and let Y be the number of houses that the developer builds. The airport's total profits are $48X - X^2$, and the developer's total profits are $60Y - Y^2 - XY$. Let us consider the outcome under various assumptions about institutional rules and about bargaining between the airport and the developer.

(a) "Free to Choose with No Bargaining": Suppose that no bargains can be struck between the airport and the developer and that each can decide on its own level of activity. No matter how many houses the developer builds, the number of planes per day that maximizes profits for the airport is 24. Given that the airport is landing this number of planes, the

number of houses that maximizes the developer's profits is 18. Total profits of the airport will be 576 and total profits of the developer will be 324. The sum of their profits will be 900.

(b) "Strict Prohibition": Suppose that a local ordinance makes it illegal to land planes at the airport because they impose an externality on the developer. Then no planes will fly. The developer will build 30 houses and will have total profits of 900.

(c) "Lawyer's Paradise": Suppose that a law is passed that makes the airport liable for all damages to the developer's property values. Since the developer's profits are $60Y - Y^2 - XY$ and his profits would be $60Y - Y^2$ if no planes were flown, the total amount of damages awarded to the developer will be XY. Therefore if the airport flies X planes and the developer builds Y houses, then the airport's profits after it has paid damages will be $48X - X^2 - XY$. The developer's profits including the amount he receives in payment of damages will be $60Y - Y^2 - XY + XY = 60Y - Y^2$. To maximize his net profits, the developer will choose to build 30 houses no matter how many planes are flown. To maximize its profits, net of damages, the airport will choose to land 9 planes. Total profits of the developer will be 900 and total profits of the airport will be 81. The sum of their profits will be 981.

Calculus **35.9 (1)** This problem continues the story of the airport and the developer from the previous problem.

(a) "Merger": Suppose that the housing developer purchases the airport. What is the profit function for the new joint entity? $48X - X^2 + 60Y - Y^2 - XY$

To maximize joint profits, it should build 24 houses and let 12 planes land. Combined profit is now 1008. Explain why each of the institutional rules proposed in the previous problem fails to achieve an efficient outcome and hence has lower combined profits. In (a), the airport does not bear the cost of damages from landing planes. In (b), planes

are prohibited to land. The source of the externality is eliminated along with the benefit of airplanes. In (c), the airport bears the full cost of damages from landing but the developer bears none of the cost. Therefore the developer overbuilds.

(b) "Dealing": Suppose that the airport and the developer remain independent. If the original situation was one of "free to choose," could the developer increase his net profits by bribing the airport to cut back one flight per day if the developer has to pay for all of the airport's lost profits? Yes. The developer decides to get the airport to reduce its flights by paying for all lost profits coming from the reduction of flights. To maximize his own net profits, how many flights per day should he pay the airport to eliminate? 12.

35.10 (1) Every morning, 6,000 commuters must travel from East Potato to West Potato. Commuters all try to minimize the time it takes to get to work. There are two ways to make the trip. One way is to drive straight across town, through the heart of Middle Potato. The other way is to take the Beltline Freeway that circles the Potatoes. The Beltline Freeway is entirely uncongested, but the drive is roundabout and it takes 45 minutes to get from East Potato to West Potato by this means. The road through Middle Potato is much shorter, and if it were un-congested, it would take only 20 minutes to travel from East Potato to West Potato by this means. But this road can get congested. In fact, if the number of commuters who use this road is N, then the number of minutes that it takes to drive from East Potato to West Potato through Middle Potato is $20 + N/100$.

(a) Assuming that no tolls are charged for using either road, in equilibrium how many commuters will use the road through Middle Potato? 2,500. What will be the total number of person-minutes per day spent by commuters travelling from East Potato to West Potato? $45 \times 6,000 = 270,000$.

(b) Suppose that a social planner controlled access to the road through Middle Potato and set the number of persons permitted to travel this way so as to minimize the total number of person-minutes per day spent by commuters travelling from East Potato to West Potato. Write an expression for the total number of person-minutes per day spent by commuters travelling from East Potato to West Potato as a function of

the number N of commuters permitted to travel on the Middle Potato road. $N(20 + \frac{N}{100}) + (6,000 - N)45$. How many commuters per day would the social planner allow to use the road through Middle Potato? 1,250. In this case, how long would it take commuters who drove through Middle Potato to get to work? 32.5 minutes. What would be the total number of person-minutes per day spent by commuters travelling from East Potato to West Potato?

$$1,250 \times 32.5 + 4,750 \times 45 = 254,375$$

(c) Suppose that commuters value time saved from commuting at $\$w$ per minute and that the Greater Potato metropolitan government charges a toll for using the Middle Potato road and divides the revenue from this toll equally among all 6,000 commuters. If the government chooses the toll in such a way as to minimize the total amount of time that people spend commuting from East Potato to West Potato, how high should it set the toll? $\$12.5w$. How much revenue will it collect per day from this toll? $\$15,625w$. Show that with this policy every commuter is better off than he or she was without the tolls and evaluate the gain per consumer in dollars. Before the toll was in place, all commuters spent 45 minutes travelling to work. With the toll in place, commuters who travel on the Beltline still spend 45 minutes travelling to work and commuters who travel through Middle Potato are indifferent between spending 45 minutes travelling on the Beltline and paying the toll to go through Middle Potato. Thus nobody would be worse off even if toll revenue were wasted. But everybody gets back about $\$2.6w$ per day from the toll revenue, so all are better off.

35.11 (2) Suppose that the Greater Potato metropolitan government rejects the idea of imposing traffic tolls and decides instead to rebuild the Middle Potato highway so as to double its capacity. With the doubled capacity, the amount of time it takes to travel from East Potato to West Potato on the Middle Potato highway is given by $20 + N/200$, where N is the number of commuters who use the Middle Potato highway. In the new equilibrium, with expanded capacity and no tolls, how many commuters will use the Middle Potato highway? 5,000 How long will it take users of the Middle Potato highway to get to work? 45 minutes How many person-minutes of commuting time will be saved by expanding the capacity of the Middle Potato highway? 0 Do you think people will think that this capacity expansion will be a good use of their tax dollars? No.

Chapter 36

NAME_____

Information Technology

Introduction. We all recognize that information technology has revolutionized the way we produce and consume. Some think that it is necessary to have a "new economics" to understand this New Economy. We think not. The economic tools that you have learned in this course can offer very powerful insights into the economics of information technology, as we illustrate in this set of problems.

36.1 (2) Bill Barriers, the president of MightySoft software company is about to introduce a new computer operating system called DoorKnobs. Because it is easier to swap files with people who have the same operating system, the amount people are willing to pay to have DoorKnobs on their computers is greater the larger they believe DoorKnobs's market share to be.

The *perceived market share* for DoorKnobs is the fraction of all computers that the public *believes* is using DoorKnobs. When the price of DoorKnobs is p, then its *actual market share* is the fraction of all computer owners that would be willing to pay at least $\$p$ to have DoorKnobs installed on their computers. Market researchers have discovered that if DoorKnobs's perceived market share is s and the price of DoorKnobs is $\$p$, then its actual market share will be x, where x is related to the price p and perceived market share s by the formula

$$p = 256s(1 - x). \tag{1}$$

In the short run, MightySoft can influence the perceived market share of DoorKnobs by publicity, advertising, giving liquor and gifts to friendly journalists, and giving away copies in conspicuous ways. In the long run, the truth will emerge, and DoorKnobs's perceived market share s must equal its actual market share x.

(a) If the perceived market share is s, then the demand curve for DoorKnobs is given by Equation 1. On the graph below, draw the demand curve relating price to actual market share in the case in which DoorKnobs's perceived market share is $s = 1/2$. Label this curve $s = 1/2$.

(b) On the demand curve that you just drew with $s = 1/2$, mark a red dot on the point at which the actual market share of DoorKnobs is $1/2$. (This is the point on the demand curve directly above $x = 1/2$.) What is the price at which half of the computer owners actually want to buy DoorKnobs, given that everybody believes that half of all computer owners want to buy DoorKnobs? $64

(c) On the same graph, draw and label a separate demand curve for the case where DoorKnobs's perceived market share s takes on each of the following values: s =1/8, 1/4, 3/4, 7/8, 1.

(d) On the demand curve for a perceived market share of $s = 1/4$, put a red dot on the point at which the actual market share of DoorKnobs is 1/4. (This is the point on this demand curve directly above $x = 1/4$.) If the perceived market share of DoorKnobs is 1/4, at what price is the actual market share of DoorKnobs also 1/4? $48

(e) Just as you did for $s = 1/2$ and $s = 1/4$, make red marks on the demand curves corresponding to $s = 1/8$, 3/4, 7/8, and 1, showing the price at which the actual market share is s, given that the perceived market share is s.

(f) Let us now draw the long-run demand curve for DoorKnobs, where we assume that computer owners' perceived market shares s are the same as the actual market shares x. If this is the case, it must be that $s = x$, so the demand curve is given by $p = 256x(1 - x)$. On the graph above, plot a few points on this curve and sketch in an approximation of the curve. (Hint: Note that the curve you draw must go through all the red points that you have already plotted.)

(g) Suppose that MightySoft sets a price of \$48 for DoorKnobs and sticks with that price. There are three different perceived market shares such that the fraction of consumers who would actually want to buy DoorKnobs for \$48 is equal to the perceived market share. One such perceived

market share is 0. What are the other two possibilities? $s = 1/4$ and $s = 3/4$

(h) Suppose that by using its advertising and media influence, MightySoft can temporarily set its perceived market share at any number between 0 and 1. If DoorKnobs's perceived market share is x and if MightySoft charges a price $p = 256x(1-x)$, the actual market fraction will also be x and the earlier perceptions will be reinforced and maintained. Assuming that MightySoft chooses a perceived market share x and a price that makes the actual market share equal to the perceived market share, what market share x should MightySoft choose in order to maximize its revenue and what price should it charge in order to maintain this market share? (Hint: Revenue is $px = 256x^2(1-x)$.) Use calculus and show your work. $x = 2/3$. The first-order condition is $\frac{d}{dx}256(x^2 - x^3) = 0$. This implies $2x = 3x^2$, which implies that $x = 2/3$ or $x = 0$. The second order condition is satisfied only when $x = 2/3$. Price should be $\$256 \times 1/3 \times 2/3 = \$56.89$.

36.2 (1) Suppose that demand for DoorKnobs is as given in the previous problem, and assume that the perceived market share in any period is equal to the actual market share in the previous period. Then where x_t is the actual market share in period t, the equation $p = 256x_{t-1}(1-x_t)$ is satisfied. Rearranging this equation, we find that $x_t = 1 - (p/256x_{t-1})$ whenever $p/256x_{t-1} \leq 1$. If $p/256x_{t-1} \leq 0$, then $x_t = 0$. With this formula, if we know actual market share for any time period, we can calculate market share for the next period.

Let us assume that DoorKnobs sets the price at $p = \$32$ and never changes this price. (To answer the following questions, you will find a calculator useful.)

(a) If the actual market share in the first period was 1/2, find the actual market share in the second period .75 , the third period .833. Write down the actual market shares for the next few periods .8529, 8534. Do they seem to be approaching a limit? If so, what? .853553.

(b) Notice that when price is held constant at p, if DoorKnobs's market share converges to a constant $\bar{x}$, it must be that $\bar{x} = 1 - (p/256\bar{x})$. Solve this equation for $\bar{x}$ in the case where $p = \$32$. What do you make of the fact that there are two solutions? This equation implies $x^2 - x + 1/8 = 0$. Solutions are $x = 0.85355$ and $x = 0.14645$. Both are equilibrium market shares with a price of \$32.

36.3 (1) A group of 13 consumers are considering whether to connect to a new computer network. Consumer 1 has an *initial value* of \$1 for hooking up to the network, consumer 2 has an *initial value* of \$2, consumer 3 has an i*initial value* of \$3, and so on up to consumer 13. Each consumer's willingness to pay to connect to the network depends on the total number of persons who are connected to it. In fact, for each i, consumer i's willingness to pay to connect to the network is i *times* the total number of persons connected. Thus if 5 people are connected to the network, consumer 1's willingness to pay is \$5, consumer 2's willingness to pay is \$10 and so on.

(a) What is the highest price at which 9 customers could hook up to the market and all of them either make a profit or break even? \$45

(b) Suppose that the industry that supplies the computer network is competitive and that the cost of hooking up each consumer to the network is \$45. Suppose that consumers are very conservative and nobody will sign up for the network unless her buyer value will be at least as high as the price she paid as soon as she signs up. How many people will sign up if the price is \$45? 0

(c) Suppose that the government offers to subsidize "pioneer users" of the system. The first two users are allowed to connect for \$10 each. After the first two users are hooked up, the government allows the next two to connect for \$25. After that, everyone who signs up will have to pay the full cost of \$45. Assume that users remain so conservative that will sign up only if their buyer values will be at least equal to the price they are charged when they connect. With the subsidy in place, how many consumers in toto will sign up for the network? 9

36.4 (2) Professor Kremepuff has written a new, highly simplified economics text, *Microeconomics for the Muddleheaded*, which will be published by East Frisian Press. The first edition of this book will be in print for two years, at which time it will be replaced by a new edition. East

Frisian Press has already made all its fixed cost investments in the book and must pay a constant marginal cost of $\$c$ for each copy that it sells.

Let p_1 be the price charged for new copies sold in the first year of publication and let p_2 be the price charged for new copies sold in the second year of publication. The publisher and the students who buy the book are aware that there will be an active market for used copies of *Microeconomics for the Muddleheaded* one year after publication and that used copies of the first edition will have zero resale value two years after publication. At the end of the first year of publication, students can resell their used textbooks to bookstores for 40% of the second-year price, p_2.

The net cost to a student of buying the book in the first year, using it for class, and reselling it at the end of the year is $p_1 - 0.4p_2$. The number of copies demanded in the first year of publication is given by a demand function, $q_1 = D_1(p_1 - 0.4p_2)$.

Some of the students who use the book in the first year of publication will want to keep their copies for future reference, and some will damage their books so that they cannot be resold. The cost of keeping one's old copy or of damaging it is the resale price $0.4p_2$. The number of books that are either damaged or kept for reference is given by a "keepers" demand function, $D^k(0.4p_2)$. It follows that the number of used copies available at the end of the first year will be $D_1(p_1 - 0.4p_2) - D^k(0.4p_2)$.

Students who buy *Microeconomics for the Muddleheaded* in the second year of publication will not be able to resell their used copies, since a new edition will then be available. These students can, however, buy either a new copy or a used copy of the book. For simplicity of calculations, let us assume that students are indifferent between buying a new copy or a used copy and that used copies cost the same as new copies in the book store. (The results would be the same if students preferred new to used copies, but bookstores priced used copies so that students were indifferent between buying new and used copies.) The total number of copies, new and used, that are purchased in the second year of publication is $q_2 = D_2(p_2)$.

(a) Write an expression for the number of new copies that East Frisian Press can sell in the second year after publication if it sets prices p_1 in year 1 and p_2 in year 2. $D_2(p_2) - D_1(p_1 - 0.4p_2) + D^k(0.4p_2)$.

(b) Write an expression for the total number of new copies of *Microeconomics for the Muddleheaded* that East Frisian can sell over two years at prices p_1 and p_2 in years 1 and 2. $D_1(p_1 - 0.4p_2) + D_2(p_2) - D_1(p_1 - 0.4p_2) + D^k(0.4p_2) = D_2(p_2) + D^k(0.4p_2)$.

(c) Would the total number of copies sold over two years increase, decrease, or remain constant if p_1 were increased and p_2 remained constant?

It would remain constant.

(d) Write an expression for the total revenue that East Frisian Press will receive over the next two years if it sets prices p_1 and p_2. $p_1 D_1(p_1 - 0.4p_2) + p_2(D_2(p_2) - D_1(p_1 - 0.4p_2) + D^k(0.4p_2)) = (p_1 - p_2)D_1(p_1 - 0.4p_2) + p_2(D_2(p_2) + D^k(0.4p_2))$.

(e) To maximize its total profits over the next two years, East Frisian must maximize the difference between its total revenue and its variable costs. Show that this difference can be written as

$$(p_1 - p_2)D_1(p_1 - 0.4p_2) + (p_2 - c)\left[D_2(p_2) + D^k(0.4p_2)\right].$$

Variable cost is $c(D_2(p_2) + D^k(0.4p_2))$.

Subtract this from the previous answer.

(f) Suppose that East Frisian has decided that it must charge the same price for the first edition in both years that it is sold. Thus it must set $p = p_1 = p_2$. Write an expression for East Frisian's revenue net of variable costs over the next two years as a function of p.

$(p - c)(D_2(p) + D^k(0.4p))$

36.5 (2) Suppose that East Frisian Press, discussed in the previous problem, has a constant marginal cost of $c = \$10$ for each copy of *Microeconomics for the Muddleheaded* that it sells and let the demand functions be

$$D_1(p_1 - 0.4p_2) = 100\,(90 - p_1 + 0.4p_2)$$
$$D_2(p_2) = 100(90 - p_2).$$

The number of books that people either damage or keep for reference after the first year is

$$D^k(0.4p_2) = 100(90 - 0.8p_2).$$

(This assumption is consistent with the assumption that everyone's willingness to pay for keeping the book is half as great as her willingness to pay to have the book while she is taking the course.) Assume that East Frisian Press is determined to charge the same price in both years, so that $p_1 = p_2 = p$.

(a) If East Frisian Press charges the same price p for *Microeconomics for the Muddleheaded* in the first and second years, show that the total sales of new copies over the two years are equal to

$$18,000 - 180p.$$

Total sales are $D_2(p) + D^k(.4p_2) = 100(90 - p) + 100(90 - .8p)) = 18,000 - 180p$

(b) Write an expression for East Frisian's total revenue, net of variable costs, over the first two years as a function of the price p. $(p - 10)(18,000 - 180p) = 19,800p - 180p^2 - 180,000$

(c) Solve for the price p that maximizes its total revenue net of variable costs over the first two years. $p = \$55$. At this price, the net cost to students in the first year of buying the text and reselling it is $33. The total number of copies sold in the first year will be 5,700. The total number of copies that are resold as used books is 1,100. The total number of copies purchased by students in the second year will be 3,500. (Remember students in the second year know that they cannot resell the book, so they have to pay the full price p for using it.) The total number of new copies purchased by students in the second year will be 2,400. Total revenue net of variable costs over the two years will be $364,500.

36.6 (2) East Frisian Press is trying to decide whether it would be profitable to produce a new edition of *Microeconomics for the Muddleheaded* after one year rather than after two years. If it produces a new edition after one year, it will destroy the used book market and all copies that are purchased will be new copies. In this case, the number of new copies that will be demanded in each of the two years will be $100(90 - p)$, where p is the price charged. The variable cost of each copy sold remains \$10.

(a) Write an expression for the total number of copies sold over the course of two years if the price is p in each year 200(90-p) . Also, write an expression for total revenue net of variable costs as a function of p.

$200(p - 10)(90 - p)$.

(b) Find the price that maximizes total revenue net of variable costs.

$50.

(c) The total number of new books sold in the first year would be 4,000 , and the total number of books sold in the second year would be 4,000.

(d) East Frisian's total revenue net of variable costs, if it markets a new edition after one year, will be $320,000.

(e) Would it be more profitable for East Frisian Press to produce a new edition after one year or after two years? After two years. Which would be better for students? (Hint: The answer is not the same for all students.) After two years is better for students who take the course in the first year of publication and plan to sell. After one year is better for the other students.

36.7 (3) Suppose that East Frisian Press publishes a new edition only after two years and that demands and costs are as in the previous problems. Suppose that it sets two different prices p_1 and p_2 in the two periods.

(a) Write an expression for the total number of new copies sold at prices p_1 and p_2 and show that this number depends on p_2 but not on p_1.

$$100\left((70 - p_1 + .4p_2) + (140 - 1.8p_2) - (70 - p_1 + .4p_2)\right) = 100(140 - 1.8p_2)$$

(b) Show that at prices p_1 and p_2, the difference between revenues and variable costs is equal to

$$100\left(90p_1 + 108p_2 + 1.4p_1p_2 - p_1^2 - 2.2p_2^2 - 1,800\right).$$

This difference is $100(p_1 - p_2)(90 - p_1 + .4p_2) + (p_2 - 10)(180 - 1.8p_2)$. Expand this expression.

(c) Calculate the prices p_1 and p_2 that maximize the difference between total revenue and variable costs and hence maximize profits. $p_1 = \$80$, $p_2 = \$50$

(d) If East Frisian Press chooses its profit-maximizing p_1 and p_2, compare the cost of using *Microeconomics for the Muddleheaded* for a student who buys the book when it is first published and resells it at the end of the first year with the cost for a student who buys the book at the beginning of the second year and then discards it. The former has a net cost of $\$80 - .4 \times 50 = \$60$ and the latter has a cost of $50.

36.8 (2) The Silicon Valley company Intoot produces checkwriting software. The program itself, Fasten, sells for $50 and includes a package of checks. Check refill packets for Fasten cost $20 to produce and Intoot sells the checks at cost. Suppose that a consumer purchases Fasten for $50 in period 1 and spends $20 on checks in each subsequent period. Assume for simplicity that the consumer uses the program for an infinite number of periods.

(a) If the interest rate is $r = .10$ per period, what is the present value of the stream of payments made by the consumer? (Hint: a stream of payments of x starting next period has a present value of x/r.) The total cost of ownership of Fasten is 50+20/.10 = $250.

(b) Fasten's competitor produces an equally effective product called Czechwriter. Czechwriter can do everything Fasten can do and vice versa except that Fasten cannot use check refill packets that are sold by anyone other than Fasten. Czechwriter also sells for $50 and sells its checks for $20 per period. A Fasten customer can switch to Czechwriter simply by purchasing the program. This means his switching costs are $50

(c) Fasten is contemplating raising the price of checks to $30 per period. If so, will its customers switch to Czechwriter? Explain. Yes, the present value of continuing to use Fasten is $300 while the costs of switching to Czechwriter are $250.

(d) Fasten contemplates raising the price of checks to \$22 per period. Will its customers switch? No. The present value of continuing to use Fasten are \$220 while the present value of switching to Czechwriter is \$250.

(e) At what price for checks will Fasten's customers just be indifferent to switching? (Hint: Let x be this amount. Compare the present value of staying with Fasten with the present value of switching to Czechwriter.)

Solve the equation $x/.10 = 250$ to find $x = 25$.

(f) If it charges the highest price that it can without making its customers switch, what profit does Fasten make on checks from each of its customers per period? \$5. What is the present value of the profit per customer that Fasten gets if it sets the price of checks equal to the number determined in the last question? $PV = 5/.10 = 50$. How does this compare to the customer switching cost? It is the same.

(g) Suppose now that the cost of switching also involves several hours of data conversion that the consumer values at \$100. The total cost of switching is the cost of the new program plus the data conversion cost which is \$150.

(h) Making allowances for the cost of data conversion, what is the highest price that Intoot can charge for its checks? Solve $x/.10 = 250 + 100$ for $x = 35$. What is the present value of profit from this price? \$150. How does this compare to total switching costs? It is the same.

(i) Suppose that someone writes a computer program that eliminates the cost of converting data and makes this program available for free. Suppose that Intoot continues to price its check refill packages at \$25. A new customer is contemplating buying Fasten at a price of \$50 and paying \$25 per period for checks, versus paying \$50 for Czechwriter and paying \$20 for checks. If the functionality of the software is identical, which will the consumer buy? Czechwriter.

(j) Intoot decides to distribute a coupon that offers a discount of \$50 off of the regular purchase price. What price would it have to set to make consumers indifferent between purchasing Fasten and Czechwriter?

Solve $50 + 25/.10 - d = 50 + 20/.10$ to find that the discount should be \$50.

(k) Suppose that consumers are shortsighted and only look at the cost of the software itself, neglecting the cost of the checks. Which program would they buy if Intoot offered this coupon? Fasten. How might Czechwriter respond to the Fasten offer? Issue its own coupon for \$50 and raise the price of its checks to \$35.

36.9 (2) Sol Microsystems has recently invented a new language, Guava, which runs on a proprietary chip, the Guavachip. The chip can only be used to run Guava, and Guava can only run on the Guavachip. Sol estimates that if it sells the chip for a price p_c and the language for a price p_g, the demand for the chip-language system will be

$$x = 120 - (p_c + p_g).$$

(a) Sol initially sets up two independent subsidiaries, one to produce the chip and one to produce the language. Each of the subsidiaries will price its product so as to maximize its profits, while assuming that a change in its own price will not affect the pricing decision of the other subsidiary. Assume that marginal costs are negligible for each company. If the price of the language is set at p_g, the chip company's profit function (neglecting fixed costs) is $[120 - p_c - p_g]p_c$.

(b) Differentiate this profit function with respect to p_c and set the result equal to zero to calculate the optimal choice of p_c as a function of p_g.

$p_c = 120 - 2p_g$.

(c) Now consider the language subsidiary's pricing decision. The optimal choice of p_g as a function of p_c is $p_g = 120 - 2p_c$.

(d) Solving these two equations in two unknowns, we find that $p_c =$ 40 and $p_g =$ 40, so that $p_c + p_g =$ 80.

(e) Sol Microsystems decides that the independent subsidiary system is cumbersome, so it sets up Guava Computing which sells a bundled system consisting of the chip and the language. Let p be the price of the bundle. Guava Computing's profit function is $[120 - p]p$

(f) Differentiate this profit with respect to p and set the resulting expression to zero to determine $p =$ 60

(g) Compare the prices charged by the integrated system and the separate subsidiaries. Which is lower? Integrated system. Which is better for consumers? Integrated system. Which makes more profit? Integrated system.

36.10 (2) South Belgium Press produces the academic journal *Nanoeconomics*, which has a loyal following among short microeconomists, and *Gigaeconomics*, a journal for tall macroeconomists. It offers a license for the electronic version of each journal to university libraries at a subscription cost per journal of $1,000 per year. The 200 top universities all subscribe to both journals, each paying $2,000 per year to South Belgium. By revealed preference, their willingness to pay for each journal is at least $1,000.

(a) In an attempt to lower costs, universities decide to form pairs, with one member of each pair subscribing to *Nanoeconomics* and one member of each pair subscribing to *Gigaeconomics*. They agree to use interlibrary loan to share the other journal. Since the copies are electronic, there is no incremental cost to doing this. Under this pairing scheme, how many subscriptions of each journal will South Belgium sell? 100

(b) In order to stem the revenue hemorrhage, South Belgium raises the price of each journal. Assuming library preferences and budgets haven't changed, how high can they set this price? They can raise the price to $2,000, since libraries have already indicated that they are willing to pay this much for the pair of journals.

(c) How does library expenditure and South Belgium's revenue compare to those of the previous regime? They remain the same.

(d) If there were a cost of interlibrary loan, how would your answer change? Assuming they still bought both journals, libraries would be worse off since they would have to pay the transactions cost for interlibrary loan.

36.11 (2) Henrietta Fowler publishes *Metropolitan Chicken*, a magazine devoted to raising poultry in urban environments. She has conducted extensive surveys from potential readers and potential advertisers. The amount that advertisers are willing to pay for an ad in *Metropolitan Chicken* is proportional to the number of subscribers. She estimates that if she charges advertisers a rate of P_A cents per-subscriber for each ad, she could attract $Q_A = 80 - P_A$ advertisements. Suppose that the cost of printing and distributing an advertisement is negligible. The number of subscribers to *Metropolitan Chicken* is determined by the demand equation $Q_S = 4,000 - P_S$ where P_S is the subscription price measured in cents. The cost of printing, handling, and mailing a copy of *Metropolitan Chicken* is 400 cents per subscriber.

(a) Write an expression for *Metropolitan Chicken*'s total revenue from advertising as a function of the variables Q_A and Q_S. (Remember that total revenue is revenue per reader *times* the number of readers.) $(80 - Q_A)Q_AQ_S$ Solve for the number of ads and the price per-subscriber that maximizes her advertising revenue. $Q_A = 40$, $P_A = 40$ cents per subscriber. Does this answer depend on the number of subscriptions sold? No. How much extra advertising revenue (in cents) would Henrietta get if she added one more subscriber? 1600 cents

(b) Write an expression for *Metropolitan Chicken*'s net revenue (in cents) from subscriptions (total subscription revenue minus costs of printing, handling, and mailing to subscribers) as a function of Q_S. $(4,000 - Q_S)Q_S - 400Q_S = 3,600Q_S - Q_S^2$. If Henrietta were to choose Q_S to maximize subscription revenue, how many subscriptions would she sell and at what price? 1800 subscriptions for $22.00 each

(c) Write an expression for Henrietta's total profit from advertising and subscriptions as a function of the two variables Q_A and Q_S. $(80 - Q_A)Q_AQ_S + 3,600Q_S - Q_S^2$. What quantities Q_A and Q_S should she choose in order to maximize profits? $Q_A = 40$, $Q_S = 2600$, $P_A = 40$ cents, $P_S = \$14.00$

Calculus **36.12 (2)** Sam Hitchem owns the dance hall in Lonely Pine, Wisconsin. He holds a dance every week. This dance hall is one of the few suitable locations in the county for unattached men and women to meet each other. Sam charges admission fee P_m for men and admission fee P_w for women. At any price, individuals are more likely to come if they expect to see more persons of the opposite sex. Sam has discovered that if he charges price P_m to men and if they expect Q_w women to be in attendance, then the number of men who will attend is

$$Q_m = 40 - P_m + \frac{3}{4}Q_w.$$

If he charges P_w to women and if they expect Q_m men to be present, the number of women who will attend is

$$Q_w = 40 - P_w + \frac{1}{4}Q_m.$$

(a) Rearrange the two demand equations above to produce "inverse demand functions" in which prices are expressed in terms of quantities. We have $P_m = 40 - Q_m + \frac{3}{4}Q_w$ and $P_w = 40 - Q_w + \frac{1}{4}Q_m$.

(b) Sam chooses prices so as to maximize his total revenue $P_mQ_m + P_wQ_w$. Using your answer to the previous question, you can write Sam's revenue simply as a function of the two variables Q_m and Q_w. Write this expression here. $P_mQ_m + P_wQ_w = (40 - Q_m + \frac{3}{4}Q_w)Q_m + (40 - Q_w + \frac{1}{4}Q_m)Q_w = 40(Q_m + Q_w) - Q_m^2 - Q_w^2 + Q_mQ_w$

(c) Sam chooses admission prices so as to maximize his revenue. We can solve for these admission prices by first finding the values of Q_m and Q_w that maximize revenue and then substituting these quantities into the demand functions to find the prices at which these quantities will be demanded. You have found an expression for revenue as a function of Q_m and Q_w. Set the partial derivatives of this expression with respect to Q_m

and Q_w equal to zero and solve these two equations for the two variables Q_m and Q_w. Sam will maximize his revenue if the number of men who attend is 40 and the number of women who attend is 40. Therefore he will maximize his profits if he sets the admission price for men equal to $30 and the admission price for women equal to $10.

Chapter 37

NAME_____

Public Goods

Introduction. In previous chapters we studied selfish consumers consuming private goods. A unit of private goods consumed by one person cannot be simultaneously consumed by another. If you eat a ham sandwich, I cannot eat the same ham sandwich. (Of course we can both eat ham sandwiches, but we must eat different ones.) Public goods are a different matter. They can be jointly consumed. You and I can both enjoy looking at a beautiful garden or watching fireworks at the same time. The conditions for efficient allocation of public goods are different from those for private goods. With private goods, efficiency demands that if you and I both consume ham sandwiches and bananas, then our marginal rates of substitution must be equal. If our tastes differ, however, we may consume different amounts of the two private goods.

If you and I live in the same town, then when the local fireworks show is held, there will be the same amount of fireworks for each of us. Efficiency does not require that my marginal rate of substitution between fireworks and ham sandwiches equal yours. Instead, efficiency requires that the *sum* of the amount that viewers are willing to pay for a marginal increase in the amount of fireworks equal the marginal cost of fireworks. This means that the sum of the absolute values of viewers' marginal rates of substitution between fireworks and private goods must equal the marginal cost of public goods in terms of private goods.

Example: A quiet midwestern town has 5,000 people, all of whom are interested only in private consumption and in the quality of the city streets. The utility function of person i is $U(X_i, G) = X_i + A_i G - B_i G^2$, where X_i is the amount of money that person i has to spend on private goods and G is the amount of money that the town spends on fixing its streets. To find the Pareto optimal amount of money for this town to spend on fixing its streets, we must set the sum of the absolute values of marginal rates of substitution between public and private goods equal to the relative prices of public and private goods. In this example we measure both goods in dollar values, so the price ratio is 1. The absolute value of person i's marginal rate of substitution between public goods and private goods is the ratio of the marginal utility of public goods to the marginal utility of private goods. The marginal utility of private goods is 1 and the marginal utility of public goods for person i is $A_i - B_i G$. Therefore the absolute value of person i's MRS is $A_i - B_i G$ and the sum of absolute values of marginal rates of substitution is $\sum_i (A_i - B_i G) = \sum_i A_i - (\sum B_i)G$. Therefore Pareto efficiency requires that $\sum_i A_i - (\sum_i B_i)G = 1$. Solving this for G, we have $G = (\sum_i A_i - 1)/\sum_i B_i$.

37.1 (0) Muskrat, Ontario, has 1,000 people. Citizens of Muskrat consume only one private good, Labatt's ale. There is one public good, the town skating rink. Although they may differ in other respects, inhabitants

have the same utility function. This function is $U(X_i, G) = X_i - 100/G$, where X_i is the number of bottles of Labatt's consumed by citizen i and G is the size of the town skating rink, measured in square meters. The price of Labatt's ale is \$1 per bottle and the price of the skating rink is \$10 per square meter. Everyone who lives in Muskrat has an income of \$1,000 per year.

(a) Write down an expression for the absolute value of the marginal rate of substitution between skating rink and Labatt's ale for a typical citizen. $100/G^2$ What is the marginal cost of an extra square meter of skating rink (measured in terms of Labatt's ale)? 10.

(b) Since there are 1,000 people in town, all with the same marginal rate of substitution, you should now be able to write an equation that states the condition that the sum of absolute values of marginal rates of substitution equals marginal cost. Write this equation and solve it for the Pareto efficient amount of G. $1,000\frac{100}{G^2} = 10$. So $G = 100$.

(c) Suppose that everyone in town pays an equal share of the cost of the skating rink. Total expenditure by the town on its skating rink will be \$10G. Then the tax bill paid by an individual citizen to pay for the skating rink is \$$10G/1,000 =$ \$$G/100$. Every year the citizens of Muskrat vote on how big the skating rink should be. Citizens realize that they will have to pay their share of the cost of the skating rink. Knowing this, a citizen realizes that if the size of the skating rink is G, then the amount of Labatt's ale that he will be able to afford is $1,000 - G/100$.

(d) Therefore we can write a voter's budget constraint as $X_i + G/100 = 1,000$. In order to decide how big a skating rink to vote for, a voter simply solves for the combination of X_i and G that maximizes his utility subject to his budget constraint and votes for that amount of G. How much G is that in our example? $G = 100$.

(e) If the town supplies a skating rink that is the size demanded by the voters will it be larger than, smaller than, or the same size as the Pareto optimal rink? The same.

(f) Suppose that the Ontario cultural commission decides to promote Canadian culture by subsidizing local skating rinks. The provincial government will pay 50% of the cost of skating rinks in all towns. The costs of this subsidy will be shared by all citizens of the province of Ontario. There are hundreds of towns like Muskrat in Ontario. It is true that to pay for this subsidy, taxes paid to the provincial government will have to be increased. But there are hundreds of towns from which this tax

is collected, so that the effect of an increase in expenditures in Muskrat on the taxes its citizens have to pay to the state can be safely neglected. Now, approximately how large a skating rink would citizens of Muskrat vote for? $G = 100\sqrt{2}$. (Hint: Rewrite the budget constraint for individuals observing that local taxes will be only half as large as before and the cost of increasing the size of the rink only half as much as before. Then solve for the utility-maximizing combination.)

(g) Does this subsidy promote economic efficiency? No.

37.2 (0) Ten people have dinner together at an expensive restaurant and agree that the total bill will be divided equally among them.

(a) What is the additional cost to any one of them of ordering an appetizer that costs \$20? \$2.

(b) Explain why this may be an inefficient system. Each pays less than full cost of own meal, so all overindulge.

37.3 (0) Cowflop, Wisconsin, has 1,000 people. Every year they have a fireworks show on the Fourth of July. The citizens are interested in only two things—drinking milk and watching fireworks. Fireworks cost 1 gallon of milk per unit. People in Cowflop are all pretty much the same. In fact, they have identical utility functions. The utility function of each citizen i is $U_i(x_i, g) = x_i + \sqrt{g}/20$, where x_i is the number of gallons of milk per year consumed by citizen i and g is the number of units of fireworks exploded in the town's Fourth of July extravaganza. (Private use of fireworks is outlawed.)

(a) Solve for the absolute value of each citizen's marginal rate of substitution between fireworks and milk. $1/(40\sqrt{g})$.

(b) Find the Pareto optimal amount of fireworks for Cowflop. 625.

37.4 (0) Bob and Ray are two hungry economics majors who are sharing an apartment for the year. In a flea market they spot a 25-year-old sofa that would look great in their living room.

Bob's utility function is $u_B(S, M_B) = (1 + S)M_B$, and Ray's utility function is $u_R(S, M_R) = (2+S)M_R$. In these expressions M_B and M_R are the amounts of money that Bob and Ray have to spend on other goods, $S = 1$ if they get the sofa, and $S = 0$ if they don't get the sofa. Bob has W_B dollars to spend, and Ray has W_R dollars.

(a) What is Bob's reservation price for the sofa? Solve $W_B = 2(W_B - p_B)$ to get $p_B = W_B/2$.

(b) What is Ray's reservation price for the sofa? Solve $2W_R = 3(W_R - p_R)$, which gives $p_R = W_R/3$.

(c) If Bob has a total of $W_B = \$100$ and Ray has a total of $W_R = \$75$ to spend on sofas and other stuff, they could buy the sofa and have a Pareto improvement over not buying it so long as the cost of the sofa is no greater than \$75.

37.5 (0) Bonnie and Clyde are business partners. Whenever they work, they have to work together. Their only source of income is profit from their partnership. Their total profit per year is $50H$, where H is the number of hours that they work per year. Since they must work together, they both must work the same number of hours, so the variable "hours of labor" is like a public "bad" for the two person community consisting of Bonnie and Clyde. Bonnie's utility function is $U_B(C_B, H) = C_B - .02H^2$ and Clyde's utility function is $U_C(C_C, H) = C_C - .005H^2$, where C_B and C_C are the annual amounts of money spent on consumption for Bonnie and for Clyde.

(a) If the number of hours that they both work is H, what is the ratio of Bonnie's marginal utility of hours of work to her marginal utility of private goods? $-.04H$. What is the ratio of Clyde's marginal utility of hours of work to his marginal utility of private goods? $-.01H$.

(b) If Bonnie and Clyde are both working H hours, then the total amount of money that would be needed to compensate them both for having to work an extra hour is the sum of what is needed to compensate Bonnie and the amount that is needed to compensate Clyde. This amount is approximately equal to the sum of the absolute values of their marginal rates of substitution between work and money. Write an expression for this amount as a function of H. $.05H$. How much extra money will they make if they work an extra hour? \$50.

(c) Write an equation that can be solved for the Pareto optimal number of hours for Bonnie and Clyde to work. $.05H = 50$.

Find the Pareto optimal H. $H = 1,000$. (Hint: Notice that this model is formally the same as a model with one public good H and one private good, income.)

37.6 (0) Lucy and Melvin share an apartment. They spend some of their income on private goods like food and clothing that they consume separately and some of their income on public goods like the refrigerator, the household heating, and the rent, which they share. Lucy's utility function is $2X_L + G$ and Melvin's utility function is $X_M G$, where X_L and X_M are the amounts of money spent on private goods for Lucy and for Melvin and where G is the amount of money that they spend on public goods. Lucy and Melvin have a total of \$8,000 per year between them to spend on private goods for each of them and on public goods.

(a) What is the absolute value of Lucy's marginal rate of substitution between public and private goods? $1/2$. What is the absolute value of Melvin's? X_M/G.

(b) Write an equation that expresses the condition for provision of the Pareto efficient quantity of the public good. $1/2 + X_M/G = 1$.

(c) Suppose that Melvin and Lucy each spend \$2,000 on private goods for themselves and they spend the remaining \$4,000 on public goods. Is this a Pareto efficient outcome? Yes.

(d) Give an example of another Pareto optimal outcome in which Melvin gets more than \$2,000 and Lucy gets less than \$2,000 worth of private goods. One example: Melvin gets \$2,500; Lucy gets \$500 and $G = \$5,000$.

(e) Give an example of another Pareto optimum in which Lucy gets more than \$2,000. Lucy gets \$5,000; Melvin gets \$1,000 and $G = \$2,000$.

(f) Describe the set of Pareto optimal allocations. The allocations that satisfy the equations $X_M/G = 1/2$ and $X_L + X_M + G = \$8,000$.

(g) The Pareto optima that treat Lucy better and Melvin worse will have (more of, less of, the same amount of) public good as the Pareto optimum that treats them equally. Less of.

37.7 (0) This problem is set in a fanciful location, but it deals with a very practical issue that concerns residents of this earth. The question is, "In a Democracy, when can we expect that a majority of citizens will favor having the government supply pure private goods publicly?" This problem also deals with the efficiency issues raised by public provision of private goods. We leave it to you to see whether you can think of important examples of publicly supplied private goods in modern Western economies.

On the planet Jumpo there are two goods, aerobics lessons and bread. The citizens all have Cobb-Douglas utility functions of the form $U_i(A_i, B_i) = A_i^{1/2} B_i^{1/2}$, where A_i and B_i are i's consumptions of aerobics lessons and bread. Although tastes are all the same, there are two different income groups, the rich and the poor. Each rich creature on Jumpo has an income of 100 fondas and every poor creature has an income of 50 fondas (the currency unit on Jumpo). There are two million poor creatures and one million rich creatures on Jumpo. Bread is sold in the usual way, but aerobics lessons are provided by the state despite the fact that they are private goods. The state gives the same amount of aerobics lessons to every creature on Jumpo. The price of bread is 1 fonda per loaf. The cost to the state of aerobics lessons is 2 fondas per lesson. This cost of the state-provided lessons is paid for by taxes collected from the citizens of Jumpo. The government has no other expenses than providing aerobics lessons and collects no more or less taxes than the amount needed to pay for them. Jumpo is a democracy, and the amount of aerobics to be supplied will be determined by majority vote.

(a) Suppose that the cost of the aerobics lessons provided by the state is paid for by making every creature on Jumpo pay an equal amount of taxes. On planets, such as Jumpo, where every creature has exactly one head, such a tax is known as a "head tax." If every citizen of Jumpo gets 20 lessons, how much will be total government expenditures on lessons? 120 million fondas. How much taxes will every citizen have to pay? 40 fondas. If 20 lessons are given, how much will a rich creature have left to spend on bread after it has paid its taxes? 60 fondas. How much will a poor creature have left to spend on bread after it has paid its taxes? 10 fondas.

(b) More generally, when everybody pays the same amount of taxes, if x lessons are provided by the government to each creature, the total cost to the government is 6 million times x and the taxes that one creature has to pay is 2 times x.

(c) Since aerobics lessons are going to be publicly provided with everybody getting the same amount and nobody able to get more lessons from another source, each creature faces a choice problem that is formally the same as that faced by a consumer, i, who is trying to maximize a Cobb-Douglas utility function subject to the budget constraint $2A + B = I$, where I is its income. Explain why this is the case. If A lessons are provided, your taxes are $2A$ fondas. After taxes, you have $I - 2A$ fondas to spend on B.

(d) Suppose that the aerobics lessons are paid for by a head tax and all lessons are provided by the government in equal amounts to everyone. How many lessons would the rich people prefer to have supplied? 25. How many would the poor people prefer to have supplied? 12.5. (Hint: In each case you just have to solve for the Cobb-Douglas demand with an appropriate budget.)

(e) If the outcome is determined by majority rule, how many aerobics lessons will be provided? 12.5. How much bread will the rich get? 75. How much bread will the poor get? 25.

(f) Suppose that aerobics lessons are "privatized," so that no lessons are supplied publicly and no taxes are collected. Every creature is allowed to buy as many lessons as it likes and as much bread as it likes. Suppose that the price of bread stays at 1 fonda per unit and the price of lessons stays at 2 fondas per unit. How many aerobics lessons will the rich get? 25. How many will the poor get? 12.5. How much bread will the rich get? 50. How much bread will the poor get? 25.

(g) Suppose that aerobics lessons remain publicly supplied but are paid for by a proportional income tax. The tax rate is set so that tax revenue pays for the lessons. If A aerobics lessons are offered to each creature on Jumpo, the tax bill for a rich person will be $3A$ fondas and the tax bill for a poor person will be $1.5A$ fondas. If A lessons are given to each creature, show that total tax revenue collected will be the total cost of A lessons. There are 2,000,000 poor and 1,000,000 rich, total revenue is $2,000,000 \times 1.5A + 1,000,000 \times 3A = 6,000,000A$.

There are 3,000,000 people in all. If each gets A lessons and lessons cost 2 fondas, total cost is $6,000,000A$.

(h) With the proportional income tax scheme discussed above, what budget constraint would a rich person consider in deciding how many aerobics lessons to vote for? $3A + B = 100$. What is the relevant budget constraint for a poor creature? $1.5A + B = 50$. With these tax rates, how many aerobics lessons per creature would the rich favor? $50/3$. How many would the poor favor? $50/3$. What quantity of aerobics lessons per capita would be chosen under majority rule? $50/3$. How much bread would the rich get? 50. How much bread would the poor get? 25.

(i) Calculate the utility of a rich creature under a head tax. $\sqrt{937.5}$ Under privatization. $\sqrt{1,250}$. Under a proportional income tax. $\sqrt{833.3}$. (Hint: In each case, solve for the consumption of bread and the consumption of aerobics lessons that a rich person gets, and plug these into the utility function.) Now calculate the utility of each poor creature under the head tax. $\sqrt{312.5}$. Under privatization. $\sqrt{312.5}$. Under the proportional income tax. $\sqrt{416.67}$. (Express these utilities as square roots rather than calculating out the roots.)

(j) Is privatization Pareto superior to the head tax? Yes. Is a proportional income tax Pareto superior to the head tax? No. Is privatization Pareto superior to the proportional income tax? No. Explain the last two answers. Rich prefer privatization, poor prefer proportional income tax.

Calculus **37.8 (2)** Three friends, Archie, Betty, and Veronica, are planning a party. They disagree about how many people to invite. Each person i has a quasilinear utility function of the form $m_i + u_i(x)$ where m_i is the

number of dollars that i has to spend and x is the number of guests invited to the party. Suppose that for each i,

$$u_i = a_i x - \frac{1}{2}x^2.$$

Everyone knows the functional form of the others' utility functions and knows his own value of a_i but does not know anyone else's value. Let us suppose that the actual values of a_i are 20 for Archie, 40 for Betty, and 60 for Veronica.

(a) How many guests should be invited to maximize the sum of the three persons' utilities? 40 guests

(b) Suppose that the three friends decide to use the VCG mechanism to determine the number of guests. If each plays his or her best strategy, how many guests will be invited? 40 guests

(c) In the VCG mechanism, if the amount of public good supplied is x, Archie would receives a sidepayment equal to the sum of Betty's and Veronica's utility for x. If Betty and Veronica play their best strategies (without colluding) and if the amount of public good is x, this sidepayment will be $100x - x^2$. If everybody plays their best strategy, the amount of this sidepayment in dollars is $2400
.

(d) In addition to receiving sidepayments, the VCG mechanism requires that each person must pay an amount equal to the maximum possible sum of the other two persons' utilities. If Betty and Veronica play their best strategies this amount is $2500. On net, Archie has to pay the difference between this amount and the sidepayment that he receives. If everybody plays their best strategy, what is the net amount that Archie must pay? $100

(e) If everybody plays their best strategy, what is the net amount that Betty has to pay? $0 What is the net amount that Veronica has to pay? $100

(f) Suppose that the party is organized not by just three people, but by a dormitory with 21 residents. All of these residents have utility functions of the same form as Archie, Betty, and Veronica. Seven of them have $a_i = 20$, seven have $a_i = 40$, and seven have $a_i = 60$. In order to maximize the sum of the residents' utilities, how many guests should be

invited? 40 guests If there were only six persons with $a_i = 20$, seven with $a_i = 40$ and seven with $a_i = 60$, how many guests would have to be invited in order to maximize the sum of utilities? 41 guests

(g) If everybody plays their best strategy in the VCG game, then after all sidepayments and taxes are collected, how much net tax will each of the people with $a_i = 20$ have to pay? $10 How much net tax will each of the people with $a_i = 40$ have to pay? $0 How much net tax will each of the people with $a_i = 60$ have to pay? $10

37.9 (2) Homeowners 1, 2, and 3 live at the end of a badly deteriorated road. Fixing the road would cost $\$C$. The value to Homeowner 1 of fixing the road is \$3,000, the value to Homeowner 2 is \$5,000, and the value to Homeowner 3 is \$8,000. Each homeowner claims that fixing the road is not worth much to him, because each wants the others to pay the cost. The local government suspects that the total value to these homeowners of fixing the road is greater than $\$C$ and has decided to require the three homeowners to use the VCG mechanism to determine whether to fix the road. Since the government had no idea of the individual values for fixing the road, it decided to allocate the costs equally among the three homeowners. Each homeowner is asked to report his value for fixing the road. If the sum of the reported values is greater than C, the road will be fixed and each homeowner will have to pay $\$C/3$ and also will have to pay an additional tax as calculated by the VCG mechanism.

(a) Suppose that $C = \$13,500$, so that each homeowner has to pay \$4,500 as his share of the cost. If homeowners report their values accurately, the sum of the reported values will be $16,000 . Since the sum of reported values is greater than \$C, the government will choose to build the road. To calculate Homeowner 3's VCG tax, we reason as follows. If the road is repaired, Homeowners 1 and 2 will have to pay a total of $9,000 while the sum of their values for the repairs is $8,000 so their net change in utility is -$1000 . Since Homeowner 1's response changes the decision about whether the road is built, she would be assessed a VCG tax of $1,000 in addition to her \$4,000 share of the costs. What would be Homeowner 1's VCG tax? 0 Homeowner 2's VCG tax? 0 Would Homeowner 1 be better off or worse off with the outcome of the VCG mechanism than if the road is left unrepaired? Worse off Would Homeowner 2 be better off or

worse off than if the road is left unrepaired? Better off Would Homeowner 3 be better off or worse off with the VCG outcome than if the road is left unrepaired? Better off When the VCG tax is used, would the sum of the utilities of the three homeowners be higher or lower than if the road were left unrepaired? Higher

(b) Suppose that $C = \$18,000$ and so each homeowner will have to pay a \$6,000 share of the cost. If homeowners report their values accurately, the sum of the reported values will be \$16,000 . Since the sum of reported values is less than \$C, the government will choose not to repair the road. Let us calculate Homeowner 1's VCG tax. If the road is repaired, the total amount that Homeowners 2 and 3 have to pay is \$12,000 and the sum of the values to Homeowners 2 and 3 of fixing the road will be \$13,000 , so their net gain from the project is \$1000 . Since Homeowner 3's response changes the outcome of whether the road is repaired, she would be assessed a VCG tax of \$1000 .

(c) Suppose that instead of 3 homeowners at the end of the road, there were 30 homeowners, 10 of type 1 who valued repairing the road at \$3,000, 10 of type 2 who valued repairing the road at \$5,000 and ten of type 3 who valued repairing the road at \$8,000. Suppose that the cost of repairing the road is $C = \$135,500$. If the road is repaired, each homeowner would have to pay a tax of $\$135,000/30 = \$4,500$. If the VCG mechanism is used and each individual reports his true valuation, the road will be repaired, since the sum of reported valuations will be \$160,000. To calculate the VCG tax of a type 3 consumer, we observe that the sum of the values of the project to all other consumers is \$152,000 . The total amount that other consumers would have to pay if the project is undertaken is \$131,000 . Would a type 3 homeowner have to pay a positive amount of VCG tax? No How about type 1 and 2 homeowners? They would have 0 VCG tax as well.

Calculus **37.10 (2)** (For this problem, you will want to use a calculator.) A developer would like to build an amusement park in the midst of the once-thriving city of Broken Axle, Michigan. In order to build this park, he must purchase all of the land in a specific 10-acre area. If he does not get all of this land, he cannot build the park. Nobody except the

developer knows exactly how much he would be willing to pay for the land. All persons other than the developer believe that the probability that he is willing to pay at least $\$p$ for the land is $G(p) = e^{-p/k}$ where $k = 200,000$.

(a) Suppose that all of the land belongs to a single owner. This owner values the land at \$50,000 in its current use. He must make a take-it-or-leave-it offer to the developer. The owner wants to make an offer p that maximizes his expected profit from the sale, which is $(p - \$50,000)G(p)$.

At what price should the owner offer the land? $\$250,000$ At this price, what is the probability that the developer will buy the land? $e^{-1.25} = .287$ How much is the owner's expected profit?

$$\$200,000 \times .287 = \$57,301$$

(b) Suppose that instead the land consists of 10 separate parcels with 10 different owners. Let v_i be owner i's value for his own parcel if he keeps it. These owners have different values v_i and only the owner knows what it is worth to himself. Let us assume that $v = \sum_{i=1}^{10} v_i = \$50,000$. Suppose that each owner i sets a price p_i for his land. Let us define p to be the sum of these individual offer prices. That is, $p = \sum_{i=1}^{10} p_i$. If the developer is willing to pay at least p for the entire 10-acre parcel, then he buys it and pays each owner i, the amount p_i. In equilibrium, each owner makes the offer that maximizes his expected profit given the offers made by the other owners. That is, he chooses p_i to maximize

$$(p_i - v_i)G(p) = (p_i - v_i)e^{-\sum p_j/k}$$

where $k = \$200,000$. What price p_i will owner i set? $v_i + k = v_i + \$200,000$ What is the sum of the prices asked by the 10 owners? $\$2,050,000$ What is the probability that the developer will buy the land at this price? $e^{-10.25} = .000035$ What is the sum over all 10 owners of expected profits? $\$70.715$

(c) The 10 separate owners recognize that if they could somehow coordinate their offers, they would make much larger expected total profits. They decide to use the VCG mechanism to do so. In this instance the VCG mechanism works as follows. Each individual states an amount r_i that the land is worth to him. All of the land will be offered to the developer at a price p that would maximize total profits of owners if each owner reports his actual value, $r_i = v_i$. If the developer chooses to purchase the land, then each owner will receive an equal share of the sales price p. In addition to receiving this share if the sale is made, each owner will pay a "tax" that depends on his response r_i and on the responses of

the other owners in the following way. Person i will receive an amount the sum of the expected profits of the other 9 owners, but must pay an amount equal to the maximum expected profits that would be received by the other owners if the price were chosen to maximize the expected total profits of these other owners. With this mechanism, the best strategy for each owner is to state his true value $r_i = v_i$. If each owner uses this best strategy, at what price will the land be offered to the developer?

$250,000 What is the probability that the developer will buy the land? $e^{-1.25} = .287$

(d) Suppose that person i has a reservation price $v_i = \$5,900$. Let us calculate the tax that the VCG mechanism would assign to i if each individual responds with $r_i = v_i$. If the land is offered at price p to the developer, then expected profits of persons other than i will be

$$\left(\frac{9}{10}p - v_{\sim i}\right) e^{-p/200,000}$$

where $v_{\sim i} = \sum_{j \neq i} v_j =$ $44,100 is the sum of the reservation prices of persons other than i. We found that if everyone responds optimally in the VCG mechanism, the land will be offered at a price of $250,000. Therefore the expected profits of persons other than i will be $(\$225,000 - \$44,100)e^{-1.25} = \$51,829$. The price p that maximizes the sum of expected profits of owners other than i is $245,556. At this price, total expected profits of these owners would be $59,014. The net tax that i pays is the difference between this amount and the actual expected profits of other owners. This difference is $14.

(e) Let us calculate total expected profit of the individual with $v_i = \$5900$ under the VCG mechanism. Whether or not there is a sale, person i pays the amount of tax that we calculated in the previous section. With probability .287, the land is sold at price $250,000 and each owner will receive 1/10 of the sales price, which is $25,000, giving him a profit of $25,000-$5900=$19,100 over his reservation value. Therefore the expected profit for person i is $.287*19,100-14=$5467

(f) If individual j has $v_j = \$5,000$, what is the net amount of tax that he pays under the VCG mechanism? $0 (Hint: there is an easy answer.)

Chapter 38 NAME_____

Asymmetric Information

Introduction. The economics of information and incentives is a relatively new branch of microeconomics, in which much intriguing work is going on. This chapter shows you a sample of these problems and the way that economists think about them.

38.1 (0) There are two types of electric pencil-sharpener producers. "High-quality" manufacturers produce very good sharpeners that consumers value at \$14. "Low-quality" manufacturers produce less good ones that are valued at \$8. At the time of purchase, customers cannot distinguish between a high-quality product and a low-quality product; nor can they identify the manufacturer. However, they can determine the quality of the product after purchase. The consumers are risk neutral; if they have probability q of getting a high-quality product and $1 - q$ of getting a low-quality product, then they value this prospect at $14q + 8(1 - q)$. Each type of manufacturer can manufacture the product at a constant unit cost of \$11.50. All manufacturers behave competitively.

(a) Suppose that the sale of low-quality electric pencil-sharpeners is illegal, so that the only items allowed to appear on the market are of high quality. What will be the equilibrium price? \$11.50.

(b) Suppose that there were no high-quality sellers. How many low-quality sharpeners would you expect to be sold in equilibrium? Sellers won't sell for less than \$11.50, consumers won't pay that much for low-quality product. So in equilibrium there would be no sales.

(c) Could there be an equilibrium in which equal (positive) quantities of the two types of pencil sharpeners appear in the market? No. Average willingness to pay would be \$11, which is less than the cost of production. So there would be zero trade.

(d) Now we change our assumptions about the technology. Suppose that each producer can *choose* to manufacture either a high-quality or a low-quality pencil-sharpener, with a unit cost of \$11.50 for the former and \$11 for the latter, what would we expect to happen in equilibrium? No trade. Producers would produce the low-quality product since it has a lower production cost. If all producers produce low-quality output, costs will be \$11 and the willingness-to-pay for low quality is \$8.

(e) Assuming that each producer is able to make the production choice described in the last question, what good would it do if the government banned production of low-quality electric pencil-sharpeners? If there is no ban, there will be no output and no consumers' surplus. If low-quality products are banned, then in equilibrium there is output and positive consumers' surplus.

38.2 (0) In West Bend, Indiana, there are exactly two kinds of workers. One kind has a (constant) marginal product worth \$10 and the other kind has a (constant) marginal product worth \$15. There are equal numbers of workers of each kind. A firm cannot directly tell the difference between the two kinds of workers. Even after it has hired them, it won't be able to monitor their work closely enough to determine which workers are of which type.

(a) If the labor market is competitive, workers will be paid the average value of their marginal product. This amount is \$12.50.

(b) Suppose that the local community college offers a microeconomics course in night school, taught by Professor M. De Sade. The high-productivity workers think that taking this course is just as bad as a \$3 wage cut, and the low-productivity workers think it is just as bad as a \$6 wage cut. The firm can observe whether or not an individual takes the microeconomics course. Suppose that the high-productivity workers all choose to take the microeconomics course and the low-productivity

workers all choose not to. The competitive wage for people who take the microeconomics course will be $15 and the wage for people who don't take the microeconomics course will be $10.

(c) If there is a separating equilibrium, with high-productivity workers taking the course and low-productivity workers not taking it, then the net benefits from taking the microeconomics course will be $2 for the high-productivity workers and $−1 for the low-productivity workers. Therefore there (will be, won't be) will be a separating equilibrium of this type.

(d) Suppose that Professor De Sade is called off to Washington, to lecture wayward representaatives on the economics of family values. His replacement is Professor Morton Kremepuff. Kremepuff prides himself on his ability to make economics "as easy as political science and as fun as the soaps on TV." Professor Kremepuff's claims are exaggerated, but at least students like him better than De Sade. High-productivity workers think that taking Kremepuff's course is as bad as a $1 wage cut, and low-productivity workers think that taking Kremepuff's course is as bad as a $4 wage cut. If the high-productivity workers all choose to take the microeconomics course and the low-productivity workers all choose not to, the competitive wage for people who take the microeconomics course will be $15 and the wage for people who don't take the microeconomics course will be $10.

(e) If there is a separating equilibrium with high-productivity workers taking the course and low-productivity workers not taking it, then the net benefits from taking Kremepuff's microeconomics course will be $4 for the high-productivity workers and $1 for the low-productivity workers. Therefore there (will be, won't be) won't be a separating equilibrium of this type.

38.3 (1) In Enigma, Ohio, there are two kinds of workers, Klutzes whose labor is worth $1,000 per month and Kandos, whose labor is worth $2,500 per month. Enigma has exactly twice as many Klutzes as Kandos. Klutzes look just like Kandos and are accomplished liars. If you ask, they will claim to be Kandos. Kandos always tell the truth. Monitoring individual work accomplishments is too expensive to be worthwhile. In the old days, there was no way to distinguish the two types of labor, so everyone was paid the same wage. If labor markets were competitive, what was this wage? $1,500

(a) A professor who loves to talk offered to give a free monthly lecture on macroeconomics and personal hygiene to the employees of one small firm. These lectures had no effect on productivity, but both Klutzes and Kandos found them to be excruciatingly dull. To a Klutz, each hour's lecture was as bad as losing $100. To a Kando, each hour's lecture was as bad as losing $50. Suppose that the firm gave each of its employees a pay raise of $55 a month but insisted that he attend the professor's lectures. What would happen to the firm's labor force? All Klutzes would leave. Kandos would stay on. More Kandos could be hired at these terms. Klutzes would not accept job. What would happen to the average productivity of the firm's employees? Rise by $1,000--from $1,500 to $2,500.

(b) Other firms noticed that those who had listened to the professor's lectures were more productive than those who had not. So they tried to bid them away from their original employer. Since all those who agreed to listen to the original lecture series were Kandos, their wage was bid up to $2,500.

(c) After observing the "effect of his lectures on labor productivity," the professor decided to expand his efforts. He found a huge auditorium where he could lecture to all the laborers in Enigma who would listen to him. If employers believed that listening to the professor's lectures improved productivity by the improvement in productivity in the first small firm and offered bonuses for attending the lectures accordingly, who would attend the lectures? Everybody. Having observed this outcome, how much of a wage premium would firms pay for those who had attended the professor's lectures? 0.

(d) The professor was disappointed by the results of his big lecture and decided that if he gave more lectures per month, his pupils might "learn more." So he decided to give a course of lectures for 20 hours a month. Would there now be an equilibrium in which the Kandos all took his course and none of the Klutzes took it and where those who took the course were paid according to their true productivity? Yes. If those who take the course get $2,500 and people who do not get $1,000 a month, then Kandos would take the course, since the

pain of 20 hours of lecture costs $1,000, but the wage premium is $1,500. Klutzes would not take the course, since the pain of lectures costs $2,000 a month and the wage premium is $1,500.

(e) What is the smallest number of hours the professor could lecture and still maintain a separating equilibrium? 15 hours

38.4 (1) Old MacDonald produces hay. He has a single employee, Jack. If Jack works for x hours he can produce x bales of hay. Each bale of hay sells for $1. The cost to Jack of working x hours is $c(x) = x^2/10$.

(a) What is the efficient number of bales of hay for Jack to cut? 5.

(b) If the most that Jack could earn elsewhere is zero, how much would MacDonald have to pay him to get him to work the efficient amount?

$5^2/10 = \$2.50$.

(c) What is MacDonald's net profit? $5 - 2.50 = \$2.50$.

(d) Suppose that Jack would receive $1 for passing out leaflets, an activity that involves no effort whatsoever. How much would he have to receive from MacDonald for producing the efficient number of bales of hay?

$3.50.

(e) Suppose now that the opportunity for passing out leaflets is no longer available, but that MacDonald decides to rent his hayfield out to Jack for a flat fee. How much would he rent it for? $2.50.

38.5 (0) In Rustbucket, Michigan, there are 200 people who want to sell their used cars. Everybody knows that 100 of these cars are "lemons" and 100 of these cars are "good." The problem is that nobody except the original owners know which are which. Owners of lemons will be happy to get rid of their cars for any price greater than $200. Owners of good used cars will be willing to sell them for any price greater than $1,500, but will keep them if they can't get $1,500. There are a large number of buyers who would be willing to pay $2,500 for a good used car, but would pay only $300 for a lemon. When these buyers are not sure of the quality of the car they buy, they are willing to pay the expected value of the car, given the knowledge they have.

(a) If all 200 used cars in Rustbucket were for sale, how much would buyers be willing to pay for a used car? $1,400. Would owners of good used cars be willing to sell their used cars at this price? No. Would there be an equilibrium in which all used cars are sold? No. Describe the equilibrium that would take place in Rustbucket. Good car owners won't sell. Lemon owners will sell. Price of a used car will be $300.

(b) Suppose that instead of there being 100 cars of each kind, everyone in town is aware that there are 120 good cars and 80 lemons. How much would buyers be willing to pay for a used car? $1,620. Would owners of good used cars be willing to sell their used cars at this price? Yes. Would there be an equilibrium in which all used cars are sold? Yes. Would there be an equilibrium in which only the lemons were sold? Yes. Describe the possible equilibrium or equilibria that would take place in Rustbucket. One equilibrium has all cars sold at a price of $1,620. There is also an equilibrium where only the lemons are sold.

38.6 (1) Each year, 1,000 citizens of New Crankshaft, Pennsylvania, sell their used cars and buy new cars. The original owners of the old cars have no place to keep second cars and must sell them. These used cars vary a great deal in quality. Their original owners know exactly what is good and what is bad about their cars, but potential buyers can't tell them apart by looking at them. Lamentably, though they are in other respects model citizens, the used-car owners in New Crankshaft have no scruples about lying about their old jalopies. Each car has a value, V, which a buyer who knew all about its qualities would be willing to pay. There is a very large number of potential buyers, any one of which would be willing to pay $\$V$ for a car of value $\$V$.

The distribution of values of used cars on the market is quite simply described. In any year, for any V between 0 and $2,000, the number of used cars available for sale that are worth less than $\$V$ is $V/2$. Potential used-car buyers are all risk-neutral. That is if they don't know the value of a car for certain, they value it at its expected value, given the information they have.

Rod's Garage in New Crankshaft will test out any used car and find its true value V. Rod's Garage is known to be perfectly accurate and perfectly honest in its appraisals. The only problem is that getting an accurate appraisal costs \$200. People with terrible cars are not going to want to pay \$200 to have Rod tell the world how bad their cars are. But people with very good cars will be willing to pay Rod the \$200 to get their cars appraised, so they can sell them for their true values.

Let's try to figure our exactly how the equilibrium works, which cars get appraised, and what the unappraised cars sell for.

(a) If nobody had their car appraised, what would the market price for used cars in North Crankshaft be and what would be the total revenue received by used-car owners for their cars? They'd all sell for \$1,000 for total revenue of \$1,000,000.

(b) If all the cars that are worth more than $\$X$ are appraised and all the cars that are worth less than $\$X$ are sold without appraisal, what will the market price of unappraised used cars be? (Hint: What is the expected value of a random draw from the set of cars worth less than $\$X$?) $\$X/2$.

(c) If all the cars that are worth more than $\$X$ are appraised and all the cars that are worth less than $\$X$ are sold without appraisal, then if your car is worth $\$X$, how much money would you have left if you had it appraised and then sold it for its true value? $\$X - 200$. How much money would you get if you sold it without having it appraised? $\$X/2$.

(d) In equilibrium, there will be a car of marginal quality such that all cars better than this car will be appraised and all cars worse than this car will be sold without being appraised. The owner of this car will be just indifferent between selling his car unappraised and having it appraised. What will be the value of this marginal car? Solve $X/2 = X - 200$ to get $X = \$400$.

(e) In equilibrium, how many cars will be sold unappraised and what will they sell for? The worst 200 cars will be unappraised and will sell for \$200.

(f) In equilibrium, what will be the total net revenue of all owners of used cars, after Rod's Garage has been paid for its appraisals? $\$1,000,000 - 800 \times 200 = 840,000$.

38.7 (2) In Pot Hole, Georgia, 1,000 people want to sell their used cars. These cars vary in quality. Original owners know exactly what their cars are worth. All used cars look the same to potential buyers until they have bought them; then they find out the truth. For any number X between 0 and 2,000, the number of cars of quality lower than X is $X/2$. If a car is of quality X, its original owner will be willing to sell it for any price greater than X. If a buyer knew that a car was of quality X, she would be willing to pay $X + 500$ for it. When buyers are not sure of the quality of a car, they are willing to pay its expected value, given their knowledge of the distribution of qualities on the market.

(a) Suppose that everybody knows that all the used cars in Pot Hole are for sale. What would used cars sell for? `$1,500.` Would every used car owner be willing to sell at this price? `No.` Which used cars would appear on the market? `Those worth less than $1,500.`

(b) Let X^* be some number between 0 and 2,000 and suppose that all cars of quality lower than X^* are sold, but original owners keep all cars of quality higher than X^*. What would buyers be willing to pay for a used car? $X^*/2 + 500$. At this price, which used cars would be for sale? `Cars worth less than` $X^*/2 + 500$.

(c) Write an equation for the equilibrium value of X^*, at which the price that buyers are willing to pay is exactly enough to induce all cars of quality less than X^* into the market. $X^*/2 + 500 = X^*$. Solve this equation for the equilibrium value of X^*. $X^* = \$1,000$.