Linear Algebra Assignment 3

Hanseul Kim

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Exercise 1:

Let $f : V \to W$ be a linear map. Show that kernel and image of f are subspaces.

Proof:
To prove that the subset is a subspace, we need to show

1. The subset is non-empty
2. The subset is closed under addition and scalar multiplication

1. Image of f

$$\begin{aligned} 0 &\in V \\ f(0) &\in W \\ \text{kernel is not empty} & \qquad \text{(a)} \\ \text{Since } f \text{ is a linear map, for every } v, w \text{ in } W & \\ f(v) + f(w) = f(v+w) &\in W \\ \forall c \in \mathbb{R}, cf(v) = f(cv) &\in W \\ f \text{ is closed under scalar multiplication and addition} & \qquad \text{(b)} \end{aligned}$$

$$\text{by (a), (b) image of } f \text{ is subspace}$$

2. Kernel of f

$$\begin{aligned} \text{Since } f \text{ is a linear map, for every } c \text{ in } \mathbb{R} & \\ f(0) = f(c * 0) = c * f(0) & \\ f(0) \text{ is a additive identity} & \\ 0 \in img(f) & \qquad \text{(a)} \\ \forall v, w \in img(f) \text{ and } c \in \mathbb{R} & \\ f(v+w) = f(v) + f(w) = 0, v + w \in img(f) & \\ f(cv) = cf(v) = c * 0 = 0, cv \in img(f) & \\ f \text{ is closed on scalar multiplication and addition} & \qquad \text{(b)} \end{aligned}$$

$$\text{by (a), (b) kernel of } f \text{ is subspace}$$

$\square$

Exercise 2:

Show that the space V/W constructed in the lecture is a vector space.

Proof:

$$\text{Elements of quotient space is defined as } v + W = \{v + w | w \in W\}$$

1) Closed under addition

$$x, y \in V/W, x = v_1 + W, y = v_2 + W$$

$$x + y = v_1 + W + v_2 + W = v_1 + v_2 + W = v_3 + W \in V/W$$

2) Closed under scalar multiplication

$$\forall c \in \mathbb{R}$$

$$c * x = cv + W = v_1 + W \in V/W$$

3) Commutativity

$$x + y = v_1 + v_2 + W = v_2 + v_1 + W = y + x$$

4) Associativity

$$(x + y) + z = (v_1 + v_2) + v_3 + W = v_1 + (v_2 + v_3) + W = x + (y + z)$$

5) Additive inverse

$$\forall x \in V/W, x = v + W$$

$$-x = -v + W$$

$$x + (-x) = v - v + W = 0 + W, \text{ which is a additive identity}$$

Since V is a vector space, scalar associativity, distribution law holds.

$$\therefore V/W \text{ is a vector space.}$$

$\square$

Exercise 3:

Show that the quotient map (η) constructed in the lecture is surjective and its kernel is subspace W.

$$\eta : V \to V/W$$
$$\eta(v) = v + W$$

Proof:

1) Surjectivity

$$\forall x \in V/W, x = \{v + w | w \in W\} = v + W, \text{ and } v \in V$$
$$\eta(v) = v + W = x$$

2) Kernel is subspace of W

$$\text{let additive identity of quotient space } v_0 + W$$
$$\forall v \in V, v + W + v_0 + W = v + v_0 + W = v + W$$
$$v + v_0 - v = v_0 \in W$$
$$\therefore \text{set of } v_0 \text{ is a subset of } W$$

□

Exercise 4:

Let A_1 be the following matrix:

$$A_1 = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -3 & -5 & 1 \end{pmatrix}$$

1) Prove that columns of A_1 are linearly independent.

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$
$$1)\ 2\lambda_1 + 3\lambda_2 + \lambda_3 = 0$$
$$2)\ \lambda_1 + 2\lambda_2 - \lambda_3 = 0$$
$$3)\ -3\lambda_1 - 5\lambda_2 + \lambda_3 = 0$$

$$1) + 2) = \quad 3\lambda_1 + 5\lambda_2 = 0$$
$$1) - 3) = \quad 5\lambda_1 + 8\lambda_2 = 0$$

$$\lambda_1 = -2\lambda_2$$
$$1) + 2) + 3) = \lambda_3 = 0$$
$$\therefore \lambda_1, \lambda_2, \lambda_3 = 0$$
$$\text{columns of } A_1 \text{ are linearly independent.}$$

□

2) coordinates for $x = (6, 2, -7)$

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -7 \end{pmatrix}$$

$$\therefore \text{coordinate is } (1, 1, 1)$$

Exercise 4:

Let $f : E \to F$ be a linear map which is also a bijection. Prove that the inverse function $f^{-1} : F \to E$ is linear.

Proof:

1) Additivity

$$\text{Because } f \text{ is a bijection}$$
$$\forall x, y \in F, \text{ there exists unique } v, w \in E \text{ where,}$$
$$f(v) = x, f(w) = y$$
$$f(v + w) = f(v) + f(w) = x + y$$
$$f^{-1}(x + y) = v + w = f^{-1}(x) + f^{-1}(y)$$

2) Scalar Multiplication

$$\text{Because } f \text{ is a bijection}$$
$$\forall x \in F, \text{ there exists unique} v \in E \text{ where,}$$
$$f(v) = x$$
$$\forall c \in \mathbb{R}, f(cv) = cf(v) = cx$$
$$f^{-1}(cx) = cv$$

$$\therefore \text{by 1), 2) } f^{-1} \text{ is linear.}$$

□

Exercise 5:

Given two vectors spaces E and F, let $(u_i)_{i\in I}$ be any basis of E and let $(v_i)_{i\in I}$ be any family of vectors in F. Prove that the unique linear map $f : E \to F$ such that $f(u_i) = v_i$ for all $i \in I$ is srujective iff $(v_i)_{i\in I}$ spans F.

Proof:

1) Forward case (surjective $\to$ spans F)

$$\text{let } f \text{ be surjective. for all } v \in F, \text{ there exists } u \in E \text{ where,}$$
$$f(u) = v$$
$$u = \lambda_1 u_1 + \cdots \lambda_n u_n \quad ((u_i)_{i\in I} \text{ are basis vectors})$$
$$\text{because } f \text{ is linear map,}$$
$$v = f(u) = f(\lambda_1 u_1 + \cdots + \lambda_n u_n) = \lambda_1 f(u_1) + \cdots + \lambda_n f(u_n)$$
$$= \lambda_1 v_1 + \cdots + \lambda_n v_n$$
$$\therefore (v_i)_{i\in I} \text{ spans } F$$

2) Backward case (spans $F \to$ surjective)

Because $(v_i)_{i\in I}$ spans F

$$\forall v \in F, \text{ there exists } \lambda_1, \ldots, \lambda_n \in \mathbb{R} \text{ where,}$$
$$\lambda_1 v_1 + \cdots + \lambda_n v_n = v$$
$$\text{because } f \text{ is linear map,}$$
$$f(\lambda_1 u_1 + \cdots + \lambda_n u_n) = \lambda_1 f(u_1) + \cdots + \lambda_n f(u_n) = \lambda_1 v_1 + \cdots + \lambda_n v_n = v$$
$$\text{a) } f \text{ is onto}$$
$$\forall x, y \in E \text{ where } x \neq y$$
$$\text{there exists unique different set of } \lambda_i^x, \lambda_i^y \text{ so that,}$$
$$\lambda_1^x u_1 + \cdots + \lambda_n^x u_n = x, \lambda_1^y u_1 + \cdots + \lambda_n^y u_n = y$$
$$f(x) - f(y) = (\lambda_1^x - \lambda_1^y)v_1 + \cdots + (\lambda_1^x - \lambda_n^y)v_n$$
$$\text{Since } \lambda_i^x \text{ and } \lambda_i^y \text{ are different, } f(x) - f(y) \neq 0$$
$$f \text{ is one to one}$$
$$b) f \text{ is surjective.}$$

$$\therefore f \text{ is srujective iff } (v_i)_{i\in I} \text{ spans } F.$$

$\square$