Assignment 07, Real Analysis MIT

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Answers

0.1 Exercise 2.6.2

Suppose both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely. Show that the product series, $\sum_{n=0}^{\infty} c_n$ where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$, also converges absolutely.

$$|a_n| \text{ and } |b_n| \text{ are both convergent and absolutely convergent series.}$$

$$\text{by the Theorem 2.6.5(Mertens' theorem, })c_n = \sum_{j=0}^{n} |a_j||b_{n-j}| \text{ converges.}$$

$$\therefore c_n = \sum_{j=0}^{n} a_j b_{n-j} \text{converges absolutely}$$

□

0.2 Exercise 2

Find all real numbers x so that the series converges.

a) $\sum_{n=0}^{\infty} 2^n x^n$
Ratio test:

$$\frac{2^{n+1}x^{n+1}}{2^n x^n} = 2x$$

- case $\frac{-1}{2} < x < \frac{1}{2}$: converges
- case $\frac{-1}{2} > x$ or $x > \frac{1}{2}$: diverges.
- $x = \frac{1}{2}, \sum_{n=0}^{\infty} 1$, diverges
- $x = \frac{-1}{2}, \sum_{n=0}^{\infty} (-1)^n$, diverges

b) $\sum_{n=0}^{\infty} nx^n$
Ratio test:

$$\frac{(n+1)x^{n+1}}{nx^n} = \frac{(n+1)x}{n}$$
$$\lim_{n\to\infty} \frac{(n+1)}{n} x = x$$

- case $-1 < x < 1$, converges
- case $x < -1$ or $x > 1$, diverges
- case $x = 1, \sum_{n=0}^{\infty} n$, diverges
- case $x = -1, \sum_{n=0}^{\infty} n(-1)^n$, diverges

c) $\sum_{n=0}^{\infty} \frac{1}{(2n)!}(x-10)^n$
Ratio test:

$$\frac{(x-10)^{n+1}(2n)!}{(x-10)^n(2n+2)!} = \frac{x-10}{(2n+1)(2n+2)}$$
$$\lim_{n\to\infty} \frac{x-10}{(2n+1)(2n+2)} = 0 \quad (\forall x \in \mathbb{R})$$
$$\therefore \forall x \in \mathbb{R}, \text{ series converges.}$$

d) $\sum_{n=0}^{\infty} n!x^n$
Ration test:

$$\frac{(n+1)!x^{n+1}}{n!x^n} = (n+1)x$$

- $x > 0, \lim_{n\to\infty}(n+1)x$ diverges, hence the series diverges.
- $x < 0, \lim_{n\to\infty}(n+1)x$ diverges, hence the series diverges.
- $x = 0, \lim_{n\to\infty}(n+1)x = 0$,hence the series converges.

0.3 Exercise 3

(Cauchy-Schwarz inequality) Prove that if $\sum |x_n|^2$ and $\sum |y_n|^2$ converge, then the series $\sum x_n y_n$ converges absolutely and

$$|\sum_{n=1}^{\infty} x_n y_n| \le \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{\frac{1}{2}}$$

Proof by induction:
Base case:

$$|\sum_{n=1}^{1} x_n y_n| = |x_1 y_1| = |x_1||y_1| = (|x_1|^2)^{\frac{1}{2}}(|y_1|^2)^{\frac{1}{2}}$$

Induction:
Let:

$$|\sum_{n=1}^{N} x_n y_n| \le \left(\sum_{n=1}^{N} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} |y_n|^2\right)^{\frac{1}{2}}$$

$$|\sum_{n=1}^{N+1} x_n y_n| \le |\sum_{n=1}^{N} x_n y_n| + |x_{N+1}y_{N+1}| \le \left(\sum_{n=1}^{N} |x_n|^2 + x_{N+1}^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} |y_n|^2 + y_{N+1}^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{N+1} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N+1} |y_n|^2\right)^{\frac{1}{2}}$$

$$\therefore |\sum_{n=1}^{\infty} x_n y_n| \le \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{\frac{1}{2}}$$

□

Let

$$A = \sum_{n=1}^{\infty} |x_n|^2, \quad B = \sum_{n=1}^{\infty} |x_n|^2$$

By the Cauchy-Schwarz inequality,

$$|\sum_{n=1}^{\infty} x_n y_n| \le \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{\frac{1}{2}}$$

thus $\sum x_n y_n$ converges absolutely.

□

0.4 Exercise 4

Prove that every real number is a cluster point of the set of irrational numbers.
Proof by contradiction
Suppose there exists a real number $r \in \mathbb{R}$ which is not a cluster point of the set of irrational numbers.

$$\exists q \in \mathbb{Q}, \quad \forall \epsilon > 0$$

$$(q - r) < \epsilon, q = r$$

r cannot be a rational number so the set satisfying $(q - r) < \epsilon$ is empty.

contradiction. the real are dense in rational.

□

0.5 Exercise 3.1.13

Suppose $S \in \mathbb{R}$ and c is a cluster point of S. Suppose $f : S \to \mathbb{R}$ is bounded. Show that there exists a sequence $\{x_n\}$ with $x_n \in S \setminus \{c\}$ and $\lim x_n = c$ such that $\{f(x_n)\}$ converges.

Proof:

Since c is a cluster point of S, by proposition 3.1.2, there exists a sequence $\{x_n\}$ with $x_n \in S$ and $\lim x_n = c$. Let,

$$b_n = f(x_n) \text{ which is bounded.}$$

Since b_n is a bounded sequence by Bolzano Weierstrass Theorem, there exist c_n subsequnce of $b_n$which converges

$$\forall \epsilon_1 > 0, \text{there exist } N \in \mathbb{N} \text{ s.t.}$$
$$n \geq N, |c_n - L| < \epsilon_1$$
$$|f(x_(n_k) - L) < \epsilon_1|$$
$$n_k > k \text{ which is not bounded, so there exist } N \text{ such that,}$$
$$\lim x_{n_k} = c$$

□

0.6 Exercise 6

Let $S \subset \mathbb{R}$, let c be a cluster point of S, and let $f : S \to \mathbb{R}$.

a) Assume $\lim_{x \to c} f(x)$ exists. Prove that there exist $B \geq 0$ and $\delta > 0$ such that if $x \in S$ and $0 < |x - c| < \delta$ then $|f(x)| \leq B$

Proof:

By definition of limit,

$$\forall \epsilon > 0, \text{there exists } \delta > 0 \text{ s.t if } 0 < |x - c| < \delta$$
$$\text{then}|f(x) - L| < \epsilon$$
$$\text{let } \epsilon = 1$$
$$|f(x) - L| < 1$$
$$L - 1 < f(x) < L + 1$$
$$|f(x)| < |L + 1|$$

$$\text{let } B = |L + 1| \geq 0$$

By definition of limit

$$\text{if } x \in S \text{ and } 0 < |x - c| < \delta_1$$
$$|(f(x) - L)| < 1 \text{ hence,}$$
$$|f(x)| < |L + 1| = B$$

□

b) Assume that $\lim_{x \to c} f(x) = L > 0$. Prove that there exists $\delta > 0$ such that if $x \in S$ and $0 < |x - c| < \delta$ then $f(x) > 0$.

Proof:

By definition of limit

Let

$$\epsilon = L/2, \exists \delta > 0 \text{ such that,}$$
$$|x - c| < \delta \text{ then } |f(x) - L| < L/2$$
$$0 < \frac{L}{2} < f(x) < \frac{3L}{2}$$
$$\therefore 0 < f(x)$$

□