Linear Algebra Assignment 8

Hanseul Kim

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Exercise 1:

Let $v_1, v_2, \dots, v_n$ be n-vectors in $\mathbb{R}^n$. Let $S(v_1, v_2, \dots, v_n)$ be the volume of n-parallelepiped form these n-vectors.

- Show that S is a multilinear function.
- Show that $S(v_1, \dots, v_i, v_j, \dots, v_n) = s(v_1, \dots, v_j, v_i, \dots, v_n)$.
- Show that $S(e_1, e_2, \dots, e_n) = 1$
- Show that $S(v_1, v_2, \dots v_n)$ is $|\det(A)|$, where A is a square matrix of size n whose i-th column is v_i.

a) Show that S is a multilinear function.

Proof:

Inductive definition of volume of n-parallelepiped form of n-vectors is given by,

- Base case (n=1):

$$V_1 = ||v_1|| = |v_1|$$

- Inductive step:

$$V_n = V_{n-1}||v_n \cdot w||$$

Where w is a unit normal vector that is orthogonal to $v_1, \dots, v_{n-1}$ and spanned by

$$v_1, \dots v_n$$

Then,

$$S(v_1, \dots, v_i + y_i, \dots, v_j) = S(v_1, \dots, v_{i-1})||v_i \cdot w_0 + y_i \cdot w_0||||v_{i+1} \cdot w_1|| \dots ||v_j \cdot w_{j-i}||$$

$$= S(v_1, \dots, v_{i-1})|||v_i|e_i \cdot w_0 + |y_i|e_i \cdot w_0||||v_{i+1} \cdot w_1|| \dots ||v_j \cdot w_{j-i}||$$

$$= S(v_1, \dots, v_{i-1})|||v_i|e_i \cdot w_0||||v_{i+1} \cdot w_1|| \dots ||v_j \cdot w_{j-i}|| + S(v_1, \dots, v_{i-1})|||y_i|e_i \cdot w_0||||v_{i+1} \cdot w_1|| \dots ||v_j \cdot w_{j-i}||$$

$$= S(v_1, \dots, v_i, \dots, v_j) + S(v_1, \dots, y_i, \dots, v_j)$$

Similar argument can be shown in scalar multiplication.

$$\therefore S(v_1, \dots, v_n) \text{ is a multilinear function.}$$

$\square$

b) Show that $S(v_1, \dots, v_i, v_j, \dots, v_n) = S(v_1, \dots, v_j, v_i, \dots, v_n)$.

Proof:

$$S(v_1, \dots, v_i, v_j, \dots, v_n) = S(v_1, \dots, v_{i-1})||v_i \cdot w_0||||v_{i+1} \cdot w_1|| \dots ||v_n \cdot w_{n-i}||$$

By definition, w_0, w_1 are orthogonal to each other and also orthogonal to $v_1, \dots v_{i-1}$

$$S(v_1, \dots, v_i, v_j, \dots, v_n) = S(v_1, \dots, v_{i-1})||v_i \cdot w_0||||v_{i+1} \cdot w_1|| \dots ||v_n \cdot w_{n-i}||$$

$$= S(v_1, \dots, v_{i-1})||v_{i+1} \cdot w_1||||v_i \cdot w_0|| \dots ||v_n \cdot w_{n-i}||$$
$$= S(v_1, \dots, v_j, v_i, \dots, v_n)$$

$\square$

c) $S(e_1, e_2, \dots, e_n) = 1$

Proof:

$$S(e_1, e_2, \dots, e_n) = |e_1||e_2 \cdot w_0| \dots |e_n \cdot w_{n-1}|$$

Since e_i is orthogonal to $e_1 \dots e_{i-1}$, $w_{i-1} = e_i$ for i in $\{2, 3, \dots n\}$

$$|e_1||e_2 \cdot w_0| \dots |e_n \cdot w_{n-1}| = |e_1||e_2 \cdot e_2| \dots |e_n \cdot e_n| = 1^n = 1$$

$\square$

d) Show that $S(v_1, v_2, \dots v_n)$ is $|\det(A)|$, where A is a square matrix of size n whose i-th column is v_i.

We have shown $S(v_1, \dots, v_n)$ is a dueling multilinear function with $S(e_1, \dots e_n) = 1$ which is same for the determinant definition. However since volume cannot be negative, we need to add *abs* function.

$$\therefore S(v_1 \dots v_n) = |\det([v_1 \dots v_n])| = |\det(A)|$$

$\square$

Exercise 2 (Problem 7.7)

Let B be the $n \times n$ matrix ($n \geq 3$) given by

$$B = \begin{pmatrix} 1 & -1 & -1 & -1 & \dots & -1 & -1 \\ 1 & -1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & -1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & -1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & -1 \end{pmatrix}$$

Prove that

$$\det(B) = (-1)^n (n-2) 2^{n-1}$$

$$\det(\begin{pmatrix} 1 & -1 & -1 & -1 & \dots & -1 & -1 \\ 1 & -1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & -1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \dots & -1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & -1 \end{pmatrix})$$

$$= \det(\begin{pmatrix} 1 & -1 & -1 & -1 & \dots & -1 & -1 \\ 2 & -2 & 0 & 0 & \dots & 0 & 0 \\ 2 & 0 & -2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 0 & 0 & 0 & \dots & -2 & 0 \\ 2 & 0 & 0 & 0 & \dots & 0 & -2 \end{pmatrix})$$

By using Cramer's rule along the first column,

$$= \det(\begin{pmatrix} -2 & 0 & 0 & \dots & 0 & 0 \\ 0 & -2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -2 \end{pmatrix}) + -2 \det(\begin{pmatrix} -1 & -1 & -1 & \dots & -1 & -1 \\ 0 & -2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -2 \end{pmatrix})$$

$$+2 \det(\begin{pmatrix} -1 & -1 & -1 & \dots & -1 & -1 \\ -2 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -2 \end{pmatrix}) \cdots + (-1)^{n-1} \det(\begin{pmatrix} -1 & -1 & -1 & \dots & -1 & -1 \\ -2 & 0 & 0 & \dots & 0 & 0 \\ 0 & -2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2 & 0 \end{pmatrix})$$

$$= (-2)^{n-1} - (-2)^{n-1} - (-2)^{n-1} - \underbrace{(-2)^{n-1} \cdots - (-2)^{n-1}}_{n-2 \text{ terms}}$$

$$= -(n-2)(-2)^{n-1} = (-1)^n(n-2)2^{n-1}$$

$\square$

0.1 Exercise 3(Problem 7.8)