Linear Algebra Assignment 10

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Exercise 1:

Prove the properties *(i)-(v)* mentioned in lecture 10.2
A real $n \times n$ matrix A is *symmetric positive definite*, for short SPD, iff it is symmetric and if

$$x^T Ax > 0 \text{ for all } x \in \mathbb{R}^n \text{ with } x \neq 0$$

The following facts about a symmetric positive definite matrix A are easily established:

(1) The matrix A is invertible
Proof:
By the rank-nullity theorem, $rank(A) + dim(null(A)) = n$

$$x \in null(A), \quad Ax = 0$$

$$x^T Ax = 0, \text{ and by definition of SPD matrix, if } x \neq 0, x^T Ax > 0$$

$$\therefore x = 0$$

$$dim(null(A)) = 0$$

$$rank(A) = n, \text{ thus invertible}$$

□

(2) We have $a_{ii} > 0$ for $i = 1, \ldots, n$.
Proof:

$$e_{ii}^T Ae_{ii} > 0 \text{ for all } i \in 1, \ldots n$$

$$e_{ii}^T Ae_{ii} = a_{ii} > 0$$

□

(3) For every $n \times n$ real invertible matrix Z, the matrix $Z^T AZ$ is real symmetric positive definite iff A is real symmetric positive definite.
Proof:

$$\text{Let } A \text{ be a SPD}$$

$$x^T (Z^T AZ)x = (x^T Z^T)A(Zx) = (Zx)^T A(Zx)$$

$$= y^T A y > 0 \ (y = Zx)$$

$$\therefore Z^T A Z \text{ is a SPD matrix.}$$

$\square$

(4) The set of $n \times n$ real symmetric positive definite matrices is convex. This means that if A and B are two $n \times n$ SPD matrices, then for any $\lambda \in \mathbb{R}$ such that $0 \leq \lambda \leq 1$, the matrix $(1-\lambda)A + \lambda B$ is also SPD.

Proof:

a) Symmetric

$$(1-\lambda)A^T + \lambda B^T = (1-\lambda)A + \lambda B$$

b) Positive definite

$$x^T((1-\lambda)A + \lambda B)x = (1-\lambda)x^T A x + \lambda x^T B x > 0 \quad (0 \leq \lambda \leq 1)$$

$$(1-\lambda)A + \lambda B \text{ is SPD.}$$

$\square$

(5) The set of $n \times n$ real symmetric positive definite matrices is a cone. This means that if A is SPD and if $\lambda > 0$ is any real, then λA is SPD.

Proof:

a) Symmetric

$$(\lambda A)^T = \lambda A^T = \lambda A$$

b) Positive definite

$$x^T(\lambda A)x = \lambda x^T A x > 0 \quad (\lambda > 0)$$

$$\therefore \lambda A \text{ is SPD.}$$

$\square$

Exercise 2 (Problem 8.4):

Solve the system using LU-factorization.

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&0&0&0\\2&1&0&0\\4&0&1&0\\3&0&0&1\end{pmatrix}\begin{pmatrix}2&1&1&0\\0&1&1&1\\0&3&5&5\\0&4&6&8\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&0&0&0\\2&1&0&0\\4&3&1&0\\3&4&0&1\end{pmatrix}\begin{pmatrix}2&1&1&0\\0&1&1&1\\0&0&2&2\\0&0&2&4\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1&0&0&0\\2&1&0&0\\4&3&1&0\\3&4&1&1\end{pmatrix}\begin{pmatrix}2&1&1&0\\0&1&1&1\\0&0&2&2\\0&0&0&2\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\2&1&0&0\\4&3&1&0\\3&4&1&1\end{pmatrix}\begin{pmatrix}2&1&1&0\\0&1&1&1\\0&0&2&2\\0&0&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

$$\begin{pmatrix}2&1&1&0\\0&1&1&1\\0&0&2&2\\0&0&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1&0&0&0\\-2&1&0&0\\2&-3&1&0\\3&-1&-1&1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\-3\\4\\6\end{pmatrix}$$

$$\begin{pmatrix}2&1&1&0\\0&1&1&1\\0&0&2&2\\0&0&0&2\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\-3\\4\\6\end{pmatrix}$$

$$x_4=3, x_3=-1, x_2=-5, x_1=3.5$$

$$x=\begin{pmatrix}3.5\\-5\\-1\\3\end{pmatrix}$$

Exercise 3 (Problem 8.13):

(1) Find a lower triangular matrix E such that

$$E\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&2&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&0\\-1&1&0&0\\0&-1&1&0\\0&0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&1&2&1\end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

(2) What is the effect of the product (on the left) with

$$E_{4,3;-1}E_{3,2;-1}E_{4,3;-1}E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$$

$$E_{i,j,\beta} = I + \beta e_{i,j}$$

on the matrix

$$Pa_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

$$E_{i,j;-1} \text{ represent changing row i} \Rightarrow \text{row i - row j}$$

Applying this from the right side to left,

$$Pa_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(3) Find the inverse of the matrix Pa_3

$$(Pa_3)^{-1} = E_{4,3;-1}E_{3,2;-1}E_{4,3;-1}E_{2,1;-1}E_{3,2;-1}E_{4,3;-1}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

$$(Pa_3)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}$$

(4) Consider the $(n+1) \times (n+1)$ Pascal matrix Pa_n whose ith row is given by the binomial coefficient

$$\binom{i-1}{j-1},$$

with $1 \leq i \leq n+1, 1 \leq j \leq n+1$ and with the usual convention that

$$\binom{0}{0} = 1, \binom{i}{j} = 0 \text{ if } j > i$$

$$Pa_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

Find n elementary matrices $E_{i_l, j_k; \beta_k}$ such that

$$E_{i_{n-1}, j_{n-1}; \beta_n} \cdots E_{i_1, j_1; \beta_1} Pa_n = \begin{pmatrix} 1 & 0 \\ 0 & Pa_{n-1} \end{pmatrix}$$

$$E_{2,1;-1} \cdots E_{n,n-1;-1},$$
$$i_k = n-k+1, j_k = n-k$$

$$\beta_k = -1 \ (\forall k = 1, \ldots, n-1)$$

$$\binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} - \binom{n-1}{k} = \binom{n-1}{k-1}$$

$$(E_{2,1;-1} \cdots E_{n,n-1;-1} Pa_n)_{i,j} = \binom{i-1}{j-1} - \binom{i-2}{j-1} = \binom{i-2}{j-2}$$

$$\therefore E_{2,1;-1} \cdots E_{n,n-1;-1} Pa_n = \begin{pmatrix} 1 & 0 \\ 0 & Pa_{n-1} \end{pmatrix}$$

Use the above to prove that the inverse of Pa_n is the lower triangular matrix whose ith row is given by the signed binomial coefficients

$$(-1)^{i+j-2} \binom{i-1}{j-1} = (-1)^{i+j} \binom{i-1}{j-1}$$

For example,

$$Pa_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{pmatrix}$$