Linear Algebra Assignment 9

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Exercise 1:

Solve the following linear system by Gaussian elimination.

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 14 \\ -1 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 2 & 3 & 2 & 3 & 14 \\ -1 & 0 & 1 & -1 & -1 \\ -2 & -1 & 4 & 0 & 2 \end{array}\right)$$

Row 2 $\rightarrow$ Row 2 - 2 Row 1

Row 3 $\rightarrow$ Row 3 + Row 1

Row 4 $\rightarrow$ Row 4 + 2Row 1

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 6 \\ 0 & 3 & 6 & 2 & 16 \end{array}\right)$$

Row 3 $\rightarrow$ Row 3 + 2Row 2

Row 4 $\rightarrow$ Row 4 + 3Row 2

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 6 & 5 & 16 \end{array}\right)$$

Row 4 $\rightarrow$ Row 4 - 3Row 3

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\therefore x_1 = 0, x_2 = 2, x_3 = 1, x_4 = 2$$

Exercise 2:

Consider the matrix

$$A = \begin{pmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{pmatrix}$$

When applying Gaussian elimination, which value of c yields zero in the scond pivot position?

$$2, \text{Row 2} \rightarrow \text{Row 2 - 2Row 1} = (0, 0, 1)$$

Which value of c yields zero in the third pivot position?

$1,$ the third row should be linearly dependent to (row1, row2) row1 + row2 = (3, 4+c,1) = (3,5,1)

In this case, what can you say about matrix A?

The matrix A is linearly dependent with rank 2

Exercise 3:

Apply **rref** to the matrix

$$A_2 = \begin{pmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{pmatrix}$$

Using a Python code, the **rref** form is,

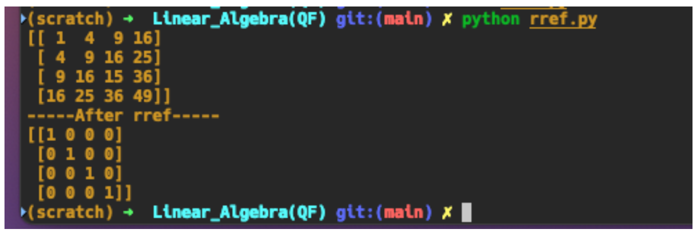

Figure 1: Rref from of A

Exercise 4:

(1) Prove that the dimension of the subspace of 2×2 matrices A, such that the sum of the entries of every row is the same (say c_1) and the sum of entries of every column is the same (say c_2) is 2.
The matrix form is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$a + b = c + d$$

$$a + c = b + d$$

Then,

$$d = a + b - c$$

$$a + c = b + a + b - c$$

$$c = b$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

$\therefore$ The number of free variable we need is 2 so the dimension is 2.

(2) Prove that the dimension of the subspace of 2×2 matrices A, such that the sum of the entries of every row is the same (say c_1), the sum of entries of

every column is the same(say c_2), and $c_1 = c_2$ is also 2. Prove that every such matrix is of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

and give a basis for this subspace.
Proof:
The proof from (1) can be used to prove this. and one basis can be,

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

(3) Prove that the dimension of the subspace of 3×3 matrices A, such that the sum of the entries of every row is the same (say c_1), the sum of entries of every column is the same (say c_2), and $c_1 = c_2$ is 5. Begin by showing that the above constraints are given by the set of equations

$$\begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \\ a_{31} \\ a_{32} \\ a_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Proof:
By looking into the matrix above, the Row 1 of 5×9 matrix represent summation of first row of A to be same as summation of second row. Similarly row 2 represent summation of row 2 to be same as that of row 3. Meaning that summation of row 1, row 2, row 3 of matrix A need to be same. Similarly by looking into row 3 row 4 of 5×9 matrix, the summation of each column needs to be same.
Also, by the final row of the 5×9 matrix,

$$a_{12} + a_{13} = a_{21} + a_{23}$$

Adding a_{11} to the each side,

$$a_{11} + a_{12} + a_{13} = a_{11} + a_{21} + a_{23}$$

Prove that every matrix satisfying the above constraints is of the form

$$\begin{pmatrix} a+b-c & -a+c+e & -b+c+d \\ -a-b+c+d+e & a & b \\ c & d & e \end{pmatrix}$$

Proof:
By using the Gaussian elimination,

$$\left(\begin{array}{ccccccccc|c} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccccccc|c} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 2 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccccccc|c} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 2 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -3 & 2 & 2 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccccccc|c} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 2 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -3 & 2 & 2 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & -2 & -2 & -2 & 0 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccccccccc|c} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & 0 \\ 0 & -2 & -1 & 2 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -3 & 2 & 2 & -1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So the free variable is last 5 indices.
Given,

$$\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a \\ b \\ c \\ d \\ e \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & -2 & -1 & 2 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -3 & 2 & 2 & -1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$a_{11} + a_{12} + a_{13} = a_{21} + a + b$$

$$a_{21} + a + b = c + d + e$$

$$-2a_{12} - a_{13} + 2a_{21} = -b - c + d$$

$$-3a_{13} + 2a_{21} = -2a + b - c - d + 2e$$

$$a_{21} = -a - b + c + d + e$$

$$-3a_{13} = 3b - 3c - 3d$$

$$a_{13} = -b + c + d$$

$$-2a_{12} = a_{13} - 2a_{21} - b - c + d = 2a - 2c - 2e$$

$$a_{12} = -a + c + e$$

$$a_{11} = -a_{12} - a_{13} + a_{21} + a + b = a + b - c$$

$$\therefore A = \begin{pmatrix} a + b - c & -a + c + e & -b + c + d \\ -a - b + c + d + e & a & b \\ c & d & e \end{pmatrix}$$

A basis is,

$$\left\{ \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

Exercise 5:

(1) Given any two permutations $\pi_1, \pi_2 : [n] \to [n]$, the permutation matrix $P_{\pi_2 \circ \pi_2}$ representing the composition of π_1 and π_2 is equal to the product $P_{\pi_2} P_{\pi_1}$ of the permutation matrices P_{π_1} and P_{π_2} representing π_1 and π_2 that is,

$$P_{\pi_2 \circ \pi_1} = P_{\pi_2} P_{\pi_1}$$

Proof: Matrix P_{π_1}

$$p^1_{ij} = \begin{cases} 1 \text{ if } i = \pi^1(j) \\ 0 \text{ if } i \neq \pi^1(j) \end{cases}$$

Matrix P_{π_2}

$$p^2_{ij} = \begin{cases} 1 \text{ if } i = \pi^2(j) \\ 0 \text{ if } i \neq \pi^2(j) \end{cases}$$

$$p^3_{ij} = \begin{cases} 1 \text{ if } i = \pi^2(\pi^1(j)) \\ 0 \text{ if } i \neq \pi^2(\pi^1(j)) \end{cases}$$

Matrix $P_{\pi_2 \circ \pi_1}$

$$= \begin{cases} 1 \text{ if } i = \pi^2 \circ \pi^1(j) \\ 0 \text{ if } i \neq \pi^2 \circ \pi^1(j) \end{cases}$$

$$\therefore P_{\pi_2 \circ \pi_1} = P_{\pi_2} P_{\pi_1}$$

(2) The matrix $P_{\pi_1^{-1}1}$ representing the inverse of the permutation π_1 is the inverse $P_{\pi_1}^{-1}$ of the matrix P_{π_1} representing the permutation π_1 that is,

$$P_{\pi_1^{-1}} = P_{\pi_1}^{-1}$$

Furthermore,

$$P_{\pi_1}^{-1} = (P_{\pi_1})^T$$

Proof:
Let $P_{\pi_1^{-1}}$ matrix where,

$$p^{-1}_{ij} = \begin{cases} 1 \text{ if } i = \pi^{-1}(j) \\ 0 \text{ if } i \neq \pi^{-1}(j) \end{cases}$$

$$P_{\pi_1} \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} \pi^{-1}(1) \\ \vdots \\ \pi^{-1}(n) \end{bmatrix}$$

$$P_{\pi_1^{-1}} P_{\pi_1} \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} \pi_1(\pi_1^{-1}(1)) \\ \vdots \\ \pi_1(\pi_1^{-1}(n)) \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ n \end{bmatrix}$$

$$\therefore P_{\pi_1^{-1}} P_{\pi_1}, P_{\pi_1^{-1}} = P_{\pi_1}^{-1}$$

(3) Prove that if P is the matrix associated with transposition, then $det(P) = -1$

Single transposition matrix is a matrix with single row exchange from identity matrix, hence $\det(P) = -1 \det(I) = -1$

(4) Prove that if P is a permutation matrix, then $\det(P) = \pm 1$.

By the proof from previous exercise, we know that permutation is multiple transposition so,

$$P = P_1 P_2 P_3 \ldots P_n \text{ where each } P_i \text{ is a transposition matrix}$$
$$\det(P) = \det(P_1) \ldots \det(P_n) = (-1)^n$$
$$\therefore \det(P) = \pm 1$$

(5) Use permutation matrices to give another proof of the fact that the parity of the number of transposition used to express a permutation π depends only on π

The parity can be expressed using det if the number of transposition is odd, $parity(\pi) = \det(P_\pi) = -1$ and if even, $parity(\pi) = \det(P_\pi) = 1$