Assignment 05, Real Analysis MIT

Student Hanseul Kim
Prof Dr. Casey Rodriguez

Answers

0.1 Exercise 2.2.9

Suppose $\{x_n\}$ is a sequence and suppose for some $x \in \mathbb{R}$, the limit

$$L := \lim_{n\to\infty} \frac{|x_{n+1} - x|}{|x_n - x|}$$

exists and $L < 1$. show that $\{x_n\}$ converges to x.

Proof:

Define

$$y_n = x_n - x$$

$$L := \lim_{n\to\infty} \frac{|y_{n+1}|}{|y_n|}$$

By ratio test lemma,

$$y_n \text{ converges and } \lim_{n\to\infty} y_n = 0$$

$$\lim_{n\to\infty} x_n - x = 0$$

$$\lim_{n\to\infty} x_n = x$$

□

0.2 Exercise 2.3.5

a) Let $x_n := \frac{(-1)^n}{n}$. Find $\limsup x_n, \liminf x_n$

$$\frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{-1}{n} \leq \lim_{n\to\infty} \frac{(-1)^n}{n} \leq \lim_{n\to\infty} \frac{1}{n}$$

$$0 \leq \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

$$\text{by squeeze theorem, limit exists and } \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

$$\therefore \limsup x_n = \liminf x_n = 0$$

b) Let $x_n := \frac{(n-1)(-1)^n}{n}$ Find $\limsup x_n, \liminf x_n$

$$\sup\{x_n \quad \forall n \geq M\} = 1$$

$$\inf\{x_n \quad \forall n \geq M\} = -1$$

$$\therefore \limsup x_n = 1, \liminf x_n = -1$$

0.3 Exercise 2.3.6

Let $\{x_n\}$ and $\{y_n\}$ be counded sequences such that $x_n \leq y_n$ for all n. Then show that

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$$

and

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$$

Proof:
By definition,

$$\limsup_{n\to\inf} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$$

$$\sup\{x_k : k \geq n\} \leq \sup\{y_k : k \geq n\}$$

$$\therefore \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$$

Similar argument can be applied to prove $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$

□

0.4 Exercise 2.3.7

Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences.
a) show that $\{x_n + y_n\}$ be bounded.
Proof:
By definition of bounded sequences,

$$\exists B_x \text{ s.t. } x_n \leq B_x \quad (\forall n \in \mathbb{N})$$

$$\exists B_y \text{ s.t. } y_n \leq B_y \quad (\forall n \in \mathbb{N})$$

$$\text{therefore, } x_n + y_n \leq B_x + B_y \quad (\forall n \in \mathbb{N})$$

$$x_n + y_n \text{ is bounded.}$$

□

b) show that

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) \leq (\liminf_{n\to\infty} x_n + y_n)$$

Proof:
Since $\{x_n + y_n\}$ is bounded sequence, by Theorem 2.3.4, there exists subsequence $\{x_{n_k} + y_{n_k}\}$ s.t,

$$\lim_{k\to\infty}(x_{n_k} + y_{n_k}) = \liminf_{n\to\infty}(x_n + y_n)$$

which is a convergence sequence. and is also bounded.
Let,

$$x_{n_{k_i}} := \inf\{x_{n_k} : n_k \geq n_{k_i}\}$$

$$x_{n_{k_i}} \leq x_{n_k}$$

$$y_{n_{k_i}} := \inf\{y_{n_k} : n_k \geq y_{k_i}\}$$

$$y_{n_{k_i}} \leq y_{n_k}$$

$$\text{both } x_{n_{k_i}}, y_{n_{k_i}} \text{ are convergent.}$$

$$\lim_{i\to\infty} x_{n_{k_i}} + \lim_{i\to\infty} y_{n_{k_i}} \leq \lim_{k\to\infty}(x_{n_k} + y_{n_k}) = \liminf_{n\to\infty}(x_n + y_n)$$

$$\therefore (\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) \leq (\liminf_{n\to\infty} x_n + y_n)$$

□

c) Find an explicit $\{x_n\}$ and $\{y_n\}$ such that

$$(\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) < (\liminf_{n\to\infty} x_n + y_n)$$

$$x_n = (-1)^n$$
$$y_n = (-1)^{n+1}$$
$$x_n + y_n = 0$$
$$\liminf_{n\to\infty} x_n = \liminf_{n\to\infty} y_n = -1$$
$$-2 = (\liminf_{n\to\infty} x_n) + (\liminf_{n\to\infty} y_n) < (\liminf_{n\to\infty} x_n + y_n) = 0$$

0.5 Exercise 5

Let $\{x_n\}$ be a bounded sequence of real numbers. Prove

$$\lim_{n\to\infty} x_n = 0$$

if and only if

$$\limsup_{n\to\infty} |x_n| = 0$$

Proof:
Forwad direction:

$$\lim_{n\to\infty} x_n = 0 \text{ which means,}$$
$$\lim n \to \infty |x_n| = 0$$
$$\text{Since limit exists and converges,}$$
$$\limsup_{n\to\infty} |x_n| = \liminf_{n\to\infty} |x_n| = 0$$
$$\therefore \limsup_{n\to\infty} |x_n| = 0$$

Backward direction:
Let

$$\limsup_{n\to\infty} |x_n| = 0$$
$$\lim_{n\to\infty} \sup_k \{|x_k| : k \geq n\} = 0$$
$$\text{let } \sup(|x_k| : k \geq n) = \alpha_n$$
$$\forall \epsilon > 0 \;\; \exists n \text{ s.t } \alpha_n < \epsilon$$
$$\forall k \geq n, \text{ since } \alpha \text{ is a superior,}$$
$$|x_k| \leq \alpha_k < \epsilon$$
$$\text{since } \epsilon \text{ does not depend on } k, \text{ it can be selected arbitrarily}$$
$$\lim_{n\to\infty} |x_n| = 0$$
$$\lim_{n\to\infty} x_n < \lim_{n\to\infty} |x_n| = 0$$

$$\therefore \lim_{n\to\infty} x_n = 0$$

□

0.6 Exercise 6

Does there exist a sequence $\{x_n\}$ such that

$$\liminf_{n\to\infty} x_n = -1, \quad \lim_{n\to\infty} x_n = 0, \quad \limsup_{n\to\infty} x_n = 1$$

No, for a limit to exist, $\lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$

Proof by contradiction:

Let such limit exists

$$\text{let } \liminf_{n\to\infty} x_n = a_n$$

$$\limsup_{n\to\infty} x_n = b_n$$

$$\text{by definition of limsup and liminf, } \forall \epsilon > 0, N \in \mathbb{N}$$

$$\exists n \geq N \text{ s.t. } |x_n - a_n| < \epsilon$$

$$\text{and by definition of limit } \exists N \in \mathbb{N} \text{ s.t. } |x_N - 0| < \epsilon$$

$$|x_n - 0| = |x_n - a_n + a_n| > |a_n| > \epsilon$$

contradiction.

□

0.7 Exercise 2.4.8

True or false, If $\{x_n\}$ is a Cauchy sequence, then there exists an M such that for all $n \geq M$ we have $|x_{n+1} - x_n| \leq |x_n - x_{n-1}|$

False

The sequence $\frac{\sin(n^2)}{n}$ is a Cauchy sequence but $|x_{n+1} - x_n|$ is not a decreasing sequence,