Abstract Algebra Bootcamp Assignment 1

Hanseul Kim

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Question 1:

Lex X, Y and Z be sets. If $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$

Answer:

Proof:
By definition,

$$\forall x \in X, x \in Y$$

$$\forall y \in Y, y \in Z$$

So,

$$\forall x \in X, x \in Y \text{ and } x \in Z$$

$$\therefore X \subseteq Z$$

$\square$

Question 2:

(1) $X \cap \emptyset = \emptyset$
Proof:

$$\forall x \in (X \cap \emptyset), x \in X \text{ and } x \in \emptyset$$

$$\therefore (X \cap \emptyset) \subseteq \emptyset \tag{a}$$

$$\forall x \in \emptyset$$

$$x \in X(\text{because there is no element!})$$

$$\therefore \emptyset \subseteq (X \cap \emptyset) \tag{b}$$

by a) and b)

$$(X \cap \emptyset) = \emptyset$$

$\square$

2) $X \cap X = X$
Proof:

$$\forall x \in (X \cap X), x \in X$$
$$\therefore (X \cap X) \subseteq X \tag{a}$$

$$\forall x \in X, x \in X \text{ and } x \in X$$
$$\therefore X \subseteq (X \cap X)$$

by a) and b)
$$X = (X \cap X)$$

3) $X \cap Y = Y \cap X$ $\square$
Proof:

$$\forall x \in X \cap Y, x \in X \text{ and } x \in Y$$
$$\text{then, } x \in Y \text{ and } x \in X$$
$$\therefore x \in Y \cap X$$
$$X \cap Y \subseteq Y \cap X$$

Similar argument can be applied to:

$$Y \cap X \subseteq X \cap Y$$

$$\therefore X \cap Y = Y \cap X$$

4) $(X \cap Y) \cap Z = X \cap (Y \cap Z)$ $\square$
Proof:
By definition,

$$\forall x \in (X \cap Y) \cap Z$$
$$x \in (X \cap Y) \text{ and } x \in Z$$
$$x \in X \text{ and } x \in Y \text{ and } x \in Z$$
$$x \in X \text{ and } x \in (Y \cap Z)$$
$$x \in X \cap (Y \cap Z)$$
$$\therefore (X \cap Y) \cap Z \subseteq X \cap (Y \cap Z)$$

Similar argument can be applied to prove:

$$X \cap (Y \cap Z) \subseteq (X \cap Y) \cap Z$$

$$\therefore (X \cap Y) \cap Z = X \cap (Y \cap Z)$$

$\square$

Question 3:

1) $X \cup \emptyset = X$ and $X \cup X = X$ **Proof**:

$$\forall x \in X \cup \emptyset, x \in X \text{ or } x \in \emptyset$$

$$\forall x, x \notin \emptyset$$

So,

$$x \in X$$

$$\therefore X \cup \emptyset \subseteq X$$

$$X \subseteq X \cup \emptyset, \text{ by definition}$$

$$\therefore X \cup \emptyset = X$$

$\square$

$$\forall x \in X \cup X, x \in X \text{ or } x \in X$$

$$x \in X$$

$$\therefore X \cup X \subseteq X$$

$$X \subseteq X \cup X, \text{ by definition}$$

$$\therefore X \cup X = X$$

$\square$

2) $X \cup Y = Y \cup X$ **Proof**:

$$\forall x \in X \cup Y, x \in X \text{ or } x \in Y$$

$$x \in Y or x \in X$$

$$\therefore X \cup Y \subseteq Y \cup X$$

Similar argument can be used to prove:

$$Y \cup X \subseteq X \cup Y$$

$$\therefore X \cup Y = Y \cup X$$

$\square$

Question 4:

Let X, Y and Z be sets. Which of the identities below are true?

1.
$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$
True.

2.
$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$
True,

3.
$$X \cup (Y \cap Z) = (X \cap Y) \cap (X \cup Z)$$
False.
$$\exists x \in X, x \notin Y \text{s.t.}$$
$$x \in X \cup (Y \cap Z) \text{ but,}$$
$$x \notin X \cap Y$$
$$x \notin (X \cap Y) \cap (X \cup Z)$$

4.
$$X \cup (Y \cap Z) = (X \cap Y) \cup (X \cup Z)$$
False.
$$\exists y \in Y, y \notin X \text{ s.t}$$
$$y \in X \cup (Y \cap Z) \text{ but,}$$
$$y \notin (X \cap Y)$$
$$t \notin (X \cap Y) \cup (X \cup Z)$$

5.
$$X \cup (Y \cap Z) = (X \cap Y) \cup (X \cap Z)$$
False.
$$\exists x \in X, x \notin Y, x \notin Z \text{ s.t,}$$
$$x \in X \cup (Y \cap Z) \text{ but,}$$
$$x \notin (X \cap Y) \cup (X \cap Z)$$

Question 5:

Let A be a subset of X. Which of the identities below are true?

1.
$$A \cap A^c = A$$

False.

Proof by Contradiction:

$$\forall x \in A \cap A^c$$

$$x \in A \text{ and } x \notin A$$

contradiction.

2.
$$A \cap A^c = \emptyset$$

True.

3.
$$A \cup A^c = \emptyset$$

Proof by Contradiction:

$$\forall x \in A \cup A^c$$

$$x \in A \text{ or } x \in A^c$$

$$x \in \emptyset$$

contradiction.

4.
$$A \cap A^c = X$$

False

by 2) from above,
$$A \cap A^c = \emptyset \neq X$$

5.
$$X \cup (Y \cap Z) = (X \cap Y) \cup (X \cap (Z)$$

False.

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

6.

$$X^c = \emptyset$$

False. (depends)

Proof by contradiction: let X be subset of natural number such that,

$$X = \{1, 2, 3, \}$$

$$4 \in X^c$$

contradiction.

Question 6:

Let A and B be subsets of X. Which of the identities below are true?

1.

$$(A \cup B)^c = A^c \cup B^c$$

False. **Proof by contradiction**: Let,

$$x \in A \text{ and } x \notin B$$
$$x \in A, x \in B^c$$
$$x \in A^c \cup B^c$$
$$x \notin (A \cup B)^c$$

contradiction.

2.

$$(A \cup B)^c = A^c \cap B^c$$

True.

3.

$$(A \cup B)^c = A^c \cap B^c$$

False. **Proof by contradiction** Let,

$$x \notin A \text{ and } x \notin B$$
$$x \in A, x \in B^c$$
$$x \notin (A^c \cap B^c)$$
$$x \in (A \cap B)^c$$

contradiction.

4.

$$(A \cap B)^c = A^c \cup B^c$$

True.