Assignment 06, Real Analysis MIT

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Answers

0.1 Exercise 2.5.3

Decide the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{3}{9n+1}$
diverges

$$\frac{3}{9n+1} > \frac{3}{10n} = \frac{3}{10} * \frac{1}{n}$$

$$\text{Since } \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, the series} \sum_{n=1}^{\infty} \frac{3}{9n+1} \text{ diverges.}$$

b) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$
diverges
similar logic to a)

$$\frac{1}{2n-1} > \frac{1}{3n} = \frac{1}{3} * \frac{1}{n}$$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$
converges

$$\frac{(-1)^n}{n^2} < \frac{1}{n^2}$$

$$\text{since the series } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges, } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ converges.}$$

d) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$
converges

$$\frac{1}{n(n+1)} < \frac{1}{n^2}$$

$$\text{since the series } \frac{1}{n^2} \text{ converges, } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converges.}$$

e) $\sum_{n=1}^{\infty} ne^{-n^2}$
converges

$$ne^{-n^2} = ne^{-n+n-n^2} = \frac{n}{e^n}(e^{-n^2+n}) < e^{-n^2+n} < e^{-n} = (\frac{1}{e})^n$$

$$\text{Since } \frac{1}{e} < 1 \text{ the series } \sum_{n=1}^{\infty} ne^{-n^2} \text{ converges.}$$

0.2 Exercise 2.5.4

a) Prove that if $\sum_{n=1}^{\infty} x_n$ converges, then $\sum_{n=1}^{\infty}(x_{2n} + x_{2n+1})$ also converges.
Proof:

$$\sum_{n=1}^{N}(x_{2n} + x_{2n+1}) = \sum_{n=2}^{2N+1} x_n$$

$$\sum_{n=2}^{\infty} x_n = \sum_{n=1}^{\infty} x_n - x_1 \text{ which converges.}$$

$$\therefore \text{ if } \sum_{n=1}^{\infty} x_n \text{ converges, then } \sum_{n=1}^{\infty}(x_{2n} + x_{2n+1}) \text{ also converges.}$$

□

b) Find and explicit example where the converse does not hold
let

$$x_n = (-1)^n$$

$$\sum_{n=1}^{\infty} x_n \text{ diverges}$$

$$\sum_{n=1}^{\infty}((-1)^{2n} + (-1)^{2n+1}) = \sum_{n=1}^{\infty}(1-1) = 0 \text{ which converges.}$$

□

0.3 2.5.10

Prove the triangle inequality for series, that is if $\sum x_n$ converges absolutely, then

$$|\sum_{n=1}^{\infty} x_n| \leq \sum_{n=1}^{\infty} |x_n|$$

Proof by induction:
Base case:

$$|x_n| = |\sum_{n=1}^{1} x_n| \leq \sum_{n=1}^{1} |x_n| = |x_n|$$

Induction
Let

$$|\sum_{n=1}^{N} x_n| \leq \sum_{n=1}^{N} |x_n|$$

$$|\sum_{n=1}^{N+1} x_n| = |\sum_{n=1}^{N} x_n + x_{N+1}| \leq |\sum_{n=1}^{N} x_n| + |x_{N+1}| \leq \sum_{n=1}^{N} |x_n| + |x_N + 1| = \sum_{n=1}^{N+1} |x_n|$$

$$\therefore \forall N \in \mathbb{N} \quad |\sum_{n=1}^{N} x_n| \leq \sum_{n=1}^{N} |x_n|$$

$$\text{Since the limit exists, } |\sum_{n=1}^{\infty} x_n| \leq \sum_{n=1}^{\infty} |x_n|$$

□

0.4 Exercise 2.6.1

Decide the convergence or divergence of the following series.

a) $\sum_{n=1}^{\infty} \frac{1}{2^{2n+1}}$
converges

$$\limsup_{n\to\infty} |\frac{1}{2^{2n+1}}|^{\frac{1}{n}} = \limsup_{n\to\infty} 2^{-2+\frac{1}{n}} = \frac{1}{4} < 1$$

$$\therefore \sum x_n \text{ converges absolutely and converges.}$$

$\square$

b) $\sum_{n=1}^{\infty} \frac{(-1)^n(n-1)}{n}$
diverges

$$x_n = \frac{n-1}{n} = 1 - \frac{1}{n} \text{ which is decreasing sequence.}$$

$$\text{By the proposition 2.6.2, the series } \sum_{n=1}^{\infty} \frac{(-1)^n(-1)}{n} \text{ converges.}$$

$$\text{but } \sum_{n=1}^{\infty}(-1^n) \text{ diverges, } \sum_{n=1}^{\infty} \frac{(-1)^n(n-1)}{n} \text{ diverges.}$$

$\square$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{10}}}$ converges

$$\text{since the sequence } x_n = \frac{1}{n^{1/10}} \text{ is a decreasing sequence and } \lim_{n\to\infty} \frac{1}{n^{1/10}} = 0$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{1}{10}}} \text{ converges.}$$

$\square$

d) $\sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{2n}}$
converges
Let,

$$x_n = \frac{n^n}{(n+1)^{2n}}$$

$$\limsup_{n\to\infty} |x_n|^{1/n} = \limsup_{n\to\infty} |\frac{n^n}{(n+1)^{2n}}|^{\frac{1}{n}} = \limsup_{n\to\infty} \frac{n}{(n+1)^2}$$

$$\lim_{n\to\infty} \frac{n}{(n+1)^2} = \lim_{n\to\infty} \frac{n}{(n+1)^2} = 0 < 1 \quad \text{(since, it converges)}$$

$$\therefore \sum_{n=1}^{\infty} \frac{n^n}{(n+1)^{2n}} \text{ converges.}$$

$\square$

0.5 Exercise 2.6.13

Find a series such that $\sum x_n$ converges but $\sum x_n^2$ diverges.

$$x_n = (-1)^n \frac{1}{\sqrt{n}}$$

$$\text{Since } y_n = \frac{1}{\sqrt{n}} \text{ is a decreasing sequence and converges to zero}$$

$$\sum x_n \text{ converges.}$$

However,

$$x_n^2 = \frac{1}{n}$$

$$\sum x_n^2 \text{ diverging sequence.}$$

$\square$