# Linear Algebra Assignment 1

Hanseul Kim

Oct 2024

## Exercise 1:

show that following are group homomorphisms.
**Key Point:**

Given two groups $(G, \cdot)$ and $(H, *)$. A function $\phi : G \to H$ is called **group homomorphism** if , for all $g_1, g_2 \in G$,

$$\phi(g_1 \cdot g_2) = \phi(g_1) * \phi(g_2)$$

1) $f : \mathbb{Z} \to \mathbb{Z}, f(n) = 2n$

$$\mathbb{Z} \text{ are groups under addition operation.}$$

$$\forall g_1, g_2 \in \mathbb{Z}, f(g_1), f(g_2) \in \mathbb{Z}$$

$$f(g_1 + g_2) = 2(g_1 + g_2) = 2g_1 + 2g_2 = f(g_1) + f(g_2)$$

$$\therefore f \text{ is a group homomorphism}$$

$\square$

2) $det : GL_n(\mathbb{C}) \to (\mathbb{C} - \{0\}, \cdot)$

$$GL_n(\mathbb{C}) \text{ forms group under matrix multiplication}$$

$$\forall g_1, g_2 \in GL_n(\mathbb{C}), f(g_1), f(g_2) \in \mathbb{R} - \{0\}$$

$$det(g_1 g_2) = det(g_1) det(g_2)$$

$$\therefore det \text{ is a group homomorphism}$$

$\square$

3) $f : (\mathbb{R}, +) \to (\mathbb{R} - \{0\}, \cdot), f(r) = e^r$

$$\forall g_1, g_2 \in \mathbb{R}, f(g_1), f(g_2) \in \mathbb{R} - \{0\}$$

$$f(g_1 + g_2) = e^{g_1 + g_2} = e^{g_1} \cdot e^{g_2} = f(g_1) \cdot f(g_2)$$

$$\therefore f \text{ is a group homomorphism}$$

$\square$

## Exercise 2:

Show that kernel and image of a homomorphism are subgroups

By definition of homomorphisms.

$$\text{let} f : G \to H \text{ be a homomorphism where } G, H \text{ are groups.}$$

$$\text{image of } f, \forall g_1, g_2 \in G,$$
$$f(g_1 g_2) = f(g_1)f(g_2)$$
$$f(g_1) = f(g_1 e_G) = f(g_1)f(e_G)$$
$$f(e_G) = e_H \text{ so identity element exists in the image}$$
$$\forall h_1, h_2 \in img(f(G)), h_1 = f(g_1), h_2 = f(g_2) \text{ for some } g_1, g_2 \in G$$
$$h_1 h_2 = f(g_1)f(g_2) = f(g_1 g_2) = f(g_3) \in img(f(G))$$
$$\text{image of } f \text{ is closed}$$
$$e_H = f(e_G) = f(gg^{-1}) = f(g)f(g^{-1})$$
$$\text{so, } f(g)^{-1} = f(g^{-1}), \text{ so inverse element does exists}$$
$$\therefore \text{ image of homomorphism are subgroups}$$

$$\text{Kernel of } f, g \in G \text{ such that } f(g) = e_H$$
$$e_G, f(e_G) = e_H \text{ so there exists identity element in kernel.}$$
$$f(g_1 g_2) = f(g_1)f(g_2) = e_H e_H = e_H, \text{ kernel is closed.}$$
$$f(g^{-1}) = f(g)^{-1} = e_H^{-1} = e_H, \text{ so the inverse element exists in kernel.}$$
$$\therefore \text{ kernel forms a subgroup}$$

□

### Exercise 3:

Show that $\mathbb{R}/\mathbb{Z}$ is isomorphic to $S^1$.

**Proof:**

$$\mathbb{R}/\mathbb{Z} \text{ forms a group under addition}$$

$$S^1 \text{ forms a group under addition}$$

Define a function from $\phi : \mathbb{R}/\mathbb{Z} \to S^1$

$$\phi(\theta) = e^{2\pi\theta i} = \cos(2\pi\theta) + i\sin(2\pi\theta)$$

$$\phi \text{ is well defined.}$$

$$\text{suppose, } \theta' - \theta = n \in \mathbb{Z}$$

$$\phi(\theta') = e^{2\pi i(\theta+n)} = e^{2\pi i\theta} \cdot e^{2\pi in} = e^{2\pi i\theta}$$

$$\therefore \phi(\theta) = \phi(\theta'), \phi \text{ is well defined.}$$

$$\phi(\theta_1 + \theta_2) = e^{2\pi i(\theta_1+\theta_2)} = e^{2\pi\theta_1} \cdot e^{2\pi\theta_2} = \phi(\theta_1) + \phi(\theta_2)$$

$$\therefore \phi \text{ is a group homomorphism.}$$

□