Assignment 03, Real Analysis MIT

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Answers

0.1 Exercise 1

. Suppose $x, y \in \mathbb{R}$ and $x < y$. Prove there exists $i \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < i < y$.
Proof:
Proof by Contradiction:
Let

$$x, y \in \mathbb{R} \text{ and } x < y, \text{ There is no irrational between } x, y \tag{a}$$
$$x - \sqrt{2} \in R, y - \sqrt{2} \in R$$
$$\text{Since } \exists r \in \mathbb{Q} \text{ s.t } x - \sqrt{2} < r < y - \sqrt{2}$$
$$x < r + \sqrt{2} < y$$
$$\text{and by a), } r + \sqrt{2} \in \mathbb{Q}$$
$$\text{Thus, } \sqrt{2} \in \mathbb{Q}$$

Contradiction.
$\therefore$ $x, y \in \mathbb{R}$ and $x < y$. there exists $i \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < i < y$

□

0.2 Exercise 2

Let $E \subset (0, 1)$ be the set of all real numbers with decimal representation using only digits 1 and 2:

$$E := \{x \in (0, 1) : \forall j \in \mathbb{N}, \exists d_{-j} \in \{1, 2\} \text{ such that} x = 0.d_{-1}d_{-2}\dots\}$$

Prove that $|E| = \mathcal{P}(\mathbb{N})$
Proof:

$$f : E \to \mathcal{P}(\mathbb{N})$$
$$x \in E, x = 0.d_{-1}d_{-2}\dots$$
$$f(x) = \{j \in: d_{-j} = 2\}$$

1) f is one-to-one

$$\forall x, y \in E \text{ where } x \neq y$$

Then

$$\exists j \in N \text{ s.t } d^x_{-j} \neq d^y_{-j} \in \{1, 2\}$$
$$\text{Then if } j \in f(x), j \notin f(y)$$
$$j \in f(y), f \notin f(x)$$
$$\therefore f(x) \neq f(y)$$

2) f is onto

$$\forall y = f(x),$$

We can define x as,

$$x = 0.d_{-1}d_{-2}\dots \text{s.t}$$

$$d_{-j} = 2 \text{ if } j \in y \text{ and } d_{-j} = 1 \text{ if } x \notin y$$

$$\therefore f \text{ is one-to-one and onto. thus,}$$

$$|E| = \mathcal{P}(\mathbb{N})$$

$\square$

0.3 Exercise 3

a) Let A and B be two disjoint, countably infinite sets. Prove that $A \cup B$ is countably infinite.

Proof:

Let

$$f \text{ be a function } f : \mathbb{N} \to (A \cup B) \text{ s.t}$$

$$f(n) = A_i \ (n = 2i + 1, i \in \mathbb{N}))$$

$$f(n) = B_j \ (n = 2j, j \in \mathbb{N}))$$

1) f is one-to-one:

$$\forall n \neq m$$

$$\text{if } n = 2i + 1 \text{ and } m = 2j$$

$$f(n) = A_i \in A \text{ and } f(m) = B_j \in B$$

$$\text{Since } A \text{ and } B \text{ are disjoint, } A \neq B$$

$$\text{if n, m are both even or both odd, } n \neq m \text{ so } i_n \neq i_m \text{ and } A_{i_n} \neq A_{i_m}$$

$$\therefore f \text{ is one-to-one}$$

2) f is onto: We will prove only set A case.

$$\forall A_i \in A$$

$$\exists n \in \mathbb{N} \text{ s.t } f(n) = A_i \text{ where } n = 2i + 1$$

$$\therefore f \text{ is onto}$$

$$\text{Since } \exists f : \mathbb{N} \to (A \cup B) \text{ s.t. } f \text{ is a bijection, } A \cup B \text{ is countable.}$$

$\square$

b) Prove that the set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, is uncountable.

Proof:

Proof by contradiction:

We know,

1. $\mathbb{Q}$ is countable infinite.
2. $\mathbb{R} \setminus \mathbb{Q}$ is infinite.
3. $\mathbb{R}$ is uncountable

Let,

$$\mathbb{R} \setminus \mathbb{Q} \text{ is countable infinite}$$

then by proposition a)

$$\mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q} \text{ is countable}$$

$$\mathbb{R} \text{ is countable}$$

Contradiction

$$\therefore \mathbb{R} \setminus \mathbb{Q} \text{ is uncountalbe infinite}$$

$\square$

0.4 Exercise 4

Let A be a subset of $\mathbb{R}$ which is bounded above, and let a_0 be an upper bound for A. Prove that $a_0 = supA$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $a_0 - \epsilon < a$

Proof:

Proof by contradiction:

Let

$$\forall \epsilon > 0, \forall a \in A, \text{ then } a_0 - \epsilon \geq a$$

$$b = a_0 - \epsilon$$

$$\text{Since } b \geq a, \ \forall a \in A$$

$$b \text{ is a upper bound}$$

$$a_0 = b + \epsilon > b$$

$$a_0 \text{ is not least upper bound}$$

Contradiction

$$\forall \epsilon > 0 \ \exists a \in A \text{ s.t } a_0 - \epsilon < a$$

□

0.5 Exercise 5

We say a set $U \subset \mathbb{R}$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subset U$$

Since the definition is vacuous for $U = \emptyset$, it follows that the empty set is open.
It is also clear from the definition that $U = \mathbb{R}$ is open. a) Let $a, b \in \mathbb{R}$ which $a < b$. Prove that the sets $(-\infty, a), (a, b)$, and (b, ∞) are open.

Proof:

$$\forall x \in (-\infty, a), \ x < a$$

$$\text{so, } \exists \epsilon > 0 \text{ s.t } x + \epsilon < a$$

$$\therefore (x - \epsilon, x + \epsilon) \subset (-\infty, a)$$

$$\forall x \in (a, b), \ a < x, x < b$$

$$\exists \epsilon_0 > 0 \text{ s.t } x - \epsilon_0 > a$$

$$\exists \epsilon_1 > 0 \text{ s.t } x + \epsilon_1 < b$$

$$\text{let } \epsilon = \min(\epsilon_0, \epsilon_1), \text{then}$$

$$a < x - \epsilon < x + \epsilon < b$$

$$\therefore (x - \epsilon, x + \epsilon) \subset U$$

$$\forall x \in (b, \infty), \ b < x$$

$$\exists \epsilon > 0 \text{ s.t } x - \epsilon > b$$

$$\therefore (x - \epsilon, x + \epsilon) \subset (b, \infty)$$

$$\therefore (-\infty, a), (a, b), \text{ and } (b, \infty) \text{ are open.}$$

□

b) Let Λ be a set (not necessarily a subset of $\mathbb{R}$), and for each $\lambda \in \Lambda$, let $U_\lambda \subset \mathbb{R}$. Prove that if U_λ is open for all $\lambda \in \Lambda$ the set

$$\bigcup_{\lambda \in \Lambda} U_\lambda = \{x \in \mathbb{R} : \exists \lambda \in \Lambda \text{ such that } x \in U_\lambda\}$$

is open.

Proof:

$$\forall x \in \bigcup_{\lambda \in \Lambda} U_\lambda,\ \exists \lambda \in \Lambda \text{ s.t.}$$

$$x \in U_\lambda$$

$$\text{then, since } U_\lambda \text{ is open}$$

$$\forall \epsilon > 0, (x - \epsilon, x + \epsilon) \subset U_\lambda$$

$$(x - \epsilon, x + \epsilon) \subset U_\lambda \subset \bigcup_{\lambda \in \Lambda} U_\lambda$$

$$\therefore \bigcup_{\lambda \in \Lambda} U_\lambda \text{ is open.}$$

$\square$

c) Let $n \in \mathbb{N}$, and let $U_1, \ldots U_n \subset \mathbb{R}$. Prove that if $U_1, \ldots, U_n$ are open then the set

$$\bigcap_{m=1}^{n} U_m = \{x \in \mathbb{R} : x \in U_m \forall m = 1, \ldots, n\}$$

is an open set.

$$\forall x \in \bigcap_{m=1}^{n} U_m$$

$$x \in U_m \ (\forall m = 1, \ldots, n)$$

$$\text{Since all } U \text{ are open, } (x - \epsilon, x + \epsilon) \subset U_m$$

$$(x - \epsilon, x + \epsilon) \subset \bigcap_{m=1}^{n} U_m$$

$$\therefore \bigcap_{m=1}^{n} U_m \text{ is open.}$$

$\square$

d) Is the set of rational $Q \subset R$ open?
False.

$$\forall x \in \mathbb{Q}$$

$$\forall \epsilon > 0$$

$$x - \epsilon, x + \epsilon \in \mathbb{R}$$

$$\text{By the completeness of reals there is irrational number in } (x - \epsilon, x + \epsilon)$$

$$(x - \epsilon, x + \epsilon) \not\subset \mathbb{Q}$$

Set of rational is not open. $\square$

0.6 Exercise 6

Prove that

$$\lim_{n\to\infty} \frac{1}{20n^2 + 20n + 2020} = 0$$

Proof:

$$\frac{1}{20n^2 + 20^n + 2020} < \frac{1}{20n^2 + 20n} < \frac{1}{20n} < \frac{1}{n} < \epsilon$$

$$\text{we can set } n > \frac{1}{\epsilon} \text{ by Archimedean principle}$$

$$\therefore \left|\frac{1}{20n^2 + 20n + 2020} - 0\right| < \epsilon \ (\forall \epsilon > 0)$$

$$\lim_{n\to\infty} \frac{1}{20n^2 + 20n + 2020} = 0$$

□