Assignment 01, Real Analysis MIT

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Answers

0.1 Exercise 0.3.6

a) Prove:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

pf:

$$\forall x \in A \cap (B \cup C)$$
$$x \in A \text{ and } x \in (B \cup C)$$
$$x \in A \text{ and } x \in B \text{ or } x \in C$$
$$x \in A \text{ and } x \in B \text{ or } x \in A \text{ and } x \in C$$
$$x \in (A \cap B) \cup (A \cap C)$$
$$\therefore A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C) \tag{a}$$

$$\forall x \in (A \cap B) \cup (A \cap C)$$
$$x \in A \cap B \text{ or } x \in A \cap C$$
$$x \in A \text{ and } x \in B \text{ or } x \in A \text{ and } x \in C$$
$$x \in A \text{ and } x \in B \text{ or } x \in C$$
$$x \in A \text{ and } x \in (B \cup C)$$
$$x \in A \cap (B \cup C)$$
$$\therefore (A \cap B) \cup (A \cap C) \subset A \cap (B \cup C) \tag{b}$$

because of a) and b)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

□

b) Prove:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$\forall x \in A \cup (B \cap C)$$
$$x \in A \text{ or } x \in B \cap C$$
$$x \in A \text{ or } x \in B \text{ and } x \in C$$
$$x \in A \text{ or } x \in B \text{ and } x \in A \text{ or } x \in C$$
$$x \in (A \cup B) \cap (A \cup C)$$
$$\therefore A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C) \tag{a}$$

$$\forall x \in (A \cup B) \cap (A \cup C)$$
$$x \in A \cup B \text{ and } x \in A \cup C$$
$$x \in A \text{ or } x \in B \text{ and } x \in A \text{ or } x \in C$$
$$x \in A \text{ or } x \in B \text{ and } x \in C$$
$$x \in A \text{ or } x \in B \cap C$$
$$x \in A \cup (B \cap C)$$
$$\therefore (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C) \tag{b}$$

because of a) and b)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

□

0.2 Exercise 0.3.11

Proof by Induction:

We wish to prove the following statement by induction:

$$n < 2^n \quad \text{for all} \quad n \in \mathbb{N}$$

Base Case $n = 1$:
For $n = 1$:

$$1 < 2^1 = 2$$

Thus, the base case holds.

Induction Hypothesis:
Assume that for some arbitrary $t \in \mathbb{N}$, the inequality holds:

$$t < 2^t$$

This is the induction hypothesis.

Inductive Step:
We now need to prove that the inequality holds for $t + 1$, i.e.,

$$t + 1 < 2^{t+1}.$$

From the induction hypothesis, we know that $t < 2^t$. To show that $t + 1 < 2^{t+1}$, we proceed as follows:

$$t + 1 < 2^t + 1$$

Since $2^t + 1 < 2^t + 2^t = 2^{t+1}$, it follows that:

$$t + 1 < 2^{t+1}$$

0.3 Exercise 0.3.12

Show that for a finite set A of cardinality n, the cardinality of $\mathcal{P}(A)$ is 2^n.

Let A be a finite set with n elements, i.e.,

$$A = \{a_1, a_2, \ldots, a_n\}.$$

The power set $\mathcal{P}(A)$ consists of all subsets of A, including the empty set and A itself. Hence, the number of elements in $\mathcal{P}(A)$ is the number of subsets of A, which we aim to show is 2^n.

Define the function f from $\{1, 2, 3, \ldots, 2^n\}$ to $\mathcal{P}(A)$

We will define a function f from the set $\{1, 2, 3, \ldots, 2^n\}$ to the power set $\mathcal{P}(A)$. This function will map each integer $x \in \{1, 2, 3, \ldots, 2^n\}$ to a subset of A.

Binary Representation Each number $x \in \{1, 2, 3, \ldots, 2^n\}$ can be uniquely represented in binary as a sequence of n bits. Each bit is either 0 or 1, and the position of the bits corresponds to the elements of A. Let $\mathcal{B} = \{2^0, 2^1, \ldots, 2^{n-1}\}$ be the set of powers of 2. Then, the number x can be written in binary as:

$$x = [v]_{\mathcal{B}} = v_1 v_2 \ldots v_n,$$

where $v_i \in \{0, 1\}$ for each $i \in \{1, 2, \ldots, n\}$.

Subset Mapping Define the function $f : \{1, 2, 3, \ldots, 2^n\} \to \mathcal{P}(A)$ by:

$$f(x) = \{a_i \mid v_i = 1\},$$

where v_i is the i-th bit of the binary representation of x. In other words, if the i-th bit of x is 1, then a_i is included in the subset $f(x)$, and if the i-th bit of x is 0, then a_i is not included in the subset.

Bijection Proof To prove that f is a bijection, we must show that:

- f is injective (one-to-one), and
- f is surjective (onto).

Injectivity Assume $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \{1, 2, 3, \dots, 2^n\}$. This means that the subsets of A corresponding to x_1 and x_2 are identical. However, since the binary representation of a number uniquely determines the subset of A, this implies that $x_1 = x_2$. Therefore, f is injective.

Surjectivity Given any subset $S \subseteq A$, we can construct a binary sequence $[v]_{\mathcal{B}} = v_1 v_2 \dots v_n$ such that $v_i = 1$ if $a_i \in S$, and $v_i = 0$ if $a_i \notin S$. The number x corresponding to this binary sequence is $f(x) = S$. Thus, every subset of A is the image of some $x \in \{1, 2, 3, \dots, 2^n\}$, and hence f is surjective.

Conclusion Since f is both injective and surjective, it is a bijection. Therefore, the cardinality of the power set $\mathcal{P}(A)$ is 2^n, which completes the proof:

$$|\mathcal{P}(A)| = 2^n.$$

0.4 Exercise 0.3.15

Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$

a) lemma: $n^3 + 5n$ is divisible by 2

Case $n = 2k, (k \in \mathbb{N})$:

$$n^3 + 5n = 8k^3 + 10k = 2 \times (4k^3 + 5k)$$

k is divisible by 2

Case $n = 2k + 1, (k \in \mathbb{N})$:

$$\begin{aligned} n^3 + 5n &= (2k+1)^3 + 10k + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 10k + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2 \times (4k^3 + 6k^2 + 3k + 3) \end{aligned}$$

$n^3 + 5n$ is divisible by 2

$\therefore$ n is divisible by 2 for all $n \in \mathbb{N}$

b) lemma: $n^3 + 5n$ is divisible by 3

Case $n = 3k, (k \in \mathbb{N})$:

$$n^3 + 5^n = 27k^3 + 15k = 3 \times (9k^3 + 5k) \tag{1}$$

k is divisible by 3

Case $n = 3k + 1, (k \in \mathbb{N})$:

$$n^3 + 5^n = 27k^3 + 27k^2 + 9k + 1 + 15k + 5 = 3 \times (9k^3 + 9k^2 + 8k + 2) \tag{2}$$

k is divisible by 3

Case $n = 3k + 2, (k \in \mathbb{N})$:

$$n^3 + 5^n = 27k^3 + 54k^2 + 36k + 8 + 15k + 10 = 3 \times (9k^3 + 18k^2 + 17k + 6) \tag{3}$$

$n^3 + 5n$ is divisible by 3

$\therefore$ n is divisible by 3 for all $n \in \mathbb{N}$

Conclusion by lemma a), b), $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$

0.5 Exercise 0.3.19

Give an example of a countably infinite collections of finite sets $A_1, A_2, \dots,$ whose union is not a finite set.

let

$$A_i = \{i\} \ (i \in \mathbb{N})$$

A_i is a finite set with cardinality of 1.
The union of A_i is $\{1, 2, \dots\}$ which is a countably infinite set.

0.6 Exercise 6

Prove that:

$$|\{q \in \mathbb{Q} \ : \ q > 0\}| = |\mathbb{N}|$$

Theorem. Let $q \in \mathbb{Q}$ with $q > 0$. Then:

1. If $q \in \mathbb{N}$ and $q \neq 1$, then there exist unique prime numbers $p_1 < p_2 < \cdots < p_N$ and unique exponents $r_1, \dots, r_N \in \mathbb{N}$ such that
$$q = p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}, \tag{†}$$
2. If $q \notin \mathbb{N}$, then there exist unique prime numbers $p_1 < p_2 < \cdots < p_N$, $q_1 < q_2 < \cdots < q_M$ with $p_i \neq q_j$ for all $i \in \{1, \dots, N\}$ and $j \in \{1, \dots, M\}$, and unique exponents $r_1, \dots, r_N, s_1, \dots, s_M \in \mathbb{N}$ such that
$$q = \frac{p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}}{q_1^{s_1} q_2^{s_2} \cdots q_M^{s_M}}. \tag{‡}$$

Define $f : \{q \in \mathbb{Q} : q > 0\} \to \mathbb{N}$ as follows: $f(1) = 1$, if $q \in \mathbb{N} \setminus \{1\}$ is given by (†), then

$$f(q) = p_1^{2r_1} \dots p_N^{2r_N}$$

and if $q \in \mathbb{Q} \setminus \mathbb{N}$ is given by (‡), then

$$f(q) = p_1^{2r_1} \dots p_N^{2r_N} q_1^{2s_1 - 1} \dots q_M^{2s_M - 1}$$

a) compute $f(4/15)$

$$\frac{4}{15} = \frac{2^2}{3 * 5}$$
$$f(4/15) = 2^{2*2} * 3^1 * 5^1 = 240$$
$$f(q) = 108 \neq k^2 \ (\forall k \in \mathbb{N})$$
$$108 = 2^2 * 3^3 = 2^{2*1} * 3^{2*2-1}$$
$$\frac{2^1}{3^2} = \frac{2}{9} = q$$

b) Prove that:

$$f \text{ is a bijection}$$

Lemma: f is one-to-one
Let:
f is not one-to-one

$$\exists x, y \in \mathbb{Q}$$

such that

$$x \neq y, f(x) = f(y)$$
$$f(x) = p_{1x}^{2r_{1x}} \dots p_{Nx}^{2r_{Nx}} q_{1x}^{2s_{1x}-1} \dots q_{Mx}^{2s_{Mx}-1}$$
$$f(y) = p_{1y}^{2r_{1y}} \dots p_{Ny}^{2r_{Ny}} q_{1y}^{2s_{1y}-1} \dots q_{My}^{2s_{My}-1}$$

$$p_{1x}^{2r_{1x}} \dots p_{Nx}^{2r_{Nx}} q_{1x}^{2s_{1x}-1} \dots q_{Mx}^{2s_{Mx}-1} = p_{1y}^{2r_{1y}} \dots p_{Ny}^{2r_{Ny}} q_{1y}^{2s_{1y}-1} \dots q_{My}^{2s_{My}-1}$$

$$\frac{p_{1x}^{2r_{1x}} \dots p_{Nx}^{2r_{Nx}} q_{1x}^{2s_{1x}-1} \dots q_{Mx}^{2s_{Mx}-1}}{p_{1y}^{2r_{1y}} \dots p_{Ny}^{2r_{Ny}} q_{1y}^{2s_{1y}-1} \dots q_{My}^{2s_{My}-1}} = 1$$

By Theorem above, exponent and the base must be same for all p, q sets of x, y. so, $r_{1x} = r_{1y} \dots r_{Nx} = r_{Ny}$ and $s_{1x} = s_{1y} \dots s_{Nx} = s_{Ny}$ and by the same theorem, each x, y is unique because it is a unique mapping.
thus,

$$x = y$$

contradiction
$\therefore$ f is one-to-one

Lemma: f is onto
Since every natural number can be decomposed into multiple of primes based on the theorem above, and there is a inverse mapping from

$$p_1^{2r_1} \dots p_N^{2r_N} q_1^{2s_1-1} \dots q_M^{2s_M-1} \quad \text{to}$$
$$q = \frac{p_1^{r_1} p_2^{r_2} \cdots p_N^{r_N}}{q_1^{s_1} q_2^{s_2} \cdots q_M^{s_M}}$$

f is onto

$$\therefore f \text{ is a bijection}$$

$\square$