Linear Algebra Assignment 2

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Problem 3.3:

Let $E = \mathbb{R} \times \mathbb{R}$, and define the addition operation

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad x_1, x_2, y_1, y_2 \in \mathbb{R}$$

and the multiplication operation $\mathbb{R} \times \mathbb{R} \to E$ by

$$\lambda \cdot (x, y) = (\lambda x, y), \quad \lambda, x, y \in \mathbb{R}$$

Show that E with the above operations $+$ and $\cdot$ is not a vector space.
Contradicts V2
V2) $(\alpha + \beta) \cdot u = (\alpha \cdot u) + (\beta \cdot v)$

$$u = (1, 2) \in E$$
$$(0 + 0) \cdot u = 0 \cdot u = (0, 2)$$
$$0 \cdot u + 0 \cdot u = (0, 2) + (0, 2) = (0, 4)$$
$$(0 + 0)u \neq 0 \cdot u + 0 \cdot u$$

□

Problem 3.4:

1) Prove that the axioms of vector spaces imply that

$$\alpha \cdot 0 = 0$$
$$0 \cdot v = 0$$
$$\alpha \cdot (-v) = -(\alpha \cdot v)$$
$$(-\alpha) \cdot v = -(\alpha \cdot v)$$

for all $v \in E$ and all $\alpha \in K$, where E is a vector space over K.
Proof:
1)

$$\alpha \cdot 0 = \alpha \cdot (0 + 0) = \alpha \cdot 0 + \alpha \cdot 0$$
$$\therefore \alpha \cdot 0 \text{ is a additive identity, } 0$$

2)

$$0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v$$
$$\therefore 0 \cdot v \text{ is a additive identity, } 0$$

3)

$$\alpha \cdot (-v) + \alpha \cdot v = \alpha \cdot (v - v) = 0$$
$$\alpha \cdot (-v) \text{ is a additive inverse of } \alpha \cdot v$$
$$\therefore \alpha \cdot (-v) = -\alpha \cdot v$$

4)

$$(-\alpha) \cdot v + \alpha \cdot v = (-\alpha + \alpha) \cdot v = 0 \cdot v = 0$$
$$(-\alpha) \cdot v \text{ is a additive inverse of } \alpha \cdot v$$
$$\therefore (-\alpha) \cdot v = -(\alpha \cdot v)$$

$\square$

2) For every $\lambda \in \mathbb{R}$ and every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define λx by

$$\lambda x = \lambda(x_1, \cdots, x_n) = (\lambda x_1, \cdots, \lambda x_n)$$

Recall that every vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ can be written uniquely as

$$x = x_1 e_1 + \cdots + x_n e_n$$

For any operation $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, if $\cdot$ satisfies the Axiom (V1) of a vector space, then prove that for any $\alpha \in \mathbb{R}$, we have,

$$\alpha \cdot x = \alpha \cdot (x_1 e_1 + \cdots + x_n e_n) = \alpha \cdot (x_1 e_1) + \cdots + \alpha \cdot (x_n e_n)$$

Proof:
Proof by induction
base case($n = 2$))

$$\text{By V1,} \alpha \cdot (x_1 e_1 + x_2 e_2) = \alpha \cdot x_1 e_1 + \alpha \cdot x_2 e_2$$

inductive step

Suppose the statement holds.

by V1,

$$\alpha \cdot (x_1 e_2 + \cdots + x_{n+1} e_{n+1})$$
$$= \alpha \cdot (x_1 e_1 + \cdots + x_n e_n) + \alpha \cdot x_{n+1} e_{n+1}$$

and by the statement,

$$= \alpha \cdot x_1 + \cdots + \alpha \cdot x_{n+1}$$

$$\therefore \alpha \cdot x = \alpha \cdot (x_1 e_1 + \cdots + x_n e_n) = \alpha \cdot (x_1 e_1) + \cdots + \alpha \cdot (x_n e_n)$$

$\square$

3) Use (2) to define operations $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfy the Axioms (V1-V3), but for which Axiom V4 fails.

V1) $\alpha \cdot (u + v) = (\alpha \cdot u) + (\alpha \cdot v)$
V2) $(\alpha + \beta) \cdot (u) = (\alpha \cdot u) + (\alpha \cdot u)$
V3) $(\alpha * \beta) \cdot u = a \cdot (\beta \cdot u)$
V4) $1 \cdot u = u$

Define operation $\cdot$ as

$$\lambda \cdot x = (\lambda x_1, 0, 0, \ldots, 0)$$

$$\alpha \cdot (u + v) = (\alpha(u_1 + v_1), 0, \ldots, 0) = \alpha \cdot u + \alpha \cdot v, \text{ satisfies V1.}$$

$$(\alpha + \beta) \cdot u = ((\alpha + \beta)u_1, 0, \ldots, 0) = (\alpha \cdot u) + (\beta \cdot u), \text{ satisfies V2.}$$

$$(\alpha * \beta) \cdot u = (\alpha\beta u_1, 0, \ldots, 0) = \alpha \cdot (\beta u_1, 0, \ldots, 0) = \alpha \cdot (\beta \cdot u), \text{ satisfies V3}$$

$$\text{However, } 1 \cdot (u) = (u_1, 0, \ldots, 0) \neq (u_1, \ldots, u_n) = u$$

$$\text{Does not satisfy V4}$$

□

4) For any operation $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfies the Axioms (V2-V3), then for every rational number $r \in \mathbb{Q}$ and every vector $x \in \mathbb{R}^n$, we have

$$r \cdot x = r(1 \cdot x)$$

V2) $(\alpha + \beta) \cdot (u) = (\alpha \cdot u) + (\alpha \cdot u)$
V3) $(\alpha * \beta) \cdot u = a \cdot (\beta \cdot u)$

Proof:

$$r(1 \cdot x) = (r * 1) \cdot x \text{ (by V3)}$$
$$= r \cdot x$$

And by (3) $1 \cdot x$ can be some vector $(y_1, \ldots, y_n) \in \mathbb{R}^n$ not necessarily equal to $x = (x_1, \ldots, x_n)$ and

$$r(1 \cdot x) = (ry_1, \ldots, ry_n)$$

□

by 2),3),4), any operation $\mathbb{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfies the Axioms (V1-V3) is completely determined by the action of 1 on the one-dimensional subsapces of $\mathbb{R}^n$ spanned by $e_1, \ldots, e_n$

Problem 3.6

Let A_2 be the following matrix:

$$A_2 = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 3 & 2 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & -1 & 3 & 0 \end{pmatrix}$$

Express the fourth column of A_2 as a linear combination of the first three columns of A_2

$$2\begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix}$$

Show $x = (7, 14, -1, 2)$ as a linear combination of column vectors.
Since the fourth column is linearly dependent to first three columns, we only need to show that it can/cannot be expressed and linear combination of first three columns.

$$\alpha\begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix} + \beta\begin{pmatrix} 2 \\ 3 \\ 0 \\ -1 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 0 \end{pmatrix}$$

No such solution exists.

Problem 3.9

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

Since H^T have the same rank as H and $rank(H^T H) \leq rank(H) \leq 4$

$$H^T H = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Which consists of scalar multiplication of e_1, e_2, e_3, e_4. So $rank(H^T H) = 4$

$$\therefore 4 = rank(H^T H) \leq rank(H) \leq 4$$

$$rank(H) = 4$$

Thus H consists of linearly independent column vectors.

$\square$