

Linear Algebra Assignment 3

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Exercise 1:

Let $f : V \rightarrow W$ be a linear map. Show that kernel and image of f are subspaces.

Proof:

To prove that the subset is a subspace, we need to show

1. The subset is non-empty
2. The subset is closed under addition and scalar multiplication

1. Image of f

$$0 \in V$$

$$f(0) \in W$$

$$\text{kernel is not empty} \tag{a}$$

Since f is a linear map, for every v, w in W

$$f(v) + f(w) = f(v + w) \in W$$

$$\forall c \in \mathbb{R}, cf(v) = f(cv) \in W$$

$$f \text{ is closed under scalar multiplication and addition} \tag{b}$$

by (a), (b) image of f is subspace

2. Kernel of f

Since f is a linear map, for every c in \mathbb{R}

$$f(0) = f(c * 0) = c * f(0)$$

$$f(0) \text{ is a additive identity}$$

$$0 \in \text{img}(f) \tag{a}$$

$$\forall v, w \in \text{img}(f) \text{ and } c \in \mathbb{R}$$

$$f(v + w) = f(v) + f(w) = 0, v + w \in \text{img}(f)$$

$$f(cv) = cf(v) = c * 0 = 0, cv \in \text{img}(f)$$

$$f \text{ is closed on scalar multiplication and addition} \tag{b}$$

by (a), (b) kernel of f is subspace

□

Exercise 2:

Show that the space V/W constructed in the lecture is a vector space.

Proof:

Elements of quotient space is defined as $v + W = \{v + w | w \in W\}$

1) Closed under addition

$$x, y \in V/W, x = v_1 + W, y = v_2 + W$$

$$x + y = v_1 + W + v_2 + W = v_1 + v_2 + W = v_3 + W \in V/W$$

2) Closed under scalar multiplication

$$\forall c \in \mathbb{R}$$

$$c * x = cv + W = v_1 + W \in V/W$$

3) Commutativity

$$x + y = v_1 + v_2 + W = v_2 + v_1 + W = y + x$$

4) Associativity

$$(x + y) + z = (v_1 + v_2) + v_3 + W = v_1 + (v_2 + v_3) + W = x + (y + z)$$

5) Additive inverse

$$\forall x \in V/W, x = v + W$$

$$-x = -v + W$$

$$x + (-x) = v - v + W = 0 + W, \text{ which is a additive identity}$$

Since V is a vector space, scalar associativity, distribution law holds.

$\therefore V/W$ is a vector space.

□

Exercise 3:

Show that the quotient map (η) constructed in the lecture is surjective and its kernel is subspace W .

$$\begin{aligned}\eta : V &\rightarrow V/W \\ \eta(v) &= v + W\end{aligned}$$

Proof:

1) Surjectivity

$$\begin{aligned}\forall x \in V/W, x &= \{v + w | w \in W\} = v + W, \text{ and } v \in V \\ \eta(v) &= v + W = x\end{aligned}$$

2) Kernel is subspace of W

$$\begin{aligned}\text{let additive identity of quotient space } &v_0 + W \\ \forall v \in V, v + W + v_0 + W &= v + v_0 + W = v + W \\ v + v_0 - v &= v_0 \in W \\ \therefore \text{ set of } v_0 &\text{ is a subset of } W\end{aligned}$$

□

Exercise 4:

Let A_1 be the following matrix:

$$A_1 = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -3 & -5 & 1 \end{pmatrix}$$

1) Prove that columns of A_1 are linearly independent.

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$1) \quad 2\lambda_1 + 3\lambda_2 + \lambda_3 = 0$$

$$2) \quad \lambda_1 + 2\lambda_2 - \lambda_3 = 0$$

$$3) \quad -3\lambda_1 - 5\lambda_2 + \lambda_3 = 0$$

$$1) + 2) = \quad 3\lambda_1 + 5\lambda_2 = 0$$

$$1) - 3) = \quad 5\lambda_1 + 8\lambda_2 = 0$$

$$\lambda_1 = -2\lambda_2$$

$$1) + 2) + 3) = \lambda_3 = 0$$

$$\therefore \lambda_1, \lambda_2, \lambda_3 = 0$$

columns of A_1 are linearly independent.

□

2) coordinates for $x = (6, 2, -7)$

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -7 \end{pmatrix}$$

\therefore coordinate is $(1, 1, 1)$

Exercise 4:

Let $f : E \rightarrow F$ be a linear map which is also a bijection. Prove that the inverse function $f^{-1} : F \rightarrow E$ is linear.

Proof:

1) Additivity

Because f is a bijection

$\forall x, y \in F$, there exists unique $v, w \in E$ where,

$$f(v) = x, f(w) = y$$

$$f(v + w) = f(v) + f(w) = x + y$$

$$f^{-1}(x + y) = v + w = f^{-1}(x) + f^{-1}(y)$$

2) Scalar Multiplication

Because f is a bijection

$\forall x \in F$, there exists unique $v \in E$ where,

$$f(v) = x$$

$$\forall c \in \mathbb{R}, f(cv) = cf(v) = cx$$

$$f^{-1}(cx) = cv$$

\therefore by 1), 2) f^{-1} is linear.

□

Exercise 5:

Given two vectors spaces E and F , let $(u_i)_{i \in I}$ be any basis of E and let $(v_i)_{i \in I}$ be any family of vectors in F . Prove that the unique linear map $f : E \rightarrow F$ such that $f(u_i) = v_i$ for all $i \in I$ is surjective iff $(v_i)_{i \in I}$ spans F .

Proof:

1) Forward case (surjective \rightarrow spans F)

let f be surjective. for all $v \in F$, there exists $u \in E$ where,

$$f(u) = v$$

$$u = \lambda_1 u_1 + \cdots + \lambda_n u_n \quad ((u_i)_{i \in I} \text{ are basis vectors})$$

because f is linear map,

$$\begin{aligned} v = f(u) &= f(\lambda_1 u_1 + \cdots + \lambda_n u_n) = \lambda_1 f(u_1) + \cdots + \lambda_n f(u_n) \\ &= \lambda_1 v_1 + \cdots + \lambda_n v_n \\ &\therefore (v_i)_{i \in I} \text{ spans } F \end{aligned}$$

2) Backward case (spans $F \rightarrow$ surjective)

Because $(v_i)_{i \in I}$ spans F

$\forall v \in F$, there exists $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ where,

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = v$$

because f is linear map,

$$f(\lambda_1 u_1 + \cdots + \lambda_n u_n) = \lambda_1 f(u_1) + \cdots + \lambda_n f(u_n) = \lambda_1 v_1 + \cdots + \lambda_n v_n = v$$

a) f is onto

$\forall x, y \in E$ where $x \neq y$

there exists unique different set of λ_i^x, λ_i^y so that,

$$\lambda_1^x u_1 + \cdots + \lambda_n^x u_n = x, \lambda_1^y u_1 + \cdots + \lambda_n^y u_n = y$$

$$f(x) - f(y) = (\lambda_1^x - \lambda_1^y) v_1 + \cdots + (\lambda_n^x - \lambda_n^y) v_n$$

Since λ_i^x and λ_i^y are different, $f(x) - f(y) \neq 0$

f is one to one

b) f is surjective.

$\therefore f$ is surjective iff $(v_i)_{i \in I}$ spans F .

□