Linear Algebra Assignment 6

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Exercise 1(Problem 6.6):

definition might be wrong
An $n \times n$ matrix H is *upper Hessenberg* if $h_{jk} = 0$ for all (j, k) such that $j - k \geq 0$.
An upper Hessenberg matrix is *unreduced* if $h_{i+1i} \neq 0$ for $i = 1, \ldots, n-1$
Prove that if H is a singular unreduced upper Hessenberg matrix,
then $dim(Ker(H)) = 1$

A example unreduced upper Hessenberg matrix is

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & h_{33} & \cdots & h_{3n} \\ 0 & 0 & h_{43} & \cdots & h_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{nn} \end{bmatrix}$$

Proof:

a) The first $n-1$ columns are linearly independent.
Let,

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & h_{33} & \cdots & h_{3n} \\ 0 & 0 & h_{43} & \cdots & h_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{nn} \end{bmatrix} = [v_1, v_2, \ldots, v_n]$$

$$\alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1} = 0$$

$$\begin{bmatrix} \sum_{i=1}^{n-1} \alpha_i h_{1i} \\ \sum_{i=1}^{n-1} \alpha_i h_{2i} \\ \sum_{i=2}^{n-1} \alpha_i h_{3i} \\ \sum_{i=3}^{n-1} \alpha_i h_{4i} \\ \vdots \\ \alpha_{n-2} h_{n-1n-2} + \alpha_{n-1} h_{n-1n-1} \\ \alpha_{n-1} h_{nn-1} \end{bmatrix}$$

$$\text{Since } h_{nn-1} \neq 0, \alpha_{n-1} = 0$$

$$\text{and by proof by strong induction, } \alpha_i = 0 \text{ for all } i = 1, 2, \ldots, n-1$$

$$\text{Hence, } v_1, \ldots, v_n \text{ column vectors are linearly independent.}$$

b)

$$dim(H) = n$$

$$dim(Ker(H)) > 0 \text{ because it's singular.}$$

$$dim(Img(H)) \geq n{-}1 \text{ because it has least } n{-}1 \text{ linearly independent column vectors.}$$

$$\text{by rank nulity theorm } dim(Ker(H)) = n - dim(Img(H)) \leq n - (n-1) = 1$$

$$0 < dim(Ker(H)) \leq 1$$

$$\therefore dim(Ker(H)) = 1$$

$\square$

Exercise 2(Problem 6.7):

Let A be any $n \times k$ matrix
1) Prove that the $k \times k$ matrix $A^T A$ and the matrix A have the same nullspace. Use this to prove that $rank(A^T A) = rank(A)$. Similarly, prove that the $n \times n$ matrix AA^T and the matrix A^T have the same nullspace. and conclude that $rank(AA^T) = rank(A^T)$.
Proof:
1) $Ker(A) = Ker(A^T A)$
a) forward case $Ker(A) \to Ker(A^T A)$

$$\forall u \in Ker(A), Au = 0$$

$$A^T Au = A^T(Au) = A^T 0 = 0$$

$$\therefore u \in Ker(A^T A)$$

b) backward case $Ker(A^T A) \to Ker(A)$

$$v \in Ker(A^T A), A^T Av = 0$$

$$v^T(A^T Av) = v^T(0) = 0$$

$$0 = v^T A^T A v = ||Av||^2$$

$$Av = 0$$

$$\therefore v \in Ker(A)$$

$$\text{by a), b) } Ker(A) = Ker(A^T A)$$

$$\text{Let } B = A^T \text{ then,} Ker(B) = Ker(B^T B)$$

$$\therefore Ker(A^T) = Ker(AA^T)$$

$\square$

2-1) Let $a_1, \ldots, a_k$ be k linearly independent vectors in $\mathbb{R}^n (1 \le k \le n)$, and let A be the $n \times k$ matrix whose ith column is a_i. Prove that $A^T A$ has rank k, and that it is invertible.
Proof:

$$A^T A \text{ is a } k \times k \text{ matrix.}$$

$$\dim(Ker(A^T A)) = \dim(Ker(A)) = 0 \text{ (since, columns of } A \text{ are linearly independent.)}$$

$$\dim(A^T A) = \dim(Img(A^T A)) + \dim(Ker(A^T A)) \text{ (rank nullity theorem)}$$

$$\therefore rank(A^T A) = \dim(Img(A^T A)) = k - 0 = k$$

$$\text{Since } A^T A : V \to V \ (V \in \mathbb{R}^k)$$

$$\text{Since } \dim(Img(A^T A)) = k = \dim(V)$$

$$A^T A \text{ is surjective.}$$

$$\dim(Ker(A^T A)) = 0, \ A^T A \text{ is injective.}$$

$$\therefore A^T A \text{ is a isomorphism thus invertible.}$$

$\square$

2-2) Let $P = A(A^T A)^{-1} A^T$ (an $n \times n$ matrix) Prove that

$$P^2 = P$$

$$P^T = P$$

What is the matrix P when $k = 1$?
Proof:

$$P^2 = A(A^T A)^{-1} A^T A (A^T A)^{-1} A^T$$

$$= A(A^T A)^{-1} (A^T A)(A^T A)^{-1} A^T$$

$$= A(A^T A)^{-1} A^T = P$$

$$P^T = (A(A^TA)^{-1}A^T)^T$$
$$= A((A^TA)^{-1})^TA^T$$

inverse of a symmetric matrix is symmetric.

$$\text{Let, } B^T = B, (B^{-1})^T = (B^{-1})^TBB^{-1}$$
$$= (B^TB^{-1})^TB^{-1} = (BB^{-1})^TB^{-1}$$
$$= (I)^TB^{-1} = B^{-1}$$
$$\text{So, } A((A^TA)^{-1})^TA^T = A(A^TA)^{-1}A^T = P$$

□

When $k = 1$,

$$P = v(v^Tv)^{-1}v^T = v \cdot \frac{v^T}{||v||^2}$$

is a vector projection.

Exercise 3(Problem 6.10):

(Affine subspaces) A subset $\mathcal{A}$ of $\mathbb{R}^n$ is called an *affine subspace* if either $\mathcal{A} = \emptyset$, or there is some vector $a \in \mathbb{R}^n$ and some subspace U of $\mathbb{R}^n$ such that,

$$\mathcal{A} = a + U = \{a + u | u \in U\}$$

We define the dimension $\dim(\mathcal{A})$ of $\mathcal{A}$ as the dimension $\dim(U)$ of U.
1) If $\mathcal{A} = a + U$, why is $a \in \mathcal{A}$?
Proof:

$$\text{Since } U \text{ is a subspace, } 0 \in U$$
$$a + 0 \in \mathcal{A}$$

□

What are affine subspaces of dimension 0?

$$\mathcal{A} = \{a\}$$

What are affine subspaces of dimension 1 (begin with $\mathbb{R}^2$)

a line in $\mathbb{R}^n$ that passes a

What are affine subspaces of dimension 2 (begin with $\mathbb{R}^3$)

a plane in $\mathbb{R}^n$ that passes a

2) Prove that if $\mathcal{A} = a + U$ is any nonempty affine subspace, then $\mathcal{A} = b + U$ for any $b \in \mathcal{A}$

Proof:

$$\text{Since } b \in \mathcal{A},$$

$$b = a + u, \text{ for some } u \in U$$

$$b - a \in U$$

$$\forall v \in \mathcal{A}, v = a + w \text{ (for some } w \in U)$$

$$v = a + w = b - b + a + w = b + u + w$$

$$u + w \in U$$

$$\therefore \mathcal{A} = b + U \text{ for any } b \in \mathcal{A}$$

3) Let $\mathcal{A}$ be any nonempty subset of $\mathbb{R}^n$ closed under affine combinations. For any $a \in \mathcal{A}$, prove that

$$U_a = \{x - a \in \mathbb{R}^n | x \in \mathcal{A}\}$$

is a (linear) subspace of $\mathbb{R}^n$ such that

$$\mathcal{A} = a + U_a$$

Proof:

Since U_a is a subset of $\mathbb{R}^n$, we only need to proof that U_a is closed under addition and scalar multiplication.

$$\text{for } u_1, u_2 \in U_a$$

$$u_1 = -a + x_1 \text{ for some } x_1 \in \mathcal{A}$$

$$u_1 = -a + a + v_1 \text{ for some } v_1 \in U$$

$$u_2 = a + x_2 \text{ for some } x_2 \in \mathcal{A}$$

$$u_2 = -a + a + v_2 \text{ for some } v_2 \in U$$

$$u_1 + u_2 = v_1 + v_2 = -a + a + (v_1 + v_2) \in U_a$$

$$cu_1 = -a + a + cu_1 \in U_a$$

$$\therefore U_a \text{ is a subspace.}$$

□

Remark: The subspace U is called the *direction* of $\mathcal{A}$

4) Two nonempty affine subspaces $\mathcal{A}$ and $\mathcal{B}$ are said to be *parallel* iff they have the same direction. Prove that if $\mathcal{A} \neq \mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ are parallel, then $\mathcal{A} \cap \mathcal{B} = \emptyset$

Proof:

Proof by contradiction

$$\text{Let two parallel affine subspace, } \mathcal{A}, \mathcal{B} \text{ where, } \mathcal{A} \neq \mathcal{B}$$

$$v \in \mathcal{A} \cap \mathcal{B}$$

$$v = a + u_1 \text{ for some } u_1 \in U$$

$$v = b + u_2 \text{ for some } u_2 \in U$$

$$a + u_1 = b + u_2$$

$$a - b = u_2 - u_1 \in U$$

$$a - b = 0, \text{ contradiction.}$$

$$\therefore \mathcal{A} \neq \mathcal{B} \text{ and } \mathcal{A} \text{ and } \mathcal{B} \text{ are parallel, then } \mathcal{A} \cap \mathcal{B} = \emptyset$$

□