Assignment 08, Real Analysis MIT

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Answers

0.1 Exercise 3.1.3

Prove the following:
Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}, g : S \to \mathbb{R}$, and $h : S \to \mathbb{R}$ are functions such that

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in S$$

Suppose the limits of $f(x)$ and $h(x)$ as x goes to c and both exist, and

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x)$$

Then the limit of $g(x)$ as x goes to c exists and

$$\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x)$$

Proof:
Since the limit exists, let

$$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$$

$$\forall \epsilon > 0, \text{ there exists } \epsilon_1, \epsilon_2 \text{ such that,}$$
$$\forall x \text{ such that } |x - c| < \delta_1, |f(x) - L| < \epsilon$$
$$\forall x \text{ such that } |x - c| < \delta_2, |h(x) - L| < \epsilon$$

$$\text{If we select } \delta = min(\delta_1, \delta_2)$$
$$L - \epsilon < f(x) < L + \epsilon$$
$$L - \epsilon < h(x) < L + \epsilon$$
$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$$
$$|g(x) - L| < \epsilon$$
$$\therefore \lim_{x \to c} g(x) = L$$
$$\lim_{x \to c} g(x) = \lim_{x \to c} f(x) = \lim_{x \to c} h(x)$$

□

0.2 Exercise 2

Let

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 2x & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that f is continuous at $x = 0$ and discontinuous at $x = 1$.
Proof:
Let

$$\forall \epsilon > 0$$
$$\delta = \frac{\epsilon}{2}$$
$$\forall |x - 0| < \delta = \frac{\epsilon}{2}$$

- $$\text{if } x \in \mathbb{Q}, \quad f(x) = 0$$
  $$|f(x) - 0| = |0 - 0| = 0 < \epsilon$$
- $$\text{if } x \notin \mathbb{Q}, \quad f(x) = 2x$$
  $$|f(x) - 0| = |2x - 0| = |2x| < \epsilon$$

$$\therefore f(x) \text{ is continuous at } x = 0$$

Proof by Contradiction:
Let $f(x)$ continuous at $x = 1$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t}$$
$$|x - 1| < \delta \rightarrow |f(x) - f(1)| < \epsilon$$
$$\forall \text{ sequences of } x_n \text{ s.t } \lim_{n\to\infty} x_n = c \quad \text{then, } \lim_{n\to\infty} f(x) = f(1)$$
$$\text{let } x_n \text{ be sequence of irrational number which converges to } 1$$
$$\lim_{n\to\infty} x_n = 1, \lim_{n\to\infty} f(x_n) = 2 \neq 0 = f(1)$$

contradiction.

$$\therefore f(x) \text{ is not continuous at } x = 1$$

0.3 Exercise 3.2.11

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, Suppose $f(c) > 0$. show that there exits an $\alpha > 0$ such that, for all $x \in (c-\alpha, c+\alpha)$ we have $f(x) > 0$
Proof:
Since f is continuous at c,

$$\forall \epsilon > 0, \ \exists \delta > 0 \text{ s.t.}$$
$$|x - c| < \delta \rightarrow |f(x) - f(c)| < \epsilon$$

Let

$$\epsilon = \frac{f(c)}{2} > 0, \alpha = \delta$$
$$|x - c| < \alpha \rightarrow |f(x) - f(c)| < \frac{f(c)}{2}$$
$$\text{meaning, } x \in (c - \alpha, c + \alpha) \rightarrow \frac{3f(c)}{2} > f(x) > \frac{f(c)}{2} > 0$$
$$\therefore \exists \alpha > 0 \text{ s.t. } \forall x \in (c - \alpha, c + \alpha) \text{ we have} f(x) > 0$$

□

0.4 Exercise 3.2.14

Let $f : [-1, 0] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous and $f(0) = g(0)$, Define $h : [-1, 1] \to \mathbb{R}$ by $h(x) := f(x)$ if $x \leq 0$ and $h(x) := g(x)$ if $x > 0$. Show that h is continuous.

Proof:

By definition

$$\forall\epsilon > 0, \exists\delta_1, \delta_2 > 0 \text{ such that}$$
$$|x - 0| < \delta_1 \to |f(x) - f(0)| < \epsilon$$
$$|x - 0| < \delta_2 \to |g(x) - g(0)| < \epsilon$$

Let

$$\delta_3 = min(\delta_1, \delta_2) > 0$$
$$x \leq 0 \text{ and } |x - 0| < \delta_3$$
$$|x - 0| < \delta_3 \leq \delta_2 \to |h(x) - h(0)| = |f(x) - f(0)| < \epsilon$$
$$x > 00 \text{ and } |x - 0| < \delta_3$$
$$|x - 0| < \delta_3 \leq \delta_1 \to |h(x) - h(0)| = |g(x) - g(0)| < \epsilon$$

$$|x - 0| < \delta_3 \to |h(x) - h(0)| < \epsilon$$
$$\therefore h \text{ is continuous at } x = 0$$

also $h(x)$ is continuous at every other point by definition.

□

0.5 Exercise 5

Let $f : \mathbb{R} \to \mathbb{R}$. Recall that if $U \subset \mathbb{R}$, the inverse image of U is the set

$$f^{-1}(U) := \{x \in \mathbb{R} : f(x) \in U\}$$

Prove that f is continuous if and only if for every open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is open.

Proof:

Forward case:

f is continuous.

$$\forall\epsilon > 0 \quad \exists\delta > 0 \text{ such that,}$$
$$|x - c| < \delta \to |f(x) - f(c)| < \epsilon$$
$$\forall x \in (c - \delta, c + \delta) \to f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$$

if $U \subset \mathbb{R}$ is a open set,

$$\forall c \in U \ \exists\epsilon > 0 \text{ such that,}$$
$$f(x) \in (f(c) - \epsilon, f(c) + \epsilon) \subset U$$

because f is a continuous function, there exists delta,

$$x \in (c - \delta, c + \delta) \subset f^{-1}(U)$$

$$\therefore \text{ if } f \text{ is continuous } f^{-1}(U) \text{ is open.}$$

Backward case:

Let,

$$\forall c \in U \subset \mathbb{R}, \text{ then,}$$
$$\exists\epsilon > 0, (c - \epsilon, c + \epsilon) \subset U$$
$$\forall f^{-1}(c), \exists\delta \text{ s.t } (f^{-1}(c) - \delta, f^{-1}(c) + \delta) \subset f^{-1}(U)$$
$$\forall x \in |x - f^{-1}(c)| < \delta \to |f(x) - c| < \epsilon$$
$$\therefore f \text{ is continuous at } f^{-1}(U) \quad \forall c \in U$$

□