Linear Algebra Assignment 7

Hanseul Kim

Jan 2025

Exercise 1(Problem 7.1):

Prove that every transposition can be written as a product of basic transpositions.

Definition: A *transposition* is a permutation $\tau : [n] \to [n]$ such that, for some $i < j$ (with $1 \leq i < j \leq n$), $\tau(i) = j, \tau(j) = i,$ and $\tau(k) = k$ for all $k \in [n] - \{i, j\}$

Proof:

Let a transformation $\tau_{i,j}$ where,

$$i < j \text{ (with } 1 \leq i < j \leq n), \tau_{i,j}(i) = j, \tau_{i,j}(j) = i,$$

$$\text{and } \tau_{i,j}(k) = k \text{ for all } k \in [n] - \{i, j\}$$

Define two transformation $\tau_{i,m}, \tau_{m,j}$ where $m \notin \{i, k\}$

$$i < m \text{ (with } 1 \leq i < m \leq n), \tau_{i,m}(i) = m, \tau_{i,m}(m) = i,$$

$$\text{and } \tau_{i,m}(k) = k \text{ for all } k \in [n] - \{i, m\}$$

$$i < j \text{ (with } 1 \leq m < j \leq n), \tau_{m,j}(m) = j, \tau_{m,j}(j) = m,$$

$$\text{and } \tau_{m,j}(k) = k \text{ for all } k \in [n] - \{m, j\}$$

Then,

$$\tau_{m,j}(\tau_{i,m}(\tau_{m,j}(\{x_1, \ldots x_i, \ldots x_m, \ldots, x_j, \ldots, x_n\})))$$

$$= \tau_{m,j}(\tau_{i,m}(\{x_1, \ldots x_i, \ldots x_j, \ldots, x_m, \ldots, x_n\}))$$

$$= \tau_{m,j}(\{x_1, \ldots, x_m, \ldots, x_j, \ldots x_i \ldots x_n\})$$

$$\{x_1, \ldots x_j, \ldots x_m, \ldots, x_i, \ldots, x_n\} = \tau_{i,j}(\{x_1, \ldots, x_i, \ldots, x_j, \ldots x_n\})$$

Exercise 2(Problem 7.2)

(1) Given two vectors in $\mathbb{R}^2$ of coordinates $(c_1 - a_1, c_2 - a_2)$ and $(b_1 - a_1, b_2 - a_2)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Proof:
a) Forward case:
Let $(c_1 - a_1, c_2 - a_2)$ and $(b_1 - a_1, b_2 - a_2)$ are linearly dependent.
Then,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 - a_1 & c_1 \\ a_2 & b_2 - a_2 & c_2 \\ 1 & 1 - 1 & 1 \end{vmatrix} + \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 - a_1 & c_1 \\ a_2 & b_2 - a_2 & c_2 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 - a_1 & c_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 \\ 1 & 0 & 1 - 1 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 - a_1 & a_1 \\ a_2 & b_2 - a_2 & a_2 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 - a_1 & c_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

b) backward case:
Let

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$0 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 - a_1 & c_1 \\ a_2 & b_2 - a_2 & c_2 \\ 1 & 1 - 1 & 1 \end{vmatrix} + \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 - a_1 & c_1 \\ a_2 & b_2 - a_2 & c_2 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 - a_1 & c_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 \\ 1 & 0 & 1 - 1 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 - a_1 & a_1 \\ a_2 & b_2 - a_2 & a_2 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 - a_1 & c_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 \\ 1 & 0 & 0 \end{vmatrix} = -1 * \begin{vmatrix} b_1 - a_1 & c_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 \end{vmatrix}$$

$\therefore (c_1 - a_1, c_2 - a_2)$ and $(b_1 - a_1, b_2 - a_2)$ are linearly dependent. $\square$

(2) Given three vectors in $\mathbb{R}^3$ of coordinates $(d_1-a_1, d_2-a_2, d_3-a_3), (c_1-a_1, c_2-a_2, c_3-a_3)$, and $(b_1-a_1, b_2-a_2, b_3-a_3)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

Proof:

Forward case:

Let $(d_1-a_1, d_2-a_2, d_3-a_3), (c_1-a_1, c_2-a_2, c_3-a_3)$, and $(b_1-a_1, b_2-a_2, b_3-a_3)$ be linearly dependent.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1-a_1 & c_1 & d_1 \\ a_2 & b_2-a_2 & c_2 & d_2 \\ a_3 & b_3-a_3 & c_3 & d_3 \\ 1 & 1-1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1-a_1 & c_1 & d_1 \\ a_2 & b_2-a_2 & c_2 & d_2 \\ a_3 & b_3-a_3 & c_3 & d_3 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1-a_1 & c_1-a_1 & d_1 \\ a_2 & b_2-a_2 & c_2-a_2 & d_2 \\ a_3 & b_3-a_3 & c_3-a_3 & d_3 \\ 1 & 0 & 1-1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1-a_1 & c_1-a_1 & d_1 \\ a_2 & b_2-a_2 & c_2-a_2 & d_2 \\ a_3 & b_3-a_3 & c_3-a_3 & d_3 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1-a_1 & c_1-a_1 & d_1-a_1 \\ a_2 & b_2-a_2 & c_2-a_2 & d_2-a_2 \\ a_3 & b_3-a_3 & c_3-a_3 & d_3-a_3 \\ 1 & 0 & 0 & 1-1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1-a_1 & c_1-a_1 & d_1-a_1 \\ a_2 & b_2-a_2 & c_2-a_2 & d_2-a_2 \\ a_3 & b_3-a_3 & c_3-a_3 & d_3-a_3 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} b_1-a_1 & c_1-a_1 & d_1-a_1 \\ b_2-a_2 & c_2-a_2 & d_2-a_2 \\ b_3-a_3 & c_3-a_3 & d_3-a_3 \end{vmatrix} = 0$$

Backward case:

Let

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$0 = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1-a_1 & c_1 & d_1 \\ a_2 & b_2-a_2 & c_2 & d_2 \\ a_3 & b_3-a_3 & c_3 & d_3 \\ 1 & 1-1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1-a_1 & c_1 & d_1 \\ a_2 & b_2-a_2 & c_2 & d_2 \\ a_3 & b_3-a_3 & c_3 & d_3 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1-a_1 & c_1-a_1 & d_1 \\ a_2 & b_2-a_2 & c_2-a_2 & d_2 \\ a_3 & b_3-a_3 & c_3-a_3 & d_3 \\ 1 & 0 & 1-1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1-a_1 & c_1-a_1 & d_1 \\ a_2 & b_2-a_2 & c_2-a_2 & d_2 \\ a_3 & b_3-a_3 & c_3-a_3 & d_3 \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 - a_1 & c_1 - a_1 & d_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 & d_2 - a_2 \\ a_3 & b_3 - a_3 & c_3 - a_3 & d_3 - a_3 \\ 1 & 0 & 0 & 1 - 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 - a_1 & c_1 - a_1 & d_1 - a_1 \\ a_2 & b_2 - a_2 & c_2 - a_2 & d_2 - a_2 \\ a_3 & b_3 - a_3 & c_3 - a_3 & d_3 - a_3 \\ 1 & 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} b_1 - a_1 & c_1 - a_1 & d_1 - a_1 \\ b_2 - a_2 & c_2 - a_2 & d_2 - a_2 \\ b_3 - a_3 & c_3 - a_3 & d_3 - a_3 \end{vmatrix}$$

$\therefore (d_1 - a_1, d_2 - a_2, d_3 - a_3), (c_1 - a_1, c_2 - a_2, c_3 - a_3)$, and $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$ are linearly dependent. □

Exercise 3 (Problem 7.3):

Let A be the $(m+n) \times (m+n)$ block matrix (over any field K) given by

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}$$

Where A_1 is an $m \times m$ matrix, A_2 is an $m \times n$ matrix, and A_4 is an $n \times n$ matrix.
a) Prove that $\det(A) = \det(A_1)\det(A_4)$
Proof:
Case I) $\det(A_1) = 0$

$$\text{There exists linearly dependent columns in } A_1$$

$$\text{Which means } \begin{pmatrix} A_1 \\ \mathbf{0} \end{pmatrix} \text{ columns are also linearly dependent}$$

$$\det(A) = 0 = 0 * \det(A_4)$$

Case II) $\det(A_1) \neq 0$

$$\det(A) = \det(\begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix}) = \det(\begin{pmatrix} A_1 & A_2 - A_2 \\ 0 & A_4 \end{pmatrix}) = \det(\begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix})$$

$$= \det(A_1)\det(A_4)$$

b) Use the above result to prove that if A is an upper triangular $n \times n$ matrix, then $\det(A) = a_{11}a_{22} \ldots a_{nn}$
Proof:

$$\det(A) = \det(\begin{pmatrix} A_{n-1,n-1} & A_{n-1,n} \\ \mathbf{0} & a_n \end{pmatrix})$$

$$= a_n \det(A_{n-1,n-1}) = a_n a_{n-1} \det(A_{n-2,n-2})$$

$$= a_n a_{n-1} \ldots a_1$$

□

0.1 Exercise 4(Problem 7.4):

Prove that if $n \geq 3$, then

$$\det \begin{pmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_1y_n \end{pmatrix} = 0$$

Proof:

$$\det \begin{pmatrix} 1+x_1y_1 & 1+x_1y_2 & \dots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \dots & 1+x_2y_n \\ \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \dots & 1+x_1y_n \end{pmatrix}$$

$$= \det \begin{pmatrix} 1+x_1y_1 & 1+x_1y_2-1-x_1y_1 & \dots & 1+x_1y_n-1-x_1y_1 \\ 1+x_2y_1 & 1+x_2y_2-1-x_2y_1 & \dots & 1+x_2y_n-1-x_2y_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & 1+x_ny_2-1-x_ny_1 & \dots & 1+x_1y_n-1-x_ny_1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1+x_1y_1 & x_1(y_2-y_1) & \dots & x_1(y_n-y_1) \\ 1+x_2y_1 & x_2(y_2-y_1) & \dots & x_2(y_n-y_1) \\ \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & x_n(y_2-y_1) & \dots & x_1(y_n-y_1) \end{pmatrix}$$

Since detetminant is a linear map,

$$= (y_2-y_1)\det \begin{pmatrix} 1+x_1y_1 & x_1 & \dots & x_1(y_n-y_1) \\ 1+x_2y_1 & x_2 & \dots & x_2(y_n-y_1) \\ \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & x_n & \dots & x_1(y_n-y_1) \end{pmatrix}$$

$$= (y_2-y_1)(y_3-y_2)\dots(y_n-y_1)\det \begin{pmatrix} 1+x_1y_1 & x_1 & \dots & x_1 \\ 1+x_2y_1 & x_2 & \dots & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & x_n & \dots & x_1 \end{pmatrix} = 0$$

□

0.2 Exercise 5(Problem 7.5):

Prove that

$$\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & -20 & -56 & -108 \\ 0 & -39 & -108 & -207 \end{vmatrix}$$

$$= \begin{vmatrix} -7 & -20 & -39 \\ -20 & -56 & -108 \\ -39 & -108 & -207 \end{vmatrix}$$

$$= -7 \begin{vmatrix} -56 & -108 \\ -108 & -207 \end{vmatrix} + 20 \begin{vmatrix} -20 & -39 \\ -108 & -207 \end{vmatrix} - 39 \begin{vmatrix} -20 & -39 \\ -56 & -108 \end{vmatrix}$$

$$= -7(56 * 207 - 108^2) + 20(20 * 207 - 39 * 108) - 39(20 * 108 - 39 * 56)$$

$$= 504 - 1440 + 936 = 0$$

$\square$