Linear Algebra Assignment 4

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Exercise 1:

1) Prove that the column vectors of matrix A_1 given by

$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \\ 1 & 3 & 1 \end{bmatrix}$$

are linearly independent.
Proof:
Solve:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 7 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1)\ \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$$

$$2)\ 2\lambda_1 + 3\lambda_2 + 7\lambda_3 = 0$$

$$3)\ \lambda_1 + 3\lambda_2 + 1\lambda_3 = 0$$

$$2 * 1) - 2)\ \ \lambda_2 - \lambda_3 = 0$$

$$2 * 3) - 2)\ \ 3\lambda_2 - 5\lambda_3 = 0$$

$$\therefore \lambda_1 = \lambda_2 = \lambda_3 = 0$$

So, the columns are linearly independent. $\square$

2) Prove that the coordinates of the column vectors of the matrix B_1 over the basis consisting of the column vectors of A_1 given by

$$B_1 = \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 4 & 3 & -6 \end{bmatrix}$$

are the columns of the matrix P_1 given by

$$\begin{bmatrix} -27 & -61 & -41 \\ 9 & 18 & 9 \\ 4 & 10 & 8 \end{bmatrix}$$

a) The first column of P_1 represents

$$\begin{bmatrix}1 & 2 & 3\\2 & 3 & 7\\1 & 3 & 1\end{bmatrix}\begin{bmatrix}-27\\9\\4\end{bmatrix}=\begin{bmatrix}3\\1\\4\end{bmatrix}$$

b) The second column of P_1 represents

$$\begin{bmatrix}1 & 2 & 3\\2 & 3 & 7\\1 & 3 & 1\end{bmatrix}\begin{bmatrix}-61\\18\\10\end{bmatrix}=\begin{bmatrix}5\\2\\3\end{bmatrix}$$

c) The third column of P_1 represents

$$\begin{bmatrix}1 & 2 & 3\\2 & 3 & 7\\1 & 3 & 1\end{bmatrix}\begin{bmatrix}-41\\9\\8\end{bmatrix}=\begin{bmatrix}1\\1\\-6\end{bmatrix}$$

So the coordinates of B_1 given basis A_1 is P_1

□

3) Give a nontrivial linear dependence of the columns of P_1. Check that $B_1 = A_1P_1$. Check that $B_1 = A_1P_1$. Is the matrix B_1 invertible?

Since P_1 is coordinate representation of B_1 given basis set A_1,

$$B_1 = A_1P_1$$

So,

$$\text{If, } P_1\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\text{ then,}$$

$$B_1\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=A_1P_1\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

So we can find nontrivial linear dependence solution to B_1 instead which is

$$\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}3\\-2\\1\end{bmatrix}$$

and since columns of B_1 are linearly dependent, matrix B_1 is not invertible.

Exercise 2:

Consider the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}$$

with $a_n \neq 0$

1) Find a matrix P such that

$$A^T = P^{-1}AP$$

Let

$$P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}$$

Find P such that

$$PA^T = AP$$

$$PA^T = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{pmatrix}$$

$$= \begin{pmatrix} -a_np_{1n} & p_{11} - a_{n-1}p_{1n} & p_{12} - a_{n-2}p_{1n} & \cdots & p_{1,n-1} - a_2p_{1n} & p_{1n} - a_1p_{1n} \\ -a_np_{2n} & p_{21} - a_{n-1}p_{2n} & p_{22} - a_{n-2}p_{2n} & \cdots & p_{2,n-1} - a_2p_{2n} & p_{2n} - a_1p_{2n} \\ -a_np_{3n} & p_{31} - a_{n-1}p_{3n} & p_{32} - a_{n-2}p_{3n} & \cdots & p_{3,n-1} - a_2p_{3n} & p_{3n} - a_1p_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_np_{n-1n} & p_{n-1,1} - a_{n-1}p_{n-1,n} & p_{n-1,2} - a_{n-2}p_{n-1,n} & \cdots & p_{n-1,n-1} - a_2p_{n-1n} & p_{n-1,n} - a_1p_{n-1,n} \\ -a_np_{nn} & p_{n1} - a_{n-1}p_{nn} & p_{n2} - a_{n-2}p_{nn} & \cdots & p_{n,n-1} - a_2p_{nn} & p_{nn} - a_1p_{nn} \end{pmatrix}$$

$$AP =$$

$$\begin{pmatrix} -a_n p_{n,1} & -a_n p_{n,2} & -a_n p_{n,3} & \cdots & -a_n p_{n,n-1} & -a_n p_{n,n} \\ p_{1,1} - a_{n-1} p_{n,1} & p_{1,2} - a_{n-1} p_{n,2} & p_{1,3} - a_{n-1} p_{n,3} & \cdots & p_{1,n-1} - a_{n-1} p_{n,n-1} & p_{1,n} - a_{n-1} p_{n,n} \\ p_{2,1} - a_{n-2} p_{n,1} & p_{2,2} - a_{n-2} p_{n,2} & p_{2,3} - a_{n-2} p_{n,3} & \cdots & p_{2,n-1} - a_{n-2} p_{n,n-1} & p_{2,n} - a_{n-2} p_{n,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n-2,1} - a_2 p_{n,1} & p_{n-2,2} - a_2 p_{n,2} & p_{n-2,3} - a_2 p_{n,3} & \cdots & p_{n-2,n-1} - a_2 p_{n,n-1} & p_{n-2,n} - a_2 p_{n,n} \\ p_{n-1,1} - a_1 p_{n,1} & p_{n-1,2} - a_1 p_{n,2} & p_{n-1,3} - a_1 p_{n,3} & \cdots & p_{n-1,n-1} - a_1 p_{n,n-1} & p_{n-1,n} - a_1 p_{n,n} \end{pmatrix}$$

So,

$$\begin{pmatrix} -a_n p_{1n} & p_{11} - a_{n-1} p_{1n} & p_{12} - a_{n-2} p_{1n} & \cdots & p_{1,n-1} - a_2 p_{1n} & p_{1n} - a_1 p_{1n} \\ -a_n p_{2n} & p_{21} - a_{n-1} p_{2n} & p_{22} - a_{n-2} p_{2n} & \cdots & p_{2,n-1} - a_2 p_{2n} & p_{2n} - a_1 p_{2n} \\ -a_n p_{3n} & p_{31} - a_{n-1} p_{3n} & p_{32} - a_{n-2} p_{3n} & \cdots & p_{3,n-1} - a_2 p_{3n} & p_{3n} - a_1 p_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_n p_{n-1n} & p_{n-1,1} - a_{n-1} p_{n-1,n} & p_{n-1,2} - a_{n-2} p_{n-1,n} & \cdots & p_{n-1,n-1} - a_2 p_{n-1n} & p_{n-1,n} - a_1 p_{n-1,n} \\ -a_n p_{nn} & p_{n1} - a_{n-1} p_{nn} & p_{n2} - a_{n-2} p_{nn} & \cdots & p_{n,n-1} - a_2 p_{nn} & p_{nn} - a_1 p_{nn} \end{pmatrix} =$$

$$\begin{pmatrix} -a_n p_{n,1} & -a_n p_{n,2} & -a_n p_{n,3} & \cdots & -a_n p_{n,n-1} & -a_n p_{n,n} \\ p_{1,1} - a_{n-1} p_{n,1} & p_{1,2} - a_{n-1} p_{n,2} & p_{1,3} - a_{n-1} p_{n,3} & \cdots & p_{1,n-1} - a_{n-1} p_{n,n-1} & p_{1,n} - a_{n-1} p_{n,n} \\ p_{2,1} - a_{n-2} p_{n,1} & p_{2,2} - a_{n-2} p_{n,2} & p_{2,3} - a_{n-2} p_{n,3} & \cdots & p_{2,n-1} - a_{n-2} p_{n,n-1} & p_{2,n} - a_{n-2} p_{n,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n-2,1} - a_2 p_{n,1} & p_{n-2,2} - a_2 p_{n,2} & p_{n-2,3} - a_2 p_{n,3} & \cdots & p_{n-2,n-1} - a_2 p_{n,n-1} & p_{n-2,n} - a_2 p_{n,n} \\ p_{n-1,1} - a_1 p_{n,1} & p_{n-1,2} - a_1 p_{n,2} & p_{n-1,3} - a_1 p_{n,3} & \cdots & p_{n-1,n-1} - a_1 p_{n,n-1} & p_{n-1,n} - a_1 p_{n,n} \end{pmatrix}$$

By looking into the diagonal enteries, we know that,

$$p_{1,2} = p_{2,3} = \cdots = p_{n-1,n} = 0$$

$$p_{n,i} = p_{i,n} \text{ (for } 1 \leq i \leq n)$$

Exercise 3:

For any matrix $A \in M_n(\mathbb{C})$, let R_A and L_A be the maps from $M_n(\mathbb{C})$ to itself defined so that

$L_A(B) = AB, R_A(B) = BA$, for all $B \in M_n(\mathbb{C})$

1) Check that L_A and R_A are linear, and that L_A and R_B commute for all A, B.

Proof:

$$L_A(B_1 + B_2) = A(B_1 + B_2) = AB_1 + AB_2 = L_A(B_1) + L_A(B_2)$$

$$L_A(cB) = AcB = cAB = cL_A(B)$$

$$\therefore L_A \text{ is linear.}$$

$$R_A(B_1 + B_2) = (B_1 + B_2)A = B_1 A + B_2 A = R_A(B_1) + R_A(B_2)$$

$$R_A(cB) = cBA = cR_A(B)$$

$$\therefore R_A \text{ is linear.}$$

$$L_A(R_B(C)) = L_A(CB) = ACB = R_B(AC) = R_B(L_A(C))$$

$$\therefore L_A, R_B \text{ are commutative.}$$

□

2) Let $ad_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the linear map given by

$$ad_A(B) = L_A(B) - R_A(B) = AB - BA = [A, B], \text{ for all } B \in M_n(\mathbb{C})$$

Prove that if A is invertible, then L_A and R_A are invertible; in fact, $(L_A)^{-1} = L_{A^{-1}}$ and $(R_A)^{-1} = R_{A^{-1}}$. Prove that if $A = PBP^{-1}$ for some invertible matrix P, then

$$L_A = L_P \circ L_B \circ L_P^{-1}, \quad R_A = R_P^{-1} \circ R_B \circ R_P$$

Proof:

Since A is invertible there exists A^{-1} so that $A^{-1}A = I$
Let,

$$L_{A^{-1}} : M_n(\mathbb{C}) \to M_n(\mathbb{C}) = A^{-1}$$

$$L_{A^{-1}}(L_A) : M_n(\mathbb{C}) \to M_n(\mathbb{C}) = A^{-1}A = I(\text{Identity map})$$

$$\therefore \text{ Inverse map } L_{A^{-1}} \text{ exists.}$$

Same can be applied to $R_{A^{-1}}$

$$L_P \circ L_B \circ L_P^{-1}(X) = L_P \circ L_B \circ L_{P^{-1}}(X) = L_P \circ L_B(P^{-1}X)$$
$$= L_P(BP^{-1}X) = PBP^{-1}X = L_A(X)$$

same can be applied to R_A □

3) Recall that the n^2 matrices E_{ij} defined such that all entries in E_ij are zero except the (i,j)th entry, which is equal to 1, form a basis of the vector space $M_n(\mathbb{C})$. Consider the partial ordering of the E_{ij} defined such that for $i = 1, \ldots, n$ if $n \geq j > k \geq 1$, then E_{ij} precedes E_{ik}, and for $j = 1, \ldots, n$ if $1 \leq i < h \leq n$ then E_{ij} precedes E_{hj}. Draw the Hasse diagram of the partial order defined above then $n = 3$

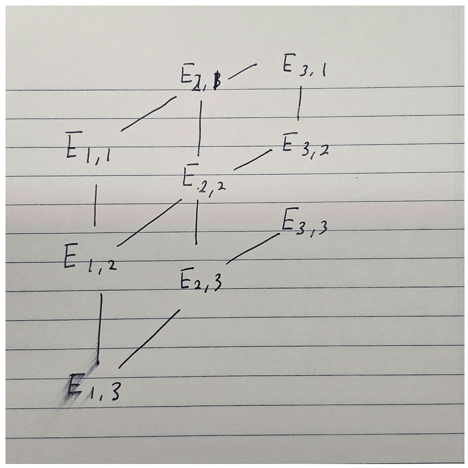

Figure 1: Hasse diagram

$$E_{1,3} < E_{1,2} < E_{1,1} < E_{2,3} < E_{2,2} < E_{2,1} < E_{3,3} < E_{3,2} < E_{3,1}$$

Total ordering if $E_{i,j} < E_{k,l}$ for all j, l if $i < k$.

4) Let the total order of the basis E_{ij} extending the partial ordering defined in (2) be given by

$$(i,j) < (h,k) \text{ iff } \begin{cases} i = h \text{ and } j > k \\ \text{or } i < h. \end{cases}$$

Let R be the $n \times n$ permutation matrix given by

$$R = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Observe that $R^{-1} = R$. Prove that for any $n \geq 1$, the matrix of L_A is given by $A \otimes I_n$, and the matrix of R_A is given by $I_n \otimes RA^T R$, where $\otimes$ is the *Kronecker product* of matrices defined in Definition 5.4.

Proof:

$$L_A(X) = AX = A(\sum_{i,j} x_{ij}E_{ij}) = \sum_{i,j} x_{ij}AE_{ij} = \sum_{ij} x_{ij}a_{ji}E_{ij} = (x_{ij}a_{ij})$$

$$(x_{ij} \to a_{ji}x_{ij})$$

$$A \otimes I_n = \begin{bmatrix} a_{11}I_n & \dots & a_{1n}I_n \\ \vdots & \ddots & \vdots \\ a_{n1}I_n & \dots & a_{nn}I_n \end{bmatrix}$$

$$A \otimes I_n(X) = \begin{bmatrix} a_{11}I_n & \dots & a_{1n}I_n \\ \vdots & \ddots & \vdots \\ a_{n1}I_n & \dots & a_{nn}I_n \end{bmatrix} \begin{bmatrix} x_{1n} \\ \vdots \\ x_{n1} \end{bmatrix} =$$