Assignment 02, Real Analysis MIT

Student Hanseul Kim
Prof Dr. Casey Rodriguez

Answers

0.1 Exercise 1.1.1

Prove:

$$\text{let } F \text{ be and ordered field and } x, y, z \in F$$
$$\text{Prove If } x < 0 \text{ and } y < z, \text{ then } xy > xz$$

Proof.

$$\begin{aligned} \text{since } x < 0 \\ x + (-x) < (-x) \\ 0 < -x \text{ (by definition)} \\ \text{likewise, } 0 < z - y \\ \therefore 0 < (-x)(z - y) \\ \text{and by definition 1.1.5.D, } 0 < -xz + xy \\ \therefore xz < xy \end{aligned}$$

□

0.2 Exercise 1.1.2

Let S be and ordered set. Let $A \subset S$ be a nonempty finite subset. Then A is bounded. *Furthermore*, inf A exists and is in A and sup A exists and is in A

Proof.

Since A is a nonempty finite subset, there exists a one-to-one function between $\{1, 2, \ldots, n\}$ and each element in A

Proof by induction

$$\text{Finite subset } A_n = a_1, a_2, \ldots, a_n \text{ is bounded}$$

Base case:

$$\begin{aligned} A_1 = a_1 \\ \forall x \in A_1, a_1 \leq x \leq a_1 \\ \therefore A_1 \text{ is bounded} \end{aligned}$$

Inductive step:

$$\begin{aligned} \text{Let } A_n \text{ bounded by } B_{min}, B_{max} \text{ s.t} \\ \forall x \in A_n, B_{min} \leq x \leq B_{max} \\ A_{n+1} = x; x \in A_n or x = a_{n+1} \\ \text{so, } \forall x \in A_{n+1}, x \leq max(B_{max}, a_{n+1}), \text{ and } min(B_{min}, a_{n+1}) \leq x \\ \therefore \text{Let } A_{n+1} \text{ bounded} \end{aligned}$$

Furthermore, since these bounds are contained inside the subset S, these are the inf, sup of the set.
Proof.

$$\begin{aligned}
\text{Let } S \text{ be bounded by } B_{\min}, B_{\max} \text{ such that } B_{\min}, B_{\max} \in S \\
\forall x \in S,\ x \leq B_{\max} \quad \text{(by definition)} \qquad \text{(a)} \\
\forall B \text{ which is an upper bound of } S,\ B \geq x \quad \forall x \in S \\
\text{Since } B_{\max} \text{ is in } S,\ B \geq B_{\max} \qquad \text{(b)} \\
\text{By a), b) } B_{\max} \text{ is a least upper bound}
\end{aligned}$$

□

Similar argument can be applied to $B_{\min}$
$\therefore$ inf S exists and is in S and sup S exists and is in S

0.3 Exercise 1.1.5

Let S be and ordered set. Let $A \subset S$ and suppose b is an upper bound for A. suppose $b \in A$ show that $b = sup\ A$
Proof.

$$\begin{aligned}
\forall x \in S,\ x \leq b \qquad \text{(a)} \\
\text{Let } B \text{ a upper bound of S, } \forall B, x \leq B\ (\forall x \in S) \\
\text{Since } b \text{ in } S,\ b \leq B \qquad \text{(b)} \\
\text{by a), b) } b \text{ is a least upper bound of S and by definition of supremum,} \\
b = sup\ A
\end{aligned}$$

□

0.4 Exercise 1.1.6

Let S be and ordered set. Let $A \subset S$ be a nonempty subset that is bounded above. Suppose $sup\ A$ exists and $sup\ A \notin A$. show that A contains a countably infinite subset.
Proof.
Proof by contradiction

Let A does not contain countably infinite subset. then A is a finite set.
(finite set $\subset$ countably infinite set $\subset$ uncountable infinite set)
Then by Exercise 1.1.2) $sup\ A$ exists and inside A, contradiction
$\therefore$ A contains a countably infinite subset.
(A contains a infinite subset which contains a countably infinite subset)

□

0.5 Exercise 1.2.7

Prove the arithmetic-geometric mean inequality. That is, for two positive real numbers x, y we have

$$\sqrt{xy} \leq \frac{x+y}{2}$$

Furthermore, equality occurs iff $x = y$
Proof.

$$\forall x, y \in \mathbb{R}$$
$$\text{by proposition 1.1.8.iv, } (x-y)^2 \geq 0$$
$$x^2 - 2xy + y^2 \geq 0$$
$$x^2 + 2xy + y^2 \geq 4xy \geq 0$$
$$\text{By extension of Example 1.2.3, there exists unique positive } r \in \mathbb{R} \text{ s.t } r^2 = s > 0 \ (s \in \mathbb{R})$$
$$(x+y)^2 \geq 4xy \geq 0$$
$$(x+y) \geq 2\sqrt{xy} \geq 0$$
$$\therefore \frac{x+y}{2} \geq \sqrt{xy}$$
$$\text{and if equality case}$$
$$\sqrt{xy} = \frac{x+y}{2} 4xy = x^2 + 2xy + y^2$$
$$x^2 - 2xy + y^2 = (x-y)^2 = 0$$
$$\text{by proposition 1.1.8.iv}$$
$$x - y = 0$$
$$\therefore x = y$$

□

0.6 Exercise 1.2.9

Let A and B bee two nonempty bounded sets of real numbers. Let $C := \{a_b : a \in A, b \in B\}$
Show that C is a bounded set and that

$$sup\ C = sup\ A + sup\ B$$

and

$$inf\ C = inf\ A + inf\ B$$

Proof.

$$\forall b \in B \subset R \text{ and } A \text{ is bounded above,}$$
$$\text{by proposition 1.2.6.i,}$$
$$sup(b + A) = b + sup(A)$$
$$\text{given set of } D = \{b + sup\ A : b \in B\},$$
$$sup\ D = sup\ B + sup\ A$$
$$\text{by definition, } \forall b \in B, sup\ D \geq b + sup\ A$$
$$\forall a \in A, sup\ D \geq b + sup\ A \geq a + b$$
$$\therefore sup\ B = sup\ C \text{ and}$$
$$sup\ C = sup\ A + sup\ B$$

□

Similar argument can be applied to infimum.

0.7 Problem 7.

Let

$$E = \{x \in \mathbb{R} : x > 0 \text{ and } x^3 < 2\}$$

a) Prove that E is bounded above
Proof. Case $x < 1$:

$$0 < x^3 < x^2 < x < 1$$

x is bounded by 1
Case $x \geq 1$:

$$x \leq x^2 \leq x^3 < 2$$

x is bounded by 2
$\therefore$ E is bounded above.

b) Let $r = sup\ E$. Prove that $r > 0$ and $r^3 = 2$

Proof.
Proof by contradiction
Case $r^3 < 2$

$$\exists \epsilon \in \mathbb{R}^+ \text{s.t}$$
$$(r+\epsilon)^3 = r^3 + 3\epsilon r^2 + 3\epsilon^2 r + \epsilon^3 < 2$$
$$3\epsilon r^2 + 3\epsilon r + \epsilon r^3 < 2 - r^3$$

since $r > 1$

$$\epsilon^3 < 3\epsilon r^2 + 3\epsilon r + \epsilon r^3 < 2 - r^3$$

since $2 - r^3 > 0$ we can set $\epsilon > 0$ satisfiys this.
r is not upper bound of E
Case $r^3 > 2$

$$\exists \epsilon \in \mathbb{R}^+$$

s.t.

$$(r-\epsilon)^2 > 2$$
$$r^3 - 3\epsilon r^2 + 3\epsilon^2 r + \epsilon^3 > 2$$
$$r^3 - 2 > 3\epsilon r^2 - 3\epsilon^2 r - \epsilon^3$$

By Archimedean property (tired) $\exists \epsilon s.t (r-\epsilon)^2 > 2$ r is not **least** upper bound.
$\therefore$ by trichotomy property, $r^3 = 2$

$\square$