Assignment 04, Real Analysis MIT

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Answers

0.1 Exercise 1

We say a set $F \subset \mathbb{R}$ is *closed* if its complement $F^c := \mathbb{R} \setminus F$ is open. Since $\emptyset$ and $\mathbb{R}$ is open, it follows that $\emptyset$ and $\mathbb{R}$ are closed as well.

a) Let $a, b \in \mathbb{R}$ with $a < b$. prove that $[a, b]$ is closed.

Proof:

$$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$$
$$\forall x \in (-\infty, a) \cup (b, \infty)$$
$$\text{if } x < a, \exists \epsilon > 0 \text{ s.t}$$
$$x - \epsilon < x < x + \epsilon < a$$
$$\therefore (x - \epsilon, x + \epsilon) \subset (-\infty, a) \subset (-\infty, a) \cup (b, \infty)$$

$$\text{if } x > b, \exists \epsilon > 0 \text{ s.t.}$$
$$b < x - \epsilon < x < x + \epsilon$$
$$\therefore (x - \epsilon, x + \epsilon) \subset (b, \infty) \subset (-\infty, a) \cup (b, \infty)$$
$$\text{Thus, } (b, \infty) \subset (-\infty, a) \cup (b, \infty) \text{ is open.}$$
$$\therefore [a, b] \text{ is closed.}$$

□

b) Is the set $\mathbb{Z} \subset \mathbb{R}$ closed?

True.

Proof:

$$\mathbb{R} \setminus \mathbb{Z} = \dots, (-1, 0), (0, 1), (1, 2), \dots$$
$$\text{Since every set } (i, i+1) \ \ (i \in \mathbb{Z}) \text{ is open.}$$
$$\text{The union of open sets } \dots, (-1, 0), (0, 1), (1, 2), \dots \text{ is open.}$$
$$\therefore \mathbb{R} \setminus \mathbb{Z} \text{ is closed.}$$

□

c) Is the set of rationals $\mathbb{Q} \subset \mathbb{R}$ closed?

Proof:

$$\text{We know that the rationals } \mathbb{Q} \subset \mathbb{R} \text{ are dense in } \mathbb{R}$$

So,

$$\forall x \in \mathbb{R} \setminus \mathbb{Q}, \forall \epsilon > 0$$
$$\exists y \in \mathbb{Q} \text{ s.t.}$$
$$y \in (x - \epsilon, x + \epsilon) \not\subset \mathbb{R} \setminus \mathbb{Q}$$
$$\therefore \mathbb{R} \setminus \mathbb{Q} \text{ is not open.}$$
$$\text{Hence, } \mathbb{Q} \text{ is not closed.}$$

□

0.2 Exercise 2

a) Let Λ be a set, and for each $\lambda \in \Lambda$, let $F_\lambda \subset \mathbb{R}$. Prove that if F_λ is closed for all $\lambda \in \Lambda$, then the set

$$\bigcap_{\lambda \in \Lambda} F_\lambda = \{x \in \mathbb{R} : x \in F_\lambda, \forall \lambda \in \Lambda\}$$

is closed.
Proof:

$$\text{By definition, } \forall \lambda \in \Lambda, \quad F_\lambda \text{ is closed hence,}$$
$$\mathbb{R} \setminus F_\lambda \text{ is open.}$$

Hence,

$$\bigcup_{\lambda \in \Lambda} \mathbb{R} \setminus F_\lambda \text{ is open}$$
$$\mathbb{R} \setminus F_\lambda = \mathbb{R} \cap F_\lambda^c$$
$$\bigcup_{\lambda \in \Lambda} \mathbb{R} \setminus F_\lambda = \bigcup_{\lambda \in \Lambda} \mathbb{R} \cap F_\lambda^c = \mathbb{R} \cap (\bigcap_{\lambda \in \Lambda} F_\lambda)^c$$
$$= \mathbb{R} \setminus \bigcap_{\lambda \in \Lambda} F_\lambda$$
$$\text{which is union of open sets,}$$
$$\therefore \mathbb{R} \setminus \bigcap_{\lambda \in \Lambda} F_\lambda \text{ is open.}$$
$$\text{Hence, } \bigcap_{\lambda \in \Lambda} F_\lambda \text{ is closed.}$$

□

b) Let $n \in \mathbb{N}$, and let $F_1, \dots F_n \subset \mathbb{R}$. Prove that if $F_1, \dots, F_n$ are closed then the set $\bigcup_{m=1}^n F_m$ is closed
Proof:

Just like Exercise 2.a) rather than using union of open sets, we can use intersection of open sets to be open.

$$\bigcap_{\lambda \in \Lambda} \mathbb{R} \setminus F_\lambda \text{ is open}$$
$$\mathbb{R} \setminus F_\lambda = \mathbb{R} \cap F_\lambda^c$$
$$\bigcap_{\lambda \in \Lambda} \mathbb{R} \setminus F_\lambda = \bigcap_{\lambda \in \Lambda} \mathbb{R} \cap F_\lambda^c = \mathbb{R} \cap (\bigcup_{\lambda \in \Lambda} F_\lambda)^c$$
$$\text{Which is intersection of open sets,}$$
$$\therefore \mathbb{R} \setminus \bigcup_{\lambda \in \Lambda} F_\lambda \text{ is open.}$$
$$\text{Hence, } \bigcup_{\lambda \in \Lambda} F_\lambda \text{ is closed.}$$

□

0.3 Exercise 3

. Let $F \subset \mathbb{R}$ be a closed set, and let $\{x_n\}$ be a sequence of elements of F converging to $x \in \mathbb{R}$. Prove that $x \in F$

Proof:

Proof by Contradiction:

Let:

$$x \in F^c$$

Since x_n converges to x

$$\forall \epsilon_0 > 0,\ \exists N \in \mathbb{N} \text{ s.t}$$
$$\left|x_n - x\right| < \epsilon_0 \ \ (\forall n \geq N)$$

And since $x \in F^c$ and F is closed (F^c is open),

$$\exists \epsilon > 0 \text{s.t}$$

$$(x - \epsilon, x + \epsilon) \subset F^c$$

Let $\epsilon_0 = \epsilon/2$

$$\left|x_n - x\right| < \epsilon/2 \ \ (\forall n \geq N)$$
$$x_n \in (x - \epsilon/2, x + \epsilon/2) \subset (x - \epsilon, x + \epsilon) \subset F^c$$
$$x_n \in F^c$$

Contradiction.

$$\therefore x \in F$$

□

0.4 Exercise 2.2.3

Prove that if $\{x_n\}$ is a convergent sequence, $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n^k = (\lim_{n\to\infty} x_n)^k$$

Proof:

Proof by induction:

Base Case (k = 1):

$$\lim_{n\to\infty} x_n^1 = (\lim_{n\to\infty} x_n) = (\lim_{n\to\infty} x_n)^1$$

Inductive Step:

Let:

$$\lim_{n\to\infty} x_n^k = (\lim_{n\to\infty} x_n)^k$$
$$\lim_{n\to\infty} x_n^{k+1} = \lim_{n\to\infty} x_n^k * x_n$$
$$= \lim_{n\to\infty} x_n^k * \lim_{n\to\infty} x_n$$
$$= (\lim_{n\to\infty} x_n)^k * \lim_{n\to\infty} x_n$$
$$= (\lim_{n\to\infty} x_n)^{k+1}$$

$$\therefore \lim_{n\to\infty} x_n^k = (\lim_{n\to\infty} x_n)^k$$

□

0.5 Exercise 2.2.5

Let $x_n := \frac{n-cos(n)}{n}$. show that $\{x_n\}$ converges and find $\lim x_n$.

Proof:

$$\text{Since,} \ -1 \leq cos(n) \leq 1$$
$$\frac{-1}{n} \leq \frac{-cos(n)}{n} \leq \frac{1}{n}$$
$$\frac{n-1}{n} \leq \frac{n-cos(n)}{n} \leq \frac{n+1}{n}$$

Since,

$$\lim \frac{n-1}{n} = 1 \text{ and } \lim \frac{n+1}{n} = 1$$

By the Squeeze lemma,

$$\lim \frac{n-1}{n} \leq \lim \frac{n-cos(n)}{n} \leq \lim \frac{n+1}{n}$$
$$1 \leq \lim \frac{n-cos(n)}{n} \leq 1$$
$$\therefore \lim \frac{n-cos(n)}{n} = 1$$

□

0.6 Exercise 6

. Let $A \subset \mathbb{R}$ be bounded above, and let a_0 be an upper bound for A. Prove that $a_0 = supA$ iff there exists a sequence $\{a_n\}$ of elements of A such that $\lim_{n\to\infty} a_n = a_0$

Proof:

Case $a_0 = supA \to \exists\{a_n\}$ s.t $\lim_{n\to\infty} a_n = a_0$

By lemma in assignment 3,

$$\text{if } a_0 = supA \text{ then } \forall n \in \mathbb{N} \ \ \exists a_n \in A \text{ s.t,}$$
$$a_0 - \frac{1}{n} < a_n \leq a_0$$
$$\forall \epsilon > 0, \ \ \exists n \in N \text{ s.t,}$$
$$\frac{1}{n} < \epsilon \text{ (By Archimedean principle)}$$
$$\left| a_n - a_0 \right| \leq \frac{1}{n} < \epsilon$$

By definition

$$\lim_{n\to\infty} a_n = a_0$$

Case $\exists\{a_n\}$ s.t $\lim_{n\to\infty} a_n = a_0 \to a_0 = supA$

By definition,

$$\forall \epsilon > 0, \ \ \exists N \in \mathbb{N}$$
$$\left| a_n - a_0 \right| < \epsilon \ \ \forall n \geq N$$
$$a_0 - \epsilon < a_n < a_0 + \epsilon$$

Since a_0 is a upper bound,

$$a_n \leq a_0$$

So,

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ s.t}$$
$$\forall n \geq N,$$
$$a_0 - \epsilon < a_n < a_0$$

By definition of Least upper bound,

$$a_0 \text{ is a least upper bound.}$$

$\therefore a_0 = supA$ iff there exists a sequence $\{a_n\}$ of elements of A such that $\lim_{n\to\infty} a_n = a_0$

□

0.7 Exercise 7

. Let $E \subset \mathbb{R}$ be a nonempty set of real numbers. We say $x \in \mathbb{R}$ is a *cluster point* of E if for every $\epsilon > 0$

$$(x-\epsilon, x+\epsilon) \cap E \setminus \{x\} \neq \emptyset$$

a) Prove that x is a cluster point of E iff there exists a sequence $\{x_n\}$ of elements of $E \setminus \{x\}$ such that $\lim_{n\to\infty} x_n = x$

Forward Proof:

By definition, if x is a cluster point of E,

$$\forall n \in \mathbb{N}, \quad \exists x_n \in E \text{ with } x_n \neq x \text{ s.t.}$$
$$x - \frac{1}{n} < x_n < x + \frac{1}{n}$$
$$\forall \epsilon > 0, \exists n \in \mathbb{N}, \text{ s.t.}$$
$$\frac{1}{n} < \epsilon \quad \text{(By Archimedean principle)}$$
$$\left|x_n - x\right| < \frac{1}{n} < \epsilon$$
$$\therefore \lim_{n\to\infty} x_n = x$$

Backward Proof:

Hence $\lim_{n\to\infty} x_n = x, x_n \neq x$

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N} \text{ s.t.}$$
$$\left|x_n - x\right| < \epsilon \ \ (\forall n \geq N)$$
$$\text{Since } x_n \neq x$$
$$x_n \in E \setminus \{x\}$$
$$x_n \in (x-\epsilon, x+\epsilon) \cap E \setminus \{x\}$$
$$\therefore (x-\epsilon, x+\epsilon) \cap E \setminus \{x\} \text{ is non-empty}$$
$$\text{By definition, } x \text{ is a cluster point of } E$$

□

b) Prove that the set of all cluster point of E is closed

Proof:

non-cluster point x

$$\exists \epsilon > 0, \quad (x-\epsilon, x+\epsilon) \cap E \setminus \{x\} = \emptyset$$

Then for the set of all of the cluster points C,

$$\exists \epsilon > 0 \text{ s.t}$$
$$(x-\epsilon, x+\epsilon) = \{x\} \subset C$$

Since we can set sufficient epsilon that contains only x.
Hence, set of non-cluster points is open

$$\therefore \text{set of all cluster points of } E \text{ is closed.}$$

□