Linear Algebra Assignment 5

Hanseul Kim

Dec 2024

Exercise 1:

Let $V_i, i \in \{1, 2, 3, \dots\}$ be a collection of vector spaces over a field F. Show that direct product of V_i for $i \in \{1, 2, 3, \dots\}$ is a vector space over F.

Let $V_1, V_2, \dots, V_n$ be vector spaces over the same field $\mathbb{F}$. The **direct product** of these vector spaces, denoted $V_1 \times V_2 \times \cdots \times V_n$, is the Cartesian product of the sets $V_1, V_2, \dots, V_n$, equipped with the following operations:

1. **Addition**: For $(v_1, v_2, \dots, v_n), (w_1, w_2, \dots, w_n) \in V_1 \times V_2 \times \cdots \times V_n$,

$$(v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n) = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n),$$

where $v_i + w_i$ is the addition operation in V_i for $i = 1, 2, \dots, n$.

2. **Scalar Multiplication**: For $\alpha \in \mathbb{F}$ and $(v_1, v_2, \dots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$,

$$\alpha \cdot (v_1, v_2, \dots, v_n) = (\alpha v_1, \alpha v_2, \dots, \alpha v_n),$$

where αv_i is the scalar multiplication in V_i for $i = 1, 2, \dots, n$.

Thus, $V_1 \times V_2 \times \cdots \times V_n$ forms a vector space over $\mathbb{F}$.

To prove that this direct product space is vector space it must satisfy $V0$ $V5$ in definition 3.1.

(V0) E is an abelian group with respect to $+$, with identity element 0;

(V1) $\alpha \cdot (u + v) = (\alpha \cdot u) + (\alpha \cdot v)$;

(V2) $(\alpha + \beta) \cdot u = (\alpha \cdot u) + (\beta \cdot u)$;

(V3) $(\alpha * \beta) \cdot u = \alpha \cdot (\beta \cdot u)$;

(V4) $1 \cdot u = u$;

Proof:

V0) By definition,

$$u + v = (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$= (v_1 + u_1, v_2 + u_2, \dots, v_n + u_n) = (v_1, v_2, \dots, v_n) + (u_1, u_2, \dots, u_n) = v + u$$

thus, the direct product space forms an abelian group under addition

Since every vector spaces have a identity element,

$$(0, \dots, 0) + (0, \dots, 0) = (0, \dots, 0) \in V_1 \times V_2 \times \cdots \times V_n$$

$$\therefore \text{ identity element is in direct product.}$$

V1)

$$\alpha \cdot (u + v) = \alpha(u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$(\alpha u_1 + \alpha v_1, \alpha u_2 + \alpha v_2, \dots, \alpha u_n + \alpha v_n) = \alpha \cdot u + \alpha \cdot v$$

V2)

$$(\alpha + \beta) \cdot u = (\alpha u_1 + \beta u_1, \alpha u_2 + \beta u_2, \dots, \alpha u_n + \beta u_n) = \alpha \cdot u + \beta \cdot u$$

V3)

$$(\alpha * \beta) \cdot u = (\alpha * \beta u_1, \alpha * \beta u_2, \dots, \alpha * \beta u_n) = \alpha \cdot (\beta \cdot u)$$

V4)

$$1 \cdot u = (1u_1, 1u_2, \dots, 1u_n) = (u_1, u_2, \dots, u_n) = u$$

$$\therefore \text{ direct product space is a vector space.}$$

□

Exercise 2:

Show that direct sum as defined in lecture is a subspace of direct product.

Let $V_1, V_2, \dots, V_n$ be vector spaces over a field $\mathbb{F}$. The **direct sum** of these vector spaces, denoted $V_1 \oplus V_2 \oplus \cdots \oplus V_n$, is the vector space consisting of all ordered tuples $(v_1, v_2, \dots, v_n)$, where $v_i \in V_i$ for $i = 1, 2, \dots, n$, with the following operations:

1. **Addition**: For $(v_1, v_2, \dots, v_n), (w_1, w_2, \dots, w_n) \in V_1 \oplus V_2 \oplus \cdots \oplus V_n$,

$$(v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n) = (v_1 + w_1, v_2 + w_2, \dots, v_n + w_n),$$

where $v_i + w_i$ is the addition operation in V_i for $i = 1, 2, \dots, n$.

2. **Scalar Multiplication**: For $\alpha \in \mathbb{F}$ and $(v_1, v_2, \dots, v_n) \in V_1 \oplus V_2 \oplus \cdots \oplus V_n$,

$$\alpha \cdot (v_1, v_2, \dots, v_n) = (\alpha v_1, \alpha v_2, \dots, \alpha v_n),$$

where αv_i is the scalar multiplication in V_i for $i = 1, 2, \dots, n$.

The direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_n$ is the vector space consisting of all tuples $(v_1, v_2, \dots, v_n)$ where each v_i belongs to V_i. The vector space is equipped with the standard addition and scalar multiplication operations defined above.

Proof:

1) Non-empty

$$0 \in V_1, \ldots, V_n$$
$$0 + 0 \cdots + 0 = 0 \in V_1 \oplus V_2 \oplus \cdots \oplus V_n$$
$$\therefore \text{direct sum is non-empty}$$

2) Closed in addition

$$u + w = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

$= (u_1 + w_1, 0, \ldots, 0) + (0, u_2 + w + 2, 0, \ldots, 0) + \ldots (0, \ldots, 0, u_n + w_n) \in V_1 \oplus V_2 \oplus \cdots \oplus V_n$

3) Closed under scalar multiplication

$\alpha u = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n) = (\alpha u_1, 0, \ldots, 0) + (0, \alpha u_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, \alpha u_n) \in V_1 \oplus V_2 \oplus \cdots \oplus V_n$

4) Subspace

$$\forall u \in V_1 \oplus V_2 \oplus \cdots \oplus V_n$$
$$u = (u_1, u_2, \ldots, u_n) \in V_1 \times V_2 \times \cdots \times V_n$$
$$\therefore \text{ direct sum is a subspace of direct product}$$

□

Exercise 3(Problem 6.1):

Let V and W be two subspaces of a vector space E. Prove that if $V \cup W$ is a subspace of E, then either $V \subseteq W$ or $W \subseteq V$

Proof:

Proof by contradiction

$$\text{let } V \not\subseteq W \text{ and } W \not\subseteq V$$
$$\text{then, there exists non-zero } v \in V, w \in W \text{ where } v \notin W, w \notin V$$
$$\text{Since } V \cup W \text{ is a vector space,}$$
$$v + w \in V \cup W$$
$$\text{suppose } v + w \in V$$
$$v + w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \text{ for some } a_1, \ldots, \alpha_n$$
$$w = (\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_n - \beta_n) v_n$$
$$\text{contradiction.}$$
$$v + w \notin V$$

Similarly

$$v + w \notin W$$
$$v + w \notin V \cup W$$
$$\text{contradiction.}$$
$$\therefore V \subseteq W \text{ or } W \subseteq V$$

□

Exercise 4(Problem 6.2):

Prove that for every vector space E, if $f : E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = Ker\ f \oplus Im\ f,$$

so that f is the projection onto its image $Im\ f$.

Proof: 1) $Ker\ f, Im\ f$ are independent.
Proof by contradiction:

$$\text{Suppose there exists non-zero vector } v \in ker\ f \cap im\ f$$

$$v = f(u)(\text{ for some } u \in E)$$
$$v = f(u) = f \circ f(u) = f(v) = 0$$
$$\text{contradiction.}$$
$$\therefore Ker\ f, Im\ f \text{ are independent.}$$

2) $E = Ker\ f + Im\ f$

$$\forall u \in E, u \in Ker\ f \text{ or } u \in Im\ f$$

$$u \in Ker\ f + Im\ f$$

$$\therefore E = Ker\ f \oplus Im\ f$$

□

Exercises 5(Problem 6.5):

Given any vector space E, a linear map, $f : E \to E$ is an involution if $f \circ f = id$.
1) Prove that an involution is invertible.

Proof:

$$\text{let, } f^{-1} = f$$
$$f^{-1} \circ f(u) = id(u) = u(\forall u \in E)$$

□2) Let E_1 and E_{-1} be the subspaces of E defined as follows:

$$E_1 = \{u \in E | f(u) = u\}$$

$$E_{-1} = \{u \in E | f(u) = -u\}$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}$$

1) E_1 and E_{-1} are independent.

Proof:

$$u \in E_1 \cap E_{-1}$$

$$f(u) = u = -u$$

$$u = 0$$

$\square$2) Surjectivity

$$\forall u \in E$$

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}$$

$$f(\frac{u + f(u)}{2}) = \frac{f(u) + id(u)}{2} = \frac{u + f(u)}{2} \in E_1$$

$$f(\frac{u - f(u)}{2}) = \frac{f(u) - id(u)}{2} = \frac{-u + f(u)}{2} \in E_{-1}$$

$$\therefore \text{ by 1), 2) } E = E_1 \oplus E_{-1}$$

$\square$3) If E is finite dimensional and f is an involution, prove that there is some basis of E with respect to which the matrix of f is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix (similarly for I_{n-k}) and $k = dim(E_1)$

Proof:

$$\text{let } e_1, e_2, \ldots, e_k \text{ be a basis of } E_1$$

$$f(e_i) = e_i \text{ for } i \leq k$$

$$I_{k,n-k}(0, \ldots, e_i, \ldots, 0) = e_i$$

$$\text{let } e_{k+1}, \ldots e_n \text{ be basis of } E_{-1}$$

$$f(e_j) = -e_j \text{ for } k+1 \leq j \leq n$$

$$I_{k,n-k}(0, \ldots, e_j, \ldots, 0) = -e_j$$

$\therefore$ there is some basis of E with respect to which the matrix of f is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}$$

$\square$

3a) Geometric Interpretation

Geometric interpretation is flip of vector with respect to $n - k$ axis. When $k = n - 1$, $n - k = 1$, so filp w.r.t final axis.