MA3676 - 2018 Past Paper

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Contents

1 Safe Answers

Question			Marks	Total Question Marks
1	a	i	[1]	
		ii	[3]	
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		v	[1]	
		vi	[6]	
	b		[3]	18/20
2	a		[4]	
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3	a	i	[10]	
		ii	[3]	13/20
Best Score				51/60 = A+

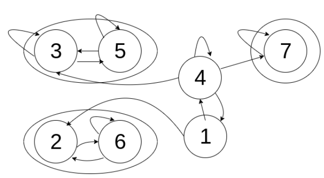
2 1

2.1 a

2.1.1 i

$$\mathbf{p} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

2.1.2 ii



States $\{2,6\}$, $\{3,5\}$, $\{7\}$ are all closed recurrent sets of recurrent states, $1,4$ are transient.

$$\mathbf{p} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}. \tag{2}$$

2.1.3 iii

There are three closed sets of recurrent states, so three of the seven eigenvalues will be equal to one. The other four will have a magnitude less than one, with one of these being exactly equal to negative 1.

2.1.4 iv

Find the equilibrium state of the closed set by solving

$$\begin{pmatrix}\pi_2 & \pi_6\end{pmatrix} = \begin{pmatrix}\pi_2 & \pi_6\end{pmatrix}\begin{pmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}, \tag{3}$$

to give you

$$\pi_2 = \frac{1}{3} \quad \pi_6 = \frac{2}{3}. \tag{4}$$

Therefore, our answer is $\pi_2 = \frac{1}{3}$.

2.1.5 v

States $1, 4$ are transient, so the answer is 0.

2.1.6 vi

Using the matrix in (2), we derive

$$\mathbf{R} = \left(\begin{array}{cc|cc|c} \frac{1}{2} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4}\end{array}\right), \tag{5}$$

$$\tilde{\mathbf{R}} = \begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & \frac{1}{4} & \frac{1}{4},\end{pmatrix}, \tag{6}$$

$$\mathbf{Q} = \begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{4},\end{pmatrix} \tag{7}$$

and solve $\tilde{\mathbf{V}} = (\mathbb{I} - \mathbf{Q})^{-1}\tilde{\mathbf{R}}$ to find

$$\tilde{\mathbf{V}} = \begin{pmatrix}\frac{3}{5} & \frac{1}{5} & \frac{1}{5}\\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5}\end{pmatrix}. \tag{8}$$

2.2 b

The transition matrix can be written as

$$\begin{pmatrix}a & b & 0 & 0\\ c & d & 0 & 0\\ e & 0 & f & 0\\ 0 & 0 & g & 0\end{pmatrix}, \tag{9}$$

where the bottom-right $2x2$-matrix is $\mathbf{Q}$. We use this to solve

$$\mu = (\mathbb{I} - \mathbf{Q})^{-1}\mathbf{e} \tag{10}$$

$$= \begin{pmatrix}1-f & 0\\ -g & 1\end{pmatrix}^{-1}\begin{pmatrix}1\\1\end{pmatrix} \tag{11}$$

$$= \begin{pmatrix}\frac{1}{1-f} & 0\\ \frac{g}{1-f} & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} \tag{12}$$

$$= \begin{pmatrix}\frac{1}{1-f}\\ 1+\frac{g}{1-f}\end{pmatrix} = \begin{pmatrix}1.4\\1.6\end{pmatrix}. \tag{13}$$

Solving this gives you the values of $f = \frac{2}{7}$ and $g = \frac{3}{7}$. Looking at the transition matrix back in (9), we can see that g must be equal to one, and therefore the information given in this question is not reliable.

3 2

3.1 a

The one-step transition matrix of the scenario is

$$\begin{pmatrix}0.8 & 0.2 & 0 & 0\\ 0.05 & 0.75 & 0.2 & 0\\ 0.05 & 0 & 0.75 & 0.2\\ 0.1 & 0 & 0 & 0.9\end{pmatrix}. \tag{14}$$

It has an equilibrial solution because there are less than two closed sets.

3.2 b

Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ be our steady state of the transition matrix (14). We then solve

$$\pi \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.05 & 0.75 & 0.2 & 0 \\ 0.05 & 0 & 0.75 & 0.2 \\ 0.1 & 0 & 0 & 0.9 \end{pmatrix} = \pi. \tag{15}$$

We obtain the linear equations:

$$\begin{align} 0.8\pi_1 + 0.05\pi_2 + 0.05\pi_3 + 0.1\pi_4 &= \pi_1 \tag{16} \\ 0.2\pi_1 + 0.75\pi_2 &= \pi_2 \tag{17} \\ 0.2\pi_3 + 0.75\pi_3 &= \pi_3 \tag{18} \\ 0.2\pi_3 + 0.9\pi_4 &= \pi_4. \tag{19} \end{align}$$

These can be solved in terms of π_4 to give

$$\pi = \left(\frac{25\pi_4}{32}, \frac{5\pi_4}{8}, \frac{\pi_4}{2}, \pi_4\right) = \pi_4\left(\frac{25}{32}, \frac{5}{8}, \frac{1}{2}, 1\right). \tag{20}$$

Using the fact that the magnitude of π is one, we solve the following

$$\pi_4 \left(\frac{25 + 20 + 16 + 32}{32}\right) = 1, \tag{21}$$

to find

$$\pi_4 = \frac{32}{93}. \tag{22}$$

This gives us the final equilibrium as

$$\pi = \left(\frac{25}{93}, \frac{20}{93}, \frac{16}{93}, \frac{32}{93}\right), \tag{23}$$

showing that the unemployment rate in the long run is $\pi_1 = \frac{25}{93} \approx 0.269\ldots$.

3.3 c

Using transition matrix (14), we note the in-goings and out-goings at each employment bracket, as well as the percentage of each that is taken into account.

Employment	Distribution	Salary	In/Outgoing
Unemployed	25/93	1500	−100%
Level 1	20/93	2000	+10%
Level 2	16/93	4000	+10%
Level 3	32/93	6000	+10%

Expand this and calculate total monthly income **I**:

$$\begin{align} \mathbf{I} &= -\left(\frac{25\mathbf{N}}{93} \times 1500 \times 1\right) + \left(\frac{20\mathbf{N}}{93} \times 2000 \times 0.1\right) + \left(\frac{16\mathbf{N}}{93} \times 4000 \times 0.1\right) + \left(\frac{32\mathbf{N}}{93} \times 6000 \times 0.1\right) \tag{24} \\ &= \frac{\mathbf{N}}{93}\left[-37500 + 4000 + 6400 + 19200\right] \tag{25} \\ &= -\frac{7900\mathbf{N}}{93}. \tag{26} \end{align}$$

The country's treasury makes a loss of $-\frac{7900\mathbf{N}}{93}$ marks each month.

3.4 d

We create a new transition matrix such that

$$\mathbf{p} = \begin{pmatrix} 0.6 & 0.4 & 0 & 0 \\ 0.04 & 0.76 & 0.2 & 0 \\ 0.04 & 0 & 0.76 & 0.2 \\ 0.1 & 0 & 0 & 0.9 \end{pmatrix}, \tag{27}$$

alongside a new tax rate to derive the following:

Employment	Distribution	Salary	In/Outgoing
Unemployed	0.146	1500	−100%
Level 1	0.244	2000	+7%
Level 2	0.203	4000	+7%
Level 3	0.407	6000	+7%

where our new distributions are found using the same method as that in question 2b (use an online calculator, I use `http://psych.fullerton.edu/mbirnbaum/calculators/Markov_Calculator.htm`). We then go through the same process as 2c to find our new income $\mathbf{I} = 42.94\mathbf{N}$ marks each month. This is definitely beneficial to the country's income.

4 3

4.1 a

4.1.1 i

Our first step decomposition is

$$g_n = \mathbb{E}[E_n| + 1] \cdot \frac{2}{3} + \mathbb{E}[E_n| + 2] \cdot \frac{1}{3}. \tag{28}$$

We then define the following:

$$\mathbb{E}[E_n| + 1] = 1 + g_{n+1}, \tag{29}$$
$$\mathbb{E}[E_n| + 2] = 1 + g_{n+2}, \tag{30}$$

allowing us to rewrite (28) as

$$g_n = 1 + \frac{2}{3}g_{n+1} + \frac{1}{3}g_{n+2}. \tag{31}$$

As this walk terminates at both k and $k+1$, the steps until termination at these positions is zero, hence

$$g_k = 0, \tag{32}$$
$$g_{k+1} = 0. \tag{33}$$

These are our boundary conditions.
Lets get the solutions to the characteristic equation of (31):

$$1 = \frac{2}{3}\lambda + \frac{1}{3}\lambda^2, \tag{34}$$
$$3 = 2\lambda + \lambda^2, \tag{35}$$
$$0 = \lambda^2 + 2\lambda - 3, \tag{36}$$
$$0 = (\lambda - 1)(\lambda + 3), \tag{37}$$

Giving us roots of $\lambda = 1, -3$. This means the solution to our inhomogeneous equation is given by

$$g_n^{(general)} = A + B(-3)^n, \tag{38}$$

where -3 is our $\lambda \neq 1$.
For the particular solution, we use a guess of αn^a, where a is the number of repeated roots (1, in our case) and substitute this into (31) is a *function* of n.

$$\alpha n = 1 + \frac{2}{3}\alpha(n+1) + \frac{1}{3}\alpha(n+2), \tag{39}$$

solves to $\alpha = -\frac{3}{4}$ after making $n = 0$ and simplifying. Therefore, the general solution is

$$g_n = A + B(-3)^n - \frac{3}{4}n. \tag{40}$$

We then use our boundaries from (32) & (33) to set the following:

$$g_k = A + B(-3)^k - \frac{3}{4}k = 0, \tag{41}$$

$$g_{k+1} = A + B(-3)^{k+1} - \frac{3}{4}(k+1) = 0. \tag{42}$$

Solving these simultaneously provides us with

$$A = \frac{3}{4}k + 3, \tag{43}$$

$$B = \frac{3}{8}\left(-\frac{1}{3}\right)^k. \tag{44}$$

Therefore, starting from $n = 0$, the probability of termination is given by

$$g_0 = A + B = \frac{3}{4}k + 3 + \frac{3}{8}\left(-\frac{1}{3}\right)^k. \tag{45}$$

4.1.2 ii

Y_n is martingale defined as $Y_n = S_n + \beta n$. For this to be true, the following equation must hold (and is subsequently solved)

$$\begin{aligned}
\mathbb{E}[Y_{n+1}|\{S_i\}_{i=0}^n] &= Y_n = S_n + \beta n, & (46)\\
&= \mathbb{E}[(S_n + X_{n+1} + \beta(n+1)], & (47)\\
&= S_n + \frac{4}{3} + \beta n + \beta = S_n + \beta n, & (48)\\
& & (49)
\end{aligned}$$

which is rearranged to show $\beta = -\frac{4}{3}$.
Note, $\mathbb{E}[X] = \frac{4}{3}$ is shown true through

$$\begin{aligned}
\mathbb{E}[X] = \mathbb{P}[X = 1] \cdot 1 + \mathbb{P}[X = 2] \cdot 2, & \quad (50)\\
= \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 2 = \frac{4}{3}. & \quad (51)
\end{aligned}$$

4.1.3 iii

4.2 b

4.2.1 i

4.2.2 ii

5 4

5.1 a

5.2 b

5.3 c

5.4 d

5.5 e

5.6 f

5.7 g