Sequences

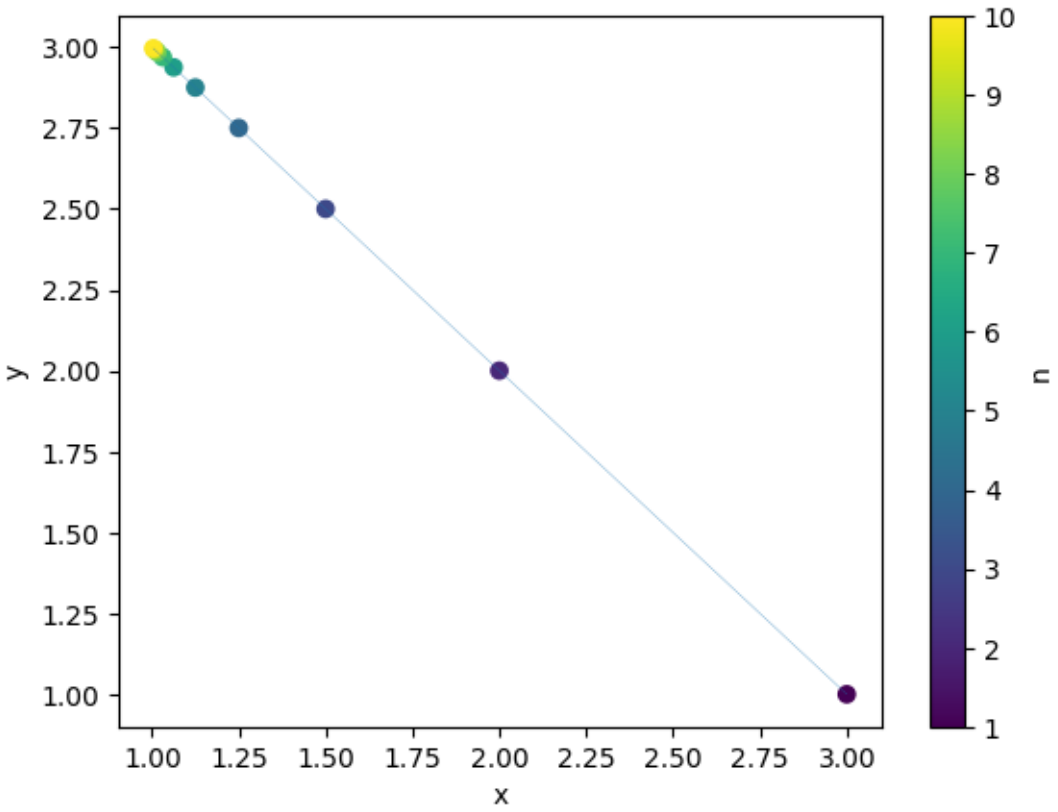
June 20, 2025

[102]:
```
import numpy as np
import matplotlib.pyplot as plt
```

[103]:
```
def plot_xn(x_n, n):
  plt.scatter(*x_n.T, c=n)
  plt.plot(*x_n.T, linewidth=0.2)
  plt.xlabel('x')
  plt.ylabel('y')
  plt.colorbar(label='n')
```

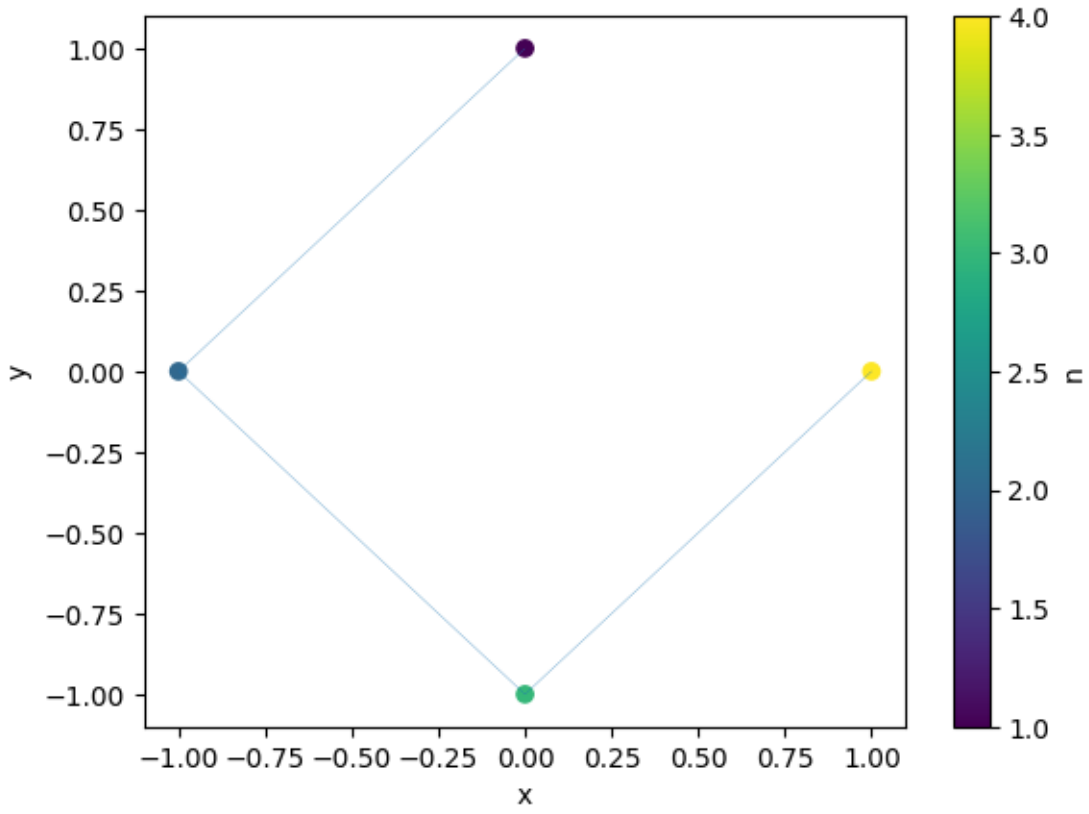
Exponential convergence to point $x_n = \left(1 + \frac{4}{2^n},\ 3 - \frac{4}{2^n}\right)$

[104]:
```
n = np.arange(1, 11)
x_n = np.column_stack((1 + 4 / 2**n, 3 - 4 / 2**n))
plot_xn(x_n, n)
```



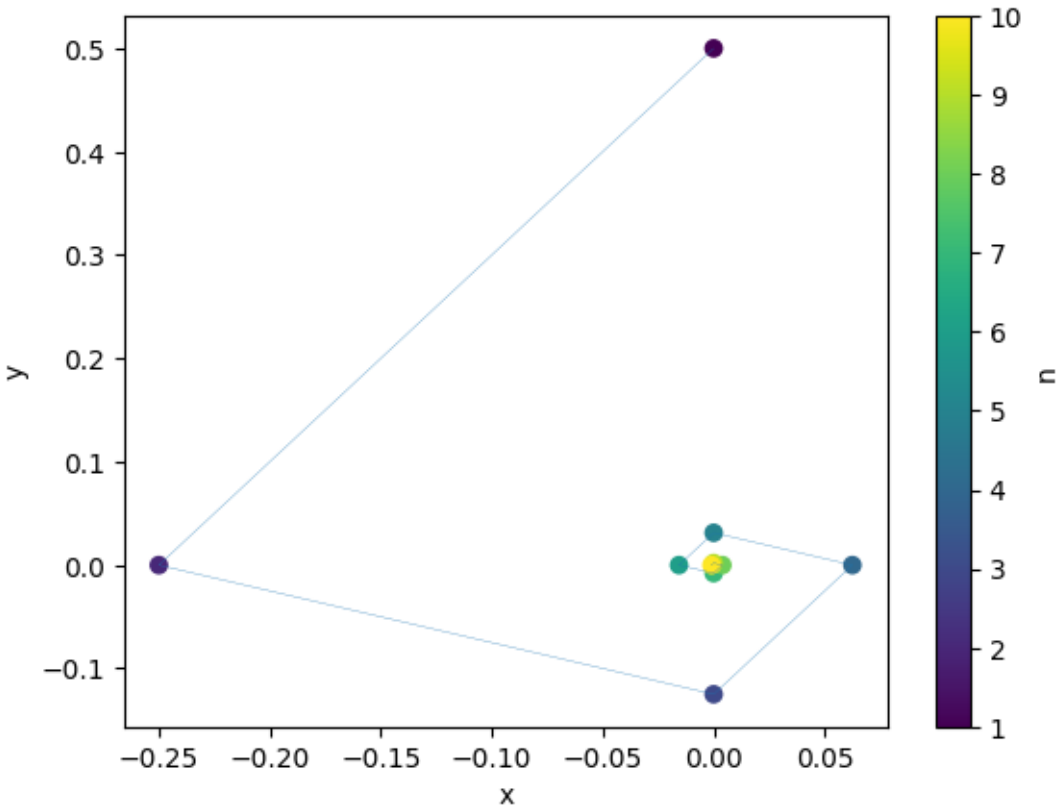
Unit circle jumps (4 points) $\quad x_n = (\cos(\frac{\pi}{2}n),\ \sin(\frac{\pi}{2}n))$

[105]:
```
n = np.arange(1, 5)
x_n = np.column_stack((np.cos(np.pi/2 * n), np.sin(np.pi/2 * n)))
plot_xn(x_n, n)
```



Shrinking spiral (4 points) $x_n = \left(\frac{1}{2^n}\cos\left(\frac{\pi}{2}n\right),\ \frac{1}{2^n}\sin\left(\frac{\pi}{2}n\right)\right)$

[106]:
```
n = np.arange(1, 11)
r = 1 / 2**n
x_n = np.column_stack((r * np.cos(np.pi/2 * n), r * np.sin(np.pi/2 * n)))
plot_xn(x_n, n)
```

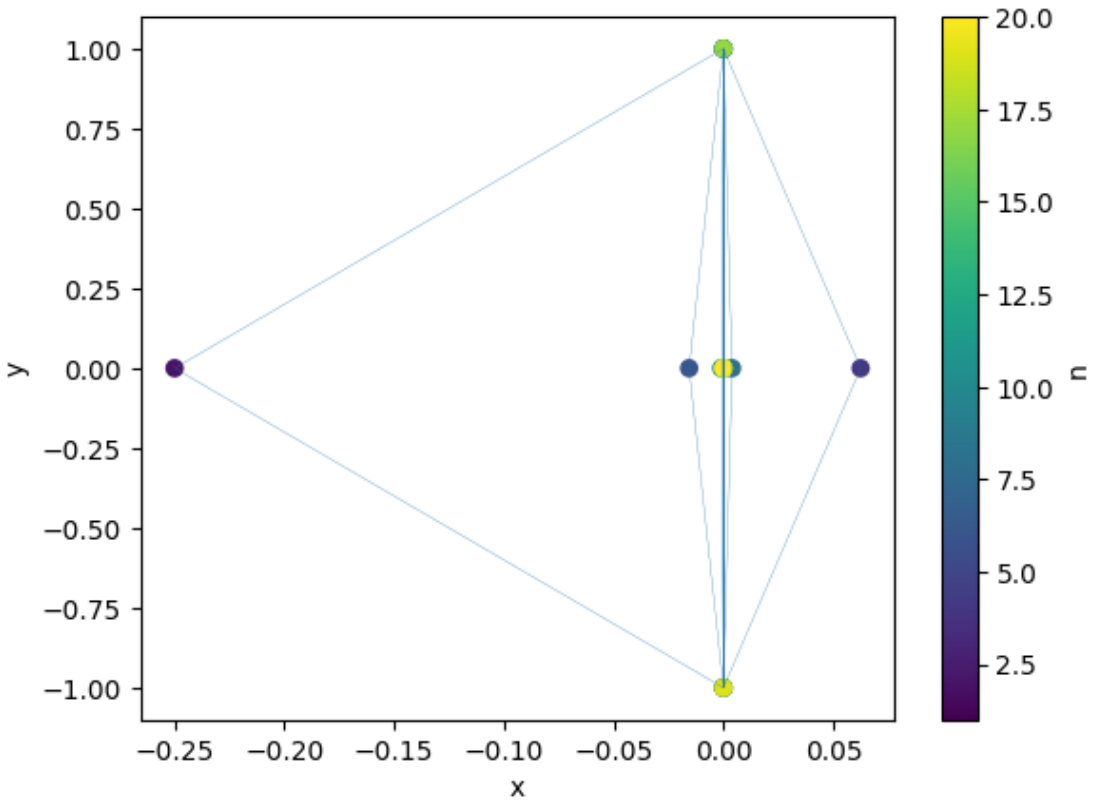


You scaled only `cos` with `r`, not `sin`.

- **x-component** shrinks with $\frac{1}{2^n}$
- **y-component** stays on unit circle (not shrinking)

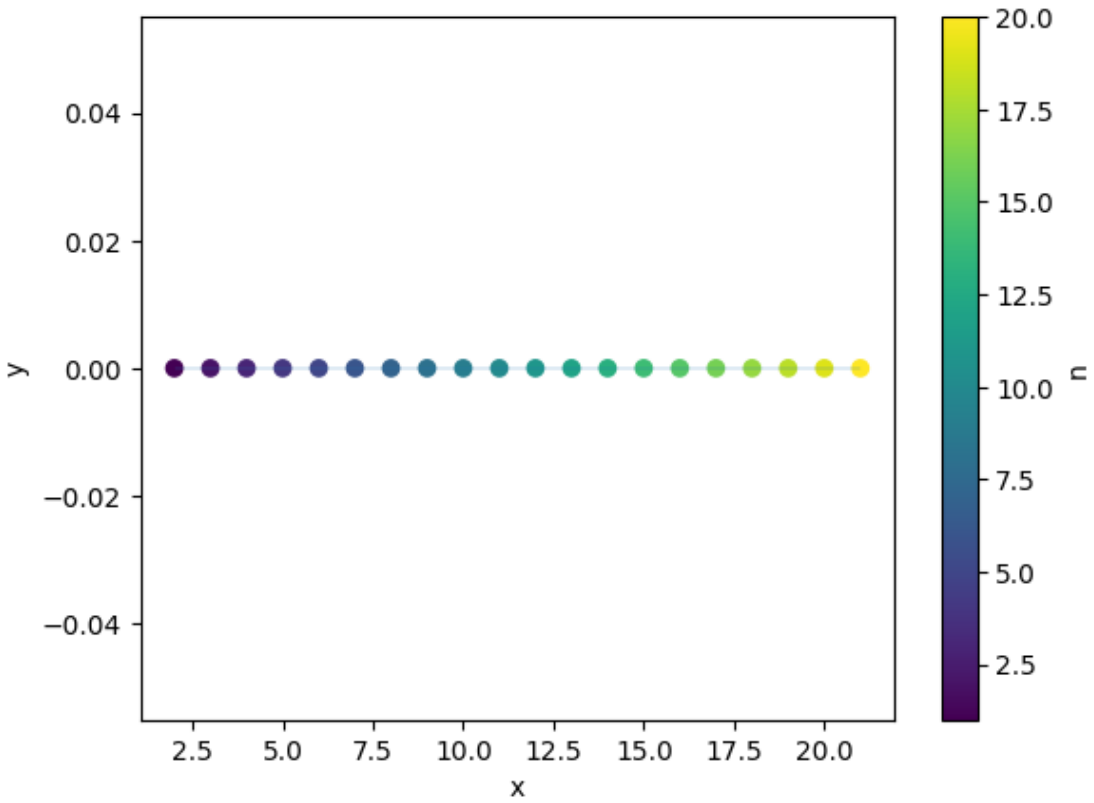
So points lie on an **ellipse**, not a spiral or circle.

[115]:
```
n = np.arange(1, 21)
r = 1 / 2**n
x_n = np.column_stack((r * np.cos(np.pi/2 * n), np.sin(np.pi/2 * n)))
plot_xn(x_n, n)
```



Linear sequence $x_n = 1 + n$

[108]:
```
n = np.arange(1, 21)
x_n = np.column_stack((1 + n, 0*n))
plot_xn(x_n, n)
```


0.04
0.02
0.00
−0.02
−0.04
y
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
x
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
n