Construction of coequalizer

1. A split coequalizer is a diagram as follow

$$\begin{tikzcd} X_1 \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & X_2 \arrow[r, "\alpha_3"] & X_3 \end{tikzcd}$$

   which has morphisms $\beta_2 \in \mathrm{Hom}(X_3, X_2)$ and $\beta_1 \in \mathrm{Hom}(X_2, X_1)$ such that the following equations hold.

$$\begin{array}{rcl} \alpha_1\beta_1 & = & \mathrm{id}_{X_2} \\ \alpha_2\beta_1 & = & \beta_2\alpha_3 \\ \alpha_3\beta_2 & = & \mathrm{id}_{X_3} \\ \alpha_3\alpha_1 & = & \alpha_3\alpha_1 \end{array}$$

2. Any split coequalizer is a coequalizer.

   Proof.
   - We assume that there exists X_4 and $\alpha_4 \in \mathrm{Hom}(X_2, X_4)$, such that $\alpha_4\alpha_1 = \alpha_4\alpha_2$. Then we construct k such that $k\alpha_3 = j$.

$$\begin{tikzcd} X_1 \arrow[r, shift left, "\alpha_1"] \arrow[r, shift right, "\alpha_2"'] & X_2 \arrow[r, "\alpha_3"] \arrow[dr, "\alpha_4"'] & X_3 \arrow[d, "k"] \\ & & X_4 \end{tikzcd}$$

     Let $k := \alpha_4\beta_2$. Then the following equation holds,

$$\begin{array}{rcl} k\alpha_3 & = & \alpha_4\beta_2\alpha_3 \\ & = & \alpha_4\alpha_2\beta_1 \\ & = & \alpha_4\alpha_1\beta_1 \\ & = & \alpha_4 \end{array}$$

     therefore we have such an arrow k.

- We next prove that such a k is unique. To do this, we prove that α_3 is epi. We assume that $k_1\alpha_3 = k_2\alpha_3$. Then, $k_1\alpha_3\beta_2 = k_2\alpha_3\beta_2$ holds, therefore we have $k_1 = k_2$.

$\square$

3. Let $P : \mathbf{Set} \to \mathbf{Set}$, *the power set functor*, be a functor such that the followings hold.

$$\begin{array}{ccc} X & \mapsto & \{S \mid S \subset X\} \\ f \in \mathrm{Hom}(X_1, X_2) & \mapsto & T \in PX_2 \mapsto \{x \in X_1 \mid f(x) \in T\} \in PX_1 \end{array}$$

4. Now, let $\varepsilon : 1 \to P^2$ be a natural transformation such that the followings hold.

$$\begin{array}{rcl} \varepsilon_X : X & \to & P^2X \\ x \in X & \mapsto & \{S \subset X \mid x \in S\} \in P^2X \end{array}$$

5. Let $\exists : \mathbf{Set} \to \mathbf{Set}$ be a functor such that the followings hold.

$$\begin{array}{rcl} \exists : A & \mapsto & PA \\ f & \in & \mathrm{Hom}(A, B) \\ \exists_f : S \subset A & \mapsto & \{f(a) \mid a \in S\} \end{array}$$

6. Let the left of the following diagrams be a pullback diagram. If i is mono, then j is also mono, and the right side of the following diagrams commutes.

$$\begin{tikzcd} Y_1 \arrow[r, "\alpha"] \arrow[d, "j"'] & Y_2 \arrow[d, "i"] \\ X_1 \arrow[r, "\alpha"'] & X_2 \end{tikzcd} \qquad \begin{tikzcd} PY_2 \arrow[r, "P(\alpha)"] \arrow[d, "\exists_i"'] & PY_1 \arrow[d, "\exists_j"] \\ PX_2 \arrow[r, "P(\alpha)"'] & PX_1 \end{tikzcd}$$

Proof. TODO $\square$

7. If $\alpha \in \mathrm{Hom}(X_1, X_2)$ is a mono, then the following diagram commutes.

$$\begin{tikzcd} PX_1 \arrow[r, "\exists_\alpha"] \arrow[dr, equal] & PX_2 \arrow[d, "P(\alpha)"] \\ & PX_1 \end{tikzcd}$$

Proof. TODO ☐

8. P is a faithful functor.

9. The following equations hold.

$$\begin{array}{rcl} P(\varepsilon_X)\varepsilon_{PX} & = & \mathrm{id} \\ P(\varepsilon_X)P(\varepsilon_{P^2X}) & = & P(\varepsilon_X)P^3(\varepsilon_X) \end{array}$$

10. P reflects isomorphism.

Proof. TODO ☐

11. Let the following diagram be a equalizer

$$\begin{tikzcd} X_1 \arrow[r, "\alpha_1"] & X_2 \arrow[r, shift left, "\alpha_2"] \arrow[r, shift right, "\alpha_3"'] & X_3 \end{tikzcd}$$

If there exists $d \in \mathrm{Hom}(X_3, X_2)$ such that $d\alpha_1 = d\alpha_2 = \mathrm{id}_{X_2}$, then the following diagram

$$\begin{tikzcd} PX_3 \arrow[r, shift left, "P(\alpha_2)"] \arrow[r, shift right, "P(\alpha_3)"'] & PX_2 \arrow[r, "P(\alpha_1)"] & PX_1 \end{tikzcd}$$

is a coequalizer.

Proof. We show that this is a split coequalizer. Now, what we prove are followings.

$$\begin{array}{rcl} P(\alpha_2)\exists_{\alpha_2} & = & id \\ P(\alpha_3)\exists_{\alpha_2} & = & \exists_{\alpha_1}P(\alpha_1) \\ P(\alpha_1)\exists_{\alpha_1} & = & id \\ P(\alpha_1)P(\alpha_2) & = & P(\alpha_1)P(\alpha_3) \end{array}$$

- $P(\alpha_2)\exists_{\alpha_2} = id$.
  We only prove that α_2 is mono. If $\alpha_2 k_1 = \alpha_2 k_2$, then $d\alpha_2 k_1 = d\alpha_2 k_2$ holds, therefore we have $k_1 = k_2$.

- $P(\alpha_3)\exists_{\alpha_2} = \exists_{\alpha_1} P(\alpha_1)$
  The following diagram is a pullback.

$$\begin{tikzcd} X_1 \arrow[r, "\alpha_1"] \arrow[d, "\alpha_1"'] & X_2 \arrow[d, "\alpha_2"] \\ X_2 \arrow[r, "\alpha_3"'] & X_3 \end{tikzcd}$$

  Now, we already proved that α_2 is mono, therefore, we have the fact that the following diagram commutes.

$$\begin{tikzcd} PX_2 \arrow[r, "P(\alpha_1)"] \arrow[d, "\exists_{\alpha_2}"'] & PX_1 \arrow[d, "\exists_{\alpha_1}"] \\ PX_3 \arrow[r, "P(\alpha_3)"'] & PX_2 \end{tikzcd}$$

  Hence, we have the equation.

- $P(\alpha_1)\exists_{\alpha_1} = id$
  α_1 is a equalizer, therefore α_1 is mono.

- $P(\alpha_1)P(\alpha_2) = P(\alpha_1)P(\alpha_3)$
  Since $\alpha_2\alpha_1 = \alpha_3\alpha_1$.

□

12. The following diagram is a coequalizer.

$$\begin{tikzcd} P^5X \arrow[r, shift left, "P^3(\varepsilon_X)"] \arrow[r, shift right, "P(\varepsilon_{P^2X})"'] & P^3X \arrow[r, "P(\varepsilon_X)"] & PX \end{tikzcd}$$

Proof. We show that it's a split coequalizer with ε_{PX} and ε_{P^3X}.

- $P(\varepsilon_{P^2X})\varepsilon_{P^3X} = \mathrm{id}$
  It's clear by 9.

- $P^3(\varepsilon_X)\varepsilon_{P^3X} = \varepsilon_{PX}P(\varepsilon_X)$
  It's clear the naturality of ε.

- $P(\varepsilon_X)\varepsilon_{PX} = \mathrm{id}$
  It's clear by 9.

- $P(\varepsilon_X)P^3(\varepsilon_X) = P(\varepsilon_X)P(\varepsilon_{P^2X})$
  It's clear by 9.

∎

13. The following diagram is a equalizer.

$$\begin{tikzcd} A \arrow[r, "\varepsilon_A"] & P^2A \arrow[r, shift left, "\varepsilon_{P^2A}"] \arrow[r, shift right, "P^2(\varepsilon_A)"'] & P^4A \end{tikzcd}$$

Proof. Let the following diagram be a equalizer.

$$\begin{tikzcd} L \arrow[r, "\kappa"] & P^2A \arrow[r, shift left, "\varepsilon_{P^2A}"] \arrow[r, shift right, "P^2(\varepsilon_A)"'] & P^4A \end{tikzcd}$$

Since equalizer, then there exists an arrow $\varepsilon'_A \in \mathrm{Hom}(A, L)$ such that left triangle of the following diagram commutes.

$$\begin{tikzcd} L \arrow[r, "\kappa"] & P^2A \arrow[r, shift left, "\varepsilon_{P^2A}"] \arrow[r, shift right, "P^2(\varepsilon_A)"'] & P^4A \\ A \arrow[u, dashed, "\varepsilon'_A"] \arrow[ur, "\varepsilon_A"'] & & \end{tikzcd}$$

Now, $P(\varepsilon_{PA}) \in \mathrm{Hom}(P^4A, P^2A)$.

Now, we can find $\eta'_A \in \mathrm{Hom}(PA, PL)$ such that the right triangle commutes.

$$\begin{tikzcd} P^5A \arrow[r, shift left, "P^3(\varepsilon_A)"] \arrow[r, shift right, "P(\varepsilon_{P^2A})"'] & P^3A \arrow[r, "P\varepsilon_A"] \arrow[dr, "P(\kappa)"'] & PA \arrow[d, dashed, "\eta'_A"] \\ & & PL \end{tikzcd}$$

Now, the following diagram commutes.

$$\begin{tikzcd} & PA \arrow[dd, shift left, "\eta'_A"] \\ P^3A \arrow[ur, "P(\varepsilon_A)"] \arrow[dr, "P(\kappa)"'] & \\ & PL \arrow[uu, shift left, "P(\varepsilon'_A)"] \end{tikzcd}$$

Notice that both $P(\varepsilon_A)$ and $P(\kappa)$ are epi. Then, we have $P(\varepsilon_A) = P(\varepsilon'_A)P(\kappa) = P(\varepsilon'_A)\eta'_A P(\varepsilon_A)$, therefore $id = P(\varepsilon'_A)\eta'_A$ holds. Similary $id = \eta'_A P(\varepsilon'_A)$ also holds. Therefore $P(\varepsilon'_A)$ is a isomorphism. Now, P reflects isomorphism, hence ε'_A is a isomorphism. $\square$

14. Now, we fix two arrows $f, g \in \mathrm{Hom}(B, A)$.

$$\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A. \end{tikzcd}$$

We will construct a coequalizer of f and g.

$$\begin{tikzcd} B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "\sigma"] & C. \end{tikzcd}$$

15. We define an object V and morphisms τ and h. Let V and τ be a equalizer of $P(f)$ and $P(g)$.

$$\begin{tikzcd} P^2V \arrow[r, "P^2(\tau)"] \arrow[d, dashed, "h"'] & P^3A \arrow[r, shift left, "P^3(f)"] \arrow[r, shift right, "P^3(g)"'] \arrow[d, "P(\varepsilon_A)"'] & P^3B \arrow[d, "P(\varepsilon_B)"] \\ V \arrow[r, dashed, "\tau"'] & PA \arrow[r, shift left, "P(f)"] \arrow[r, shift right, "P(g)"'] & PB \end{tikzcd}$$

The right two rectangles of this diagram commute, since the naturality of ε. Thus we have $P(f)P(\varepsilon_A)P^2(\tau) = P(g)P(\varepsilon_A)P^2(\tau)$ since $P^3(f)P^2(\tau) = P^3(g)P^2(\tau)$ holds. Therefore, there exists $h \in \mathrm{Hom}(P^2V, V)$ such that the left rectangle of this diagram commutes.

16. Next, let us consider the following diagram.

$$\begin{tikzcd} C \arrow[r, dashed, "v"] & PV \arrow[r, shift left, "\varepsilon_{PV}"] \arrow[r, shift right, "P(h)"'] & P^3V \\ A \arrow[u, dashed, "\sigma"] \arrow[r, "\varepsilon_A"'] & P^2A \arrow[u, "P(\tau)"] \arrow[r, shift left, "\varepsilon_{P^2A}"] \arrow[r, shift right, "P^2(\varepsilon_A)"'] & P^4A \arrow[u, "P^3(\tau)"'] \end{tikzcd}$$

Let C and v be a equalizer of ε_{PV} and $P(h)$. Now, the two right rectangles commute, since the naturality of ε and the left rectangle of the diagram in 15. We already know that $\varepsilon_{P^2A}\varepsilon_A = P^2(\varepsilon_A)\varepsilon_A$ since the lower three morphisms $\varepsilon_A, \varepsilon_{P^2A}$ and $P^2(\varepsilon_A)$ form a equalizer. Therefore we have $\varepsilon_{PV}P(\tau)\varepsilon_A = P(h)P(\tau)\varepsilon_A$. Hence we have $\sigma \in \mathrm{Hom}(A, C)$ such that the left rectangle commutes, since the uniqueness of the equalizer.

17. Now, we prove that $\sigma f = \sigma g$. Let us consider the following diagram.

$$\begin{tikzcd}
C \arrow[r, "v"] & PV \\
A \arrow[u, "\sigma"] \arrow[r, "\varepsilon_A"'] & P^2A \arrow[u, "P(\tau)"'] \\
B \arrow[u, shift left, "g"'] \arrow[u, shift right, "f"] \arrow[r, "\varepsilon_B"'] & P^2B \arrow[u, shift right, "P^2(f)"] \arrow[u, shift left, "P^2(g)"']
\end{tikzcd}$$

We are already know that the upper rectangle of this diagram commutes. And the two lower rectangles of this diagram also commute by the naturality of ε. Since $P(\tau)P^2(f) = P(\tau)P^2(g)$ holds, we have $v\sigma f = v\sigma g$. The arrow v is mono, hence we have $\sigma f = \sigma g$.

18. Let us consider the following diagram

$$\begin{tikzcd}
B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "r"] & D
\end{tikzcd}$$

such that $rf = gr$ holds. Next, we consider the following diagram.

$$\begin{tikzcd}
V \arrow[r, "\tau"] & PA \arrow[r, shift left, "P(f)"] \arrow[r, shift right, "P(g)"'] & PB \\
PD \arrow[u, dashed, "s"] \arrow[ur, "P(r)"'] & &
\end{tikzcd}$$

Since the upper three morphisms $\tau, P(f)$ and $P(g)$ form a equalizer, there exists $s \in \mathrm{Hom}(PD, V)$ such that the left triangle commutes.

19. Next, let us consider the following diagram.

$$\begin{tikzcd}
D \arrow[r, "\varepsilon_D"] & P^2D \arrow[r, shift left, "\varepsilon_{P^2D}"] \arrow[r, shift right, "P^2(\varepsilon_D)"'] & P^4D \\
C \arrow[u, dashed, "u"] \arrow[r, "v"'] & PV \arrow[u, "P(s)"] \arrow[r, shift left, "\varepsilon_{PV}"] \arrow[r, shift right, "P(h)"'] & P^3V \arrow[u, "P^3(s)"']
\end{tikzcd}$$

The one of right rectangle commutes since the naturality of ε. Next we prove that $P^2(\varepsilon_D)P(s) = P^3(s)P(h)$. To do this, let us consider the following diagrams.

$$\begin{tikzcd}
P^3D \arrow[r, "P^2(s)"] \arrow[d, "P(\varepsilon_D)"'] & P^2V \arrow[r, "P^2(\tau)"] \arrow[d, "h"'] & P^3A \arrow[d, "P(\varepsilon_A)"] \\
PD \arrow[r, "s"'] & V \arrow[r, "\tau"'] & PA
\end{tikzcd}$$

We already know that the right rectangle commutes. And, the naturality of the $P\varepsilon$,

$$\begin{tikzcd}
P^3D \arrow[r, "P^3(r)"] \arrow[d, "P(\varepsilon_D)"'] & P^3A \arrow[d, "P(\varepsilon_A)"] \\
PD \arrow[r, "P(r)"'] & PA
\end{tikzcd}$$

and considering $P(r) = \tau s$, the outer diagram commutes. Now, τ is mono, since τ is a equalizer. Hence the left rectangle also commutes.

Therefore, the two right rectangles of above diagram commute. Thus $\varepsilon_{P^2D}P(s)v = P^2(\varepsilon_D)P(s)v$ holds. Hence, we have $u \in \mathrm{Hom}(C, D)$ such that left rectangle commutes, $\varepsilon_D u = P(s)v$, since the uniqueness of equalizer.

20. Now, we prove that $r = u\sigma$. Let us consider the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\varepsilon_A"] \arrow[d, "\sigma"'] \arrow[dd, bend right=60, "r"'] & P^2A \arrow[d, "P(\tau)"] \arrow[dd, bend left=60, "P^2(r)"] \\
C \arrow[r, "v"'] \arrow[d, "u"'] & PV \arrow[d, "P(s)"] \\
D \arrow[r, "\varepsilon_D"'] & P^2D
\end{tikzcd}$$

Now, the right triangle commutes. And, we already know that the upper and lower rectangles of the center rectangle commte. Therefore the center diagram commutes. And the outer diagram also commutes since the naturality of ε. Now, since ε_D is mono, hence we have the left triangle commutes.

21. Now we prove that $h\varepsilon_V = \mathrm{id}$ holds. To do this, let us consider the following diagram.

$$
\begin{tikzcd}
V \arrow[r, "\tau"] \arrow[d, "\varepsilon_V"'] & PA \arrow[d, "\varepsilon_{PA}"] \\
P^2V \arrow[r, "P^2(\tau)"] \arrow[d, "h"'] & P^3A \arrow[d, "P(\varepsilon_A)"] \\
V \arrow[r, "\tau"'] & PA
\end{tikzcd}
$$

The upper rectangle commutes since the naturality of ε, and we already know that the lower rectangle commutes. Thus the outer diagram commutes. Therefore $\tau = \tau h\varepsilon_V$ holds. Since τ is mono, we have $h\varepsilon_V = id$.

22. Next, we prove that $hP^2(h) = hP(\varepsilon_{PV})$ holds. Let us consider the following diagram.

$$
\begin{tikzcd}
P^4V \arrow[rrr, "P^2(h)"] \arrow[dr, "P^4(\tau)"] \arrow[ddd, "P(\varepsilon_{PV})"'] & & & P^2V \arrow[dl, "P^2(\tau)"'] \arrow[ddd, "h"] \\
& P^5A \arrow[r, "P^3(\varepsilon_A)"] \arrow[d, "P(\varepsilon_{P^2A})"'] & P^3A \arrow[d, "P(\varepsilon_A)"] & \\
& P^3A \arrow[r, "P(\varepsilon_A)"'] & PA & \\
P^2V \arrow[rrr, "h"'] \arrow[ur, "P^2(\tau)"] & & & V \arrow[ul, "\tau"']
\end{tikzcd}
$$

Now, it is clearly proved that five inner diagrams commte. Therefore, we have $\tau hP^2(h) = \tau hP(\varepsilon_{PV}$. The homomorphism τ is mono since equalizer, hence $hP^2(h) = hP(\varepsilon_{PV})$ holds.

23. The followign diagram is a coequalizer.

$$
\begin{tikzcd}
P^4V \arrow[r, shift left, "P^2(h)"] \arrow[r, shift right, "P(\varepsilon_{PV})"'] & P^2V \arrow[r, "h"] & V
\end{tikzcd}
$$

Proof. We show the four following euqations.

- $P(\varepsilon_{PV})\varepsilon_{P^2V} = \mathrm{id}$
  It's clear by 9.
- $P^2(h)\varepsilon_{P^2V} = \varepsilon_V h$
  It's clear by the naturality of ε.
- $h\varepsilon_V = \mathrm{id}$
  It's clear by 21
- $hP^2(h) = hP(\varepsilon_{PV})$
  It's clear by 22

□

24. The following diagram is a coequalizer.

$$\begin{tikzcd} P^4V \arrow[r, shift left, "P^2(h)"] \arrow[r, shift right, "P(\varepsilon_{PV})"'] & P^2V \arrow[r, "P(v)"] & PC \end{tikzcd}$$

Proof. By 11, we show that $P(\varepsilon_V)\varepsilon_{PV} = \mathrm{id}$ and $P(\varepsilon_V)P(h) = \mathrm{id}$ hold. First equation is proved in 9. The second equation is proved in 21. □

25. Now, the uniqueness of coequalizers, the following diagram commutes.

$$\begin{tikzcd} & V \arrow[dd, shift left, "q^{-1}"] \\ P^2V \arrow[ur, "h"] \arrow[dr, "P(v)"'] & \\ & PC \arrow[uu, shift left, "q"] \end{tikzcd}$$

Now, we consider the following diagram.

$$\begin{tikzcd} P^2V \arrow[r, "qP(v)"] \arrow[d, "P^2(\tau)"'] & V \arrow[r, shift left, "q^{-1}"] \arrow[d, "\tau"'] & PC \arrow[l, shift left, "q"] \arrow[dl, "P(\sigma)"] \\ P^3A \arrow[r, "P(\varepsilon_A)"'] & PA & \end{tikzcd}$$

Now, the left rectangle commutes since $qP(v) = h$ holds. And, we already know that the outer diagram also commutes. Since $P(v)$ is epi, $\tau q = P(\sigma)q^{-1}q$ holds, therefore $\tau q = P(\sigma)$ holds. Moreover,

$$\begin{aligned} \tau &= \tau q q^{-1} \\ &= P(\sigma)q^{-1} \end{aligned}$$

holds, therefore the right rectangle commutes.

26. We assume that $u_1\sigma = u_2\sigma = r$. Let us remind that there uniquely exists s such that $P(r) = \tau s$, since V, τ is a equalizer. We consider the following diagram.

$$\begin{tikzcd}
& PC \arrow[d, shift left, "q"] \arrow[dr, "P(\sigma)"] & \\
& V \arrow[u, shift left, "q^{-1}"] \arrow[r, "\tau"] & PA \\
& PD \arrow[u, "s"] \arrow[ur, "P(r)"'] \arrow[uu, bend left=70, "P(u_1)"] \arrow[uu, bend left=50, "P(u_2)"'] &
\end{tikzcd}$$

The outer diagram commutes. We know already that the left up diagram commutes, therefore we have $\tau q P(u_1) = P(r)$ and $\tau q P(u_2) = P(r)$. Since the uniqueness of s, we have $qP(u_1) = qP(u_2)$ therefore $P(u_1) = P(u_2)$. We have already proved that P is a faithful functor in 8, hence $u_1 = u_2$ holds.

27. Hence, the following diagram is a coequalizer.

$$\begin{tikzcd}
B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "\sigma"] & C
\end{tikzcd}$$

References

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