Numerical method of solving a problem of calculus of variation with boundary conditions

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1 Problem Statement

Let $y(x) : \mathbb{R} \to \mathbb{R}$ be a C^1 function which represents a curve in $\mathbb{R}^2$, with boundary conditions of $y(x_a) = y_a$ and $y(x_b) = y_b$. Then, for a function $f(y, y', x)$, a functional

$$F[y] = \int_{x_a}^{x_b} f(y, y', x)dx, \tag{1}$$

can be defined in a given domain. Our goal is to find the curve y(x) at which F[y] is at its extremum.

2 Numerical method

Let $y(x)$ be an arbitrary curve that satisfies the boundary condition. If $y_0(x)$ is a straight line from (x_a, y_a) to (x_b, y_b), then y_0 is

$$y_0(x) = \frac{y_b - y_a}{x_b - x_a}(x - x_a) + y_a. \tag{2}$$

Then, we can express the curve $y(x)$ as $y(x) = y_0(x) + \eta(x)$. Since $\eta(x) = y(x) - y_0(x)$ and both $y(x)$ and $y_0(x)$ are C^1 functions, $\eta(x)$ is also a C^1 function. Also, $\eta(x)$ must satisfy a boundary condition $\eta(x_a) = 0$ and $\eta(x_b) = 0$, as $y_0(x_a) = y_a$ and $y_0(x_b) = y_b$.

Using the representation of the curve as $y(x) = y_0(x) + \eta(x)$, we can express any admissible that satisfies the boundary condition by setting an appropriate $\eta(x)$. Then, the problem of finding $y(x)$ reduces down to the problem of finding $\eta(x)$ such that $y(x) = y_0(x) + \eta(x)$ makes $F[y]$ to be its minimum.

To find $\eta(x)$, we can parameterize $\eta(x)$ using Fourier series as

$$\eta(x) = \sum_{n=1}^{\infty} C_n sin\left(\frac{n\pi}{x_b - x_a}(x - x_a)\right), \tag{3}$$

which satisfies the boundary conditions of $\eta(x)$. Using the parameterization, we can further reduce the problem to finding an appropriate sequence C_n which minimizes $F[y]$. To find such a sequence we use the method of gradient descent.

For each element C_n, we can observe how the value of $F[y]$ changes as C_n changes. Figure 1 is an example of how $F[y]$ depends on C_1, for $F[y]$ defined with $f(y, y', x) = y'(1 + x^2 y')dx$. If each C_n is the value that minimizes $F[y]$, then $y_0(x) + \eta(x)$ must be the curve that minimizes $F[y]$.

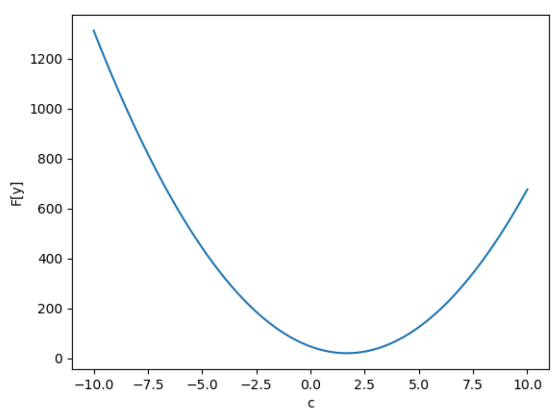

Figure 1: Plot of $F[y]$ as a function of C_n for a fixed n.

If C_n minimizes $F[y]$, the necessary condition for such C_n is that

$$\frac{\partial F}{\partial C_n} = 0. \tag{4}$$

To find the point where the above condition is satisfied, we use the method of gradient descent, starting from the arbitrary value of C_n. In our method, we used the initial value of $C_n = 0$. As described in Figure 2, If the value of $\frac{\partial F}{\partial C_n}$ is smaller than 0, it means that the value of F is decreasing, and C_n should be bigger to get a smaller value of $F[y]$. Therefore, we increase C_n by the amount proportional to the value of $\frac{\partial F}{\partial C_n}$ scaled by some constant, namely a descending rate. Therefore, the change of C_n after one iteration will be

$$C'_n = C_n - \frac{\partial F}{\partial C_n} \times r, \tag{5}$$

where $0 < r < 1$ is a constant descending rate. By changing the value of C_n through iteration, C_n will converge to a point where $\frac{\partial F}{\partial C_n} = 0$, with appropriately small value of r. If we wanted to find the maximum of $F[y]$, then we can use the method of gradient ascent, in which we update the value of C_n to an opposite direction.

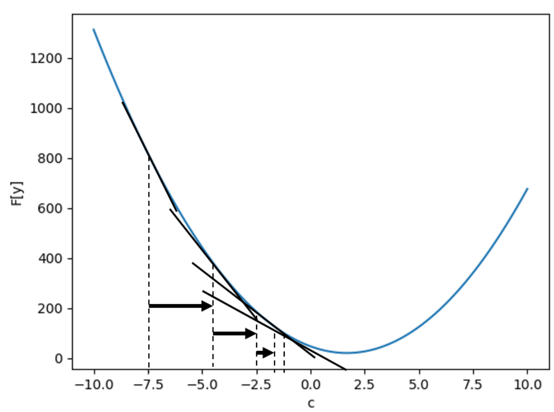

Figure 2: Gradient descent method.

By iterating the gradient descent on many values of n, the function $y(x) = y_0(x)+\eta(x)$ will converge to a curve which minimizes $F[y]$. The overall algorithm of finding $\eta(x)$ is described in Figure 3.

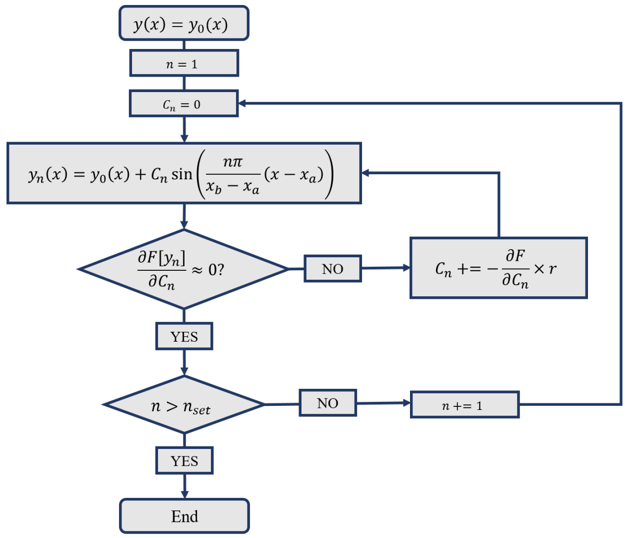

Figure 3: Flowchart of the numerical method. We start with $y_0(x)$, which is a line that connects two boundary points. For n=1, we test by adding the 1st order term of Fourier series. After finding an optimized value of C_1, we move on to the next n, and start optimizing next term of Fourier series. The process continues until n reaches the limit we set. This iteration of gradient descent isi repeated finite times.

3 Comparison to analytical solutions

3.1 Geodesics on a plane

For $F[y] = \int_1^2 \sqrt{1 + (y')^2} dx$, $y(1) = 1$ and $y(2) = 5$, find $y(x)$ such that $F[y]$ is minimized.

The analytical solution is $y(x) = 4x + 1$. Figure 4 shows both the plot of the analytical solution and the numerical solution. $F[y]$ of both curve was calculated to be 4.12, and mean squared error between two curves was 0.

3.2 Hyperbola

For $F[y] = \int_1^2 y'(1 + x^2 y') dx$, $y(1) = 1$ and $y(2) = 0.5$, find $y(x)$ such that $F[y]$ is minimized.

The analytical solution is $y(x) = \frac{1}{x}$. Figure 6 shows both the plot of the

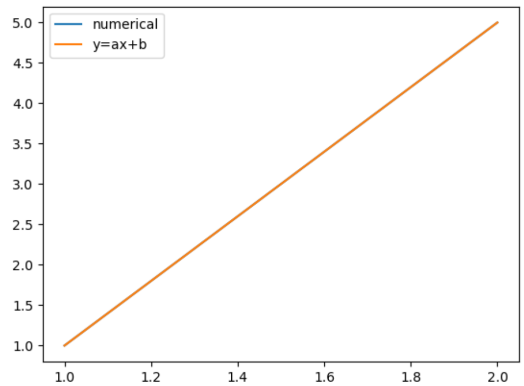

Figure 4: Analytical and numerical solution to the geodesics problem.

analytical solution and the numerical solution. $F[y]$ of the analytical solution was 0.00019, that of the numerical solution was 0.00034, and mean squared error between two curves was 0.015.

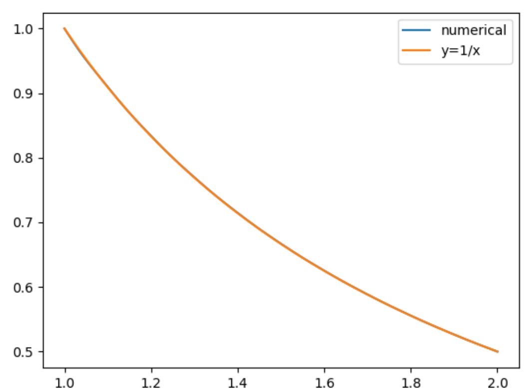

Figure 5: Analytical and numerical solution to the problem.

3.3 Brachistochrone

For $F[y] = \int_0^\pi \sqrt{\frac{1+(y')^2}{-y}}$, $y(0) = 0$ and $y(\pi) = -2$, find $y(x)$ such that $F[y]$ is minimized.

The analytical solution is a cycloid. Figure 6 shows both the plot of the analytical solution and the numerical solution. $F[y]$ of the analytical solution was calculated to be 4.337, and that of the numerical solution was 4.308.

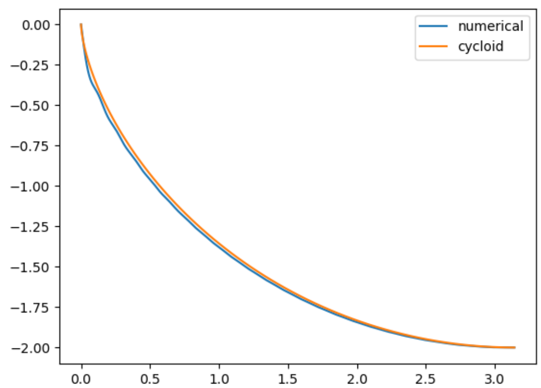

Figure 6: Analytical and numerical solution to the brachistochrone problem.