An analysis of a chain falling faster by collision

SeBeom Lee

Oberlin College, Oberlin, OH 44074

(January 30th, 2016)

When a simple chain or a rope is falling on a floor, its collision with the floor does not affect the time it takes to fall completely. It is because the part of the chain that is colliding with the floor can be considered independent from the rest of the chain, as there exists no force between two parts. However, if the chain consists of objects that are tilted in an oblique angle, the collision of the chain is different from that of a rope in that the velocity of the bottom part does not go to zero after collision. Therefore, when a chain made of sticks inclined relative to horizontal is falling on a floor, the stick colliding with the floor exerts force on the rest of the sticks, and the chain falls faster than when it's falling without collision. This paper shows an analytical explanation of the chain's collision with the floor and suggests a theoretical prediction of the chain's motion, using a numerical simulation.

I. Introduction

A simple chain problem we encounter in a classical mechanics deals with a chain that consists of a series of point masses, or what can be approximated as point mass, connected by a massless thread. When such a chain falls on a floor, will the top part of the chain be affected by the collision with the floor? When the mass attached to the chain collides with floor, assuming a non-elastic collision, its velocity goes to zero almost instantly, while rest of the chain tends to fall downward. Therefore, a tension on a string that connects the colliding mass and the rest of the chain becomes zero, and two parts can be considered independent. As the collision has no influence on the falling of the chain, we can say that the chain will fall in a same fashion regardless of the collision. However, if the chain does not consist of point masses but long sticks that are tilted at an angle, the velocity of the bottom stick does not go to zero when it collides with the floor, as it rotates rather than stops. In this case, there can exist a tension between the bottom stick and the rest of the chain. To analyse this motion, we first need to see what happens when a single stick collides with the floor.

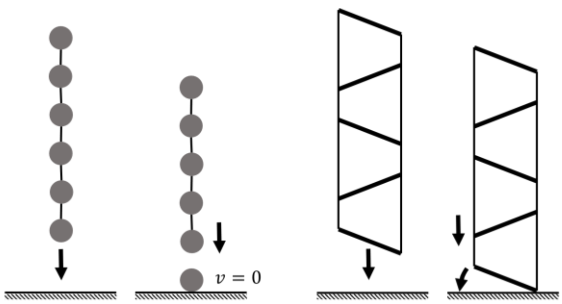

FIG. 1: The difference between simple chain and fast-falling chain: A chain consists of point masses collides with the floor without inducing any tension between masses, but the chain made of tilted sticks induces tension on a string connecting sticks.

II. Collision of a single stick

A stick of length L collides with a floor at a certain angle. Before the collision, the stick moves with a linear velocity of v and does not rotate. After the collision, assuming a non-elastic collision, the stick undergoes a motion of rotating with respect to its one end, which is in contact with the floor. In this case, what is the velocity of the stick after the collision?

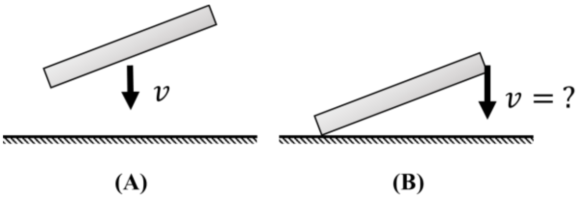

FIG. 2: Collision of a single stick: (A) A stick is moving with linear velocity of v without rotation, (B) After the collision, the stick is rotating with respect to its contact point to the floor.

A force acting on the stick during the collision is a normal force applied by the floor, and this force is a major cause of the stick's change of the momentum. On a reference frame located on the floor, the change in stick's linear momentum will be as follows.

$$\Delta \mathrm{p} = \int_0^{\Delta t} N\,dt - \int_0^{\Delta t} mg\,dt \tag{1}$$

However, since the collision happens in a very short period, so we can say that the gravity cannot change the momentum of the stick as $\Delta \mathrm{t}$ goes to zero. Therefore, the momentum change is as follows.

$$\Delta \mathrm{p} = \int_0^{\Delta t} N\,dt \tag{2}$$

Above relationship tells us how the linear momentum changes as a result of the collision, but it is useless if we do not know the normal force. Therefore, we observe how the sticks angular momentum changes.

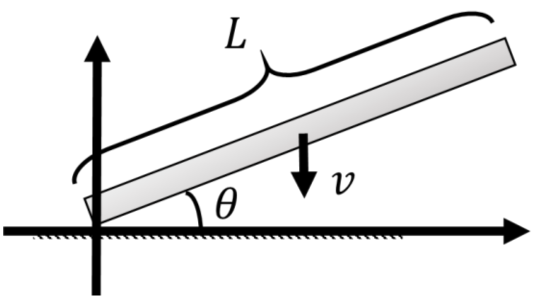

FIG. 3: Reference frame on the contact point: The normal force is applied on a contact point between the stick and the floor, so it makes no torque on this reference frame.

On a reference frame located on a contact point between the stick and the floor, an impulse made by torque equals the change in its angular momentum. Since the normal force has its point of action on the contact point, the only torque on the stick is that made by gravity. Therefore, the change in angular momentum after the collision is as follows.

$$\Delta \mathrm{l} = \int_0^{\Delta t} -mgL\cos\theta\,dt \tag{3}$$

Since the collision happens in a very short period, we can say that the angle of the stick remains constant during the collision. Therefore, the change in angular momentum due to gravity also goes to zero.

$$\Delta \mathrm{l} = \lim_{\Delta t \to 0} \int_0^{\Delta t} -mgL\cos\theta\,dt = 0 \tag{4}$$

From above analysis, we can know that the angular momentum of the stick before and after the collision are the same.

Before the collision, the angular momentum of a falling stick is as follows.

$$\mathrm{l_i} = mv\frac{L}{2}\cos\theta \tag{5}$$

After the collision, the angular momentum of the rotating stick is

$$\mathrm{l_f} = I\omega \tag{6}$$

where I is a moment of inertia and ω is an angular velocity. In this context, when calculating the moment of inertia of a stick, the angular momentum becomes

$$\mathrm{l_f} = \frac{1}{3}mL^2\omega \tag{7}$$

By equation (4), we can set a relationship of the following.

$$mv\frac{L}{2}\cos\theta = \frac{1}{3}mL^2\omega \tag{8}$$

From equation (8), we can derive the angular velocity of the stick after the collision as follows.

$$\omega = \frac{3v}{2L}\cos\theta \tag{9}$$

When the angle is small, we can approximate that the velocity of the endpoint of the stick is

$\frac{3v}{2}$, which is bigger than its velocity before the collision. Therefore, we can conclude that when a chain is connected above the stick, it pulls the chain downward as it tends to move toward the floor. Then, a next question is how the stick behaves when connected to a chain.

III. Collision of a chain of sticks

Suppose that a chain consists of n sticks that are tilted by a constant angle is falling to the floor. Then, we can divide the chain into two systems; one is a single stick at the bottom, and the other is the rest of the sticks.

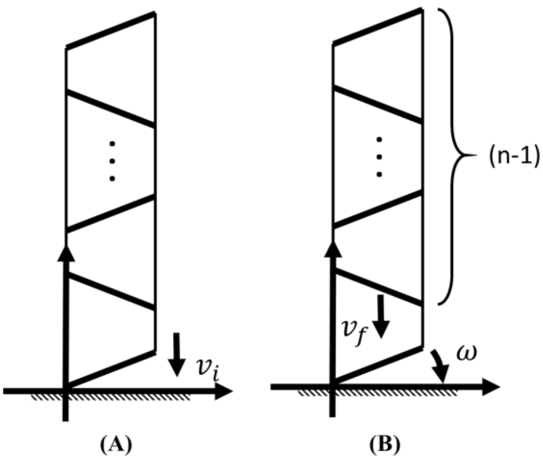

FIG. 4: A chain of sticks falling on a floor: If a chain is made of n number of sticks, we can divide it into two systems of a stick colliding with the floor and the rest of n-1 sticks. (A) Before the collision, every stick is falling with same velocity. (B) After the collision, the bottom stick starts rotating and rest of the chain keeps falling.

Before the collision, each stick is falling with a same velocity of v. Therefore, on a reference frame located on the contact point, the angular momentum of the chain is as follows.

$$\mathrm{l_i} = \mathrm{nmv_i}\frac{\mathrm{L}}{2}\cos\theta \qquad (10)$$

m is a mass of a single stick, L is a length of the stick, and θ is an angle that each stick is tilted from horizontal.

After the collision, a bottom stick, which collided with the floor, starts rotating, and rest of the sticks keeps falling, now with different velocity. Therefore, we can express the angular momentum of the chain after the collision like a following.

$$\mathrm{l_f} = (\mathrm{n}-1)\mathrm{mv_f}\frac{\mathrm{L}}{2}\cos\theta + \frac{1}{3}\mathrm{mL^2}\omega \qquad (11)$$

Assuming that the string connecting the sticks is not deformed, we can set a constraint that the falling velocity of the chain will be same as the downward velocity of the endpoint of the rotating stick.

$$\mathrm{v_f} = L\omega\cos\theta \qquad (12)$$

Using this relationship, equation (11) can be expressed as follows.

$$\mathrm{l_f} = \left(\frac{\mathrm{n}-1}{2}\cos\theta + \frac{1}{3\cos\theta}\right)\mathrm{mLv_f} \qquad (13)$$

With the same reasoning we used in the case of a single stick, and considering that a tension between sticks is inner force, we can conclude that the angular momentum of a whole chain does not change after its collision with a floor.

$$\mathrm{nmv_i}\frac{\mathrm{L}}{2}\cos\theta = \left(\frac{\mathrm{n}-1}{2}\cos\theta + \frac{1}{3\cos\theta}\right)\mathrm{mLv_f} \qquad (14)$$

Therefore, the falling velocity of the chain after the collision is

$$\mathrm{v_f} = \frac{nv_i\cos^2\theta}{\frac{2}{3}+(n-1)\cos^2\theta} \qquad (15)$$

and by equation (12), the angular velocity of the bottom stick is

$$\omega = \frac{\frac{n}{L}v_i\cos\theta}{\frac{2}{3}+(n-1)\cos^2\theta} \qquad (16)$$

When n = 1, the equation yields a same result of equation (9).

When θ is small, equation (14) can be approximated as follows.

$$\mathrm{v_f} = \frac{n}{n-\frac{1}{3}}v_i > v_i \qquad (17)$$

Therefore, the chain gets faster and faster after the collision of each stick.

IV. Existence of a critical angle

Equation (14) gives us an insight about why a chain gets faster after collision. However, when we observe how the velocity changes at different angles, the chain gets slower as the stick collides with the floor at larger angle.

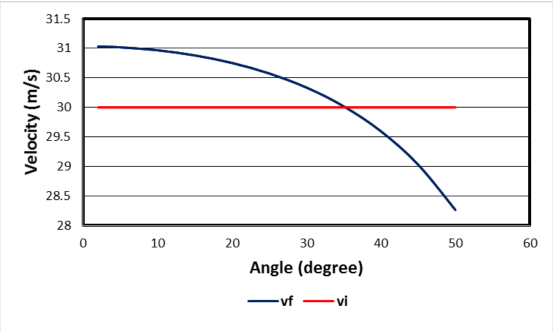

FIG. 5: Plot of angle vs. velocity for a chain made of 10 sticks: A blue curve represents the velocity after a collision, and a red line represents the velocity before a collision, which is 30 m/s.

From equation (14), we can derive that the velocity after a collision is not bigger than that before a collision at angles bigger than 35.2°. Then, what would happen if a chain made of sticks inclined more than 35.2° about horizontal falls on the floor? In the previous discussion, we determined that a main reason the chain falls faster is that the bottom chain pulls the rest of the chain. However, if the bottom chain does not fall faster after collision, it does not pull the upper stick. As a result, as there exists no tension between sticks, it becomes as same as a normal chain, so it does not fall faster. Therefore, 35.2° is a critical angle under which chain can fall faster, but at larger angle, chain does not fall faster.

V. Numerical Simulation

To make a numerical prediction of the chain's motion, we need to consider three different phase of the motion. First is a falling phase, in which all sticks fall with same velocity. During this phase, the chain falls in a uniform acceleration of g. Second is a collision phase, in which velocity of the chain changes based on equation (14) and (15). Third is rotation phase, where the bottom stick rotates with respect to its contact with the floor, and the rest of the chain falls. During this phase, the only torque that causes a change in angular momentum is a torque made by gravity, so we can say

$$\frac{\mathrm{dl}}{\mathrm{dt}} = -\frac{1}{2} mgL\cos\theta_0 \qquad (18)$$

Therefore, from equation (11), we can derive the equation of motion in this phase as follows.

$$\left(\frac{n-1}{2}\cos^2\theta_0 + \frac{1}{3}\right) L\ddot{\theta} = -\frac{1}{2} ng\cos\theta_0 \qquad (19)$$

In this equation, θ_0 is an angle that the sticks are tilted originally, and θ is an angle that the bottom stick is tilted from horizontal. We can also get an equation of the upper sticks' motion by simply using equation (12). By using a 4th order Runge Kutta method, we can simulate the motion of the chain in this phase.

In addition to above three phases, we need some constraints. First, we assume that the air resistance is negligible. Second, we assume that the sticks that are not colliding with the floor do not rotate. Third, all sticks undergo a non-elastic collision.

Based on above analysis, we can design a numerical simulation of the fast falling chain. FIG. 6 is the algorithm of the simulation.

With the numerical simulation, we can get two types of graphs: position - time graph and position – velocity squared graph. Both plots calculate the values of the top stick of a chain, which consists of 20 sticks that are 50 centimeters long, 50 centimeters away from each other, and inclined 10 degrees from horizontal. Also, the chain is falling from 2 meters away from the floor. FIG 7 is a plot of the position of the chain about time. When compared a chain falling on the floor, which will be noted as a "fast falling chain," and a chain falling without collision, which will be noted as a "free falling chain," two chains fall with a same velocity, but from the moment of collision, two chains start to have different velocities. FIG. 8 shows a plot of position vs. velocity squared. The graph of a free falling chain shows a straight line with a slope of -1/2g, but the graph of a fast falling chain shows a stair-like plot, of which the chain gets faster for each collision.

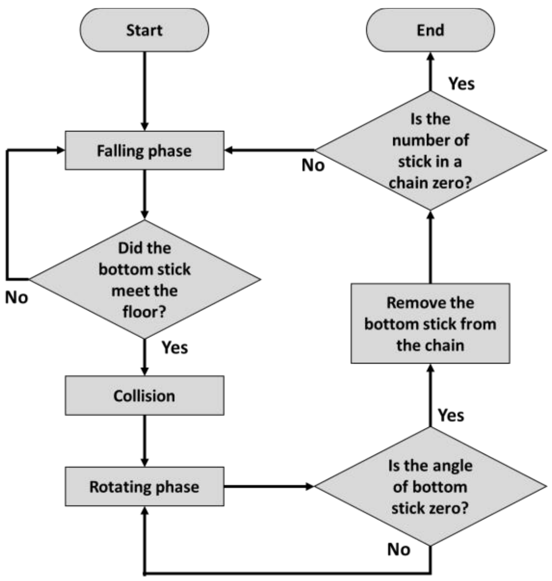

FIG. 6: Algorithm of the numerical simulation: The calculation includes a class object of "chain," which consists of several sticks. The chain falls with uniform acceleration first, and when it meets a floor, it undergoes a collision and moves with a different translational and angular velocity. After the bottom stick completely falls on the floor, it is removed from the chain, and the chain starts falling again. This process is repeated until the chain completely falls on the ground.

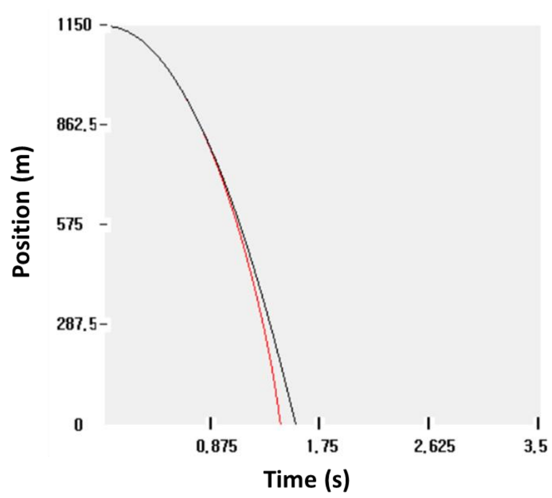

FIG. 7: Position – Time graph of the top stick: The red curve represents a chain falling on the floor, and the black curve represents a chain falling without collision. The chain falling on the floor falls about 0.13 seconds earlier.

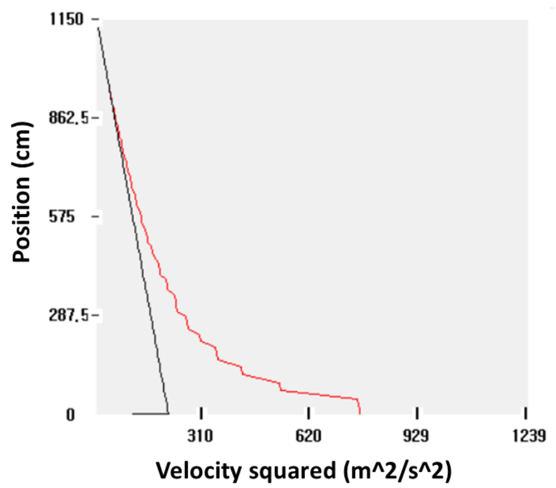

FIG. 8: Position – Velocity Squared graph of the top stick: The red curve represents a chain falling on the floor, and the black curve represents a chain falling without collision. While a free falling chain shows a straight line, the chain falling on the floor is getting faster as its sticks collides with the floor.

VI. Experiment

For the experiment, we used a chain made of 15 sticks, which are 20 centimeters long, 8 centimeters away from each other, and about 10 degrees inclined from horizontal. The chain was dropped from 5 centimeters away from the floor. Two identical chains were dropped simultaneously, and their motions were analyzed by a video analysis.

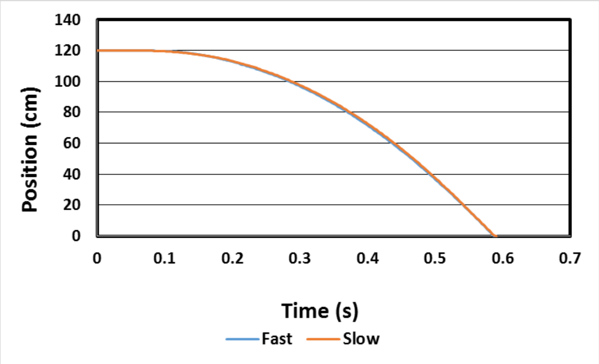

FIG. 9: Position vs. Time plot of falling chains: The blue curve represents a fast falling chain and the red curve represents a free falling chain.

Two chains seemed to fall almost identically, but a fast falling chain fell about 0.02 seconds earlier. This difference is very small, but it is not a surprise, because a numerical simulation gives a result of 0.04 seconds of difference.

VII. Discussion

A better comparison between a fast falling chain and a free falling chain can be done by an experiment done by Dr. A. Grewal. In his experiment, a chain consists of 25 sticks with average length of 10.5 centimeters, inclination of 13 degrees from horizontal, and distance of 5.21 centimeters from each other. The chains were dropped at a position where a top chain is 2.01 meters away from the floor, and it took about 0.59 seconds for a fast falling chain to fall completely. [1]

Under above circumstance, the numerical simulation predicts that it will take 0.601 seconds for a fast falling chain to fall completely, and 0.642 seconds for a free falling chain to fall completely. Therefore, we can conclude that our model of a fast falling chain is fairly accurate.

Acknowledgement

[1] A. Grewal, Philip Johnson, Andy Ruina, “A chain that accelerates, rather than slows, due to collisions: how compression can cause tension,” American Journal of Physics 79, 723 (2011)