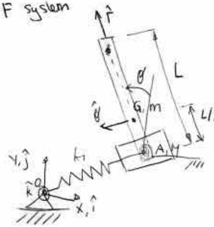
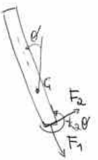
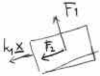
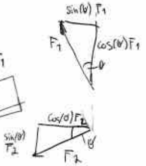
3DOF system

14.7.2021
Andreas









① $\sum F_{x,M} = M\ddot{x}\hat{i} = (-k_1 x - \sin(\theta)F_1 - \cos(\theta)F_2)\hat{i}$

② $\sum F_{y,M} = M\ddot{y}\hat{j} = (-k_1 y + \cos(\theta)F_1 - \sin(\theta)F_2)\hat{j}$

③ $\sum F_{r,m} = m a_{G/O}\hat{r} = -F_1\hat{r}$

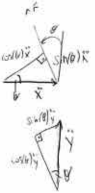
④ $\sum F_{\theta,m} = m a_{G/O}\hat{\theta} = -F_2\hat{\theta}$

⑤ $\sum \tau_{/G} = \frac{dH_{/G}}{dt} = \left[-k_2\theta + F_2\frac{L}{2}\right]\hat{k} = \frac{d}{dt}\left([I]\begin{pmatrix}0\\0\\\omega_z\end{pmatrix}\right) = I_{zz/G}\ddot{\theta}\hat{k} = \frac{mL^2}{12}\ddot{\theta}\hat{k}$

$$a_{G/O} = \underbrace{a_{A/O}}_{\ddot{x}\hat{i}+\ddot{y}\hat{j}} + \overset{0}{a_{G/A}}\Big|_{\omega_{/O}=0} + \underbrace{\dot{\omega}_{/O}\times r_{G/A}}_{\ddot{\theta}\frac{L}{2}\hat{\theta}} + \underbrace{\omega_{/O}\times(\omega_{/O}\times r_{G/A})}_{-\dot{\theta}^2\frac{L}{2}\hat{r}} + 2\,\omega_{/O}\times(\overset{0}{v_{G/A}})\Big|_{\omega_{/O}=0}$$

$$= \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{\theta}\frac{L}{2}\hat{\theta} - \dot{\theta}^2\frac{L}{2}\hat{r}$$

$$= \left[\sin(\theta)\ddot{x} + \cos(\theta)\ddot{y} - \dot{\theta}^2\frac{L}{2}\right]\hat{r} + \left[-\cos(\theta)\ddot{x} - \sin(\theta)\ddot{y} + \ddot{\theta}\frac{L}{2}\right]\hat{\theta}$$



③' $m\left[-\sin(\theta)\ddot{x}+\cos(\theta)\ddot{y}-\dot{\theta}^2\frac{L}{2}\right]=-F_1$

$\rightarrow F_1 = m\left[\sin(\theta)\ddot{x}-\cos(\theta)\ddot{y}+\dot{\theta}^2\frac{L}{2}\right]$

④' $m\left[-\cos(\theta)\ddot{x}-\sin(\theta)\ddot{y}+\ddot{\theta}\frac{L}{2}\right]=-F_2$

$\rightarrow F_2 = m\left[\cos(\theta)\ddot{x}+\sin(\theta)\ddot{y}-\ddot{\theta}\frac{L}{2}\right]$

⑤' $\frac{mL^2}{12}\ddot{\theta} - m\frac{L}{2}\left[\cos(\theta)\ddot{x}+\sin(\theta)\ddot{y}-\ddot{\theta}\frac{L}{2}\right]+k_2\theta=0$

$$\rightarrow \underbrace{\left(\frac{mL^2}{12}+\frac{mL^2}{4}\right)}_{\frac{mL^2}{3}}\ddot{\theta} - m\frac{L}{2}\cos(\theta)\ddot{x} - \cancel{\sin}\, m\frac{L}{2}\sin(\theta)\ddot{y}+k_2\theta=0$$

①' $M\ddot{x}+k_1x+\sin(\theta)F_1+\cos(\theta)F_2=0$

$$= M\ddot{x}+k_1x+\sin(\theta)m\left[\sin(\theta)\ddot{x}-\cos(\theta)\ddot{y}+\dot{\theta}^2\frac{L}{2}\right]+\cos(\theta)m\left[\cos(\theta)\ddot{x}+\sin(\theta)\ddot{y}-\ddot{\theta}\frac{L}{2}\right]$$

~~with $\sin^2(\theta)+\cos$~~

$$= M\ddot{x}+k_1x+m\left[\sin^2(\theta)\ddot{x}-\sin(\theta)\cos(\theta)\ddot{y}+\sin(\theta)\dot{\theta}^2\frac{L}{2}\right]$$
$$+\, m\left[\cos^2(\theta)\ddot{x}+\cos(\theta)\sin(\theta)\ddot{y}-\cos(\theta)\ddot{\theta}\frac{L}{2}\right]$$

with $\sin^2(\theta)+\cos^2(\theta)=1$

$$= (M+m)\ddot{x}-\cos(\theta)\ddot{\theta}\frac{L}{2}m+\sin(\theta)m\frac{L}{2}\dot{\theta}^2+k_1x=0$$

$$= (M+m)\ddot{x}-\cos(\theta)m\frac{L}{2}\ddot{\theta}+\sin(\theta)m\frac{L}{2}\dot{\theta}^2+k_1x=0$$

② ' $M\ddot{y} + k_1 y - \cos(\theta) F_1 + \sin(\theta) F_2 = 0$

$$M\ddot{y} + k_1 y - \cos(\theta)\, m\left[\sin(\theta)\ddot{x} - \cos(\theta)\ddot{y} + \dot{\theta}^2 \frac{L}{2}\right]$$
$$+ \sin(\theta)\, m\left[\cos(\theta)\ddot{x} + \sin(\theta)\ddot{y} - \ddot{\theta}\frac{L}{2}\right]$$

with $\cos^2(\theta) + \sin^2(\theta) = 1$

$$\rightarrow M\ddot{y} + k_1 y + m\ddot{y} - \cos(\theta)\, m\dot{\theta}^2 \frac{L}{2} - \sin(\theta)\, m\ddot{\theta}\frac{L}{2}$$

$$\rightarrow (M+m)\ddot{y} - \sin(\theta)\, m\frac{L}{2}\ddot{\theta} - \cos(\theta)\, m\frac{L}{2}\dot{\theta}^2 + k_1 y = 0$$

→ 3 equations of motion

$$\boxed{\begin{aligned}
x&: \ (M+m)\ddot{x} - \cos(\theta)\, m\frac{L}{2}\ddot{\theta} + \sin(\theta)\, m\frac{L}{2}\dot{\theta}^2 + k_1 x = 0\\
y&: \ (M+m)\ddot{y} - \sin(\theta)\, m\frac{L}{2}\ddot{\theta} - \cos(\theta)\, m\frac{L}{2}\dot{\theta}^2 + k_1 y = 0\\
\theta&: \ \frac{mL^2}{3}\ddot{\theta} - \cos(\theta)\, m\frac{L}{2}\ddot{x} - \sin(\theta)\, m\frac{L}{2}\ddot{y} + k_2\theta = 0
\end{aligned}}$$

$$\boxed{\begin{aligned}
x&: \ \ddot{x} = \frac{1}{M+m}\left[\cos(\theta)\, m\frac{L}{2}\ddot{\theta} - \sin(\theta)\, m\frac{L}{2}\dot{\theta}^2 - k_1 x\right]\\
y&: \ \ddot{y} = \frac{1}{M+m}\left[\sin(\theta)\, m\frac{L}{2}\ddot{\theta} + \cos(\theta)\, m\frac{L}{2}\dot{\theta}^2 - k_1 y\right]\\
\theta&: \ \ddot{\theta} = \frac{3}{mL^2}\left[\cos(\theta)\, m\frac{L}{2}\ddot{x} + \sin(\theta)\, m\frac{L}{2}\ddot{y} - k_2\theta\right]
\end{aligned}}$$