8

SYNTHESIS OF HUMAN MOVEMENT—FORWARD SOLUTIONS

8.0 INTRODUCTION

The vast majority of kinetic analyses of human movement have been inverse dynamics. As detailed in Chapters 5 and 7, this type of analysis took the kinematic measures and combined them with measured external forces (i.e., ground reaction forces) to estimate the internal (joint) reaction forces and moments. We used the outcome measures plus a link-segment model to predict forces that were the cause of the movement. This, of course, is the inverse of what really happens. The real sequence of events begins with a varying neural drive to the muscles, resulting in varying levels of recruitment of agonist and antagonist muscles. The net effect of all muscle forces acting at each joint is the generation of a time-varying moment, which in turn accelerates (or decelerates) the adjacent segments and ultimately causes the displacements that our cameras record. If we model this approach in the computer, we are doing what is called a *forward solution*.

The constraints of forward solution models are considerable when compared with inverse solutions. For example, if we wish to calculate the ankle and knee moments on one limb, we do not need data from anything but the segments concerned (in this case, the foot and the leg). No kinematic or kinetic data are required from the thigh, the opposite limbs, the trunk, the arms, or the head. For a forward solution, we must model the entire body before we start, or in the case where part of the body is fixed in space, all segments that are capable of moving must be modeled. If the link segment that we use is not a valid replication of the anatomical situation, there will likely be major errors in our predictions. The reason why we must model the entire link system is the interlimb coupling of forces. Our inputs could be net muscle moments at each joint plus the initial conditions of positions and velocity.

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A given joint moment acts on the two adjacent segments, and they, in turn, create reaction forces on segments further away from the original cause. For example, in gait the pushoff muscles at the ankle (plantarflexors) create a large plantarflexor moment, which results in a rapid ankle plantarflexion. The horizontal and vertical reaction forces at the knee are drastically altered by this plantarflexor moment, and they, in turn, act on the thigh to alter its acceleration. In turn, the reaction forces at the hip, contralateral hip, and trunk are also altered. Thus, the acceleration of all body segments is affected by the ankle muscles at this time. If we have errors in any part of our anatomical model, we will have errors in our prediction. If any segment has the wrong mass or a joint has a missing or unrealistic constraint, the entire link system will start to generate displacement errors, and these errors will accumulate with time. Thus, a poor anatomical model, even with valid time histories of moments, will start to accumulate trajectory errors very quickly.

8.0.1 Assumptions and Constraints of Forward Solution Models

1. The link-segment model has the same assumptions as those presented in Section 5.0.1 for the inverse solution.
2. There must be no kinematic constraints whatsoever; the model must be permitted to fall over, jump, or collapse as dictated by the motor inputs.
3. The initial conditions must include the position and velocity of every segment.
4. The only inputs to the model are externally applied forces and internally generated muscle forces or moments.
5. The model must incorporate all important degrees of freedom and constraints. For example, the hip and shoulder joints must have three axes of rotation, but with limitations on the range of movements (because of the passive internal structures such as ligaments), modeled as passive internal forces and moments.
6. External reaction forces must be calculated. For example, the ground reaction forces would be equal to the algebraic summation of the mass—acceleration products of all segments when the feet are on the ground. Partitioning of the reaction forces when two or more points of the body are in contact with external objects is a separate and possibly major problem.

8.0.2 Potential of Forward Solution Simulations

The research and practical potential of simulations are tremendous, but because of the severe constraints described in the previous section, the potential unfortunately has not been realized. The kinds of questions that can be posed are prefaced with "What would happen if . . . ?" For example,

a surgeon might ask the question, "What would happen if I transferred a muscle from one insertion point to another, or what would happen if a muscle tendon were released (lengthened) to reduce specific muscle spasticity?" Or a coach might ask, "Is the movement pattern of my runner optimal, and if not, what changes in motor pattern might improve it?" Or a basic researcher might have a certain theory of the motor control of gait and might wish to test that theory.

However, before any valid answers are forthcoming, the link-segment representation of the anatomy must first be valid. A necessary (but not sufficient) condition that has been tested by researchers is that of internal validity. Such a test requires an inverse solution to calculate the moments at each of the joints in each of the required planes (sagittal, coronal, etc.). Then, by using these motor patterns as inputs along with measured initial conditions, the forward solution should reproduce the originally measured kinematics. If the model does not pass that test, it is fruitless to use the model to answer any functionally related questions. All that will result will be an erroneous set of motor patterns overlaid on an erroneous biomechanical model.

8.1 REVIEW OF FORWARD SOLUTION MODELS

Human locomotion is the movement that has attracted the most attention with researchers. Because of the complexity of the movement and the link-segment model, certain oversimplifications were made or the simulation was confined to short periods of the movements. Townsend and Seireg (1972) modeled the human with massless rigid lower limbs with 1 degree of freedom at each hip (flexion/extension). Hemami (1980) proposed a three-segment three-dimensional model with rigid legs and no feet, and Pandy and Berme (1988) simulated single support only, using a five-segment planar model with no feet. Obviously, such serious simplifications would not produce valid answers. Even with more complete models, many researchers constrained parts of their models kinematically (Beckett and Chang, 1968; Chao and Rim, 1973; Townsend, 1981) by assuming sinusoidal trajectories of the trunk or pelvic segments. Such constraints violate one of the major requirements of a true simulation. Initial work in our laboratory (Onyshko and Winter, 1980) modeled the body as a seven-segment system (two feet, two legs, two thighs, and an HAT segment), but the model did not satisfy the requirements of internal validity because of certain anatomical constraints (sagittal plane movement only at all three joints plus a rigid foot segment). The model was eventually made to walk but only by altering the moment patterns. Such results should alert researchers that two wrongs can make a right. An incomplete model will result in a valid movement only if faulty motor patterns are used. More complex models (more segments, more degrees of freedom at each joint) have been introduced (Hemami et al., 1982*b*; Chen et al., 1986), but as yet no internal validity has been attempted.

Simple motions have been modeled reasonably successfully. Phillips et al. (1983) modeled the swinging limbs of a human using the accelerations of the swing hip along with the moments about the hip. Hemami et al. (1982*a*) modeled the sway of the body in the coronal plane with each knee locked. With adductor/abductor actuators at the hip and ankle as input, the stability of the total system was defined.

More recent modeling of three-dimensional (3D) gait has been somewhat more successful. One of the major problems with previous attempts was the modeling of initial contact. These earlier models employed springs to represent the elastic characteristics of the bottom of the feet or shoes, but this resulted in extremely large accelerations of the foot segment and similarly large spikes in the ground reaction forces. This was solved with a viscoelastic model of the foot with an array of parallel springs and dampers under the rigid foot segment (Gilchrist and Winter, 1996). A 3D nine-segment model of walking that used ADAMS software (discussed in Section 8.2) demonstrated some success using the inverse dynamics joint moments as inputs (Gilchrist and Winter, 1997). Nonlinear springs at the knee, ankle, and metatarsal-phalangeal joints constrained those joints to their anatomical range. Linear springs were used at the hips and dampers at all joints to ensure a smooth motion. However, it became apparent that any small errors in the joint moments after about 500 ms resulted in accumulating kinematic errors that ultimately became too large, and the model either became unbalanced or collapsed. The build-up of these errors is an inherent characteristic of any forward solution; the double integration of the segment accelerations caused by the input and reaction moments and forces cause displacement errors that increase over time and that can only be corrected by continuous fine-tuning of the input joint moments. Thus, we are forced to violate some of the constraints of forward solution modeling as listed in Section 8.0.1.

8.2 MATHEMATICAL FORMULATION

For the dynamic analysis of connected segment systems, mathematical models consisting of interconnected mass elements, springs, dampers, and actuators (motion generators) are often used. The motion of such models may be determined by defining the time history of the position of individual segments, or by applications of motor forces, in which case the motion of the segments is determined by the laws of physics.

Until quite recently, the nonlinear nature of the problem has been an impediment to the solution of the general dynamic model. With the advancement in computers, researchers are in a position to address the problem of nonlinearity and utilize the computer in creating models that consist of many elements. In the following pages, a systematic method for writing the equations of motion for a general model configuration is explained. The method is suitable for hand derivation of the equations of motion of simple to moderate model configurations. For complex models, the method can be adapted easily for writing

computer programs using symbolic computer languages such as LISP, PROLOG, and MAPLE and to generate the equations of motion. It is also possible to use computer languages such as C or BASIC in writing self-formulating computer programs general enough to accept model description as an input and provide model response as an output. In fact, Mechanical Dynamics, Inc., of Ann Arbor, Mich., has developed a computer program that automates the dynamic simulation process. Their product is marketed under the name ADAMS (Automatic dynamic analysis of mechanical systems; Chace, 1984).

Formulating the equations of motion may be done in several ways. The first and most direct, but possibly the least efficient way, is to apply Newton's laws of dynamics to each segment in the model. Although the reaction forces and torques are obtained as by-products of the solution, the method is cumbersome and does not lend itself easily to a general dynamic simulation program. However, if some concepts of graph theory are incorporated into Newton's laws of motion, the result is a methodical procedure that can be used for writing self-formulating dynamic simulation programs. Three computer programs that utilize these ideas have been developed at the University of Waterloo: VECENT (vector network), a three-dimensional package for particles (Andrews and Kesavan, 1975); PLANET (plane network) for planar mechanisms; and ADVNET (advanced network) for 3D systems (Andrews, 1977; Singhal and Kesavan, 1983).

The second method to formulate the equations of motion is to utilize Lagrangian dynamics (Wells, 1967). Lagrange's equations require the concepts of virtual displacement and employ system energy and work as functions of the generalized coordinates to obtain a set of second-order differential equations of motion. To a large extent, the method reduces the entire field of dynamics to a single procedure involving the same basic steps, regardless of the number of segments considered, the type of coordinates employed, the number of constraints on the model, and whether or not the constraints are in motion. Alternative methods have also been used in this research, such as methods of virtual work extended using D'Alembert principles as in DYMAC (Paul, 1978).

Which method is more suitable? Each method has its advantages and disadvantages. However, the Lagrangian method is characterized by simplicity and is applicable in any suitable coordinates. The task of this chapter is to encode a procedure based on Lagrange's dynamics and suitable for computer implementation using symbolic manipulation language. In doing so, system elements are described by lists that are linked together. Each list has its name and index number and a stack of parameters associated with it. The first element in a given list is usually an integer that establishes a link between the list and other relevant lists in the system. The equations of motion are then obtained by systematic manipulation of these lists. Before we go any further, a review of Lagrange's method is necessary.

8.2.1 Lagrange's Equations of Motion

The following lines are a brief description of Lagrange's equations of motion of the second type. The derivation of these equations can be found in any advanced classical dynamics textbook, such as Greenwood (1977) or Wells (1967).

8.2.2 The Generalized Coordinates and Degrees of Freedom

One of the first things to be determined in a given system is the number of coordinates that represent the model. It is important to realize that any set of time-dependent parameters that give an unambiguous representation of the system configuration will serve as system coordinates. These parameters are known as generalized coordinates, and for the sake of generality, they are denoted by $[\mathbf{q}]^t$,

$$[\mathbf{q}]^t = [q_1, q_2, \ldots, q_n] \tag{8.1}$$

The symbol $[]^t$ is used to indicate the transpose of an array or a matrix. For example, consider the particle p in the plane xy, as shown in Figure 8.1. The point p may be located by several pairs of quantities, such as (x, y), (r, θ), $(A, \sin\theta)$, $(r, \sin\theta)$, and so on. There is, in fact, no limit to the number of different coordinate systems that can be defined, which could be used to indicate the position. Furthermore, the relations between these pairs can be found easily. Using the first and last pairs, and letting $q_1 = r$ and $q_2 = \sin\theta$, it can be proven that $x = q_1[1 - (q_2)^2]^{0.5}$ and $y = q_1 q_2$.

A spatial segment can have as many as six degrees of freedom (DOF) (i.e., possible coordinates), while a point mass has a maximum of three DOF. A system degrees of freedom is defined as the sum of the degrees of freedom of its elements. The degrees of freedom of a system consisting of segments, S, particles, P, and constraints, C, is given by:

$$\text{DOF} = 6S + 3P - C \tag{8.2}$$

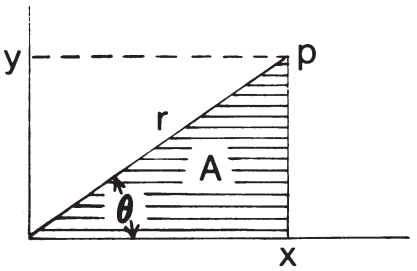

Figure 8.1 Point p whose position can be specified as a function of several other variables: r, x, y, θ, and A.

The constraints in a system are described by equations of constraint $[\mathbf{\Phi}] = [0]$. They may be caused by joints (i.e., a pin joint forces two points on adjacent segments to have the same trajectory) or by an externally imposed motion pattern. The constraints enter the equations of motion in the form of constraint forces $[\boldsymbol{\lambda}]^t$ rather than in geometric terms. There exists a constraint force associated with each constraint equation and analogous to a reaction force. The constraint forces $[\boldsymbol{\lambda}]^t$ are known as Lagrange multipliers. In the previous example, the particle has two DOF since it is restricted to planar motion. Hence, the constraint equation $\mathbf{\Phi}$ is in the form $z = 0$. The constraint force associated with this constraint equation is the force in the z axis that keeps the particle in the xy plane of motion. A totally constrained system (DOF $= 0$) can be solved kinematically. It is impossible to solve an overconstrained system (DOF < 0) without removing the redundant constraints.

The independent generalized coordinates are those coordinates that can be varied independently without violating the constraints, and must equal the degrees of freedom of the model. The use of the independent generalized coordinates allows the analysis of most models to be made without solving for the forces of constraint. Any additional coordinates in the model are known as superfluous or dependent coordinates. The relations between the independent and the superfluous coordinates, are in fact, constraint equations $[\mathbf{\Phi}] = [0]$. If the dependent coordinates can be eliminated, then the system is called *holonomic*. Nonholonomic systems always require more coordinates for their description than there are degrees of freedom. By definition, the first and second derivatives of a generalized coordinate q_i with respect to time are called the *generalized velocity*, $\dot{q}_i$, and the *generalized acceleration*, $\ddot{q}_i$, respectively. The relation between the position vector $\mathbf{r}_i$ of a point i in the system and the generalized coordinates $[\mathbf{q}]^t$ are called the *transformation equations*. It is assumed that these equations are in the form:

$$\begin{aligned} x_i &= f_{xi}(q_1, q_2, q_3, \ldots, q_n, t) \\ y_i &= f_{yi}(q_1, q_2, q_3, \ldots, q_n, t) \\ z_i &= f_{zi}(q_1, q_2, q_3, \ldots, q_n, t) \end{aligned} \tag{8.3}$$

where some or all of the generalized coordinates may be present. The velocity component in the x direction is obtained by taking the time derivative of the x_i equation,

$$\dot{x}_i = \sum_{j=1}^{n} \left(\frac{\partial x_i}{\partial q_j} \right) \left(\frac{\partial q_i}{\partial t} \right) + \frac{\partial x_i}{\partial t} \tag{8.4}$$

Similar relations can be written for the $\dot{y}_i$ and $\dot{z}_i$ components.

8.2.3 The Lagrangian Function L

The Lagrangian function L is defined as the difference between the total kinetic energy KE and the total potential energy PE in the system,

$$L = \text{KE} - \text{PE} \tag{8.5}$$

The kinetic energy for a segment is defined as the work done on the segment to increase its velocity from rest to some value **v**, where **v** is measured relative to a global (inertial) reference system. The existence of an inertial reference system is a fundamental postulate of classical dynamics. Potential energy exists if the system is under the influence of conservative forces. Hence, segment potential energy is defined as the energy possessed by virtue of a segment (or particle) position in a gravity field relative to a selected datum level (usually ground level) in the system. In case of a spring, potential energy is the energy stored in the spring because of its elastic deformation.

8.2.4 Generalized Forces [Q]

A nonconservative force $\mathbf{F}_j$ acting on a segment can be resolved into components corresponding to each generalized coordinate $(q_i, i = 1, \ldots, n)$ in the system. This is also true for constraint forces. A generalized force Q_i is the component of the forces that do work when q_i is varied and all other generalized coordinates are kept constant. In more useful terms, if f forces are acting on the system, then:

$$Q_i = \sum_{j=1}^{f} \lambda_j \left(\frac{F_{xj} \partial R_{xj}}{\partial q_i} + \frac{F_{yj} \partial R_{yj}}{\partial q_i} + \frac{F_{zj} \partial R_{zj}}{\partial q_i} \right) \tag{8.6}$$

where $\mathbf{R}_j$ is the position vector of the force $\mathbf{F}_j$. Moments that are generated by these forces or externally applied moments are greatly affected by the choice of the angular system. More on this topic later. In the case of m constraint equations,

$$Q_i = \sum_{j=1}^{m} \lambda_j \frac{\partial \Phi_j}{\partial q_i} \tag{8.7}$$

Equations (8.6) and (8.7) are added together before they are used.

8.2.5 Lagrange's Equations

One of the principal forms of the Lagrange equations for a system with n generalized coordinates and m constraint equations is:

$$\frac{\partial\,(\partial L/\partial \dot{q}_i)}{\partial t} - \frac{\partial L}{\partial q_i} = Q_i, \qquad (i = 1, \ldots, n) \tag{8.8}$$

$$[\mathbf{\Phi}] = [0] \tag{8.9}$$

As we shall demonstrate, these equations consist of n second-order nonlinear differential equations and m constraint equations. The set has $n + m$ unknowns in the form of $[\mathbf{q}]^t$ and $[\boldsymbol{\lambda}]^t$.

8.2.6 Points and Reference Systems

A moving point pt in a given Cartesian reference system (RS) has a maximum of three degrees of freedom (DOF). If a point is constrained to a certain motion, its DOF are reduced accordingly. A given RS, in addition to the translational DOF of its original point, has a maximum of three rotational DOF. In this context, points can represent the origin of an RS, a segment's center of mass, a point where an external force is applied, a joint center, a muscle insertion, and other points of interest. An RS that is conveniently chosen represents either a segment's local reference system (LRS) or the global reference system (GRS).

A moving reference frame LRS is defined if its point of origin and its orientation relative to an already defined GRS are given. On the other hand, a moving point is defined if its RS is given and its local coordinates are known. By using the notation of linked lists and the indices i, j, k, a point $pt(i)$ moving relative to an RS_j is represented by the list:

$$pt(i) = [j, x_i, y_i, z_i] \tag{8.10}$$

In a similar way, the moving $\mathrm{LRS}(j)$ with origin at $pt(k)$ and given orientation (usually relative to the zero reference system GRS) is represented by the list:

$$\mathrm{LRS}(j) = [k, \theta_{1j}, \theta_{2j}, \theta_{3j}, 0] \tag{8.11}$$

For a two-dimensional (2D) system, Equations (8.10) and (8.11) are reduced to the forms

$$pt(i) = [j, x_i, y_i] \quad \text{and} \quad \mathrm{LRS}(j) = [k, \theta_j]$$

Starting at some convenient stationary point $pt(0)$ in the system (point zero), the GRS is constructed. It is given the index zero. Other points in the domain of the GRS can be specified by Equation (8.10), where the index j

is set to zero. Some of these points are origins of all or some of the LRSs in the model. Each point should have its unique index number. The model's LRSs are specified by Equation (8.11); they are also given their own unique index numbers. The process is repeated until all the points and LRSs in the model are represented. Thus, the model configuration is reduced to a vector network in the form of two groups of linked lists.

As an illustrative example, consider a block of mass m_2 slides on another block of mass m_1, which, in turn, slides on a horizontal surface, as shown in Figure 8.2. In this example, the blocks are treated as particles. The system has two DOFs, and hence two independent generalized coordinates, q_1, q_2, are selected as shown. The system is described by one LRS and two *pt* as follows:

$$\begin{aligned} \text{LRS}(1) &= [1, \theta] \\ pt(1) &= [0, q_1, 0] \\ pt(2) &= [1, d_1 - q_2, 0] \end{aligned} \tag{8.12a}$$

where x and y of Equation (8.10) are replaced by the appropriate transformations. It should be noted that the q are treated as implicit time-dependent variables, while other symbols are considered constants. To calculate the force of the constraint that keeps the two blocks together at a distance d_2 between the center of mass m_2 and the sliding surface of m_1, the mass m_2 is given an additional degree of freedom using the generalized coordinate q_3. A constraint is then added to restore the system to its original state as follows:

$$\begin{aligned} pt(2) &= [1, d_1 - q_2, q_3] \\ 0 &= q_3 - d_2 \end{aligned} \tag{8.12b}$$

Once these lists are established, with the aid of the first element in each list, the displacement and velocity vectors of any point in the system can be derived easily. The system is completely specified when the mass and the center of mass of the two blocks are given. This is done using the lists:

$$\begin{aligned} \text{seg}(1) &= [1, m_1] \\ \text{seg}(2) &= [2, m_2] \end{aligned} \tag{8.12c}$$

The term *segment* seg is used to describe a group of lists that represent any mass element in the system, including particles and rigid segments. More will be said about this example (see Figure 8.2) later, with the specific lists or equations numbered (8.12*d*) to (8.12*g*).

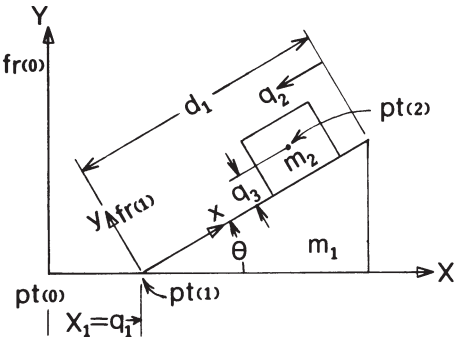

Figure 8.2 Sliding mass example where mass m_1 moves horizontally and mass m_2 moves along the top of the sloped surface of m_1.

8.2.7 Displacement and Velocity Vectors

In the following discussion, the notations $\mathbf{r}, \mathbf{v}, x, y, z, \ldots$ are used for LRS while $\mathbf{R}, \mathbf{V}, X, Y, Z, \ldots$ are used for GRS. The letters c and s are used to denote cosine and sine, respectively. The derived displacement and velocity vectors can be saved in two groups of lists, disp and velo, where an entry (i) is coded as follows:

$$\text{disp}(i) = [x_i, y_i, z_i] \tag{8.13}$$

$$\text{velo}(i) = [\dot{x}_i, \dot{y}_i, \dot{z}_i] \tag{8.14}$$

Equations (8.13) and (8.14) represent the displacement and the velocity, respectively, of $pt(i)$. Saving the derived components rather than deriving them when they are needed will save considerable time, especially in the case of complex systems. The angular velocity vectors of LRSs can also be saved in a group, omga. An entry (j) in this group represents components of the angular velocity of LRS(j):

$$\text{Omga}(j) = [\omega_{jx}, \omega_{jy}, \omega_{jz}] \tag{8.15}$$

8.2.7.1 Two-Dimensional Systems. Let $pt(a)$ be a moving point in the LRS(i) (Figure 8.3). Let $\mathbf{R}_a$ be the absolute position vector for the same point transformed to the directions of the GRS. From vector algebra,

$$\mathbf{R}_a = \mathbf{R}_i + [\phi]\mathbf{r}_{ia} \tag{8.16}$$

where ϕ is the angle between x_i and X.

The same equation in expanded form is:

$$\begin{bmatrix} X \\ Y \end{bmatrix}_a = \begin{bmatrix} X \\ Y \end{bmatrix}_i + \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}_{ia} \tag{8.16$'$}$$

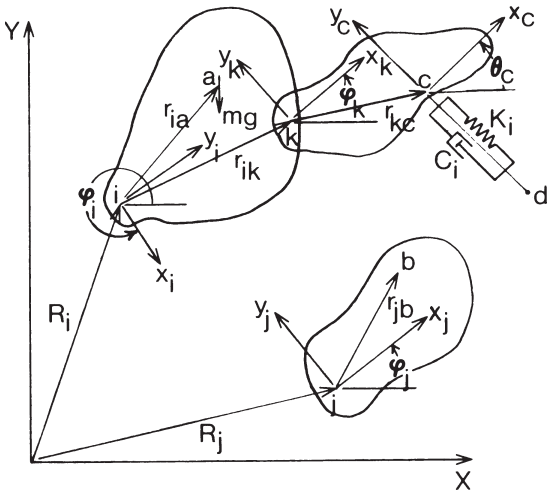

Figure 8.3 Link-segment diagram showing planar modeling of three segments and a spring damper. Three local references systems are shown. See the text for details.

The velocity vector of the same point is obtained by taking the time derivative of Equation (8.16). Remembering that $\omega_z = d\theta_z/dt$,

$$\mathbf{V}_a = \mathbf{V}_i + [\phi]\,\mathbf{v}_{ia} + [\phi]\left[\tilde{\boldsymbol{\omega}}\right]\mathbf{r}_{ia} \tag{8.17}$$

where $[\tilde{\boldsymbol{\omega}}]$ is a 2×2 skew symmetric angular velocity matrix generated from the angular velocity vector,

$$\left[\tilde{\boldsymbol{\omega}}\right] = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \tag{8.18}$$

Now let us return to the sliding block example (see Figure 8.2). Setting $\theta = 45^\circ$, then $c = \cos(45) = \sin(45) = 0.707$, and $\omega = 0$. The displacement and velocity lists are:

$$\begin{aligned} \text{DISP}(1) &= [q_1, 0] \\ \text{VELO}\,(1) &= \left[\dot{q}_1, 0\right] \end{aligned} \tag{8.12d}$$

Since the origin of LRS(1) is $pt(1)$, then, from Equation (8.16),

$$\begin{aligned} \text{DISP}(2) &= \text{DISP}(1) + [\phi][d_1 - q_2 q_3]^t \\ &= [q_1 + 0.707(d_1 - q_2) - 0.707q_3, 0.707(d_1 - q_2) + 0.707q_3] \\ \text{VELO}\,(2) &= \left[\dot{q}_1 - 0.707\,(\dot{q}_2 + \dot{q}_3)\,, 0.707\,(\dot{q}_3 - \dot{q}_2)\right] \end{aligned} \tag{8.12e}$$

From lists (8.12*c*), (8.12*d*), and (8.12*e*), system energy and, hence the Lagrangian [Equation (8.5)], are:

$$\begin{aligned}
\text{KE} &= \tfrac{1}{2}m_1\,(\dot{q}_1)^2 + \tfrac{1}{2}m_2\left\{\left[q_1 - 0.707\,(\dot{q}_2 + \dot{q}_3)\right]^2 + \left[0.707\,(\dot{q}_3 - \dot{q}_2)\right]^2\right\} \\
\text{PE} &= 0.707m_2g\,(d_1 + q_3 - q_2) \\
L &= \tfrac{1}{2}\,(m_1 + m_2)\,(\dot{q}_1)^2 + \tfrac{1}{2}m_2\left[(\dot{q}_2)^2 + (\dot{q}_3)^2 - 1.414\,(\dot{q}_1)\,(\dot{q}_2 + \dot{q}_2)\right] \\
&\quad - 0.707\text{m}_2\text{g}(d_1 + q_3 - q_2) \qquad (8.12f)
\end{aligned}$$

From Equations (8.7) and (8.12*b*), the constraint forces [**Q**] are:

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_2 = \lambda_3, \qquad (8.12g)$$

From Equations (8.8), (8.9), (8.12*f*), and (8.12*g*), the equations of motion are for q_1,

$$0 = (m_1 + m_2)\,\ddot{q}_1 - (0.707m_2)\,\ddot{q}_2 - (0.707m_2)\,\ddot{q}_3$$

for q_2,

$$0 = -(0.707m_2)\,\ddot{q}_1 + (m_2)\,\ddot{q}_2 - (0.707m_2g)$$

and for q_3,

$$\lambda_3 = -(0.707m_2)\,\ddot{q}_1 + (m_2)\,\ddot{q}_2 + (0.707m_2g)$$

$$0 = q_3 - d_2$$

These four equations are in four variables $(q_1, q_2, q_3, \lambda_3)$. If the constraint force is not required, then it can be seen that the first two equations less the q_3 term are the equations needed.

For computer implementation using symbolic computer language, it is a simple task to encode a general program that accepts system variables and parameters in the form of linked lists as an input and gives the equations of motions as an output. If the task is hand derivation of the equations, tables have to be created, each containing one group of relevant lists. Some of these tables are used for intermediate derivations. The task is then as simple as filling in the blanks in these tables.

8.2.7.2 Three-Dimensional Systems. For 3D systems, Equations (8.16) and (8.17) are rewritten for a moving $pt(j)$ in the domain of an LRS with an origin at $pt(i)$, as shown in Figure 8.4,

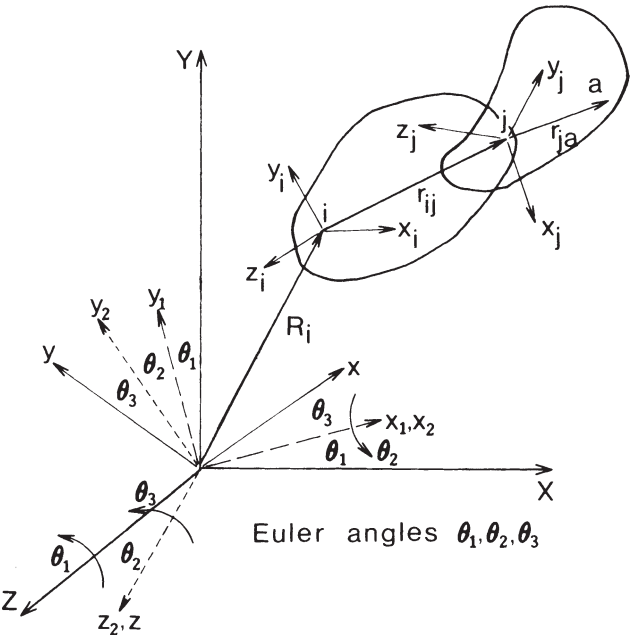

Figure 8.4 Three-dimensional link-segment system showing the three Euler angles θ_1, θ_2, and θ_3 and the two LRS for the ith and jth segments.

$$\mathbf{R}_j = \mathbf{R}_i + [\phi]\mathbf{r}_{ij}$$
$$\mathbf{V}_j = \mathbf{V}_i + [\phi]\ \mathbf{v}_{ij} + [\phi]\left[\tilde{\boldsymbol{\omega}}\right]\mathbf{r}_{ij}$$

where $[\phi]$ is a 3×3 transformation matrix known as the direction cosines matrix (DCM) and $\left[\tilde{\boldsymbol{\omega}}\right]$ is a 3×3 angular velocity matrix. The elements of both matrices are greatly dependent on the angular system used. Among the common parameters used to describe the angular orientation of a segment in space are Euler angles. For Euler angles that follow the order of rotation *zxz*, the angular orientation of an LRS (reference *xyz* in Figure 8.4) is represented as the result of a sequence of three rotations. The first rotation θ_1 is carried out about the Z axis. It results in the auxiliary reference (x_1, y_1, z_1). The second rotation through the angle θ_2 about the x_1 axis produces a second reference (x_2, y_2, z_2). The third rotation through the angle θ_3 about the z_2 axis gives the final orientation. Let c_1 and s_1 be $\cos(\theta_1)$ and $\sin(\theta_1)$, respectively. It can be seen that the transformation matrices associated with these rotations are:

$$[\phi_1] = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\phi_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{bmatrix}, \quad [\phi_3] = \begin{bmatrix} c_3 & s_3 & 0 \\ -s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The DCM is then obtained by combining the effect of the three rotations in the same order, $[\phi] = [\phi_3][\phi_2][\phi_1]$. In expanded form,

$$[\phi] = \begin{bmatrix} c_1c_3 - s_1c_2s_3 & -c_3s_1 + c_1c_2c_3 & s_2s_3 \\ s_1s_3 + c_1c_2s_3 & -s_1s_3 + c_1c_2c_3 & c_3c_2 \\ s_2s_3 & -c_1s_2 & c_2 \end{bmatrix} \tag{8.19}$$

The first time derivatives $(\omega_1, \omega_2, \omega_3)$ of Euler angles are in fact vectors in the directions Z, x_1, and z_2, respectively. These components expressed in the xyz directions are more useful for segment energy calculation. They are given by the transformation:

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} s_2s_3 & c_3 & 0 \\ s_2c_3 & -s_3 & 0 \\ c_3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{8.20}$$

Finally, the skew-symmetric matrix associated with $\boldsymbol{\omega} = [\omega_x, \omega_y, \omega_z]^t$ is in the form:

$$[\tilde{\boldsymbol{\omega}}] = \begin{bmatrix} 0 & -\omega_2 & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{8.21}$$

It should be noted that the externally applied torques τ (or $\mathbf{T}$) are always expressed as components in the directions xyz. The relations between these torque components and the generalized torques required by Lagrange equations are given by:

$$\begin{bmatrix} Q_{\theta 1} \\ Q_{\theta 2} \\ Q_{\theta 3} \end{bmatrix} = \begin{bmatrix} s_2s_2 & s_2c_3 & c_3 \\ c_3 & -s_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tau_x \\ \tau_y \\ \tau_z \end{bmatrix} \tag{8.22}$$

8.3 SYSTEM ENERGY

At any given time, the system energy may consist of the energy of segments due to their motion and position in the system, the energy stored in springs because of their elastic deformation, and the dissipation energy due to friction in the system. The first and second types of energy are included in the Lagrangian L of the system. The third type can be treated as an external force applied to the system at the proper points rather than energy, and hence it is covered under external forces. In the case of dampers, it is possible to write an energy expression that may be included in Lagrangian equations. The three types of energy are covered in more detail in the sections that follow.

8.3.1 Segment Energy

As was previously discussed, a segment is replaced by an arbitrary LRS. Let $pt(j)$ be the origin of the segment LRS and $pt(c)$ be its center of mass. The segment has a mass M and inertia tensor [**J**]. The formula for calculating the kinetic energy of the segment has the form (Wittenburg, 1977):

$$\mathbf{KE} = \tfrac{1}{2}M\left[\boldsymbol{v}_j\right]^t\left[\boldsymbol{v}_j\right] + M\left[\boldsymbol{v}_j\right]^t\left[\tilde{\boldsymbol{\omega}}\right]\left[\mathbf{r}_{jc}\right] + \tfrac{1}{2}\left[\boldsymbol{\omega}\right]^t\left[\mathbf{J}\right]\left[\boldsymbol{\omega}\right] \tag{8.23}$$

A closer look at the first and third terms in this formula reveals the well-known kinetic energy formula $\frac{1}{2}(m\upsilon^2 + I\omega^2)$. The second term reflects the selection of the LRS origin at a point rather than the center of mass of the segment. When $pt(c)$ and $pt(j)$ coincide, the second term vanishes. The inertia tensor [**J**] is a 3×3 matrix containing the mass moments of inertia (I_{xx}, I_{yy}, I_{zz}) and the mass products of inertia (I_{xy}, I_{xz}, I_{yz}) of the segment relative to its LRS. The inertia tensor has the general form:

$$\left[\mathbf{J}\right] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \tag{8.24}$$

It is possible to select LRS in such a way that the products of inertia vanish and the tensor [**J**] becomes a diagonal matrix. The axes of the selected LRS are then called the *principal axes of the segment*.

If the Z axis of the GRS is selected in the direction of the gravity field and its origin is taken as the zero level of the system, then the potential energy of a segment j is defined as

$$\mathrm{PE}_j = MgZ_c \tag{8.25}$$

where g is the gravity constant and Z_c is the z component of $\mathbf{R}_c$. It should be noted that the first element of the $pt(c)$ list is the segment LRS index [Equation (8.11)]. Hence the pt(c) list contains the necessary information about segment kinematics. It follows that a segment i is completely specified when its mass, inertia, and the index of its center point are given. This information can be stacked in the seg(i) list as follows:

$$\mathrm{seg}(i) = [c, M, I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{xz}, I_{yz}] \tag{8.26}$$

There are many variations of this list. When segment LRS is a principal system, the last three elements are not required. A segment in a 2D system is completely specified by the first three elements in the list, while a particle is specified by two elements only. It should be noted that a massless segment (i.e., a rigid link that joins two points) need not be specified. Hence, its LRS replaces it completely.

8.3.2 Spring Potential Energy and Dissipative Energy

In case of a linear spring damper with spring constant k_l, damping coefficient c_l, free length l_s, and end terminals at $pt(e)$ and $pt(f)$, as shown in Figure 8.3, the potential energy is calculated by:

$$PE_s = \tfrac{1}{2}k_l\,(l - l_s)^2 \tag{8.27}$$

where l is the length of $\mathbf{r}_{ef}$. A linear spring j is thus defined by its end points and properties in the form of the list:

$$spl(j) = [e, f, k_t, c_1, l_s] \tag{8.28}$$

For a torsional spring with spring constant k_t and end terminals attached to two adjacent references LRS(g) and LRS(h), respectively, the potential energy is calculated by $\mathrm{PE}_s = \frac{1}{2}k_t(\theta - \theta_s)^2$, where θ is the angle between the two LRS about a common axis of rotation and θ_s is the angle when the spring is unloaded. The torsional spring list $spt(j)$ is then:

$$spt(j) = [g, h, k_t, c_t, \theta_s] \tag{8.29}$$

The dissipative energy (DE) associated with dampers is obtained by the aid of Rayleigh's dissipation function $p = c\upsilon^{n+1}/(n+1)$, where $n = 1$ for viscous friction and υ is the relative velocity between damper end points. For a system of d dampers, the DE is defined by:

$$DE = \sum_{j=1}^{d} \tfrac{1}{2}c_j\,\left(v_j\right)^2 \tag{8.30}$$

The generalized force Q_i due to the dampers in the system is given by

$$Q_i = -\frac{\delta DE}{\delta \dot{q}_i} \tag{8.31}$$

Either the left-hand side or the right-hand side of this equation is added to the Lagrange equations.

8.4 EXTERNAL FORCES AND TORQUES

As part of the model description, an external force j acting at a point a is given by a force list *frc* that contains force components and the point of application,

$$frc(j) = [a, F_x, F_y, F_z] \tag{8.32}$$

Bidirectional forces (force motors and actuators) are treated as two forces equal in magnitude and opposite in direction, acting on two segments. Equation (8.32) is then modified to include the other end point. Similarly, an external torque or torque motor i acting between frames j and k is given by the list:

$$trq(i) = [j, k, \tau_x, \tau_y, \tau_z] \tag{8.33}$$

where (τ_x, τ_y, τ_z) are the components of the externally applied torque. The contribution of the externally applied forces and torques to the generalized forces of the system is obtained by:

$$Q_i = \frac{\delta W}{\delta q_i} \tag{8.34}$$

where W is the total work done by the external forces and torques [similar to Equation (8.6)].

8.5 DESIGNATION OF JOINTS

Under Lagrangian dynamics, there are two ways of treating joints in a given model because of the fact that action and reaction at joints do not perform work. If joint reaction forces are not required, then a joint can be viewed as a common point between two adjacent segments. To obtain joint forces, a joint is set apart into two points, each on the appropriate segment LRS. An equation, known as *loop closure* (or *constraint*) *equation* $\mathbf{\Phi}$, is then written to state the relation between these points. The reaction forces are those forces λ that maintain the constraint. In the case of pin joints or ball and socket joints, the relation is that the vector between the two points is zero. Other joint types are beyond the limitations of this introductory topic.

8.6 ILLUSTRATIVE EXAMPLE

A standing human is modeled as an inverted pendulum. The three elements shown in Figure 8.5*a* represent the leg, thigh, and trunk segments. The segment lengths are represented by l. The locations of the segment centers of mass are represented by r, all measured from the distal end. A horizontal disturbance force F is applied at the hip. Horizontal feet disturbances were simulated as a fixed acceleration $\ddot{x}$ of the ground contact point. The joint torques T_1, T_2, and T_3 represent the ankle, knee, and hip torques, respectively.

The system has four variables, $q_1(= x)$, $q_2(= \theta_1)$, $q_3(= \theta_2)$, and $q_4(= \theta_3)$. Therefore, four differential equations are expected. As a first step, the model is replaced by the appropriate references LRS and *points pt*. Model parameters

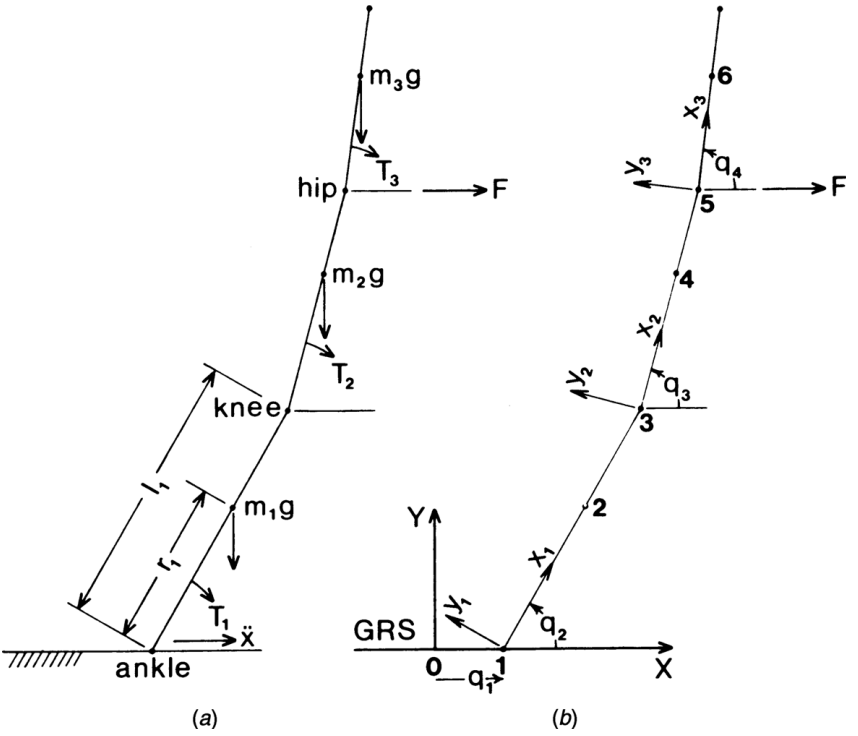

Figure 8.5 (*a*) Three-segment model of the leg, thigh, and trunk during standing balance. The feet (ankle) can be perturbed with a horizontal acceleration $\ddot{x}$ and the hip with a horizontal force F. The system responds with three joint torques, T_1, T_2, and T_3. (*b*) Link-segment model of (*a*) showing the global reference system and three local reference systems along with output displacement variables q_1, q_2, q_3, and q_4.

are described relative to these points and references. Selecting LRS such that their origins are segments cg simplifies the formulation of the equations, but for the sake of generality let the LRS origins be at the distal ends of the segments. Also, the variables $(\theta_2, \theta_3, \theta_4)$ can be selected to represent joint angles, but we'd rather let these variables be segment angles. It is also assumed that the reaction forces at the joints are not needed. Based on these choices, the model is completely specified using three LRSs, six points, three segments, three torques, and one force, as shown in Figure 8.5*b*. They are stated in the following lists:

$$\text{LRS}(1) = [1, q_2],\ \text{LRS}(2) = [3, q_3],\ \text{LRS}(3) = [5, q_4]$$
$$pt(1) = [0, q_1, 0], pt(2) = [1, r_1, 0], pt(3) = [1, l_1, 0],$$
$$pt(4) = [2, r_2, 0] pt(5) = [2, l_2, 0] pt(6) = [3, r_3, 0]$$

$$\text{seg}(1) = [2, m_1, I_1], \text{seg}(2) = [4, m_2, I_2], \text{seg}(3) = [6, m_3, I_3]$$
$$trq(1) = [0, 1, T_1], trq(2) = [1, 2, T_2], trq(3) = [2, 3, T_3],$$
$$frc(1) = [5, F, 0]$$

The steps that follow are to create displacement and velocity lists that contain all points and use the results to obtain system Lagrangian and the generalized forces that are necessary for the Lagrange equations. The following is a pseudo-code for the derivation procedures:

For $p = 1, \ldots, 6$, derive displacement and velocity expressions for point (p) [Equations (8.16), (8.17)].
For $s = 1, \ldots, 3$, calculate KE and PE for $seg(s)$ and update the Lagrangian [Equations (8.5), (8.23), (8.25)].
For $t = 1, \ldots, 3$, find the work done by torque $trq(t)$ and update system work W [Equation (8.34)].
For $f = 1$, find the work done by the force $frc(f)$ and update W [Equation (8.34)].
For $i = 1, \ldots, 4$, apply Lagrangian equations for coordinate q_i [Equations (8.8), (8.9)].

The intermediate results for the relevant parts of these steps are as follows:

1. Displacement vectors,

$$\text{disp}(1) = [q_1, 0]$$
$$\text{disp}(2) = [q_1 + r_1 \cos(q_2), r_1 \sin(q_2)]$$
$$\text{disp}(3) = [q_1 + l_1 \cos(q_2), l_1 \sin(q_2)]$$
$$\text{disp}(4) = [q_1 + l_1 \cos(q_2) + r_2 \cos(q_3), l_1 \sin(q_2) + r_2 \sin(q_3)]$$
$$\text{disp}(5) = [q_1 + l_1 \cos(q_2) + l_2 \cos(q_3), l_1 \sin(q_2) + l_2 \sin(q_3)]$$
$$\text{disp}(6) = [q_1 + l_1 \cos(q_2) + l_2 \cos(q_3) + r_3 \cos(q_4), l_1 \sin(q_2) + l_2 \sin(q_3) + r_3 \sin(q_4)]$$

2. Velocity vectors for selected points,

$$\text{velo}\,(2) = \left[\dot{q}_1 - r_1 \dot{q}_2 \sin\,(q_2)\,, r_1 \dot{q}_2 \cos\,(q_2)\right]$$
$$\text{velo}\,(4) = [\dot{q}_1 - l_1 \dot{q}_2 \sin\,(q_2) - r_1 \dot{q}_3 \sin\,(q_3)\,, l_1 \dot{q}_2 \cos(q_2) + r_2 \dot{q}_3 \cos\,(q_3)]$$
$$\text{velo}(6) = [\dot{q}_1 - l_1 \dot{q}_2 \sin(q_2) - l_2 \dot{q}_3 \sin(q_3) - r_3 \dot{q}_4 \sin(q_4), l_1 \dot{q}_2 \cos(q_2) + l_2 \dot{q}_3 \cos(q_3) + r_3 \dot{q}_4 \cos(q_4)]$$

3. Lagrangian $L = \text{KE} - \text{PE}$

$$
\begin{aligned}
L = {} & \tfrac{1}{2}m_1\left\{\left[\dot{q}_1 - r_1\dot{q}_2\sin(q_2)\right]^2 + \left[r_1\dot{q}_2\cos(q_2)\right]^2\right\} + \tfrac{1}{2}I_1(\dot{q}_2)^2 \\
& - m_1 g\left[r_1\sin(q_2)\right] + \tfrac{1}{2}m_2\left\{\left[\dot{q}_1 - l_1\dot{q}_2\sin(q_2) - r_2\dot{q}_3\sin(q_3)\right]^2\right. \\
& \left. + \left[l_1\dot{q}_2\cos(q_2) + r_2\dot{q}_3\cos(q_3)\right]^2\right\} + \tfrac{1}{2}I_2(\dot{q}_3)^2 - m_2 g\left[I_1\sin(q_2)\right. \\
& \left. + r_2\sin(q_3)\right] + \tfrac{1}{2}m_3\left\{\left[\dot{q}_1 - l_1\dot{q}_2\sin(q_2) - l_2\dot{q}_3\sin(q_3)\right.\right. \\
& \left.\left. - r_3\dot{q}_4\sin(q_4)\right]^2 + \left[l_1\dot{q}_2\cos(q_2) + l_2\dot{q}_3\cos(q_3)\right.\right. \\
& \left.\left. + r_3\dot{q}_4\cos(q_4)\right]^2\right\} + \tfrac{1}{2}I_3(\dot{q}_4)^2 - m_3 g\left[l_1\sin(q_2) + l_2\sin(q_3)\right. \\
& \left. + r_3\sin(q_4)\right]
\end{aligned}
$$

The Lagrangian L is simplified by expanding the previous expression and collecting terms,

$$
\begin{aligned}
L = {} & \tfrac{1}{2}(\dot{q}_1)^2(m_1 + m_2 + m_3) \\
& + \tfrac{1}{2}(\dot{q}_2)^2\left[m_1(r_1)^2 + I_1 + m_2(l_1)^2 + m_3(l_1)^2\right] \\
& + \tfrac{1}{2}(\dot{q}_3)^2\left[m_2(r_2)^2 + I_2 + m_3(l_2)^2\right] + \tfrac{1}{2}(\dot{q}_4)^2\left[m_3(r_3)^2 + I_3\right] \\
& - \dot{q}_1\dot{q}_2\sin(q_2)(m_1 r_1 + m_2 l_1 + m_3 l_1) - \dot{q}_1\dot{q}_3\sin(q_3) \\
& \cdot (m_2 r_2 + m_3 l_2) - \dot{q}_1\dot{q}_4(m_3 r_3) + \dot{q}_2\dot{q}_3 l_1\cos(q_3 - q_2) \\
& \cdot (m_2 r_2 + m_3 l_2) + \dot{q}_2\dot{q}_4 l_1\cos(q_4 - q_2)(m_3 r_3) \\
& + \dot{q}_3\dot{q}_4 l_2\cos(q_4 - q_3)(m_3 r_3) - m_1 g r_1\sin(q_2) - m_2 g\left[l_1\sin(q_2)\right. \\
& \left. + r_2\sin(q_3)\right] - m_3 g\left[l_1\sin(q_2) + l_2\sin(q_3) + r_3\sin(q_4)\right]
\end{aligned}
$$

4. External work W,

$$
\begin{aligned}
W = {} & T_1(q_2 - 0) + T_2(q_4 - q_3) + T_3(q_4 - q_3) + F(q_1 \\
& + l_1\cos(q_2) + l_2\cos(q_3))
\end{aligned}
$$

5. Finally, the equations of motion are for the generalized coordinates q_1, q_2, q_3, q_4, respectively,

(a)
$$
\begin{aligned}
F = {} & \ddot{q}_1(m_1 + m_2 + m_3) \\
& - \left[(\dot{q}_2)^2\cos(q_2) + \ddot{q}_2\sin(q_2)\right]\cdot(m_1 r_1 + m_2 l_2 + m_3 l_1) \\
& - \left[(\dot{q}_3)^2\cos(q_3) + \ddot{q}_3\sin(q_3)\right]\cdot(m_2 r_2 + m_3 l_2) - \ddot{q}_4 m_3 r_3
\end{aligned}
$$

$$
\begin{aligned}
\text{(b)}\quad T_1 - T_2 - Fl_1 \sin(q_2) = & -\ddot{q}_1 \sin(q_2)(m_1 r_1 + m_2 l_1 + m_3 l_1) \\
& + \ddot{q}_2 \left[m_1 (r_1)^2 + I_1 + m_2 (l_1)^2 + m_3 (l_1)^2\right] \\
& + \left[\ddot{q}_3 \cos(q_3 - q_2) + (\dot{q}_3)^2 \right. \\
& \left. \cdot \sin(q_3 - q_2)\right](m_2 r_2 l_1 + m_3 l_1 l_2) \\
& + \left[\ddot{q}_4 \cos(q_4 - q_2) + (\dot{q}_4)^2 \right. \\
& \left. \cdot \sin(q_4 - q_2)\right](m_3 r_3 l_1) \\
& + g(m_1 r_1 + m_2 l_1 + m_3 l_1) \cos(q_2) \\
\text{(c)}\quad T_2 - T_3 - Fl_2 \sin(q_3) = & -\ddot{q}_1 \sin(q_3)(m_2 r_2 + m_3 l_2) \\
& + [\ddot{q}_2 \cos(q_3 - q_2) - (\dot{q}_2)^2 \\
& \cdot \sin(q_3 - q_2)](m_2 r_2 l_1 + m_3 l_1 l_2) \\
& + \ddot{q}_3 \left[m_2 (r_2)^2 + m_3 (l_2)^2 + I_2\right] \\
& + \left[\ddot{q}_4 \cos(q_4 - q_3)\right] - (\dot{q}_4)^2 \\
& \cdot \sin(q_4 - q_3)](m_3 r_3 l_2) \\
& + g(m_2 r_2 + m_3 l_2) \cos(q_3)
\end{aligned}
$$

$$
\begin{aligned}
\text{(d)}\quad T_3 = & -\ddot{q}_1 m_3 r_3 \sin(q_4) + \ddot{q}_2 \cos(q_4 - q_2) \\
& + (\dot{q}_2)^2 \sin(q_4 - q_2)\big](m_3 r_3 l_1) + \left[\ddot{q}_3 \cos(q_4 - q_3)\right. \\
& \left. + (\dot{q}_3)^2 \cdot \sin(q_4 - q_3)\right](m_3 r_3 l_2) \\
& + \ddot{q}_4 \left[m_3 (r_3)^2 + I_3\right] + m_3 g r_3 \cos(q_4)
\end{aligned}
$$

These four equations are the minimum number of equations that describe the model. They are highly nonlinear, tightly coupled, lengthy, and prone to errors. By breaking up the model at joints and introducing quasi-DOF in the form of additional generalized coordinates and constraint equations that counteract these superfluous coordinates, a larger set of equations is obtained with equations that are shorter, manageable, and contain more information about the model.

The tight coupling between these four equations demonstrates the potential errors that were mentioned in Section 8.0.2. An error in the mass of any segment, for example, will result in errors in all four coordinates. Mass m_3, the mass of the HAT, appears in all four equations and will affect the calculation of q_1, q_2, q_3, and q_4. The errors will also accumulate over time. Thus, it is essential that, before the model is used to answer a research question, all the anthropometrics and internal joint constraints be almost perfect such that an internal validation is achieved.

8.7 CONCLUSIONS

In this chapter, a procedure for generating the equations of motion for a given model was presented. The method replaces model configuration and gathers parameters by groups of linked lists. The first element of each list links the list to other relevant lists in the model. Once these lists are written correctly, deriving the equations of motion follows the same systematic procedure. This makes the method more suitable for computer implementation. As for hand derivation, the only variation from a simple model to a complicated one is the amount of work involved.

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