Study and Simulations of Wiener Process with Applications to Single and Multispecies Interactions

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submitted by

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1 Summary

We look at the random walk with analogies to walking on a linear habitat broken into discrete zones and the movement of the net change in number of males and females, with the condition that in every next time interval, there is some change, that is, either there is a female or male birth. We notice that the random variable representing whether there is a male birth or female birth follows Bernoulli distribution and the net change in the population size then follows the Binomial distribution. Further on applying the central limit theorem we notice that it asymptotically follows the normal distribution. This concept is extended to a limiting case, where we consider continuous space and time. As the space and time indices tend to 0, we obtain the expressions for the transition probabilities in terms of the mean(drift) and variance of the process. A natural application of the central limit theorem to the Bernoulli process in continuous space and time shows that the process of net changes, is asymptotically normal too. We call this rescaled random walk the Wiener process, which is a diffusion process, and is nothing but the Brownian motion at mean 0. A study of this diffusion process using the Kolmogorov equations and the definition of MGF confirms theoretically that the process is asymptotically normal. We then see what happens if we place barriers at certain locations. We see two types of barriers, reflecting and absorption. By placing reflecting barriers, we mean that we want the process to stay within those physical limits, and by placing an absorbing barrier we mean that once the process reaches it, it stays there.

While studying these processes theoretically, we employ a very powerful tool, the method of images, to easily obtain the solution. We then move on to a more general setting where we allow the process to stay in the same state in the next time index. On using the three transition probabilities, we again use the forward Kolmogorov equation to obtain a more general SDE, which the Fokker-Planc equation. We notice that the variance of the Wiener process is not bounded as the time step goes to 0. In situations when the velocity of the process is crucial, this unbounded nature must be controlled, and we do exactly that.

In an uncorrelated random walk, the direction and so the velocity of the process does not change unless there is a collision, with values +/-1. The OU-process generalises this, and the velocity is unrestricted to continuous values. Theoretically, it too is normal, and we then find a transformation to and from the Wiener process, and then conclude our study of the chapter.

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2 Introduction

Think about walking along a linear habitat like a seashore. Assume that it is broken up into zones. Also assume that every time there is movement into the next or previous zone, i.e. we are not in the same zone in the next time period. This process can be thought of as a Bernoulli trial. Let Z denotes the random variable representing the movement of the person. If $Z = 1$, then the person moves to the next zone and if $Z = -1$, the person goes to the previous zone. Let us assume that $\Pr(Z = 1) = p$ and $\Pr(Z = -1) = 1 - p$. Now, if we keeping walking along the habitat, $\sum_{i=0}^{n} Z_i = X_n$ tells us our net position at time instant n. This means that X_n follows $\text{Bin}(n, p)$. We compare this geographic process to that of a population process. Let the population have males and females. At the nth step, $Z_n = 1$ with probability p if there is a male birth or a female death. At the nth step, $Z_n = -1$ with probability $1 - p$ if there is a female birth or a male death. So, X_n = net difference between males and females at time n. Probability that process moved $n - i$ steps up, and i steps down, is given by

$$\Pr(X_n = n - 2i) = \binom{n}{i} p^{n-i} q^i,$$

where $q = 1 - p, i = 0, 1, \cdots, n$. Let, $j = n - 2i, j = -n, -n + 2, ..., n - 2, n$. So,

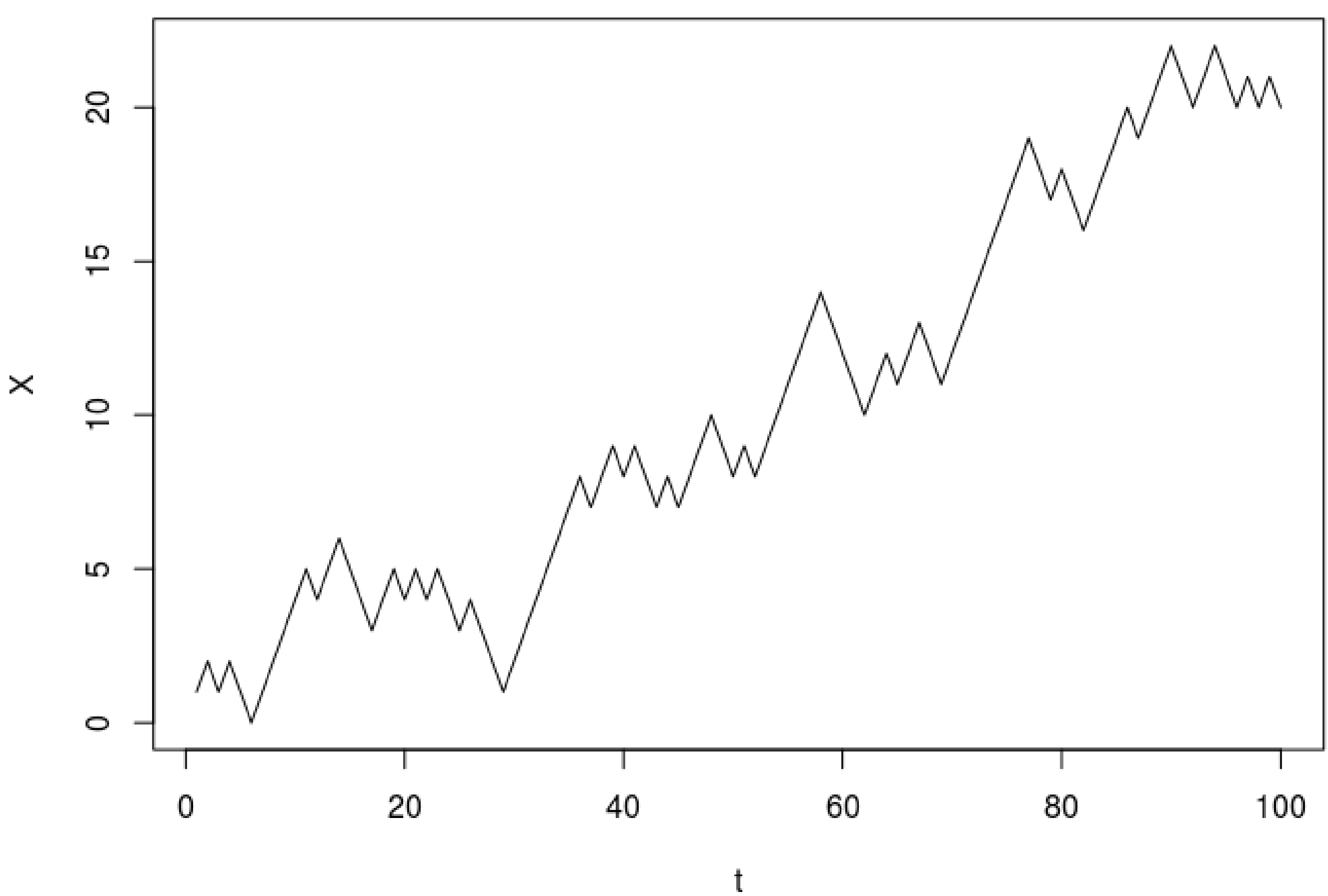
$$\Pr(X_n = j) = \binom{n}{\frac{n-j}{2}} p^{\frac{n+j}{2}} q^{\frac{n-j}{2}} \tag{1}$$

At $p = q = 0.5$, we have the symmetric random walk. Figure 1 illustrates a simple random walk.

```
# R-code for Figure 1.
par(mfrow=c(1,1))
set.seed(37)
n <- 100 # time = 0,1,2,...,n
p <- 0.6
q <- 1-p

z0 <- numeric(n)  # generating zi.
for(i in 1:n){
  z0[i] <- rbinom(1,1,p)
  if(z0[i]==0)
  {
```

```
    z0[i] <- -1
  }
}


z0
x0 = cumsum(z0) # Xn
plot(1:n,x0,type = "l",xlab="t",ylab="X") # Random walk
```

Figure 1: A simulation of Random walk

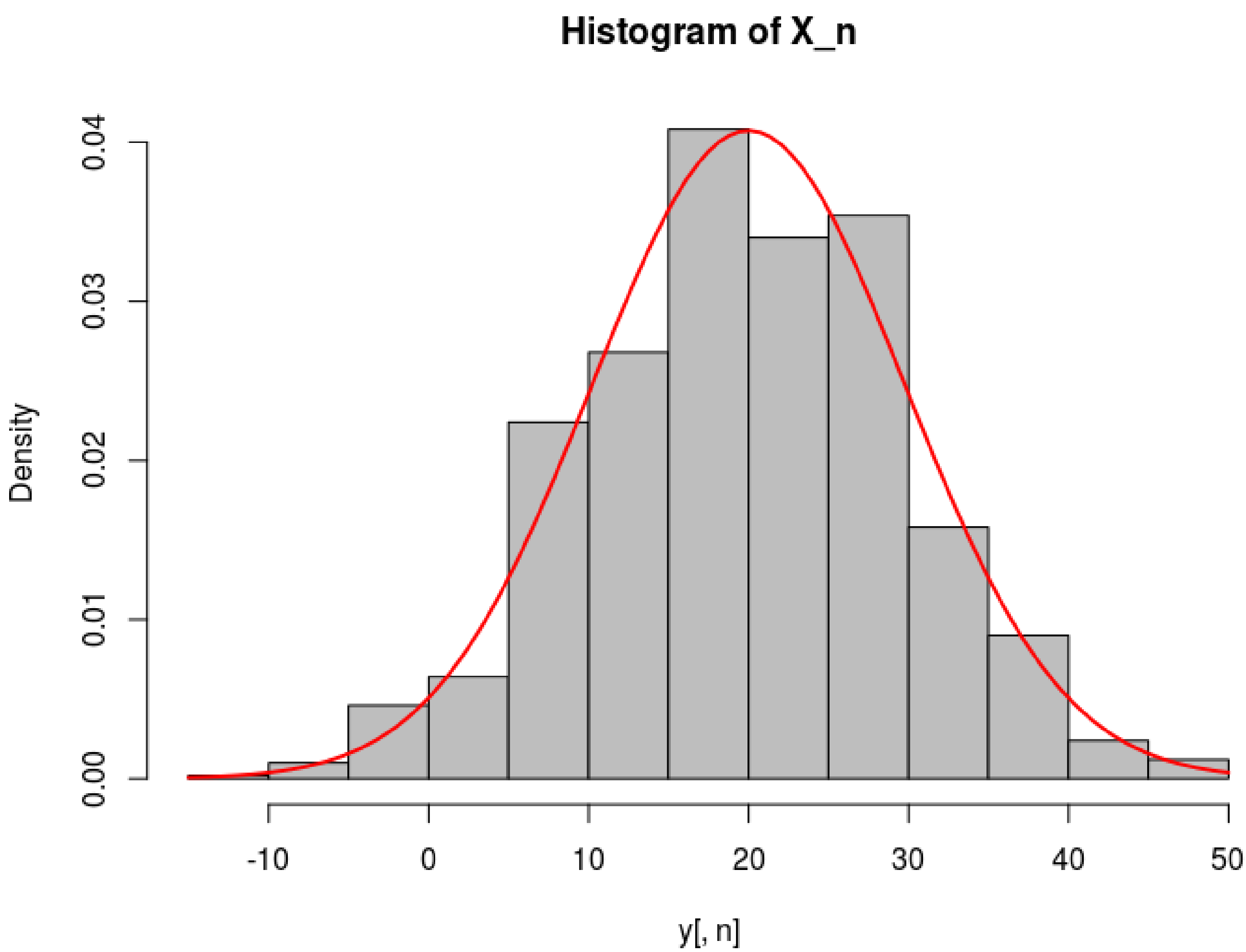


Let μ be the mean of a jump and σ be its variance. $\mu = E[Z]$. Therefore, $\mu = (+1)(p) + (-1)(q) = p - q$ $\sigma^2 = \text{Var}(Z)$. Therefore $\sigma^2 = \text{E}[X^2] - (\text{E}[X])^2 = (1(p) + (-1)^2(q)) - (p-q)^2 = p + q - p^2 - q^2 + 2pq$. Using $p + q = 1$, we get $\sigma^2 = 4pq$. Now, $\text{E}[X_n] = n\mu$ and $\text{Var}(X_n) = n\sigma^2$. An application of the central limit theorem yields, $\frac{\sum Z_i - \mu}{n\sigma} \to \mathcal{N}(n\mu, n\sigma^2)$ in distribution. For a normal random variable, we know that the probability that it lies beyond $4\sigma < 0.0001$ i.e. $\Pr(X_n - n\mu < |4\sqrt{n}\sigma|) > 0.9999$. This means that as $n \to \infty$, X_n has drift $n\mu$, and the deviation from $n\mu$ increases with $\sqrt{n}$. Note that here both space and time are discrete. Figure 2 shows how the random walk tends to normal distribution.

```
# R-code for Figure 2.
set.seed(37)
par(mfrow=c(1,1))

m <- 1000 # number of replication
n <- 100 # time = 0,1,2,...,n
p <- 0.6
q <- 1-p

z <- numeric(n) # Bernoulli trial
x <- numeric(n) # sum Zn
y <- matrix(0,nrow=m,ncol=n) # collect each process X(n) for m replications.

for(j in 1:m){
  for(i in 1:n){
    z[i] <- rbinom(1,1,p)
    if(z[i]==0)
    {
      z[i] <- -1
    }
  }
  x = cumsum(z)
  if(j==1){
    #plot(1:n,x,type="l")
```

```r
  }else{
    #lines(1:n,x,lwd=2,col="turquoise4")
    y[j,] <- x
  }
}
#View(y)
# Xn of all m reps  are expected to follow normal dist.
hist(y[,n], prob = T, col = "grey",main = paste("Histogram of X_n"))
mean = n*(p-q)
sd = sqrt(4*n*p*q)
curve(dnorm(x, mean = mean, sd=sd), col = "red", lwd=2, add =TRUE)
```

Figure 2: RANDOM WALK TENDS TO NORMAL DISTRIBUTION



3 Basic Wiener Process

The basic wiener process is in general a rescaled or limiting random walk. When studying the motion of particles suspended in a fluid, and under random collision from other particles, a graph plot showed a random but continuous path of small line segments. The particles seemed to diffuse through the medium. The phenomenon is called Brownian motion. We deal with continuous time and space, and limit each space and time increment.

Previously, we moved in space and time on integers, now we move in infinitesimal terms $\mathrm{d}x$, $\mathrm{d}t$. In case of random walk, $X_n \in \{-n, -n+2, \cdots, n-2, n\}$. Now we use a limiting argument over $x \in \mathbb{R}$ with $X \in \{\cdots, -\mathrm{d}x, 0, \mathrm{d}x, \cdots\}$ in the limit $\mathrm{d}x \to 0$. In the Wiener process $\mathrm{d}x \to 0, \mathrm{d}t \to 0$ in some way. The occupation still assumes a Binomial distribution form as above. Hence, in the diffusion limit, the occupation probabilities can be approximated by using the normal approximation. Now, at time t with a step size $\mathrm{d}t$, the number of steps will be $n = t/\mathrm{d}t$, and space step size will be $\mathrm{d}x$. Mean of each step is $(p-q)\mathrm{d}x$ with variance $4pq(\mathrm{d}x)^2$, where $p + q = 1$. Let $X(0) = 0$, that is the process starts from 0. So, mean of the process $X(t)$ at time t is given by $m(t) = n(p-q)\mathrm{d}x = (p-q)t\mathrm{d}x/\mathrm{d}t$ with variance $\sigma^2(t) = 4tpq(\mathrm{d}x)^2/\mathrm{d}t$. Although we want $\mathrm{d}x$ and $\mathrm{d}t$ to reach 0 we want $m(t)$ and $\sigma(t)$ to be finite. Suppose for $t = 1$, we want the process to have mean μ and variance σ^2, i.e.

$$\begin{aligned} m(1) &= (p-q)(\mathrm{d}x/\mathrm{d}t) \to \mu, \\ \sigma^2(1) &= 4pq(\mathrm{d}x^2/\mathrm{d}t) \to \sigma^2. \end{aligned}$$

Now, we let $\frac{\mathrm{d}x^2}{\mathrm{d}t} = \sigma^2$ and we want $\lim_{\mathrm{d}x\to 0}\lim_{\mathrm{d}t\to 0}\frac{x}{\mathrm{d}t} = \frac{\mu}{p-q}$ and we want $\lim_{\mathrm{d}x\to 0}\lim_{\mathrm{d}t\to 0}\frac{\mathrm{d}x^2}{\mathrm{d}t} = \frac{\sigma^2}{4pq}$. In the first limit we use our substitution to get $\lim_{\mathrm{d}t\to 0}\frac{\sigma(p-q)}{\sqrt{\mathrm{d}t}} = \mu$ which must be finite. Let, $\frac{\sigma(p-q)}{\sqrt{\mathrm{d}t}} = \mu$, and on solving for p we get

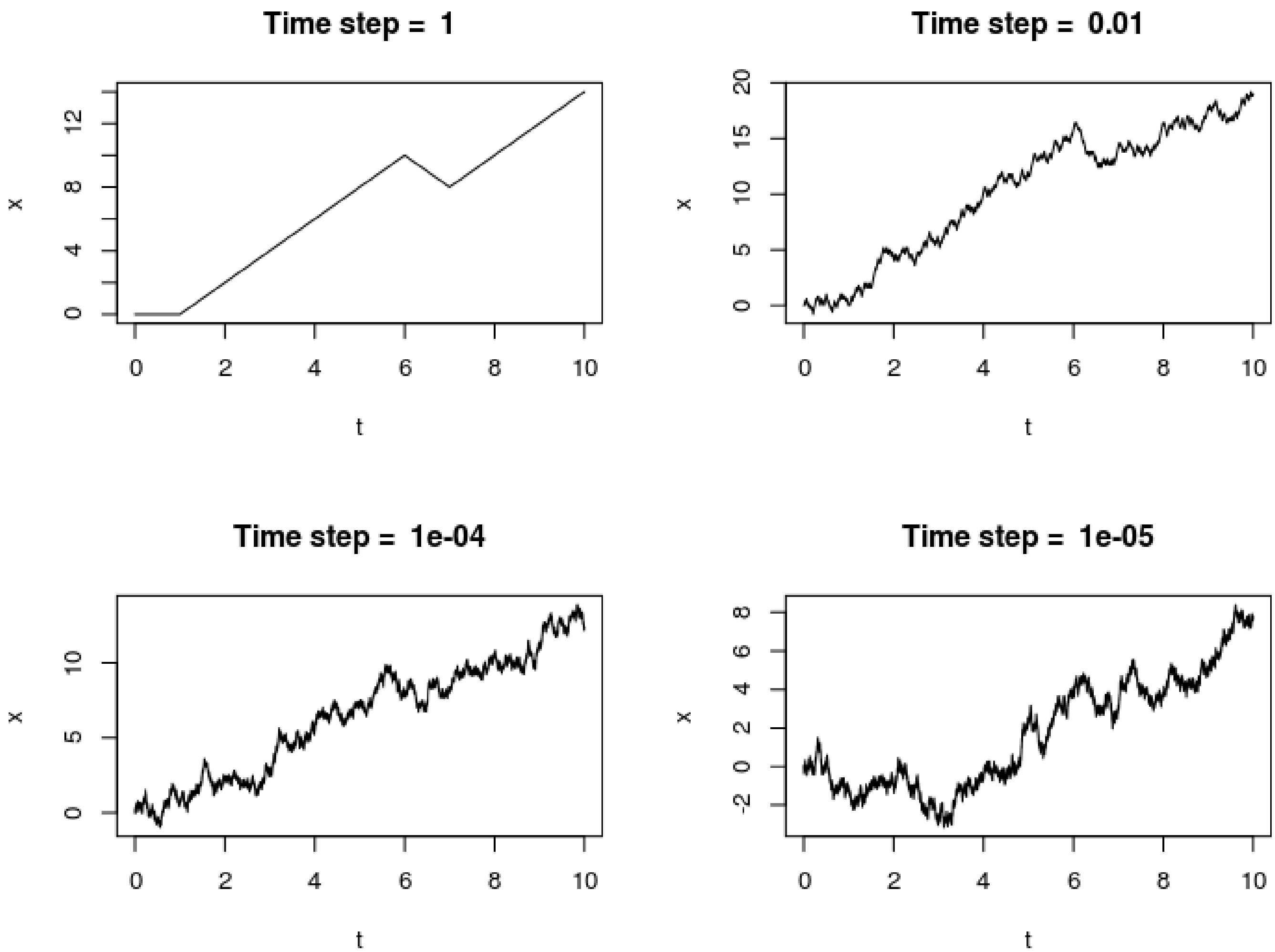
$$p = \frac{1}{2}\left(1 + \frac{\mu\sqrt{\mathrm{d}t}}{\sigma}\right) \quad \text{and} \quad q = \frac{1}{2}\left(1 - \frac{\mu\sqrt{\mathrm{d}t}}{\sigma}\right) \tag{2}$$

Then, using the central limit theorem to $Z(t)$, the process $X(t) \sim \mathcal{N}(t\mu, t\sigma^2)$, which is the Wiener process with drift μ and variance σ^2. The Brownian motion is obtained for $\mu = 0$ so that $m(t) = 0$. Let us see the paths of a simple random walk versus that of a rescaled random walk i.e Wiener process. Figure 3 shows how the Wiener process is a rescaled random walk, by setting dt closer to 0.

```
# R-code for Figure 3.
set.seed(37)
m <- 1
dt <- c(1,0.01,1e-04,1e-05)
n <- c(10,1000,1e+05,1e+06)
t <- n*dt
seqlen <- t/n
mu <- 1
```

```
sd <- 2
p <- (1+mu*sqrt(dt)/sd)/2
q <- 1-p
dx <- sd*sqrt(dt)

par(mfrow=c(2,2))
for(k in 1:length(dx)){
  z <- numeric(n[k]+1)
  x <- numeric(n[k]+1)
  z[1] <- 0
  x[1] <- 0
  y <- matrix(0,nrow=m,ncol=n[k]+1)

  for(j in 1:m){
    for(i in 2:n[k]+1){
      z[i] <- rbinom(1,1,p[k])
      if(z[i]==0)
      {
        z[i] <- -dx[k]
      }else{
        z[i] <- dx[k]
      }
    }
    x = cumsum(z)
    if(j==1){
      plot(seq(0,t[k],dt[k]),x,type="l",xlab="t",main=paste("Time step = ",dt[k]))
    }else{
      lines(seq(0,t[k],dt[k]),x,lwd=2,col="turquoise4")
      y[j,] <- x
    }
  }
}
```

Figure 3: WIENER PROCESS AS RESCALED RANDOM WALK



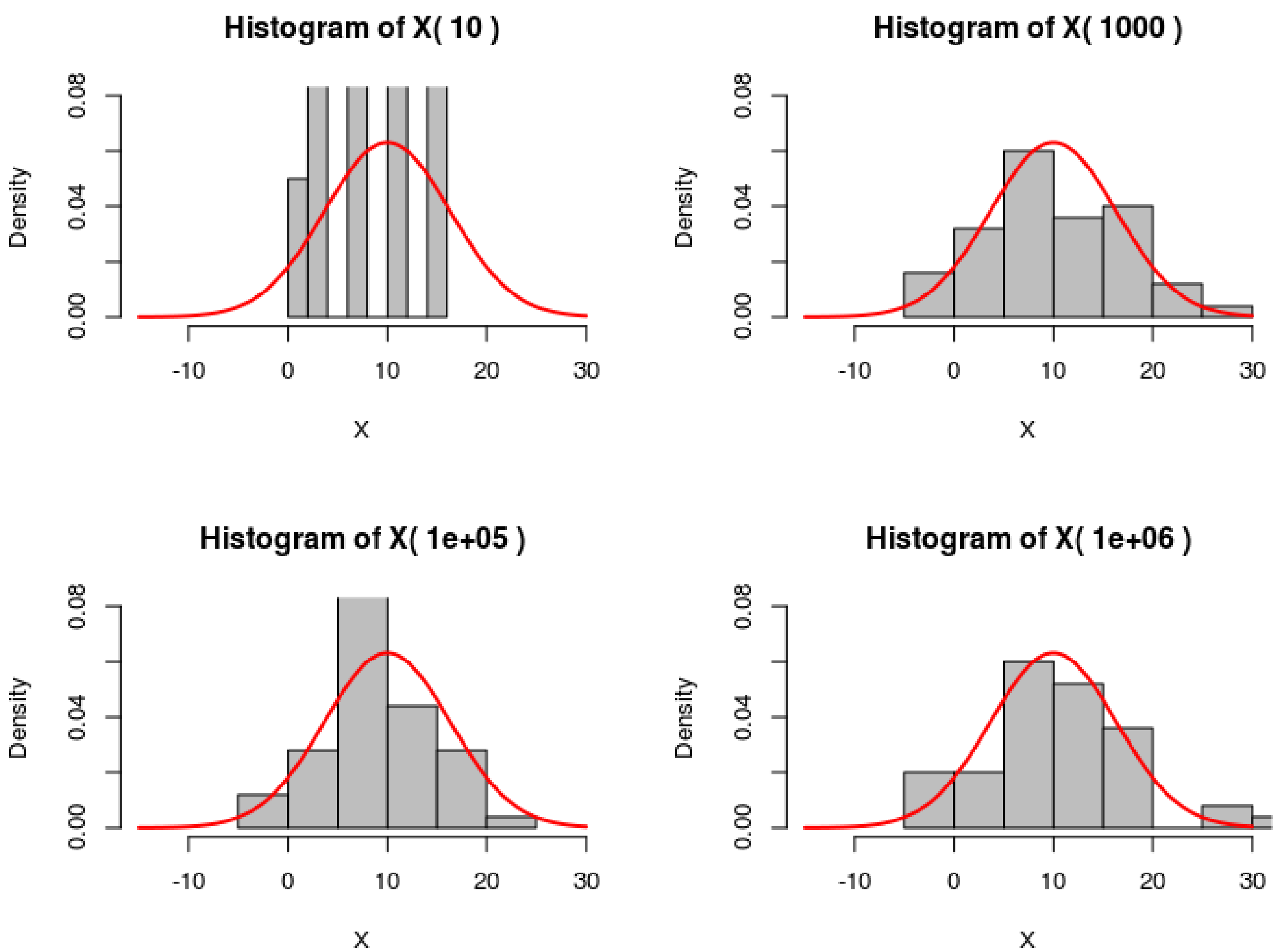
Now we see the limiting property in Figure 4.

```
# R-code for Figure 4.
set.seed(37)
m <- 50
dt <- c(1,0.01,1e-04,1e-05)
n <- c(10,1000,1e+05,1e+06)
t <- n*dt
```

```
seqlen <- t/n
mu <- 1
sd <- 2
p <- (1+mu*sqrt(dt)/sd)/2
q <- 1-p
dx <- sd*sqrt(dt)

par(mfrow=c(2,2))
for(k in 1:length(dx)){
  z <- numeric(n[k]+1)
  x <- numeric(n[k]+1)
  z[1] <- 0
  x[1] <- 0
  y <- matrix(0,nrow=m,ncol=n[k]+1)

  for(j in 1:m){
    for(i in 2:n[k]+1){
      z[i] <- rbinom(1,1,p[k])
      if(z[i]==0)
      {
        z[i] <- -dx[k]
      }else{
        z[i] <- dx[k]
      }
    }
    x = cumsum(z)
    if(j==1){
      #plot(seq(0,t[k],dt[k]),x,type="l",xlab="t",main=paste("Time step = ",dt[k]))
    }else{
      #lines(seq(0,t[k],dt[k]),x,lwd=2,col="turquoise4")
      y[j,] <- x
    }
```

```
  }
  hist(y[,n[k]], prob = T, col = "grey",main = paste("Histogram␣of␣X(",n[k],")"),
        xlim = c(-15,30),ylim=c(0,0.08),xlab = "X")
  # Xn of all m reps  are expected to follow normal dist.
  mean = mu*t
  stddev = sd*sqrt(t)
  curve(exp(-0.5*((x-mean)/(stddev))^2)/(stddev*sqrt(2*pi)),
        col = "red", lwd=2, add =TRUE)
  #curve(dnorm(x, mean = mean, sd=sd), col = "red", lwd=2, add =TRUE)
}
```

Figure 4: WIENER PROCESS TENDS TO NORMAL DISTRIBUTION



3.1 Diffusion equations for Wiener process

We have seen the normal solution using the central limit theorem. In this section we obtain the same thing using the Forward Kolmogorov equation (FKE) for the simple random walk, which is given by,

$$p_{j,k}^{(n+1)} = pp_{j,k-1}^{(n)} + qp_{j,k+1}^{(n)}$$

where p and q are as above. Let $x_0 = j\mathrm{d}x, x = k\mathrm{d}x, t = n\mathrm{d}t$. Define $p(x_0, x; t)\mathrm{d}x$ be the conditional probability that the process is at x at time t given it started at x_0 at time 0. We rescale the integer state space in the diffusion limit. So, then the FKE assumes it's limit form as

$$p(x_0, x; t + \mathrm{d}t)\mathrm{d}x = pp(x_0, x - \mathrm{d}x; t)\mathrm{d}x + qp(x_0, x + \mathrm{d}x; t)\mathrm{d}x.$$

Now we take the Taylor's series expansion up to second order terms to obtain

$$p + \mathrm{d}t\frac{\partial p}{\partial t} \approx p\left[p - \mathrm{d}x\frac{\partial p}{\partial x} + 0.5\mathrm{d}x^2\frac{\partial^2 p}{\partial x^2} - \cdots\right] + q\left[p + \mathrm{d}x\frac{\partial p}{\partial x} + 0.5\mathrm{d}x^2\frac{\partial^2 p}{\partial x^2} + \cdots\right].$$

Then on using $p + q = 1$,

$$\mathrm{d}t\frac{\partial p}{\partial t} \approx (q - p)\mathrm{d}x\frac{\partial p}{\partial x} + 0.5\mathrm{d}x^2\frac{\partial^2 p}{\partial x^2} + O(\mathrm{d}x^3).$$

From the above, $q - p = -\mu\frac{\mathrm{d}x}{\sigma^2}$ and $\mathrm{d}t = \mathrm{d}x^2/\sigma^2$ and in the limit $\mathrm{d}x \to 0$,

$$\frac{\partial p(x_0, x; t)}{\partial t} \approx -\mu\frac{\partial p(x_0, x; t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 p(x_0, x; t)}{\partial x^2}, \tag{3}$$

which is the required counterpart. It is a very crucial equation in our studies of these processes, as it's solution can be obtained using the method of images, as we will see further on. On comparing with the general second order partial differential equation, we see that $A = \sigma^2/2$ and $B = C = 0$. So $B^2 - 4AC = 0$, which means that the equation is parabolic in nature. Our use of method of images is facilitated by the occurrence of this equation in heat conduction and electrostatics. The backward Kolmogorov equation with respect to the initial point x_0, with drift term $+\mu$, is

$$\frac{\partial p(x_0, x; t)}{\partial t} \approx \mu\frac{\partial p(x_0, x; t)}{\partial x_0} + \frac{1}{2}\sigma^2\frac{\partial^2 p(x_0, x; t)}{\partial x_0^2}. \tag{4}$$

Both forward and backward Kolmogorov equations are duals of each other. To solve these equations, we use the moment generating function. Since the MGF is unique to a distribution, it is a very good option in this case, as it nicely wraps things up. A little ways down the road, we see that this is not always going to be so elegant, and we will have to adopt other methods, like the method of images, as informed earlier.

The moment generating function (MGF) of the probability density function $p(x_0, x; t)$ is given by

$$M(\theta, t) = \int_{-\infty}^{\infty} e^{\theta x} p(x_0, x; t)\,\mathrm{d}x.$$

We multiply the equation $e^{\theta x}$ and then integrate with respect to x. So, the left hand side of the equation would become

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{\theta x} p(x_0, x; t)\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{\theta x}\frac{\partial p(x_0, x; t)}{\partial t}\,\mathrm{d}x.$$

So Eq. (3) is reduced to

$$\frac{\partial M}{\partial t} = \int_{-\infty}^{\infty} e^{\theta x}\left(-\mu\frac{\partial p(x_0, x; t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 p(x_0, x; t)}{\partial x^2}\right)\mathrm{d}x$$

$$T\phi_j(s) = \int_0^{2\pi} k(s,t)e^{ijt}dt$$

$$< T\phi_l, \phi_j >= \int_0^{2\pi}\int_0^{2\pi} k(s,t)e^{ilt}e^{-ijs}dtds$$

To obtain a solution, we need to integrate this, which can be easily carried out. One has to keep in mind the $p(x_0, x; t)$ is a probability density function, so it tends to zero as $|x| \to \infty$. After some manipulation we obtain the partial differential equation for M as

$$\frac{\partial M}{\partial t} = M\left(\theta\mu + \frac{\sigma^2\theta^2}{2}\right),$$

which is further solved as a linear differential equation: $\ln(Mc) = \left(\theta\mu + \frac{\sigma^2\theta^2}{2}\right)t$ with $Mc = 1, t = 0$ where $M = e^{x_0\theta}, t = 0$ and so $c = e^{-x_0\theta}$ and finally, the expression for M is obtained as

$$M(\theta, t) = e^{(x_0+t\mu)\theta + \frac{1}{2}t\sigma^2\theta^2},$$

which is the MGF of $\mathcal{N}(x_0 + \mu t, \sigma^2 t)$, as expected. It may be tempting to cross verify the result again, that is to show that the normal density function solves the forward and backward Kolmogorov equation.

3.2 Wiener Process with reflecting barriers

If we place a reflecting barrier at a certain position, then the process cannot cross that position and then stays there for a while and then reflects back into other states. If we place two reflecting barriers, one at 0 and the other at a, then we expect the process to infinitely oscillate between these 0 and a. We go back to the discrete time random walk.

As $t \to \infty$ if the process settles down, then we neither have extinction, nor explosion. So, we assume that as $t \to \infty$ the population size probabilities $\{p_N(t)\}$ approach constant values $\{\pi_N\}$. Because they are probabilities and form a probability mass function, so $\sum_{N=0}^{\infty} \pi_N = 1$. For $\Pr(N(t) = i|N(0) = k)$ we have the FKE :

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{k,i}(t) = \lambda_{i-1} p_{k,i-1}(t) - (\lambda_i + \mu_i) p_{k,i}(t) + \mu_{i+1} p_{k,i+1}(t),$$

where λ_i is the birth rate and μ_i is the death rate at time population size i, with $\lambda_{-1} = \mu_0 = 0$. Why? Substituting $p(t) = \pi_N$ and as $t \to \infty$ we get that

$$\frac{d}{\mathrm{d}t} \pi_N = 0 = \lambda_{N-1} \pi_{N-1} - (\lambda_N + \mu_N) \pi_N + \mu_{N+1} \pi_{N+1},$$

for fixed k and $N = 0, 1, 2 \cdots$. Rearranging the terms, we get

$$\mu_{N+1} \pi_{N+1} - \lambda_N \pi_N = \mu_N \pi_N - \lambda_{N-1} \pi_{N-1}, \; N = 0, 1, 2, \cdots$$

Clearly $LHS(N) = RHS(N-1)$, Therefore, $\mu_N \pi_N - \lambda_{N-1} \pi_{N-1} =$ constant. At $N = 0$, $\mu_0 \pi_0 - \lambda_{-1} \pi_{-1} = 0$. This means that the constant is 0. Therefore $\mu_N \pi_N = \lambda_{N-1} \pi_{N-1}$, which is the balance equation. In equilibrium, the "probability flow" in between neighboring states must be in "balance". This means the following : LHS : The probability of having a population of size N followed by a death i.e. $\mu_N \pi_N$ is the same as RHS : The probability of having a population of size $N-1$ followed by a birth i.e. $\pi_{N-1} \pi_{N-1}$

From the balance equation : $\pi_N = \frac{\pi_{N-1} \lambda_{N-1}}{\mu_N}$ but, at $N-1$ the balance equation gives, $\pi_{N-1} = \frac{\pi_{N-2} \lambda_{N-2}}{\mu_{N-1}}$ So then, $\pi_N = \frac{\pi_{N-2} \lambda_{N-2} \lambda_{N-1}}{\mu_N \mu_{N-1}}$ Using this recurrence, we finally get : $\pi_N = \frac{\lambda_0 \lambda_1 \cdots \lambda_{N-2} \lambda_{N-1}}{\mu_1 \mu_2 \cdots \mu_{N-1} \mu_N} \pi_0$ Let $\omega_N = \frac{\lambda_0 \lambda_1 \cdots \lambda_{N-2} \lambda_{N-1}}{\mu_1 \mu_2 \cdots \mu_{N-1} \mu_N}$, $\quad N > 0$ and $\omega_0 = 1$. We have to choose π_0 to satisfy $\sum_{N=0}^{\infty} \pi_N = 1$.

Now, $\omega_1 = \lambda_0/\mu_1 = \pi_1/\pi_0$ $\omega_2 = \lambda_0 \lambda_1/\mu_1 \mu_2 = \pi_1 \pi_2/\pi_0 \pi_1 = \pi_2/\pi_0$ and so on. Then $\sum_{i=0}^{\infty} \omega_i = 1 + \pi_1/\pi_0 + \pi_2/\pi_0 + \cdots = \frac{1}{\pi_0}(1 + \pi_1 + \pi_2 + \cdots) = \frac{\sum_{i=0}^{\infty} \pi_i}{\pi_0} = \frac{1}{\pi_0}$. So we choose $\pi_0 = \frac{1}{\sum_{i=0}^{\infty} \omega_i}$. Therefore

$$\pi_N = \omega_N \pi_0 = \frac{\omega_N}{\sum_{i=0}^{\infty} \omega_i}, \quad N = 0, 1, 2, \cdots . \tag{5}$$

We extend this to the continuous time and space system. Now as the number of steps increase, it really does not matter where the process began. Let π_i be the equilibrium probability. $\lim_{n \to \infty} \Pr(X_n = i | X_0 = j) = \pi_i, \; i, j = 0, 1, \cdots, a$. We define, p_i as the transition probability from state $i \to i+1$ q_i as the transition probability from state $i \to i-1$ And so $1 - p_i - q_i$ is the probability that the state is not changed. Observe

: $q_0 = 0 = p_a$. Letπ_i be the equilibrium probability. $\lim_{n\to\infty} \Pr(X_n = i|X_0 = j) = \pi_i,\ i, j = 0, 1, \cdots, a$. On paralleling the above construction, $\pi_i = p_{i-1}\pi_{i-1} + (1 - p_i - q_i)\pi_i + q_{i+1}\pi_{i+1}$ which gives

$$q_{i+1}\pi_{i+1} - p_i\pi_i = q_i\pi_i - p_{i-1}\pi_{i-1} = \text{constant} \tag{6}$$

which is our balance equation. Clearly here there is a comparison of the transition probabilities and the birth and death rates in the above construction. So quite naturally : $\pi_i = \frac{p_0 p_1 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}\pi_0, \quad i = 1, 2, \cdots, a$. We want to choose π_0 so that $\sum_{i=0}^{a} \pi_i = 1$ (why?). The following relation

$$\pi_0 + \pi_1 + \cdots + \pi_a = 1 = \pi_0 + p_0\pi_0/q_0 + \cdots + \frac{p_0 p_1 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}\pi_0 = \pi_0 \left(1 + \sum_{i=0}^{a} \frac{p_0 p_1 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}\right),$$

gives $\pi_0 = \frac{1}{\left(1 + \sum_{i=0}^{a} \frac{p_0 p_1 \cdots p_{i-1}}{q_1 q_2 \cdots q_i}\right)}$. Now, in simple random walk $p_i = p = 1 - q = 1 - q_i, \pi_0 = \frac{1}{\left(1 + \sum_{i=0}^{a} \left(\frac{p}{q}\right)^i\right)} = \frac{1 - p/q}{1 - (p/q)^{a+1}}$. Therefore, $\pi_i = \left(\frac{p}{q}\right)^i \left(\frac{1 - p/q}{1 - (p/q)^{a+1}}\right), \quad i = 0, 1, \cdots, a$.

Now, we extend this to Wiener process. $p = \frac{1}{2}\left(1 + \frac{\mu\sqrt{\mathrm{d}t}}{\sigma}\right)$ $q = \frac{1}{2}\left(1 - \frac{\mu\sqrt{\mathrm{d}t}}{\sigma}\right)$ $\sigma\sqrt{\mathrm{d}t} = \mathrm{d}x$ $i \to x/\mathrm{d}x$ and $a \to a/\mathrm{d}x$ As $\mathrm{d}x \to 0$ we have : $p/q = \frac{\left(1 + \frac{\mu \mathrm{d}x}{\sigma^2}\right)}{\left(1 - \frac{\mu \mathrm{d}x}{\sigma^2}\right)}$ $\left(\frac{p}{q}\right)^i = \left(\frac{\left(1 + \frac{\mu \mathrm{d}x}{\sigma^2}\right)}{\left(1 - \frac{\mu \mathrm{d}x}{\sigma^2}\right)}\right)^{x/\mathrm{d}x}$. Let $\lim_{\mathrm{d}x\to 0} \left(\frac{p}{q}\right)^i = L$ Then $\ln L = \frac{x}{\mathrm{d}x} \ln\left(\frac{\left(1 + \frac{\mu dx}{\sigma^2}\right)}{\left(1 - \frac{\mu dx}{\sigma^2}\right)}\right)$. Taking the limit and observing, we see it is an indeterminate form $\frac{0}{0}$. Using L'Hopital's rule we get $\lim_{\mathrm{d}x\to 0} \ln L = 2x\mu/\sigma^2$ So then $L = e^{2x\mu/\sigma^2}$ Similarly $\left(\frac{p}{q}\right)^a \to e^{2a\mu/\sigma^2}$

Also, $\pi_i \to \pi(x)\mathrm{d}x$ since $P(x \leq X \leq x + \mathrm{d}x) = \int_x^{x+\mathrm{d}x} f(t)\,\mathrm{d}t \approx f(x)[x + \mathrm{d}x - x] \approx f(x)\mathrm{d}x$. So, in the limit $\mathrm{d}x \to 0$ we get, $\lim_{\mathrm{d}x\to 0} \pi_i = \lim_{\mathrm{d}x\to 0} \left(\frac{p}{q}\right)^i \left(\frac{1 - p/q}{1 - (p/q)^{a+1}}\right)$ which solves to

$$\begin{aligned} \pi(x)\mathrm{d}x &= e^{2x\mu/\sigma^2} \lim_{\mathrm{d}x\to 0} \left(\frac{1 - p/q}{1 - (p/q)^{a+1}}\right) && (7) \\ &= e^{2x\mu/\sigma^2} \lim_{\mathrm{d}x\to 0} \left(\frac{\frac{-\mu \mathrm{d}x/\sigma^2}{0.5 * (1 - \mu \mathrm{d}x/\sigma^2)}}{1 - (p/q)^{a+1}}\right) && (8) \end{aligned}$$

which finally reduces to $\pi(x) = \frac{e^{(2\mu/\sigma^2)x}(2\mu/\sigma^2)}{e^{(2\mu/\sigma^2)a} - 1}$. Now, there is quite a bit of a difference in continuous state space as compared to discrete. The proportion of time spent at $x = 0$ is $\pi(0)\mathrm{d}x \to 0, \mathrm{d}x \to 0$.

If we want to spend non-zero amount of time at our barriers 0 and a, then what do we do? At reflecting barrier $x = 0$: $p_{01} = p'$ is the transition probability of moving from state 0 to $\mathrm{d}x$. Since we cannot go below $x = 0$ due to the barrier, $p_{00} = 1 - p'$. The balance equation is $\pi_0 p_{01} = \pi_1 q_{10}$ i.e. $\pi_0 p' = \pi_1 q$. So we get that

$p' = \pi_1 q/\pi_0$ where $\pi_0 = \pi(0)$ our target probability, i.e. proportion that we want process to stay at x =0, and $\pi_1 = \pi(\mathrm{d}x) = \frac{e^{(2\mu/\sigma^2)\mathrm{d}x}(2\mu/\sigma^2)}{e^{(2\mu/\sigma^2)a}-1}$.

At reflecting barrier $x = a$, $p_{a,a-1} = q'$ is the transition probability of moving from state a to $a - \mathrm{d}x$. Since we cannot go above $x = a$ due to the barrier, $p_{a,a} = 1 - q'$. The balance equation is $\pi_{a-1}p_{a-1,a} = \pi_a q_{a,a-1}$ i.e. $\pi_{a-1}p = \pi_a q'$. So we get that $q' = \pi(a - \mathrm{d}x)p/\pi_a$ where $\pi_a = \pi(a)$ our target probability, i.e. proportion that we want process to stay at $x = a$, and $\pi_{a-1} = \pi(a - \mathrm{d}x) = \pi(\mathrm{d}x) = \frac{e^{(2\mu/\sigma^2)(a-dx)}(2\mu/\sigma^2)}{e^{(2\mu/\sigma^2)a}-1}$.

Since we require that $\pi(x)$ to be a pdf, we rescale it as $\pi(x) = (1 - \pi_0 - \pi_a)\frac{e^{(2\mu/\sigma^2)x}(2\mu/\sigma^2)}{e^{(2\mu/\sigma^2)a}-1}, 0 < x < a$, and $\pi(0) = \pi_0$, $\pi(a) = \pi_a$. So, we have a continuous pdf in $(0, a)$ and a discrete on at $0, a$, which is given by

$$\begin{aligned} \pi(x) &= \pi_0, & x = 0, \quad (9)\\ &= (1 - \pi_0 - \pi_a)\frac{e^{(2\mu/\sigma^2)x}(2\mu/\sigma^2)}{e^{(2\mu/\sigma^2)a} - 1}, & 0 < x < a, \quad (10)\\ &= \pi_a, & x = a. \quad (11) \end{aligned}$$

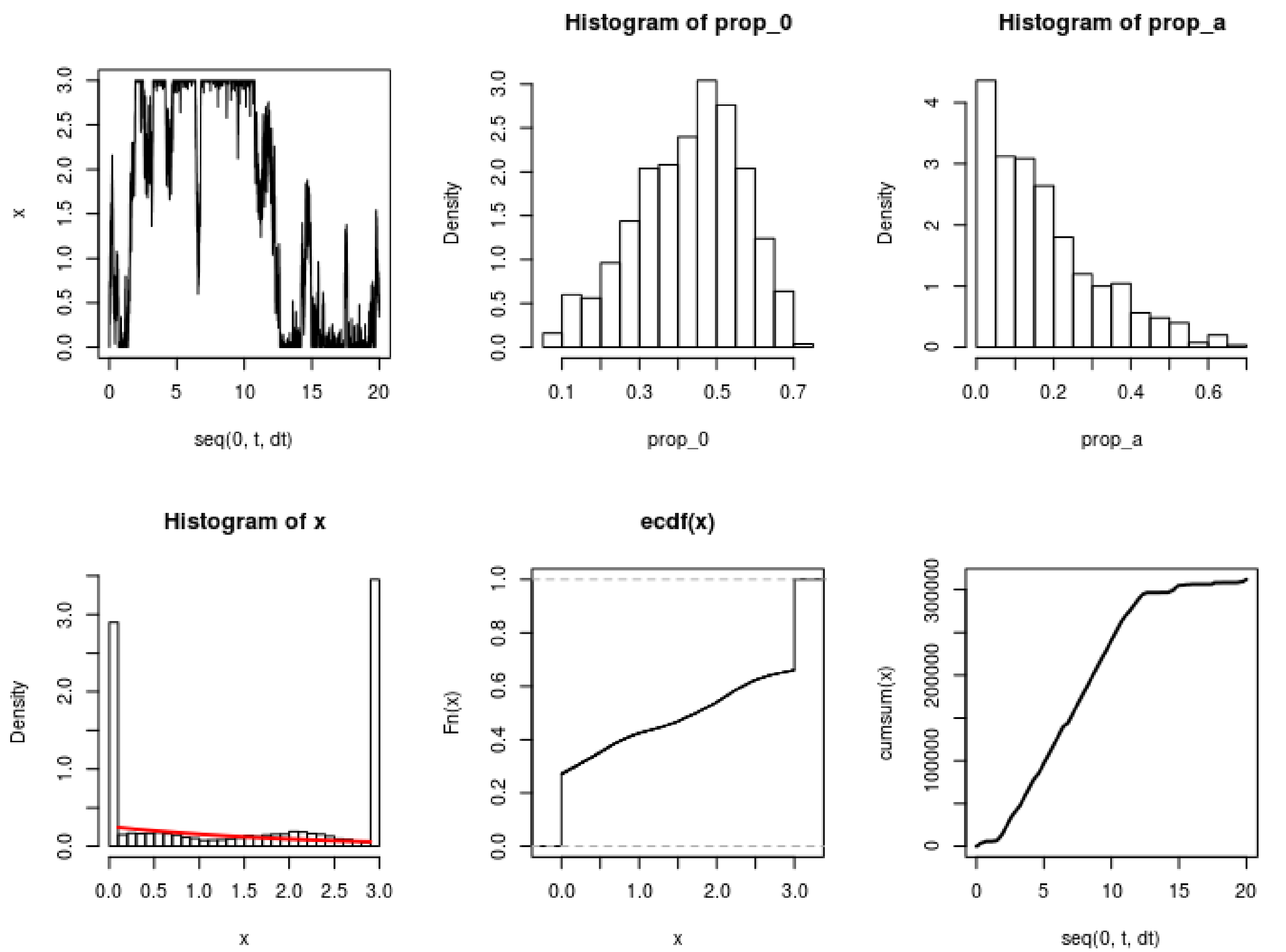
Figure 5 shows the process limited by two reflecting barriers.

```
# R-code for Figure 5.
set.seed(37)
rep <- 500
a <- 3
pi0 <- 0.4
pia <- 0.2
mu <- -1 # mu is negative because we have used forward kolmogorov diffusion equation.
sd <- 2
t <- 20
dt <- 0.0001
n <- t/dt
dx <- sd*sqrt(dt)
p <- (1+mu*sqrt(dt)/sd)/2
q <- (1-mu*sqrt(dt)/sd)/2
p_dash <- ((1-pi0-pia)*(2*mu/sd^2)(exp(2*dx*mu/sd^2))
        /(exp(2*a*mu/sd^2)-1)*q/pi0)*dx
q_dashdash <- ((1-pi0-pia)*(2*mu/sd^2)*(exp(2*(a-dx)*mu/sd^2))
```

```
        /(exp(2*a*mu/sd^2)-1)*p/pia)*dx
p_dashdash <- 1 - q_dashdash

z <- numeric(n+1)  # generating zi.
x <-numeric(n+1)
z[1] <- 0
x[1] <- 0
perc_x_zero <- numeric(100)
perc_x_a <- numeric(100)
o <- numeric(rep)

prop_a <- numeric(rep)
prop_0 <- numeric(rep)


for(k in 1:rep){
  for(i in 1:n){
    if(x[i] <= 0){
      z[i] = rbinom(1, 1, p_dash)
      if(z[i] == 0){
        x[i+1] <- 0
      }
      else{
        x[i+1] <- dx
      }
    }
    if(x[i]>= a){
      z[i] <- rbinom(1, 1,p_dashdash)
      if(z[i] == 0){
        x[i+1] <- x[i]-dx
      }
      else{
```

```
        x[i+1] <- a
      }
    }
    if( (0 < x[i]) & (x[i]< a)){
      z[i] = rbinom(1, 1,p)
      if(z[i] == 1){
        x[i+1] <- x[i] + dx
      }
      else{
        x[i + 1] <- x[i] - dx
      }
    }
  }
  prop_0[k] <- sum(x==0)/length(x)
  prop_a[k] <- sum(x==a)/length(x)
}

mean(prop_0)*100 # should be approx pi0
mean(prop_a)*100 # should be approx pia
par(mfrow=c(2,3))
plot(seq(0,t,dt),x,type = "l")
hist(prop_0, prob = T)
hist(prop_a, prob = T)
hist(x, prob = T,breaks = 10*a)
curve((1-pi0-pia)*(2*mu/sd^2)*exp((2*mu/sd^2)*x)/(exp((2*mu/sd^2)*a)-1),
        add=T,col="red",lwd=2,xlim = c(0.1,a-0.1)) # match our pdf with hist
plot(ecdf(x),verticals=T,do.points=F)
plot(seq(0,t,dt),cumsum(x),type = "l",lwd=2)
```

Figure 5: Wiener Process with two reflecting barriers. One barrier is at $x = 0$ and another one is at $x = a$.



You may run this code and observe that the mean in percentage of the variable prop_0 is 42.78 which is very close to the set target probability "$\pi_0 = 0.4$ or 40 percent" and that mean of prop_a is 17.18 which is close to the target probability "$\pi_a = 0.2$ or 20 percent". You may change a to a higher value, but also must increase the variable "rep" considerably to get satisfying results.

Let us now limit ourself to a single reflecting barrier at $x = 0$. Let the Wiener process start at $X(0) = x_0 > 0$. At $t = 0$ we have the initial condition $p(x; 0) = \delta(x - x_0)$. For an infinite number of trajectories starting at x_0, the proportion of them lying in $(x, x + dx)$ during $(t, t + dt)$ is given by the probability $p(x; t)dx$. Also note that $\int_0^\infty p(x;t)dx = 1$. Why?

Taking derivate with respect to t we obtain

$$\frac{\partial}{\partial t}\int_0^\infty p(x;t)\mathrm{d}x = 0 \text{ or } \int_0^\infty \frac{\partial}{\partial t}p(x;t)\mathrm{d}x = 0$$

which on substituting $\frac{\partial}{\partial t}p(x;t)$ from the FKE gives

$$\left[-\mu p(x_0, x; t) + \frac{1}{2}\sigma^2\frac{\partial p(x_0, x; t)}{\partial x}\right]_{x=0} = 0,$$

which is a boundary condition. As mentioned earlier, we now use the method of images to solve the following equation,

$$\frac{\partial p(x_0, x; t)}{\partial t} \approx -\mu\frac{\partial p(x_0, x; t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 p(x_0, x; t)}{\partial x^2},$$

subject to the initial condition $p(x;0) = \delta(x - x_0)$, and subject to the boundary condition

$$\left[-\mu p(x_0, x; t) + \frac{1}{2}\sigma^2\frac{\partial p(x_0, x; t)}{\partial x}\right]_{x=0} = 0.$$

Let us first try to understand what exactly the method of images is. Think of a point charge located at $(0, 0, a) \in \mathbb{R}^3$ which is above a grounded plate located in the XY plane. Now we place a similar point charge with opposite polarity at $(0, 0, -a)$. Clearly our plane of symmetry is the XY plane. The boundary condition is that the plate is grounded. By symmetry this new point charge will produce the same electric field as the original one along the Z-axis, and will also satisfy the boundary condition. So now, the force at any point can be calculated as the superposition of the two charges in free space.

The method of images is a way to solve a differential equation. The function is allowed to take values in the reverse direction of its domain. As we have restricted ourself to a single reflecting barrier at $x = 0$ our domain is $x \geq 0$. So if we were to use the method of images, we would extend the domain to $x \in \mathbb{R}$, i.e. we added its mirror image with respect to y axis, that is the line $x = 0$.

This method easily solves the boundary conditions due to the allowable values the function on the mirror image of its domain. This method is very useful when the boundary is flat, since then there linear reflection about it.

To solve a inhomogeneous linear differential equation $Ly = f$, we may use Green's function, which is the impulse response of the LDE. It is an integral kernel representing L^{-1}. We want to solve for y. So, $y = L^{-1}f$.

We write : $(L^{-1})_{x,\xi} = G(x,\xi)$, with $LG(x,\xi) = \delta(x-\xi)$, where L acts on x. Then, $y(x) = G(x,\xi)f(\xi)$ And we also then get, $Ly = \int LG(x,\xi)f(\xi)d\xi = \int \delta(x-\xi)f(\xi)d\xi = f(x)$, where, $G(x,\xi)$ has the following properties:

1. must have some discontinuity to satisfy delta function .

2. The function is symmetric, $G(x,\xi) = G(\xi,x)$.

3. The function is such that $LG(x,\xi) = 0$ when $x \neq \xi$.

In view of our problem, we allow our process to operate over $x \in \mathbb{R}$, by reflecting it over $x = 0$. To satisfy the boundary condition $\left[-\mu p(x_0,x;t) + \frac{1}{2}\sigma^2 \frac{\partial p(x_0,x;t)}{\partial x}\right]_{x=0} = 0$, we place a point image at $-x_0$ the reflected initial point. We notice that a single point $-x_0$, i.e. the isolated image source will be insufficient, we have reflection, we also place a continuous source of image points in $x < -x_0$.

We know that the system has its principal solution as a normal solution, let us call it $U(x_0)$. So let,

$$p(x;t) = \frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(x-x_0-\mu t)^2/2\sigma^2 t} + ce^{-(x+x_0-\mu t)^2/2\sigma^2 t} + \int_{-\infty}^{-x_0} e^{-(x-\xi-\mu t)^2/2\sigma^2 t} h(\xi)d\xi\right].$$

That is, the solution is basically a superposition (linear combination) of the normal solution $U(x_0)$, the normal solution with respect to the isolated negative initial point $U(-x_0)$ and the continuous sources $U(\xi), \xi < -x_0$, where we can think of c and $h(\xi)d\xi$ as yields of these sources, or their contribution to the solution. The initial condition $p(x,0) = \delta(x-x_0)$ is therefore automatically satisfied. Now, we need $h(\xi)$ and c so as to satisfy the boundary condition, $\left[-\mu p(x_0,x;t) + \frac{1}{2}\sigma^2 \frac{\partial p(x_0,x;t)}{\partial x}\right]_{x=0} = 0$. We differentiate the solution with respect to x, multiply $\sigma^2/2$, subtract $\mu p(x,t)$ and then place $x = 0$.

$$\frac{\sigma^2}{2}\frac{\partial}{\partial x}p(0;t) = \frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(-x_0-\mu t)^2/2\sigma^2 t}\left(\frac{x_0+\mu t}{2t}\right) - ce^{-(x_0-\mu t)^2/2\sigma^2 t}\left(\frac{x_0-\mu t}{2t}\right)\right. \tag{12}$$

$$\left. + \int_{-\infty}^{-x_0} e^{-(-\xi-\mu t)^2/2\sigma^2 t}\left(\frac{\xi+\mu t}{2t}\right)h(\xi)d\xi\right] \tag{13}$$

$$\mu p(0;t) = \frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(-x_0-\mu t)^2/2\sigma^2 t} + ce^{-(x_0-\mu t)^2/2\sigma^2 t} + \int_{-\infty}^{-x_0} e^{-(-\xi-\mu t)^2/2\sigma^2 t}h(\xi)d\xi\right] \tag{14}$$

On subtracting,

$$\left[-\mu p(x_0,x;t) + \frac{1}{2}\sigma^2\frac{\partial p(x_0,x;t)}{\partial x}\right]_{x=0} = \tag{15}$$

$$= \frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(x_0+\mu t)^2/2\sigma^2 t}\left(\frac{x_0-\mu t}{2t}\right) - ce^{-(x_0-\mu t)^2/2\sigma^2 t}\left(\frac{x_0+\mu t}{2t}\right)\right. \tag{16}$$

$$\left. + \int_{-\infty}^{-x_0} e^{-(\xi+\mu t)^2/2\sigma^2 t}\left(\frac{\xi+\mu t}{2t} - \mu\right)h(\xi)\mathrm{d}\xi\right] = 0 \tag{17}$$

Now, we split the bracket in the integral, solve the first integral by parts, and let the second one be as is.

$$\int_{-\infty}^{-x_0} e^{-(\xi+\mu t)^2/2\sigma^2 t}\left(\frac{\xi+\mu t}{2t}-\mu\right)h(\xi)d\xi = \int_{-\infty}^{-x_0} e^{-(\xi+\mu t)^2/2\sigma^2 t}\left(\frac{\xi+\mu t}{2t}\right)h(\xi)d\xi - \int_{-\infty}^{-x_0} e^{-(\xi+\mu t)^2/2\sigma^2 t}(\mu)\,h(\xi)d\xi$$

To solve the first integral, we use the fact that $\int f'(x)e^{f(x)}\mathrm{d}x = e^{f(x)}$. By multiplying and dividing $\sigma^2/2$, we obtain

$$\frac{\sigma^2}{2}\int_{-\infty}^{-x_0}\frac{2}{\sigma^2}e^{-(\xi+\mu t)^2/2\sigma^2 t}\left(\frac{\xi+\mu t}{2t}\right)h(\xi)d\xi = \frac{\sigma^2}{2}\left[e^{-(-x_0+\mu t)^2/2\sigma^2 t}h(-x_0) - \int_{-\infty}^{-x_0} e^{-(\xi+\mu t)^2/2\sigma^2 t}\frac{dh(\xi)}{d\xi}d\xi\right]$$

So we have

$$\frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(x_0+\mu t)^2/2\sigma^2 t}\left(\frac{x_0-\mu t}{2t}\right) + ce^{-(x_0-\mu t)^2/2\sigma^2 t}\left(\frac{x_0+\mu t}{2t}\right)\right. \tag{18}$$

$$+\quad \frac{\sigma^2}{2}\left[-e^{-(-x_0+\mu t)^2/2\sigma^2 t}h(-x_0)\right] + \left.\int_{-\infty}^{-x_0} e^{-(\xi+\mu t)^2/2\sigma^2 t}\left(\frac{\sigma^2}{2}\frac{dh(\xi)}{d\xi}-\mu h(\xi)\right)d\xi\right] = 0 \tag{19}$$

On expanding power of exponential term and taking the common part of their first three terms out,

$$e^{-(x_0^2+\mu^2t^2)^2/2\sigma^2 t}\left[e^{-x_0\mu/\sigma^2}\left(\frac{x_0-\mu t}{2t}\right) - ce^{-x_0\mu/\sigma^2}\left(\frac{x_0+\mu t}{2t}\right) - \frac{\sigma^2}{2}e^{x_0\mu/\sigma^2}h(-x_0)\right] \tag{20}$$

$$+\quad \int_{-\infty}^{-x_0} e^{-(\xi+\mu t)^2/2\sigma^2 t}\left(\frac{\sigma^2}{2}\frac{dh(\xi)}{d\xi}-\mu h(\xi)\right)d\xi = 0 \tag{21}$$

On taking derivative with respect to ξ we get that $\frac{dh(\xi)}{d\xi} - \mu h(\xi) = 0$, which on solving gives $h(\xi) = c_1 e^{2\mu\xi/\sigma^2}$. Since, $\frac{dh(\xi)}{d\xi} - \mu h(\xi) = 0$ we get

$$e^{-(x_0^2+\mu^2t^2)^2/2\sigma^2 t}\left[e^{-x_0\mu/\sigma^2}\left(\frac{x_0-\mu t}{2t}\right) - ce^{-x_0\mu/\sigma^2}\left(\frac{x_0+\mu t}{2t}\right) - \frac{\sigma^2}{2}e^{x_0\mu/\sigma^2}h(-x_0)\right] = 0$$

Which means that

$$e^{-x_0\mu/\sigma^2}\left(\frac{x_0-\mu t}{2t}\right) - ce^{-x_0\mu/\sigma^2}\left(\frac{x_0+\mu t}{2t}\right) - \frac{\sigma^2}{2}e^{x_0\mu/\sigma^2}h(-x_0) = 0$$

On solving we see that $c = e^{-2x_0\mu/\sigma^2}$, and then $c_1 = 2\mu/\sigma^2$, i.e. $h(\xi) = \frac{2\mu}{\sigma^2}e^{2\mu\xi/\sigma^2}$. Now, we put this value of $h(\xi)$ in our solution, and get the integral as $\frac{2\mu}{\sigma^2}e^{2x\mu t/\sigma^2}\phi\left[\frac{-x_0-\mu t-x}{\sigma\sqrt{t}}\right]$, where $\phi(.)$ is the cumulative distribution function of the standard normal random variable. This is easy to verify and left as an exercise. Then on finally substituting all the values we get the solution to as

$$p(x;t) = \frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(x-x_0-\mu t)^2/2\sigma^2 t} + e^{(-4x_0\mu t-(x+x_0-\mu t)^2)/2\sigma^2 t}\right] + \frac{2\mu}{\sigma^2}e^{2\mu x/\sigma^2}\Phi\left[\frac{-x_0-\mu t-x}{\sigma\sqrt{t}}\right]. \tag{22}$$

Let us observe the values of drift μ. If $\mu > 0$, then the process drifts towards ∞, and $p(x;t) \to 0$ as $t \to \infty$, $x > 0$ and if $\mu < 0$, then the process drifts towards the reflecting barrier. Letting $t \to \infty$, we get : $p(x;\infty) = \frac{2|\mu|}{\sigma^2} e^{-2|\mu|x/\sigma^2}, x > 0$.

3.3 Wiener Process with absorbing barriers

We place an absorption barrier at $x = a$. We say that once the process reaches the absorption barrier at $x = a$, it is going to stay there. Take $X(0) = x_0 = 0$. Our FKE is : $\frac{\partial p(x_0,x;t)}{\partial t} \approx -\mu \frac{\partial p(x_0,x;t)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 p(x_0,x;t)}{\partial x^2}$, subject to IC $p(x;0) = \delta(x)$.

We also require a boundary condition to solve the above partial differential equation. Consider the diffusion equation $p(x_0, x; t + \mathrm{d}t) = pp(x_0, x - \mathrm{d}x; t) + qp(x_0, x + \mathrm{d}x; t)$, so therefore : $p(0, x; t + \mathrm{d}t) = pp(0, x - \mathrm{d}x; t) + qp(0, x + \mathrm{d}x; t)$. At $x = a$, $p(0, a; t + \mathrm{d}t) = pp(0, a - \mathrm{d}x; t) + qp(0, a + \mathrm{d}x; t)$, and since we have an absorption barrier at $x = a$, the equation reduces to $p(a; t + \mathrm{d}t) = pp(a - \mathrm{d}x; t) + p(a; t)$. So what does $q = 1$ mean ? Also, $p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\mathrm{d}t}\right)$ and $\mathrm{d}t = \frac{\mathrm{d}x^2}{\sigma^2}$. Now taking the Taylor's series expansion we get

$$p + \mathrm{d}t\frac{\partial p}{\partial t} + \cdots = p(p - \mathrm{d}x\frac{\partial p}{\partial x} + \frac{\mathrm{d}x^2}{2!}\frac{\partial p}{\partial x} - \cdots) + p(a;t) \tag{23}$$

$$p + \mathrm{d}t\frac{\partial p}{\partial t} + \cdots = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\mathrm{d}t}\right)(p - \mathrm{d}x\frac{\partial p}{\partial x} + \frac{\mathrm{d}x^2}{2!}\frac{\partial p}{\partial x} -]\cdots) + p \tag{24}$$

The p in the Taylor's series is $p(a;t)$.

$$p + \frac{\mathrm{d}x^2}{\sigma^2}\frac{\partial p}{\partial t} + \cdots = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\mathrm{d}t}\right)(p - \mathrm{d}x\frac{\partial p}{\partial x} + \frac{\mathrm{d}x^2}{2!}\frac{\partial p}{\partial x} - \cdots) + p. \tag{25}$$

Then, on rearranging terms in the above expansion we get,

$$\frac{\mathrm{d}x^2}{\sigma^2}\frac{\partial p}{\partial t} + \cdots = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\mathrm{d}t}\right)(p - \mathrm{d}x\frac{\partial p}{\partial x} + \frac{\mathrm{d}x^2}{2!}\frac{\partial p}{\partial x} - \cdots), \tag{26}$$

$$\frac{\mathrm{d}x^2}{\sigma^2}\frac{\partial p}{\partial t} + \cdots = \frac{p}{2} - \frac{\mathrm{d}x}{2}\frac{\partial p}{\partial x} + \cdots + \frac{\mu}{\sigma^2}\mathrm{d}x\left(p - \mathrm{d}x\frac{\partial p}{\partial x} + \cdots\right) \tag{27}$$

which on taking the $p/2$ as subject gives, $\frac{p}{2} - O(\mathrm{d}x) = 0$, and in the limit $dx \to 0$, we have $p(a,t) = 0$, which is the boundary condition. We will base the solution on the normal solution as before $\frac{1}{\sigma\sqrt{t}\sqrt{2\pi}} e^{-(x-x_0-\mu t)^2/2\sigma^2 t}$.

Using the method of images, our mirror is now at $x = a$, and the point is reflected from initial point $x_0 = 0$. So, the perceived image point is at $x = 2a$. Then, let the solution be

$$p(x;t) = \frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(x-x_0-\mu t)^2/2\sigma^2 t} + Ae^{-(x+x_0-\mu t)^2/2\sigma^2 t}\right]$$

Note that the continuous system of images is not required, as we have an absorption barrier. The principal source is at $x_0 = 0$, and the image source at $x_0 = 2a$.

$$p(x;t) = \frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(x-\mu t)^2/2\sigma^2 t} + Ae^{-(x-2a-\mu t)^2/2\sigma^2 t}\right].$$

It is easy to verify that the equation satisfies the Fokker-Plank equation. This must also satisfy the boundary condition. Let $x = a$. Then

$$p(x;t) = \frac{1}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{-(a-\mu t)^2/2\sigma^2 t} + Ae^{-(-a-\mu t)^2/2\sigma^2 t}\right] = 0. \tag{28}$$

$$\frac{e^{-(a^2+\mu^2 t^2)/2\sigma^2 t}}{\sigma\sqrt{t}\sqrt{2\pi}}\left[e^{a\mu t/\sigma^2 t} + Ae^{-a\mu t/\sigma^2 t}\right] = 0. \tag{29}$$

which clearly requires $A = -e^{2a\mu/\sigma^2}$. Observe that $p(x;t)$ is a superposition of (1) source of strength 1 at $x_0 = 0$, and (2) sink of strength A at $x = 2a$. We claim that the absorption is a certain event. To show this we go back to the discrete time situation and inspect the probability of absorption at time n. Pr(Eventual absorbtion) = 1. We want to evaluate the probability of absorption at $x = a$ at time n. $p(X_{n+1} = a|X_n = a) = 1$

In general, let us place two absorption barriers at $x = a$ and $x = b$. Then the process stops when it reaches either a or b. As $n \to \infty$, one of three things can happen: (1) Absorption at a, (2) Absorption at b and (3) the process oscillates infinitely between the two barriers, and this outcome has probability 0. So, absorption is certain.

Let the process be in motion at time n. This means that $\{X_n\}$ takes a value in $\{b+1, b+2, ..., a-2, a-1\}$.

$$\Pr(\text{Process in motion at time } n) \leq \Pr(\text{Unrestricted process occupies one of these states st time } n).$$

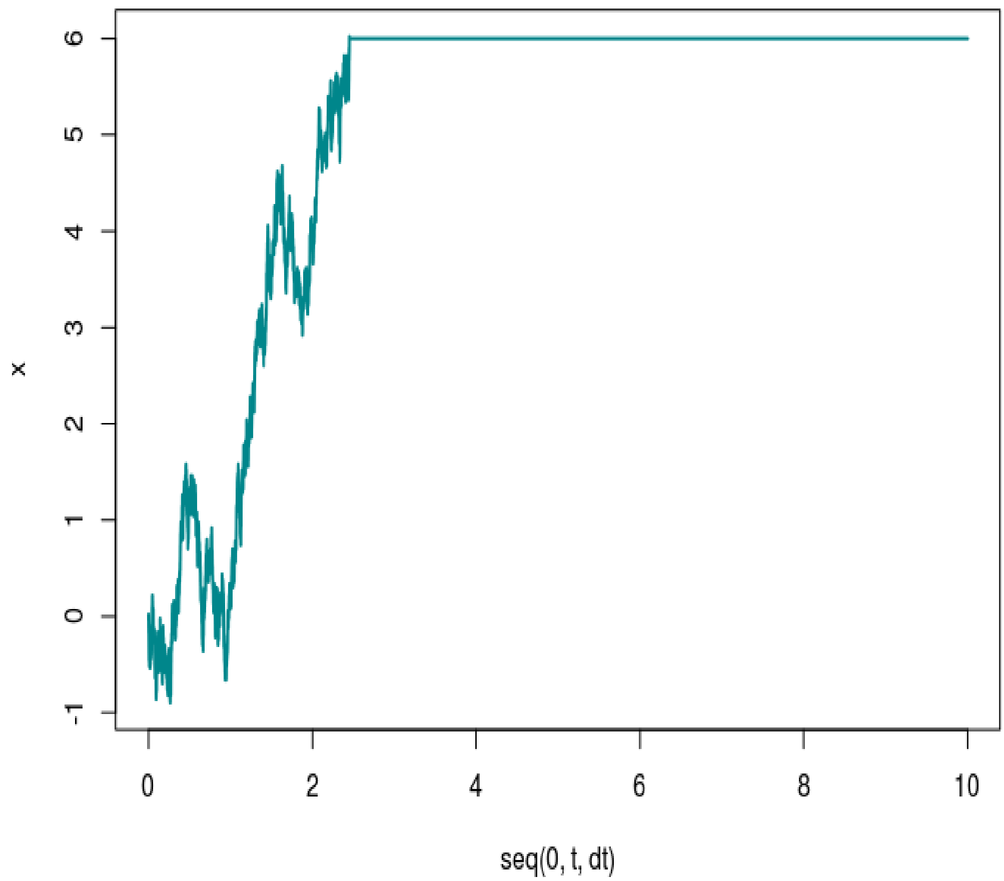
LHS: We must exclude all paths containing states outside of $[b+1, a-1]$. RHS: Recall that $\Pr(X_n = j) = \binom{n}{(n-j)/2} p^{(n+j)/2} q^{(n-j)/2}$, $j = \{-n, -n+2, ..., n-2, n\}$, where p is the transition probability to a higher state. If $p > q$, then the mean of the jump is $\mu = p - q > 0$, and $E[X_n] = n(p-q) > 0$. On applying CLT to the process $\{Z\}$, or the unrestricted random walk, and using continuity correction, $Pr(a \leq X_n \leq b) \approx \frac{1}{\sigma\sqrt{n}\sqrt{2\pi}}\int_{a-\frac{1}{2}}^{b+\frac{1}{2}} e^{-\frac{1}{2}\left(\frac{x-n\mu}{\sigma\sqrt{n}}\right)^2} dx$. This normal approximation is useful to give us a range in which $\{X_n\}$ will lie. We know that $Pr(-4\sigma \leq x - \mu \leq 4\sigma) > 0.9999$ for a normal RV. in our case that is : $\Pr(-4\sigma\sqrt{n} \leq X_n - n\mu \leq 4\sigma\sqrt{n}) > 0.9999$ The random walk has drift $n\mu$ and the deviation from it is proportional to $\sqrt{n}$. This gives : $Pr(|X_n - n\mu| \leq 4\sigma\sqrt{n}) > 0.9999$, i.e. $X_n = n\mu + O(\sqrt{n})$ or $X_n = n\mu(1 + O(\sqrt{1/n}))$ In the limit $n \to \infty$, $X_n = n\mu$. Therefore $Pr(X_n \; becomes \; arbitrarily \; big) \to 1$ as $n \to \infty$ Let us fix j. Then, $Pr(j \leq X_n \leq \infty) = 1 - \phi\left[\frac{j-0.5-\mu n}{\sigma\sqrt{n}}\right]$ which as $n \to \infty$ becomes $Pr(j \leq X_n) \to 1$. Using the strong law of large numbers : for an $\epsilon > 0$ the probability that X_n in $(n(\mu-\epsilon), n(\mu+\epsilon))$ for all $n > m$ can be made as close to 1 as we want it to be by choosing a large enough m. So for any j we have : $Pr(X_n > j, X_{n+1} > j, ...) \to 1$ as $m \to \infty$. So, as $n \to \infty$ so does j. And then, since $\mu > 0$, X_n drifts to ∞ with probability 1. What if $\mu < 0$? So, as $n \to \infty$ the unrestricted random walk drifts to $+/-\infty$ with probability 1, depending on μ. So, the probability that it assumes values in $[b+1, a-1]$ as $n \to \infty$ is 0.

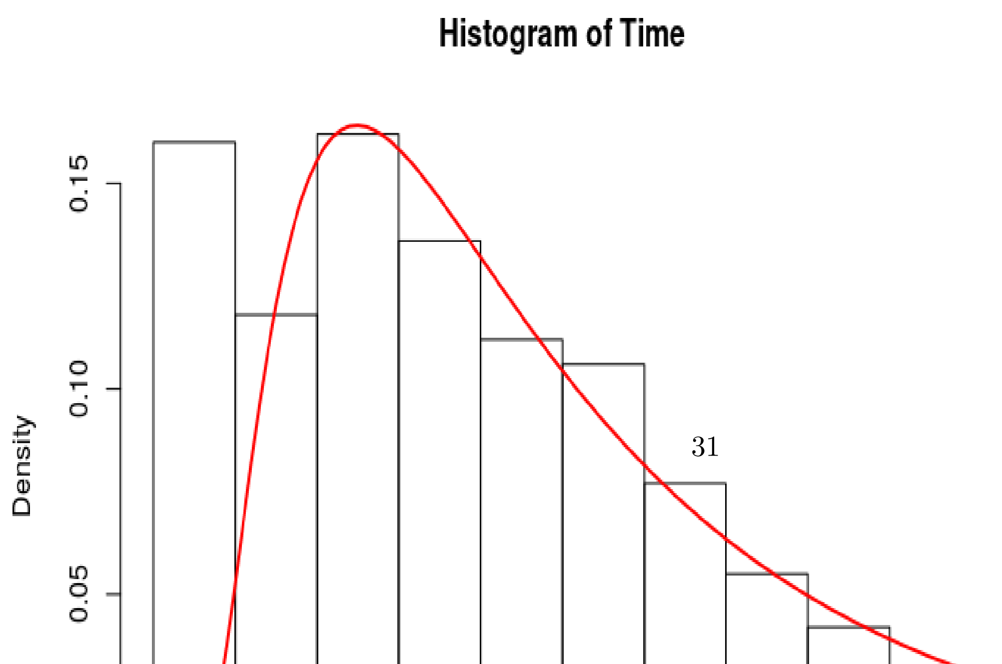
Therefore the probability that the process is still in motion as $n \to \infty$ is not more than the probability that the unrestricted process will take a value in $[b+1, a-1]$. The RHS is 0. Therefore the LSH is 0. This means the probability that the process is still in motion as $n \to \infty$ is 0. This means that absorption is certain! Figure 6 illustrates that, while also showing how time is normally distributed.

```
# R-code for Figure 6.
set.seed(37)
rep <- 1000
par(mfrow=c(1,1))
a <- 6 # Absorbing barrier, start at x0 = 0 < a = 6.

dt <- 1e-04
t <- 10
n <- t/dt
mu <- 1
sd <- 2
p <- (1+mu*sqrt(dt)/sd)/2
q <- 1-p
dx <- sd*sqrt(dt)

z <- numeric(n+1)
x <- numeric(n+1)
index <- numeric(rep)
z[1] <- 0
```

```r
x[1] <- 0

for(k in 1:rep){
  z <- numeric(n+1)
  x <- numeric(n+1)
  for(i in 2:(n+1)){
    if(x[i-1]<a){
    z[i] <- rbinom(1,1,p)
      if(z[i]==0){
        z[i] <- -dx}else{
            z[i] <- dx}
      x[i] <- x[i-1]+z[i]
      if(abs(x[i]-a)<10e-06){index[k] <- i-1}
          }else{
      x[i] <- a}
  }
}
match(a,x)
plot(seq(0,t,dt),x,lwd=2,col="turquoise4",type = "l")
Time <- index*dt
hist(Time,probability = T)
curve(exp(-(a-mu*x)^2/(2*sd^2*x))*a/(sd*(sqrt(2*pi*x^3))),add=T,col="red",lwd=2)
```

Figure 6: Weiner process with absorption barrier





3.4 Fokker-Planck diffusion equation

Till now we have assumed that the process is moving. Now, we consider a more general approach. Also, we have said that the transition probabilities are independent of x. In practice, they do depend on x. Let θ_k be the transition probability from $k \to k+1$ and ϕ_k be the transition probability of moving from $k \to k-1$. Then, $1-\theta_k-\phi_k$ is the probability of staying in k. The diffusion equation is : $p_{j,k}^{(n+1)} = pp_{j,k-1}^{(n)} + qp_{j,k+1}^{(n)}$, which now becomes : $p_{j,k}^{(n+1)} = \theta_{k-1}p_{j,k-1}^{(n)} + (1-\theta_k-\phi_k)p_{j,k}^{(n)} + \phi_{k+1}p_{j,k+1}^{(n)}$. For now, we fix dx and dt. If the particle is at x at time t, then at time $t+dt$ the probability of being at : $x+dx$ is $\theta(x)$ $x-dx$ is $\phi(x)$ x is $1-\theta(x)-\phi(x)$.

Denote : $p(x_0, x; t)$ be the conditional probability that the particle is at x at time t, given that it started at x_0 at time 0.

Then; $p(x_0, x; t+dt)dx = p(x_0, x-dx, t)dx(\theta(x-dx)) + p(x_0, x; t)dx(1-\theta(x)-\phi(x)) + p(x_0, x+dx, t)(\phi(x+dx))$.

We want to make sure the process makes sense as $dx, dt \to 0$. Let $\beta(x)$ be the instantaneous mean per unit time of the change in $X(t) = x$. Let $\alpha(x)$ be the instantaneous variance per unit time of the change in $X(t) = x$. Therefore,

$$\beta(x) = \lim_{dt \to 0} \frac{E[X(t+dt) - X(t)|X(t) = x]}{dt} \tag{30}$$

$$\alpha(x) = \lim_{dt \to 0} \frac{Var[X(t+dt) - X(t)|X(t) = x]}{dt} \tag{31}$$

Now in $(t, t+dt)$, E$[X] = \sum xp(x) = dx(\theta(x)) + 0(1-\theta(x)-\phi(x)) - dx(\phi(x)) = (\theta(x)-\phi(x))dx$ is the mean change in position in time interval $(t, t+dt)$.

$Var(x) = E[(X-E[x])^2] = E[x^2] - (E[x])^2 = [dx^2(\theta(x)) + 0^2(1-\theta(x)-\phi(x)) + dx^2(\phi(x))] - [(\theta(x) - \phi(x))dx]^2 = [\theta(x)+\phi(x)-(\theta(x)-\phi(x))^2]dx^2$

And this we place in the limit to get : $\beta(x) = \lim_{dx,dt\to 0} \frac{(\theta(x)-\phi(x))dx}{dt}$ and $\alpha(x) = \lim_{dx,dt\to 0} \frac{[\theta(x)+\phi(x)-(\theta(x)-\phi(x))^2]dx^2}{dt}$.

Let $\alpha(x) \leq A$, for all $x, A > 0$. Take $(dx)^2 = Adt$. We substitute this in the above limits to get : $\frac{\beta(x)dx}{A} = \theta(x) - \phi(x)$ and $\frac{\alpha(x)}{A} + (\frac{\beta dx}{A})^2 = \theta(x)+\phi(x) \approx \frac{\alpha(x)}{A}$. Verify

On adding the two we get : $\theta(x) = \frac{1}{2A}(\alpha(x)+\beta(x))dx$ and $\phi(x) = \frac{1}{2A}(\alpha(x)-\beta(x))$.

We have imposed that $\alpha(x) \leq A$, i.e. $\frac{\alpha(x)}{A} \leq 1$, i.e. $\theta(x)+\phi(x) \leq 1$, which is obvious as that is the probability of movement.

We now use the Taylor's series expansion on our FKE with $p(x_0, x; t) = p$: $p+dt\frac{\partial p}{\partial t} \approx \left(p - dx\frac{\partial p}{\partial x} + \frac{dx^2}{2!}\frac{\partial^2 p}{\partial x^2}\right)\left(\theta - dx\theta' -\right.$ $p(1-\theta-\phi) + \left(p + dx\frac{\partial p}{\partial x} + \frac{dx^2}{2!}\frac{\partial^2 p}{\partial x^2}\right)\left(\phi - dx\phi' + \frac{dx^2}{2}\phi''\right)$. What does the $'$ mean?

$dt\frac{\partial p}{\partial t} \approx p\left(dx(-\theta'+\phi') + \frac{dx^2}{2}(\theta''+\phi'')\right) + \frac{\partial p}{\partial x}\left(dx(-\theta+\phi) + dx^2(\theta'+\phi') + \frac{dx^3}{2}(-\theta''+\phi'')\right) + \frac{\partial^2 p}{\partial x^2}\left(\frac{dx^2}{2}(\theta+\phi) - \frac{dx^3}{2}(-\right.$

Substitute $Adt = dx^2$ $\frac{dx^2}{A}\frac{\partial p}{\partial t} \approx p\left(dx(-\theta'+\phi') + \frac{dx^2}{2}(\theta''+\phi'')\right) + \frac{\partial p}{\partial x}\left(dx(-\theta+\phi) + dx^2(\theta'+\phi') + \frac{dx^3}{2}(-\theta''+\phi'')\right) + \frac{\partial^2 p}{\partial x^2}\left(\frac{dx^2}{2}(\theta+\phi) - \frac{dx^3}{2}(-\theta'+\phi') + \frac{dx^4}{4}(\theta''+\phi'')\right)$.

Divide by dx^2 to get : $\frac{1}{A}\frac{\partial p}{\partial t} \approx p\left(\frac{1}{dx}(-\theta'+\phi') + \frac{1}{2}(\theta''+\phi'')\right) + \frac{\partial p}{\partial x}\left(\frac{1}{dx}(-\theta+\phi) + (\theta'+\phi') + \frac{dx}{2}(-\theta''+\phi'')\right) + \frac{\partial^2 p}{\partial x^2}\left(\frac{1}{2}(\theta+\phi) - \frac{dx}{2}(-\theta'+\phi') + \frac{dx^2}{4}(\theta''+\phi'')\right)$.

We got : $\theta = \frac{\alpha+\beta dx}{2A}$ and $\phi = \frac{\alpha-\beta dx}{2A}$ So then, $\theta' = \frac{\alpha'+\beta' dx}{2A}$ and $\phi' = \frac{\alpha'-\beta' dx}{2A}$ $\theta'' = \frac{\alpha''+\beta'' dx}{2A}$ and $\phi'' = \frac{\alpha''-\beta'' dx}{2A}$. Using these values in the above expansion : $\frac{\partial p}{\partial t} \approx p\left(-\beta' + \frac{1}{2}(\alpha'')\right) + \frac{\partial p}{\partial x}\left(-\beta + \alpha' + \frac{dx}{2}(-\beta'' dx)\right) + \frac{\partial^2 p}{\partial x^2}\left(\frac{1}{2}(\alpha) - \frac{dx}{2}(-\theta'+\phi') + \frac{dx^2}{4}(\theta''+\phi'')\right)$

As $dx, dt \to 0$: $\frac{\partial p}{\partial t} = p\left(-\beta' + \frac{1}{2}(\alpha'')\right) + \frac{\partial p}{\partial x}\left(-\beta+\alpha'\right) + \frac{\partial^2 p}{\partial x^2}\left(\frac{1}{2}(\alpha)\right) = p\left(\frac{1}{2}\frac{d^2\alpha}{dx^2} - \frac{d\beta}{dx}\right) + \frac{\partial p}{\partial x}\left(\frac{d\alpha}{dx} - \beta\right) + \frac{\partial^2 p}{\partial x^2}\left(\frac{\alpha}{2}\right) = \frac{1}{2}\left(p\frac{d^2\alpha}{dx^2} + 2\frac{\partial p}{\partial x}\frac{d\alpha}{dx} + \frac{\alpha}{2}\frac{\partial^2 p}{\partial x^2}\right) - \left(p\frac{d\beta}{dx} + \beta\frac{\partial p}{\partial x}\right) = \frac{1}{2}\frac{\partial^2}{\partial x^2}(p\alpha(x)) - \frac{\partial}{\partial x}(p\beta(x))$.

Therefore $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(p\alpha(x)) - \frac{\partial}{\partial x}(p\beta(x))$, which is the Forward Fokker-Planck equation.

The solution $p(x_0, x; t)$ is defined by $\alpha(x), \beta(x)$. Comparing to the WP : $\theta(x) + \phi(x) = 1$ and $A = \sigma^2$.

The backward equation follows similarly, and is with respect to x_0.

Now, we simulate the simple immigration death diffusion process. Population birth rate $= \lambda_N = \alpha$ Population death rate $= \mu_N = \mu N$. On making population a diffusion process i.e. replacing $N = 0, 1, 2, ...$ with $dx \geq 0$ we get : $E[X(t+dt) - X(t)|X(t) = x] = (\lambda_x - \mu_x)dt = (\alpha - \mu x)dt$ and $Var[X(t+dt) - X(t)|X(t) = x] = (\lambda_x + \mu_x)dt - [(\lambda_x + \mu_x)dt]^2$. $\beta(x) = \lim_{dt\to 0}(\alpha - \mu x)dt/dt = \alpha - \mu x$ $\alpha(x) = \lambda_x + \mu_x = \alpha + \mu_x$.

$\theta(x) = \frac{1}{2A}(\alpha(x) + \beta(x))dx = \frac{1}{2A}(\alpha + \mu(x) + (\alpha - \mu(x))dx)$ and $\phi(x) = \frac{1}{2A}(\alpha(x) - \beta(x)) = \frac{1}{2A}(\alpha + \mu(x) - (\alpha - \mu(x))dx)$

We also need $\alpha(x) \leq A$. Here it is $\alpha + \mu x \leq A$, which is clearly unbounded. We obviously must bound it. To convert a continuous time population process on $N = 0, 1, 2, \cdots$ to a diffusion process on $x \geq 0$ with an unbounded $\alpha(x)$, we need to define a pragmatic maximum. $X^{\max} = \max\{x(0), k_1(\infty) + 6\sqrt{k_2(\infty)}\}$, where $k_i's$ are the equilibrium means.

On simulation the diffusion and population processes broad structure seems similar at the macro level. At the micro level there are rapid fluctuations of the diffusion process are very different from the $exp(\lambda_N + \mu_N)$ distributed horizontal steps of the population process.

We parallel diffusion process to the population process for continuous time general birth-death. In the population process the time to the next event in exponentially distributed with parameter $\lambda_N + \mu_N$. In diffusion the number of steps M to the next event follows geometric distribution $\Pr(M = m) = pq^{m-1}, m = 1, 2, \cdots,$ where $p = \theta(x) + \phi(x)$ is the probability of change and $q = 1 - p$.

Then $\Pr(M \leq m) = p(1 + q + \cdots + q^{m-1}) = 1 - q^m$. Let $\{U\}$ be a sequence of i.i.d. $U(0, 1)$. We choose $M = i$ if $U \leq 1 - q^i$. Therefore, $M = m$ if $1 - q^{m-1} < U \leq 1 - q^m$. On replacing U with $1 - U$; $(m - 1)\ln(q) > 1 - U \geq m\ln(q)$, Which gives $m - 1 < \ln(1 - U)/\ln(q) \leq m$, which is the same as : $m - 1 < \ln(U)/\ln(q) \leq m$, since even $1 - U$ is a $U(0, 1)$ random variable. Then we may take :

$$m = 1 + [(\ln(U)/\ln(q))].$$

Figures 7 and 8 show the immigration death processes at different parameter values.

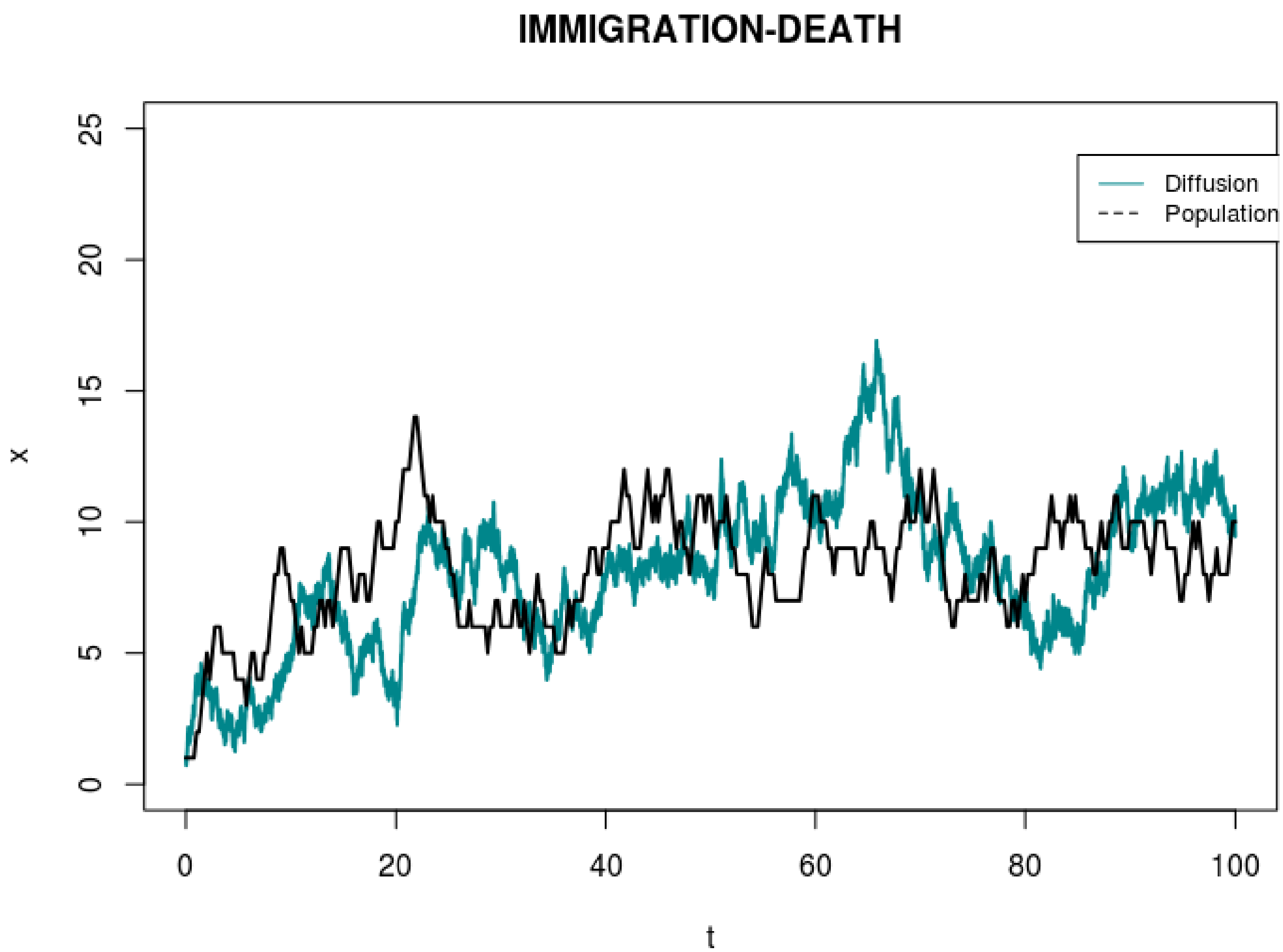
3.5 (a)

```
# R-code for Figure 7.
set.seed(37)
par(mfrow=c(1,1))
alpha <- 1 # Constant birth rate.
mu <- 0.1 # death rate.
x0 <- 1
mean_inf <- alpha/mu # since at eqbm prob ~ poi(alpha/mu)
var_inf <- alpha/mu
x_max <- max(x0,mean_inf+6*sqrt(var_inf)) # pragmatic max
a <- alpha+mu*x_max # to impose alpha(x) <= A.
A <- ceiling(a) # choose closest upper integer to a.
dx <- c(0.01,1)
dt <- dx^2/A
t <- 100
n <- t/dt

for(k in 1:length(dx)){
```

```
  U <- numeric(n[k]) # store U(0,1) rv
  x <- numeric(n[k])
  theta <- numeric(n[k]) # store transition prob to birth.
  phi <- numeric(n[k]) # store transition prob to die.
  U[1] <- runif(1,0,1)
  x[1] <- x0

  theta[1] <- (alpha+mu*x[1]+(alpha-mu*x[1])*dx[k])/(2*A)
  # diffusion transition prob to go up
  phi[1] <- (alpha+mu*x[1]-(alpha-mu*x[1])*dx[k])/(2*A)
  # diffusion transition prob to go down
for(i in 1:n[k]){
U[i+1] <- runif(1,0,1)
theta[i+1] <- (alpha+mu*x[i]+(alpha-mu*x[i])*dx[k])/(2*A)
phi[i+1] <- (alpha+mu*x[i]-(alpha-mu*x[i])*dx[k])/(2*A)
if(U[i+1]<=theta[i+1]){
  x[i+1] <- x[i]+dx[k]
}else{
  if(U[i+1]<=theta[i+1]+phi[i+1]){
    x[i+1] <- x[i]-dx[k]
  }else{
    x[i+1] <- x[i]
  }
}
}
# shows similar broad structure, even though the rapid diffusion fluctuations are very smal
if(k==1){  # exponential(lamdan + mun) population horizontal steps.
plot(seq(0,t,dt[k]),x,lwd=2,col="turquoise4",type = "l",main=paste("IMMIGRATION-DEATH"),xla
  legend(85,24, legend=c("Diffusion", "Population"),col=c("turquoise4", "black"), lty=1:2,
}else{
  lines(seq(0,t,dt[k]),x,lwd=2,col="black")
}
```

```
}
```

Figure 7: IMMIGRATION-DEATH



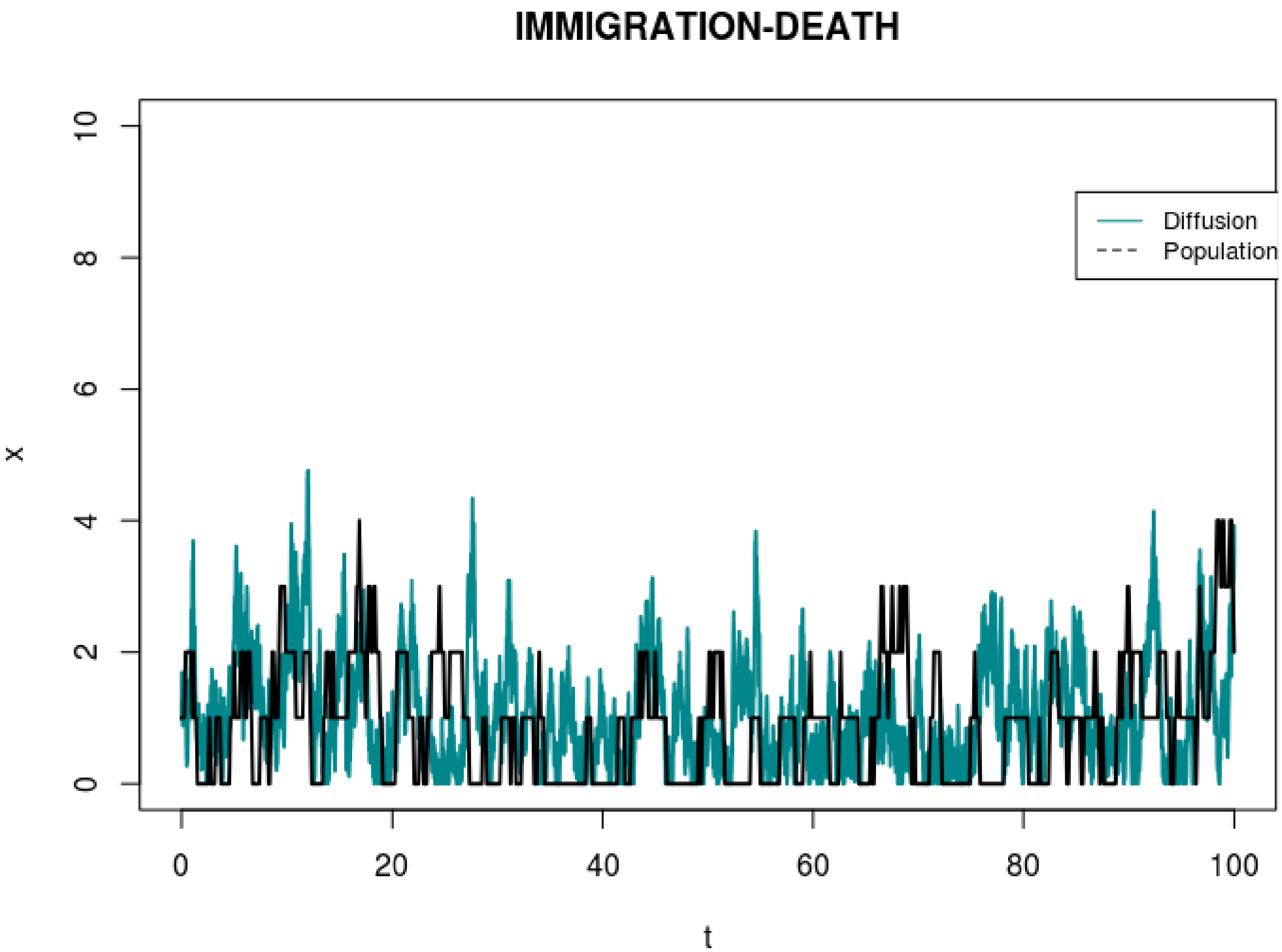
We have replaced death rate $\phi(0)$ with 0, since we do not want excursions below 0.

```
# R-code for Figure 8.
set.seed(37)
par(mfrow=c(1,1))
alpha <- 1 # Constant birth rate.
```

```
mu <- 1 # death rate.
x0 <- 1
mean_inf <- alpha/mu # since at eqbm prob ~ poi(alpha/mu)
var_inf <- alpha/mu
x_max <- max(x0,mean_inf+6*sqrt(var_inf)) # pragmatic max
a <- alpha+mu*x_max # to impose alpha(x) <= A.
A <- ceiling(a) # choose closest upper integer to a.
dx <- c(0.01,1)
dt <- dx^2/A
t <- 100
n <- t/dt

for(k in 1:length(dx)){
  U <- numeric(n[k]) # store U(0,1) rv
  x <- numeric(n[k])
  theta <- numeric(n[k]) # store transition prob to birth.
  phi <- numeric(n[k]) # store transition prob to die.
  U[1] <- runif(1,0,1)
  x[1] <- x0

  theta[1] <- (alpha+mu*x[1]+(alpha-mu*x[1])*dx[k])/(2*A) # diffusion transition prob to go
  phi[1] <- (alpha+mu*x[1]-(alpha-mu*x[1])*dx[k])/(2*A)  # diffusion transition prob to go
  for(i in 1:n[k]){
    U[i+1] <- runif(1,0,1)
    theta[i+1] <- (alpha+mu*x[i]+(alpha-mu*x[i])*dx[k])/(2*A)
    phi[i+1] <- (alpha+mu*x[i]-(alpha-mu*x[i])*dx[k])/(2*A)
    if(U[i+1]<=theta[i+1]){
      x[i+1] <- max(0,x[i]+dx[k])
    }else{
      if(U[i+1]<=theta[i+1]+phi[i+1]){
        x[i+1] <- max(0,x[i]-dx[k])
      }else{
```

```
        x[i+1] <- max(0,x[i])
      }
    }
  }   # shows similar broad structure, even though the rapid diffusion fluctuations are ver
  if(k==1){  # exponential(lamdan + mun) population horizontal steps.
    plot(seq(0,t,dt[k]),x,lwd=2,col="turquoise4",type = "l",main=paste("IMMIGRATION-DEATH")
    legend(85,9, legend=c("Diffusion", "Population"),col=c("turquoise4", "black"), lty=1:2,
  }else{
    lines(seq(0,t,dt[k]),x,lwd=2,col="black")
  }
}
```

Figure 8: IMMIGRATION-DEATH WITH NO EXCURSIONS BELOW 0



Parallel results with WP reflecting barrier at 0, to prevent excursions for $X(t)$ below 0.

We wish to regard i) restarting the process from empty and ii) immigration as two different parts. We hold empty process for Exp(gamma) time before restarting it from delta, an interior point. Placing gamma = alpha, we get the Exp(alpha), corresponding to a new immigrant restarting the population process from empty. delta = 1, is like that an immigrant arrives and immediately $X(t) = 0$ becomes $\delta > 1$ means mass immigration.

4 Ornstein-Uhlenbeck(OU) process

In the Wiener process, $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\sigma}{\sqrt{\mathrm{d}t}}$. As $\mathrm{d}t \to 0$, the velocity drifts to ∞. In cases where the velocity of the process is crucial to the dynamics of the process, then this is not going to be okay.

So, let us model the velocity $U(t)$ of the process. In a correlated randomn walk, it is assumed that a particle continues to move in it's direction with velocity 1 unless it suffers an impact from a neighbouring particle, which cause a reverse of direction and therefore velocity becomes -1. Ornstein-Uhlenbeck designed a process that generalises this, where the velocity is not binary anymore.

In time interval (t,t+dt), 2 things affect change in momentum : 1. Frictional resistance = β of the surrounding medium induces change in $U(t)$ proportional to itself. 2. The particle suffers from random impacts with neighbouring particles whose effect in successive time intervals can be represented by independent random variables with mean 0, i.e. increments of a WP with no drift.

The stochastic differential equation (SDE) is : $\mathrm{d}U(t) = -\beta U(t)\mathrm{d}t + \mathrm{d}Y(t)$, where $Y(t)$ is the unrestricted Wiener process, with variance parameter σ^2. Let $X(t)$ be the displacement at time t. Then $\mathrm{d}X(t) = X(t + \mathrm{d}t) - X(t)$ is the change in time $\mathrm{d}t$. $\mathrm{d}U(t) = U(t+\mathrm{d}t) - U(t)$ is the corresponding change in velocity. We know that $Y(t) \sim \mathcal{N}(\mu t, \sigma^2)$. Then, $\mathrm{d}Y(t) = \mu\mathrm{d}t + \sigma Z(t)\sqrt{\mathrm{d}t}$, where $Z(t)$ is a random Gaussian process with mean 0 and variance 1. This means that the change is $X(t)$ in time $\mathrm{d}t$ is a normal variate with mean $E[\mathrm{d}Y(t)] = \mu\mathrm{d}t$ and variance $Var[\mathrm{d}Y(t)] = \sigma^2\mathrm{d}t$. These changes are independent of $X(t)$ and also of the change during another time interval. Therefore $Cov[Y(t), Y(s)] = 0$. Does it hold true if $t = s$?

Since $\mu = 0$, we have $\mathrm{d}U(t) = -\beta U(t)\mathrm{d}t + \sigma Z(t)\sqrt{\mathrm{d}t}$. The general SDE is : $\mathrm{d}X(t) = \beta(x,t)\mathrm{d}t + Z(t)\sqrt{\alpha(x,t)\mathrm{d}t}$. On comparing the two, we see that $U(t)$ is a diffusion process with mean $\beta(u,t) = -\beta u$ and variance $\alpha(u,t) = \sigma^2$. The forward Fokker Planck equation is : $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(p\alpha(x)) - \frac{\partial}{\partial x}(p\beta(x))$, which here becomes : $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}(p\sigma^2) - \frac{\partial}{\partial x}(p(-\beta u))$ i.e. $\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}(p) + \beta\frac{\partial}{\partial x}(pu)$.

Now, $\frac{\partial M}{\partial \theta} = -\int_{-\infty}^{\infty} e^{u\theta}updu$, where M is the MGF of U(t). As we did to find a solution to the WP, using the MGF, we reduce the Forward Fokker-Planck equation by multiplying it with $e^{u\theta}$, and integrating over wrt u to get : $\frac{\partial M}{\partial t} = \frac{1}{2}\sigma^2\theta^2 M - \beta\theta\frac{\partial M}{\partial \theta}$ Now, $\mathrm{d}U(t) = -\beta U(t)\mathrm{d}t + \sigma Z(t)\sqrt{\mathrm{d}t}$ i.e. $\mathrm{d}U(t) = -\beta U(t)\mathrm{d}t + \mathrm{d}Y(t)$ Let $V = e^{\beta t}U$. Then, $dV = \beta e^{\beta t}U\mathrm{d}t + e^{\beta t}\mathrm{d}U$, and on inserting $\mathrm{d}U$ from above, we get that : $dV = e^{\beta t}\mathrm{d}Y = e^{\beta t}\sigma Z(t)\sqrt{\mathrm{d}t} = e^{\beta t}\sigma\mathrm{d}t$. So then, $V = V(0) + \sigma\int_0^t e^{\beta s}\mathrm{d}N$ On taking expectation we see that the integral term vanishes because $E[\mathrm{d}N] = 0$, and so $E[V(t)] = E[V(0)] = u_0$, which then means that $E[U(t)] = u_0 e^{\beta t}$.

Similarly, on taking variance, $Var[V(t)] = \sigma^2 var\left[\int_0^t e^{\beta t}\mathrm{d}N\right] = \sigma^2 E\left[\int_0^t e^{2\beta t}\mathrm{d}t - 0\right]$. Finally on solving we get that $Var[U(t)] = \frac{\sigma^2}{2\beta}(1 - e^(-2\beta t))$

The solution is a signature of Normal distribution. We then get $E[U(t)] = u_0 e^{-\beta t}$ and $Var[U(t)] = \frac{\sigma^2}{2\beta}(1 - e^{-2\beta t})$. $t \to \infty$, $U(\infty) \sim \mathcal{N}(0, \sigma^2/2\beta)$. $U(t)$ is the OU process. Since $X(t)$ is displacement and $U(t)$ is it's velocity, $dX(t) = U(t)dt$. Then $X(t) - X(0) = \int_0^t U(s)ds$.

Since U(t) is normal, so is X(t). $E[X(t) - X(0)] = \int_0^t E[U(s)]ds = u_0(1 - e^{-\beta t})/\beta$, and $Var[X(t) - X(0)] = \int_0^t \int_s^t Cov[U(s)U(w)]dwds$. Work the expectation out.

Let the process be in equilibrium, then U(t) ahs 0 mean, and the so does X(t). $Cov(U(s), U(w)) = E\left[U(s)E[U(w)|U(s)]\right], w > s$ Since $E[U(t)] = u_0 e^{-\beta t}$, we have $E[U(s)|U(w)] = U(s)e^{-\beta(w-s)}$ which means that $Cov(U(s), U(w)) = E\left[(U(s))^2 e^{-\beta(w-s)}\right] = Var[U(s)]e^{-\beta(w-s)} = \frac{\sigma^2}{2\beta}e^{-\beta(w-s)}$.

In equilibrium, correlation coefficient between $U(s)$ and $U(w)$ is

$$\frac{E\left[U(w) - E[U(w)])(U(s) - E[U(s)])\right]}{\sqrt{Var[U(w)]Var[U(s)]}} = E[U(w)U(s)]/\sqrt{Var[U(w)]Var[U(s)]} = \sigma^2 e^{-\beta(w-s)}/(2\beta\sqrt{(\sigma^2/2\beta)^2}) = e^{-\beta(w-$$

Therefore $Var[X(t) - X(0)] = \int_0^t \int_s^t Cov[U(s)U(w)]dwds = \sigma^2/2\beta \int_0^t \int_s^t e^{-\beta(w-s)}dwds = \frac{\sigma^2}{2\beta}(\beta t - 1 + e^{-\beta t})$.

When t is small, $Var[X(t) - X(0)] \approx (\sigma^2\beta/4)t^2$. Therefore, as $t \to \infty$, in WP the velocity $\to \infty$, here it is bounded.

```
# R-code for Figure 9.
rep <- 100
t <- 500
beta <- 1
sd <- 1
dx <- 0.01
dt <- 0.01
x0 <- 0
u0 <- 0
n <- t/dt

x <- numeric(n+1)
u <- numeric(n+1)
X <- numeric(rep)
```

```
U <- numeric(rep)

for(j in 1: rep){
for(i in 2:(n+1)){
  u[i] <- rnorm(1,u0*exp(-beta*t),sqrt(0.5*(1/beta)*sd^2*(1-exp(-2*beta*t))))
  x[i] <- rnorm(1,0,sqrt((sd^2/(2*beta))*(beta*t-1+exp(-beta*t))))
}
  X[j] <- x[n+1]
  U[j] <- u[n+1]
}
plot(seq(0,t,dt),u,lwd=2,col="turquoise4",type = "l",main=paste("O-U␣PROCESS"),xlab="t",yli
points(seq(0,t,dt),x,lwd=2,col="black")
legend(85,24, legend=c("Diffusion", "Population"),col=c("turquoise4", "black"), lty=1:2, ce

hist(X,probability = T)
sigx <- sqrt(sd^2/(2*beta)*(beta*t-1+exp(-beta*t)))
curve(1/(sqrt(2*pi)*sigx)*exp(-0.5*(x/sigx)^2),add=T,col="turquoise4",lwd=2)

hist(U,probability = T)
meanu <- u0*exp(-beta*t)
sigu <- sqrt(sd*2/(2*beta)*(1-exp(-2*beta*t)))
curve(1/(sqrt(2*pi)*sigu)*exp(-0.5*((x-meanu)/sigu)^2),add=T,col="turquoise4",lwd=2)

incx <- numeric(n)
incu <- numeric(n)

for(i in 1:n){
  incx[i] <- x[i+1] - x[i]
  incu[i] <- u[i+1] - u[i]
}
plot(seq(0,t,dt)[-1],incu,lwd=2,col="turquoise4",type = "l",
        main=paste("O-U␣PROCESS"), xlab="t",ylim=c(-15,15))
```

```r
plot(seq(0,t,dt)[-1],incx,lwd=2,col="turquoise4",type = "l",
        main=paste("O-U PROCESS"), xlab="t",ylim=c(-15,15))
hist(incx,probability = T,ylim = c(0,0.03))
hist(incu,probability =T)
```

Now, since both WP and OU have normal solutions, there must exist a transform that interchanges the 2. Let $Y(t)$ be a WP, with $Y(t) \sim \mathcal{N}(0,1)$. Then for continuous functions of t, g and h, $X(t) = g(t)Y(h(t))$ is also normal.

$$\mathrm{d}X(t) = g'(t)Y(h(t))\mathrm{d}t + g(t)\mathrm{d}Y(h(t)).$$

Hint : Y is a WP. On direct comparison with the general SDE, the infinitesimal mean $\beta(x,t) = (g'/g)x$ and the infinitesimal variance is g^2h' Let $g(t) = e^{-\beta t}$ and $h(t) = \frac{\sigma^2}{2\beta}e^{2\beta t}$ Then we observe : The infinitesimal mean of $X(t)$ is: $e^{-\beta t}(-\beta x)/e^{-\beta t} = -\beta x$ and the infinitesimal variance of $X(t)$ is : $e^{-2\beta t}\sigma^2/(2\beta)e^{2\beta t}2\beta = \sigma^2$. Therefore X(t) is OU. The inverse transform

$$Y(t) = \sqrt{\sigma^2/2\beta}X(1/2\beta \ln(2\beta t/\sigma^2))$$

gives us the WP process. Verify.