Modified projection method for second kind Fredholm integral equations with trignometric polynomials.

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Abstract

We consider a second kind Fredholm integral equation with a periodic kernel. Modified projection method is applied to such an integral equation with the approximating space to be the space of trigonometric polynomials and orders of convergence are obtained. This method proves to be better than the classical Galerkin method. A numerical example which arises from a Dirichlet problem for the Laplace equation is considered to illustrate our theoretical results.

Key Words : Fredholm integral operator, trigonometric polynomials, modified projection method.

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1 Introduction

Let $X = L^2[0,1]$, be the space of Lebesgue measurable square integrable functions. Consider the integral operator

$$Tx(s) = \int_0^1 k(s,t)x(t)dt,\ s \in [0,1],$$

with $k(\cdot,\cdot) \in C([0,1]\times[0,1])$, the space of continuous functions on $[0,1]\times[0,1]$. There are many classical methods such as the Nyström method and projection methods available to solve the equation

$$u - Tu = f,\ f \in X,$$

approximately. An account of this can be found in Atkinson [1]. Let X_n be the space of piecewise polynomials of degree $\leq (r-1)$, with respect to a uniform partition

$$0 < \frac{1}{n} < \frac{2}{n} < \ldots < \frac{n-1}{n} < 1,$$

of $[0,1]$. Let $h = \frac{1}{n}$. Let $\pi_n : X \to X_n$ be the orthogonal projection. Let u_n^G denote the Galerkin solution and u_n^S denote the iterated Galerkin or the Sloan solution. u_n^G solves the equation

$$u_n^G - \pi_n T u_n^G = \pi_n f \tag{1.1}$$

and u_n^S solves

$$u_n^S - T\pi_n u_n^S = f.$$

A modified projection method was introduced in Kulkarni [2], which involves solving

$$u_n^M - T_n^M u_n^M = f, \tag{1.2}$$

where

$$T_n^M = \pi_n T \pi_n + \pi_n T(I-\pi_n) + (I-\pi_n)T\pi_n.$$

Define the iterated modified projection solution by

$$\tilde{u}_n^M = Tu_n^M + f.$$

Let $\Delta \subset \mathbb{R}$. Let $C^r(\Delta)$ be the space of continuous functions on Δ. Assume that $k(\cdot,\cdot) \in C^r([0,1]\times[0,1])$, $f \in C^r[0,1]$, so that $u \in C^r[0,1]$. Then it is known that

$$\|u - u_n^G\| = O(h^r)$$

and

$$\|u - u_n^S\| = O(h^{2r}).$$

(See Atkinson [1].) But for the modified projection method, we have

$$\|u - u_n^M\| = O(h^{3r})$$

and

$$\|u - \widetilde{u}_n^M\| = O(h^{4r}).$$

(See Kulkarni [2].)

Thus, the modified projection solution and its iterated version provides a better approximation to the actual solution than the classical Galerkin or the Sloan solution. Dirichlet problem with Laplace's equation gives rise to integral operators with periodic kernels. If the kernel $k(s,t)$ is periodic in both the variables with period 2π and f is periodic, then the solution u is 2π periodic. Instead of using local approximation by piecewise polynomials, quite often it is preferable to use global approximation in the form of trigonometric polynomials, since this leads to rapid convergence. Trigonometric polynomial approximations is used by Atkinson [1] to solve boundary integral equations on smooth planar regions. Sometimes such methods are referred as Spectral methods. Atkinson [1] has considered the Galerkin and the iterated Galerkin method, when the approximating space X_n is the space of trigonometric polynomials. We study the orders of convergence of the modified projection solution and its iterated version, when the approximating space is the space of trigonometric polynomials. In this paper, we shall see that the new method with trigonometric polynomials has an improved rate of convergence than the Galerkin method. Another important example of globally defined smooth polynomial approximations is the use of spherical polynomials to functions defined on the unit sphere in $\mathbb{R}^3$. In future we will like to study the modified projection method with spherical polynomial approximations to solve a boundary integral equation on smooth closed surfaces in $\mathbb{R}^3$.

The paper is organised as follows. In Section 2, we define the orthogonal projection onto the space of trigonometric polynomials. We obtain the orders of convergence of the modified and the iterated modified projection solution in the Sobolov space setting. Section 3 is devoted, in obtaining the orders of convergence of the modified projection solution in the uniform norm. A numerical example is considered in Section 4.

2 Orders of Convergence

In this Section, we assume that the kernel $k(s,t)$ is periodic in both the variables with period 2π. Let $X = L^2[0, 2\pi]$, with the usual norm given by

$$\|x\| = \left(\int_0^{2\pi} |x(t)|^2 dt\right)^{\frac{1}{2}}, \ \ x \in L^2[0, 2\pi],$$

Consider the following integral operator

$$(Tx)(s) = \int_0^{2\pi} k(s,t)x(t)dt, \; s \in [0,2\pi],$$

with $k(\cdot,\cdot) \in L^2([0,2\pi]\times[0,2\pi])$. Assume that the operator equation

$$u(s) - \int_0^{2\pi} k(s,t)u(t)dt = f(s), \; s \in [0,2\pi], \tag{2.1}$$

compactly written as

$$u - Tu = f$$

has a unique solution. Let r be a non negative integer. For $j = 1,2,\ldots,r$, let $x^{(j)}$ denote the jth derivative of x (distributional sense).
Define:

$$H^r[0,2\pi] := \{x \in L^2[0,2\pi] : x \text{ is } 2\pi \text{ periodic, } \; x^{(j)} \in L^2[0,2\pi], j = 1,2,\ldots,r\}.$$

We consider the following norm

$$\|x\|_r := [\sum_{j=0}^{r} \|x^{(j)}\|^2]^{\frac{1}{2}},$$

on $H^r[0,2\pi]$. In addition, assume kernel $k(\cdot,\cdot) \in H^r([0,2\pi]\times[0,2\pi])$ and $f \in H^r[0,2\pi]$. Then the exact solution $u \in H^r[0,2\pi]$. Let X_n be the space of all trigonometric polynomials of degree $\le n$. Then the dimension of X_n is $2n+1$ and a basis of X_n is given by

$$\varphi_j(s) = e^{ijs}, \quad j = 0,\pm1,\pm2,\ldots,\pm n, \quad (i = \sqrt{-1}). \tag{2.2}$$

Let $\pi_n : L^2[0,2\pi] \to X_n$ be the orthogonal projection given by

$$\pi_n x = \frac{1}{2\pi}\sum_{j=-n}^{n} < x, \varphi_j > \varphi_j. \tag{2.3}$$

Then π_n converges to the Identity operator pointwise and for $x \in H^r[0,2\pi]$ we have

$$\|x - \pi_n x\| \le \frac{C_1}{n^r}\|x\|_r. \tag{2.4}$$

(See Chapter 3, Page 69, Atkinson [1].) The above estimate shall be crucially used in what follows. We consider the modified projection approximation given by

$$T_n^M = \pi_n T \pi_n + \pi_n T(I-\pi_n) + (I-\pi_n)T\pi_n. \tag{2.5}$$

The operator equation (2.1) is approximated by

$$u_n^M - T_n^M u_n^M = f. \tag{2.6}$$

The iterated solution is defined as

$$\tilde{u}_n^M = T u_n^M + f. \tag{2.7}$$

In the subsequent section, we shall observe that

$$\|T_n^M - T\| = \|(I-\pi_n)T(I-\pi_n)\| \to 0.$$

As a consequence, we can use the following Theorem from Kulkarni [2].

Theorem 2.1. *For all large n,*

$$\|u - u_n^M\| \leq C_2 \|(I-\pi_n)T(I-\pi_n)u\|$$

and

$$\|u - \widetilde{u}_n^M\| \leq C_3 \|(I-T)^{-1}\| (\|T(I-\pi_n)T(I-\pi_n)u\| + \|T(I-\pi_n)T(I-\pi_n)\| \|u - u_n^M\|),$$

where C_2 and C_3 are constants independent of n.

In order to obtain the rates of convergence of $\|u - u_n^M\|$ and $\|u - \widetilde{u}_n^M\|$ we need a bound for the norms of $\|(I-\pi_n)T(I-\pi_n)u\|$, $\|T(I-\pi_n)T(I-\pi_n)u\|$ and the operator norm $\|T(I-\pi_n)T(I-\pi_n)\|$. We first prove a preliminary result using standard techniques. We now set some notations. For $0 \leq j \leq r$, let

$$l_j(s,t) = \frac{\partial^j}{\partial s^j} k(s,t), \; s,t \in [0, 2\pi].$$

For a fixed $s \in [0, 2\pi]$, we denote

$$l_{s,j}(t) = l_j(s,t), \; t \in [0, 2\pi].$$

The complex conjugate of $l_{s,j}(t)$ is denoted by $\bar{l}_{s,j}(t)$. Assume that $l_{s,j} \in H^r[0, 2\pi]$, for $j = 0, 1, \ldots, r$.

Proposition 2.2. For $x \in H^r[0, 2\pi]$, we have

$$\|T(I-\pi_n)x\|_r \leq \frac{C_1^2}{n^{2r}} \|x\|_r (\sum_{j=0}^{r} \int_0^{2\pi} \|\bar{l}_{s,j}\|_r ds)^{\frac{1}{2}}. \tag{2.8}$$

Proof. Note that

$$\|T(I-\pi_n)x\|_r = \left[\sum_{j=0}^{r} \| \; [T(I-\pi_n)x]^{(j)} \; \|^2 \right]^{\frac{1}{2}}.$$

By definition,

$$T(I-\pi_n)x(s) = \int_0^{2\pi} k(s,t)(I-\pi_n)x(t)dt,\ s\in[0,2\pi].$$

Hence for $0 \le j \le r$,

$$[T(I-\pi_n)x]^j(s) = \int_0^{2\pi} \frac{\partial^j}{\partial s^j} k(s,t)(I-\pi_n)x(t)dt.$$

Since $(I-\pi_n)$ is an orthogonal projection, for $s\in[0,2\pi]$,

$$\begin{aligned}
[T(I-\pi_n)x]^j(s) &= \int_0^{2\pi} l_{s,j}(t)(I-\pi_n)x(t)dt \\
&= <(I-\pi_n)x,\ \bar{l}_{s,j}> \\
&= <(I-\pi_n)^2x,\ \bar{l}_{s,j}> \\
&= <(I-\pi_n)x,\ (I-\pi_n)\bar{l}_{s,j}>.
\end{aligned}$$

Now using Cauchy Schwarz inequality and the estimate (2.4), we get

$$\begin{aligned}
|[T(I-\pi_n)x]^j(s)| &\leqslant \|(I-\pi_n)x\|\|(I-\pi_n)\bar{l}_{s,j}\| \\
&\leqslant \frac{C_1}{n^r}\|x\|_r \frac{C_1}{n^r}\|\bar{l}_{s,j}\|_r \\
&\leqslant \frac{C_1^2}{n^{2r}}\|x\|_r\|\bar{l}_{s,j}\|_r,\ s\in[0,2\pi].
\end{aligned}$$

Thus,

$$\|[T(I-\pi_n)x]^j\|^2 = \int_0^{2\pi} |[T(I-\pi_n)x]^j(s)|^2 ds \leqslant \frac{C_1^4}{n^{4r}}\|x\|_r^2 \int_0^{2\pi} \|\bar{l}_{s,j}\|_r^2 ds.$$

As a result, we have

$$\sum_{j=0}^{r} \|[T(I-\pi_n)x]^j\|^2 \leqslant \frac{C_1^4}{n^{4r}}\|x\|_r^2 \sum_{j=0}^{r}\int_0^{2\pi} \|\bar{l}_{s,j}\|_r^2 ds.$$

By taking square roots of both the sides, we obtain

$$\|T(I-\pi_n)x\|_r = \left(\sum_{j=0}^{r} \|[T(I-\pi_n)x]^j\|^2\right)^{\frac{1}{2}} \leqslant \frac{C_1^2}{n^{2r}}\|x\|_r\Big(\sum_{j=0}^{r}\int_0^{2\pi} \|\bar{l}_{s,j}\|_r^2 ds\Big)^{\frac{1}{2}},$$

which completes the proof. □

We now obtain the bounds for $\|(I-\pi_n)T(I-\pi_n)x\|$, $\|T(I-\pi_n)T(I-\pi_n)x\|$ and $\|T(I-\pi_n)T(I-\pi_n)\|$

Proposition 2.3. For $x \in H^r[0,2\pi]$, we have

$$\|(I-\pi_n)T(I-\pi_n)x\| \leq \frac{C_1^3}{n^{3r}}\|x\|_r(\sum_{j=0}^{r}\int_0^{2\pi}\|\bar{l}_{s,j}\|_r ds)^{\frac{1}{2}}$$

and

$$\|T(I-\pi_n)T(I-\pi_n)x\| \leq \frac{C_1^4}{n^{4r}}\|x\|_r^2\sum_{j=0}^{r}\int_0^{2\pi}\|\bar{l}_{s,j}\|_r^2 ds.$$

Also

$$\|T(I-\pi_n)T(I-\pi_n)\| = O(\|(I-\pi_n)T\|) = O(\frac{1}{n^r}).$$

Proof. From (2.4) we have

$$\|(I-\pi_n)T(I-\pi_n)x\| \leqslant \frac{C_1\|T(I-\pi_n)x\|_r}{n^r}. \tag{2.9}$$

From Proposition 2.2 and the above estimate, it follows that

$$\|(I-\pi_n)T(I-\pi_n)x\| \leq \frac{C_1^3}{n^{3r}}\|x\|_r(\sum_{j=0}^{r}\int_0^{2\pi}\|\bar{l}_{s,j}\|_r ds)^{\frac{1}{2}}.$$

Using (2.8), we get

$$\begin{aligned}\|T(I-\pi_n)T(I-\pi_n)x\| &\leqslant \frac{C_1^2}{n^{2r}}\|T(I-\pi_n)x\|_r(\sum_{j=0}^{r}\int_0^{2\pi}\|\bar{l}_{s,j}\|_r ds)^{\frac{1}{2}}\\ &\leqslant \frac{C_1^4}{n^{4r}}\|x\|_r^2\sum_{j=0}^{r}\int_0^{2\pi}\|\bar{l}_{s,j}\|_r ds.\end{aligned}$$

This completes the proof. Next we need to compute an upper bound for the operator norm $\|T(I-\pi_n)T(I-\pi_n)\|$. Since π_n is a sequence of projections converging to identity pointwise. $\|\pi_n\|$ are uniformly bounded. Since $T : L^2[0,2\pi] \to L^2[0,2\pi]$ is a bounded operator, to compute an upper bound for $\|T(I-\pi_n)T(I-\pi_n)\|$ it is enough to compute an upper bound for $\|(I-\pi_n)T\|$. We have

$$\|(I-\pi_n)Tx\| \leq \frac{C_1}{n^r}\|Tx\|_r = \frac{C_1}{n^r}\left(\sum_{j=0}^{r}\|(Tx)^{(j)}\|^2\right)^{\frac{1}{2}}$$

Since

$$(Tx)^{(j)}(s) = \int_0^{2\pi} l_{j,s}(t)x(t)dt$$

we have by Cauchy Schwarz inequality,

$$|(Tx)^{(j)}(s)|^2 \leq \int_0^{2\pi}|l_{j,s}(t)|^2 dt\int_0^{2\pi}|x(t)|^2 dt$$

and hence

$$\int_0^{2\pi} |(Tx)^{(j)}(s)|^2 ds \leq \int_0^{2\pi}\int_0^{2\pi} |l_{j,s}(t)|^2 dt ds \int_0^{2\pi} |x(t)|^2 dt$$

So

$$\sum_{j=0}^{r} \|(Tx)^{(j)}\|^2 \leq \sum_{j=0}^{r} \left(\int_0^{2\pi}\int_0^{2\pi} |l_{j,s}(t)|^2 dt ds \right) \|x\|^2.$$

Let $M^2 = \sum_{j=0}^{r} \left(\int_0^{2\pi}\int_0^{2\pi} |l_{j,s}(t)|^2 dt ds \right)$ and as a consequence

$$\|(I-\pi_n)Tx\| \leq M\frac{C_1}{n^r}\|x\|.$$

□

We now state the main theorem.

Theorem 2.4. *Let $\pi_n : L^2[0,2\pi] \to X_n$ be the orthogonal projection. Assume that $k(.,.) \in H^r([0,2\pi] \times [0,\pi])$ and $f \in H^r([0,2\pi])$. Then*

$$\|u - u_n^M\| = O(\frac{1}{n^{3r}})$$

and

$$\|u - \widetilde{u}_n^M\| = O(\frac{1}{n^{4r}}).$$

Proof. The proof follows from Theorem 2.2 and Proposition 2.3. □

3 Uniform Convergence

We are often interested in obtaining uniform convergence of the solution u_n^M to u. Let $C_p(2\pi)$ be the set of all 2π-periodic and continuous functions on $x(s)$, $-\infty < s < \infty$. $C_p(2\pi)$ with the uniform norm $\|.\|_\infty$ is a Banach space. We consider the solution of (2.1) that is

$$u(s) - \int_0^{2\pi} k(s,t)u(t)dt = f(s),\ s \in [0,2\pi].$$

It is no longer true that π_n converges to identity pointwise. We have

$$\|\pi_n\| = O(\log n).$$

See Zygmund [4]. We do not have

$$\|(I-\pi_n)T(I-\pi_n)\| \to 0,$$

and as a consequence $\|(I-\pi_n)T(I-\pi_n)\|$ has to be estimated differently. We need the following results to prove that $\|(I-\pi_n)T(I-\pi_n)\| \to 0$. Given $x \in C_p(2\pi)$, define the minimax error in approximating x by elements of X_n as follows :

$$\rho_n(x) = \inf_{z \in X_n} \|x-z\|_\infty.$$

The minimax approximation to x by elements of X_n is that element $q_n \in X_n$ for which

$$\|x-q_n\|_\infty = \rho_n(x).$$

It can be shown that there is a unique such q_n and for any $x \in C_p(2\pi)$ it can be shown that $\rho_n(x) \to 0$ as $n \to \infty$. We have the following result from Section 3.3.3 of Atkinson [1].

Theorem 3.1. $\rho_n(x) = \inf_{z \in X_n} \|x-z\|_\infty = \|x-q_n\|_\infty,\ q_n \in X_n$

$$\|x-\pi_n x\|_\infty \leq O(\log n)\rho_n(x)$$

If x is Hölder continuous, of exponent α then

$$\|x-\pi_n x\|_\infty \leq c\, O\left(\frac{\log n}{n^\alpha}\right),$$

for some constant c. This converges to 0 as $n \to \infty$. Higher speeds of convergence can be obtained by assuming greater smoothness for $x(t)$. We quote the following result from Atkinson [1]

Theorem 3.2. *If $k(s,t)$ satisfies the Hölder condition*

$$|k(s_1,t)-k(s_2,t)| \leq c(k)|s_1-s_2|^\alpha$$

then

$$\|(I-\pi_n)T\|_\infty \leq C\frac{\log n}{n^\alpha}$$

and

$$\|u-u_n^G\|_\infty = O\left(\frac{\log n}{n^\alpha}\right),$$

where u_n^G is the Galerkin solution.

For the modified projection method we have the following result.

Theorem 3.3. *If $k(s,t)$ satisfies the Hölder condition*

$$|k(s_1,t)-k(s_2,t)| \leq c(k)|s_1-s_2|^\alpha$$

and x satisfies satisfies the Hölder condition, then

$$\|(I-\pi_n)T(I-\pi_n)x\|_\infty \le C\frac{(\log n)^2}{n^{2\alpha}}$$

and

$$\|u-u_n^M\|_\infty = O(\frac{(\log n)^2}{n^{2\alpha}}).$$

Proof. For $s \in [0,2\pi]$, we have

$$[(I-\pi_n)T(I-\pi_n)x](s) = \int_0^{2\pi}(I-\pi_n)K(s,t)[(I-\pi_n)x](t)dt$$

$$[(I-\pi_n)T(I-\pi_n)x](s) = \int_0^{2\pi}(I-\pi_n)K_t(s)[(I-\pi_n)x](t)dt.$$

We have

$$\|(I-\pi_n)T(I-\pi_n)x\|_\infty \le \max_{s\in[0,2\pi]}\int_0^{2\pi}|(I-\pi_n)K_t(s)|\ \|(I-\pi_n)x\|_\infty dt$$

If x and $K(s,t)$ satisfies the Hölder continuous of exponent α then

$$\|(I-\pi_n)K_t\|_\infty \le c\ O\left(\frac{\log n}{n^\alpha}\right)$$

and

$$\|x-\pi_n x\|_\infty \le c\ O\left(\frac{\log n}{n^\alpha}\right)$$

which gives

$$\|(I-\pi_n)T(I-\pi_n)x\|_\infty \le \frac{C(\log n)^2}{n^{2\alpha}},$$

and the result follows from Theorem 2.1. □

In actual computations, integrations have to be replaced by numerical integration. In general we do not expect iteration to improve the order of convergence. The analysis suggests that in the uniform norm, iterated Galerkin is not an improvement over Galerkin solution and iterated modified solution is not an improvement over modified solution. Modified projection solution is better than the Galerkin solution. The next section illustrates this fact.

4 Numerical Results

4.1 Example 1, true solution known

We consider the example (12.15) presented in Kress [3]. Consider the Fredholm integral equation of the second kind given by

$$u(s)+\frac{ab}{\pi}\int_0^{2\pi}\frac{u(t)}{a^2+b^2-(a^2-b^2)\cos(s+t)}dt = f(s),\ s\in[0,2\pi], a\ge b>0. \qquad (3.1)$$

This integral equation arises from the solution of the Dirichlet problem for the Laplace equation in an ellipse with semi-axis a and b.
The exact solution is given by:

$$u(s) = e^{\cos s}\cos(\sin s), s \in [0, 2\pi] \tag{3.2}$$

We take

$$f(s) = u(s) + e^{c\cos s}\cos(c\sin s).$$

The above integral operator in (3.1) is approximated by the modified integral operator given by equation (1.2). Computing the solution in this case involves solving a system of linear equations, which further involves computing inner products and hence integrals. These integrals are evaluated numerically using the composite trapezoidal rule for single, double and triple variables for varying grids of size N.
In our analysis, we have observed that the modified projection method has proved to be better than the Galerkin method which reconciles with the theory. We approximate the solution using both the Galerkin method given by operator equation (1.1) and the Modified Projection method given by operator equation (1.2) and then compare the two with the exact solution given by equation (3.2). We provide a comparison of errors between the Galerkin and Modified Projection errors, at points $s = 0, \frac{\pi}{2}$ and π for $n = 3, 7$ and 15. The errors as defined as follows:

$$Galerkin\ Error = |u(s) - u_n^G(s)|$$

$$Modified\ Error = |u(s) - u_n^M(s)|$$

Table 4.1: Error Comparison for s = 0

s = 0		
n	Galerkin Error	Modified Error
3	5.1×10^{-2}	5.51×10^{-4}
7	2.79×10^{-5}	3.92×10^{-9}
15	6.13×10^{-14}	1.29×10^{-14}

Table 4.2: Error Comparison for s = $\pi/2$

s = $\pi/2$		
n	Galerkin Error	Modified Error
3	4.03×10^{-2}	5.12×10^{-4}
7	2.45×10^{-5}	3.77×10^{-9}
15	4.63×10^{-14}	1.66×10^{-15}

Table 4.3: Error Comparison for s = π

s = π		
n	Galerkin Error	Modified Error
3	-3.45×10^{-2}	4.82×10^{-4}
7	-2.23×10^{-5}	3.64×10^{-9}
15	-4.45×10^{-14}	-2.33×10^{-15}

4.2 Example 2, true solution unknown

We consider example (13.2.1) presented in Atkinson and Han [5]. Consider the boundary of an ellipse, $r(t) = (a\cos t, b\sin t)$, $t \in [0, 2\pi]$. We now consider the integral equation:

$$-\pi\, u(s) + \int_0^{2\pi} u(t) K\left(\frac{s+t}{2}\right) dt = f(s),\ s \in [0, 2\pi] \tag{3.3}$$

where

$$K(\theta) = \frac{-ab}{2\left(a^2\sin^2(\theta) + b^2\cos^2(\theta)\right)} \tag{3.4}$$

and

$$f(x,y) = e^x \cos y,\ (x,y) \in S \tag{3.5}$$

We take $(a,b) = (1,2)$. The true solution is unknown, but we obtain a highly accurate solution by using a large value of $N = 128$ and a fixed value of $n = 9$, N and n as described above. We compute approximate errors assuming the above computed solution to be true. We see that the Modified method converges faster than Galerkin, again serving as a confirmation of the theoretical results. The errors as defined as follows:

$$Galerkin\ Error = |u(s) - u_n^G(s)|$$

$$Modified\ Error = |u(s) - u_n^M(s)|$$

Table 4.4: Error Comparison for s = 0

s = 0		
N	Galerkin Error	Modified Error
16	0.136	1.58×10^{-4}
24	6.17×10^{-4}	5.61×10^{-10}
32	6.17×10^{-4}	4.61×10^{-11}
64	6.17×10^{-4}	2.92×10^{-12}

Table 4.5: Error Comparison for s = $\pi/2$

s = $\pi/2$		
N	Galerkin Error	Modified Error
16	0.114	1.06×10^{-4}
24	2.85×10^{-3}	7.09×10^{-10}
32	2.85×10^{-3}	2.62×10^{-11}
64	2.85×10^{-3}	2.01×10^{-11}

Table 4.6: Error Comparison for s = π

s = π		
N	Galerkin Error	Modified Error
16	0.955	2.64×10^{-3}
24	4.78×10^{-3}	9.91×10^{-8}
32	4.78×10^{-3}	4.33×10^{-11}
64	4.78×10^{-3}	5.21×10^{-11}

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