

# How to solve: Graph theory

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## 1 Matchings

We define the following:

- $\alpha(G)$  is the maximum cardinality of an independent set,
- $\beta(G)$  is the maximum cardinality of a vertex cover (subset  $T \subseteq V$  which covers all edges),
- $\alpha'(G)$  is the cardinality of the largest matching in  $G$ ,
- $\beta'(G)$  is the maximum cardinality of an edge cover.

We note the following easy relations:

$$\begin{aligned}\alpha(G) + \beta(G) &= n(G) & \alpha'(G) &\leq \beta(G) \\ \alpha'(G) &\leq \beta'(G) & \alpha(G) &\leq \beta'(G)\end{aligned}$$

**Theorem 1.1** (Gallai). *If  $\delta(G) \geq 1$ , then  $\alpha'(G) + \beta'(G) = n(G)$ .*

There is much we can say about matchings. Let  $M$  be a matching in  $G$ . We say that a path  $P$  is an  $M$ -AUGMENTING PATH if it alternates between edges in  $M$  and in  $\overline{M}$ , and if the end vertices are not covered by  $M$ . It is worth noting that there is an  $M$ -augmenting path if and only if  $M$  is not a maximum matching.

**Theorem 1.2** (Tutte). *A graph  $G$  has a perfect matching if and only if Tutte's condition holds, so if for any  $S \subseteq V$ , we have  $|S| > o(G - S)$ , where  $o(G - S)$  is the number of odd components in  $G - S$ .*

This is a special case of the Berge-Tutte formula

$$\alpha'(G) = \frac{1}{2} \left( n - \max_{S \subseteq V} (o(G - S) - |S|) \right).$$

## 1.1 Bipartite graphs

**Theorem 1.3** (König). *Let  $G$  be a bipartite graph. Then  $\alpha'(G) = \beta(G)$ . Additionally, if  $M$  is a matching in  $G$  and there is no  $M$ -augmenting path,  $M$  is a maximum matching.*

As a corollary, if  $G$  is bipartite, then  $\alpha(G) = \beta'(G)$ . We can say more about bipartite graphs, as the next theorem states.

**Theorem 1.4** (Hall). *If  $G$  is bipartite with partite classes  $A, B$ , then there exists a matching that covers  $A$  if and only if Hall's condition holds for  $A$ , so if for every  $S \subseteq A$ ,  $|S| \leq |N(S)|$ .*

So in a bipartite graph, there is a perfect matching if and only if  $|A| = |B|$  and Hall's condition holds. Also,

$$\alpha'(G) = |A| - \max_{S \subseteq A} (|S| - |N(S)|).$$

We may also say the following.

**Theorem 1.5.** *If  $G$  is a regular bipartite graph, then  $G$  has a perfect matching.*

## 1.2 Factors

A  $k$ -FACTOR is a  $k$ -regular spanning subgraph (so a  $k$ -regular subgraph which contains all vertices). Note that a 1-factor corresponds to a perfect matching.

**Theorem 1.6** (Petersen). *Every bridgeless cubic graph has a 1-factor.*

We also have the following results about  $k$ -regular graphs.

**Theorem 1.7.** *If  $G$  is a  $k$ -regular graph for an even  $k$ , then  $G$  has a 2-factor.*

**Theorem 1.8.** *If  $G$  is a  $k$ -regular bipartite graph, then  $G$  can be decomposed into 1-factors.*

## 2 Connectivity

The (vertex) connectivity of a graph  $G$  is the minimum number of vertices  $S$  such that  $G - S$  is either isomorphic to  $K_1$  or is disconnected. We denote it by  $\kappa(G)$ . Note that  $\kappa(G) \leq \delta(G)$  and  $\kappa(G) \leq \beta(G)$ . Similarly, we can define the edge-connectivity number  $\kappa'(G)$  as the minimum number of edges  $F$  such that  $G - F$  is disconnected. If  $G$  has at least 2 vertices, then  $\kappa(G) \leq \kappa'(G) \leq \delta(G)$ .

We know that  $\kappa(K_n) = n - 1$  and  $\kappa(K_{a,b}) = \min\{a, b\}$ .

If we have a  $k$ -connected graph, then we may build new graphs with the following result.

**Theorem 2.1** (expansion lemma). *If  $G$  is a  $k$ -connected graph and we add a new vertex  $v$  and  $k$  incident edges to the graph, then we obtain a  $k$ -connected graph.*

We have several results about 2-connected and 2-edge-connected graphs.

**Theorem 2.2** (Whitney). *If  $G$  is a 2-connected graph, then for every  $u, v \in V(G)$ , there are two internally disjoint  $u, v$ -paths. The converse also holds.*

**Proposition 2.3** (subdivision lemma). *Suppose  $G'$  is obtained from  $G$  by subdividing an edge  $uv \in E(G)$  with a vertex  $w$ . Then  $G$  is 2-connected if and only if  $G'$  is 2-connected.*

An (open) ear decomposition of  $G$  is a sequence  $P_0, P_1, \dots, P_k$ , where  $P_0$  is a cycle in  $G$  and every other  $P_i$  is an ear in the graph  $G_i = P_0 \cup P_1 \cup \dots \cup P_i$ . We also require  $G_k = G$ . Here, an ear is a path where all internal vertices are of degree 2, but end vertices are of degree at least 3.

**Theorem 2.4.** *A graph  $G$  is 2-connected if and only if it has an ear decomposition.*

Similarly, we can define a closed ear as a cycle in which all but one vertex have degree 2, with the exceptional vertex having degree at least 4. A closed ear decomposition then is a sequence  $P_0, P_1, \dots, P_k$  where  $P_i$  is either an open or closed ear in  $G_i$  (defined as before), and  $G_k = G$ .

**Theorem 2.5.** *A graph  $G$  is 2-edge-connected if and only if it has a closed ear decomposition.*

**Theorem 2.6** (Robbins). *An undirected graph  $G$  is 2-edge-connected if and only if it has a strong orientation, so if we can choose the orientation of each edge in such a way that we get a strongly connected digraph.*

A set  $S \subseteq V(G)$  is an  $x, y$ -cut if  $x$  and  $y$  belong to different components in  $G - S$ . We label the minimum size of such a cut with  $\kappa_G(x, y)$ . We also define  $\lambda_G(x, y)$  as the maximum number of internally vertex-disjoint  $x, y$ -paths in  $G$ . We then have the following result.

**Theorem 2.7** (Menger's theorem for vertex cuts). *If  $x$  and  $y$  are nonadjacent vertices in  $G$ , then  $\kappa_G(x, y) = \lambda_G(x, y)$ .*

We can also define an  $x, y$ -edge cut as an edge set  $R$  such that  $G - R$  is disconnected and  $x$  and  $y$  are in different components. The minimum size of such a set is denoted by  $\kappa'_G(x, y)$ . Also, the maximum number of edge-disjoint  $x, y$ -paths is denoted by  $\lambda'_G(x, y)$ .

**Theorem 2.8** (Menger's theorem for edge cuts). *Let  $x, y \in V(G)$ . Then  $\kappa'_G(x, y) = \lambda'_G(x, y)$ .*

### 3 Coloring

The chromatic number of  $G$  is the minimum number of colors in a proper coloring of  $G$ . It is denoted by  $\chi(G)$ . We can remark that in any graph, the following holds:

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1, \quad \chi(G) \geq \frac{n(G)}{\alpha(G)}.$$

We can use the greedy coloring algorithm to find a proper coloring of a graph, but depending on the choice of vertex order, the result may be arbitrarily bad. A slightly better idea than an arbitrary order is to order vertices by their degrees, decreasing. We then find

$$\chi(G) \leq 1 + \max_{i=1, \dots, n} \{\min\{d_i, i - 1\}\}.$$

We gave a higher bound  $\chi(G) \leq \Delta(G) + 1$ , but we can improve it slightly.

**Theorem 3.1** (Brooks). *If  $G$  is connected and not a complete graph or odd cycle, then  $\chi(G) \leq \Delta(G)$ .*

We also gave a lower bound of  $\omega(G)$ , which is sharp, but the difference between  $\chi$  and  $\omega$  can be arbitrarily large, as can be shown from the following.

**Theorem 3.2** (Mycielski's construction). *If  $G$  is a graph with at least one edge, then  $\chi(M(G)) = \chi(G) + 1$  and  $\omega(M(G)) = \omega(G)$ , where  $M(G)$  is a graph derived from  $G$  by the following construction:*

- label the vertices of  $G$  as  $v_1, \dots, v_n$ ,
- create  $n + 1$  new vertices  $u_1, \dots, u_n, z$ ,
- add connections  $u_i v_j$  for all pairs  $v_i v_j \in E(G)$ ,
- add connections  $u_i z$  for all  $i$ .

A graph is CHORDAL if there is no induced subgraph isomorphic to a cycle of size  $\geq 4$ . In a chordal graph,  $\chi(G) = \omega(G)$ .

**Theorem 3.3.** *A graph  $G$  is chordal if and only if there is a simplicial elimination ordering of the vertices of  $G$ , so if there exists an ordering  $v_1, v_2, \dots, v_n$  such that the closed neighbourhood of  $v_i$  in  $G - \{v_1, \dots, v_{i-1}\}$  is a clique.*

A graph  $G$  is PERFECT if  $\chi(H) = \omega(H)$  holds for every induced subgraph  $H$  of  $G$ . All chordal graphs are perfect, as are bipartite graphs. We also know that the line graph of a bipartite graph is perfect.

**Theorem 3.4** (Perfect graph theorem). *A graph is perfect if and only if its complement is perfect.*

There is also a stronger result:

**Theorem 3.5** (Strong perfect graph theorem). *A graph is perfect if and only if neither it nor its complement have an induced cycle of size 5 or greater.*

### 3.1 Edge coloring

The edge colour number  $\chi'(G)$  is the smallest number of colors in a proper edge coloring. By Vizing's theorem,  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$  for any graph  $G$ .

**Proposition 3.6.** *If  $G$  is bipartite,  $\chi'(G) = \Delta(G)$ .*

## 4 Planar graphs

Simply put, a graph is planar if you can draw it on a plane without edges intersecting. If we have a plane graph (that is, a planar graph which is embedded into the plane), we can form a dual graph by switching the role of vertices and faces, with two former faces being connected once for each component of their common boundary. Dual graphs are always connected, and a double dual of a connected plane graph is isomorphic to the original.

The length of a face is the number of edges along a walk at the face's boundary, where we count an edge twice if we must go through it twice. We denote the length by  $l(F)$ . In any plane graph,

$$\sum_{F \text{ face}} l(F) = 2m(G).$$

**Theorem 4.1.** *Let  $G$  be a plane graph. Then the following are equivalent:*

- $G$  is bipartite,
- every face of  $G$  has an even length,
- $G^*$  is Eulerian (connected and all vertices are of even degree).

A planar graph is OUTERPLANAR if there is an embedding in which all vertices are on the boundary of the outside face. It turns out that a simple outerplanar graph has  $\delta(G) \leq 2$ .

**Theorem 4.2** (Euler). *If  $G$  is a plane graph, then  $n(G) + f(G) - m(G) = 2$ .*

As an easy corollary, in a planar graph,  $m(G) \leq 3n(G) - 6$ .

**Theorem 4.3** (Kuratowski). *A graph is planar if and only if it contains no Kuratowski subgraph (a subgraph which is a subdivision of  $K_{3,3}$  or  $K_5$ ).*

**Theorem 4.4** (Wagner). *A graph  $G$  is planar if and only if neither  $K_5$  nor  $K_{3,3}$  are minors of  $G$  (so if we cannot obtain either by deleting or contracting edges of  $G$ ).*

**Theorem 4.5** (four-colour theorem). *If  $G$  is planar, then  $\chi(G) \leq 4$ .*