Zbrani zapiski za 1. letnik magistrskega študija

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Opomba k notaciji: Včasih je produkt vektorja s skalarjem napisan kot $\lambda.x$, s piko. Tudi oprator $\vec{\nabla}$ včasih obravnavamo kot vektor, in pišemo gradient s piko spodaj, $\vec{\nabla}.f$. Divergenco označimo z $\vec{\nabla} \cdot f$.

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1.1.1 A note on fields

A FIELD is defined as a quantity that takes a value at every point in space and time, the value being some element of a particular vector space. Fields are usually LOCAL, which means they only depend on one point in space-time. They influence particles, and particles influence fields.

There are four known interactions:

- ELECTROMAGNETISM is the force between charged particles. Quantum mechanically, it is carried by photons, and it is responsible for most of our experience of the world.
- The WEAK and STRONG NUCLEAR FORCES govern the behaviour inside atomic cores.
- GRAVITY is a geometric force, which we model slightly differently. Compared to the other forces, it is extremely weak.

Electromagnetism is a unified theory connecting the electric and magnetic fields. There is also a successful unification of EM and the weak force, called ELECTROWEAK THEORY. We may further unify this with the strong force, giving the GRAND UNIFICATION THEORY (GUT). It is mathematically consistent, but none of its predictions have ever been measured. This is a real shame, since among other things, it predicts magnetic monopoles and proton decay.

All these unifications are performed with the same techniques, but these fail when trying to unify GUT with gravity.

1.1.2 A Note on Notation

We use Einstein's summation convention: Whenever there is a repeated index in the expression, assume that we are summing over it. For example, the dot product of two vectors $\vec{w}$ and $\vec{v}$ can be written as

$$\langle \vec{v}, \vec{w} \rangle = v^i w^i = \sum_{i=1}^{3} v^i w^i,$$

and similarly, the divergence of a vector field $\vec{E}$ is

$$\vec{\nabla} \cdot \vec{E} = \partial_i E^i = \sum_{i=1}^{3} \partial_i E^i.$$

Note that here, v^i denotes the i-th component of vector $\vec{v}$.

We also define two symbols, the KRONECKER DELTA

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & \text{otherwise} \end{cases}$$

and the LEVI-CHIVITA SYMBOL

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{one of the indices is repeated,} \\ \operatorname{sgn}(i\,j\,k) & \text{all indices are distinct.} \end{cases}$$

Here, $\operatorname{sgn}\pi$ denotes the sign of permutation π as an element of $\{\pm 1\}$. With these definitions, the following useful identities hold:

- for two vectors $\vec{v}, \vec{w}$, the dot product is

  $$\langle \vec{v}, \vec{w} \rangle = \delta_{ij} v^i w^j$$

  and the cross product is, component-wise

  $$(\vec{v} \times \vec{w})^i = \varepsilon_{ijk} v^j w^k,$$

- $\delta_{ii} = 3$,
- $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta kn - \delta_{jn}\delta_{km}$,
- $\varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}$,
- $\varepsilon_{ijk}\varepsilon_{ijk} = 6$.

1.1.3 Maxwell's Equations

We define two fields, the electric field $\vec{E}(\vec{x}, t)$ and the magnetic field $\vec{B}(\vec{x}, t)$. For these fields, we assume the following equations to be true:

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} & \vec{\nabla} \times \vec{E} &= -\partial_t \vec{B} \\ \vec{\nabla} \cdot \vec{B} &= 0 & \vec{\nabla} \times \vec{B} &= \varepsilon_0 \mu_0 \partial_t \vec{E} + \mu_0 \vec{J} \end{aligned}$$

These are called MAXWELL'S EQUATIONS. Along with the aforementioned $\vec{E}$ and $\vec{B}$, they also contain the following fields;

- the electric charge density ρ is a scalar field,
- the electric current density $\vec{J}$ is a vector field,

and two constants, the PERMITTIVITY OF FREE SPACE ε_0 and the PERMEABILITY OF FREE SPACE μ_0.

2 Uvod v funkcionalno analizo

2.1 Normirani in Banachovi prostori

Definicija 2.1.1. Naj bo X vektorski prostor nad poljem $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Preslikava $\|\cdot\| : X \to \mathbb{R}$ je NORMA, če velja:

- $\|x\| \geq 0$,
- $\|x\| = 0 \Leftrightarrow x = 0$,
- $\|\lambda x\| = |\lambda| \, \|x\|$,
- $\|x + y\| \leq \|x\| + \|y\|$.

Opomba. Velja $|\|x\| - \|y\|| \leq \|x - y\|$, iz česar sledi, da je norma zvezna (celo Lipschitzova za $L = 1$).

Norma porodi metriko $d(x, y) = \|x - y\|$ na prostoru X, ki je invariantna na translacije, in za katero velja

$$d(\lambda x, \lambda y) = |\lambda| \, d(x, y).$$

Zaprto kroglo radija r s središčem v točki x označimo z $B(x, r)$, odprto kroglo pa z $\mathring{B}(x, r)$. Zaradi zveznosti norme je zaprtje odprte krogle natanko pripadajoča zaprta krogla.

Definicija 2.1.2. Normiran prostor je BANACHOV, če je poln za inducirano metriko.

Trditev 2.1.3. *Seštevanje in množenje vektorjev s skalarjem sta zvezni operaciji.*