How to solve: Graph theory

Patrik Žnidaršič

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1 Matchings

We define the following:

- $\alpha(G)$ is the maximum cardinality of an independent set,
- $\beta(G)$ is the maximum cardinality of a vertex cover (subset $T \subseteq V$ which covers all edges),
- $\alpha'(G)$ is the cardinality of the largest matching in G,
- $\beta'(G)$ is the maximum cardinality of an edge cover.

We note the following easy relations:

$$\begin{aligned} \alpha(G) + \beta(G) &= n(G) \qquad & \alpha'(G) &\leq \beta(G) \\ \alpha'(G) &\leq \beta'(G) \qquad & \alpha(G) &\leq \beta'(G) \end{aligned}$$

Theorem 1.1 (Gallai)**.** *If $\delta(G) \geq 1$, then $\alpha'(G) + \beta'(G) = n(G)$.*

There is much we can say about matchings. Let M be a matching in G. We say that a path P is an M-AUGMENTING PATH if it alternates between edges in M and in $\overline{M}$, and if the end vertices are not covered by M. It is worth noting that there is an M-augmenting path if and only if M is not a maximum matching.

Theorem 1.2 (Tutte)**.** *A graph G has a perfect matching if and only if Tutte's condition holds, so if for any $S \subseteq V$, we have $|S| > o(G - S)$, where $o(G - S)$ is the number of odd components in $G - S$.*

This is a special case of the Berge-Tutte formula

$$\alpha'(G) = \frac{1}{2}\left(n - \max_{S \subseteq V}(o(G - S) - |S|)\right).$$

1.1 Bipartite graphs

Theorem 1.3 (König). *Let G be a bipartite graph. Then $\alpha'(G) = \beta(G)$. Additionally, if M is a matching in G and there is no M-augmenting path, M is a maximum matching.*

As a corollary, if G is bipartite, then $\alpha(G) = \beta'(G)$. We can say more about bipartite graphs, as the next theorem states.

Theorem 1.4 (Hall). *If G is bipartite with partite classes A, B, then there exists a matching that covers A if and only if Hall's condition holds for A, so if for every $S \subseteq A$, $|S| \leq |N(S)|$.*

So in a bipartite graph, there is a perfect matching if and only if $|A| = |B|$ and Hall's condition holds. Also,

$$\alpha'(G) = |A| - \max_{S \subseteq A}(|S| - |N(S)|).$$

We may also say the following.

Theorem 1.5. *If G is a regular bipartite graph, then G has a perfect matching.*

1.2 Factors

A k-FACTOR is a k-regular spanning subgraph (so a k-regular subgraph which contains all vertices). Note that a 1-factor corresponds to a perfect matching.

Theorem 1.6 (Petersen). *Every bridgeless cubic graph has a 1-factor.*

We also have the following results about k-regular graphs.

Theorem 1.7. *If G is a k-regular graph for an even k, then G has a 2-factor.*

Theorem 1.8. *If G is a k-regular bipartite graph, then G can be decomposed into 1-factors.*

2 Connectivity

The (vertex) connectivity of a graph G is the minimum number of vertices S such that $G - S$ is either isomorphic to K_1 or is disconnected. We denote it by $\kappa(G)$. Note that $\kappa(G) \leq \delta(G)$ and $\kappa(G) \leq \beta(G)$. Similarly, we can define the edge-connectivity number $\kappa'(G)$ as the minimum number of edges F such that $G - F$ is disconnected. If G has at least 2 vertices, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

We know that $\kappa(K_n) = n - 1$ and $\kappa(K_{a,b}) = \min\{a, b\}$.

If we have a k-connected graph, then we may build new graphs with the following result.

Theorem 2.1 (expansion lemma). *If G is a k-connected graph and we add a new vertex v and k incident edges to the graph, then we obtain a k-connected graph.*

We have several results about 2-connected and 2-edge-connected graphs.

Theorem 2.2 (Whitney)**.** *If G is a 2-connected graph, then for every $u, v \in V(G)$, there are two internally disjoint u, v-paths. The converse also holds.*

Proposition 2.3 (subdivision lemma)**.** *Suppose G' is obtained from G by subdividing an edge $uv \in E(G)$ with a vertex w. Then G is 2-connected if and only if G' is 2-connected.*

An (open) ear decomposition of G is a sequence $P_0, P_1, \ldots, P_k$, where P_0 is a cycle in G and every other P_i is an ear in the graph $G_i = P_0 \cup P_1 \cup \ldots \cup P_i$. We also require $G_k = G$. Here, an ear is a path where all internal vertices are of degree 2, but end vertices are of degree at least 3.

Theorem 2.4. *A graph G is 2-connected if and only if it has an ear decomposition.*

Similarly, we can define a closed ear as a cycle in which all but one vertex have degree 2, with the exceptional vertex having degree at least 4. A closed ear decomposition then is a sequence $P_0, P_1, \ldots, P_k$ where P_i is either an open or closed ear in G_i (defined as before), and $G_k = G$.

Theorem 2.5. *A graph G is 2-edge-connected if and only if it has a closed ear decomposition.*

Theorem 2.6 (Robbins)**.** *An undirected graph G is 2-edge-connected if and only if it has a strong orientation, so if we can choose the orientation of each edge in such a way that we get a strongly connected digraph.*

A set $S \subseteq V(G)$ is an x, y-CUT if x and y belong to different components in $G - S$. We label the minimum size of such a cut with $\kappa_G(x, y)$. We also define $\lambda_G(x, y)$ as the maximum number of internally vertex-disjoint x, y-paths in G. We then have the following result.

Theorem 2.7 (Menger's theorem for vertex cuts)**.** *If x and y are nonadjacent vertices in G, then $\kappa_G(x, y) = \lambda_G(x, y)$.*

We can also define an x, y-edge cut as an edge set R such that $G - R$ is disconnected and x and y are in different components. The minimum size of such a set is denoted by $\kappa'_G(x, y)$. Also, the maximum number of edge-disjoint x, y-paths is denoted by $\lambda'_G(x, y)$.

Theorem 2.8 (Menger's theorem for edge cuts)**.** *Let $x, y \in V(G)$. Then $\kappa'_G(x, y) = \lambda'_G(x, y)$.*

3 Coloring

The chromatic number of G is the minimum number of colors in a proper coloring of G. It is denoted by $\chi(G)$. We can remark that in any graph, the following holds:

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1, \qquad \chi(G) \geq \frac{n(G)}{\alpha(G)}.$$

We can use the greedy coloring algorithm to find a proper coloring of a graph, but depending on the choice of vertex order, the result may be arbitrarily bad. A slightly better idea than an arbitrary order is to order vertices by their degrees, decreasing. We then find

$$\chi(G) \leq 1 + \max_{i=1,\dots,n} \{\min\{d_i, i-1\}\}.$$

We gave a higher bound $\chi(G) \leq \Delta(G) + 1$, but we can improve it slightly.

Theorem 3.1 (Brooks)**.** *If G is connected and not a complete graph or odd cycle, then $\chi(G) \leq \Delta(G)$.*

We also gave a lower bound of $\omega(G)$, which is sharp, but the difference between χ and ω can be arbitrarily large, as can be shown from the following.

Theorem 3.2 (Mycielski's construction)**.** *If G is a graph with at least one edge, then $\chi(M(G)) = \chi(G) + 1$ and $\omega(M(G)) = \omega(G)$, where $M(G)$ is a graph derived from G by the following construction:*

- *label the vertices of G as $v_1, \dots, v_n$,*
- *create $n+1$ new vertices $u_1, \dots, u_n, z$,*
- *add connections $u_i v_j$ for all pairs $v_i v_j \in E(G)$,*
- *add connections $u_i z$ for all i.*

A graph is CHORDAL if there is no induced subgraph isomorphic to a cycle of size ≥ 4. In a chordal graph, $\chi(G) = \omega(G)$.

Theorem 3.3. *A graph G is chordal if and only if there is a simplicial elimination ordering of the vertices of G, so if there exists an ordering $v_1, v_2, \dots, v_n$ such that the closed neighbourhood of v_i in $G - \{v_1, \dots, v_{i-1}\}$ is a clique.*

A graph G is PERFECT if $\chi(H) = \omega(H)$ holds for every induced subgraph H of G. All chordal graphs are perfect, as are bipartite graphs. We also know that the line graph of a bipartite graph is perfect.

Theorem 3.4 (Perfect graph theorem)**.** *A graph is perfect if and only if its complement is perfect.*

There is also a stronger result:

Theorem 3.5 (Strong perfect graph theorem)**.** *A graph is perfect if and only if neither it nor its complement have an induced cycle of size 5 or greater.*

3.1 Edge coloring

The edge colour number $\chi'(G)$ is the smallest number of colors in a proper edge coloring. By Vizing's theorem, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for any graph G.

Proposition 3.6. *If G is bipartite, $\chi'(G) = \Delta(G)$.*

4 Planar graphs

Simply put, a graph is planar if you can draw it on a plane without edges intersecting. If we have a plane graph (that is, a planar graph which is embedded into the plane), we can form a dual graph by switching the role of vertices and faces, with two former faces being connected once for each component of their common boundary. Dual graphs are always connected, and a double dual of a connected plane graph is isomorphic to the original.

The length of a face is the number of edges along a walk at the face's boundary, where we count an edge twice if we must go through it twice. We denote the length by $l(F)$. In any plane graph,

$$\sum_{F \text{ face}} l(F) = 2m(G).$$

Theorem 4.1. *Let G be a plane graph. Then the following are equivalent:*

- *G is bipartite,*
- *every face of G has an even length,*
- *G^* is Eulerian (connected and all vertices are of even degree).*

A planar graph is OUTERPLANAR if there is an embedding in which all vertices are on the boundary of the outside face. It turns out that a simple outerplanar graph has $\delta(G) \leq 2$.

Theorem 4.2 (Euler)**.** *If G is a plane graph, then $n(G) + f(G) - m(G) = 2$.*

As an easy corollary, in a planar graph, $m(G) \leq 3n(G) - 6$.

Theorem 4.3 (Kuratowski)**.** *A graph is planar if and only if it contains no Kuratowski subgraph (a subgraph which is a subdivision of $K_{3,3}$ or K_5).*

Theorem 4.4 (Wagner)**.** *A graph G is planar if and only if neither K_5 nor $K_{3,3}$ are minors of G (so if we cannot obtain either by deleting or contracting edges of G).*

Theorem 4.5 (four-colour theorem)**.** *If G is planar, then $\chi(G) \leq 4$.*