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UNIT I- LOGIC

Statements and Notations,Connectives,Negation,Conjunction,Disjunction, statement, Formulae and TruthTables ,Conditional and Bi-conditional,Well–formed Formulae, Tautologies, EquivalenceofFormulae,DualityLaw, Tautological Implications.

Definition: Propositional Logic

A proposition is a collection of declarative statements that has either a truth value "true" or a truth value "false". A propositional consists of propositional variables and connectives. We denote the propositional variables by capital letters (A, B, C etc). The connectives connect the propositional variables.

Some examples of Propositions are given below:

- "Man is Mortal", it returns truth value"TRUE" as T.
- "12 + 9 = 3 − 2", it returns truth

value "FALSE" as F.

The following is not a Proposition

- "A is less than 2". It is because unless we give a specific value of A, we cannot say whether the statement is true orfalse.

Connectives

In propositional logic generally we use five connectives which are OR $(\vee)$, AND $(\wedge)$, Negation/ NOT (¬), Conditional or Implication / if-then (→), Bi conditional or If and only if $(\leftrightarrow)$.

Negation (¬) − The negation of a proposition A (written as ¬A) is false when A is true and is true when A is false.

The truth table is as follows –

A	**¬A**
True	False

False	True

AND (∧) − The AND operation of two propositions A and B (written as A ∧ B) is true if both the propositional variable A and B is true.

The truth table is as follows −

A	B	A∧ B
True	True	False
True	False	False
False	True	False
False	False	True

OR (∨)− The OR operation of two propositions A and B (written as A ∨ B) is true if at least any of the propositional variable A or B is true.

The truth table is as follows −

A	B	A ∨ B
T	T	T
T	F	T
F	T	T
F	F	F

Implication / if-then (→) − An implication A→B is False if A is true and B is false. The rest cases are true.

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The truth table is as follows −

A	B	A → B
True	True	True
True	False	False
False	True	True
False	False	True

If and only if (↔)− A ↔B is bi-conditional logical connective which is true when p and q are both false or both are true.

The truth table is as follows −

A	B	A ↔ B
True	True	True
True	False	False
False	True	False
False	False	True

Well Formed Formulas(WFFs)

The well formed formulas(WFFs) or statement formulas or logic formulas are defined recursively (or inductively) as below.

1. Propositional variables p,q,r,… and propositional constants F,T are well formed formulas. They are known as primitive WFFs.

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2. If P and Q are WFFs then $\neg P, \neg Q, P \wedge Q, P \vee Q, P \rightarrow Q$ *and* $P \leftrightarrow Q$ are also WFFs.
3. All WFFs are obtained by the above procedures applied a finite number of times.
   For example, the following are WFFs
   $p,\ p \wedge q,\ p \rightarrow q,\ p \wedge (q \rightarrow r) \equiv (p \wedge q) \rightarrow r, (p \rightarrow q) \rightarrow (q \rightarrow p)$

Note: In order to avoid excessive use of parenthesis, we adopt an order of precedence for logical Operators.

$\neg, \wedge, \vee, \rightarrow$ *and* $\leftrightarrow$

Tautologies

A Tautology is a formula which is always true for every value of its propositional variables.

Example − Prove [(A → B) ∧ A] → B is a tautology

The truth table is as follows −

A	B	A → B	(A → B) ∧ A	[(A → B) ∧ A] → B
True	True	True	True	True
True	False	False	False	True
False	True	True	False	True
False	False	True	False	True

As we can see every value of [(A → B) ∧ A] → B is "True", it is a tautology.

Contradictions

A Contradiction is a formula which is always false for every value of its propositional variables.

Example − Prove (A VB) ∧ [(¬A) ∧ (¬B)] is a contradiction

The truth table is as follows −

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A	B	A ∨ B	¬A	¬B	(¬A) ∧ (¬B)	(A V B) ∧ [(¬A) ∧ (¬B)]
True	True	True	False	False	False	False
True	False	True	False	True	False	False
False	True	True	True	False	False	False
False	False	False	True	True	True	False

As we can see every value of (A ∨ B) ∧ [(¬A) ∧ (¬B)] is "False", it is a Contradiction.

Contingency

A Contingency is a formula which has both some true and some false values for every value of its propositional variables.

Example − Prove (A ∨ B) ∧ (¬A) a contingency

The truth table is as follows −

A	B	A ∨ B	¬A	(A ∨ B) ∧ (¬A)
True	True	True	False	False
True	False	True	False	False
False	True	True	True	True
False	False	False	True	False

As we can see every value of (A ∨ B) ∧ (¬A) has both "True" and "False", it is a contingency.

Propositional Equivalences

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Two statements X and Y are logically equivalent if any of the following two conditions −

- The truth tables of each statement have the same truthvalues.
- The bi-conditional statement X $\leftrightarrow$Y is a tautology.

Example−Prove¬(A ∨ B)and[(¬A) ∧ (¬B)]areequivalent

Testing by 1st method (Matching truth table)

A	**B**	**A ∨ B**	**¬ (A ∨ B)**	**¬A**	**¬B**	**[(¬A) ∧ (¬B)]**
True	True	True	False	False	False	False
True	False	True	False	False	True	False
False	True	True	False	True	False	False
False	False	False	True	True	True	True

Here, we can see the truth values of ¬ (A ∨ B) and [(¬A) ∧ (¬B)] are same, hence the statements are equivalent.

Testing by 2nd method (Bi-conditionality)

A	**B**	**¬ (A ∨ B)**	**[(¬A) ∧ (¬B)]**	**[¬ (A ∨ B)] ↔ [(¬A) ∧ (¬B)]**
True	True	False	False	True
True	False	False	False	True
False	True	False	False	True
False	False	True	True	True

As [¬ (A ∨ B)] ↔ [(¬A) ∧ (¬B)] is a tautology, the statements are equivalent.

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Laws of Propositional Logic:

S.No	Name of Laws	Primal Form	Dual Form
1	Idempotent Law	$p \vee p \equiv p$	$p \wedge p \equiv p$
2	Identity Law	$p \vee F \equiv p$	$p \wedge T \equiv p$
3	Dominant Law	$p \vee T \equiv T$	$p \wedge F \equiv F$
4	Complement Law	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
5	Commutative Law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
6	Associative Law	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
7	Distributive Law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \wedge (p \wedge r)$
8	Absorption Law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9	De Morgan's Law	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
10	Double Negation Law	$\neg(\neg p) \equiv p$	-

Logical Equivalences involving Conditional Statements

$$p \rightarrow q \equiv \neg p \vee q$$
$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$
$$p \vee q \equiv \neg p \rightarrow q$$
$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$
$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$
$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$
$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$
$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$
$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Logical Equivalences involving Biconditional Statements

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Inverse, Converse, and Contra-positive

A conditional statement has two parts − **Hypothesis** and **Conclusion**.

Example of Conditional Statement − "If you do your homework, you will not be punished." Here, "you do your homework" is the hypothesis and "you will not be punished" is the conclusion.

Inverse –An inverse of the conditional statement is the negation of both the hypothesis and the conclusion. If the statement is “If p, then q”, the inverse will be “If not p, then not q”. The inverse of “If you do your homework, you will not be punished” is “If you do not do your homework, you will be punished.”

Converse–The converse of the conditional statement is computed by interchanging the hypothesis and the conclusion. If the statement is “If p, then q”, the inverse willbe “If *q*, then*p*”. The converse of "If you do your homework, you will not be punished" is "If you will not be punished, you do not do your homework”.

Contra-positive –The contra-positive of the conditional is computed by interchanging the hypothesis and the conclusion of the inverse statement. If the statement is “If p, then q”, the inverse will be “If not *q*, then not *p*”. The Contra-positive of "If you do your homework, you will not be punished” is "If you will be punished, you do yourhomework”.

Duality Principle

Duality principle set states that for any true statement, the dual statement obtained by interchanging unions into intersections (and vice versa) and interchanging Universal set into Null set (and vice versa) is also true. If dual of any statement is the statement itself, it is said **self-dual**statement.

DUALITY LAW

The *dual* of a compound proposition that contains only the logical operators $\vee$, $\wedge$ and $\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T, where T and F are special variables representing compound propositions that are tautologies and contradictions respectively. The dual of a proposition A is denoted by A^*.

DUALITY THEOREM

If $A(p_1, p_2, \ldots, p_n) \equiv B(p_1, p_2, \ldots, p_n)$, where A and B are compound propositions, then $A^*(p_1, p_2, \ldots, p_n) \equiv B^*(p_1, p_2, \ldots, p_n)$.

Proof

In Table (1.7), we have proved that

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \text{ or } p \vee q \equiv \neg(\neg p \wedge \neg q) \quad (1)$$

Similarly we can prove that

$$p \wedge q \equiv \neg(\neg p \vee \neg q) \quad (2)$$

Note (1) and (2) are known as *De Morgan's laws.*

Using (1) and (2), we can show that

$$\neg A(p_1, p_2, \ldots, p_n) \equiv A^*(\neg p_1, \neg p_2, \ldots, \neg p_n) \quad (3)$$

Equation (3) means that the negation of a proposition is equivalent to its dual in which every variable (primary proposition) is replaced by its negation. From Eq. (3), it follows that

$$A(p_1, p_2, \ldots, p_n) \equiv \neg A^*(\neg p_1, \neg p_2, \ldots, \neg p_n) \quad (4)$$

Now since $A(p_1, p_2, \ldots, p_n) \equiv B(p_1, p_2, \ldots, p_n)$, we have $A(p_1, p_2, \ldots, p_n) \leftrightarrow B(p_1, p_2, \ldots, p_n)$ is tautology

$\therefore \quad A(\neg p_1, \neg p_2, \ldots, \neg p_n) \leftrightarrow B(\neg p_1, \neg p_2, \ldots, \neg p_n)$ is also a tautology (5)

Using (4) in (5), we get

$\neg A^*(p_1, p_2, \ldots, p_n) \leftrightarrow \neg B^*(p_1, p_2, \ldots, p_n)$ is a tautology.

$\therefore A^* \leftrightarrow B^*$ is a tautology.

$\therefore A^* \equiv B^*$

Examples

(i) The dual of $(P \wedge \neg Q) \vee R$ is $(P \vee \neg Q) \wedge R$

(ii) The dual of $(T \vee \neg P) \wedge Q$ is $(F \wedge \neg P) \vee Q$

(iii) The dual of $(P \rightarrow Q) \wedge (R \vee F) \Leftrightarrow (\neg P \vee Q) \wedge (R \vee F)$ is $(\neg P \wedge Q) \vee (R \wedge T)$

NAND OPERATOR

The operator **NAND** is a combination of **'NOT'** and 'AND' where NOT stands for negation and AND stands for conjunction.

Definition 16 : Let p and q be propositions. The proposition p NAND q is denoted by $p \mid q$. $p \mid q$ is true when either p or q, or both are false and it is false when both p and q are true.

Note : The NAND operator, denoted by $\mid$, is called Sheffer stroke after the logician H.M.Sheffer. NAND operator is also known as **'alternative denial'**.

The truth table for $p \mid q$ is

p	q	$p \mid q$
T	T	F
T	F	T
F	T	T
F	F	T

NOR OPERATOR

The operator NOR is a combination of 'NOT' and 'OR' where NOT stands for negation and OR stands for disjunction.

Definition 17 : Let p and q be propositions. The proposition p NOR q is denoted by $p \downarrow q$. $p \downarrow q$ is true when both p and q are false and false otherwise.

The truth table for $p \downarrow q$ is

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

The connective NAND $|$ has the following equivalence.

$$p \mid p \Leftrightarrow \neg (p \wedge p) \Leftrightarrow \neg p \vee \neg p \Leftrightarrow \neg p$$

Thus $\quad p \mid p \Leftrightarrow \neg p; \; p \mid q \Leftrightarrow \neg (p \wedge q)$

$$(p \mid p) \mid (q \mid q) \Leftrightarrow p \vee q$$

The connective NAND is commutative but not associative.

$$p \mid q \Leftrightarrow q \mid p \text{ but}$$

$$p \mid (q \mid r) \Leftrightarrow \neg p \vee (q \wedge r)$$

$$\text{and } (p \mid q) \mid r \Leftrightarrow \neg (p \wedge q) \mid r$$

$$\Leftrightarrow \neg \left(\neg(p \wedge q) \wedge r \right)$$

$$\Leftrightarrow p \wedge q \vee \neg r$$

The connective NOR has the following equivalence.

$$p \downarrow p \Leftrightarrow \neg p$$

$$(p \downarrow p) \downarrow (p \downarrow q) \Leftrightarrow p \vee q$$

The connective $\downarrow$ is commutative but not associative.

Functionally Complete set of Connectives

Definition 17. A set S of connectives is said to be functionally complete if every statement formula can be expressed in terms of an equivalent formula containing the connectives only from S.

The set of connectives $\{\neg, \wedge\}$, $\{\neg, \vee\}$ are minimal functionally complete sets. Because, to eliminate the conditional $\rightarrow$ we use the equivalence $p \rightarrow q \equiv \neg p \vee q$ ... (1) and to eliminate the biconditional we use $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

or $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv (\neg p \vee q) \wedge (\neg q \vee p)$

Thus conditional and biconditional can be replaced by the three connectives $\vee, \wedge, \neg$. Now using De-Morgan's laws $p \wedge q \equiv \neg(\neg p \vee \neg q)$ and $p \vee q \equiv \neg(\neg p \wedge \neg q)$, we can rewrite any formula in terms of $\{\vee, \neg\}$ or $\{\wedge, \neg\}$. Hence these are the minimal functionally complete sets of connectives.