CHAPTER 3
ELECTROMAGNETIC THEORY

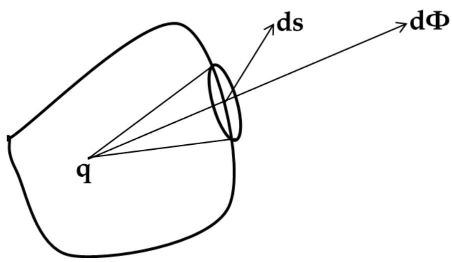
Maxwell's Equations:

Gauss Law:

"Total normal electric induction over a closed surface is equal to sum of all the charges enclosed by that surface."

Consider charge q enclosed in volume V.

Let dΦ be the electric flux through area ds as shown in fig.



By definition of electric field,

$$d\phi = E\,.\hat{n}\,ds$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q\,\hat{r}.\hat{n}\,ds\cos\theta}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0}d\omega$$

Total flux passing through whole area,

$$\phi = \int d\phi = \oint E.ds = \frac{q}{4\pi\varepsilon_0}\int d\omega = \frac{q}{\varepsilon_0} \qquad \left(\because \int d\omega = 4\pi\right)$$

If ρ is volume density of charge,

$$\oint E.ds = \frac{1}{\varepsilon_0}\int \rho\, dv$$

According to Gauss' divergence theorem,

$$\oint E.ds = \int_v \nabla.E\, dv = \frac{1}{\varepsilon_0}\int \rho\, dv$$

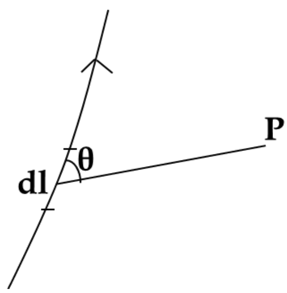
$$\therefore \nabla.E = \frac{\rho}{\varepsilon_0} \qquad \text{.....................(Eq. 3.9)}$$

This is Maxwell's first equation.

3.2.2 Biot - Savart Law:

The magnetic induction B at a point P shown in fig., due to current element I.dl is,

$$dB = \frac{\mu_0}{4\pi}\frac{I\, dl \sin\theta}{r^2} \qquad \text{............(Eq.3.10)}$$



Let r be the position vector of P.

$$\therefore dB = \frac{\mu_0}{4\pi}\frac{I\, d\bar{l}\times\bar{r}}{r^3}$$

Total induction at P due to whole conductor,

$$B = \frac{\mu_0}{4\pi}\int I\,\frac{d\bar{l}\times\bar{r}}{r^3} \qquad \text{............(Eq.3.11)}$$

Direction of B is that of $d\bar{l} \times \bar{r}$

Taking divergence of equation (3.11)

$$\nabla . B = \frac{\mu_0 I}{4\pi} \int \nabla . \frac{d\bar{l} \times \bar{r}}{r^3} \qquad \text{............(Eq.3.12)}$$

$$\overline{\nabla} . (d\bar{l} \times \frac{\bar{r}}{r^3}) = \frac{\bar{r}}{r^3} . \left(\overline{\nabla} \times d\bar{l}\right) - d\bar{l} . \left(\overline{\nabla} \times \frac{\bar{r}}{r^3}\right) \qquad \text{............(Eq.3.13)}$$

dl is independent of coordinates of P, hence $\overline{\nabla} \times d\bar{l} = 0$. Equation (3.13) becomes

$$\overline{\nabla} . (d\bar{l} \times \frac{\bar{r}}{r^3}) = -d\bar{l} . \overline{\nabla} \times \left(\frac{\bar{r}}{r^3}\right)$$

$$= d\bar{l} . \overline{\nabla} \times \overline{\nabla}\left(\frac{1}{r}\right)$$

$$= 0$$

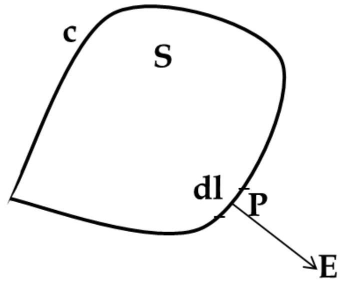
Putting in equation (3.),

$$\nabla . B = 0 \qquad \text{....................(Eq. 3.14)}$$

This is Maxwell’s second equation.

Faraday’s Law of Electromagnetic Induction:

Consider surface S having closed circuit c as shown in figure. If the electric intensity at point p is E, work done in taking unit positive charge through element dl is E.dl. Total work done in taking unit positive charge once round the circuit $= \int E.dl$. By definition, it is e.m.f.



If B is magnetic flux density, total flux enclosed by circuit c $= \int_s B.n\,ds$

By Faraday's law,

$$e.m.f. = \int_c E.dl = -\frac{\partial}{\partial t}\int_s B.n\,ds \qquad \text{.....................(Eq. 3.15)}$$

$$\therefore \int_c E.dl = -\int_s \frac{\partial B}{\partial t} n\,ds$$

By Stoke's theorem,

$$\int_c E.dl = \int_s \nabla \times E.n\,ds$$

$$\therefore \int_s \nabla \times E.n\,ds = -\int_s \frac{\partial B}{dt} n\,ds$$

$$\therefore \nabla \times E = -\frac{\partial B}{dt} \qquad \text{.....................(Eq. 3.16)}$$

This is Maxwell's third equation.

Ampere's Theorem:

"Magnetic flux associated with any closed path is μ_0 times current enclosed by that path."

If B is flux density, total flux,

$$\oint B.dl = \mu_0 I = \mu_0 \int J.ds \qquad \text{.....................(Eq. 3.17)}$$

By Stoke's theorem,

$$\oint B.dl = \int_s \nabla \times B\,ds = \mu_0 \int J.ds$$

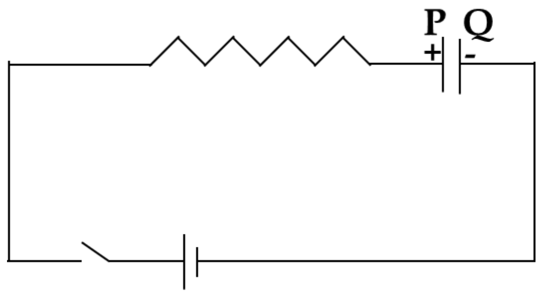
$$\therefore \nabla \times B = \mu_0 J \qquad \text{.....................(Eq. 3.18)}$$

Maxwell's concept of displacement current:

Divergence of curl is zero.

$$\therefore \nabla.(\nabla \times B) = \mu_0 \nabla.J = 0 \qquad \text{.....................(Eq. 3.19)}$$

This equation demands $\nabla.J = 0$. i.e. current density is divergenceless. Current density is produced by moving charge. For steady current in closed loop, $\nabla.J = 0$ and equation (3.19) is applicable. But, if there is accumulation of charge anywhere in the circuit, $\nabla.J \neq 0$. Simple practical example of this situation is capacitor in circuit during charging and discharging phase. Consider circuit as shown in figure.



When switch is closed, current flows through connecting wire from positive terminal of battery to plate P of capacitor and from plate Q of capacitor to negative terminal of battery, and it is conduction current. No charge and hence no current is flowing between the plates of capacitor. However experimental proof of Ampere's law led to conclusion that varying electric field between two plates of capacitor is equivalent to current called displacement current. Displacement current produces magnetic field like conduction current.

Surface charge density on capacitor is σ.

Displacement current $= \dfrac{d\sigma}{dt} = \dfrac{dD}{dt}$, $\qquad D = \varepsilon_0 E$

Now, equation (3.18) becomes,

$$\nabla \times B = \mu_0 \left[J + \frac{\partial D}{\partial t} \right]$$

$$\nabla \times B = \mu_0 \left[J + \varepsilon_0 \frac{\partial E}{\partial t} \right] \qquad \text{.....................(Eq. 3.20)}$$

This is Maxwell's fourth equation.

Maxwell's equations,

$$\nabla . E = \frac{\rho}{\varepsilon_0}, \qquad \nabla . B = 0,$$

$$\nabla \times E = -\frac{\partial B}{dt} \qquad \nabla \times B = \mu_0 \left[J + \varepsilon_0 \frac{\partial E}{\partial t} \right]$$

Electromagnetic wave nature of light:

'Varying electric and magnetic fields in Y and Z – directions are given as,

$E_y = E_0 \sin(x - ct)$ ....(Eq. 3.21)

$B_z = B_0 \sin(x - ct)$ ....(Eq. 3.22)

Maxwell's third equation,

$$\frac{\partial E_y}{dx} = -\frac{\partial B_z}{dt}$$

Significance of this equation is, 'Time varying magnetic field produces electric field varying in space.' Differentiating equations (3.21) and (3.22)

$$\frac{\partial E_y}{dx} = E_0 \cos(x - ct)$$

$$\frac{\partial B_z}{dt} = -B_0 c \cos(x - ct)$$

$$\therefore\ E_0 = cB_0 \qquad c = \frac{E_0}{B_0} \qquad \text{.....................(Eq. 3.23)}$$

Maxwell's fourth equation,

$$\frac{\partial B_z}{dx} = \mu_0 \varepsilon_0 \frac{\partial E_y}{dt}$$

'Time varying electric field produces magnetic field varying in space.

$$\frac{\partial B_z}{dx} = B_0 \cos(x - ct)$$

$$\frac{\partial E_y}{dt} = -E_0 c \cos(x - ct)$$

$$\therefore B_0 = -c\mu_0\varepsilon_0 E_0$$

$$1 = -c^2\mu_0\varepsilon_0$$

$$\therefore c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8\, m/s \qquad \text{.....................(Eq. 3.24)}$$

It comes out to be velocity of light in vacuum. Thus it proves electromagnetic wave nature of light.

Electromagnetic wave propagation:

Free space:

Maxwell's equations for free space can be written as,

$$\nabla.\overline{E} = 0, \qquad \nabla.\overline{H} = 0,$$

$$\nabla\times\overline{E} = -\mu_0\frac{\partial\overline{H}}{\partial t}, \qquad \nabla\times\overline{H} = \varepsilon_0\frac{\partial\overline{E}}{\partial t}$$

Taking curl of Maxwell's third equation for free space,

$$\nabla\times(\nabla\times E) = -\mu_0\frac{\partial}{\partial t}(\nabla\times H)$$

$$\therefore \nabla(\nabla.E) - \nabla^2 E = -\mu_0\frac{\partial}{\partial t}\left(\varepsilon_0\frac{\partial E}{\partial t}\right) \qquad \text{Using fourth equation.}$$

$$\therefore \nabla^2 E = \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}$$

$$\therefore \nabla^2 E - \mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2} = 0 \qquad \text{.....................(Eq. 3.25)}$$

Comparing with general wave equation,

$$\therefore \nabla^2 y - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0$$

Thus equation (3.25) is wave equation governing the field E.

Similarly,

$$\therefore \nabla^2 H - \mu_0\varepsilon_0\frac{\partial^2 H}{\partial t^2} = 0 \qquad \text{.....................(Eq. 3.26)}$$

In view of dimensions, velocity, $v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

The plane wave solution for equations (3.25) and (3.26),

$\overline{E}(r,t) = E_0 e^{i(\overline{k}.\overline{r}-\omega t)}$ and $\overline{H}(r,t) = H_0 e^{i(\overline{k}.\overline{r}-\omega t)}$

Where ω is angular frequency and k is wave vector. ω/k is phase velocity.

$\overline{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$, $\quad E_0 = E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k}$,

$k = k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$, $\quad r = x\hat{i} + y\hat{j} + z\hat{k}$

$\overline{k}.\overline{r} = (k_x x + k_y y + k_z z)$

$$\overline{\nabla} \times \overline{E} = \overline{\nabla} \times \left(E_0 e^{i(\overline{k}.\overline{r} - \omega t)} \right)$$

$$= i\left[\overline{k} \times E_0 e^{i(\overline{k}.\overline{r}-\omega t)}\right]$$

$$= i\left[\overline{k} \times \overline{E}\right] \quad \text{.....................(Eq. 3.27)}$$

From Maxwell's third equation and equation (3.27)

$$-\mu_0 \frac{\partial H}{\partial t} = i(\overline{k} \times \overline{E})$$

$$-\mu_0 \frac{\partial}{\partial t}\left(H_0 e^{i(\overline{k}.\overline{r}-\omega t)}\right) = i(\overline{k} \times \overline{E})$$

$$i\mu_0 \omega \overline{H} = i(\overline{k} \times \overline{E})$$

$$(\overline{k} \times \overline{E}) = \mu_0 \omega \overline{H} \quad \text{.....................(Eq. 3.28)}$$

Similarly, it can be shown that,

$$(\overline{k} \times \overline{H}) = -\varepsilon_0 \omega \overline{E} \quad \text{.....................(Eq. 3.29)}$$

From equation (3.28), it is obvious that magnetic field is perpendicular to both wave vector k and electric field E and from equation (3.29), it is obvious that electric field is perpendicular to both wave vector k and magnetic field E. Thus, it may be concluded that electric and magnetic field vectors are perpendicular to each other and perpendicular to direction of propagation. Thus electromagnetic waves are transverse in nature.

Dielectric Medium:

For dielectric medium, current density J and volume charge density ρ are zero.

$\bar{J}=0$, ρ = 0, $\bar{D}=\varepsilon\bar{E}$ and $\bar{B}=\mu\bar{H}$,

Where, ε and μ are absolute permittivity and permeability of the medium.

Now, Maxwell's equations becomes,

$\nabla.\bar{E}=0$, $\quad\nabla.\bar{H}=0$,

$\nabla\times\bar{E}=-\mu\frac{\partial\bar{H}}{\partial t}$, $\quad\nabla\times\bar{H}=\varepsilon\frac{\partial\bar{E}}{\partial t}$

Taking curl of third and fourth equations,

$$\nabla^2E-\mu\varepsilon\frac{\partial^2E}{\partial t^2}=0 \qquad \text{.....................(Eq. 3.30)and}$$

$$\nabla^2H-\mu\varepsilon\frac{\partial^2H}{\partial t^2}=0 \qquad \text{.....................(Eq. 3.31)}$$

Comparing these equations with general wave equation,

$$v=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{1}{\sqrt{\mu_0\mu_r\,\varepsilon_0\varepsilon_r}}=\frac{c}{\sqrt{\mu_r\,\varepsilon_r}}$$

Thus velocity of electromagnetic wave in a dielectric medium is less than that in free space.

Refractive index of medium, $n=\frac{c}{v}=\sqrt{\mu_r\varepsilon_r}$

For non magnetic dielectric medium, $\mu_r=1$, $\therefore n=\sqrt{\varepsilon_r}$

Boundary conditions:

Boundary conditions for Electric Field:

For Dielectric – Dielectric Interface:

Consider the interface between two dielectrics with permittivity ε_1 and ε_2. Electric field intensities for the two media can be written as,

$E_1 = E_{1t} + E_{1n}$

$E_2 = E_{2t} + E_{2n}$

E_t and E_n represents tangential and normal components of E respectively.

Consider closed path abcda as shown in figure 3.5.

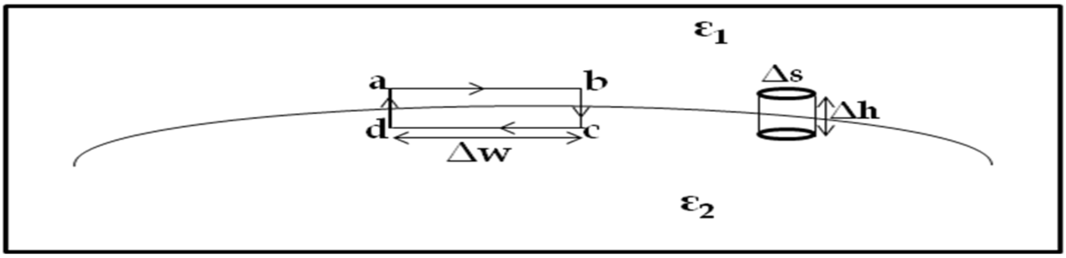


Fig.3.5

Applying Maxwell's equation to this path,

$$\oint E.dl = 0$$

$$E_{1t}\Delta w - E_{1n}\frac{\Delta h}{2} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta w + E_{2n}\frac{\Delta h}{2} + E_{1n}\frac{\Delta h}{2} = 0$$

$$\therefore (E_{1t} - E_{2t})\Delta w = 0$$

$$E_{1t} = E_{2t} \qquad \text{.....................(Eq. 3.32)}$$

Thus tangential component is same on both the sides of boundary. The potential difference between any two points on the boundary separated by a distance Δw is same immediately above and below the boundary.

$$D = \varepsilon E$$

$$\frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2}$$

Thus D_t is discontinuous across the interface. Consider small pillbox as shown in figure 3.5. Applying Gauss law,

$$D_{1n}\Delta s - D_{2n}\Delta s = \Delta Q = \sigma$$

where σ is surface density of charge. If no free charges exist at the interface, σ = 0.

$$D_{1n} = D_{2n} \qquad \text{.....................(Eq. 3.33)}$$

The normal component of D is continuous across the interface.

$$\varepsilon_1 E_1 = \varepsilon_2 E_2 \qquad \text{.....................(Eq. 3.34)}$$

The normal component of E is discontinuous across the boundary.

The equations (3.32), (3.33) and (3.34) are collectively called boundary conditions for electric field at the boundary between two different dielectric media. These boundary conditions are used to find electric field on one side of boundary, if field on other side is known. These conditions can also be used to study refraction of electric field at the boundary.

Consider figure 3.6.

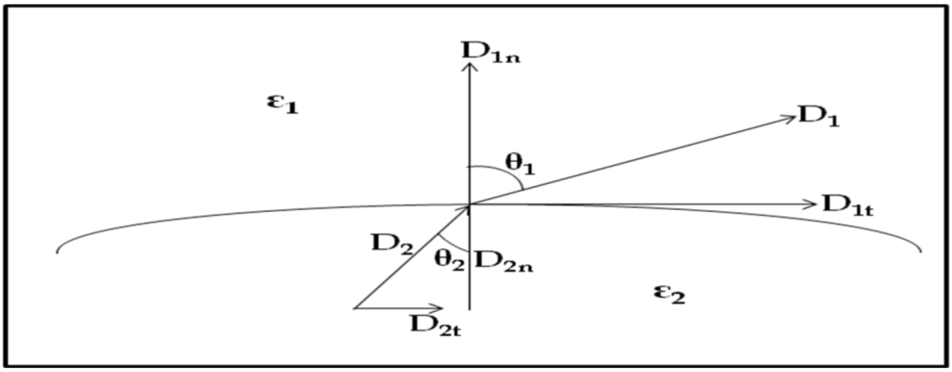


Fig.3.6

Let D1 (and E1) make angle θ1 with a normal to the surface. Since normal components of D are continuous,

$$D_{1n}=D_1\cos\theta_1 = D_{2n} = D_2\cos\theta_2$$

$$\therefore D_{1n}=\varepsilon_1 E_1\cos\theta_1 = D_{2n} = \varepsilon_2 E_2\cos\theta_2 \qquad \text{.....................(Eq. 3.35)}$$

Tangential components,

$$\mathrm{E}_{1t} = E_1 \sin\theta_1 = \mathrm{E}_{2t} = E_2\sin\theta_2 \qquad \text{.....................(Eq. 3.36)}$$

Dividing equation (3.36) by (3.35)

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} \qquad \text{.....................(Eq. 3.37)}$$

If ε1 > ε2, θ1 > θ2.

This is law of refraction for the electric field at the interface between two dielectrics. Thus there is bending of electric flux lines due to unequal polarization charges that accumulates on opposite sides of the interface.

For Conductor – Dielectric Interface:

There is no electric field inside the conductor. Hence electric field must be external to the conductor and normal to the surface.

$$D_t = \varepsilon E_t = \varepsilon_0\varepsilon_r E_t = 0, \qquad D_n = \varepsilon_0\varepsilon_r E_n = \sigma \qquad \text{.....................(Eq. 3.38)}$$

For conductor – free space interface,

$D_t = \varepsilon_0 E_t = 0, \quad D_n = \varepsilon_0 E_n = \sigma$ (For free space, $\varepsilon_r = 1$)

Boundary conditions for Magnetic Field:

Magnetic field satisfies certain boundary conditions at interface between two different media.

$$\oint B.ds = 0$$

$$B_{1n}\Delta s - B_{2n}\Delta s = 0$$

$$B_{1n} = B_{2n} \quad \text{OR} \quad \mu_1 H_{1n} = \mu_2 H_{2n} \qquad \text{.....................(Eq. 3.39)}$$

Normal component of B is continuous across the boundary, but that of H is discontinuous. Tangential component of H is continuous, but that of B is discontinuous at the boundary. It can be shown that,

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} \qquad \text{.....................(Eq. 3.40)}$$

This is law of refraction for magnetic flux at the boundary.

Poynting Theorem:

Energy density is associated with electric and magnetic fields of electromagnetic wave and wave carries energy when it propagates through medium. The amount of energy flowing through the medium per unit area per unit time, perpendicular to direction of propagation is called Poynting vector.

$$\overline{P} = \overline{E} \times \overline{H}\,, \qquad \text{.....................(Eq. 3.41)}$$

E and H are instantaneous electric and magnetic field vectors.

$$\nabla.\left(\overline{E} \times \overline{H}\right) = -\overline{E}.\left(\nabla \times \overline{H}\right) + \overline{H}.\left(\nabla \times \overline{E}\right)$$

Using Maxwell's third and fourth equations,

$$\nabla.\left(\overline{E} \times \overline{H}\right) = -\overline{E}.\frac{\partial \overline{D}}{\partial t} - \overline{H}.\frac{\partial \overline{B}}{\partial t}$$

$$\nabla.\left(\overline{E} \times \overline{H}\right) = -\left[\overline{E}.\frac{\partial}{\partial t}\left(\varepsilon_0 E\right) + \overline{H}.\frac{\partial}{\partial t}\left(\mu_0 H\right)\right]$$

$$\therefore \nabla.\left(\overline{E} \times \overline{H}\right) = -\frac{1}{2}\left[\frac{\partial}{\partial t}\left(\varepsilon_0 E^2 + \mu_0 H^2\right)\right] \qquad \text{.....................(Eq. 3.42)}$$

Right hand side of this equation represents the energy stored in electric and magnetic fields respectively and their sum is total energy stored in electromagnetic field.

Questions for Practice

Q.1. Derive Maxwell's first equation from Gauss law.

Q.2. Using Biot – Savart law, obtain expression for magnetic field produced by long thin current carrying conductor.

Q.3. Explain the Maxwell's concept of displacement current.

Q.4. Derive Maxwell's second equation.

Q.5. Derive Maxwell's third equation from Faraday's law of electromagnetic induction.

Q.6. What is physical significance of Maxwell's second equation.

Q.7. Derive Maxwells' fourth equation.

Q.8. Write Maxwell's electromagnetic equations and their physical significance.

Q.9. Show that light is electromagnetic wave in nature from Maxwell's equations.

Q.10. Deduce expression foe velocity of propagation of plane electromagnetic wave in a medium with permittivity ε and permeability μ.

Q.11. Write boundary conditions for electric and magnetic fields.

Q.12. Obtain law of refraction for the electric field at the interface between two dielectrics.

Q.13. Obtain law of refraction for the magnetic field at the interface between two media.

Q.14. Define Poynting vector. Derive expression for it and explain its physical significance for electromagnetic wave in free space.

Problems for Practice

1. If 3000 lines of force enter a given volume of space and 5000 lines diverge from it, find the total charge confined in the volume. (1.77×10^{-8}C)

2. A point charge of 13.5 μC is placed at the centre of cube of side 6 cm. Find the electric flux through (i) whole volume and (ii) one face of cube.

(1.525×10^{6} Nm^2/C, 0.254×10^{6} Nm^2/C)

3. Calculate the value of Poynting vector at the surface of Sun if the power radiated by sun is 3.8 X 10^{33}erg/s and its radius is 7 X 10^{8}m. (6.174 X 10^{7}W/m^{2})

4. If the relative permittivity of liquid is 81, calculate refractive index of liquid and velocity of light in liquid. (0.33 X 10^{8}m/s)

5. Refractive index of medium is 1.5 and permeability 5X10^{-7}H/m. Find relative permittivity. (5.67)

6. Flux linked with coil is $4t^{2}$ at t = 2s, find emf.

7. Determine magnetic field applied to system of Electrons moving with velocity 4 m/s when subjected to electric field 20 units. (5 units)

8. For free space, E = 20 cos (ωt- 50 x) V/m. Calculate displacement current.