UNIT - II
NORMAL FORMS

Disjunctive Normal forms:

Definition: (Elementary Product)

The product of the variables and their negation in a formula is called an Elementary product

Definition: (Elementary sum)

The sum of the variables and their negation in a formula is called an Elementary sum.

Note:

Sum means disjunction and product means conjunction

Example:

Let P and Q be two statement tautology then

i) $P\vee\neg P$ is an elementary sum
ii) $P\vee Q$ is an E.S
iii) $P\vee\neg Q$ is an E.S
iv) $\neg P\vee Q$ is an E.S
v) $\neg P\vee\neg Q$ is an E.S
vi) $P\wedge\neg P$ is an E.P

Disjunctive normal form (DNF)

Definition

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula

Procedure to obtain DNF

1. An equivalent formula can be obtained by replacing $\rightarrow$ and $\leftrightarrow$ with $\wedge$, $\vee$ and $\neg$
2. Apply negation to the formula or to a part of the formula and not to the variables
3. Using Demorgan's laws, apply negation to variables
4. Repeated application of distributive laws will give +ve

Remark 2:

A given formula is false if every elementary product in DNF is identically false .

DNF = (elementary product)$\vee$ (elementary product)$\vee$….(..)

Problem 1:

Obtain disjunctive normal form of $P\wedge(P\rightarrow Q)$

Solution

$P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

Which is a sum of elementary product

$\therefore$DNF is $(P \wedge \neg P) \vee (P \wedge Q)$

Problem 2:

Obtain disjunctive normal form of $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$ is $(P \wedge \neg Q) \vee (Q \wedge \neg P)$

Solution:

We know that $P \leftrightarrow Q \leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$ ...(1)

Now, $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$\Leftrightarrow [\neg(P \vee Q) \wedge (P \wedge Q)] \vee [(P \vee Q) \wedge (P \wedge Q)]$ ...(by (1))

$\Leftrightarrow [(\neg P \vee \neg Q) \wedge (P \wedge Q)] \vee [(P \vee Q) \wedge (\neg P \wedge \neg Q)]$

$\Leftrightarrow [(\neg P \vee P) \wedge (\neg Q \wedge Q)] \vee [P \wedge (\neg P \vee \neg Q) \vee (Q \wedge (\neg P \wedge \neg Q))]$

$\Leftrightarrow [F \wedge F] \vee [(P \wedge \neg P) \vee (P \wedge \neg Q)] \vee [(Q \wedge \neg P) \vee (Q \wedge \neg Q)]$

$\Leftrightarrow F \vee [(F \vee (P \wedge \neg Q)) \vee ((Q \wedge \neg P) \vee F)]$

$\Leftrightarrow F \vee [(P \wedge \neg Q)) \vee (Q \wedge \neg P)]$

$\Leftrightarrow (P \wedge \neg Q)) \vee (Q \wedge \neg P)$

The disjunctive normal form of $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$ is $(P \wedge \neg Q) \vee (Q \wedge \neg P)$

Solution:

We know that $P \leftrightarrow Q \leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$ ...(1)

$\Leftrightarrow [\neg(P \vee Q) \wedge (P \wedge Q)] \vee [(P \vee Q) \wedge (P \wedge Q)]$ ...(by (1))

$\Leftrightarrow [(\neg P \vee \neg Q) \wedge (P \wedge Q)] \vee [(P \vee Q) \wedge (\neg P \vee \neg Q)]$ (by Demorgan's Law)

$\Leftrightarrow [(\neg P \vee \neg Q) \wedge (P \wedge Q)] \vee [(P \wedge (\neg P \vee \neg Q) \vee (Q \wedge (\neg P \wedge \neg Q)]$

$\Leftrightarrow [(P \vee Q) \wedge (\neg P \vee \neg Q)] \vee [(P \wedge \neg P) \vee (P \wedge \neg Q)] \vee [(Q \wedge \neg P) \wedge (Q \wedge \neg Q)]$

$\Leftrightarrow [P \vee Q \wedge \neg P \vee Q] \vee [P \wedge \neg P \vee P \wedge \neg Q] \vee [Q \wedge \neg P \wedge Q \wedge \neg Q]$

Which is a sum of elementary product

Obtain a disjunctive normal form of $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

Solution:

$\Leftrightarrow P \rightarrow ((\neg P \vee Q) \wedge \neg\neg(Q \vee P))$

$\Leftrightarrow P \rightarrow ((\neg P \vee Q) \wedge (Q \vee P))$

$\Leftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \vee P))$

$\Leftrightarrow [\neg P \vee (\neg P \vee Q)] \wedge [\neg P \vee (Q \vee P)]$

$\Leftrightarrow [(\neg P \vee Q)] \wedge [(\neg P \vee Q) \wedge (\neg P \vee P)]$

$\Leftrightarrow [(\neg P \vee Q)] \wedge [(\neg P \vee Q) \wedge T]$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee Q)$

$\Leftrightarrow \neg P \vee Q$

Alter

$\Leftrightarrow [\neg P \wedge (\neg P \vee Q)] \vee [Q \wedge (\neg P \vee Q)]$

$\Leftrightarrow [(\neg P \wedge \neg P) \vee (\neg P \wedge Q)] \vee [(Q \wedge \neg P) \vee (Q \wedge Q)]$

$\Leftrightarrow [\neg P \vee (\neg P \wedge Q)] \vee [(Q \wedge \neg P) \vee Q]$

Principal disjunctive normal forms:

Let P and Q be two statement variables. Then, we can write 2^2 formulas $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$ and $\neg P \wedge \neg Q$

These formulas are called minter or Boolean conjunctions of P and Q

A formula which is equivalent to a given formula and which consists of sum of its minterms is called "principal disjunctive normal form" (PDNF)

Construction of PDNF without truth table

- To replace conditionals and bi-conditionals by their equivalent formula involving $\wedge$, $\vee$, $\neg$ only
- To use Demorgan's law and distributive law
- To drap any elementary product which is a contradiction
- To obtain min-terms in the disfunction by introducing missing factors
- To delete identical minterms keeping only one, that appear in the disjunction

Find the minterms of P, Q, R

Solution:

There 2^3 minterms. They are $P \wedge Q \wedge R$, $P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge R$, $\neg P \wedge Q \wedge R$, $\neg P \wedge \neg Q \wedge R$, $\neg P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge \neg R$

Note:

i) $P \wedge Q$ or $Q \wedge P$ is include but not both
ii) $P \wedge \neg P$ and $Q \wedge \neg Q$ are not allowed
iii) No two minterms are equivalent

iv) Each minterm has the truth value T. for every exactly one combination of the truth values of the varivbles Pand Q

Note 2:

In general for a given n-number of variables, there will be 2^n minterms

Minterms of P and Q

P	**Q**	**¬P**	**¬Q**	**P∧Q**	**¬P∧Q**	**P∧¬Q**	**¬P∧¬Q**
T	T	F	F	**T**	F	F	F
T	F	F	T	F	F	**T**	F
F	T	T	F	F	**T**	F	F
F	F	T	T	F	F	F	**T**

Problem 1:

Obtain the PDNF for ¬P∧Q

Solution:

¬P∧Q

⇔[¬P∧(Q∨¬Q)]∨[Q∧(P∨¬P)] (∵Q∨¬Q⇒T)

⇔ [(¬P∧Q)∨(¬P∧¬Q)]∨[(Q∧P)∨(Q∧¬P)] (by distributive law)

⇔ (¬P∧Q)∨(¬P∧¬Q)∨(Q∧P)∨(Q∧¬P) (∵(P∧Q)∨(P∧Q)⇔P∧Q)

Which is a product of minterms

The PDNF is (¬P∧Q)∨(¬P∧¬Q)∨(Q∧P)

Problem 2:

Obtain PDNF for (P∧Q)∨(¬P∧R)∨(Q∧R)

Solution:

(P∧Q)∨(¬P∧R)∨(Q∧R)

⇔[(P∧Q)∧(R∨¬R)]∨[(¬P∧R)∧(Q∨¬Q)]∨[(Q∧R)∧(P∨¬P)] (∵P∧T=T)

⇔[((P∧Q)∧R)∨((P∧Q)∧¬R)]∨[((¬P∧R)∧Q)∨((¬P∧R)∧¬Q)]∨[((Q∧R)∧P)∨((Q∧R)∧¬P)]

⇔[(P∧Q∧R)∨(P∧Q∧¬R)]∨[(¬P∧R∧Q)∨(¬P∧R∧¬Q)]∨[(Q∧R∧P)∨(Q∧R∧¬P)]

⇔(P∧Q∧R)∨(¬P∧R∧Q)∨(P∧Q∧¬R)∨(¬P∧R∧¬Q)

Which is a product of P,Q

Problem 3:

Obtain PDNF for P→[(P→Q)∧¬(¬Q∨¬P)]

Solution:

$P\rightarrow[(P\rightarrow Q)\wedge\neg(\neg Q\vee\neg P)]$

$P\rightarrow[(\neg P\vee Q)\wedge (Q\wedge P)]$

$\neg P\vee[(\neg P\vee Q)\wedge (Q\wedge P)]$

$\neg P\vee[(\neg P\vee Q)\wedge(Q\wedge P)]$

$[\neg P\wedge(Q\vee\neg Q)]\vee[(\neg P\vee Q) \wedge (Q\wedge P)]$

$[(\neg P\wedge Q)\vee(\neg P\wedge\neg Q)]\vee[\neg P\wedge(Q\wedge P)\vee Q\wedge(Q\wedge P)]$

$(\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(\neg P\wedge(Q\wedge P))\vee(Q\wedge P)$

$\therefore$ The PDNF is $(\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(\neg P\wedge(Q\wedge P))\vee(Q\wedge P)$

Conjunctive normal form

A formula which is equivalent to a given formula and which consists of a product of elementary sum is called a conjunctive normal form of the given formulas

Remark

CNF = product of elementary sums

= (elementary sum)$\wedge$ (elementary sum)$\wedge$….. (elementary sum)

Principal conjunctive normal form

Let P and Q be two statement variables. Then we can write 2^2 formula $P\vee Q$, $P\vee\neg Q$, $\neg P\vee Q$, $\neg P\vee\neg Q$.

These formulas are called maxterms of Boolean disjunction of P and Q

A formula which is equivalent to a given formula and which consists of product of its maxterm is called "Principal of cinjunctive normal form" (PCNF)

Problem 1:

Obtain a conjunctive normal form of $P\wedge(P\rightarrow Q)$

Proof

$P\wedge(P\rightarrow Q)\Leftrightarrow P\wedge(\neg P\vee Q)$

Which is a product of elementary sums

$\therefore$ The CNF of $P\wedge(P\rightarrow Q)$is $P\wedge(\neg P\vee Q)$

Problem 2:

Obtain a conjunctive form $[Q\vee(P\wedge R)]\wedge\neg[(P\vee R)\wedge Q]$

Solution:

$[Q\vee(P\wedge R)]\wedge\neg[(P\vee R)\wedge Q]$

$[(Q \vee P) \wedge (Q \vee R)] \wedge [\neg(P \vee R) \vee \neg Q]$

$[(Q \vee P) \wedge (Q \vee R)] \wedge [(\neg P \wedge \neg R) \vee \neg Q]$

$(Q \vee P) \wedge (Q \vee R) \wedge (\neg P \vee \neg Q) \wedge (\neg R \vee \neg Q)$

Product of elementary sums

The CNF is $(Q \vee P) \wedge (Q \vee R) \wedge (\neg P \vee \neg Q) \wedge (\neg R \vee \neg Q)$

Maxterms

For a given number of variables the max terms consists of conjunctions in which each variable or its negation, but not both appears only once

Remark

1. The maxterms are duals of minterms
2. Each of the maxterms has the truth value F for exactly one combination of the truth values of the variables.
3. Different max terms have the truth table F for different combinations of the truth values of the variable

P	**Q**	**¬P**	**¬Q**	**P∨Q**	**¬P∨Q**	**P∨¬Q**	**¬P∨¬Q**
T	T	F	F	T	T	T	**F**
T	F	F	T	T	**F**	T	T
F	T	T	F	T	T	**F**	T
F	F	T	T	**F**	T	T	T

Obtain the principal conjunctive normal form of the formula S given by $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

Solution:

$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$(\neg P \rightarrow R) \wedge [(Q \wedge R) \vee (\neg Q \wedge \neg P)]$

$[\neg (\neg P) \vee R] \wedge [(Q \wedge R) \vee (\neg Q \wedge \neg P)]$

$(P \vee R) \wedge [(Q \vee (\neg P \wedge \neg P) \wedge P \vee (\neg Q \wedge \neg P)]$

$(P \vee R) \wedge [(Q \vee \neg P) \wedge (Q \vee \neg P) \wedge (P \vee \neg Q) \wedge (P \vee \neg P)]$

$(P \vee R) \wedge (Q \vee \neg P) \wedge (Q \vee \neg P) \wedge (P \vee \neg Q) \wedge T$

$(P \vee R) \wedge (Q \vee \neg P) \wedge (P \vee \neg Q)$

Alter

$(\neg P \rightarrow R) \wedge [(Q \rightarrow P) \wedge (P \rightarrow Q)]$

$[\neg(\neg P)\vee R]\wedge[(\neg Q\vee P)\wedge(\neg P\vee Q)]$

$(P\vee R)\wedge(\neg Q\vee P)\wedge(\neg P\vee Q)$

$[(P\vee R)\wedge(Q\vee\neg P)]\wedge(\neg Q\vee P)\vee(R\wedge\neg R)]\wedge[(\neg P\vee Q)\vee(R\wedge\neg R)]$

$(P\vee R\vee Q)\wedge(P\vee R\vee\neg Q)\wedge(\neg P\vee R\vee Q)\wedge(\neg P\vee R\vee\neg Q)\wedge(\neg P\vee R\vee Q)\wedge(\neg P\vee\neg R\vee Q)$

Which is a product of max.terms