Recall,

$(S^1, \mathcal{B}(S^1))$

$\Phi : [0,1) \longrightarrow S^1$

$\Phi(t) = e^{2\pi i t}$

If λ is the Lebesgue measure on $[0,1)$, then we define the measure :-

$$\mu := \lambda \circ \Phi^{-1} \text{ on } (S^1, \mathcal{B}(S^1))$$

The map $M_\omega : S^1 \longrightarrow S^1$ given by $z \longmapsto \omega z$ (where $\omega \in S^1$) is a mpt.

We define $\mathbb{Z} \overset{\alpha}{\curvearrowright} S^1$ where

$\alpha(n) \in MP(\mu)$ is given by

$$\alpha(n) = M_{\omega_0^n} = (M_{\omega_0})^n \quad \forall n \in \mathbb{Z}$$

where $\omega_0 \in S^1$ is fixed.

<u>Prop.</u> $\mathbb{Z} \overset{\alpha}{\curvearrowright} S^1$. $\alpha(n) = M_{\omega_0^n}$. TFAE:-

(1) $(\Phi^{-1}(\omega_0)) \in \mathbb{R} \setminus \mathbb{Q}$

(2) $\omega_0^n \neq 1 \quad \forall\, n \in \mathbb{N}$

(3) α is ergodic.

<u>Example</u>:- Action of S_∞ on $\mathbb{R}^{\mathbb{N}}$.

$$S_\infty = \{\tau : \mathbb{N} \longrightarrow \mathbb{N} \mid \tau \text{ is a bijection, and } \exists\, \underset{\text{finite}}{F \subseteq \mathbb{N}} \text{ s.t } \tau(n) = n \ \forall\, n \in \mathbb{N} \setminus F\}$$

To see S_∞ is countable, note that $\forall\, \underset{\text{finite}}{F \subseteq \mathbb{N}}$,

$$S_F = \{\tau : \mathbb{N} \longrightarrow \mathbb{N} \mid \tau \text{ is a bijection, } \tau(n) = n \ \forall\, n \in \mathbb{N} \setminus F\}$$

is finite.

And note that $S_\infty = \bigcup_{\substack{F \subseteq \mathbb{N} \\ F \text{ finite}}} S_F$

Hence S_∞ is countable.

Let P be a prob. measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

<u>Result</u> (Kolmogorov Consistency Thm.)

P : prob. measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Consider $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$.

$\exists!$ prob. measure $\tilde{P}$ on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$ s.t. $\forall n \in \mathbb{N}$,

$\forall E_1, \dots, E_n \in \mathcal{B}(\mathbb{R})$

$\tilde{P}\left(\prod_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} P(E_i)$ where $E_i = \mathbb{R}$ $\forall i > n$.

———— X ————

Coming back to our example, consider $\tilde{P}$ on $\mathbb{R}^{\mathbb{N}}$.

Consider the action

$$\alpha : S_\infty \longrightarrow MP(\tilde{P})$$

Recall, $\mathbb{R}^{\mathbb{N}} = \{ x : \mathbb{N} \longrightarrow \mathbb{R} \mid x \text{ is a function} \}$

$$((\alpha(\tau))(x))(n) = x(\tau^{-1}(n))$$

<u>Fact</u>: $\alpha(\tau)$ is indeed mpt on $\mathbb{R}^{\mathbb{N}}$.

$$\begin{aligned}(\alpha(\tau_1\tau_2))(x)(n) &= x(\tau_2^{-1}\tau_1^{-1}(n)) \\ &= (\alpha(\tau_2)(x))(\tau_1^{-1}(n)) \\ &= ((\alpha(\tau_1)(\alpha(\tau_2))(x))(n)\end{aligned}$$

So, $\alpha(\tau_1\tau_2) = \alpha(\tau_1)\alpha(\tau_2)$

So, α is mpa

<u>Result</u>: α is ergodic

This result is called Hewitt Savage 0-1 law.

Recall: Ergodic Representations :-

$\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$: unitary repn.

π is called ergodic if it has no non-zero invariant vectors

$(\xi \in \mathcal{H}$ s.t $\pi(g)\xi = \xi \ \forall g \in G \implies \xi = 0)$

Unitary Representations from (Discrete) Group Action :-

G - group ; X set $\quad G \overset{\beta}{\curvearrowright} X$

$\ell^2(X)$: Hilbert space

$\overset{\shortparallel}{\{f : X \longrightarrow \mathbb{C} \mid \sum_{x \in X} |f(x)|^2 < \infty\}}$ with $\langle f, g \rangle = \sum_{x \in X} f(x)\overline{g(x)}$

$\pi : G \longrightarrow \mathcal{U}(\ell^2(X))$ by

$(\pi(g)(f))(x) = f(g^{-1}x)$

$$\sum_{x \in X} |f(g^{-1}x)|^2 = \sum_{x \in X} |f(x)|^2 = \|f\|^2$$

$$\shortparallel$$

$$\|\pi(g)(f)\|^2$$

$(\pi(g_1 g_2))(f)(x) = f(g_2^{-1} g_1^{-1} x)$

$= (\pi(g_2)(f))(g_1^{-1}x)$

$= (\pi(g_1)\pi(g_2)(f))(x)$

$\forall g \in G$, $\pi(g)\pi(g^{-1}) = \pi(g^{-1})\pi(g) = \pi(e) = id$

Hence $\pi(g)$ is an invertible isometry. So $\pi(g)$ is unitary.

<u>Prop.</u>: π is ergodic $\Leftrightarrow$ Every orbit (under β) is infinite.

Pf:- ($\Longrightarrow$) Given that π is ergodic, let if possible $F \subseteq X$ be a finite nonempty orbit. Then $gx \in F$ $\forall g \in G$, $\forall x \in F$

Consider $1\!\!1_F : X \longrightarrow \mathbb{C}$ given by

$$1\!\!1_F(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \notin F \end{cases}$$

$1\!\!1_F \in l^2(X)$ (as F is finite)

Since $F \neq \emptyset$, $1\!\!1_F \neq 0$

$$(\pi(g))(1\!\!1_F)(x) = 1\!\!1_F(g^{-1}x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \notin F \end{cases} = 1\!\!1_F(x)$$

Hence $1\!\!1_F$ $(\neq 0)$ is π-invariant and thus π is not ergodic (contradiction)

($\Longleftarrow$) Given every orbit is infinite. Let $f \in l^2(X)$ be π-invariant. We'll show $f = 0$. Fix $x_0 \in X$ arbitrary. Since f is π-invariant, $\pi(g)f = f$ $\forall g \in G$.

Hence $(\pi(g)(f))(x) = f(x) \ \forall g \in G \ \forall x \in X$.

$$f(g^{-1}x) = f(x) \quad \forall g \in G \quad \forall x \in X$$

So $f(g^{-1}x_0) = f(x_0) \quad \forall g \in G \qquad \text{--- } (*)$

Let Y be the orbit of x_0

Then $f(x) = f(x_0) \quad \forall x \in Y \quad (\text{by } (*))$

Hence $\sum\limits_{x \in Y} |f(x)|^2 \leq \sum\limits_{x \in X} |f(x)|^2 < \infty$

Hence $\sum\limits_{x \in Y} |f(x_0)|^2 < \infty$

But Y is infinite. Hence $f(x_0) = 0$. As $x_0 \in X$ was arbitrary $f \equiv 0$. $\blacksquare$