3 Lecture 3

3.1 Transitivity property of $M\ddot{o}b^+$

A set of maps from $X \to X$ is called *transitive* if for any two points $x, y \in X, \exists m$ in the set such that $m(x) = y$.

The $M\ddot{o}b^+$ group is *uniquely triply transitive*.

Proposition 3.1.1. *Given distinct points $z_1, z_2, z_3 \in \mathbb{C}_\infty$ and another triple of distinct points $w_1, w_2, w_3 \in \mathbb{C}_\infty, \exists! m \in M\ddot{o}b^+$ st $m(z_i) = w_i$ for $i = 1, 2, 3$.*

Proof. If we prove that $(0, 1, \infty)$ can be taken to any (z_1, z_2, z_3) we are done with the existence part of the proof.

Consider $m(z) = \dfrac{z - z_1}{z - z_3} \times \dfrac{z_2 - z_3}{z_2 - z_1}$. We now have

$$m(z_1) = 0, m(z_2) = 1, m(z_3) = \infty$$

, so m^{-1} works.

For uniqueness it is enough to prove that there is a unique map taking $(0, 1, \infty)$ to itself which is the identity map.

Let $m = \frac{az+b}{cz+d}$ be such that $m(0) = 0, m(1) = 1, m(\infty) = \infty$. Then we have $m(0) = \frac{b}{d} = 0$, $m(\infty) = \frac{a}{c} = \infty$ and $m(1) = \frac{a+b}{c+d}$ which gives $b = 0, c = 0, a = d$. Hence $m = Id$. □

Proposition 3.1.2. *Mob^+ acts transitively on the set of circles in $\mathbb{C}_\infty$*

Proof. Given any circle pick three points on it.

Now we get a Mobius Transformation taking the first set of points to the second. Since it preserves circles, the circle through the first three points must go to the circle through the second three points.

Since three points in $\mathbb{C}_\infty$ determines an unique circle, we are done. □

Proposition 3.1.3. *Mob^+ is transitive on disks*

Proof. WLOG let us assume that the boundary of the given disks is the unit circle S^1.

If both disks are the bounded or unbounded components of $\mathbb{C}_\infty$ we can use identity map.

If one is bounded and the other is unbounded we can use the $\frac{1}{z}$ map. □

3.2 Matrices and Mobius Transformation

Every matrix in $\mathrm{GL}_2(\mathbb{C})$ gives a Mobius transformation, but this map is not *one-one*.

Eg : $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ give the same Mobius transformation.

More generally $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$ give the same.

We will not distinguish between the matrices in Mob^+ as they give the same function.

3.3 Mobius Transforms preserving $\mathbb{H}$

Since our interest is in the upper half plane, we try to find which MObius maps take $\mathbb{H}$ to $\mathbb{H}$.

Look at the Monius transforms of the form $\frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$

- Where does this take $\mathbb{R}$?
- Where does it map $\mathbb{H}$ to?

Proposition 3.3.1. *Mobius maps coming from $GL_2^+\mathbb{R}$ map $\bar{\mathbb{H}}$ to $\bar{\mathbb{H}}$.*

We can multiply our matrices by $\alpha \in \mathbb{R}$ but the Mobius map doesn't change. So we can restrict to $\mathrm{SL}_2(\mathbb{R})$.

But $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and it gives the identity Mobius map. So we can quotient out by this too.

Definition 3.3.1. $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R}) / \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$

Proposition 3.3.2. *$PSL_2(\mathbb{R})$ is triply transitive on $\partial\mathbb{H}$*

Proof. For any distinct $a_1, a_2, a_3 \in \partial\mathbb{H}$ (taken in clockwise orientation) look at

$$z \longrightarrow \frac{z - a_1}{z - a_2} . \frac{a_3 - a_2}{a_3 - a_1}$$

This takes (a_1, a_2, a_3) to $(0, \infty, 1)$. □

3.4 Length and Distances in $\mathbb{H}$

We have a Riemannian metric $ds^2_{hyp} = \frac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$ called the *Hyperbolic metric.*

Let γ be a smooth curve in $\mathbb{H}$ i.e. $\gamma : [0, 1] \to \mathbb{H}$, then the length of γ is defined by

$$l(\gamma) := \int_0^1 ||\gamma'(t)||_{hyp} dt = \int_0^1 \frac{||\gamma'(t)||_{euc}}{Im(\gamma(t))} dt$$

If $\gamma(t) = (x(t), y(t))$, $l(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt$