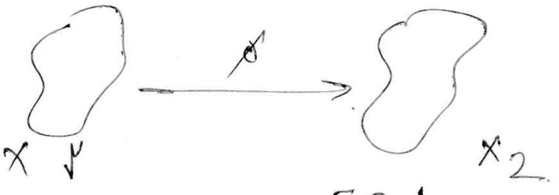
ERGODIC THEORY

G: countable infinite group.

$(\Omega, \mathcal{A}, \mu)$. probability measure space.

Change of variable theorem: $(X_1, \mathcal{A}_1, \mu) \xrightarrow{\phi} (X_2, \mathcal{A}_2)$.

then i) if $f: X_2 \to [0, \infty]$ m'ble.



then $\int f \, d(\mu \circ \phi^{-1}) = \int f \circ \phi \, d\mu$.

$E \subseteq A_2$

ii) if $f: X_2 \to \mathbb{C}$ then $f \in L^1(\mu \circ \phi^{-1})$ $\quad E \mapsto \mu(\phi^{-1}(E)) \to \mu \circ \phi^{-1}$.

$\Leftrightarrow$ $f \circ \phi \in L^1(\mu)$ in which case. $\quad$ this is measure on X_2.

$$\int f \, d(\mu \circ \phi^{-1}) = \int f \circ \phi \, d\mu.$$

$\square. (\Omega, \mathcal{A}, \mu)$.

<u>Defn:</u> A bijection $\phi: \Omega \to \Omega$, is called measure preserving if ϕ, ϕ^{-1} are measurable, and $\forall E \in \mathcal{A}$, $\mu(\phi^{-1}(E)) = \mu(E)$.

<u>Rmk:</u> if ϕ is a mpt $\Leftrightarrow$ ϕ^{-1} is mpt.

If $MP(\mu)$: space of all mpt, then $MP(\mu)$ is a group.

<u>Rmk:</u> for "nice enough" f.

$$\int f \, d\mu = \int (f \circ \phi) \, d\mu.$$

<u>Defn:</u> A measure preserving action is group homomorphism from $G \to MP(\mu)$.

H: Hilbert spaces.

$x: H_1 \to H_2$ is bdd linear operator.

$x^*: H_2 \to H_1$ $\quad \langle x^* \eta, \xi \rangle = \langle \eta, x \xi \rangle$

adjoint of operator. $\quad \forall \eta \in H_2 \; \forall \xi \in H_1$.

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$u : H_1 \to H_2$ is a unitary if it is an isometric, linear isomorphism.

u is unitary $\Leftrightarrow$ $u^* u = Id|_{H_1}$ and $u u^* = Id|_{H_2}$.

$\boxed{u^* = u^{-1}}$

$U(H)$ = space of all unitaries from H to H forms a group.

Defn: A ~~unitary representation~~ of G on H is a group homomorphism from G to $U(H)$.

Remk: $x \in B(H)$ $K \subseteq H$ closed subspace of H. K is invariant under x then $K^\perp$ is invariant under x^*.

~~$\xi \in x^\perp$~~ $\xi \in K^\perp$ then $\forall$ $\xi' \in K$ we have

$$\langle x^* \xi, \xi' \rangle = \langle \xi, x\xi' \rangle = 0. \text{ Hence } x^* \xi \in K^\perp.$$

$u : H \to H$ unitary, K is invariant under u then $u_K : K \to K$ is an isometry.

if K is also invariant under u^* then $u_K : K \to K$ is a unitary.

Pf/ We just need to prove onto.

if $\xi \in K$. $u(u^* \xi) = \xi$.

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let $\pi : G \to U(H)$ be a unitary repn. ~~then~~ Suppose $K \subseteq H$ closed subspace be π-invariant then

$$\tilde{\pi} : G \to U(K)$$
$$g \mapsto \pi(g)|_K \text{ is unitary repn.}$$

In this case, $K^\perp$ is also π-invariant.

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$(\Omega, \mathcal{A}, \mu)$ $\quad G \curvearrowright \Omega$ mpt.

$\mathcal{M}(\mathcal{A})$ = space of measurable $\mathbb{C}$-valued function on Ω.

$\phi : G \to \mathcal{L}(\mathcal{M}(\mathcal{A}))$.

$(\phi(g)f)(x) = f(g^{-1}x)$.

$(\phi(g_1g_2)f)(x) = f(g_2^{-1}g_1^{-1}x) = (\phi(g_2)f)(g_1^{-1}x) = (\phi(g_1)\phi(g_2)f)(x)$.

so ϕ is a representation.

<u>Rmk</u>: $f_1 = f_2$ μ-a.e. then $\phi(g)f_1 = \phi(g)f_2$ μ-a.e.

$\exists$ $\pi : G \to \mathcal{U}(L^2(\mu))$. $\qquad \pi(g_1g_2) = \pi(g_1)\pi(g_2)$.

$(\pi(g)f)(x) = f(g^{-1}x)$.

$$\|\pi(g)f\|^2 = \int |(\pi(g)f)(x)|^2 d\mu = \int |f(g^{-1}x)|^2 d\mu$$
$$= \int |f(x)|^2 d\mu = \|f\|^2.$$

so $\pi(g)$ is an isometry.

$\pi(g)\pi(g^{-1}) = \pi(g^{-1})\pi(g) = \pi(e) = Id_{L^2(\mu)}$.

so $\pi(g)$ is unitary.

<u>Conclusion</u>: $G \curvearrowright \Omega$ mpt then we get

$\pi : G \to \mathcal{U}(L^2(\mu))$

This rep^n is called <u>Koopman</u> rep^n.

If $c \in \mathbb{C}$. then $c \in L^2(\mu)$ is π-invariant.

$\pi(g)c = c \quad \forall g \in G$.

$\therefore 1$ is π invariant.

Ergodicity:

Def^n: $G \curvearrowright (\Omega, \mathcal{A}, \mu)$ mpa. This action is called ergodic if $\forall E \in \mathcal{A}$ $\mu(E) > 0 \Rightarrow \mu\left(\bigcup_{g \in G} gE\right) = 1$.

Def^n: $G \curvearrowright (\Omega, \mathcal{A}, \mu)$ mpa. $E \in \mathcal{A}$ is called invariant if $\mu(E \Delta gE) = 0$ $\forall g \in G$.

Def^n: $G \curvearrowright \Omega$ mpa. $E \in \mathcal{A}$ is called strictly invariant if $gE = E$ $\forall g \in G$.

Prop: $G \curvearrowright \Omega$ mpa. TFAE.

(i) The action is ergodic.

(ii) If $A, B \in \mathcal{A}$ satisfy $\mu(A), \mu(B) > 0$, then $\exists g \in G$ s.t. $\mu(gA \cap B) > 0$.

(iii) Any invariant $E \in \mathcal{A}$ has measure 0 or 1.

(iv) Any strictly invariant $E \in \mathcal{A}$ has measure 0 or 1.

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