Thm: Any simply connected Riemannian 2-manifold with -1 Gaussian curvature everywhere ~~and~~ and which is complete wrt the metric is isometric to $\mathbb{H}$.

Trigonometry

Thm: Consider the geodesic triangle with sides lengths a, b, c and angles α, β, γ in hyperbolic space.

The following hold.

1. $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$

2. $\cos \gamma = \sin \alpha \sin \beta \cosh c - \cos \alpha \cos \beta$.

3. $\dfrac{\sinh a}{\sin \alpha} = \dfrac{\sinh b}{\sin \beta} = \dfrac{\sinh c}{\sin \gamma}$

Classification of isometries.

Isometries of $\mathbb{H}$ can be considered cts maps between $\overline{\mathbb{D}} \longrightarrow \overline{\mathbb{D}}$. By Brouwer's fixed point thm there is atleast one fixed point.

- If there are atleast two fixed points in $\mathbb{D}$ (interior) then by pre and post composing by some Mobius map ~~we get that~~ we can take 0 to be one of the fixed points WLOG. Now by Schwarz lemma the map is identity Hence there can be at most one fixed point in the interior.

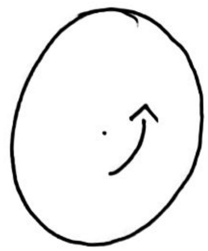
- If there are no fixed points in the interior there will be fixed points on $\partial\mathbb{D}$ the boundary.

Now if there are atleast 3 fixed points, since Möb$^+$ maps are uniquely triply transitive, the map has to be identity.

Def: Every isometry of the hyperbolic space which is not the identity can be classified into the following three categories:

- Elliptic: Exactly one fixed point in $\mathbb{D}$
- Parabolic: No fixed points in $\mathbb{D}$, one in $\partial\mathbb{D}$
- Hyperbolic: No fixed points in $\mathbb{D}$, two in $\partial\mathbb{D}$.

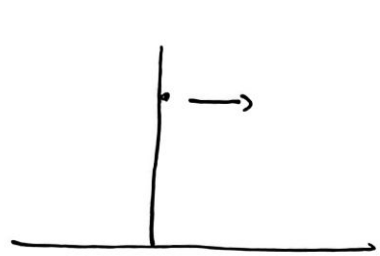
Eg: 1.



$z \to e^{i\theta} z \quad , \theta \in \mathbb{R}$

is Elliptic

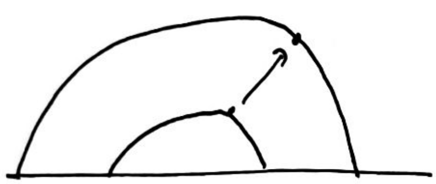
2.



$z \to z + a \quad , a \in \mathbb{R}$

is Parabolic.

3.



$z \to \lambda z \;,\; \lambda > 0$

is Hyperbolic.

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$. Solving for z in $\frac{az+b}{cz+d} = z$ will tell us the type of isometry $\frac{az+b}{cz+d}$ is.

- If $\frac{az+b}{cz+d} = z$ has no real solution, it is elliptic
- 1 real solution, parabolic
- 2 real solutions, hyperbolic. Note $c=0$ is a separate case.

Thm. If we take a representative of $\mathrm{Isom}^+(\mathbb{H})$ from $PSL_2(\mathbb{R})$, ~~the~~ $|\mathrm{tr}\, A|$ is well defined

- If $|\mathrm{tr}\, A| < 2$, A is elliptic
- If $|\mathrm{tr}\, A| = 2$, A is parabolic
- If $|\mathrm{tr}\, A| > 2$, A is ~~hyp~~ hyperbolic.

The terminology comes from the fact that these maps preserve ellipses, parabolas or hyperbolas respectively.

Axis of hyperbolic isometry: Since 2 points on $\partial\mathbb{D}$ are fixed, the line through them is too. This is the axis.

Lemma: The translation distance d of a hyperbolic isometry satisfies $\mathrm{tr}^2(A) = 4\cosh^2(d/2)$

Pf: Conjugate ~~t~~ the isometry so that the axis is the imaginary axis in $\mathbb{H}$, so it looks like

$\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}$, $\lambda \neq 1$. Then $\mathrm{tr}(A) = \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}$ and i is mapped to λi

Lemma: Isometries commute iff they have the same fixed point set

Pf: (=>) If S, T satisfy $ST = TS$ and p is a fixed point of T, then $S(p) = TST^{-1}(p) = TS(p)$ => $S(p)$ is also fixed. Follow through by cases. ~~We also need that the two hyp isometries with different axis don't commute~~

This is incomplete

(<=) Do by cases.

Parabolic looks like $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ in $\mathbb{H}$

Elliptic " $e^{i\theta} z$ in $\mathbb{D}$

Hyperbolic " $z \to \mu z$ in $\mathbb{H}$.

Cor: Hyperbolic isometries commute iff the have the same axis