Euclidean Geometry

Euclid's Postulates

1. A straight line segment can be drawn joining any two points.

2. Any straight line segment can be extended indefinitely in a straight line.

3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

4. All right angles are congruent.

5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. ~~The~~ *postulate* is

Parallel postulate: Given any straight line and a point not on it, there "exists" one and only one straight line which passes" through that point and never intersects the first line.

This postulate is equivalent the Euclids 5th postulate. It is also equivalent to the equidistance postulate, angle sum property of Δ^{le} and many more.

For the most time mathematicians thought that the 5th postulate is a consequence of the first four. They tried to prove it for 2000 years. The answer finally came in around 1830s by Carl. F. Gauss, Janos Bolyai and N.I. Lobachevsky. Lobachevsky was the first to publish about non-Euclidean geometry. Non-Euclidean geometries are models which satisfy Euclid's first 4 postulates but not the 5th.

The mathematics community did not take his discovery well and Lobachevsky faced backlash. Gauss never published his findings fearing the same. ~~Hyper~~ Non-Euclidean geometry was not popularized ~~after~~ until after 1862 when a private letter written by Gauss about "Hyperbolic Geometry" was ~~released~~ published.

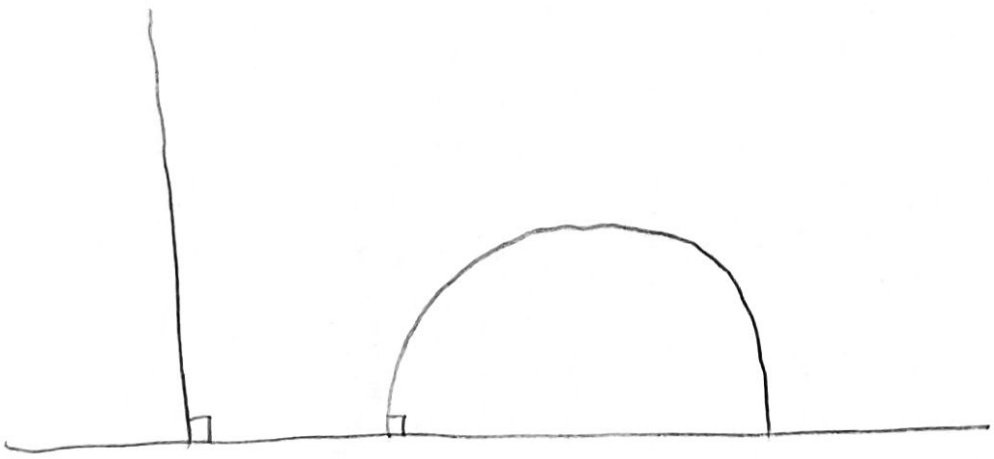
Non-Euclidean Geometry

To prove that the 5th postulate is independent of the first 4, we have to construct an example satisfying the first four but not the 5th.

Upper Half-Space model.

Consider the upper half space $\mathbb{H} = \{z \in \mathbb{C} : \text{im}(z) > 0\}$

Define "lines" in this space to be all vertical lines and all semicircles with centers on the real line



In this space for any vertical line and a point not on it we can find infinitely many semicircles through the point which don't intersect the vertical line.

Translating Everything to Modern Language.

We need the following properties from our space.

- It should be a surface
- A metric to measure distances
- A way to measure angles between curves.
- Orientation to talk about sides.

Riemann Manifold (2D)

A 2-D Riemannian manifold is a smooth oriented surface with a smoothly varying inner product at each tangent space.

- Lines will be geodesics
- Parallel will mean not intersecting.

The Hyperbolic Path Element

$\gamma : [0,1] \longrightarrow \mathbb{H}$ be a smooth path in $\mathbb{H}$, the length of γ is defined to be $\text{len}_{\mathbb{H}}(\gamma) := \int_0^1 \frac{|\gamma'(t)|}{\text{Im}(\gamma(t))}\, dt$.

The path length element is $\frac{|dz|}{\text{Im}(z)}$