

5 Lecture 5

Lemma 5.0.1. *The orientation preserving isometries of \mathbb{H} are given by $Isom^+(\mathbb{H}) = PSL_2(\mathbb{R})$.*

Proof. We know that $PSL_2(\mathbb{R}) \subseteq Isom^+(\mathbb{H})$. Now to prove $Isom^+(\mathbb{H}) \subseteq PSL_2(\mathbb{R})$. If $f \in Isom^+(\mathbb{H})$ we can consider f as an isometry of \mathbb{D} . Isometries are conformal (angle preserving). It is enough now to prove that all conformal automorphisms of \mathbb{D} are in $PSL_2(\mathbb{R})$.

Fact: Conformal maps are also biholomorphisms and $Aut(\mathbb{D}) = PSL_2(\mathbb{R})$ (use Schwarz lemma). \square

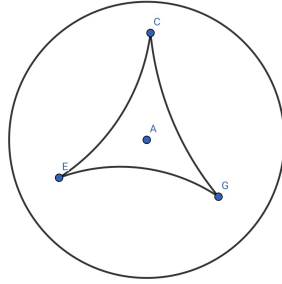
Lemma 5.0.2. *The hyperbolic metric is the unique metric (upto scaling) invariant under $Aut(\mathbb{D})$.*

5.1 Area and Curvature

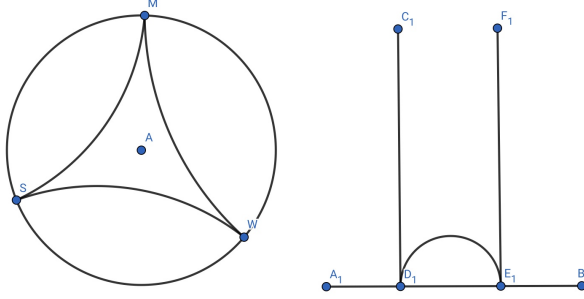
The area form on \mathbb{H} is given by $\frac{dx.dy}{y^2}$. The area form on \mathbb{D} is given by $\frac{4rdr.d\theta}{(1-r^2)^2}$. In general if the metric is $Edx^2 + 2Fdx.dy + Gdy^2$ the area form is $\sqrt{EG - F^2}dx.dy$.

5.1.1 Triangles

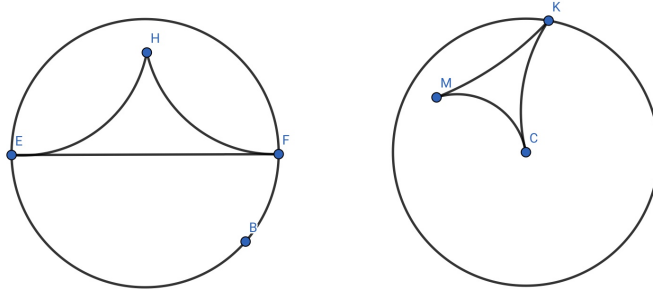
Geodesic triangles are triangles with geodesic sides.



An ideal triangles is a triangles with "vertices" on the boundary $\partial\mathbb{D}$.



There are other triangles too



Proposition 5.1.1. *Ideal triangles are unique upto isometry.*

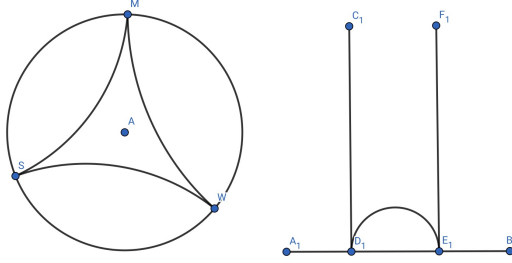
Proof. First note that given any two points on the boundary $\partial\mathbb{D}$ there is a unique geodesic such that the end points of the geodesics are the given points.

So given three points on $\partial\mathbb{D}$, there is a unique ideal triangle determined. Similarly every ideal triangle gives three boundary points. All ideal triangles can be identified by a triple of $\partial\mathbb{D} (\cong \mathbb{H})$.

We know that $Isom^+(\mathbb{H}) = PSL_2(\mathbb{R})$ acts triply transitively on $\partial\mathbb{H}$. SO we can find a map in $Isom^+(\mathbb{H})$ taking any triple to any triple. Hence any ideal triangle can be taken to any other by isometries. \square

Isometries preserve area, hence all ideal triangles have the same (hyperbolic) area.

Proposition 5.1.2. *Ideal triangles have area π*



Proof. Area of an ideal triangle =

$$\iint_D \frac{dx \cdot dy}{y^2} = \int_{-1}^1 \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} dx = \pi$$

□

Theorem 5.1.1. *The area of a hyperbolic triangle with angles $\alpha_1, \alpha_2, \alpha_3$ is $\pi - \alpha_1 - \alpha_2 - \alpha_3$.*

Note: We can take $\alpha_i = 0$ to make it an ideal triangle. Also for any triangle $\alpha_1 + \alpha_2 + \alpha_3 < \pi$ (this is strongly related to the fifth postulate).

Note: Any two similar triangles are congruent as they will have the same area.

Theorem 5.1.2. *For any conformal metric $\rho(z)|dz|$ or $\rho^2(x, y)(dx^2 + dy^2)$ the Gaussian curvature is given by*

$$K(z) = -\frac{\Delta \ln \rho}{\rho^2}(z)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Lemma 5.1.3. *The hyperbolic metric has curvature -1 everywhere.*

This is the defining property of hyperbolic geometry

Theorem 5.1.4. *Any simply connected Riemannian 2-manifold with -1 Gaussian curvature everywhere and which is complete with respect to the metric is isometric to \mathbb{H} .*