Lemma: A Fuchsian group has no $\mathbb{Z} \oplus \mathbb{Z}$ - subgroup.

Pf: The elements of such a subgroup commute so all the elements have the same fixed point set

- Hyperbolic case: WLOG the fixed points can be $0, \infty$ so these can be maps of the form $z \to \lambda z$, $\lambda > 0$. This is ~~a~~ the group $\mathbb{R}$. The only discrete subgroups here ~~are~~ is $\mathbb{Z}$

- Parabolic case: WLOG ∞ is the fixed point and the maps are $z \to z + a$, $a \in \mathbb{R}$. Argue as before

- Elliptic case: Γ will be a discrete subgroup of $SO(2)$ which is finite

Defn:

Lemma : A subgroup ~~Γ~~ of $PSL_2\mathbb{R}$ is Fuchsian $\Leftrightarrow$ it acts properly discontinuously on $\mathbb{H}^2$.

Pf: $(\Leftarrow)$ Let $T_k \longrightarrow Id$ then $T_k z \longrightarrow z$. Properly discontinuous action will imply $T_k = Id$ $\forall k \geq N$ for some large N.

$(\Rightarrow)$ Let $z_0 \in \mathbb{H}$ and $K \subseteq \mathbb{H}$ a cpt set. It is enough to prove that the set $\{T \in PSL_2(\mathbb{R}) : T(z_0) \in K\}$ is cpT.

The evaluation map
$$\Psi_{z_0} : PSL_2(\mathbb{R}) \longrightarrow \mathbb{H}$$
$$A \longrightarrow A(z_0)$$
is cts.

So $\Psi_{z_0}^{-1}(K)$ is closed. We need to prove it is bounded too.

K is cpT, hence $\left|\dfrac{az_0+b}{cz_0+d}\right| < M_1$ for some $M_1 > 0$

and $\mathrm{Im}\left(\dfrac{az_0+b}{cz_0+d}\right) > M_2 > 0$ for some $M_2 > 0$

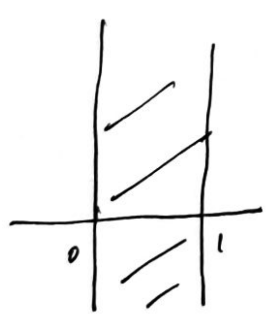
$\therefore$ a, b, c, d are bdd

Fundamental Domain

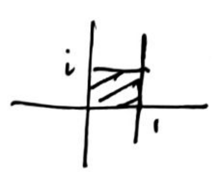
Defn: For any group Γ acting properly discontinuously on a metric space X, a fundamental domain is a closed set $C \subseteq X$, st

1. the interior $C^\circ \neq \phi$
2. $T \neq Id \Rightarrow T(C^\circ) \cap C^\circ = \phi$
3. Γ-translates of C tessellates X, i.e. $\bigcup_{T \in \Gamma} T(C) = X$.

Eg: For $\Gamma = \langle z \to z+1 \rangle$



For $\Gamma = \langle z \to z+1, z \to z+i \rangle$



- **Dirichlet Domain**

  Let Γ be a Fuchsian group. The Dirichlet Domain of Γ centered at $z_0 \in \mathbb{H}^2$ is

$$D_\Gamma(z_0) = \{ z \in \mathbb{H}^2 : d(x, z_0) \le d(x, Tz_0) \ \forall T \in \Gamma \}$$

$$= \bigcap_{T \in \Gamma} \text{half planes determined by bisector of } \overline{z_0, Tz_0}$$

- **Lemma:** the Dirichlet domain $D_\Gamma(z_0)$ is a convex fundamental domain for the action of Γ on $\mathbb{H}^2$.

Eg: $D_{PSL_2(\mathbb{Z})}(2i)$ ~~= {z~~ is