5 Lecture 5

Lemma 5.0.1. *The orientation preserving isometries of H are given by $Isom^+(\mathbb{H}) = PSL_2(\mathbb{R})$.*

Proof. We know that $PSL_2(\mathbb{R}) \subseteq Isom^+(\mathbb{H})$. Now to prove $Isom^+(\mathbb{H}) \subseteq PSL_2(\mathbb{R})$. If $f \in Isom^+(\mathbb{H})$ we can consider f as an isometry of $\mathbb{D}$. Isometries are conformal (angle preserving). It is enough now to prove that all conformal automorphisms of $\mathbb{D}$ are in $PSL_2(\mathbb{R})$.

Fact: Conformal maps are also biholomorphisms and $Aut(\mathbb{D}) = PSL_2(\mathbb{R})$ (use Schwarz lemma). □

Lemma 5.0.2. *The hyperbolic metric is the unique metric (upto scaling) invariant under $Aut(\mathbb{D})$.*

5.1 Area and Curvature

The area form on $\mathbb{H}$ is given by $\dfrac{dx.dy}{y^2}$. The area form on $\mathbb{D}$ is given by $\dfrac{4rdr.d\theta}{(1-r^2)^2}$. In general if the metric is $Edx^2 + 2Fdx.dy + Gdy^2$ the area form is $\sqrt{EG - F^2}dx.dy$.

5.1.1 Triangles

Geodesic triangles are triangles with geodesic sides.
Eg: **********Insert image************
An ideal triangles is a triangles with "vertices" on the boundary $\partial\mathbb{D}$.
Eg: **********Insert images************
There are other triangles too
Eg: **********Insert images************

Proposition 5.1.1. *Ideal triangles are unique upto isometry.*

Proof. First note that given any two points on the boundary $\partial\mathbb{D}$ there is a unique geodesic such that the end points of the geodesics are the given points.

So given three points on $\partial\mathbb{D}$, there is a unique ideal triangle determined. Similarly every ideal triangle gives three boundary points. All ideal triangles can be identified by a triple of $\partial\mathbb{D}$ ($\cong \mathbb{H}$).

We know that $Isom^+(\mathbb{H}) = PSL_2(\mathbb{R})$ acts triply transitively on $\partial\mathbb{H}$. SO we can find a map in $Isom^+(\mathbb{H})$ taking any triple to any triple. Hence any ideal triangle can be taken to any other by isometries. □

Isometries preserve area, hence all ideal triangles have the same (hyperbolic) area.

Proposition 5.1.2. *Ideal triangles have are* π

Proof. ********Insert image**********
Area of an ideal triangle =

$$\iint_D \frac{dx.dy}{y^2} = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} dx = \pi$$

□

Theorem 5.1.1. *The area of a hyperbolic triangle with angles* $\alpha_1, \alpha_2, \alpha_3$ *is* $\pi - \alpha_1 - \alpha_2 - \alpha_3$.

Note: We can take $\alpha_i = 0$ to make it an ideal triangle. Also for and triangle $\alpha_1 + \alpha_2 + \alpha_3 < \pi$ (this is strongly related to the fifth postulate).

Note: Any two similar triangles are congruent as they will have the same area.

Theorem 5.1.2. *For any conformal metric* $\rho(z)|dz|$ *or* $\rho^2(x,y)(dx^2 + dy^2)$ *the Gaussian curvature is given by*

$$K(z) = -\frac{\Delta ln\rho}{\rho^2}(z)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Lemma 5.1.3. *The hyperbolic metric has curvature -1 everywhere.*

This is the defining property of hyperbolic geometry

Theorem 5.1.4. *Any simply connected Riemannian 2- manifold with -1 Gaussian curvature everywhere and which is complete with respect to the metric is isometric to* $\mathbb{H}$.