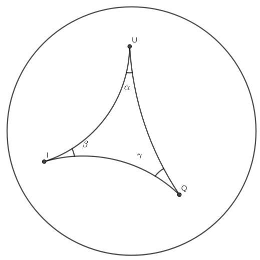
6 Lecture 6

6.1 Trigonometry

Theorem 6.1.1. *Consider the geodesic triangle with side lengths a, b, c and angles α, β, γ in hyperbolic space.*



Then the following hold

1. $cosh(c) = cosh(a)cosh(b) - sinh(a)sinh(b)cos(\gamma)$
2. $cos(\gamma) = sin(\alpha)sin(\beta)cosh(c) - cos(\alpha)cos(\beta)$
3. $\dfrac{sinh(a)}{sin(\alpha)} = \dfrac{sinh(b}{sin(\beta)} = \dfrac{sinh(c}{sin(\gamma)}$

6.2 Classification of isometries

Isometries of $\mathbb{H}$ can be considered as continuous maps between $\bar{\mathbb{D}} \to \bar{\mathbb{D}}$. By Brouwer's fixed point theorem there is at least one fixed point.

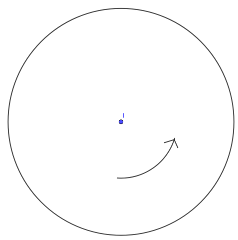
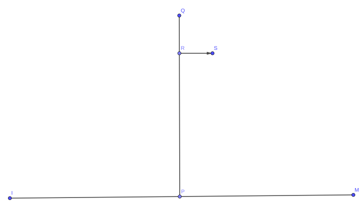
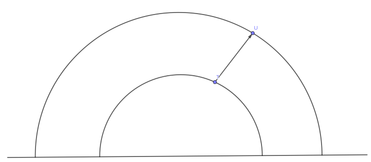
If there are at least two fixed points in $\mathbb{D}$ then by pre and post composing by some Möbius map we can take 0 to be the fixed point. Now by Schwarz lemma the map is the identity. Hence there can be at most one fixed point in the interior.

If there are no fixed points in the interior, there will be fixed points on $\partial\bar{\mathbb{D}}$, the boundary. Now if there are at least three fixed points the map has to be the identity as $m\ddot{o}b^+$ acts uniquely triply transitively on $\mathbb{C}_\infty$.

Definition 6.2.1. Every isometry of the hyperbolic space which is not the identity can be classified into the following

1. Elliptic: Exactly one fixed point in $\mathbb{D}$
2. Parabolic: No fixed points in $\mathbb{D}$, one in $\partial\mathbb{D}$
3. Hyperbolic: No fixed points in $\mathbb{D}$, two in $\partial\mathbb{D}$

Examples:

1.  $z \to e^{i\theta}, \theta \in \mathbb{R}$ is Elliptic
2.  $z \to z + a, a \in \mathbb{R}$ is Parabolic
3.  $z \to \lambda z, \lambda > 0$ is Hyperbolic

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$, solving for z in $\dfrac{az+b}{cz+d} = z$ will tell us the type of the isometry $\dfrac{az+b}{cz+d}$.

1. If $\dfrac{az+b}{cz+d}$ has no real solution, it is elliptic
2. If it has one real solution, parabolic
3. If it has two real solutions, hyperbolic

Note: $c = 0$ is a separate case.

Theorem 6.2.1. *If we tale a representative of $A \in Isom^+(\mathbb{H})$ from $PSL_2(\mathbb{R})$, $|tr(A)|$ is well defined*

1. *If $|tr(A)| < 2$, A is elliptic*
2. *If $|tr(A)| = 2$, A is parabolic*
3. *If $|tr(A)| > 2$, A is hyperbolic*

The terminology comes from the fact that these maps preserve ellipses, parabolas or hyperbolas respectively in the hyperbolloid model.

Definition 6.2.2. Axis of a hyperbolic isometry: Since two points on $\partial\mathbb{D}$ are fixed, the line through then is too. This is the axis.

Lemma 6.2.2. *The translation distance d of a hyperbolic isometry satisfies*

$$tr^2(A) = 4cosh^2\left(\frac{d}{2}\right)$$

Proof. Conjugate the isometry so that the axis is the imaginary axis in $\mathbb{H}$ so that the map looks like $\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}$, $\lambda \neq 1$. Then $tr(A) = \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}$ and i is mapped to λi. □

Lemma 6.2.3. *Isometries commute if and only if they have the same fixed point set*

Proof. If S, T,isometries, satisfy $ST = TS$ and p is a fixed point of T, then $S(p) = TST^{-1}(p) = TS(p)$. Hence $S(p)$ is also a fixed point of T. The proof now follows by looking at different cases separately.

The other direction follows by dealing with the type of the isometry is separate cases. □

Corollary 6.2.3.1. *Hyperbolic isometries commute if and only if they have the same axis*