Transitivity Property of Mob$^+$

~~A map $m: X \longrightarrow X$ is transitive if it co~~

A set of maps from $X \longrightarrow X$ ~~are~~ is called transitive if ~~$\exists$ a map m who~~ for any two points $x, y \in X$, $\exists$ m in the set such that $m(x) = y$.

The Mob$^+$ group is ~~trip~~ uniquely triply transitive

Propn: Given distinct points $z_1, z_2, z_3 \in \mathbb{C}_\infty$ and another triple of distinct points ~~$\exists$~~ $w_1, w_2, w_3 \in \mathbb{C}_\infty$, $\exists!$ $m \in$ Mob$^+$ ST $m(z_i) = w_i$ $\forall i = 1, 2, 3$.

Pf: If we prove $(0, 1, \infty)$ can be taken to any (z_1, z_2, z_3) we are done with the existence part.

Consider
$$m(z) = \frac{z - z_1}{z - z_3} \times \frac{z_2 - z_3}{z_2 - z_1}$$

Then $m(z_1) = 0$, $m(z_2) = 1$, $m(z_3) = \infty$. m^{-1} works for existence.

For uniqueness it is enough to prove ~~any~~ there is only one map taking $(0, 1, \infty) \longrightarrow (0, 1, \infty)$ ~~is~~.

Let m be such that $m(0)=0$, $m(1)=1$, $m(\infty)=\infty$

$$m(z) = \frac{az+b}{cz+d}$$

$$m(0) = \frac{b}{d} = 0 \Rightarrow b = 0$$

$$m(\infty) = \frac{a}{c} = \infty \Rightarrow c = 0$$

$$m(1) = \frac{a+b}{c+d} = 1 \Rightarrow a+b = c+d \Rightarrow a = d$$

So $m = \dfrac{az+0}{0+a} = z \Rightarrow m = Id$.

$\blacksquare$

Propn: Mob^+ acts transitively on the set of circles in $\mathbb{C}_\infty$

Pf: Given any circle pick three points on it.

Now we get a Möbius transfor taking the first set of points to the second. Since it preserves circles, the circle through the first three points must go to the circle through the second three points. Since 3 points determine a unique circle we are done.

Propn: Mob^+ is transitive on disks.

Pf: ~~Given two disks~~ WLOG let us assume that the boundary of the given disks is the unit circle S^1.

If ~~the one~~ both disks are the bounded or unbounded components of $\mathbb{C}_\infty \setminus S^1$ we can use Id map.

If one is bounded and the other is unbounded we can use the $1/z$ map.

Matrices and Möbius transforms.

Every matrix in $GL_2(\mathbb{C})$ gives a Möbius transform, but this map is not ~~unique~~ 1-1.

Eg: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ give the same ~~map~~ Möbius transform

More generally

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} \text{ give the same}$$

~~Möbius maps~~. ~~So we can choose α such that the det of our matrix is ± 1.~~

We will not distinguish between these matrices in $Möb^+$ as they give the same function.

Möbius Transforms preserving $\mathbb{H}$.

Since our interest is in the upper half plane, we try to find which Möbius maps ~~so~~ take $\mathbb{H}$ to $\mathbb{H}$.

Look at the Möbius transforms of the form $\frac{az+b}{cz+d}$ where $a,b,c,d \in \mathbb{R}$ and $ad-bc>0$.

- Where does this take $\mathbb{R}$?
- Where does it map $\mathbb{H}$ to?

Propn: Möbius maps coming from $GL_2^+(\mathbb{R})$ map $\overline{\mathbb{H}}$ to $\overline{\mathbb{H}}$

We can multiply our matrix by $\alpha \in \mathbb{R}$ and the Möbius map doesn't change. So we can ~~assume~~ restrict to $SL_2(\mathbb{R})$

But $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is in $SL_2(\mathbb{R})$ and it gives the Id Möbius map. So we can quotient out by this too

Defn: $PSL_2(\mathbb{R}) := SL_2(\mathbb{R}) \Big/ \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$

Propn: $PSL_2(\mathbb{R})$ is triply transitive on $\partial\mathbb{H}$.

Pf: ~~Look at~~ For any distinct $a_1, a_2, a_3 \in \partial\mathbb{H}$ (in clockwise orientation), look at

$$z \longrightarrow \frac{z-a_1}{z-a_2} \cdot \frac{a_3-a_2}{a_3-a_1}$$

This takes (a_1, a_2, a_3) to $(0, \infty, 1)$

Lenghts and Distances in $\mathbb{H}$.

We have a Riemannian metric $ds^2_{hyp} = \dfrac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$ called the hyperbolic metric.

Let ~~k~~ γ be a smooth curve in $\mathbb{H}$. i.e. $\gamma: [0,1] \to \mathbb{H}$, sm

then length of $\ell = \ell(\gamma) := \displaystyle\int_0^1 \|\gamma'(t)\|_{hyp}\, dt$.

$$= \int_0^1 \frac{\|\gamma'(t)\|_{euc}}{Im(\gamma(t))}\, dt$$

If $\gamma(t) = (x(t), y(t))$

$$\ell(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2+y'(t)^2}}{y(t)}\, dt.$$