HG 101

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1 Lecture 1

1.1 Euclid's Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended infinitely in a straight line.
3. Given any straight line segment a circle can be drawn having the segment as radius and one endpoint as center.
4. all right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Parallel Postulate : Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line.

This postulate is equivalent to the "Euclid's fifth postulate". It is also equivalent to the equidistance postulate, angle sum property and many more.

For the most time mathematicians thought that the 5^{th} postulate is a consequence of the first four. They tried to prove it for 2000 years. The answer finally came around $1830's$ by Carl. F. Gauss, Janos Bolyai and N. I. Lobachevsky.

Lobachevsky was the first to publish about non-Euclidean geometry. Non-Euclidean geometry are models which satisfy Euclid's first four postulates but not the fifth.

The mathematics community did not take this discovery well and Lobachevsky faced backlash. Gauss never published his findings fearing the same. Non-Euclidean geometry was not popularized until after 1862 when a private letter written by Gauss about "Hyperbolic Geometry" was published.

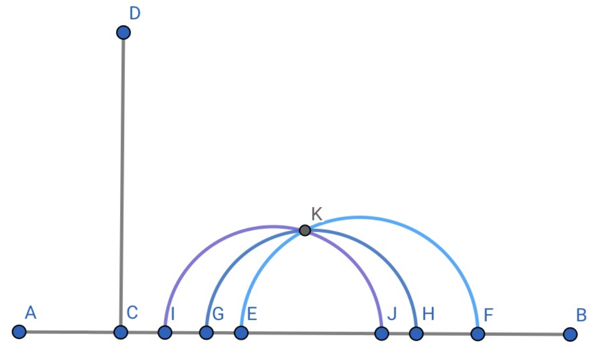
1.2 Non-Euclidean Geometry

To prove that the fifth postulate is independent of the first four, we have to construct an example satisfying the first four but not the fifth.

1.3 Upper Half-Space Model

Consider the upper half space $\mathbb{H} = \{z \in \mathbb{C} : \ im(z) > 0\}$.
Define "lines" in this space to be all vertical lines and all semicircles with center

on the real line.



In this space for any vertical line and a point not on it we can find infinitely many semicircles through the point which doesn't intersect the vertical line.

So in "Modern Language" we need the following:

- It should be a surface.
- A metric to measure distance.
- A way to measure angles between curves.
- Orientation to talk about sides.

1.4 Riemannian Manifold(2D)

A 2-D Riemannian manifold is a smooth oriented surface with a smoothly varying inner product at each tangent space.

- Lines will be geodesics.
- Parallel will mean not intersecting

1.5 The Hyperbolic Path Element

$\gamma : [0, 2] \longrightarrow \mathbb{H}$ be a smooth path in $\mathbb{H}$, the *length* of γ is defined to be

$$len_{\mathbb{H}}(\gamma) := \int_0^1 \frac{|\gamma'(t)|}{Im(\gamma(t))} dt$$

The path length element id $\frac{|dz|}{Im(z)}$.

2 Lecture 2

2.1 The Hyperbolic Metric

The Hyperbolic Metric on $\mathbb{H}$ is $ds^2_{hyp} = \frac{dx^2+dy^2}{y^2}$.
We interpret this as an inner product at each tangent space.

Let $p = (x_0, y_0) \in \mathbb{H}$. the Tangent Space at this point is $T_p\mathbb{H} = \mathbb{R}^2$.Thus to specify an inner product on it we only need to give the inner product on the basis $\{e_1, e_2\}$.

Now (x, y) are coordinate for $\mathbb{H}$ (it is covered by a single chart). If the metric is given to be $\frac{dx^2+dy^2}{y^2}$ at the point p, the inner product on $T_p\mathbb{H}$ is defined by:
$\langle e_1, e_1\rangle = \text{coeff of } dx^2 \;=\; \frac{1}{y_0^2}$
$\langle e_2, e_2\rangle = \text{coeff of } dy^2 \;=\; \frac{1}{y_0^2}$
$\langle e_1, e_2\rangle = \frac{1}{2}\text{ coeff of } dxdy \;=\; 0$

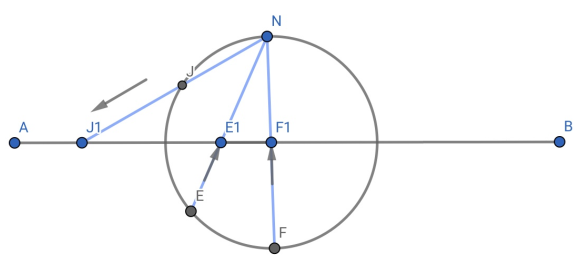
2.2 The Riemann Sphere $\mathbb{C}_\infty$

The Riemann Sphere is defined to be $\mathbb{C}_\infty := \mathbb{C}\bigcup\{\infty\}$
We give a topology on it by declaring the following sets as open in $\mathbb{C}_\infty$:

1. $U \subset \mathbb{C}$ open in $\mathbb{C}$
2. $(\mathbb{C} \setminus K) \cup \{\infty\}$ where K is compact in $\mathbb{C}$.

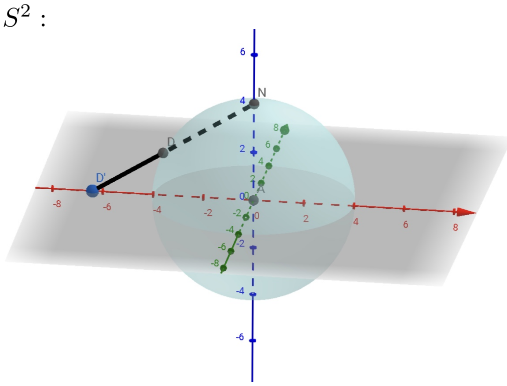
2.3 Stereographic Projection

For S^1 :



$$S^1 \setminus \{i\} \cong \mathbb{R} \text{ and } \mathbb{R} \cup \{\infty\} \cong S^1$$

For S^2 :



$$S^2 \setminus \{n\} \cong \mathbb{R}^2 \text{ and } \mathbb{R}^2 \cup \{\infty\} \cong S^2$$

Hence we have $\mathbb{C}_\infty \cong S^2$

2.4 Continuous functions on $\mathbb{C}_\infty$

- $f : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$
  $z \longmapsto z^n$ when $z \in \mathbb{C}$
  $\infty \longmapsto \infty$

- $g : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$
  $z \longmapsto \frac{1}{z}$ when $z \in \mathbb{C} \setminus \{0\}$
  $\infty \longmapsto 0$
  $0 \longmapsto \infty$

- $h : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$
  $z \longmapsto az + b$ when $z \in \mathbb{C}$ where $a, b \in \mathbb{C}$
  $\infty \longmapsto \infty$

2.5 Circles in $\mathbb{C}_\infty$

We call the following as circles in $\mathbb{C}_\infty$

1. All circles in $\mathbb{C}_\infty$
2. $\{$All lines in $\mathbb{C}\} \cup \infty$

Now all circles in $\mathbb{C}_\infty$ are homeomorphic to S^1.

Consider $\mathbb{R} \cup \{\infty\}$ a circle. It is S^1. By Jordan Brouwer Theorem $\mathbb{R} \cup \{\infty\}$ splits $\mathbb{C}_\infty$ into two disks. namely the upper and lower half plane. So $\mathbb{H}$ is homeomorphic to a disk and $\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ is a closed disk.

2.6 Equation of Circles

The equation of an Euclidean Circle is $(x-x_0)^2+(y-y_0)^2=r^2$. Substitute $x=\frac{z+\bar{z}}{2}$ and $y=\frac{z-\bar{z}}{2i}$ and by rearranging we get $\alpha z\bar{z}+\beta z+\bar{\beta}\bar{z}+\gamma=0$ where $\alpha,\gamma\in\mathbb{R}$ and $\beta\in\mathbb{C}$. Similarly the equation of an Euclidean line is $\beta z+\bar{\beta}\bar{z}+\gamma=0$ where $\gamma\in\mathbb{R}$ and $\beta\in\mathbb{C}$.

2.7 Mobius Transformations

A Mobius transformation is a map of the form $m(z)=\frac{az+b}{cz+d}$ with $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$

It can so happen that the denominator is zero for some point $z=\frac{-d}{c}$. In such a case the numerator is non zero *(hint:$ad-bc\neq 0$)*. So we define $m(\frac{-d}{c})=\infty$

Remark. Algebra with ∞

- $\frac{1}{0}=\infty$
- $\frac{1}{\infty}=0$

Definition 2.7.1 (Mobius Transformation)**.** Let $m:\mathbb{C}_\infty\longrightarrow\mathbb{C}_\infty$ be a function defined by:
$z\longmapsto\frac{az+b}{cz+d}$ when $z\in\mathbb{C}$
$\infty\longmapsto\frac{a}{c}$
where $a,b,c,d\in\mathbb{C}$ and $ad-bc\neq 0$. Then m is called a *Mobius Transformation*

Proposition 2.7.1. *$m(z)$ is a homeomorphism.*

Proof. Composition rule:

Let $m_1(z)=\frac{a_1z+a_2}{a_3z+a_4}$, $m_2(z)=\frac{b_1z+b_2}{b_3z+b_4}$ be Mobius Transformations.

$$\begin{aligned}
m_1\circ m_2(z) &= \frac{a_1\frac{b_1z+b_2}{b_3z+b_4}+a_2}{a_3\frac{b_1z+b_2}{b_3z+b_4}+a_4}\\
&= \frac{a_1b_1z+a_1b_2+a_2b_3z+a_2b_4}{a_3b_1z+a_3b_2+a_4b_3z+a_4b_4}\\
&= \frac{(a_1b_1+a_4b_3)z+(a_1b_2+a_2b_4)}{(a_3b_1+a_4b_3)z+(a_3b_2+a_4b_4)}
\end{aligned}$$

Consider the matrices

A= $\begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}$, B= $\begin{pmatrix}b_1 & b_2\\ b_3 & b_4\end{pmatrix}$ AB= $\begin{pmatrix}a_1b_1+a_2b_3 & a_3b_1+a_4b_2\\ a_3b_1+a_4b_3 & a_3b_2+a_4b_4\end{pmatrix}$

Notice that the terms of $m_1\circ m_2(z)$ and AB match. So the composition of Mobius Transformation can be thought of as matrix multiplication.

Now since $ad-bc\neq 0$, m becomes invertible and the inverse is also a Mobius Transformation. Hence $m:\mathbb{C}_\infty\longrightarrow\mathbb{C}_\infty$ is a homeomorphism. $\square$

2.8 Generators of Mobius Transformations

Definition 2.8.1. Mob^+ is the set of all Mobius Transformations.

Proposition 2.8.1. *f, g, J generate Mob^+*

Proof. Using $f(z) = z + a,\ g(z) = bz,\ a, b \in \mathbb{C}$ we can generate the elements of the form $az + b$.
Using $az + b$ and $J(z) = \frac{1}{z}$ we can generate any element of Mob^+ □

Equations of circles are of the form $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$

1.Circles are invariant under f:

$\alpha(z + a)(z \bar{+} a) + \beta(z + a) + \bar{\beta}(z \bar{+} a) + \gamma = 0$
$\alpha z\bar{z} + \alpha a\bar{a} + \alpha z\bar{a} + \alpha\bar{z}a + \beta z + \bar{\beta}\bar{z} + \beta a + \bar{\beta}\bar{a} + \gamma = 0$
$\alpha z\bar{z} + (\beta\alpha\bar{a})z + (\bar{\beta} + \bar{\alpha}\bar{z} + \alpha|a|^2 + \gamma = 0$

2.Circles are invariant under g:

$\alpha(bz)((\bar{bz}) + \beta(bz) + \bar{\beta}(\bar{bz}) + \gamma = 0$
$\alpha b\bar{b}z\bar{z} + \beta bz + \bar{\beta}\bar{b}\bar{z} + \gamma = 0$

3.Circles are invariant under J:

$\frac{\alpha}{z(\bar{z}} + \frac{\beta}{z} + \frac{\bar{\beta}}{\bar{z}} + \gamma = 0$
$\gamma z\bar{z} + \bar{\beta}z + \beta\bar{z} + \alpha = 0$

And these three map generates Mob^+.We proved:

Proposition 2.8.2. *Circles in $\mathbb{C}_\infty$ are invariant under Mob^+*

3 Lecture 3

3.1 Transitivity property of $M\ddot{o}b^+$

A set of maps from $X \to X$ is called *transitive* if for any two points $x, y \in X, \exists m$ in the set such that $m(x) = y$.

The $M\ddot{o}b^+$ group is *uniquely triply transitive*.

Proposition 3.1.1. *Given distinct points $z_1, z_2, z_3 \in \mathbb{C}_\infty$ and another triple of distinct points $w_1, w_2, w_3 \in \mathbb{C}_\infty, \exists! m \in M\ddot{o}b^+$ st $m(z_i) = w_i$ for $i = 1, 2, 3$.*

Proof. If we prove that $(0, 1, \infty)$ can be taken to any (z_1, z_2, z_3) we are done with the existence part of the proof.

Consider $m(z) = \dfrac{z - z_1}{z - z_3} \times \dfrac{z_2 - z_3}{z_2 - z_1}$. We now have

$$m(z_1) = 0, m(z_2) = 1, m(z_3) = \infty$$

, so m^{-1} works.

For uniqueness it is enough to prove that there is a unique map taking $(0, 1, \infty)$ to itself which is the identity map.

Let $m = \frac{az+b}{cz+d}$ be such that $m(0) = 0, m(1) = 1, m(\infty) = \infty$. Then we have $m(0) = \frac{b}{d} = 0$, $m(\infty) = \frac{a}{c} = \infty$ and $m(1) = \frac{a+b}{c+d}$ which gives $b = 0, c = 0, a = d$. Hence $m = Id$. □

Proposition 3.1.2. *Mob^+ acts transitively on the set of circles in $\mathbb{C}_\infty$*

Proof. Given any circle pick three points on it.

Now we get a Mobius Transformation taking the first set of points to the second. Since it preserves circles, the circle through the first three points must go to the circle through the second three points.

Since three points in $\mathbb{C}_\infty$ determines an unique circle, we are done. □

Proposition 3.1.3. *Mob^+ is transitive on disks*

Proof. WLOG let us assume that the boundary of the given disks is the unit circle S^1.

If both disks are the bounded or unbounded components of $\mathbb{C}_\infty$ we can use identity map.

If one is bounded and the other is unbounded we can use the $\frac{1}{z}$ map. □

3.2 Matrices and Mobius Transformation

Every matrix in $\mathrm{GL}_2(\mathbb{C})$ gives a Mobius transformation, but this map is not *one-one*.

Eg : $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ give the same Mobius transformation.

More generally $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$ give the same.

We will not distinguish between the matrices in Mob^+ as they give the same function.

3.3 Mobius Transforms preserving $\mathbb{H}$

Since our interest is in the upper half plane, we try to find which MObius maps take $\mathbb{H}$ to $\mathbb{H}$.

Look at the Monius transforms of the form $\frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$

- Where does this take $\mathbb{R}$?
- Where does it map $\mathbb{H}$ to?

Proposition 3.3.1. *Mobius maps coming from $GL_2^+\mathbb{R}$ map $\bar{\mathbb{H}}$ to $\bar{\mathbb{H}}$.*

We can multiply our matrices by $\alpha \in \mathbb{R}$ but the Mobius map doesn't change. So we can restrict to $\mathrm{SL}_2(\mathbb{R})$.

But $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and it gives the identity Mobius map. So we can quotient out by this too.

Definition 3.3.1. $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\langle\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\rangle$

Proposition 3.3.2. *$PSL_2(\mathbb{R})$ is triply transitive on $\partial\mathbb{H}$*

Proof. For any distinct $a_1, a_2, a_3 \in \partial\mathbb{H}$ (taken in clockwise orientation) look at

$$z \longrightarrow \frac{z - a_1}{z - a_2} . \frac{a_3 - a_2}{a_3 - a_1}$$

This takes (a_1, a_2, a_3) to $(0, \infty, 1)$. □

3.4 Length and Distances in $\mathbb{H}$

We have a Riemannian metric $ds^2_{hyp} = \frac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$ called the *Hyperbolic metric.*

Let γ be a smooth curve in $\mathbb{H}$ i.e. $\gamma : [0, 1] \to \mathbb{H}$, then the length of γ is defined by

$$l(\gamma) := \int_0^1 ||\gamma'(t)||_{hyp} dt = \int_0^1 \frac{||\gamma'(t)||_{euc}}{Im(\gamma(t))} dt$$

If $\gamma(t) = (x(t), y(t))$, $l(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt$

4 Lecture 4

4.1 The Hyperbolic distance

For $z, w \in \mathbb{H}$ define the distance between them to be

$$d_h(z,w) = inf\{l(\gamma) : \gamma \text{ is a curve between z and w}\}$$

Proposition 4.1.1. *d_h is a metric on $\mathbb{H}$.*

Definition 4.1.1. A curve γ between z and w is called a *geodesic* if $d_h(z,w) = l(\gamma)$.

Warning: This is not the actual definition of geodesics, but in our case it is equivalent.

Proposition 4.1.2. *Vertical lines are geodesics*

Proof. Let $p_1 = (x_0, a)$ and $p_2 = (x_0, b)$ be two points on the line $x = x_0$ and WLOG $0 < a < b$. Let $\gamma(t) = (x(t), y(t))$ be any path connecting them, then

$$\begin{aligned} l(\gamma) &= \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt \\ &\geq \int_0^1 \frac{|y'(t)|}{y(t)} dt \\ &\geq \int_0^1 \frac{y'(t)}{y(t)} dt \\ &\geq ln\left(\frac{b}{a}\right) \end{aligned} \tag{1}$$

Look at $\alpha : [0,1] \to \mathbb{H}$ where $\alpha(t) = i((b-a)t + a)$, then $l(\alpha) = ln(\frac{b}{a})$. But $d_h(p_1, p_2) = inf\{l(\gamma) : \gamma \text{ connects } p_1 \text{ and } p_2\} \geq l(\frac{b}{a})$ and α achieves the equality. Hence $d_h(p_1, p_2) = ln(\frac{b}{a})$ and α is a geodesic. □

Corollary 4.1.0.1.

$$d(ia, ib) = \left| ln\left(\frac{b}{a}\right) \right|$$

Proposition 4.1.3. *The hyperbolic metric d_h is complete.*

Remark: There is a proof using Hopf-Rinow theorem.

4.2 Isometries of the hyperbolic space

Definition 4.2.1. Any diffeorphism $T : \mathbb{H} \to \mathbb{H}$ such that

$$ds^2_{hyp} = \frac{|dz|^2}{Im(z)^2} = \frac{|dT(z)|^2}{Im(T(z))^2}$$

is an *isometry* of $\mathbb{H}$.

Example: $T(z) = z + a$ where $a \in \mathbb{R}$

$$\frac{|dT(z)|^2}{Im(T(z))^2} = \frac{|dz|^2}{Im(z)^2}$$

so T is an isometry.
The following are isometries

- $T(z) = -\bar{z}$
- $T(z) = \frac{1}{z}$
- $T(z) = \lambda z, \lambda > 0$