19/09

$G \overset{\beta}{\curvearrowright} X$.

$\pi : G \to \mathcal{U}(\ell^2(X))$

$\pi(g)(f)(x) = f(g^{-1}x)$.

<u>Propⁿ:</u> $G \overset{\beta}{\curvearrowright} X$. Then π is ergodic iff Every orbit (under β) is infinite.

Left/Right regular representations of G.

1) $G \overset{\beta_1}{\curvearrowright} G$. $\qquad$ $G \overset{\beta_2}{\curvearrowright} G$.

$g.h = gh$ $\qquad$ $g.h = hg^{-1}$

$\lambda : G \to \mathcal{U}(\ell^2(G))$ Left regular repⁿ

$((\lambda(g))(f))(h) = f(g^{-1}.h) = f(g^{-1}h)$.

$\rho : G \to \mathcal{U}(\ell^2(G))$ Right regular repⁿ.

$((\rho(g))(f))(h) = f(g^{-1}.h) = f(hg)$.

Note: Both β_1 and β_2 are transitive actions. The only orbit is G itself.

Hence, if G is infinite ~~then~~ $\Leftrightarrow$ λ is ergodic.

$\Leftrightarrow$ ρ is ergodic.

2) Conjugation action of G on G.

$G \overset{\beta}{\curvearrowright} G$.

$g.h = ghg^{-1}$. $\qquad$ $\pi : G \to \mathcal{U}(\ell^2(G))$

$((\pi(g))(f))(h) = f(g^{-1}.h) = f(g^{-1}hg)$.

we see π is <u>not</u> ergodic. because $\{e\}$ is an orbit.

$G \overset{\beta}{\curvearrowright} G \setminus \{e\}$.

$g.h := ghg^{-1}$ $\qquad$ $\pi_1 : G \to \mathcal{U}(\ell^2(G\setminus\{e\}))$.

$((\pi_1(g))(f))(h) = f(g^{-1}h) = f(g^{-1}hg)$.

π_1 is ergodic $\Leftrightarrow$ all nontrivial conjugacy classes of G are infinite.

In above case G is called an ICC group. (Infinite Conjugacy classes)

Eg: i) S_∞ is an ICC group.

ii). Free group over finitely many generators.

What happens when G is finite?

Recall: $G \overset{\alpha}{\curvearrowright} (\Omega, \mathcal{A}, \mu)$ mpa. α is ergodic iff. (μ prob measure)

(i) $\forall E \in \mathcal{A}$, $\mu(E) > 0 \Rightarrow \mu\left(\bigcup_{g \in G} gE\right) = 1$.

or equivalently.

(ii) $\forall A, B \in \mathcal{A}$. $\mu(A), \mu(B) > 0 \Rightarrow \exists g \in G$ s.t. $\mu(gA \cap B) > 0$.

We will consider G to be finite, α ergodic.

$E \in \mathcal{A}$; $\mu(E) > 0$, then $\mu\left(\bigcup_{g \in G} gE\right) = 1$

$$\mu\left(\bigcup_{g \in G} gE\right) \leq \sum_{g \in G} \mu(E) = |G| \mu(E).$$

We get $\mu(E) \geq 1/|G|$. so we cannot have sets with very small measure.

Now we can classify all the finite ergodic actions.

Propⁿ: G finite. $G \overset{\alpha}{\curvearrowright} (\Omega, \mathcal{A}, \mu)$ mpa. is ergodic. Then $\exists E \in \mathcal{A}$ such that (i) $\forall g, h \in G$, $gE \cap hE = \emptyset$ or $gE = hE$.

(ii) $\forall X \in \mathcal{A}$, $\exists F \subseteq G$ such that $\mu\left(X \Delta \left(\bigcup_{g \in F} gE\right)\right) = 0$.

Pf/ Consider the set $M = \{\mu(E) \mid E \in \mathcal{A}$ and $\mu(E) > 0\}$.

From (*), we see that $\forall M \in M$ we have $1/|G| \leq M$.

Hence $\inf(M) > 0$.

let $c = \inf(M)$.

We claim $c \in M$.

We recursively construct a decreasing sequence of measurable sets $E_1 \supseteq E_2 \supseteq \cdots$, $c < \mu(E_n) < c + 1/n$

$E_1 := \Omega$.

Suppose $E_1 \supseteq \cdots \supseteq E_n$ have been defined. Since $c = \inf(m)$.

$\exists X \in A$ s.t.

$$c \leq \mu(X) < c + 1/n+1$$

$\mu(E_n), \mu(X) > 0$, $\exists g \in G$ s.t. $\mu(gX \cap E_n) > 0$.

Define, $E_{n+1} := gX \cap E_n$. ~~Then~~ $E_{n+1} \subseteq E_n$.

$$0 \leq \mu(E_{n+1}) \leq \mu(gX) = \mu(X) < c + 1/n+1$$

Hence we get

$$E_1 \supseteq E_2 \supseteq \cdots$$

s.t. $c < \mu(E_i) \leq c + 1/i+1$. $\forall i \in \mathbb{N}$.

If $E = \bigcap_{i \in \mathbb{N}} E_i$ then $\mu(E) = \lim_{i \to \infty} \mu(E_i)$.

Hence $c \in M$.

Consider $E \in A$ s.t. $\mu(E) = c$.

Hence $\forall X \subseteq E$ measurable then

$$\mu(X) \in \{0, c\}.$$

Then $\forall g \in G$ we get $\mu(gE \cap E) \in \{0, c\}$. Consider $F = \{g \in G \mid \mu(gE \cap E) = 0\}$. let $E_0 = E \setminus \left(\bigcup_{g \in F_0} gE \cap E\right)$.

But $\mu\left(\bigcup_{g \in F_0} gE \cap E\right) \leq \sum_{g \in F_0} \mu(gE \cap E) = 0$.

Hence $\mu(E_0) = c$.

<u>Claim:</u> $\forall g \in G$, $gE_0 = E_0$ or $gE_0 \cap E_0 = \emptyset$.

Sam

Case 1: $\mu(gE_0 \cap E_0) = 0$. ~~$\mu(gE \cap E) = \mu(g(E \setminus \bigcup_{g \notin F_0} gE \cap E)$~~

$gE \cap E \subseteq (gE_0 \cap E_0) \cup (E \setminus E_0) \cup (gE \setminus gE_0)$.

so we get $\mu(gE \cap E) = 0$. Hence $g \notin F_0$.

hence $(gE \cap E) \cap E_0 = \emptyset$

$gE \cap E_0 = \emptyset$ and hence $gE_0 \cap E_0 = \emptyset$.

Case 2: $\mu(gE_0 \cap E_0) > 0$.

Then $\mu(gE_0 \cap E_0) = 0$.

To be continued....