Propn: $PSL_2(\mathbb{R})$ is triply transitive on $\partial\mathbb{H}$.

Pf: ~~Look at~~ For any distinct $a_1, a_2, a_3 \in \partial\mathbb{H}$, look at
(In clockwise orientation)

$$z \longrightarrow \frac{z-a_1}{z-a_2} \cdot \frac{a_3-a_2}{a_3-a_1}$$

This takes (a_1, a_2, a_3) to $(0, \infty, 1)$

Lenghts and Distances in $\mathbb{H}$.

We have a Riemannian metric $ds^2_{hyp} = \dfrac{dx^2+dy^2}{y^2}$ on $\mathbb{H}$ called the hyperbolic metric.

Let γ be a smooth curve in $\mathbb{H}$. i.e. $\gamma : [0,1] \to \mathbb{H}$, sm

then length of $l = l(\gamma) := \displaystyle\int_0^1 \|\gamma'(t)\|_{hyp}\, dt$.

$$= \int_0^1 \frac{\|\gamma'(t)\|_{euc}}{Im(\gamma(t))}\, dt$$

If $\gamma(t) = (x(t), y(t))$

$$l(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt.$$

The hyperbolic distance

For $z, w \in \mathbb{H}$ define

$$d_h(z,w) = \inf\{l(\gamma) : \gamma \text{ is a curve between } z \,\&\, w\}$$

Propn: d_h is a metric on $\mathbb{H}$.

Defn: A curve γ between w and z is called a geodesic if

$$d_h(z,w) = l(\gamma).$$

Note: This is not the actual defn of geodesics, but in our case it is ~~correc~~ equivalent.

Propn: Vertical lines are geodesics

Pf: Let $p_1 = (x_0, a)$, $p_2 = (x_0, b)$ be two ~~tree~~ points on the line $x = x_0$. WLOG $0 < a < b$. Let $\gamma(t) = (x(t), y(t))$ be any path connecting them.

$$l(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt \geq \int_0^1 \frac{|y'(t)|}{y(t)}\,dt \geq \int_0^1 \frac{y'(t)}{y(t)}\,dt \geq \ln\left(\frac{b}{a}\right) \quad (1)$$

Look at $\alpha : [0,1] \longrightarrow \mathbb{H}$

$$\alpha(t) = i\big((b-a)t + a\big)$$

Then $l(\alpha) = \ln\left(\frac{b}{a}\right)$ — 2

$$d_h(p_1, p_2) = \inf\{l(\gamma) : \gamma \text{ connects } p_1 \,\&\, p_2\}$$

$$\geq l\left(\frac{b}{a}\right)$$

α achieves the equality.

$$\therefore \quad d_h(p_1, p_2) = \ln\left(\frac{b}{a}\right)$$

and α is a geodesic.

Cor: $d(ia, ib) = \left| \ln\left(\frac{b}{a}\right) \right|$

Propn: The hyperbolic metric is complete.

Pf: Omitted.

Remark: We can conclude this using Hopf-Rinow thm as well.

~~2~~ Isometries of the hyperbolic space.

Any diffeo ~~map~~ $T: \mathbb{H} \to \mathbb{H}$ ST

$$ds^2_{hyp} = \frac{|dz|^2}{(Im(z))^2} = \frac{|d\,T(z)|^2}{Im(T(z))^2}$$ is an isometry of

$\mathbb{H}$.

~~Note: This is not the~~

Eg: $T(z) = z + a$ where $a \in \mathbb{R}$

$$\frac{|dT(z)|^2}{Im(T(z))^2} = \frac{|dz|^2}{(Im(z))^2}$$

$\therefore$ T is an isometry

Ex: The following are isometries

- $z \to -\bar{z}$
- $z \to -\frac{1}{z}$
- $z \to \lambda z$, $\lambda > 0$

Propn: $PSL_2(\mathbb{R}) \subseteq Isom(\mathbb{H}^2)$

Pf: Let $T(z) = \dfrac{az+b}{cz+d}$ be an element of $PSL_2(\mathbb{R})$

$$w = T(z) = \frac{az+b}{cz+d} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

$$Im(w) = \frac{Im(z)}{|cz+d|^2}$$

$$\frac{dT}{dz} = \frac{1}{(cz+d)^2}$$

So $$\frac{|dT(z)|^2}{Im(T(z))^2} = \frac{\frac{1}{|cz+d|^4}|dz|^2}{\frac{Im(z)^2}{|cz+d|^4}} = \frac{|dz|^2}{Im(z)^2}.$$

Hence T is an isometry

Ex: PT $l(T(\gamma)) = l(\gamma)$ for any curve γ.

We have proved that vertical lines are geodesics and Möbius maps (in $PSL_2(\mathbb{R})$) are isometries. This means that the images of these lines are ~~are~~ geodesics.

In particular we can find Möbius maps taking vertical lines to ~~are~~ semicircles with center on the real line.

Hence these are geodesics.

Propn: the vertical lines and semi circles with center on $\mathbb{R}$ are the only geodesics of $\mathbb{H}$.

Pf: ~~B~~ Comes from Riemannian geometry.

The disk model.

The disk model will be another model for Hyperbolic geometry

$\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$. The hyperbolic metric on this will

be $$ds^2_{hyp} = \frac{4\,(dx^2 + dy^2)}{(1-|z|^2)^2} = \frac{4\, ds^2_{euc}}{(1-|z|^2)^2}$$

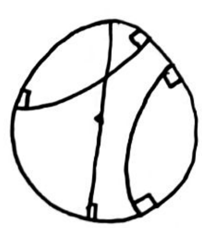
Ex: ~~Very~~ Verify that the map $\varphi : \mathbb{H} \longrightarrow \mathbb{D}$

$$z \longrightarrow \frac{z-i}{z+i}$$

is an isometry

The geodesics in $\mathbb{D}$ will be ~~arcs~~ circular arcs

which are perpendicular to the boundary $\partial \mathbb{D}$



~~Let us calculate the distance between~~

Ex: Calculate/Verify that $d_h(0,r) = \ln\left(\frac{1+r}{1-r}\right)$ where ~~$r \in \mathbb{R}$~~, $r \in (0,1)$

Notice that the metric ds^2_{hyp} has rotational symmetry about the origin.

$$\therefore\ d_h(0, re^{i\theta}) = d_h(0, r) = \ln\left(\frac{1+r}{1-r}\right).$$

Cor: The open ball $B_0^h\left(\cancel{r}\ \ln\left(\frac{1+r}{1-r}\right)\right) := \left\{ z \in \mathbb{D} : d_h(0,z) < \cancel{r}\ \ln\left(\frac{1+r}{1-r}\right) \right\}$ is actually the set $S = \left\{ z \in \mathbb{D} : |z| < \cancel{\ln\left(\frac{1+r}{1-r}\right)}\ r \right\}$.

That is the hyperbolic ball is a disk in $\mathbb{D}$.

Ex: PT the topology induced by the metric d_h is compatible with the Euclidean metric.

We know that $\rho = d_h(0, r) = \ln\left(\frac{1+r}{1-r}\right)$ so

$r = \tanh(\rho/2)$

Hence if we want a ball of hyperbolic radius ρ centered at 0, it will be the set

$$B_0^h(\rho) = \{ z \in \mathbb{C} : |z| < \tanh(\rho/2) \} = B_0^{Euc}(\tanh(\rho/2))$$

Ex: Circumference of $B_0^h(\beta)$ has length $2\pi \sinh(\rho)$

In the Euclidean setting the circumference is $2\pi\rho$.

Notice that the circumference grows exponentially in hyperbolic space whereas it grows linearly in Euclidean space.