1/09/20

Unitary representation:

- $\mathcal{H}$ is a hilbert space.
- $\mathcal{U}(\mathcal{H}) = \{u : \mathcal{H} \to \mathcal{H} \mid u$ is linear unitary$\}$ is a group.

Def^n: A unitary rep^n of a group G on $\mathcal{H}$ is a group homo $\pi : G \to \mathcal{U}(\mathcal{H})$

Def^n: $\xi \in \mathcal{H}$. Then ξ is called π-invariant if $\pi(g)\xi = \xi \ \forall g \in G$.

- Let K be a linear subspace of $\mathcal{H}$. Then K is called π-invariant if every vector of K is π-invariant.

Prop: let $K \subseteq \mathcal{H}$ be a closed subspace. $\pi : G \to \mathcal{U}(\mathcal{H})$ and if K is π invariant
$\pi_K : G \to \mathcal{U}(K)$ by $\pi_K(g) = \pi(g)|_K$ then π_K is a well defined unitary rep^n of G on K. Further $K^\perp$ is also π-invariant and hence $\pi_{K^\perp} : G \to \mathcal{U}(K^\perp)$ is a unitary rep^n of G on $K^\perp$.

Measure preserving actions:

G: countably infinite group.

$(\Omega, \mathcal{A}, \mu)$: probability space.

Def^n: A bijection $\phi : \Omega \to \Omega$ is called measure preserving transformation (mpt) if ϕ and ϕ^{-1} are measurable and $\mu(\phi^{-1}(E)) = \mu(E)$ $\forall E \in \mathcal{A}$.

Rmk: $MP(\mu) = \{\phi : \Omega \to \Omega \mid \phi$ is mpt$\}$ then this is a group.

Def^n: A measure preserving action of G on Ω is a group hom $\alpha : G \to MP(\mu)$.

Notation: if $g \in G$, $x \in \Omega$, $gx := (\alpha(g))(x)$.

Propⁿ: $G \overset{\alpha}{\curvearrowright} (\Omega, \mathcal{A}, \mu)$ mpa.

$\pi : G \longrightarrow \mathcal{U}(L^2(\mu))$ given by $(\pi(g))(f) = f \circ \alpha(g)^{-1}$ gives unitary representation of G on $L^2(\mu)$.

$(\pi(g)(f))(x) = f(g^{-1}x)$. Koopman representation.

I) 1 is an invariant vector π and consequently $\mathbb{C}1$ is π invariant.

II) $L^2_0(\mu) = (\mathbb{C}1)^{\perp}$ and we know.

$\pi_0 : G \longrightarrow \mathcal{U}(L^2_0(\mu))$ is again a unitary repⁿ.

$(\pi_0(g)f)(x) = f(g^{-1}x)$.

$\pi_0 \rightarrow$ deleted Koopman representation.

Recall: $f, g \in L^2(\mu)$, $\langle f, g \rangle := \int f\bar{g}\, d\mu$.

Hence $f \in L^2_0(\mu) \Leftrightarrow \langle f, 1 \rangle = 0 \Leftrightarrow \int f\, d\mu = 0$. [expected value is zero]

Ergodicity:

Defⁿ: $G \overset{\alpha}{\curvearrowright} (\Omega, \mathcal{A}, \mu)$. ~~Then~~ mpa. Then.

1) α is called ergodic if $\forall E \in \mathcal{A}$ if $\mu(E) > 0$, then $\mu\left(\bigcup_{g \in G} gE\right) = 1$.

2) $E \in \mathcal{A}$ is called α-invariant if ~~$\mu(g)$~~ $\mu(gE \Delta E) = 0$ $\forall g \in G$.

3) $E \in \mathcal{A}$ is strictly α-invariant if $gE = E$ $\forall g \in G$.

Propⁿ: $G \overset{\alpha}{\curvearrowright} (\Omega, \mathcal{A}, \mu)$ mpa. then. TFAE.

1) α is ergodic.

2) $\forall A, B \in \mathcal{A}$ if $\mu(A), \mu(B) > 0$ then $\exists g \in G$ s.t. $\mu(gA \cap B) > 0$.

3) if $E \in \mathcal{A}$ is α-invariant, then $\mu(E) \in \{0, 1\}$.

4) if $E \in \mathcal{A}$ is strictly α-invariant, then $\mu(E) \in \{0, 1\}$.

Proof:

(1)$\Rightarrow$(2) $\mu\left(\bigcup_{g\in G} gA\right)=1$.

$\mu(B) = \mu\left(\left(\bigcup_{g\in G} gA\right)\cap B\right) + \mu\left(B\cap\left(\bigcup_{g\in G} gA\right)^c\right)$.

$= \mu\left(\left(\bigcup_{g\in G} gA\right)\cap B\right) = \mu\left(\bigcup_{g\in G}(gA\cap B)\right)$

$0<\mu(B)\le \sum_{g\in G}\mu(gA\cap B)$.

Hence $\exists g\in G$ s.t $\mu(gA\cap B)>0$.

(2)$\Rightarrow$(3) $E\in\mathcal{A}$ is α-invariant.

suppose $\mu(E)\notin\{0,1\}$ Then $\mu(E)>0$, $\mu(E^c)>0$.

Hence $\exists g\in G$ s.t. $\mu(gE\cap E^c)>0$.

But $gE\cap E^c\subseteq gE\,\Delta\, E$.

Hence $\mu(gE\,\Delta\, E)>0$ which is a contradiction.

(3)$\Rightarrow$(4) trivial.

(4)$\Rightarrow$(1) let $E\in\mathcal{A}$ with $\mu(E)>0$.

To show $\mu\left(\bigcup_{g\in G} gE\right)=1$.

But note that $\bigcup_{g\in G} gE$ is strictly α-invariant.

Hence $\mu\left(\bigcup_{g\in G} gE\right)\in\{0,1\}$. But $E\subseteq\bigcup_{g\in G} gE$.

so $\mu\left(\bigcup_{g\in G} gE\right)\ge\mu(E)>0$.

so $\mu\left(\bigcup_{g\in G} gE\right)=1$.

X: Topological space.

$\mathcal{B}(X)$ is the smallest σ-alg containing all open subsets of X.

<u>Lemma</u>: X second countable, T' topological space.

ν: probability measure on $(X, \mathcal{B}(X))$

Suppose $\nu(E) \in \{0,1\}$ $\forall E \in \mathcal{B}(X)$.

Then $\exists x_0 \in X$ s.t. $\nu(\{x_0\}) = 1$.

Pf/ let S be a countable basis for the topology.

let wlog $X \in S$.

let $S_1 = \{U \in S \mid \nu(U) = 1\}$.

$S_1 \neq \phi$ because $X \in S_1$.

let $E_0 = \bigcap_{U \in S_1} U$. $E_0^c = \bigcup_{U \in S_1} U^c$. Hence $\nu(E_0^c) \leq \sum_{U \in S_1} \nu(U^c)$

$= \sum_{U \in S_1} (1 - \nu(U)) = 0$.

so $\nu(E_0^c) = 0$ hence $\nu(E_0) = 1$.

let $x_0 \in E_0$. Then we claim $\nu(\{x_0\}) = 1$.

Suppose not then $\nu(\{x_0\}) = 0$. Hence $\nu(X \setminus \{x_0\}) = 1$.

But $X \setminus \{x_0\}$ is open in X. Hence $\exists$ $S_2 \subseteq S$ s.t. ~~$X \setminus \{x_0\}$~~.

$X \setminus \{x_0\} = \bigcup_{U \in S_2} U$.

$1 = \nu(X \setminus \{x_0\}) \leq \sum_{U \in S_2} \nu(U)$. Hence $\exists$ $U_0 \in S_2$ s.t. $\nu(U_0) \neq 0$.

Hence $\nu(U_0) = 1$.

Hence $U_0 \in S_1$.

But $x_0 \in E_0 \subseteq U_0$.

But $U_0 \in S_2$ and hence $U_0 \subseteq X \setminus \{x_0\}$. This is a contradiction.

Hence $\nu(\{x_0\}) = 1$.

<u>Recall</u>: $(\Omega_1, \mathcal{A}, \mu)$: measure space, $(\Omega_2, \mathcal{A}_2)$ measurable space.

$\phi: \Omega_1 \to \Omega_2$ mble. Then

$\mu \circ \phi^{-1}: \mathcal{A}_2 \to [0, \infty]$ will give a measure on $(\Omega_2, \mathcal{A}_2)$.

Lemma 2: $(\Omega, \mathcal{A}, \mu)$ is probability space.
X: 2nd countable, T^1 topo space.
$f: \Omega \to X$ measurable.
Suppose $\mu(f^{-1}(E)) = \{0,1\}$ $\forall E \in \mathcal{B}(X)$.
then f is constant a.e.

Pf/ $\mu \circ f^{-1}$ is a measure on $(X, \mathcal{B}(X))$ which satisfies hypothesis of lemma 1. Hence $\exists x_0 \in X$ s.t. $\mu(f^{-1}(\{x_0\})) = 1$.

Prop^n: $G \overset{\alpha}{\curvearrowright} (\Omega, \mathcal{A}, \mu)$. mpa. TFAE. (let $1 \le p \le \infty$).
1) α is ergodic
2) $\forall f: \Omega \to \mathbb{C}$ m'ble, if $f \circ \alpha(g^{-1}) = f$ a.e.
then f is constant a.e.
3) $\forall f \in L^p(\mu)$ if $f \circ \alpha(g^{-1}) = f$, then $f = c$ for some $c \in \mathbb{C}$.

Pf/
(1) $\Rightarrow$ (2).
let $E \subseteq \mathbb{C}$ be m'ble.
We claim: $\mu(g(f^{-1}(E)) \Delta f^{-1}(E)) = 0$.
We will show $g(f^{-1}(E)) \Delta f^{-1}(E) \subseteq \{x \in \Omega \mid f(x) \neq f(g^{-1}(x))\}$

If $x \in g(f^{-1}(E)) \Delta f^{-1}(E)$.
case 1: $x \in f^{-1}(E)$. $x \notin g(f^{-1}(E))$.
$x = g(g^{-1}x)$. so $g^{-1}x \notin f^{-1}(E)$ so. $f(g^{-1}x) \notin E$.
hence $f(x) \neq f(g^{-1}(x))$.
hence $x \in \{x \in \Omega \mid f(x) \neq f(g^{-1}(x))\}$.
Case 2: $x \notin f^{-1}(E)$. then. $x \in g(f^{-1}(E))$ hence $g^{-1}x \in f^{-1}(E)$.
$x \in \{y \in \Omega \mid f(y) \neq f(g^{-1}(y))\}$
hence $g(f^{-1}(E)) \Delta f^{-1}(E) \subseteq \{y \in \Omega \mid f(y) \neq f(g^{-1}y)\}$.

Hence $\mu(g(f^{-1}(E)) \Delta f^{-1}(E)) = 0$. $\forall g \in G$.

Hence $\mu(f^{-1}(E)) \in \{0,1\}$ since α is ergodic.

Hence by lemma 2, f is constant a.e.

$(2) \Rightarrow (3)$. Trivial.

$(3) \Rightarrow (1)$.

$E \in \mathcal{A}$. claim: $\mathbb{1}_E \circ \alpha(g^{-1}) = \mathbb{1}_{gE}$ for any $E \in \mathcal{A}$.

$$(\mathbb{1}_E \circ \alpha(g^{-1}))(x) = \mathbb{1}_E(g^{-1}x) = \begin{cases} 1 & \text{if } g^{-1}x \in E \\ 0 & \text{if } g^{-1}x \notin E \end{cases}$$

$$= \begin{cases} 1 & \text{if } x \in gE \\ 0 & \text{if } x \notin gE \end{cases} = \mathbb{1}_{gE}(x).$$

let $E \in \mathcal{A}$ be strictly α-invariant.

$\mathbb{1}_E \in L^p(\mu)$ $\mu \to$ prob measure.

Then $\mathbb{1}_E \circ \alpha(g^{-1}) = \mathbb{1}_{gE} = \mathbb{1}_E$.

hence $\mathbb{1}_E$ is constant almost everywhere

Hence $\mu(E) \in \{0,1\}$.