4 Lecture 4

4.1 The Hyperbolic distance

For $z, w \in \mathbb{H}$ define the distance between them to be

$$d_h(z,w) = inf\{l(\gamma) : \gamma \text{ is a curve between z and w}\}$$

Proposition 4.1.1. *d_h is a metric on $\mathbb{H}$.*

Definition 4.1.1. A curve γ between z and w is called a *geodesic* if $d_h(z,w) = l(\gamma)$.

Warning: This is not the actual definition of geodesics, but in our case it is equivalent.

Proposition 4.1.2. *Vertical lines are geodesics*

Proof. Let $p_1 = (x_0, a)$ and $p_2 = (x_0, b)$ be two points on the line $x = x_0$ and WLOG $0 < a < b$. Let $\gamma(t) = (x(t), y(t))$ be any path connecting them, then

$$\begin{aligned} l(\gamma) &= \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt \\ &\geq \int_0^1 \frac{|y'(t)|}{y(t)} dt \\ &\geq \int_0^1 \frac{y'(t)}{y(t)} dt \\ &\geq ln\left(\frac{b}{a}\right) \end{aligned} \tag{1}$$

Look at $\alpha : [0,1] \to \mathbb{H}$ where $\alpha(t) = i((b-a)t + a)$, then $l(\alpha) = ln(\frac{b}{a})$. But $d_h(p_1, p_2) = inf\{l(\gamma) : \gamma \text{ connects } p_1 \text{ and } p_2\} \geq l(\frac{b}{a})$ and α achieves the equality. Hence $d_h(p_1, p_2) = ln(\frac{b}{a})$ and α is a geodesic. $\square$

Corollary 4.1.0.1.

$$d(ia, ib) = \left| ln\left(\frac{b}{a}\right) \right|$$

Proposition 4.1.3. *The hyperbolic metric d_h is complete.*

Remark: There is a proof using Hopf-Rinow theorem.

4.2 Isometries of the hyperbolic space

Definition 4.2.1. Any diffeorphism $T : \mathbb{H} \to \mathbb{H}$ such that

$$ds^2_{hyp} = \frac{|dz|^2}{Im(z)^2} = \frac{|dT(z)|^2}{Im(T(z))^2}$$

is an *isometry* of $\mathbb{H}$.

Example: $T(z) = z + a$ where $a \in \mathbb{R}$

$$\frac{|dT(z)|^2}{Im(T(z))^2} = \frac{|dz|^2}{Im(z)^2}$$

so T is an isometry.
The following are isometries

- $T(z) = -\bar{z}$
- $T(z) = \frac{1}{z}$
- $T(z) = \lambda z, \lambda > 0$

Proposition 4.2.1.

$$PSL_2(\mathbb{R}) \subseteq Isom(\mathbb{H})$$

Proof. Let $T(z) = \dfrac{az+b}{cz+d}$ be an element of $PSL_2(\mathbb{R})$.

$$w = T(z) = \frac{az+b}{cz+d} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz+d|^2}$$

$$Im(w) = \frac{Im(z)}{|cz+d|^2}$$

$$\frac{dT}{dz} = \frac{1}{(cz+d)^2}$$

So

$$\frac{|dT(z)|^2}{Im(T(z))^2} = \frac{|dz|^2}{Im(z)^2}$$

Hence T is an isometry. □

Ex: Prove that $l(T(\gamma)) = l(\gamma)$ for any curve γ.

We have proved that vertical lines are geodesics and Möbius maps (in $PSL_2(\mathbb{R})$) are isometries. This means that the images of these lines are geodesics. In particular we can find Möbius maps taking vertical lines to semicircles with center on the real line. Hence these are geodesics as well.

Proposition 4.2.2. *The vertical lines and the semicircles with center on $\mathbb{R}$ are the only geodesics of H.*

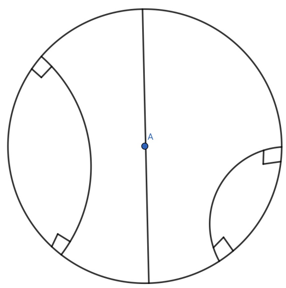
Proof. Comes from Riemannian geometry. □

4.3 The Disk model

The disk model will be another model for hyplerbolic geometry $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. The hyperbolic metric on this will be $ds^2 = \dfrac{4(dx^2+dy^2)}{(1-|z|^2)^2} = \dfrac{4ds^2_{euc}}{(1-|z|^2)^2}$

Ex: Verift that the map $\varphi : \mathbb{H} \to \mathbb{D}$, $\varphi(z) = \dfrac{z-i}{z+i}$ is an isometry.

The geodesics in $\mathbb{D}$ will be circular arcs which are perpendicular to the boundary $\partial\bar{\mathbb{D}}$



Ex: Verify that $d_h(0,r) = ln\Big(\dfrac{1+r}{1-r}\Big)$ where $r \in (0,1)$.

Notice that the metric ds^2_{hyp} has rotational symmetry about the origin. Therefore, $d_h(0, re^{i\theta}) = d_h(0,r) = ln\Big(\dfrac{1+r}{1-r}\Big)$.

Corollary 4.3.0.1. *The open ball $B^h_0\Big(ln\Big(\dfrac{1+r}{1-r}\Big)\Big)$ is equal to the set $S = \{z \in \mathbb{D} : |z| < r\}$. The hyperbolic ball is a disk in $\mathbb{D}$.*

Ex: Prove that the topology induced by the metric d_h is the same as the standard topology.

We know that $\rho = d_h(0,r) = ln\Big(\dfrac{1+r}{1-r}\Big)$ so $r = tanh(\frac{\rho}{2})$. Hence if we want a ball of hyperbolic radius ρ centered at 0, it willl be the set

$$B^h_0(\rho) = \{z \in \mathbb{C} : |z| < tanh\Big(\frac{\rho}{2}\Big)\} = B^{euc}_0\Big(tanh\Big(\frac{\rho}{2}\Big)\Big)$$

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