Lemma: The orientation preserving isometries of $\mathbb{H}^2$ are given by $\text{Isom}^+\mathbb{H}^2 = PSL_2(\mathbb{R})$

Pf: We know that $PSL_2(\mathbb{R}) \subseteq \text{Isom}^+\mathbb{H}^2$.

Now to prove $\text{Isom}^+\mathbb{H}^2 \subseteq PSL_2(\mathbb{R})$.

If $f \in \text{Isom}^+\mathbb{H}^2$, we can consider f as an isometry of $\mathbb{D}$ as well, so $f: \mathbb{D} \to \mathbb{D}$.

Isometries are conformal (preserves angle).

It is enough now to prove that all conformal automorphisms of $\mathbb{D}$ are in $PSL_2(\mathbb{R})$.

Take ~~g~~ $\in$ ~~$\text{Aut}(\mathbb{D})$, these are biholomorphic maps~~

(A): Conformal maps are also biholomorphisms and $\text{Aut}(\mathbb{D}) = PSL_2(\mathbb{R})$ (Use Schwarz Lemma)

Lemma: The ~~typ~~ hyperbolic metric is the unique metric (up to scaling) invariant under $Aut(\mathbb{D})$.

~~Pf~~

Area and Curvature

The area form on $\mathbb{H}$ is given by $\dfrac{dx\,dy}{y^2}$

The area form on $\mathbb{D}$ is given by $\dfrac{4r\,dr\,d\theta}{(1-r^2)^2}$

In general if the metric is $E\,dx^2 + 2F\,dx\,dy + G\,dy^2$ the area form is $\sqrt{EG-F^2}\,dx\,dy$

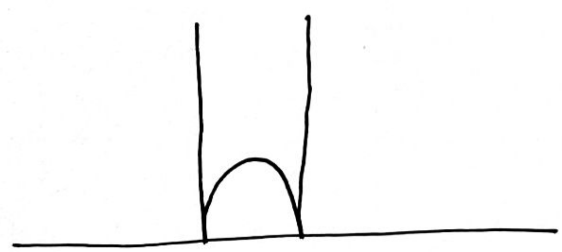
Triangles

Geodesic triangles are triangles with geodesic sides.

Eg:

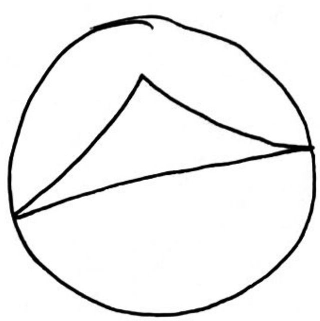
Ideal triangle. An ideal triangle is a triangle with verticies on the boundary $\partial\mathbb{D}$.

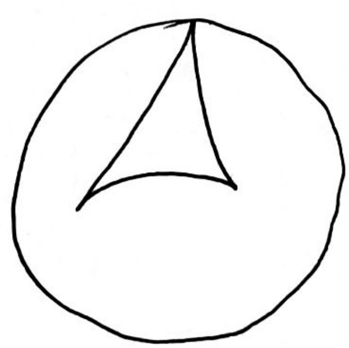
Eg:



There are other triangles too

Eg





Propn: Ideal Triangles are unique upto isometry.

Pf: First note that given any two points on the boundary $\partial\mathbb{D}$ there is a unique geodesic such that the end points of the geodesic are the given points.

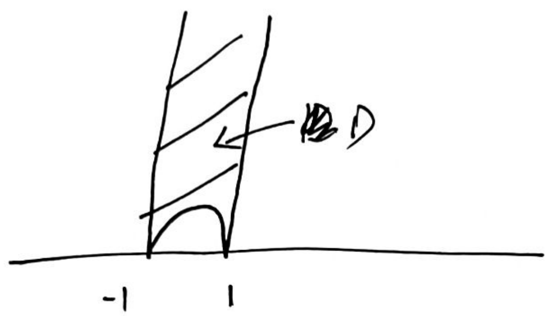
So given three points on $\partial\mathbb{D}$, there is a unique ideal Triangle ~~at~~ determined. IIIly every ideal triangle gives three boundary points. All ideal triangles can be identified by ~~three point to~~ a triple on $\partial\mathbb{D}$ $(\cong \partial\mathbb{H})$

We know that $\mathrm{Isom}^+(\mathbb{H}^2) = PSL_2(\mathbb{R})$ acts triply transitively on $\partial\mathbb{H}$. So we can find a map in $\mathrm{Isom}^+(\mathbb{H})$ taking any triple to any triple. Hence any ideal Triangle can be taken to any other by isometries. $\square$

Isometries preserve area, hence all ideal triangles have the same ~~area~~ area (hyperbolic area).

Propn: Ideal triangles have area π.

Pf:



Area of ideal triangle $= \iint_D \frac{dx\,dy}{y^2}$

$$= \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2}\, dx = \pi.$$

Thm: The area of a hyperbolic triangle ~~of~~ with angles $\alpha_1, \alpha_2, \alpha_3$ is $\pi - \alpha_1 - \alpha_2 - \alpha_3$

Note: We can't take $\alpha_i = 0$ to make it an ideal Δ^{le}.
Also for any Δ^{le} $\alpha_1 + \alpha_2 + \alpha_3 < \pi$ by the 5^{th} postulate

Note that any two similar triangles are also congruent (because area will remain same if angles are same)

Thm: For any conformal metric $\rho(z)|dz|$ or $\rho^2(x,y)(dx^2+dy^2)$, the Gaussian curvature is given by.

$$K(z) = -\frac{\Delta \ln \rho}{\rho^2}(z)$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Lemma: The hyperbolic metric has curvature -1 everywhere.

Note: This is a defining property of hyperbolic geometry

Thm: Any simply connected Riemannian 2-manifold with -1 Gaussian curvature everywhere ~~and~~ and which is complete wrt the metric is isometric to $\mathbb{H}$.

Trigonometry

Thm: Consider the geodesic triangle with sides lengths a, b, c and angles α, β, γ in hyperbolic space.

The following hold.

1. $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$

2. $\cos \gamma = \sin \alpha \sin \beta \cosh c - \cos \alpha \cos \beta$.

3. $\dfrac{\sinh a}{\sin \alpha} = \dfrac{\sinh b}{\sin \beta} = \dfrac{\sinh c}{\sin \gamma}$