2 Lecture 2

2.1 The Hyperbolic Metric

The Hyperbolic Metric on $\mathbb{H}$ is $ds^2_{hyp} = \frac{dx^2+dy^2}{y^2}$.
We interpret this as an inner product at each tangent space.

Let $p = (x_0, y_0) \in \mathbb{H}$. the Tangent Space at this point is $T_p\mathbb{H} = \mathbb{R}^2$.Thus to specify an inner product on it we only need to give the inner product on the basis $\{e_1, e_2\}$.

Now (x, y) are coordinate for $\mathbb{H}$ (it is covered by a single chart). If the metric is given to be $\frac{dx^2+dy^2}{y^2}$ at the point p, the inner product on $T_p\mathbb{H}$ is defined by:
$\langle e_1, e_1\rangle$ = coeff of $dx^2 \;=\; \frac{1}{y_0^2}$
$\langle e_2, e_2\rangle$ = coeff of $dy^2 \;=\; \frac{1}{y_0^2}$
$\langle e_1, e_2\rangle = \frac{1}{2}$ coeff of $dxdy \;=\; 0$

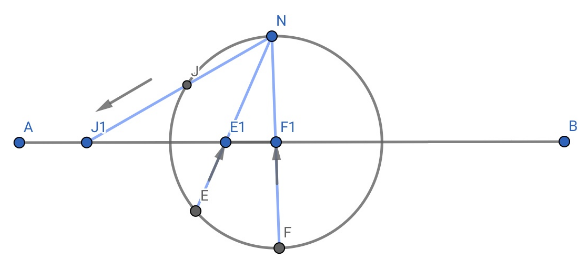
2.2 The Riemann Sphere $\mathbb{C}_\infty$

The Riemann Sphere is defined to be $\mathbb{C}_\infty := \mathbb{C}\bigcup\{\infty\}$
We give a topology on it by declaring the following sets as open in $\mathbb{C}_\infty$:

1. $U \subset \mathbb{C}$ open in $\mathbb{C}$
2. $(\mathbb{C} \setminus K) \cup \{\infty\}$ where K is compact in $\mathbb{C}$.

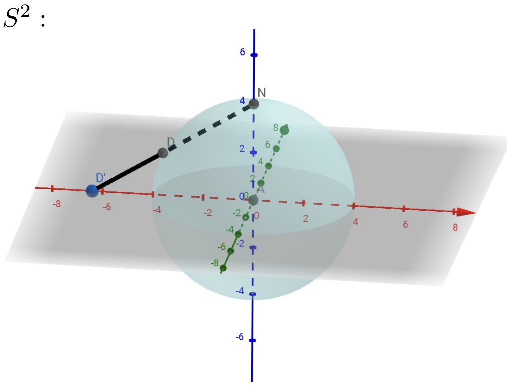
2.3 Stereographic Projection

For S^1 :



$$S^1 \setminus \{i\} \cong \mathbb{R} \text{ and } \mathbb{R} \cup \{\infty\} \cong S^1$$

For S^2 :



$$S^2 \setminus \{n\} \cong \mathbb{R}^2 \text{ and } \mathbb{R}^2 \cup \{\infty\} \cong S^2$$

Hence we have $\mathbb{C}_\infty \cong S^2$

2.4 Continuous functions on $\mathbb{C}_\infty$

- $f : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$
  $z \longmapsto z^n$ when $z \in \mathbb{C}$
  $\infty \longmapsto \infty$

- $g : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$
  $z \longmapsto \frac{1}{z}$ when $z \in \mathbb{C} \setminus \{0\}$
  $\infty \longmapsto 0$
  $0 \longmapsto \infty$

- $h : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$
  $z \longmapsto az + b$ when $z \in \mathbb{C}$ where $a, b \in \mathbb{C}$
  $\infty \longmapsto \infty$

2.5 Circles in $\mathbb{C}_\infty$

We call the following as circles in $\mathbb{C}_\infty$

1. All circles in $\mathbb{C}_\infty$
2. $\{$All lines in $\mathbb{C}\} \cup \infty$

Now all circles in $\mathbb{C}_\infty$ are homeomorphic to S^1.

Consider $\mathbb{R} \cup \{\infty\}$ a circle. It is S^1. By Jordan Brouwer Theorem $\mathbb{R} \cup \{\infty\}$ splits $\mathbb{C}_\infty$ into two disks. namely the upper and lower half plane. So $\mathbb{H}$ is homeomorphic to a disk and $\bar{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ is a closed disk.

2.6 Equation of Circles

The equation of an Euclidean Circle is $(x - x_0)^2 + (y - y_0)^2 = r^2$. Substitute $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ and by rearranging we get $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$. Similarly the equation of an Euclidean line is $\beta z + \bar{\beta}\bar{z} + \gamma = 0$ where $\gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

2.7 Mobius Transformations

A Mobius transformation is a map of the form $m(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$

It can so happen that the denominator is zero for some point $z = \frac{-d}{c}$. In such a case the numerator is non zero *(hint:$ad - bc \neq 0$)*. So we define $m(\frac{-d}{c}) = \infty$

Remark. Algebra with ∞

- $\frac{1}{0} = \infty$
- $\frac{1}{\infty} = 0$

Definition 2.7.1 (Mobius Transformation)**.** Let $m : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ be a function defined by:
$z \longmapsto \frac{az+b}{cz+d}$ when $z \in \mathbb{C}$
$\infty \longmapsto \frac{a}{c}$
where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Then m is called a *Mobius Transformation*

Proposition 2.7.1. *$m(z)$ is a homeomorphism.*

Proof. Composition rule:

Let $m_1(z) = \frac{a_1z+a_2}{a_3z+a_4}$, $m_2(z) = \frac{b_1z+b_2}{b_3z+b_4}$ be Mobius Transformations.

$$\begin{aligned} m_1 \circ m_2(z) &= \frac{a_1\frac{b_1z+b_2}{b_3z+b_4} + a_2}{a_3\frac{b_1z+b_2}{b_3z+b_4} + a_4} \\ &= \frac{a_1b_1z + a_1b_2 + a_2b_3z + a_2b_4}{a_3b_1z + a_3b_2 + a_4b_3z + a_4b_4} \\ &= \frac{(a_1b_1 + a_4b_3)z + (a_1b_2 + a_2b_4)}{(a_3b_1 + a_4b_3)z + (a_3b_2 + a_4b_4)} \end{aligned}$$

Consider the matrices

A= $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, B= $\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ AB= $\begin{pmatrix} a_1b_1 + a_2b_3 & a_3b_1 + a_4b_2 \\ a_3b_1 + a_4b_3 & a_3b_2 + a_4b_4 \end{pmatrix}$

Notice that the terms of $m_1 \circ m_2(z)$ and AB match. So the composition of Mobius Transformation can be thought of as matrix multiplication.

Now since $ad - bc \neq 0$, m becomes invertible and the inverse is also a Mobius Transformation. Hence $m : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ is a homeomorphism. $\square$

2.8 Generators of Mobius Transformations

Definition 2.8.1. Mob^+ is the set of all Mobius Transformations.

Proposition 2.8.1. *f, g, J generate Mob^+*

Proof. Using $f(z) = z + a,\ g(z) = bz,\ a, b \in \mathbb{C}$ we can generate the elements of the form $az + b$.
Using $az + b$ and $J(z) = \frac{1}{z}$ we can generate any element of Mob^+ □

Equations of circles are of the form $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$

1.Circles are invariant under f:

$\alpha(z + a)(z \bar{+} a) + \beta(z + a) + \bar{\beta}(z \bar{+} a) + \gamma = 0$
$\alpha z\bar{z} + \alpha a\bar{a} + \alpha z\bar{a} + \alpha\bar{z}a + \beta z + \bar{\beta}\bar{z} + \beta a + \bar{\beta}\bar{a} + \gamma = 0$
$\alpha z\bar{z} + (\beta\alpha\bar{a})z + (\bar{\beta} + \bar{\alpha}\bar{z} + \alpha|a|^2 + \gamma = 0$

2.Circles are invariant under g:

$\alpha(bz)((\bar{bz}) + \beta(bz) + \bar{\beta}(\bar{bz}) + \gamma = 0$
$\alpha b\bar{b}z\bar{z} + \beta bz + \bar{\beta}\bar{b}\bar{z} + \gamma = 0$

3.Circles are invariant under J:

$\frac{\alpha}{z(\bar{z}} + \frac{\beta}{z} + \frac{\bar{\beta}}{\bar{z}} + \gamma = 0$
$\gamma z\bar{z} + \bar{\beta}z + \beta\bar{z} + \alpha = 0$

And these three map generates Mob^+.We proved:

Proposition 2.8.2. *Circles in $\mathbb{C}_\infty$ are invariant under Mob^+*