6/9/20

$G \overset{\alpha}{\curvearrowright} (\Omega, \mathcal{A}, \mu)$ mpa.

Thm: TFAE:

(i) The action α is ergodic.

(ii) $\forall A, B \in \mathcal{A} \cdot \mu(A), \mu(B) > 0 \Rightarrow \exists g \in G$ s.t. $\mu(gA \cap B) = 0$.

(iii) if $E \in \mathcal{A}$ is α invariant then $\mu(E) \in \{0, 1\}$.

(iv) if $E \in \mathcal{A}$ is strictly α invariant

(v) if $f : \Omega \to \mathbb{C}$ mble and $f \circ \alpha(g^{-1}) = f$ a.e. $\forall g \in G$, then f is constant a.e.

(vi) if $f \in L^P(\mu)$ is such that $f \circ \alpha(g^{-1}) = f$ then $f = c$ for some $c \in \mathbb{C}$.

Rmk: if (vi) is true for some $1 \leq P \leq \infty$ then it is true for every $1 \leq P < \infty$

If $\pi : G \to \mathcal{U}(L^2(\mu))$ is the Koopman representation then any constant function in $L^2(\mu)$ is π-invariant.

Action α is ergodic $\Leftrightarrow$ no non constant function is π-invariant

Consider $\pi_0 : G \to L_0^2(\mu)$: deleted Koopman rep^n.

Prop^n: α is ergodic $\Leftrightarrow$ π_0 has no non zero invariant vectors.

Pf/ ($\Rightarrow$) let $f \in L_0^2(\mu)$ is π_0 invariant. Then $f \in L^2(\mu)$ is π-invariant. Then f is constant. But $L_0^2(\mu) = (\mathbb{C} \cdot 1)^\perp$ and hence $f = 0$.

($\Leftarrow$) We will show any $f \in L^2(\mu)$ which is π invariant is constant.

let $c = \int f \, d\mu$. Consider the function $f - c \in L^2(\mu)$.

then $\int (f - c) \, d\mu = \int f \, d\mu - \int c \, d\mu = 0$.

Hence $f - c \in L_0^2(\mu)$.

But $\forall g \in G$. $\pi(g)(f-c) = \pi(g)f - \pi(g)c = f-c$.
Hence $f-c$ is π-invariant and hence $f-c$ is π_0 invariant
Hence $f-c=0$, hence $f=c$.
Hence (vi) $\Rightarrow$ (i) in previous thm, α is ergodic.

Defn: A unitary repn. $\pi: G \to \mathcal{U}(H)$ is called ergodic.
if π has no nonzero invariant vectors.

Cor: An action $G \overset{\alpha}{\curvearrowright} (\Omega, \mathcal{A}, \mu)$ (mpa) is ergodic $\Leftrightarrow$ The deleted Koopman repn π_0 is ergodic.

Examples:

i) Rotation:

$([0,1), \mathcal{B}([0,1)), \lambda)$.

$\forall x \in [0,1)$ we define

$t_x: [0,1) \longrightarrow [0,1)$ by

$$t_x(y) = \{y+x\}.$$

t_{1-x} is the inverse of t_x.

$$t_x(E) = t_x([0,1-x)\cap E) \cup t_x([1-x,1)\cap E)$$
$$= \cancel{(x+E) \cup ((x-1)+E)}\ (x+([0,1-x)\cap E)) \cup ((x-1)+([1-x,1)\cap E))$$

$$\lambda(t_x(E)) = \lambda(x+[0,1-x)\cap E) + \lambda((x-1)+[1-x,1)\cap E)$$
$$= \lambda([0,1-x)\cap E) + \lambda((x-1)+([1-x,1)\cap E)) = \mu(E).$$

$\Phi: [0,1) \to S^1$

$$\Phi(t) = e^{2\pi i t}$$

$\forall E \in \mathcal{B}(S^1)$, $\mu(E) = \lambda(\Phi^{-1}(E))$

$\forall w \in S^1$ then $M_w: S^1 \to S^1$ by $M_w(z) = wz$.

Note that $\forall w_1, w_2 \in S^1$ $M_{w_1 w_2} = M_{w_1} \circ M_{w_2}$.

and $M_w = \Phi \circ t_{\Phi^{-1}(w)} \circ \Phi^{-1}$.

We can check by defn of μ M_w is a mpt.

Fix $w_0 \in S^1$

let $\alpha : \mathbb{Z} \longrightarrow MP(\mu)$

by $\alpha(n) = M_{w_0}^n = (M_{w_0})^n$.

α is a measure preserving action.

Recall: $\forall n \in \mathbb{Z}$. define $e_n : S^1 \to \mathbb{C}$ by $e_n(z) = z^n$. Then.

$e_n \in L^2(\mu)$. $\int z^n \, d\mu = \int e_n(z) \, d\mu = \int (e_n \circ \Phi) \, d\lambda$

$$= \int_0^1 e^{2\pi i n t} \, d\lambda = \begin{cases} 1 & \text{if } n = 0 \\ \left(\frac{1}{2\pi i n} e^{2\pi i n t}\right)\Big|_0^1 = \frac{1}{2\pi i n} - \frac{1}{2\pi i n} = 0. & \text{if } n \neq 0. \end{cases}$$

$\forall m, n \in \mathbb{Z}$.

$$\langle e_n, e_m \rangle = \int e_n(z) \overline{e_m(z)} \, d\mu = \int z^n z^{-m} \, d\mu = \int z^{n-m} \, d\mu.$$

$$= \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Hence $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal set.

We claim. span $\{e_n \mid n \in \mathbb{Z}\}$ is dense in $C(S^1)$ in L^2 norm and hence dense in $L^2(\mu)$.

We can see, by Stone Weirestrass' thm. span $\{e_n \mid n \in \mathbb{Z}\}$ is a dense (uniformly) in $C(S^1)$ and hence is dense in $C(S^1)$ in L^2 norm.

Hence $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mu)$.

~~Proof~~:

Prop^n: $\omega_0 \in S^1$ and we define $(\alpha(n))(z) = \omega_0^n z$ Then α is mpa

TFAE

(i) $\Phi_n^{-1}(\omega_0) \in \mathbb{R} \setminus \mathbb{Q}$.

(ii) $\omega_0^n \neq 1 \quad \forall n \in \mathbb{N}$

(iii) α is ergodic.

Pf/ (i) $\Leftrightarrow$ (ii)

$\omega_0^n = 1$ for some $n \in \mathbb{N} \Leftrightarrow e^{2\pi i \Phi^{-1}(\omega_0^n)} = 1$

$\Leftrightarrow e^{2\pi i n \Phi^{-1}(\omega_0)} = 1. \Leftrightarrow n\Phi^{-1}(\omega_0) \in \mathbb{Z} \Leftrightarrow \Phi^{-1}(\omega_0) \in \mathbb{Q}$.

(ii) $\Rightarrow$ (iii) ~~If any~~ let $f \in L^2(\mu)$. satisfy $f \circ \alpha(n^{-1}) = f$ a.e $\forall n \in \mathbb{Z}$.

let $n \in \mathbb{Z} \setminus \{0\}$ then $\omega_0^n \neq 1$.

then $\langle f, e_n \rangle = \int f(z) \overline{e_n(z)}\, d\mu$.

by change of variable $= \int f(\alpha(-1)(z))\, e_n(\alpha(-1)(z))\, d\mu$.

$= \int f(z)\, e_n(\omega_0^{-1} z)\, d\mu$.

$= \int f(z) (\omega_0^{-1} z)^n\, d\mu = \omega_0^{-n} \int f(z) z^{-n}\, d\mu$.

$= \omega_0^{-n} \int f(z) \overline{e_n(z)}\, d\mu = \omega_0^{-n} \langle f, e_n \rangle$

But $\omega_0^{-n} \neq 1$ hence $\langle f, e_n \rangle = 0$.

hence f is constant a.e.

(iii) $\Rightarrow$ (ii) let if possible $\omega_0^N = 1$ for some $N \in \mathbb{N}$.

Pf ① Consider $e_N \in L^2(\mu)$. Then e_N is nonconstant

But $(e_N \circ \alpha(-n))(z) = e_N(\omega_0^{-n} z) = \omega_0^{-nN} z^N = z^N = e_N(z)$

Hence $e_N \circ \alpha(-n) = e_N \ \forall n \in \mathbb{Z}$. hence α is not ergodic. $\Rightarrow \Leftarrow$

Pf ② $E = \Phi([0, \mu))$ where $\mu < 1/N$ then $\mu(E) = \mu > 0$. $\bigcup_{n \in \mathbb{Z}} \alpha(n) E = \bigcup_{n=0}^{N-1} \alpha(n) E$.

Hence $\mu\left(\bigcup_{n \in \mathbb{Z}} \alpha(n) E\right) = \mu\left(\bigcup_{n=0}^{N-1} \alpha(n) E\right) \leq \sum_{n=0}^{N-1} \mu(\alpha(n) E) = \sum_{n=0}^{N-1} \mu(E) = N\mu < 1$.

Hence α is not ergodic $\Rightarrow \Leftarrow$