The Hyperbolic Metric.

The Hyperbolic Metric on $\mathbb{H}$ is $ds^2_{hyp} = \dfrac{dx^2 + dy^2}{y^2}$

We interpret this as an inner product at each tangent space.

Let $p = (x_0, y_0) \in \mathbb{H}$. The tangent space at this point $T_p\mathbb{H} = \mathbb{R}^2$. To specify an inner product on it we only need to give the inner product on the basis $\{e_1, e_2\}$.

Now (x,y) are coordinates for $\mathbb{H}$ (i.e. it is covered by a single chart). If the metric is given to be $\dfrac{dx^2+dy^2}{y^2}$, at the point p, the inner product on $T_p\mathbb{H}$ is

$$\langle e_1, e_1\rangle = \text{coeff of } dx^2 = \frac{1}{y_0^2}$$

$$\langle e_2, e_2\rangle = \text{coeff of } dy^2 = \frac{1}{y_0^2}$$

$$\langle e_1, e_2\rangle = \frac{1}{2}\times \text{coeff of } dx\,dy = 0$$

- The Riemann Sphere $\mathbb{C}_\infty$

The Riemann Sphere is defined to be $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$

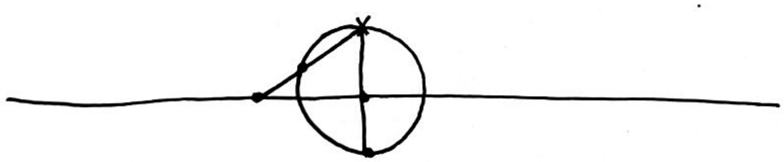
We give a topology on it by declaring the following sets as open in $\mathbb{C}_\infty$

1. $U \subseteq \mathbb{C}$ open in $\mathbb{C}$

2. $(\mathbb{C} \setminus K) \cup \{\infty\}$ where $K \subseteq \mathbb{C}$ is compact in $\mathbb{C}$.

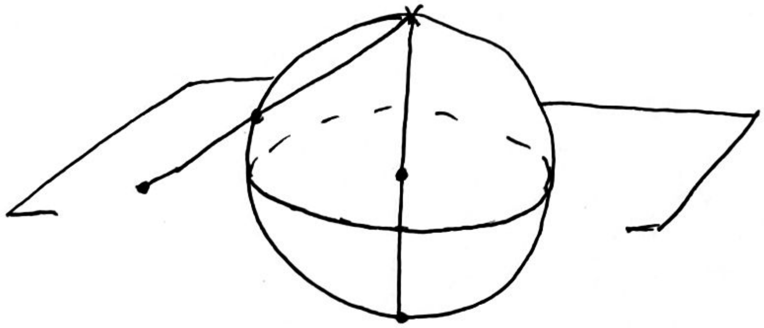
- Stereographic projection

For S^1



$S^1 \setminus \{i\} \cong \mathbb{R}$ and $\mathbb{R} \cup \{\infty\} \cong S^1$

For S^2



$S^2 \setminus \{n\} \cong \mathbb{R}^2$ and $S^2 \cong \mathbb{R}^2 \cup \{\infty\}$

Hence we have $\mathbb{C}_\infty \cong S^2$

Cts functions on $\mathbb{C}_\infty$

- $f(z) = \begin{cases} z^n & \text{when } z \in \mathbb{C} \\ \infty & \text{when } z = \infty \end{cases}$ is a cts fn on $\mathbb{C}_\infty$

- $g : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$

  $z \longrightarrow \frac{1}{z}$ if $z \in \mathbb{C} \setminus \{0\}$

  $0 \longrightarrow \infty$

  $\infty \longrightarrow 0$

  is also a cts fn

- $h : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$

  $z \longrightarrow az+b$ if $z \in \mathbb{C}$ where $a, b \in \mathbb{C}$

  $\infty \longrightarrow \infty$

  is a cts fn.

Circles in $\mathbb{C}_\infty$

We call the following as circles in $\mathbb{C}_\infty$

1. All circles in $\mathbb{C}$
2. All lines in $\mathbb{C} \cup \{\infty\}$.

Now all circles in $\mathbb{C}_\infty$ are homeomorphic to S^1

Consider $\mathbb{R} \cup \{\infty\}$ a circle. It is S^1. By Jordan-Brouwer thm $\mathbb{R} \cup \{\infty\}$ splits $\mathbb{C}_\infty$ into two disks, namely the upper and lower half plane. So $\mathbb{H}$ is homeomorphic to a disk and $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ is a closed disk.

Equations of circles

The equation of an Euclidean circle is $(x-x_0)^2+(y-y_0)^2 = r^2$
Substitute $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ and rearrange to get

$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

III^{ly} the equation of an Euclidean line is $\beta z + \bar{\beta}\bar{z} + \gamma = 0$ where $\beta \in \mathbb{C}$, $\gamma \in \mathbb{R}$.

Möbius Transformations.

A ~~non~~ Möbius Transform is a map of the form $m(z) = \frac{az+b}{cz+d}$

$a, b, c, d \in \mathbb{C}$ & $ad - bc \neq 0$. ~~This is actu~~

It can so happen that the denominator is zero for some point $\left(z = \frac{-d}{c}\right)$. In such a case the numerator is non zero (as $az + b = 0 \Rightarrow z = \frac{-b}{a} = \frac{-d}{c} \Rightarrow ad - bc = 0$)

So we define $m\left(-\frac{d}{c}\right) = \infty$

Algebra with ∞

- $\frac{1}{0} = \infty$

- $\frac{1}{\infty} = 0$

Def:

$$\begin{aligned} m: \mathbb{C}_\infty &\longrightarrow \mathbb{C}_\infty \\ z &\longrightarrow \frac{az+b}{cz+d} \qquad z \in \mathbb{C} \\ \infty &\longrightarrow \frac{a}{c} \end{aligned}$$

for $a, b, c, d \in \mathbb{C}$ & $ad - bc \neq 0$ is called a Möbius transform.

Propn: $m(z)$ is cts.

Composition rule.

Let $m_1(z) = \dfrac{a_1 z + a_2}{a_3 z + a_4}$

$m_2(z) = \dfrac{b_1 z + b_2}{b_3 z + b_4}$ be Möbius Transforms.

$$m_1 \circ m_2(z) = \frac{a_1\left(\dfrac{b_1 z + b_2}{b_3 z + b_4}\right) + a_2}{a_3\left(\dfrac{b_1 z + b_2}{b_3 z + b_4}\right) + a_4}$$

$$= \frac{a_1 b_1 z + a_1 b_2 + a_2 b_3 z + a_2 b_4}{a_3 b_1 z + a_3 b_2 + a_4 b_3 z + a_4 b_4}$$

$$= \frac{(a_1 b_1 + a_2 b_3) z + (a_1 b_2 + a_2 b_4)}{(a_3 b_1 + a_4 b_3) z + (a_3 b_2 + a_4 b_4)}$$

Consider the matrices

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$AB = \begin{pmatrix} a_1 b_1 + a_2 b_3 & a_3 b_1 + a_4 b_2 \\ a_3 b_1 + a_4 b_3 & a_3 b_2 + a_4 b_4 \end{pmatrix}$$

Notice that the terms of $m_1 \circ m_2(z)$ and AB match.

So the composition of Möbius Transforms can be thought of as matrix multiplication.

Now since $ad - bc \neq 0$, m becomes invertible and the inverse is also a Möbius transform

$\therefore$ $m : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is a homeomorphism.

Generators of Möbius Transforms

Def: Mob^+ is the set of all Möbius Transforms.

Usinge $f(z) = z + a$, $g(z) = \cancel{a}bz$, $a, b \in \mathbb{C}$ we can generate ~~of~~ elements of the form $az+b$, $a, b \in \mathbb{C}$.

Using $az+b$ and $J(z) = \frac{1}{z}$ we can generate any element of Mob^+

$$\left(\text{Look at} \quad \begin{pmatrix} b - \frac{ad}{c} & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \quad \text{for } c \neq 0 \right)$$

Propn: f, g, J generate Mob^+

Equation of a circle in $\mathbb{C}_\infty$ is of the form

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0, \quad \alpha, \gamma \in \mathbb{R} \ \& \ \beta \in \mathbb{C}$$

1. Circles are invariant under $z+a$

$$\alpha(z+a)(\overline{z+a}) + \beta(z+a) + \bar{\beta}(\overline{z+a}) + \gamma = 0$$

$$\alpha z\bar{z} + \alpha a\bar{a} + \alpha z\bar{a} + \alpha\bar{z}a + \beta z + \bar{\beta}\bar{z} + \beta a + \bar{\beta}\bar{a} + \gamma = 0$$

$$\alpha z\bar{z} + (\beta + \alpha\bar{a})z + (\bar{\beta} + \bar{\alpha}a)\bar{z} + \alpha|a|^2 + \gamma = 0$$

2. Circles are invariant under bz, $b \neq 0$

$$\alpha(bz)(\bar{b}\bar{z}) + \beta(bz) + \bar{\beta}(\bar{b}\bar{z}) + \gamma = 0$$

$$\alpha b\bar{b}z\bar{z} + \beta bz + \bar{\beta}\bar{b}\bar{z} + \gamma = 0$$

3. Circles are invariant under $1/z$

$$\frac{\alpha}{z\bar{z}} + \frac{\beta}{z} + \frac{\bar{\beta}}{\bar{z}} + \gamma = 0$$

$$\gamma z\bar{z} + \bar{\beta}z + \beta\bar{z} + \alpha = 0$$

Propn: Circles in $\mathbb{C}_\infty$ are invariant under Möb^+

Pf: The above three maps generate Möb^+

Transitivity Property of Mob^+

~~A map $m: X \to X$ is transitive if it co~~

A set of maps from $X \to X$ is called transitive if ~~$\exists$ a map m who~~ for any two points $x, y \in X$, $\exists$ m in the set such that $m(x)=y$.

The Mob^+ group is ~~trip~~ uniquely triply transitive

Propn: Given distinct points $z_1, z_2, z_3 \in \mathbb{C}_\infty$ and another triple of distinct points $w_1, w_2, w_3 \in \mathbb{C}_\infty$, $\exists!\, m \in Mob^+$ ST $m(z_i) = w_i \;\; \forall i = 1,2,3$.

Pf: If we prove $(0,1,\infty)$ can be taken to any (z_1, z_2, z_3) we are done with the existence part.

Consider $$m(z) = \frac{z - z_1}{z - z_3} \times \frac{z_2 - z_3}{z_2 - z_1}$$

Then $m(z_1) = 0$, $m(z_2) = 1$, $m(z_3) = \infty$. m^{-1} works for existence.

For uniqueness it is enough to prove there is only one ~~~~ map taking $(0,1,\infty) \longrightarrow (0,1,\infty)$ ~~~~.

Let m be such that $m(0)=0$, $m(1)=1$, $m(\infty)=\infty$

$$m(z) = \frac{az+b}{cz+d}$$

$$m(0) = \frac{b}{d} = 0 \Rightarrow b=0$$

$$m(\infty) = \frac{a}{c} = \infty \Rightarrow c=0$$

$$m(1) = \frac{a+b}{c+d} = 1 \Rightarrow a+b=c+d \Rightarrow a=d$$

So $$m = \frac{az+0}{0+a} = z \Rightarrow m = Id.$$

$\blacksquare$

Propn: Mob^+ acts transitively on the set of circles in $\mathbb{C}_\infty$

Pf: Given any circle pick three points on it.

Now we get a Möbius transfor taking the first set of points to the second. Since it preserves circles, the circle through the first three points must go to the circle through the second three points. Since 3 points determine a unique circle we are done.

Propn: Mob^+ is transitive on disks.

Pf: ~~Given two disks~~ WLOG let us assume that the boundary of the given disks is the unit circle S^1.

If ~~the one~~ both disks are the bounded or unbounded components of $\mathbb{C}_\infty \setminus S^1$ we can use Id map.

If one is bounded and the other is unbounded we can ~~use~~ use the $1/z$ map.