Group actions

Let G be a group and X be a set. Then G acts on X if there is an operation '$\cdot$' such that

$\cdot : G \times X \longrightarrow X$

i) $\forall x \in X$, $e \cdot x = x$ where $e \in G$ is identity

ii) $\forall g, h \in G$, $gh \cdot x = g \cdot (h \cdot x)$ $\forall x$

This is left action.

Eg: Let X be a top space. Homeo(X) are all the homeomorphisms between $X \to X$. This is a group under composition. It acts on X as well

Take $f \in$ Homeo(X), then $f \cdot x := f(x)$

*: Homeo(X) has a topology given by the compact open topology

If G acts on X any subgroup of G acts on X.

Equivalently a group action on X by G can be given by a homomorphism $G \longrightarrow \mathrm{Bij}(X)$

More eg

- X be a smooth manifold. Then $\mathrm{Diffeo}(X)$ acts on X

  $(\mathrm{Diff}(X) \hookrightarrow \mathrm{Homeo}(X) \hookrightarrow \mathrm{Bij}(X))$

- X be a Riemannian manifold, then $\mathrm{Isom}(X)$ acts on X

Orbits of an action

Let G act on X. Let $x \in X$, then the orbit of x is

$\mathrm{orb}(x) := \{ g \cdot x : g \in G \}$.

The orbits of an action partition X.

Define an equiv relation $\sim$ on X such that

$x \sim y \iff x$ & y are in the same orbit. i.e. $y = g.x$ for some $g \in G$

We now get the quotient $X/\sim$

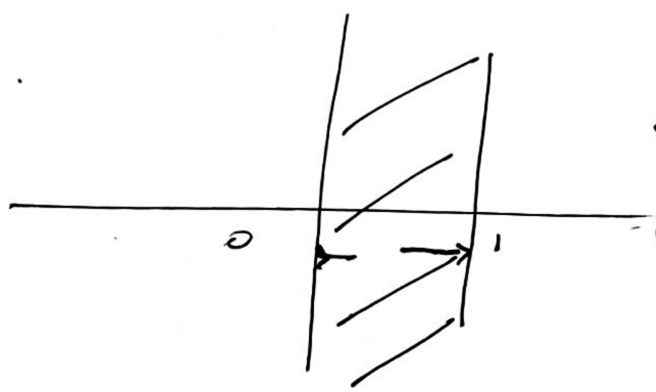
If X was a top space we can give $X/\sim$ the quotient topology.

We will be interested in the space $X/\sim$ given by different group actions.

Eg: Consider $\mathbb{C}$ with the action given by the group $\{z+n : n \in \mathbb{Z}\}$. This group is generated by $G := \langle z+1 \rangle$.

Now consider $\mathbb{C}/G$ (or $\mathbb{C}/\sim_G$) what is it topologically?

$\mathbb{C}/G$ has a unique representative from the set $\{z \in \mathbb{C} : 0 \le \mathrm{Re} < 1\}$.



Think of this as a fundamental domain.

This is a cylinder $S^1 \times \mathbb{R}$.

Ex: look at the action by $G = \langle z+i, z+1 \rangle$ and $\mathbb{C}/G$

It so happened that $\mathbb{C}/G$ was a manifold of same dim.

If you look at $\mathbb{C}/G$ where $G = \{ T(z) = z + \alpha : \alpha \in \mathbb{C} \}$

then $\mathbb{C}/G$ is a single point

We ask when do we get quotient as a manifold of same dim.

Defn:

Properly discontinuous action

A group Γ acts properly discontinuously on a locally compact metric space X if for any compact set $K \subseteq X$

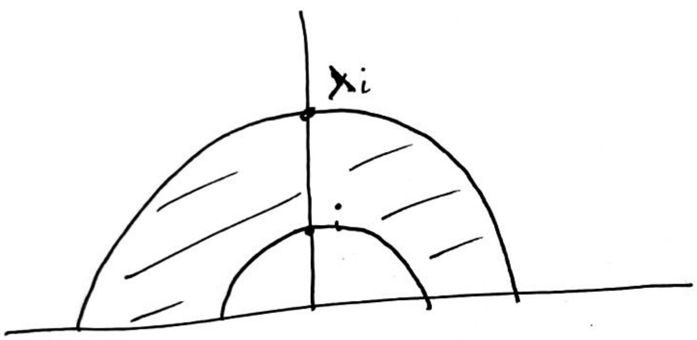
$$\# \{ T \in \Gamma \mid T(K) \cap K \neq \phi \} < \infty .$$

Equivalently, for any $x \in X$, $\Gamma_x \cap K$ is finite

Fuchsian groups

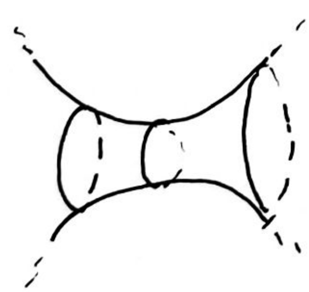
Defn: A ~~f~~ Fuchsian group is a discrete subgroup of $PSL_2(\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$. i.e. for any sequence $T_n \to Id$ in the group $T_n = Id$ for all large ~~n~~ n.

Eg i) $\Gamma = \langle z \to \lambda z \rangle$ $\lambda > 1$

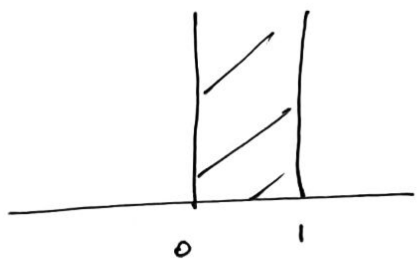


$\mathbb{H}/\Gamma$ is homeomorphic to $S^1 \times \mathbb{R}$.

This is called a hyperbolic cylinder

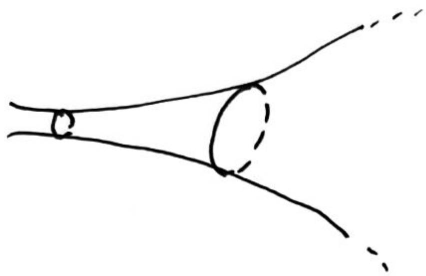


ii) $\Gamma = \langle z \to z+1 \rangle$

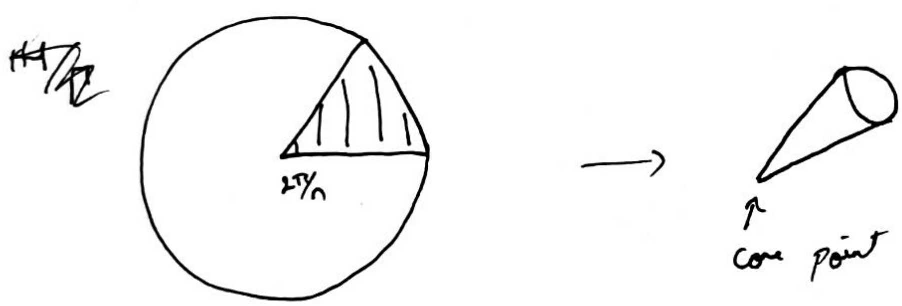


$\mathbb{H}/\Gamma$ is again homeomorphic to $S^1 \times \mathbb{R}$

This is called a cusp



iii) $\Gamma = \langle z \to e^{i2\pi/n} z \rangle$ $n \in \mathbb{N}$



This will not get a smooth structure directly.

Such spaces are called orbifolds.

iv) $\Gamma = PSL_2(\mathbb{Z})$ ← This is a discrete subgroup of $PSL_2(\mathbb{R})$.