1 Lecture 1

1.1 Euclid's Postulates

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended infinitely in a straight line.
3. Given any straight line segment a circle can be drawn having the segment as radius and one endpoint as center.
4. all right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

Parallel Postulate : Given any straight line and a point not on it, there "exists one and only one straight line which passes" through that point and never intersects the first line.

This postulate is equivalent to the "Euclid's fifth postulate". It is also equivalent to the equidistance postulate, angle sum property and many more.

For the most time mathematicians thought that the 5^{th} postulate is a consequence of the first four. They tried to prove it for 2000 years. The answer finally came around $1830's$ by Carl. F. Gauss, Janos Bolyai and N. I. Lobachevsky.

Lobachevsky was the first to publish about non-Euclidean geometry. Non-Euclidean geometry are models which satisfy Euclid's first four postulates but not the fifth.

The mathematics community did not take this discovery well and Lobachevsky faced backlash. Gauss never published his findings fearing the same. Non-Euclidean geometry was not popularized until after 1862 when a private letter written by Gauss about "Hyperbolic Geometry" was published.

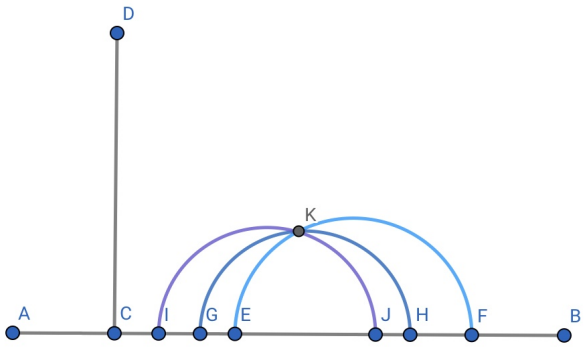
1.2 Non-Euclidean Geometry

To prove that the fifth postulate is independent of the first four, we have to construct an example satisfying the first four but not the fifth.

1.3 Upper Half-Space Model

Consider the upper half space $\mathbb{H} = \{z \in \mathbb{C} : \ im(z) > 0\}$.
Define "lines" in this space to be all vertical lines and all semicircles with center

on the real line.



In this space for any vertical line and a point not on it we can find infinitely many semicircles through the point which doesn't intersect the vertical line.

So in "Modern Language" we need the following:

- It should be a surface.
- A metric to measure distance.
- A way to measure angles between curves.
- Orientation to talk about sides.

1.4 Riemannian Manifold(2D)

A 2-D Riemannian manifold is a smooth oriented surface with a smoothly varying inner product at each tangent space.

- Lines will be geodesics.
- Parallel will mean not intersecting

1.5 The Hyperbolic Path Element

$\gamma : [0, 2] \longrightarrow \mathbb{H}$ be a smooth path in $\mathbb{H}$, the *length* of γ is defined to be

$$len_{\mathbb{H}}(\gamma) := \int_0^1 \frac{|\gamma'(t)|}{Im(\gamma(t))} dt$$

The path length element id $\frac{|dz|}{Im(z)}$.