Loop Invariant Using Alt-Ergo

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Recap: Weakest Preconditions

- The Weakest Preconditions (WP) methodology involves generating **Verification Conditions (VCs).**
- These VC are proved using a **Theorem Prover (Alt-Ergo).**
- Given program P with pre-condition I and post-condition O, the overall VC to be proved is:

  I ==> wp (P, O)

- If P consists only of assignment statements, sequencing, and if/else, only one VC needs to be proved I ==> **wp (P, O)**
  - **Assignment statement, x = expr:**

    wp (x = expr, O) = O [x <-- expr]

    i.e., replace all occurrences of x in O by expr.

  - **Statement sequence, S1; S2** :

    wp (S1; S2, O) = wp (S1, wp (S2, O))

  - **If-Else, if(B) S1 else S2 :**
    - **wp (if (B) S1 else S2, O) = [B ==> wp (S1, O) && not(B) ==> wp (S2, O)]**
    - **Wp(if(B) S1 else S2, O) = [B && wp (S1, O) || not(b) && wp (S2, O)]**

If, if(B) S1 :

- **wp (if (B) S1, O) = [B ==> wp (S1, O) && not(B) ==> O]**
- **Wp (if (B) S1, O) = [B && wp (S1, O) || not(B) && O]**

Loops, while (B) S :

Loops are verified using Loop invariant {Inv}:

Given a while statement,

while (B) S,

since S can be executed an unbounded number of times (0,1, 2,...), wp cannot be derived simply from B, S, and O.

Hence, we define **Loop Invariant {Inv} such that**

wp (while (B) S, O) = Inv

where **Inv** is a loop invariant condition to be supplied by the user and satisfying:

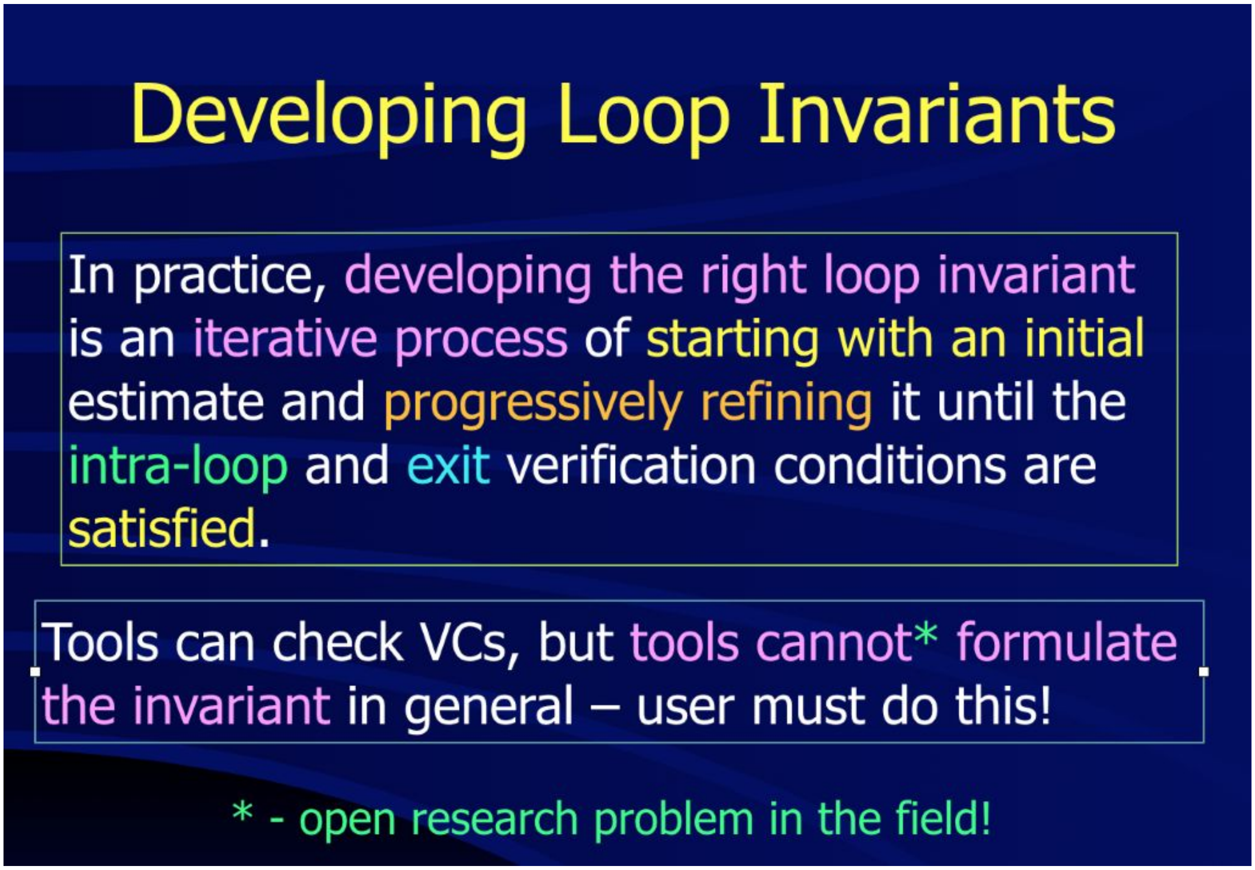
- **Start VC: P ==> wp(initstmts, Inv)**
- **An intra-loop VC: Inv && B ==> wp (LoopStmts, Inv)**
- **An exit VC: Inv && not(B) ==> Q**

P – Precondition, Q – Postcondition, B – Loop condition (or Loop Guard)

Hence to verify a loop, the above mentioned 3 verification conditions need to be checked.

Three Verification Conditions state that:

1. Start VC: {I} holds immediately before the loop, i.e., immediately after the initialization steps.
2. Intra-Loop VC: {I} holds at the end of the loop body, given that {I Λ B} hold at the beginning of the loop body.
3. Exit VC: {I Λ !B}=>{Q} holds after exit of loop.



Steps to find the Loop Invariant:

STEP 1: Trace our program

STEP 2: Assume an invariant based on the relation between the values of the variables

STEP 3: Use Alt-ergo to check the 3 Verification Conditions (VC) [Intra loop VC, Exit VC, Start VC].

> **STEP 3a:** If all the 3 VCs are valid, then the INVARIANT assumed is correct. Hence, we can stop.
>
> **STEP 3b:** If any one of the VC is invalid (unknown), then it means the loop invariant we assumed is insufficient. Repeat step 3c, till all the VCs become valid.
>
> **STEP 3c:** STRENGTHEN THE INVARIANT, generate the 3 VCs and check again in Alt-ergo till all the VCs become valid.

Annotations used in program verification :

@requires - Used to state the Pre-condition or the Input condition for the program.

@ensures – Used to state the Post-condition for the program.

@program – Used to provide the program statements.

@var – used to list the variables and their datatypes used.

@invariant – used to specify the loop invariant.

While Loop General Form:

P – Precondition,

Q – Postcondition,

B – Loop condition (or Loop Guard)

S1, S2 – Loop Statements

```
 {P}
[ initalization statements]
While (B) {
        S1;
        S2;
 }
{Q}
```

Rewriting using verification annotations:

@requires P

@ensures Q

@program

@var comma – separated variables: datatype

[initalization statements]

@invariant: loop-invariant

While (B) {

S1;

S2;

}

@end

Example 1 : Sum of n integers

@requires n >= 1

@ensures s = n*(n+1)/2

@program

```
    @var i, n, s : int
    s = 1;
    i = 1;
    @invariant :
    while (i < n) {
    i = i+1;
    s = s+i;
    }
@end
```

STEP 1: Trace our program

Precondition: n > =1. Let us assume n = 5

I	S
1	1
2	3
3	6
4	10
5	15

S = I * (I+1)/2

STEP 2: Assume an invariant based on the relation between the values of the variables:

Inv: S = I * (I+1)/2

STEP 3: Use Alt-ergo to check the 3 Verification Conditions (VC) [Intra loop VC, Exit VC, Start VC].

1. **Intra-loop VC: Inv && B ==> wp (LoopStmts, Inv)**

Inv: S = I * (I+1)/2

B: I < n

Wp (Loopstmts,Inv) :

Loop Stmts:

S1: i = i+1;

S2: s = s+i;

Wp(Loopstmts, Inv)

=> wp (S1; S2, Inv) = wp (S1, wp(S2, Inv))

=> wp (i = i+1; s = s+i; , s = i * (i+1)/2)

=> wp (i=i+1, **wp (s=s+i, s=i*(i+1)/2)**

[Substituting for s = s+i in s=i*(i+1)/2]

=> s+i = i * (i+1)/2

[Substituting for i=i+1]

=> s+(i+1) = (i+1) *(i+1+1)/2

=> s+(i+1) = (i+1) *(i+2)/2

=> s+(i+1) = (i^2 + 2*i+1*i+2)/2

=> s+(i+1) = (I^2 + 3*i +2)/2

=> 2*(s+(i+1)) = (i^2 + 3*i +2)

=> 2*s + 2*(i+1) = i^2 + 3*i + 2

=> 2*s = i^2 +3*i+2 - (2*(i+1))

=> 2*s= i^2 +I

=> 2*s = i*(i+1)

WP (LoopStmts, Inv) => s = i*(i+1)/2

Condition to be checked in alt-ergo :

Inv && B ==> wp (LoopStmts, Inv)

S = i * (i+1)/2 && i <n ==> s = i*(i+1)/2

2*s = i*(i+1) && i<n ==> 2*s = i*(i+1)

Alt-ergo : Intraloop Verification Condition is Valid

```
goal intraloop:

forall i,n,s: int.

2*s = i*(i+1) and i<n
     ->
     2*s = i*(i+1)
```

```
# [answer] Valid (0.1240 seconds) (3 steps)
```

2. **Exit VC: Inv && not(B) ==> Q**

   Inv: S = I * (I+1)/2

   Not(B) : not(i <n) => i >=n

   Q: s = n*(n+1)/2

   Exit VC:

   S= i*(i+1)/2 && (i>=n) ==> s=n*(n+1)/2

   2*s = i*(i+1) && (i>=n) ==> 2*s = n*(n+1)

Alt-ergo: Exit Verification Condition is unknown

Inv && not(B) ==> Q is not valid

```
goal exit:

forall i,n,s: int.

  2*s = i*(i+1) and i>=n  ->  2*s = n*(n+1)
```

```
# [answer] unknown (0.0700 seconds) (4 steps)
```

STEP 3b: If any one of the VC is invalid (unknown), then it means the loop invariant we assumed is insufficient. Repeat step 3c, till all the VCs become valid.

STEP 3c: STRENGTHEN THE INVARIANT, generate the 3 VCs and check again in Alt-ergo till all the VCs become valid.

STEP 3c: STRENGTHENING THE INVARIANT:

Question : How to **strengthen** the invariant such that **Inv && not(B) ==> Q becomes valid?**

2*s = i*(i+1) && (i>=n) ==> 2*s = n*(n+1)

Question: Why is the following code "**unknown**" as per alt-ergo?

goal exit:

forall i,n,s: int.

2*s = i*(i+1) and i>=n -> 2*s = n*(n+1)

forall i,n,s: int.

LHS : 2*s = i*(i+1) && (i>=n)

If (i=n),

LHS ==> 2*s = n*(n+1) == RHS

If(i>n), say n+1,

LHS ==> 2*s = (n+1) *(n+2)! = RHS

So the condition is valid only for i =n, for all other i>n, the condition is invalid. Hence alt-ergo says its unknown.

So, our invariant should ensure that the value of "i" should not exceed "n" in the condition

Inv && not(B).

Inv: s = i * (i+1)/2 && i<=n

We are strengthening the Invariant by adding another condition to limit the value of n.

Stronger conditions --> reduces the number of items in the matching set

Weaker conditions --> increases the number of items in the matching set

Revisiting - Exit VC: Inv && not(B) ==> Q

Inv: s = i * (i+1)/2 && i<=n

Not(B) : not(i <n) => i >=n

Q: s = n*(n+1)/2

Exit VC : s= i * (i+1)/2 && i<=n && i >=n ==> s = n*(n+1)/2

LHS ==> 2*s = i * (i+1) && i<=n && i >=n

==> 2*s = i * (i+1) && i=n

==> 2*s= n*(n+1) = RHS.

Hence Valid.

Alt-Ergo Verification for Exit VC, Intra Loop Vc and Start VC:

1. **Exit Verification Condition:**

Inv && not(B) ==> Q **is valid**

```
goal new_exitvc:

forall i,n,s: int.

2*s = i*(i+1) and i<=n and i>=n  ->  2*s = n*(n+1)
```

```
goal new_exitvc:

forall i,n,s: int.

  2*s = i*(i+1) and i<=n and i>=n  ->  2*s = n*(n+1)
```

```
# [answer] Valid (0.0680 seconds) (4 steps)
```

2. Intra-loop VC: Inv && B ==> wp (LoopStmts, Inv)

Inv: s = i * (i+1)/2 && i<=n

B: i < n

Wp (Loopstmts,Inv) :

Loop Stmts:

S1: i = i+1;

S2: s = s+i;

Wp(Loopstmts, Inv)

=> wp (S1; S2, Inv) = wp (S1, wp(S2, Inv))

=> wp (i = i+1; s = s+i; , s = i * (i+1)/2 && i<=n)

=> wp (i=i+1, **wp (s=s+i, s=i*(i+1)/2 && i<=n)**

wp (s=s+i, s=i*(i+1)/2&&i<=n)

[Substituting for s = s+i in s=i*(i+1)/2]

=> s+i = i * (i+1)/2 && i<=n

[Substituting for i = i+1]

$\Rightarrow s+(i+1) = (i+1) *(i+1+1)/2 \ \&\& \ i+1<=n$

$\Rightarrow s+(i+1) = (i+1) *(i+2)/2 \ \&\& \ i<=n-1$

$\Rightarrow s+(i+1) = (i^2 + 2*i+1*i+2)/2 \ \&\& \ i<=n-1$

$\Rightarrow s+(i+1) = (i^2 + 3*i +2)/2 \ \&\& \ i<=n-1$

$\Rightarrow 2*(s+(i+1)) = (i^2 + 3*i +2) \ \&\& \ i<=n-1$

$\Rightarrow 2*s + 2*(i+1) = i^2 + 3*i + 2 \ \&\& \ i<=n-1$

$\Rightarrow 2*s = i^2 +3*i+2 - (2*(i+1)) \ \&\& \ i<=n-1$

$\Rightarrow 2*s= i^2 + i\&\& \ i<=n -1$

$\Rightarrow 2*s = i*(i+1) \ \&\& \ i<=n-1$

$\Rightarrow s = i*(i+1)/2 \ \&\& \ i<=n-1$

Wp(Loopstmts, Inv) = s = i*(i+1)/2 && i<=n-1

Condition to be checked in alt-ergo :

Inv && B ==> wp (LoopStmts, Inv)

s = i * (i+1)/2 && i<=n && i < n ==> s = i*(i+1)/2 && i<=n-1

2*s = i*(i+1) && i<=n && i < n ==> 2*s = i*(i+1) && i<=n-1

Alt-ergo: Intraloop Verification Condition is Valid

```
goal new_intraloopvc:

forall i,n,s: int.

2*s = i*(i+1) and i<=n and i < n
     ->
     2*s = i*(i+1) and i<=n-1
```

```
# [answer] Valid (0.0790 seconds) (4 steps)
```

3. **Start VC: P ==> wp(initstmts, Inv)**

   Inv: s = i * (i+1)/2 && i<=n

   P: n >= 1

   wp(initstmts, I) = wp (s=1; i=1, s=(i*(i+1)/2 && i<=n)

   - S1: S=1
   - S2: i=1

   Inv: S=i*(i+1)/2 && i<=n

   Wp (S2, Inv) ==>   2*s=1*(1+1) && 1 <= n [substitute for i=1]

   Wp (S1, Wp (S2, Inv)) ==> 2*1 = 1*(1+1) && 1<=n [Substitute for s=1]

   P ==> wp(initstmts, Inv)

Substituting for P, wp(initstmts,P) in the verification Condition (VC) for start :

P--> Wp(initstmts, Inv) =

Start VC :   (n>=1) -> 2*1 = 1*(1+1) && 1<=n

```
goal new_startvc:

forall i,n,s: int.

(n>=1) -> (2*1) = (1*(1+1)) and 1<=n
```

```
# [answer] Valid (0.0800 seconds) (2 steps)
```

Hence, We have got our required Loop invariant: **s = i * (i+1)/2 && i<=n**

Exercise Problems

Find the loop invariant and prove the verification conditions using Alt-Ergo :

1. Finding Sum of squares of n integers.
2. Find the post-condition and loop invariant. Assume appropriate pre-condition and prove the VCs.

```
x=c;
y=0;
While (x > 0) {
        x = x -1 ;
        y = y+1;
}
```

3. Find the post-condition and loop invariant, for the pre-condition x=10.

```
i =0
j =0
k =0
while (j < x)
{
        j = j + k + 3 * i +1
        k = k + 6 * i + 3
        i = i + 1
}
```

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