
A Computation and Communication Efficient Method for Distributed Nonconvex Problems in the Partial Participation Setting

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Abstract

1 We present a new method that includes three key components of distributed opti-
2 mization and federated learning: variance reduction of stochastic gradients, partial
3 participation, and compressed communication. We prove that the new method has
4 optimal oracle complexity and state-of-the-art communication complexity in the
5 partial participation setting. Regardless of the communication compression feature,
6 our method successfully combines variance reduction and partial participation: we
7 get the optimal oracle complexity, never need the participation of all nodes, and do
8 not require the bounded gradients (dissimilarity) assumption.

9 1 Introduction

10 Federated and distributed learning have become very popular in recent years (Konečný et al., 2016;
11 McMahan et al., 2017). The current optimization tasks require much computational resources and
12 machines. Such requirements emerge in machine learning, where massive datasets and computations
13 are distributed between cluster nodes (Lin et al., 2017; Ramesh et al., 2021). In federated learning,
14 nodes, represented by mobile phones, laptops, and desktops, do not send their data to a server due to
15 privacy and their huge number (Ramaswamy et al., 2019), and the server remotely orchestrates the
16 nodes and communicates with them to solve an optimization problem.

17 As in classical optimization tasks, one of the main current challenges is to find **computationally**
18 **efficient** optimization algorithms. However, the nature of distributed problems induces many other
19 (Kairouz et al., 2021), including i) **partial participation** of nodes in algorithm steps: due to stragglers
20 (Li et al., 2020) or communication delays (Vogels et al., 2021), ii) **communication bottleneck**: even
21 if a node participates, it can be costly to transmit information to a server or other nodes (Alistarh
22 et al., 2017; Ramesh et al., 2021; Kairouz et al., 2021; Sapio et al., 2019; Narayanan et al., 2019). It
23 is necessary to develop a method that considers these problems.

24 2 Optimization Problem

25 Let us consider the nonconvex distributed optimization problem

$$\min_{x \in \mathbb{R}^d} \left\{ f(x) := \frac{1}{n} \sum_{i=1}^n f_i(x) \right\}, \quad (1)$$

26 where $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth nonconvex function for all $i \in [n] := \{1, \dots, n\}$. The full
27 information about function f_i is stored on i^{th} node. The communication between nodes is maintained
28 in the parameters server fashion (Kairouz et al., 2021): we have a server that receives compressed

information from nodes, updates a state, and broadcasts an updated model.¹ Since we work in the nonconvex world, our goal is to find an ε -solution (ε -stationary point) of (1): a (possibly random) point $\hat{x} \in \mathbb{R}^d$, such that $\mathbb{E}[\|\nabla f(\hat{x})\|^2] \leq \varepsilon$.

We consider three settings:

1. **Gradient Setting.** The i^{th} node has only access to the gradient $\nabla f_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ of function f_i . Moreover, the following assumptions for the functions f_i hold.

Assumption 1. *There exists $f^* \in \mathbb{R}$ such that $f(x) \geq f^*$ for all $x \in \mathbb{R}$.*

Assumption 2. *The function f is L -smooth, i.e., $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^d$.*

Assumption 3. *The functions f_i are L_i -smooth for all $i \in [n]$. Let us define $\hat{L}^2 := \frac{1}{n} \sum_{i=1}^n L_i^2$.²*

2. **Finite-Sum Setting.** The functions $\{f_i\}_{i=1}^n$ have the finite-sum form

$$f_i(x) = \frac{1}{m} \sum_{j=1}^m f_{ij}(x), \quad \forall i \in [n], \quad (2)$$

where $f_{ij} : \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth nonconvex function for all $j \in [m]$. We assume that Assumptions 1, 2 and 3 hold and the following assumption.

Assumption 4. *The function f_{ij} is L_{ij} -smooth for all $i \in [n], j \in [m]$. Let $L_{\max} := \max_{i \in [n], j \in [m]} L_{ij}$.*

3. **Stochastic Setting.** The function f_i is an expectation of a stochastic function,

$$f_i(x) = \mathbb{E}_{\xi} [f_i(x; \xi)], \quad \forall i \in [n], \quad (3)$$

where $f_i : \mathbb{R}^d \times \Omega_{\xi} \rightarrow \mathbb{R}$. For a fixed $x \in \mathbb{R}$, $f_i(x; \xi)$ is a random variable over some distribution \mathcal{D}_i , and, for a fixed $\xi \in \Omega_{\xi}$, $f_i(x; \xi)$ is a smooth nonconvex function. The i^{th} node has only access to a stochastic gradients $\nabla f_i(\cdot; \xi_{ij})$ of the function f_i through the distribution \mathcal{D}_i , where ξ_{ij} is a sample from \mathcal{D}_i . We assume that Assumptions 1, 2 and 3 hold and the following assumptions.

Assumption 5. *For all $i \in [n]$ and for all $x \in \mathbb{R}^d$, the stochastic gradient $\nabla f_i(x; \xi)$ is unbiased and has bounded variance, i.e., $\mathbb{E}_{\xi} [\nabla f_i(x; \xi)] = \nabla f_i(x)$, and $\mathbb{E}_{\xi} [\|\nabla f_i(x; \xi) - \nabla f_i(x)\|^2] \leq \sigma^2$, where $\sigma^2 \geq 0$.*

Assumption 6. *For all $i \in [n]$ and for all $x, y \in \mathbb{R}$, the stochastic gradient $\nabla f_i(x; \xi)$ satisfies the mean-squared smoothness property, i.e., $\mathbb{E}_{\xi} [\|\nabla f_i(x; \xi) - \nabla f_i(y; \xi)\|^2] \leq L_{\sigma}^2 \|x - y\|^2$.*

We compare algorithms using the *oracle complexity*, i.e., the number of (stochastic) gradients that each node has to calculate to get ε -solution, and the *communication complexity*, i.e., the number of bits that each node has to send to the server to get ε -solution.

2.1 Unbiased Compressors

We use the concept of unbiased compressors to alleviate the communication bottleneck. The unbiased compressors quantize and/or sparsify vectors that the nodes send to the server.

Definition 1. A stochastic mapping $\mathcal{C} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an *unbiased compressor* if there exists $\omega \in \mathbb{R}$ such that

$$\mathbb{E}[\mathcal{C}(x)] = x, \quad \mathbb{E}[\|\mathcal{C}(x) - x\|^2] \leq \omega \|x\|^2, \quad (4)$$

for all $x \in \mathbb{R}^d$.

¹Note that this strategy can be used in peer-to-peer communication, assuming that the server is an abstraction and all its algorithmic steps are performed on each node.

²Note that $L \leq \hat{L}$, $\hat{L} \leq L_{\max}$, and $\hat{L} \leq L_{\sigma}$.

Table 1: Summary of methods that solve the problem (1) in the stochastic setting (3). Abbr.: *VR* (Variance Reduction) = Does a method have the optimal oracle complexity $\mathcal{O}\left(\frac{\sigma^2}{\varepsilon} + \frac{\sigma}{\varepsilon^{3/2}}\right)$? *PP* (Partial Participation) = Does a method support partial participation from Section 2.2? *CC* = Does a method have the communication complexity equals to $\mathcal{O}\left(\frac{d}{\sqrt{n\varepsilon}}\right)$?

Method	VR	PP	CC	Limitations
SPIDER, SARAH, PAGE, STORM (Fang et al., 2018; Nguyen et al., 2017) (Li et al., 2021a; Cutkosky and Orabona, 2019)	✓	✗	✗	—
MARINA (Gorbunov et al., 2021)	✓	✗ ^(a)	✓ ^(b)	Suboptimal convergence rate (see (Tyurin and Richtárik, 2023)).
FedPAGE (Zhao et al., 2021b)	✗	✗ ^(a)	✗	Suboptimal oracle complexity $\mathcal{O}\left(\frac{\sigma^2}{\varepsilon^2}\right)$.
FRECON (Zhao et al., 2021a)	✗	✓	✓	—
FedAvg (McMahan et al., 2017; Karimireddy et al., 2020b)	✗	✓	✗	Bounded gradients (dissimilarity) assumption of f_i .
SCAFFOLD (Karimireddy et al., 2020b)	✗	✓	✗	Suboptimal convergence rate ^(e) .
MIME^(c) (Karimireddy et al., 2020a)	✗ ^(d)	✓	✗	Calculates full gradient. Bounded gradients (dissimilarity) assumption of f_i . Suboptimal oracle complexity $\mathcal{O}(1/\varepsilon^{3/2})$ in the setting (2).
CE-LSGD (for Partial Participation)^(c) (Patel et al., 2022) (concurrent work)	✓	✓	✗	Bounded gradients (dissimilarity) assumption of f_i . Suboptimal oracle complexity $\mathcal{O}(1/\varepsilon^{3/2})$ in the setting (2).
DASHA (Tyurin and Richtárik, 2023)	✓ ✗	✗ or ✓	✓ ✓	—
DASHA-PP (new)	✓	✓	✓	—

^(a) **MARINA** and **FedPAGE**, with a small probability, require the participation of all nodes so that they can not support partial participation from Section 2.2. Moreover, these methods provide suboptimal oracle complexities.

^(b) On average, **MARINA** provides the compressed communication mechanism with complexity $\mathcal{O}\left(\frac{d}{\sqrt{n\varepsilon}}\right)$. However, with a small probability, this method sends non-compressed vectors.

^(c) Note that **MIME** and **CE-LSGD** can not be directly compared with **DASHA-PP** because **MIME** and **CE-LSGD** consider the online version of the problem (1), and require more strict assumptions.

^(d) Although **MIME** obtains the convergence rate $\mathcal{O}\left(\frac{1}{\varepsilon^{3/2}}\right)$ of a variance reduced method, it requires the calculation of the full (exact) gradients.

^(e) It can be seen when $\sigma^2 = 0$. Let us consider the s -nice sampling of the nodes, then **SCAFFOLD** requires $\mathcal{O}\left(\frac{n^{3/2}}{\varepsilon s^{3/2}}\right)$ communication rounds to get ε -solution, while **DASHA-PP** requires $\mathcal{O}\left(\frac{\sqrt{n}}{\varepsilon s}\right)$ communication rounds (see Theorem 4 with $\omega = 0$, $b = \frac{p_a}{2-p_a}$, and $p_a = \frac{s}{n}$).

We denote a set of stochastic mappings that satisfy Definition 1 as $\mathbb{U}(\omega)$. In our methods, the nodes make use of unbiased compressors $\{\mathcal{C}_i\}_{i=1}^n$. The community developed a large number of unbiased compressors, including *RandK* (see Definition 5) (Beznosikov et al., 2020; Stich et al., 2018), Adaptive sparsification (Wangni et al., 2018) and Natural compression and dithering (Horváth et al., 2019a). We are aware of correlated compressors by Szlendak et al. (2021) and quantizers by Suresh et al. (2022) that help in the homogeneous regimes, but in this work, we are mainly concentrated on generic heterogeneous regimes, though, for simplicity, assume the independence of the compressors.

Assumption 7. $\mathcal{C}_i \in \mathbb{U}(\omega)$ for all $i \in [n]$, and the compressors are statistically independent.

Table 2: Summary of methods that solve the problem (1) in the finite-sum setting (2). Abbr.: *VR* (Variance Reduction) = Does a method have the optimal oracle complexity $\mathcal{O}\left(m + \frac{\sqrt{m}}{\varepsilon}\right)$? *PP* and *CC* are defined in Table 1.

Method	VR	PP	CC	Limitations
SPIDER, PAGE (Fang et al., 2018; Li et al., 2021a)	✓	✗	✗	—
MARINA (Gorbunov et al., 2021)	✓	✗ ^(a)	✓ ^(b)	Suboptimal convergence rate (see (Tyurin and Richtárik, 2023)).
ZeroSARAH (Li et al., 2021b)	✓	✓	✗	Only homogeneous regime, i.e., the functions f_i are equal.
FedPAGE (Zhao et al., 2021b)	✗	✗ ^(a)	✗	Suboptimal oracle complexity $\mathcal{O}\left(\frac{m}{\varepsilon}\right)$.
DASHA (Tyurin and Richtárik, 2023)	✓	✗	✓	—
DASHA-PP (new)	✓	✓	✓	—

(a), (b) : see Table 1.

2.2 Nodes Partial Participation Assumptions

We now try to formalize the notion of partial participation. Let us assume that we have n events $\{i^{\text{th}} \text{ node is participating}\}$ with the following properties.

Assumption 8. *The partial participation of nodes has the following distribution: exists constants $p_a \in (0, 1]$ and $p_{aa} \in [0, 1]$, such that*

1. $\text{Prob}(i^{\text{th}} \text{ node is participating}) = p_a, \quad \forall i \in [n],$
2. $\text{Prob}(i^{\text{th}} \text{ and } j^{\text{th}} \text{ nodes are participating}) = p_{aa},$
for all $i \neq j \in [n].$
3. $p_{aa} \leq p_a^2,$

and these events from different communication rounds are independent.

We are not fighting for the full generality and believe that more complex sampling strategies can be considered in the analysis. For simplicity, we settle upon Assumption 8. Standard partial participation strategies, including s -nice sampling, where the server chooses uniformly s nodes without replacement ($p_a = s/n$ and $p_{aa} = s(s-1)/n(n-1)$), and independent participation, where each node independently participates with probability p_a (due to independence, we have $p_{aa} = p_a^2$), satisfy Assumption 8. In the literature, s -nice sampling is one of the most popular strategies (Zhao et al., 2021a; Richtárik et al., 2021; Reddi et al., 2020; Konečný et al., 2016).

3 Motivation and Related Work

The main goal of our paper is to develop a method for the nonconvex distributed optimization that will include three key features: variance reduction of stochastic gradients, compressed communication, and partial participation. We now provide an overview of the literature (see also Table 1 and Table 2).

1. Variance reduction of stochastic gradients

It is important to consider finite-sum (2) and stochastic (3) settings because, in machine learning tasks, either the number of local functions m is huge or the functions f_i is an expectation of a stochastic function due to the batch normalization (Ioffe and Szegedy, 2015) or random augmentation (Goodfellow et al., 2016), and it is infeasible to calculate the full gradients analytically. Let us recall the results from the nondistributed optimization. In the gradient setting, the optimal oracle complexity

is $\mathcal{O}(1/\varepsilon)$, achieved by the vanilla gradient descent (GD) (Carmon et al., 2020; Nesterov, 2018). In the finite-sum setting and stochastic settings, the optimal oracle complexities are $\mathcal{O}\left(m + \frac{\sqrt{m}}{\varepsilon}\right)$ and $\mathcal{O}\left(\frac{\sigma^2}{\varepsilon} + \frac{\sigma}{\varepsilon^{3/2}}\right)$ (Fang et al., 2018; Li et al., 2021a; Arjevani et al., 2019), accordingly, achieved by methods SPIDER, SARAH, PAGE, and STORM from (Fang et al., 2018; Nguyen et al., 2017; Li et al., 2021a; Cutkosky and Orabona, 2019).

2. Compressed communication

In distributed optimization (Ramesh et al., 2021; Xu et al., 2021), lossy communication compression can be a powerful tool to increase the communication speed between the nodes and the server. Different types of compressors are considered in the literature, including unbiased compressors (Alistarh et al., 2017; Beznosikov et al., 2020; Szlendak et al., 2021), contractive (biased) compressors (Richtárik et al., 2021), 3PC compressors (Richtárik et al., 2022). We will focus on unbiased compressors because methods DASHA and MARINA (Tyurin and Richtárik, 2023; Szlendak et al., 2021; Gorbunov et al., 2021) that employ unbiased compressors provide the current theoretical state-of-the-art (SOTA) communication complexities.

Many methods analyzed optimization methods with the unbiased compressors (Alistarh et al., 2017; Mishchenko et al., 2019; Horváth et al., 2019b; Gorbunov et al., 2021; Tyurin and Richtárik, 2023). In the gradient setting, the methods MARINA and DASHA by Gorbunov et al. (2021) and Tyurin and Richtárik (2023) establish the current SOTA communication complexity, each method needs $\frac{1+\omega/\sqrt{n}}{\varepsilon}$ communication rounds to get an ε -solution. In the finite-sum and stochastic settings, the current SOTA communication complexity is attained by the DASHA method, while maintaining the optimal oracle complexities $\mathcal{O}\left(m + \frac{\sqrt{m}}{\varepsilon\sqrt{n}}\right)$ and $\mathcal{O}\left(\frac{\sigma^2}{\varepsilon n} + \frac{\sigma}{\varepsilon^{3/2}n}\right)$ per node.

3. Partial participation

From the beginning of federated learning era, the partial participation has been considered to be the essential feature of distributed optimization methods (McMahan et al., 2017; Konečný et al., 2016; Kairouz et al., 2021). However, previously proposed methods have limitations: i) methods MARINA and FedPAGE from (Gorbunov et al., 2021; Zhao et al., 2021b) still require synchronization of all nodes with a small probability. ii) in the stochastic settings, methods FedAvg, SCAFFOLD, and FRECON with the partial participation mechanism (McMahan et al., 2017; Karimireddy et al., 2020b; Zhao et al., 2021a) provide results without variance reduction techniques from (Fang et al., 2018; Li et al., 2021a; Cutkosky and Orabona, 2019) and, therefore, get suboptimal oracle complexities. Note that FRECON and DASHA reduce the variance *only from compressors* (in the partial participation and stochastic setting). iii) in the finite-sum setting, the ZeroSARAH method by Li et al. (2021b) focuses on the homogeneous regime only (the functions f_i are equal). iv) The MIME method by Karimireddy et al. (2020a) and the CE-LSGD method (for Partial Participation) by the concurrent paper (Patel et al., 2022) consider the online version of the problem (1). Therefore, MIME and CE-LSGD (for Partial Participation) require stricter assumptions, including the bounded inter-client gradient variance assumption. In the finite-sum setting (2), MIME and CE-LSGD obtain a suboptimal oracle complexity $\mathcal{O}(1/\varepsilon^{3/2})$ while, in the full participation setting, it is possible to get the complexity $\mathcal{O}(1/\varepsilon)$.

4 Contributions

We propose a new method DASHA-PP for the nonconvex distributed optimization.

- As far as we know, this is the first method that includes three key ingredients of federated learning methods: *variance reduction of stochastic gradients, compressed communication, and partial participation*.
- Moreover, this is the first method that combines *variance reduction of stochastic gradients and partial participation* flawlessly: i) it gets the optimal oracle complexity ii) does not require the participation of all nodes iii) does not require the bounded gradients assumption of the functions f_i .
- We prove convergence rates and show that this method has *the optimal oracle complexity and the state-of-the-art communication complexity in the partial participation setting*. Moreover, in our work, we observe a nontrivial side-effect from mixing the variance reduction of stochastic gradients and partial participation. It is a general problem not related to our methods or analysis that we discuss in Section 7.

Algorithm 1 DASHA-PP

- 1: **Input:** starting point $x^0 \in \mathbb{R}^d$, stepsize $\gamma > 0$, momentum $a \in (0, 1]$, momentum $b \in (0, 1]$, probability $p_{\text{page}} \in (0, 1]$ (only in **DASHA-PP-PAGE**), batch size B (only in **DASHA-PP-PAGE**, **DASHA-PP-FINITE-MVR** and **DASHA-PP-MVR**), probability $p_a \in (0, 1]$ that a node is *participating*^(a), number of iterations $T \geq 1$
 - 2: Initialize $g_i^0 \in \mathbb{R}^d$, $h_i^0 \in \mathbb{R}^d$ on the nodes and $g^0 = \frac{1}{n} \sum_{i=1}^n g_i^0$ on the server
 - 3: Initialize $h_{ij}^0 \in \mathbb{R}^d$ on the nodes and take $h_i^0 = \frac{1}{m} \sum_{j=1}^m h_{ij}^0$ (only in **DASHA-PP-FINITE-MVR**)
 - 4: **for** $t = 0, 1, \dots, T - 1$ **do**
 - 5: $x^{t+1} = x^t - \gamma g^t$
 - 6: Broadcast x^{t+1}, x^t to all *participating*^(a) nodes
 - 7: **for** $i = 1, \dots, n$ in parallel **do**
 - 8: **if** i^{th} node is *participating*^(a) **then**
 - 9: Calculate k_i^{t+1} using Algorithm 2, 3, 4 or 5
 - 10: $h_i^{t+1} = h_i^t + \frac{1}{p_a} k_i^{t+1}$
 - 11: $m_i^{t+1} = C_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right)$
 - 12: $g_i^{t+1} = g_i^t + m_i^{t+1}$
 - 13: Send m_i^{t+1} to the server
 - 14: **else**
 - 15: $h_{ij}^{t+1} = h_{ij}^t$ (only in **DASHA-PP-FINITE-MVR**)
 - 16: $h_i^{t+1} = h_i^t, \quad g_i^{t+1} = g_i^t, \quad m_i^{t+1} = 0$
 - 17: **end if**
 - 18: **end for**
 - 19: $g^{t+1} = g^t + \frac{1}{n} \sum_{i=1}^n m_i^{t+1}$
 - 20: **end for**
 - 21: **Output:** \hat{x}^T chosen uniformly at random from $\{x^t\}_{k=0}^{T-1}$
- (a): For the formal description see Section 2.2.
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Algorithm 2 Calculate k_i^{t+1} for **DASHA-PP** in the gradient setting. See line 9 in Alg. 1

- 1: $k_i^{t+1} = \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))$
-

Algorithm 3 Calculate k_i^{t+1} for **DASHA-PP-PAGE** in the finite-sum setting. See line 9 in Alg. 1

- 1: Generate a random set I_i^t of size B from $[m]$ *with replacement*
 - 2: $k_i^{t+1} = \begin{cases} \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)), \\ \text{with probability } p_{\text{page}} \text{ on all } \textit{participating} \text{ nodes,} \\ \frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)), \\ \text{with probability } 1 - p_{\text{page}} \text{ on all } \textit{participating} \text{ nodes} \end{cases}$
-

Algorithm 4 Calc. k_i^{t+1} for **DASHA-PP-FINITE-MVR** in the finite-sum setting. See line 9 in Alg. 1

- 1: Generate a random set I_i^t of size B from $[m]$ *without replacement*
 - 2: $k_{ij}^{t+1} = \begin{cases} \frac{m}{B} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))), & j \in I_i^t, \\ 0, & j \notin I_i^t \end{cases}$
 - 3: $h_{ij}^{t+1} = h_{ij}^t + \frac{1}{p_a} k_{ij}^{t+1}$
 - 4: $k_i^{t+1} = \frac{1}{m} \sum_{j=1}^m k_{ij}^{t+1}$
-

Algorithm 5 Calculate k_i^{t+1} for **DASHA-PP-MVR** in the stochastic setting. See line 9 in Alg. 1

- 1: Generate i.i.d. samples $\{\xi_{ij}^{t+1}\}_{j=1}^B$ of size B from \mathcal{D}_i .
 - 2: $k_i^{t+1} = \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^t; \xi_{ij}^{t+1}) - b \left(h_i^t - \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^t; \xi_{ij}^{t+1}) \right)$
-

5 Algorithm Description and Main Challenges Towards Partial Participation

We now present **DASHA-PP** (see Algorithm 1), a family of methods to solve the optimization problem (1). When we started investigating the problem, we took **DASHA** as a baseline method for two reasons: the family of algorithms **DASHA** provides the current state-of-the-art communication complexities in the *non-partial participation* setting, and, unlike **MARINA**, it does not send non-compressed gradients and does not synchronize all nodes. Let us briefly discuss the main idea of **DASHA**, its problem in the *partial participation* setting, and why the refinement of **DASHA** is not an exercise.

In fact, **DASHA** supports the partial participation of nodes *in the gradient setting*. Since the nodes only do the following steps (see full algorithm in Algorithm 6):

$$g_i^{t+1} = g_i^t + C_i (\nabla f_i(x^{t+1}) - (1 - a)\nabla f_i(x^t) - ag_i^t).$$

The partial participation mechanism (independent participation from Section 2.2) can be easily implemented here if we redefine the compressor and use another one instead:

$$C_i^p := \begin{cases} \frac{1}{p}C_i, & \text{with pr. } p_a, \\ 0, & \text{with pr. } 1 - p_a. \end{cases} \Rightarrow g_i^{t+1} = \begin{cases} g_i^t + \frac{1}{p_a}C_i (\nabla f_i(x^{t+1}) - (1 - a)\nabla f_i(x^t) - ag_i^t), & p_a \\ g_i^t, & 1 - p_a. \end{cases}$$

With probability $1 - p$, a node does not update g_i^t and does not send anything to the server. The main observation is that we can do this trick since g_i^{t+1} depends only on the vectors x^{t+1} , x^t , and g_i^t .

However, we focus our attention on partial participation *in the finite-sum and stochastic settings*. Consider the nodes' steps in **DASHA-MVR** (see Algorithm 7) that is designed for the stochastic setting:

$$h_i^{t+1} = \nabla f_i(x^{t+1}; \xi_i^{t+1}) + (1 - b)(h_i^t - \nabla f_i(x^t; \xi_i^{t+1})), \quad (6)$$

$$g_i^{t+1} = g_i^t + C_i (h_i^{t+1} - h_i^t - a(g_i^t - h_i^t)). \quad (7)$$

Even if we use the same trick for (7), we still have to update (6) in every iteration of the algorithm since g_i^{t+1} additionally depends on h_i^{t+1} and h_i^t . In other words, if a node does not update g_i^t and does not send anything to the server, it still has to update h_i^t , what is impossible without the points x^{t+1} and x^t . One of the main challenges was to “guess” how to generalize (6) and (7) to the partial participation setting. We now provide a solution:

$$h_i^{t+1} = h_i^t + \frac{1}{p_a}k_i^{t+1}, \quad k_i^{t+1} = \nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - b(h_i^t - \nabla f_i(x^t; \xi_i^{t+1})), \quad (8)$$

$$g_i^{t+1} = g_i^t + C_i \left(\frac{1}{p_a}k_i^{t+1} - \frac{a}{p_a}(g_i^t - h_i^t) \right) \text{ with pr. } p_a, \text{ and } h_i^{t+1} = h_i^t, \quad g_i^{t+1} = g_i^t \text{ with pr. } 1 - p_a.$$

Now both control variables g_i^t and h_i^t do not change with the probability p_a . However, this required changing the update rule of g_i^{t+1} and h_i^{t+1} to make the proof work.

i) The theoretical analysis of **DASHA-PP** is more complicated: while in **DASHA**, the randomness from compressors is independent of the randomness from stochastic gradients, in **DASHA-PP**, these two randomnesses are coupled by the randomness from the partial participation. Moreover, the new methods have to reduce the variance from partial participation.

ii) In the gradient setting, comparing the structure of algorithms **DASHA-PP** and **DASHA**, one can see that in **DASHA-PP** we added at least two crucial things: the momentum b , which helps to reduce the variance of partial participation randomness, and the proper scaling by $1/p_a$. Note that in finite-sum and stochastic settings, in **DASHA-PP-FINITE-MVR** and **DASHA-PP-MVR**, accordingly, the momentum b plays the dual role; it also helps to reduce the variance of stochastic gradients.

iii) In the finite-sum setting, we present two methods: **DASHA-PP-PAGE** and **DASHA-PP-FINITE-MVR**. The former is based on **PAGE** (Li et al., 2021a) and with small probability p_{page} calculates the full gradients of the functions f_i . The latter always calculates mini-batches, but it needs extra memory $\mathcal{O}(dm)$ per node to store vectors h_{ij}^t .

At the first reading of the proofs, we suggest the reader follow the proof of Theorem 2 in the gradient setting, which takes a small part of the paper. Although the proof seems to be dense and large, the size of the appendix is justified by the fact that we consider different settings and PL-condition.

6 Theorems

We now present the convergence rates theorems of **DASHA-PP** in different settings. We will compare the theorems with the results of the current state-of-the-art methods, **MARINA** and **DASHA**, that work in the full participation setting. Suppose that **MARINA** or **DASHA** converges to ε -solution after T communication rounds. Then, ideally, we would expect the convergence of the new algorithms to ε -solution after up to T/p_a communication rounds due to the partial participation constraints³. The detailed analysis of the algorithms under Polyak-Łojasiewicz condition we provide in Section E. Let us define $\Delta_0 := f(x^0) - f^*$.

6.1 Gradient Setting

Theorem 2. Suppose that Assumptions 1, 2, 3, 7 and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{p_a}{2-p_a}$,

$$\gamma \leq \left(L + \left[\frac{48\omega(2\omega+1)}{np_a^2} + \frac{16}{np_a^2} \left(1 - \frac{p_{aa}}{p_a} \right) \right]^{1/2} \hat{L} \right)^{-1},$$

and $g_i^0 = h_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ in Algorithm 1 (**DASHA-PP**), then $\mathbb{E} [\|\nabla f(\hat{x}^T)\|^2] \leq \frac{2\Delta_0}{\gamma T}$.

Let us recall the convergence rate of **MARINA** or **DASHA**, the number of communication rounds to get ε -solution equals $\mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega}{\sqrt{n}} \hat{L} \right] \right)$, while the rate of **DASHA-PP** equals $\mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega+1}{p_a \sqrt{n}} \hat{L} \right] \right)$. Up to Lipschitz constants factors, we get the degeneration up to $1/p_a$ factor due to the partial participation.

6.2 Finite-Sum Setting

Theorem 3. Suppose that Assumptions 1, 2, 3, 4, 7, and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{p_{page} p_a}{2-p_a}$, probability $p_{page} \in (0, 1]$,

$$\gamma \leq \left(L + \left[\frac{48\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{(1-p_{page})L_{\max}^2}{B} \right) + \frac{16}{np_a^2 p_{page}} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1-p_{page})L_{\max}^2}{B} \right) \right]^{1/2} \right)^{-1}$$

and $g_i^0 = h_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ in Algorithm 1 (**DASHA-PP-PAGE**) then $\mathbb{E} [\|\nabla f(\hat{x}^T)\|^2] \leq \frac{2\Delta_0}{\gamma T}$.

We now choose p_{page} to balance heavy full gradient and light mini-batch calculations. Let us define $\mathbb{1}_{p_a} := \sqrt{1 - \frac{p_{aa}}{p_a}} \in [0, 1]$. Note that if $p_a = 1$ then $p_{aa} = 1$ and $\mathbb{1}_{p_a} = 0$.

Corollary 1. Let the assumptions from Theorem 3 hold and $p_{page} = B/(m+B)$. Then **DASHA-PP-PAGE** needs

$$T := \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + \frac{1}{p_a} \sqrt{\frac{m}{n}} \left(\frac{\mathbb{1}_{p_a} \hat{L}}{\sqrt{B}} + \frac{L_{\max}}{B} \right) \right] \right) \quad (9)$$

communication rounds to get an ε -solution and the expected number of gradient calculations per node equals $\mathcal{O}(m + BT)$.

The convergence rate the rate of the current state-of-the-art method **DASHA-PAGE** without partial participation equals $\mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega}{\sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + \sqrt{\frac{m}{n}} \frac{L_{\max}}{B} \right] \right)$. Let us closer compare it with (9). As expected, we see that the second term w.r.t. ω degenerates up to $1/p_a$. Surprisingly, the third term

³We check this numerically in Section A.

218 w.r.t. $\sqrt{m/n}$ can degenerate up to \sqrt{B}/p_a when $\hat{L} \approx L_{\max}$. Hence, in order to keep degeneration up to
 219 $1/p_a$, one should take the batch size $B = \mathcal{O}(L_{\max}^2/\hat{L}^2)$. This interesting effect we analyze separately
 220 in Section 7. The fact that the degeneration is up to $1/p_a$ we check numerically in Section A.

221 In the following corollary, we consider RandK compressors (see Definition 5) and show that with
 222 the particular choice of parameters, up to the Lipschitz constants factors, DASHA-PP-PAGE gets the
 223 optimal oracle complexity and SOTA communication complexity. The choice of the compressor is
 224 driven by simplicity, and the following analysis can be used for other unbiased compressors.

225 **Corollary 2.** Suppose that assumptions of Corollary 1 hold, $B \leq \min \left\{ \frac{1}{p_a} \sqrt{\frac{m}{n}}, \frac{L_{\max}^2}{\frac{1}{p_a} \hat{L}^2} \right\}^4$, and we
 226 use the unbiased compressor RandK with $K = \Theta(Bd/\sqrt{m})$. Then the communication complexity of
 227 Algorithm 1 is

$$\mathcal{O} \left(d + \frac{L_{\max} \Delta_0 d}{p_a \varepsilon \sqrt{n}} \right), \quad (10)$$

228 and the expected number of gradient calculations per node equals

$$\mathcal{O} \left(m + \frac{L_{\max} \Delta_0 \sqrt{m}}{p_a \varepsilon \sqrt{n}} \right). \quad (11)$$

229 The convergence rate of DASHA-PP-FINITE-MVR is provided in Section D.5. The conclusions are the
 230 same for the method.

231 6.3 Stochastic Setting

232 We define $h^t := \frac{1}{n} \sum_{i=1}^n h_i^t$.

233 **Theorem 4.** Suppose that Assumptions 1, 2, 3, 5, 6, 7 and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$,
 234 $b \in \left(0, \frac{p_a}{2-p_a} \right]$,

$$\begin{aligned} \gamma \leq & \left(L + \left[\frac{48\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{(1-b)^2 L_\sigma^2}{B} \right) \right. \right. \\ & \left. \left. + \frac{12}{np_a b} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1-b)^2 L_\sigma^2}{B} \right) \right]^{1/2} \right)^{-1}, \end{aligned}$$

235 and $g_i^0 = h_i^0$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP-MVR). Then

$$\begin{aligned} \mathbb{E} \left[\left\| \nabla f(\hat{x}^T) \right\|^2 \right] \leq & \frac{1}{T} \left[\frac{2\Delta_0}{\gamma} + \frac{2}{b} \left\| h^0 - \nabla f(x^0) \right\|^2 \right. \\ & \left. + \left(\frac{32b\omega(2\omega+1)}{np_a^2} + \frac{4 \left(1 - \frac{p_{aa}}{p_a} \right)}{np_a} \right) \left(\frac{1}{n} \sum_{i=1}^n \left\| h_i^0 - \nabla f_i(x^0) \right\|^2 \right) \right] \\ & + \left(\frac{48b^2\omega(2\omega+1)}{p_a^2} + \frac{12b}{p_a} \right) \frac{\sigma^2}{nB}. \end{aligned}$$

236 In the next corollary, we choose momentum b and initialize vectors h_i^0 to get ε -solution. Let us define

237 $\mathbb{1}_{p_a} := \sqrt{1 - \frac{p_{aa}}{p_a}} \in [0, 1]$.

238 **Corollary 3.** Suppose that assumptions from Theorem 4 hold, momentum $b =$
 239 $\Theta \left(\min \left\{ \frac{p_a}{\omega} \sqrt{\frac{n\varepsilon B}{\sigma^2}}, \frac{p_a n \varepsilon B}{\sigma^2} \right\} \right)$, $\frac{\sigma^2}{n\varepsilon B} \geq 1$, and $h_i^0 = \frac{1}{B_{\text{init}}} \sum_{k=1}^{B_{\text{init}}} \nabla f_i(x^0; \xi_{ik}^0)$ for all $i \in [n]$,
 240 and batch size $B_{\text{init}} = \Theta \left(\frac{\sqrt{p_a B}}{b} \right)$, then Algorithm 1 (DASHA-PP-MVR) needs

$$T := \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right] \right)$$

⁴If $\mathbb{1}_{p_a} = 0$, then $\frac{L_\sigma^2}{\frac{1}{p_a} \hat{L}^2} = +\infty$

$$+ \frac{\sigma}{p_a \sqrt{\varepsilon n}} \left(\frac{\mathbb{1}_{p_a} \hat{L}}{\sqrt{B}} + \frac{L_\sigma}{B} \right) \Big] + \frac{\sigma^2}{\sqrt{p_a n \varepsilon B}} \Big)$$

241 communication rounds to get an ε -solution and the number of stochastic gradient calculations per
242 node equals $\mathcal{O}(B_{\text{init}} + BT)$.

243 The convergence rate of the **DASHA-SYNC-MVR**, the state-of-the-art method without partial participa-
244 tion, equals $\mathcal{O}\left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega}{\sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma}{\sqrt{\varepsilon n}} \frac{L_\sigma}{B} \right] + \frac{\sigma^2}{n \varepsilon B} \right)$. Similar to Section 6.2, we see that in
245 the regimes when $\hat{L} \approx L_\sigma$ the third term w.r.t. $1/\varepsilon^{3/2}$ can degenerate up to \sqrt{B}/p_a . However, if we take
246 $B = \mathcal{O}(L_\sigma^2/\hat{L}^2)$, then the degeneration of the third term will be up to $1/p_a$. This effect we analyze in
247 Section 7. The fact that the degeneration is up to $1/p_a$ we check numerically in Section A.

248 In the following corollary, we consider **RandK** compressors (see Definition 5) and show that with
249 the particular choice of parameters, up to the Lipschitz constants factors, **DASHA-PP-MVR** gets the
250 optimal oracle complexity and SOTA communication complexity of **DASHA-SYNC-MVR** method.

251 **Corollary 4.** Suppose that assumptions of Corollary 3 hold, batch size $B \leq \min \left\{ \frac{\sigma}{p_a \sqrt{\varepsilon n}}, \frac{L_\sigma^2}{\mathbb{1}_{p_a}^2 \hat{L}^2} \right\}$,
252 we take **RandK** compressors with $K = \Theta\left(\frac{B d \sqrt{\varepsilon n}}{\sigma}\right)$. Then the communication complexity equals

$$\mathcal{O}\left(\frac{d\sigma}{\sqrt{p_a n \varepsilon}} + \frac{L_\sigma \Delta_0 d}{p_a \sqrt{n \varepsilon}}\right), \quad (12)$$

253 and the expected number of stochastic gradient calculations per node equals

$$\mathcal{O}\left(\frac{\sigma^2}{\sqrt{p_a n \varepsilon}} + \frac{L_\sigma \Delta_0 \sigma}{p_a \varepsilon^{3/2} n}\right). \quad (13)$$

254 We are aware that the initial batch size B_{init} can be suboptimal w.r.t. ω in **DASHA-PP-MVR** in some
255 regimes (see also (Tyurin and Richtárik, 2023)). This is a side effect of mixing the variance reduction
256 of stochastic gradients and compression. However, Corollary 4 reveals that we can escape these
257 regimes by choosing the parameter K of **RandK** compressors in a particular way. To get the complete
258 picture, we analyze the same phenomenon under PL condition (see Section E) and provide a new
259 method **DASHA-PP-SYNC-MVR** (see Section F).

260 7 The Problem of Estimating the Mean in the Partial Participation Setting

261 We now provide the example to explain why the only choice of $B = \mathcal{O}\left(\min \left\{ \frac{1}{p_a} \sqrt{\frac{m}{n}}, \frac{L_{\max}^2}{\mathbb{1}_{p_a}^2 \hat{L}^2} \right\}\right)$ and

262 $B = \mathcal{O}\left(\min \left\{ \frac{\sigma}{p_a \sqrt{\varepsilon n}}, \frac{L_\sigma^2}{\mathbb{1}_{p_a}^2 \hat{L}^2} \right\}\right)$ in **DASHA-PP-PAGE** and **DASHA-PP-MVR**, accordingly, guarantees
263 the degeneration up to $1/p_a$. This is surprising, because in methods with the variance reduction of
264 stochastic gradients (Li et al., 2021a; Tyurin and Richtárik, 2023) we can take the size of batch size
265 $B = \mathcal{O}\left(\sqrt{\frac{m}{n}}\right)$ and $B = \mathcal{O}\left(\frac{\sigma}{\sqrt{\varepsilon n}}\right)$ and guarantee the optimality. Note that the smaller the batch size
266 B , the more the server and the nodes have to communicate to get ε -solution.

267 Let us consider the task of estimating the mean of vectors in the distributed setting. Suppose that we
268 have n nodes, and each of them contains m vectors $\{x_{ij}\}_{j=1}^m$, where $x_{ij} \in \mathbb{R}^d$ for all $i \in [n], j \in [m]$.
269 First, let us consider that each node samples a mini-batch I^i of size B with replacement and sends it
270 to the server. Then the server calculates the mean of the mini-batches from nodes. One can easily
271 show that the variance of the estimator is

$$\begin{aligned} \mathbb{E} \left[\left\| \frac{1}{nB} \sum_{i=1}^n \sum_{j \in I^i} x_{ij} - \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m x_{ij} \right\|^2 \right] \\ = \frac{1}{nB} \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \left\| x_{ij} - \frac{1}{m} \sum_{j=1}^m x_{ij} \right\|^2. \end{aligned} \quad (14)$$

Next, we consider the same task in the partial participation setting with s -nice sampling, i.e., we sample a random set $S \subset [n]$ of $s \in [n]$ nodes without replacement and receive the mini-batches only from the sampled nodes. Such sampling of nodes satisfy Assumption 8 with $p_a = s/n$ and $p_a = s(s-1)/n(n-1)$. In this case, the variance of the estimator (See Lemma 1 with $r_i = 0$ and $s_i = \sum_{j \in I^i} x_{ij}$) is

$$\begin{aligned} & \mathbb{E} \left[\left\| \frac{1}{sB} \sum_{i \in S} \sum_{j \in I^i} x_{ij} - \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m x_{ij} \right\|^2 \right] \\ &= \frac{1}{sB} \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \underbrace{\left\| x_{ij} - \frac{1}{m} \sum_{j=1}^m x_{ij} \right\|^2}_{\mathcal{L}_{\max}^2} \\ & \quad + \frac{n-s}{s(n-1)} \frac{1}{n} \sum_{i=1}^n \underbrace{\left\| \frac{1}{m} \sum_{j=1}^m x_{ij} - \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m x_{ij} \right\|^2}_{\widehat{\mathcal{L}}^2}. \end{aligned} \tag{15}$$

Let us assume that $s \leq n/2$. Note that (14) scales with any $B \geq 1$, while (15) only scales when $B = \mathcal{O}(\mathcal{L}_{\max}^2/\widehat{\mathcal{L}}^2)$. In other words, for large enough B , the variance in (15) does not significantly improves with the growth of B due to the term $\widehat{\mathcal{L}}^2$. In our proof, due to partial participation, the variance from (15) naturally appears, and we get the same effect. As was mentioned in Sections 6.2 and 6.3, it can be seen in our convergence rate bounds.

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421 A Numerical Verification of Theoretical Dependencies

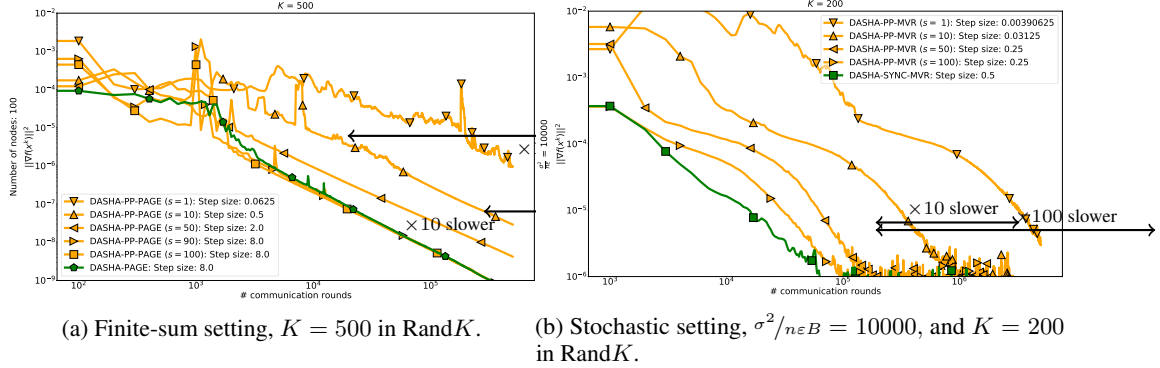


Figure 1: Classification task with the *real-sim* dataset.

422 Our main goal is to verify the dependeces from the theory. We compare **DASHA-PP** with **DASHA**.
 423 Clearly, **DASHA-PP** can not generally perform better than **DASHA**. In different settings, we verify
 424 that the bigger p_a , the closer **DASHA-PP** is to **DASHA**, i.e., **DASHA-PP** converges no slower than $1/p_a$
 425 times.

In all experiments, we take the *real-sim* dataset with dimension $d = 20,958$ and the number of samples equals 72,309 from LIBSVM datasets (Chang and Lin, 2011) (under the 3-clause BSD license), and randomly split the dataset between $n = 100$ nodes equally, ignoring residual samples. In the finite-sum setting, we solve a classification problem with functions

$$f_i(x) := \frac{1}{m} \sum_{j=1}^m \left(1 - \frac{1}{1 + \exp(y_{ij} a_{ij}^\top x)} \right)^2,$$

426 where $a_{ij} \in \mathbb{R}^d$ is the feature vector of a sample on the i^{th} node, $y_{ij} \in \{-1, 1\}$ is the corresponding
 427 label, and m is the number of samples on the i^{th} node for all $i \in [n]$. In the stochastic setting, we
 428 consider functions

$$f_i(x_1, x_2) := \mathbb{E}_{j \sim [m]} \left[-\log \left(\frac{\exp(a_{ij}^\top x_1 y_{ij})}{\sum_{y \in \{1, 2\}} \exp(a_{ij}^\top x_y y)} \right) + \lambda \sum_{y \in \{1, 2\}} \sum_{k=1}^d \frac{\{x_y\}_k^2}{1 + \{x_y\}_k^2} \right],$$

429 where $x_1, x_2 \in \mathbb{R}^d$, $\{\cdot\}_k$ is an indexing operation, $a_{ij} \in \mathbb{R}^d$ is a feature of a sample on the i^{th} node,
 430 $y_{ij} \in \{1, 2\}$ is a corresponding label, m is the number of samples located on the i^{th} node, constant
 431 $\lambda = 0.001$ for all $i \in [n]$.

432 The code was written in Python 3.6.8 using PyTorch 1.9 (Paszke et al., 2019). A distributed
 433 environment was emulated on a machine with Intel(R) Xeon(R) Gold 6226R CPU @ 2.90GHz and
 434 64 cores.

435 We use the standard setting in experiments⁵ where all parameters except step sizes are taken as
 436 suggested in theory. Step sizes are finetuned from a set $\{2^i \mid i \in [-10, 10]\}$. We emulate the partial
 437 participation setting using s -nice sampling with the number of nodes $n = 100$. We consider the
 438 $\text{Rand}K$ compressor and take the batch size $B = 1$. We plot the relation between communication
 439 rounds and values of the norm of gradients at each communication round.

440 In the finite-sum (Figure 1a) and in the stochastic setting (Figure 1b), we see that the bigger probability
 441 $p_a = s/n$ to 1, the closer **DASHA-PP** to **DASHA**. Moreover, **DASHA-PP** with $s = 10$ and $s = 1$
 442 converges approximately $\times 10$ ($= 1/p_a$) and $\times 100$ ($= 1/p_a$) times slower, accordingly. Our theory
 443 predicts such behavior.

⁵Code: <https://github.com/mysteryresearcher/dasha-partial-participation>

444 B Original DASHA and DASHA-MVR Methods

445 To simplify the discussion and explanation from the main part, we present the algorithms from (Tyurin
446 and Richtárik, 2023)

Algorithm 6 DASHA

```

1: Input: starting point  $x^0 \in \mathbb{R}^d$ , stepsize  $\gamma > 0$ , momentum  $a \in (0, 1]$ , number of iterations  $T \geq 1$ 
2: Initialize  $g_i^0 \in \mathbb{R}^d$  on the nodes and  $g^0 = \frac{1}{n} \sum_{i=1}^n g_i^0$  on the server
3: for  $t = 0, 1, \dots, T - 1$  do
4:    $x^{t+1} = x^t - \gamma g^t$ 
5:   Broadcast  $x^{t+1}, x^t$  to all participating(a) nodes
6:   for  $i = 1, \dots, n$  in parallel do
7:      $m_i^{t+1} = \mathcal{C}_i(\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - a(g_i^t - \nabla f_i(x^t)))$ 
8:      $g_i^{t+1} = g_i^t + m_i^{t+1}$ 
9:     Send  $m_i^{t+1}$  to the server
10:  end for
11:   $g^{t+1} = g^t + \frac{1}{n} \sum_{i=1}^n m_i^{t+1}$ 
12: end for
13: Output:  $\hat{x}^T$  chosen uniformly at random from  $\{x^t\}_{k=0}^{T-1}$ 

```

Algorithm 7 DASHA-MVR (with batch size $B = 1$)

```

1: Input: starting point  $x^0 \in \mathbb{R}^d$ , stepsize  $\gamma > 0$ , momentums  $a, b \in (0, 1]$ , number of iterations  $T \geq 1$ 
2: Initialize  $g_i^0 \in \mathbb{R}^d$  on the nodes and  $g^0 = \frac{1}{n} \sum_{i=1}^n g_i^0$  on the server
3: for  $t = 0, 1, \dots, T - 1$  do
4:    $x^{t+1} = x^t - \gamma g^t$ 
5:   Broadcast  $x^{t+1}, x^t$  to all participating(a) nodes
6:   for  $i = 1, \dots, n$  in parallel do
7:      $h_i^{t+1} = \nabla f_i(x^{t+1}; \xi_i^{t+1}) + (1 - b)(h_i^t - \nabla f_i(x^t; \xi_i^{t+1})), \quad \xi_i^{t+1} \sim \mathcal{D}_i$ 
8:      $m_i^{t+1} = \mathcal{C}_i(h_i^{t+1} - h_i^t - a(g_i^t - h_i^t))$ 
9:      $g_i^{t+1} = g_i^t + m_i^{t+1}$ 
10:    Send  $m_i^{t+1}$  to the server
11:  end for
12:   $g^{t+1} = g^t + \frac{1}{n} \sum_{i=1}^n m_i^{t+1}$ 
13: end for
14: Output:  $\hat{x}^T$  chosen uniformly at random from  $\{x^t\}_{k=0}^{T-1}$ 

```

447 C Auxiliary facts

448 We list auxiliary facts that we use in our proofs:

449 1. For all $x, y \in \mathbb{R}^d$, we have

$$\|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 \quad (16)$$

450 2. Let us take a *random vector* $\xi \in \mathbb{R}^d$, then

$$\mathbb{E} \left[\|\xi\|^2 \right] = \mathbb{E} \left[\|\xi - \mathbb{E}[\xi]\|^2 \right] + \|\mathbb{E}[\xi]\|^2. \quad (17)$$

451 C.1 Sampling Lemma

452 This section provides a lemma that we regularly use in our proofs, and it is useful for samplings that
453 satisfy Assumption 8.

454 **Lemma 1.** *Suppose that a set S is a random subset of a set $[n]$ such that*

455 1. $\text{Prob}(i \in S) = p_a, \quad \forall i \in [n],$

456 2. $\text{Prob}(i \in S, j \in S) = p_{aa}, \quad \forall i \neq j \in [n],$

457 3. $p_{aa} \leq p_a^2,$

458 where $p_a \in (0, 1]$ and $p_{aa} \in [0, 1]$. Let us take random independent vectors $s_i \in \mathbb{R}^d$ for all $i \in [n]$,
459 nonrandom vector $r_i \in \mathbb{R}^d$ for all $i \in [n]$, and random vectors

$$v_i = \begin{cases} r_i + \frac{1}{p_a} s_i, & i \in S, \\ r_i, & i \notin S, \end{cases}$$

460 then

$$\begin{aligned} & \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n v_i - \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n v_i \right] \right\|^2 \right] \\ &= \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E} \left[\|s_i - \mathbb{E}[s_i]\|^2 \right] + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\mathbb{E}[s_i]\|^2 + \frac{p_{aa} - p_a^2}{p_a^2} \left\| \frac{1}{n} \sum_{i=1}^n \mathbb{E}[s_i] \right\|^2 \\ &\leq \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E} \left[\|s_i - \mathbb{E}[s_i]\|^2 \right] + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\mathbb{E}[s_i]\|^2. \end{aligned}$$

461 *Proof.* Let us define additional constants p_{an} and p_{nn} , such that

462 1. $\text{Prob}(i \in S, j \notin S) = p_{an}, \quad \forall i \neq j \in [n],$

463 2. $\text{Prob}(i \notin S, j \notin S) = p_{nn}, \quad \forall i \neq j \in [n].$

464 Note, that

$$p_{an} = p_{aa} - p_a \quad (18)$$

465 and

$$p_{nn} = 1 - p_{aa} - 2p_{an}. \quad (19)$$

466 Using the law of total expectation and

$$\mathbb{E}[v_i] = p_a \left(r_i + \mathbb{E} \left[\frac{1}{p_a} s_i \right] \right) + (1 - p_a) r_i = r_i + \mathbb{E}[s_i],$$

467 we have

$$\begin{aligned}
& \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n v_i - \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n v_i \right] \right\|^2 \right] \\
&= \frac{1}{n^2} \sum_{i=1}^n \mathbb{E} \left[\|v_i - (r_i + \mathbb{E}[s_i])\|^2 \right] \\
&\quad + \frac{1}{n^2} \sum_{i \neq j}^n \mathbb{E} [\langle v_i - (r_i + \mathbb{E}[s_i]), v_j - (r_j + \mathbb{E}[s_j]) \rangle] \\
&= \frac{p_a}{n^2} \sum_{i=1}^n \mathbb{E} \left[\left\| r_i + \frac{1}{p_a} s_i - (r_i + \mathbb{E}[s_i]) \right\|^2 \right] \\
&\quad + \frac{1-p_a}{n^2} \sum_{i=1}^n \|r_i - (r_i + \mathbb{E}[s_i])\|^2 \\
&\quad + \frac{p_{aa}}{n^2} \sum_{i \neq j}^n \mathbb{E} \left[\left\langle r_i + \frac{1}{p_a} s_i - (r_i + \mathbb{E}[s_i]), r_j + \frac{1}{p_a} s_j - (r_j + \mathbb{E}[s_j]) \right\rangle \right] \\
&\quad + \frac{2p_{an}}{n^2} \sum_{i \neq j}^n \mathbb{E} \left[\left\langle r_i + \frac{1}{p_a} s_i - (r_i + \mathbb{E}[s_i]), r_j - (r_j + \mathbb{E}[s_j]) \right\rangle \right] \\
&\quad + \frac{p_{nn}}{n^2} \sum_{i \neq j}^n \langle r_i - (r_i + \mathbb{E}[s_i]), r_j - (r_j + \mathbb{E}[s_j]) \rangle.
\end{aligned}$$

468 From the independence of random vectors s_i , we obtain

$$\begin{aligned}
& \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n v_i - \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n v_i \right] \right\|^2 \right] \\
&= \frac{p_a}{n^2} \sum_{i=1}^n \mathbb{E} \left[\left\| \frac{1}{p_a} s_i - \mathbb{E}[s_i] \right\|^2 \right] \\
&\quad + \frac{1-p_a}{n^2} \sum_{i=1}^n \|\mathbb{E}[s_i]\|^2 \\
&\quad + \frac{p_{aa}(1-p_a)^2}{n^2 p_a^2} \sum_{i \neq j}^n \langle \mathbb{E}[s_i], \mathbb{E}[s_j] \rangle \\
&\quad + \frac{2p_{an}(p_a-1)}{n^2 p_a} \sum_{i \neq j}^n \langle \mathbb{E}[s_i], \mathbb{E}[s_j] \rangle \\
&\quad + \frac{p_{nn}}{n^2} \sum_{i \neq j}^n \langle \mathbb{E}[s_i], \mathbb{E}[s_j] \rangle.
\end{aligned}$$

469 Using (18) and (19), we have

$$\begin{aligned}
& \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n v_i - \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n v_i \right] \right\|^2 \right] \\
&= \frac{p_a}{n^2} \sum_{i=1}^n \mathbb{E} \left[\left\| \frac{1}{p_a} s_i - \mathbb{E}[s_i] \right\|^2 \right] \\
&\quad + \frac{1-p_a}{n^2} \sum_{i=1}^n \|\mathbb{E}[s_i]\|^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{p_{aa} - p_a^2}{n^2 p_a^2} \sum_{i \neq j}^n \langle E[s_i], E[s_j] \rangle \\
& \stackrel{(17)}{=} \frac{1}{n^2 p_a} \sum_{i=1}^n E \left[\|s_i - E[s_i]\|^2 \right] \\
& + \frac{1 - p_a}{n^2 p_a} \sum_{i=1}^n \|E[s_i]\|^2 \\
& + \frac{p_{aa} - p_a^2}{n^2 p_a^2} \sum_{i \neq j}^n \langle E[s_i], E[s_j] \rangle \\
& = \frac{1}{n^2 p_a} \sum_{i=1}^n E \left[\|s_i - E[s_i]\|^2 \right] \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|E[s_i]\|^2 \\
& + \frac{p_{aa} - p_a^2}{p_a^2} \left\| \frac{1}{n} \sum_{i=1}^n E[s_i] \right\|^2.
\end{aligned}$$

470 Finally, using that $p_{aa} \leq p_a^2$, we have

$$\begin{aligned}
& E \left[\left\| \frac{1}{n} \sum_{i=1}^n v_i - E \left[\frac{1}{n} \sum_{i=1}^n v_i \right] \right\|^2 \right] \\
& \leq \frac{1}{n^2 p_a} \sum_{i=1}^n E \left[\|s_i - E[s_i]\|^2 \right] + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|E[s_i]\|^2.
\end{aligned}$$

471

□

472 C.2 Compressors Facts

473 We define the *RandK* compressor that chooses without replacement K coordinates, scales them by a
474 constant factor to preserve unbiasedness and zero-out other coordinates.

Definition 5. Let us take a random subset S from $[d]$, $|S| = K$, $K \in [d]$. We say that a stochastic mapping $\mathcal{C} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is *RandK* if

$$\mathcal{C}(x) = \frac{d}{K} \sum_{j \in S} x_j e_j,$$

475 where $\{e_i\}_{i=1}^d$ is the standard unit basis.

476 **Theorem 6.** If \mathcal{C} is *RandK*, then $\mathcal{C} \in \mathbb{U} \left(\frac{d}{K} - 1 \right)$.

477 See the proof in (Beznosikov et al., 2020).

478 D Proofs of Theorems

479 There are three different sources of randomness in Algorithm 1: the first one from vectors $\{k_i^{t+1}\}_{i=1}^n$,
480 the second one from compressors $\{\mathcal{C}_i\}_{i=1}^n$, and the third one from availability of nodes. We define
481 $E_k[\cdot]$, $E_C[\cdot]$ and $E_{p_a}[\cdot]$ to be conditional expectations w.r.t. $\{k_i^{t+1}\}_{i=1}^n$, $\{\mathcal{C}_i\}_{i=1}^n$, and availability,
482 accordingly, conditioned on all previous randomness. Moreover, we define $E_{t+1}[\cdot]$ to be a conditional
483 expectation w.r.t. all randomness in iteration $t+1$ conditioned on all previous randomness. Note,
484 that $E_{t+1}[\cdot] = E_k[E_C[E_{p_a}[\cdot]]]$.

485 In the case of **DASHA-PP-PAGE**, there are two different sources of randomness from $\{k_i^{t+1}\}_{i=1}^n$.
486 We define $E_{p_{\text{page}}}[\cdot]$ and $E_B[\cdot]$ to be conditional expectations w.r.t. the probabilistic switching and
487 mini-batch indices I_i^t , accordingly, conditioned on all previous randomness. Note, that $E_{t+1}[\cdot] =$
488 $E_B[E_C[E_{p_a}[E_{p_{\text{page}}}[\cdot]]]]$ and $E_{t+1}[\cdot] = E_B[E_{p_{\text{page}}}[E_C[E_{p_a}[\cdot]]]]$.

489 D.1 Standard Lemmas in the Nonconvex Setting

490 We start the proof of theorems by providing standard lemmas from the nonconvex optimization.

491 **Lemma 2.** Suppose that Assumption 2 holds and let $x^{t+1} = x^t - \gamma g^t$. Then for any $g^t \in \mathbb{R}^d$ and
 492 $\gamma > 0$, we have

$$f(x^{t+1}) \leq f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2}\right) \|x^{t+1} - x^t\|^2 + \frac{\gamma}{2} \|g^t - \nabla f(x^t)\|^2. \quad (20)$$

493 *Proof.* Using L -smoothness, we have

$$\begin{aligned} f(x^{t+1}) &\leq f(x^t) + \langle \nabla f(x^t), x^{t+1} - x^t \rangle + \frac{L}{2} \|x^{t+1} - x^t\|^2 \\ &= f(x^t) - \gamma \langle \nabla f(x^t), g^t \rangle + \frac{L}{2} \|x^{t+1} - x^t\|^2. \end{aligned}$$

494 Next, due to $-\langle x, y \rangle = \frac{1}{2} \|x - y\|^2 - \frac{1}{2} \|x\|^2 - \frac{1}{2} \|y\|^2$, we obtain

$$f(x^{t+1}) \leq f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2}\right) \|x^{t+1} - x^t\|^2 + \frac{\gamma}{2} \|g^t - \nabla f(x^t)\|^2.$$

495

□

496 **Lemma 3.** Suppose that Assumption 1 holds and

$$\mathbb{E} [f(x^{t+1})] + \gamma \Psi^{t+1} \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] + \gamma \Psi^t + \gamma C, \quad (21)$$

497 where Ψ^t is a sequence of numbers, $\Psi^t \geq 0$ for all $t \in [T]$, constant $C \geq 0$, and constant $\gamma > 0$.
 498 Then

$$\mathbb{E} [\|\nabla f(\hat{x}^T)\|^2] \leq \frac{2\Delta_0}{\gamma T} + \frac{2\Psi^0}{T} + 2C, \quad (22)$$

499 where a point \hat{x}^T is chosen uniformly from a set of points $\{x^t\}_{t=0}^{T-1}$.

500 *Proof.* By unrolling (21) for t from 0 to $T - 1$, we obtain

$$\frac{\gamma}{2} \sum_{t=0}^{T-1} \mathbb{E} [\|\nabla f(x^t)\|^2] + \mathbb{E} [f(x^T)] + \gamma \Psi^T \leq f(x^0) + \gamma \Psi^0 + \gamma TC.$$

501 We subtract f^* , divide inequality by $\frac{\gamma T}{2}$, and take into account that $f(x) \geq f^*$ for all $x \in \mathbb{R}$, and
 502 $\Psi^t \geq 0$ for all $t \in [T]$, to get the following inequality:

$$\frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} [\|\nabla f(x^t)\|^2] \leq \frac{2\Delta_0}{\gamma T} + \frac{2\Psi^0}{T} + 2C.$$

503 It is left to consider the choice of a point \hat{x}^T to complete the proof of the lemma. □

Lemma 4. If $0 < \gamma \leq (L + \sqrt{A})^{-1}$, $L > 0$, and $A \geq 0$, then

$$\frac{1}{2\gamma} - \frac{L}{2} - \frac{\gamma A}{2} \geq 0.$$

504 The lemma can be easily checked with the direct calculation.

505 D.2 Generic Lemmas

506 **Lemma 5.** Suppose that Assumptions 7 and 8 hold and let us consider sequences g_i^{t+1} , h_i^{t+1} , and
 507 k_i^{t+1} from Algorithm 1, then

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \\ & \leq \frac{2\omega}{n^2 p_a} \sum_{i=1}^n \|k_i^{t+1}\|^2 + \frac{a^2((2\omega + 1)p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2, \end{aligned} \quad (23)$$

508 and

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g_i^{t+1} - h_i^{t+1}\|^2 \right] \right] \\ & \leq \frac{2\omega}{p_a} \|k_i^{t+1}\|^2 + \left(\frac{a^2(2\omega + 1 - p_a)}{p_a} + (1-a)^2 \right) \|g_i^t - h_i^t\|^2 \quad \forall i \in [n]. \end{aligned} \quad (24)$$

509 *Proof.* First, we estimate $\mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right]$:

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \\ & = \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g^{t+1} - h^{t+1} - \mathbb{E}_{p_a} [g^{t+1} - h^{t+1}]\|^2 \right] \right] + \|\mathbb{E}_{\mathcal{C}} [\mathbb{E}_{p_a} [g^{t+1} - h^{t+1}]]\|^2, \end{aligned}$$

510 where we used (17). Due to Assumption 8, we have

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} [\mathbb{E}_{p_a} [g_i^{t+1}]] \\ & = p_a \mathbb{E}_{\mathcal{C}} \left[g_i^t + \mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) \right] + (1-p_a) g_i^t \\ & = g_i^t + p_a \mathbb{E}_{\mathcal{C}} \left[\mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) \right] \\ & = g_i^t + k_i^{t+1} - a (g_i^t - h_i^t), \end{aligned}$$

511 and

$$\mathbb{E}_{\mathcal{C}} [\mathbb{E}_{p_a} [h_i^{t+1}]] = p_a \mathbb{E}_{\mathcal{C}} \left[h_i^t + \frac{1}{p_a} k_i^{t+1} \right] + (1-p_a) h_i^t = h_i^t + k_i^{t+1}.$$

512 Thus, we can get

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \\ & = \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g^{t+1} - h^{t+1} - \mathbb{E}_{p_a} [g^{t+1} - h^{t+1}]\|^2 \right] \right] + (1-a)^2 \|g^t - h^t\|^2. \end{aligned}$$

513 Due to the independence of compressors, we can use Lemma 1 with $r_i = g_i^t - h_i^t$ and $s_i =$

514 $p_a \mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - k_i^{t+1}$, and obtain

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \\ & \leq \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_{\mathcal{C}} \left[\left\| p_a \mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - k_i^{t+1} - \mathbb{E}_{\mathcal{C}} \left[p_a \mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - k_i^{t+1} \right] \right\|^2 \right] \\ & \quad + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \mathbb{E}_{\mathcal{C}} \left[p_a \mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - k_i^{t+1} \right] \right\|^2 \end{aligned}$$

$$\begin{aligned}
& + (1-a)^2 \|g^t - h^t\|^2 \\
& = \frac{p_a}{n^2} \sum_{i=1}^n \mathbb{E}_{\mathcal{C}} \left[\left\| \mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) \right\|^2 \right] \\
& \quad + \frac{a^2 (p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2.
\end{aligned}$$

515 From Assumption 7, we have

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \\
& \leq \frac{\omega p_a}{n^2} \sum_{i=1}^n \left\| \frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right\|^2 + \frac{a^2 (p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2 \\
& = \frac{\omega}{n^2 p_a} \sum_{i=1}^n \|k_i^{t+1} - a (g_i^t - h_i^t)\|^2 + \frac{a^2 (p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2 \\
& \stackrel{(16)}{\leq} \frac{2\omega}{n^2 p_a} \sum_{i=1}^n \|k_i^{t+1}\|^2 + \frac{a^2 ((2\omega + 1)p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2.
\end{aligned}$$

516 The second inequality can be proved almost in the same way:

$$\begin{aligned}
& \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g_i^{t+1} - h_i^{t+1}\|^2 \right] \right] \\
& = \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g_i^{t+1} - h_i^{t+1} - \mathbb{E}_{\mathcal{C}} [\mathbb{E}_{p_a} [g_i^{t+1} - h_i^{t+1}]]\|^2 \right] \right] + \|\mathbb{E}_{\mathcal{C}} [\mathbb{E}_{p_a} [g_i^{t+1} - h_i^{t+1}]]\|^2 \\
& = \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g_i^{t+1} - h_i^{t+1} - g_i^t + a (g_i^t - h_i^t) + h_i^t\|^2 \right] \right] + (1-a)^2 \|g_i^t - h_i^t\|^2 \\
& = p_a \mathbb{E}_{\mathcal{C}} \left[\left\| \mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - \frac{1}{p_a} k_i^{t+1} + a (g_i^t - h_i^t) \right\|^2 \right] \\
& \quad + a^2 (1-p_a) \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g_i^t - h_i^t\|^2 \\
& \stackrel{(17)}{=} p_a \mathbb{E}_{\mathcal{C}} \left[\left\| \mathcal{C}_i \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - \left(\frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) \right\|^2 \right] \\
& \quad + a^2 \frac{(1-p_a)^2}{p_a} \|g_i^t - h_i^t\|^2 \\
& \quad + a^2 (1-p_a) \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g_i^t - h_i^t\|^2 \\
& \leq \frac{\omega}{p_a} \|k_i^{t+1} - a (g_i^t - h_i^t)\|^2 \\
& \quad + \frac{a^2 (1-p_a)}{p_a} \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g_i^t - h_i^t\|^2 \\
& \stackrel{(16)}{\leq} \frac{2\omega}{p_a} \|k_i^{t+1}\|^2 + \frac{a^2 (2\omega + 1 - p_a)}{p_a} \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g_i^t - h_i^t\|^2.
\end{aligned}$$

517

□

518 **Lemma 6.** Suppose that Assumptions 2, 7, and 8 hold and let us take $a = \frac{p_a}{2\omega+1}$, then

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{n p_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& \quad + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{n p_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] + \frac{4\gamma\omega(2\omega+1)}{n p_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right].
\end{aligned}$$

519 *Proof.* Due to Lemma 2 and the update step from Line 4 in Algorithm 1, we have

$$\begin{aligned}
& \mathbb{E}_{t+1} [f(x^{t+1})] \\
& \leq \mathbb{E}_{t+1} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \frac{\gamma}{2} \|g^t - \nabla f(x^t)\|^2 \right] \\
& = \mathbb{E}_{t+1} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \frac{\gamma}{2} \|g^t - h^t + h^t - \nabla f(x^t)\|^2 \right] \\
& \stackrel{(17)}{\leq} \mathbb{E}_{t+1} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \left(\|g^t - h^t\|^2 + \|h^t - \nabla f(x^t)\|^2 \right) \right].
\end{aligned}$$

520 Let us fix some constants $\kappa, \eta \in [0, \infty)$ that we will define later. Combining the last inequality,
521 bounds (23), (24) and using the law of total expectation, we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] \\
& + \kappa \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& = \mathbb{E} [\mathbb{E}_{t+1} [f(x^{t+1})]] \\
& + \kappa \mathbb{E} [\mathbb{E}_C [\mathbb{E}_{p_a} [\|g^{t+1} - h^{t+1}\|^2]]] + \eta \mathbb{E} \left[\mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \right] \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \left(\|g^t - h^t\|^2 + \|h^t - \nabla f(x^t)\|^2 \right) \right] \\
& + \kappa \mathbb{E} \left[\frac{2\omega}{n^2 p_a} \sum_{i=1}^n \|k_i^{t+1}\|^2 + \frac{a^2((2\omega+1)p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2 \right] \\
& + \eta \mathbb{E} \left[\frac{2\omega}{n p_a} \sum_{i=1}^n \|k_i^{t+1}\|^2 + \left(\frac{a^2(2\omega+1-p_a)}{p_a} + (1-a)^2 \right) \frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& = \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \left(\gamma + \kappa(1-a)^2 \right) \mathbb{E} [\|g^t - h^t\|^2] \\
& + \left(\frac{\kappa a^2((2\omega+1)p_a - p_{aa})}{n p_a^2} + \eta \left(\frac{a^2(2\omega+1-p_a)}{p_a} + (1-a)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(\frac{2\kappa\omega}{n p_a} + \frac{2\eta\omega}{p_a} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right].
\end{aligned}$$

522 Now, by taking $\kappa = \frac{\gamma}{a}$, we can see that $\gamma + \kappa(1-a)^2 \leq \kappa$, and thus

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] \\
& + \frac{\gamma}{a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma}{a} \mathbb{E} [\|g^t - h^t\|^2] \\
& + \left(\frac{\gamma a((2\omega+1)p_a - p_{aa})}{n p_a^2} + \eta \left(\frac{a^2(2\omega+1-p_a)}{p_a} + (1-a)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(\frac{2\gamma\omega}{a n p_a} + \frac{2\eta\omega}{p_a} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right].
\end{aligned}$$

523 Next, by taking $\eta = \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2}$ and considering the choice of a , one can show that
 524 $\left(\frac{\gamma a((2\omega+1)p_a - p_{aa})}{np_a^2} + \eta \left(\frac{a^2(2\omega+1-p_a)}{p_a} + (1-a)^2 \right) \right) \leq \eta$. Thus

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] \\ & + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\ & + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ & + \left(\frac{2\gamma(2\omega+1)\omega}{np_a^2} + \frac{2\gamma((2\omega+1)p_a - p_{aa})\omega}{np_a^3} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right]. \end{aligned}$$

525 Considering that $p_{aa} \geq 0$, we can simplify the last term and get

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] \\ & + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\ & + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ & + \frac{4\gamma(2\omega+1)\omega}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right]. \end{aligned}$$

526 □

527 **D.3 Proof for DASHA-PP**

528 **Lemma 7.** Suppose that Assumptions 3 and 8 hold. For h_i^{t+1} and k_i^{t+1} from Algorithm 1 (DASHA-PP)
 529 we have

1.

$$\begin{aligned} & \mathbb{E}_{p_a} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] \\ & \leq \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \|x^{t+1} - x^t\|^2 + \frac{2b^2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + (1-b)^2 \|h^t - \nabla f(x^t)\|^2. \end{aligned}$$

2.

$$\begin{aligned} & \mathbb{E}_{p_a} [\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2] \\ & \leq \frac{2(1-p_a)}{p_a} L_i^2 \|x^{t+1} - x^t\|^2 + \left(\frac{2b^2(1-p_a)}{p_a} + (1-b)^2 \right) \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n]. \end{aligned}$$

3.

$$\|k_i^{t+1}\|^2 \leq 2L_i^2 \|x^{t+1} - x^t\|^2 + 2b^2 \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n].$$

530 *Proof.* First, let us proof the bound for $\mathbb{E}_k [\mathbb{E}_{p_a} [\|h^{t+1} - \nabla f(x^{t+1})\|^2]]$:

$$\mathbb{E}_{p_a} [\|h^{t+1} - \nabla f(x^{t+1})\|^2]$$

$$= \mathbb{E}_{p_a} \left[\left\| h^{t+1} - \mathbb{E}_{p_a} [h^{t+1}] \right\|^2 \right] + \left\| \mathbb{E}_{p_a} [h^{t+1}] - \nabla f(x^{t+1}) \right\|^2.$$

531 Using

$$\mathbb{E}_{p_a} [h_i^{t+1}] = h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))$$

532 and (17), we have

$$\begin{aligned} & \mathbb{E}_{p_a} \left[\left\| h^{t+1} - \nabla f(x^{t+1}) \right\|^2 \right] \\ &= \mathbb{E}_{p_a} \left[\left\| h^{t+1} - \mathbb{E}_{p_a} [h^{t+1}] \right\|^2 \right] + (1-b)^2 \left\| h^t - \nabla f(x^t) \right\|^2. \end{aligned}$$

533 We can use Lemma 1 with $r_i = h_i^t$ and $s_i = k_i^{t+1}$ to obtain

$$\begin{aligned} & \mathbb{E}_{p_a} \left[\left\| h^{t+1} - \nabla f(x^{t+1}) \right\|^2 \right] \\ & \leq \frac{1}{n^2 p_a} \sum_{i=1}^n \left\| k_i^{t+1} - k_i^{t+1} \right\|^2 + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| k_i^{t+1} \right\|^2 + (1-b)^2 \left\| h^t - \nabla f(x^t) \right\|^2 \\ &= \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)) \right\|^2 + (1-b)^2 \left\| h^t - \nabla f(x^t) \right\|^2 \\ & \stackrel{(16)}{\leq} \frac{2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 + \frac{2b^2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \left\| h_i^t - \nabla f_i(x^t) \right\|^2 + (1-b)^2 \left\| h^t - \nabla f(x^t) \right\|^2 \\ & \leq \frac{2(p_a - p_{aa}) \hat{L}^2}{n p_a^2} \left\| x^{t+1} - x^t \right\|^2 + \frac{2b^2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \left\| h_i^t - \nabla f_i(x^t) \right\|^2 + (1-b)^2 \left\| h^t - \nabla f(x^t) \right\|^2. \end{aligned}$$

534 In the last in inequality, we used Assumption 3. Now, we prove the second inequality:

$$\begin{aligned} & \mathbb{E}_{p_a} \left[\left\| h_i^{t+1} - \nabla f_i(x^{t+1}) \right\|^2 \right] \\ &= \mathbb{E}_{p_a} \left[\left\| h_i^{t+1} - \mathbb{E}_{p_a} [h_i^{t+1}] \right\|^2 \right] + \left\| \mathbb{E}_{p_a} [h_i^{t+1}] - \nabla f_i(x^{t+1}) \right\|^2 \\ &= \mathbb{E}_{p_a} \left[\left\| h_i^{t+1} - (h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))) \right\|^2 \right] + (1-b)^2 \left\| h_i^t - \nabla f_i(x^t) \right\|^2 \\ &= \frac{(1-p_a)^2}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)) \right\|^2 \\ & \quad + (1-p_a) \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)) \right\|^2 + (1-b)^2 \left\| h_i^t - \nabla f_i(x^t) \right\|^2 \\ &= \frac{(1-p_a)}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)) \right\|^2 + (1-b)^2 \left\| h_i^t - \nabla f_i(x^t) \right\|^2 \\ & \leq \frac{2(1-p_a)}{p_a} L_i^2 \left\| x^{t+1} - x^t \right\|^2 + \left(\frac{2b^2(1-p_a)}{p_a} + (1-b)^2 \right) \left\| h_i^t - \nabla f_i(x^t) \right\|^2. \end{aligned}$$

535 Finally, the third inequality of the theorem follows from (16) and Assumption 3. \square

536 **Theorem 2.** Suppose that Assumptions 1, 2, 3, 7 and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{p_a}{2-p_a}$,

$$\gamma \leq \left(L + \left[\frac{48\omega(2\omega+1)}{n p_a^2} + \frac{16}{n p_a^2} \left(1 - \frac{p_{aa}}{p_a} \right) \right]^{1/2} \hat{L} \right)^{-1},$$

537 and $g_i^0 = h_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP), then $\mathbb{E} \left[\left\| \nabla f(\hat{x}^T) \right\|^2 \right] \leq \frac{2\Delta_0}{\gamma T}$.

538 *Proof.* Let us fix constants $\nu, \rho \in [0, \infty)$ that we will define later. Considering Lemma 6, Lemma 7,
539 and the law of total expectation, we obtain

$$\mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\left\| g^{t+1} - h^{t+1} \right\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{n p_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \left\| g_i^{t+1} - h_i^{t+1} \right\|^2 \right]$$

$$\begin{aligned}
& + \nu \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
= & \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} \left[\mathbb{E}_{p_a} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] \right] + \rho \mathbb{E} \left[\mathbb{E}_{p_a} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \\
\leq & \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{4\gamma\omega(2\omega + 1)}{np_a^2} \mathbb{E} \left[2\hat{L}^2 \|x^{t+1} - x^t\|^2 + 2b^2 \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \nu \mathbb{E} \left[\frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \|x^{t+1} - x^t\|^2 + \frac{2b^2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \rho \mathbb{E} \left[\frac{2(1-p_a)\hat{L}^2}{p_a} \|x^{t+1} - x^t\|^2 + \left(\frac{2b^2(1-p_a)}{p_a} + (1-b)^2 \right) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

540 After rearranging the terms, we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
\leq & \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{8\gamma\omega(2\omega + 1)\hat{L}^2}{np_a^2} - \nu \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} - \rho \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + (\gamma + \nu(1-b)^2) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b^2\gamma\omega(2\omega + 1)}{np_a^2} + \nu \frac{2b^2(p_a - p_{aa})}{np_a^2} + \rho \left(\frac{2b^2(1-p_a)}{p_a} + (1-b)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

541 By taking $\nu = \frac{\gamma}{b}$, one can show that $(\gamma + \nu(1-b)^2) \leq \nu$, and

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
\leq & \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{8\gamma\omega(2\omega+1)\widehat{L}^2}{np_a^2} - \frac{2\gamma(p_a - p_{aa})\widehat{L}^2}{bnp_a^2} - \rho \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \mathbb{E} \left[\|x^{t+1} - x^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} \left[\|h^t - \nabla f(x^t)\|^2 \right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma b(p_a - p_{aa})}{np_a^2} + \rho \left(\frac{2b^2(1-p_a)}{p_a} + (1-b)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

542 Note that $b = \frac{p_a}{2-p_a}$, thus

$$\begin{aligned}
& \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma b(p_a - p_{aa})}{np_a^2} + \rho \left(\frac{2b^2(1-p_a)}{p_a} + (1-b)^2 \right) \right) \\
& \leq \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma b(p_a - p_{aa})}{np_a^2} + \rho(1-b) \right).
\end{aligned}$$

543 And if we take $\rho = \frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2}$, then

$$\left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma b(p_a - p_{aa})}{np_a^2} + \rho(1-b) \right) \leq \rho,$$

544 and

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^{t+1} - h^{t+1}\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} \left[\|\nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{8\gamma\omega(2\omega+1)\widehat{L}^2}{np_a^2} - \frac{2\gamma(p_a - p_{aa})\widehat{L}^2}{bnp_a^2} \right. \\
& \quad \left. - \frac{16b\gamma\omega(2\omega+1)(1-p_a)\widehat{L}^2}{np_a^3} - \frac{4\gamma(p_a - p_{aa})(1-p_a)\widehat{L}^2}{np_a^3} \right) \mathbb{E} \left[\|x^{t+1} - x^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} \left[\|h^t - \nabla f(x^t)\|^2 \right] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

545 Let us simplify the last inequality. First, note that

$$\frac{16b\gamma\omega(2\omega+1)(1-p_a)\widehat{L}^2}{np_a^3} \leq \frac{16\gamma\omega(2\omega+1)\widehat{L}^2}{np_a^2},$$

546 due to $b \leq p_a$. Second,

$$\frac{2\gamma(p_a - p_{aa})\widehat{L}^2}{bnp_a^2} \leq \frac{4\gamma(p_a - p_{aa})\widehat{L}^2}{np_a^3},$$

547 due to $b \geq \frac{p_a}{2}$. All in all, we have

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^{t+1} - h^{t+1}\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
&\quad + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{24\gamma\omega(2\omega+1)\hat{L}^2}{np_a^2} - \frac{8\gamma(p_a - p_{aa})\hat{L}^2}{np_a^3}\right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
&\quad + \frac{\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right].
\end{aligned}$$

548 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned}
&\mathbb{E}[f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
&\quad + \frac{\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
&\leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
&\quad + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad + \frac{\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right].
\end{aligned}$$

549 It is left to apply Lemma 3 with

$$\begin{aligned}
\Psi^t &= \frac{(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad + \frac{1}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\omega(2\omega+1)}{np_a^2} + \frac{2(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right]
\end{aligned}$$

550 to conclude the proof. \square

551 **D.4 Proof for DASHA-PP-PAGE**

552 Let us denote

$$\begin{aligned}
k_{i,1}^{t+1} &:= \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)), \\
k_{i,2}^{t+1} &:= \frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)), \\
h_{i,1}^{t+1} &:= \begin{cases} h_i^t + \frac{1}{p_a} k_{i,1}^{t+1}, & i^{\text{th}} \text{ node is participating,} \\ h_i^t, & \text{otherwise,} \end{cases} \\
h_{i,2}^{t+1} &:= \begin{cases} h_i^t + \frac{1}{p_a} k_{i,2}^{t+1}, & i^{\text{th}} \text{ node is participating,} \\ h_i^t, & \text{otherwise,} \end{cases}
\end{aligned}$$

553 $h_1^{t+1} := \frac{1}{n} \sum_{i=1}^n h_{i,1}^{t+1}$, and $h_2^{t+1} := \frac{1}{n} \sum_{i=1}^n h_{i,2}^{t+1}$. Note, that

$$h^{t+1} = \begin{cases} h_1^{t+1}, & \text{with probability } p_{\text{page}}, \\ h_2^{t+1}, & \text{with probability } 1 - p_{\text{page}}. \end{cases}$$

554 **Lemma 8.** Suppose that Assumptions 3, 4, and 8 hold. For h_i^{t+1} and k_i^{t+1} from Algorithm 1
 555 (DASHA-PP-PAGE) we have

1.

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right] \\ & \leq \left(\frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} + \frac{(1 - p_{\text{page}})L_{\max}^2}{np_a B} \right) \|x^{t+1} - x^t\|^2 \\ & \quad + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{page}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1 - p_{\text{page}}) \right) \|h^t - \nabla f(x^t)\|^2. \end{aligned}$$

2.

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \\ & \leq \left(\frac{2(1 - p_a)L_i^2}{p_a} + \frac{(1 - p_{\text{page}})L_{\max}^2}{p_a B} \right) \|x^{t+1} - x^t\|^2 \\ & \quad + \left(\frac{2(1 - p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1 - p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n]. \end{aligned}$$

3.

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_{\text{page}}} \left[\|k_i^{t+1}\|^2 \right] \right] \\ & \leq \left(2L_i^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \|x^{t+1} - x^t\|^2 + \frac{2b^2}{p_{\text{page}}} \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n]. \end{aligned}$$

556 *Proof.* First, we prove the first inequality of the theorem:

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right] \\ & = p_{\text{page}} \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h_1^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] + (1 - p_{\text{page}}) \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h_2^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right]. \end{aligned}$$

557 Using

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right] = \\ & = p_a h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) + (1 - p_a) h_i^t \\ & = h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)). \end{aligned}$$

558 and

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,2}^{t+1} \right] \right] = \\ & = p_a h_i^t + \mathbb{E}_B \left[\frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) \right] + (1 - p_a) h_i^t \\ & = h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t), \end{aligned}$$

559 we obtain

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right] \\ & \stackrel{(17)}{=} p_{\text{page}} \mathbb{E}_{p_a} \left[\|h_1^{t+1} - \mathbb{E}_{p_a} [h_1^{t+1}]\|^2 \right] + (1 - p_{\text{page}}) \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h_2^{t+1} - \mathbb{E}_{p_a} [h_2^{t+1}]\|^2 \right] \right] \end{aligned}$$

$$\begin{aligned}
& + p_{\text{page}} \left\| \mathbb{E}_{p_a} [h_1^{t+1}] - \nabla f(x^{t+1}) \right\|^2 + (1 - p_{\text{page}}) \left\| \mathbb{E}_B [\mathbb{E}_{p_a} [h_2^{t+1}]] - \nabla f(x^{t+1}) \right\|^2 \\
& = p_{\text{page}} \mathbb{E}_{p_a} \left[\left\| h_1^{t+1} - \mathbb{E}_{p_a} [h_1^{t+1}] \right\|^2 \right] + (1 - p_{\text{page}}) \mathbb{E}_B \left[\left\| h_2^{t+1} - \mathbb{E}_{p_a} [h_2^{t+1}] \right\|^2 \right] \\
& + \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1 - p_{\text{page}}) \right) \left\| h^t - \nabla f(x^t) \right\|^2. \tag{25}
\end{aligned}$$

560 Next, we consider $\mathbb{E}_{p_a} \left[\left\| h_1^{t+1} - \mathbb{E}_{p_a} [h_1^{t+1}] \right\|^2 \right]$. We can use Lemma 1 with $r_i = h_i^t$ and $s_i = k_{i,1}^{t+1}$
561 to obtain

$$\begin{aligned}
& \mathbb{E}_{p_a} \left[\left\| h_1^{t+1} - \mathbb{E}_{p_a} [h_1^{t+1}] \right\|^2 \right] \\
& \leq \frac{1}{n^2 p_a} \sum_{i=1}^n \left\| k_{i,1}^{t+1} - k_{i,1}^{t+1} \right\|^2 + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| k_{i,1}^{t+1} \right\|^2 \\
& = \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) \right\|^2 \\
& \stackrel{(16)}{\leq} \frac{2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{page}}^2} \sum_{i=1}^n \left\| h_i^t - \nabla f_i(x^t) \right\|^2.
\end{aligned}$$

562 From Assumption 3, we have

$$\begin{aligned}
& \mathbb{E}_{p_a} \left[\left\| h_1^{t+1} - \mathbb{E}_{p_a} [h_1^{t+1}] \right\|^2 \right] \\
& \leq \frac{2(p_a - p_{aa})\hat{L}^2}{n p_a^2} \left\| x^{t+1} - x^t \right\|^2 + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{page}}^2} \sum_{i=1}^n \left\| h_i^t - \nabla f_i(x^t) \right\|^2. \tag{26}
\end{aligned}$$

563 Now, we prove the bound for $\mathbb{E}_B \left[\left\| h_2^{t+1} - \mathbb{E}_B [\mathbb{E}_{p_a} [h_2^{t+1}]] \right\|^2 \right]$. Considering that mini-
564 batches in the algorithm are independent, we can use Lemma 1 with $r_i = h_i^t$ and $s_i = k_{i,2}^{t+1}$
565 to obtain

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_2^{t+1} - \mathbb{E}_{p_a} [h_2^{t+1}] \right\|^2 \right] \right] \\
& \leq \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_B \left[\left\| k_{i,2}^{t+1} - \mathbb{E}_B [k_{i,2}^{t+1}] \right\|^2 \right] + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \mathbb{E}_B [k_{i,2}^{t+1}] \right\|^2 \\
& = \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_B \left[\left\| \frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 \\
& = \frac{1}{n^2 p_a B^2} \sum_{i=1}^n \mathbb{E}_B \left[\sum_{j \in I_i^t} \left\| (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 \\
& = \frac{1}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \left\| (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 \\
& \leq \frac{1}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \left\| \nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) \right\|^2 + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2.
\end{aligned}$$

566 Next, we use Assumptions 3 and 4 to get

$$\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_2^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_2^{t+1} \right] \right] \right\|^2 \right] \right] \leq \left(\frac{L_{\max}^2}{np_a B} + \frac{(p_a - p_{aa}) \hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2. \quad (27)$$

567 Applying (26) and (27) into (25), we get

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\left\| h^{t+1} - \nabla f(x^{t+1}) \right\|^2 \right] \right] \right] \\ & \leq p_{\text{page}} \left(\frac{2(p_a - p_{aa}) \hat{L}^2}{np_a^2} \|x^{t+1} - x^t\|^2 + \frac{2(p_a - p_{aa}) b^2}{n^2 p_a^2 p_{\text{page}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right) + \\ & \quad + (1 - p_{\text{page}}) \left(\frac{L_{\max}^2}{np_a B} + \frac{(p_a - p_{aa}) \hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \\ & \quad + \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1 - p_{\text{page}}) \right) \|h^t - \nabla f(x^t)\|^2 \\ & \leq \left(\frac{2(p_a - p_{aa}) \hat{L}^2}{np_a^2} + \frac{(1 - p_{\text{page}}) L_{\max}^2}{np_a B} \right) \|x^{t+1} - x^t\|^2 \\ & \quad + \frac{2(p_a - p_{aa}) b^2}{n^2 p_a^2 p_{\text{page}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1 - p_{\text{page}}) \right) \|h^t - \nabla f(x^t)\|^2. \end{aligned}$$

568 The proof of the second inequality almost repeats the previous one:

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\left\| h_i^{t+1} - \nabla f_i(x^{t+1}) \right\|^2 \right] \right] \right] \\ & = p_{\text{page}} \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,1}^{t+1} - \nabla f_i(x^{t+1}) \right\|^2 \right] \right] + (1 - p_{\text{page}}) \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,2}^{t+1} - \nabla f_i(x^{t+1}) \right\|^2 \right] \right] \\ & \stackrel{(17)}{=} p_{\text{page}} \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,1}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right] \right\|^2 \right] \right] + (1 - p_{\text{page}}) \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,2}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,2}^{t+1} \right] \right] \right\|^2 \right] \right] \\ & \quad + p_{\text{page}} \left\| \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right] - \nabla f_i(x^{t+1}) \right\|^2 + (1 - p_{\text{page}}) \left\| \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,2}^{t+1} \right] \right] - \nabla f_i(x^{t+1}) \right\|^2 \\ & = p_{\text{page}} \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,1}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right] \right\|^2 \right] \right] + (1 - p_{\text{page}}) \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,2}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,2}^{t+1} \right] \right] \right\|^2 \right] \right] \\ & \quad + \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1 - p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2. \quad (28) \end{aligned}$$

569 Let us consider $\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,1}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right] \right\|^2 \right] \right]$:

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,1}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right] \right\|^2 \right] \right] \\ & = \mathbb{E}_{p_a} \left[\left\| h_{i,1}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right] \right\|^2 \right] \\ & = p_a \left\| h_i^t + \frac{1}{p_a} k_{i,1}^{t+1} - \left(h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) \right) \right\|^2 \\ & \quad + (1 - p_a) \left\| h_i^t - \left(h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) \right) \right\|^2 \\ & = \frac{(1 - p_a)^2}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) \right\|^2 \\ & \quad + (1 - p_a) \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) \right\|^2 \\ & = \frac{1 - p_a}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) \right\|^2. \end{aligned}$$

570 Considering (16) and Assumption 3, we obtain

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,1}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right] \right\|^2 \right] \right] \\ & \leq \frac{2(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2 + \frac{2(1-p_a)b^2}{p_a p_{\text{page}}^2} \|h_i^t - \nabla f_i(x^t)\|^2. \end{aligned} \quad (29)$$

571 Next, we obtain the bound for $\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,2}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,2}^{t+1} \right] \right] \right\|^2 \right] \right]$:

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{i,2}^{t+1} - \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[h_{i,2}^{t+1} \right] \right] \right\|^2 \right] \right] \\ & = p_a \mathbb{E}_B \left[\left\| h_i^t + \frac{1}{p_a} k_{i,2}^{t+1} - (h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\ & \quad + (1-p_a) \mathbb{E}_B \left[\left\| h_i^t - (h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\ & = p_a \mathbb{E}_B \left[\left\| \frac{1}{p_a} k_{i,2}^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\ & \quad + (1-p_a) \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 \\ & \stackrel{(17)}{=} \frac{1}{p_a} \mathbb{E}_B \left[\left\| k_{i,2}^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] + \frac{(1-p_a)^2}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 \\ & \quad + (1-p_a) \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 \\ & = \frac{1}{p_a} \mathbb{E}_B \left[\left\| k_{i,2}^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] + \frac{1-p_a}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 \\ & \leq \frac{1}{p_a} \mathbb{E}_B \left[\left\| k_{i,2}^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] + \frac{(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2, \end{aligned} \quad (30)$$

572 where we used Assumption 3. By plugging (29) and (30) into (28), we get

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\left\| h_i^{t+1} - \nabla f_i(x^{t+1}) \right\|^2 \right] \right] \right] \\ & \leq p_{\text{page}} \left(\frac{2(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2 + \frac{2(1-p_a)b^2}{p_a p_{\text{page}}^2} \|h_i^t - \nabla f_i(x^t)\|^2 \right) \\ & \quad + (1-p_{\text{page}}) \left(\frac{1}{p_a} \mathbb{E}_B \left[\left\| k_{i,2}^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] + \frac{(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2 \right) \\ & \quad + \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \\ & \leq \frac{2(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2 + \frac{1-p_{\text{page}}}{p_a} \mathbb{E}_B \left[\left\| k_{i,2}^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\ & \quad + \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2. \end{aligned}$$

573 From the independence of elements in the mini-batch, we obtain

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\left\| h_i^{t+1} - \nabla f_i(x^{t+1}) \right\|^2 \right] \right] \right] \\ & \leq \frac{2(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2 + \frac{1-p_{\text{page}}}{p_a} \mathbb{E}_B \left[\left\| \frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\ & \quad + \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{2(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2 + \frac{1-p_{\text{page}}}{p_a B^2} \mathbb{E}_B \left[\sum_{j \in I_i^t} \|(\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
&\quad + \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}}\right)^2 + (1-p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \\
&= \frac{2(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2 + \frac{1-p_{\text{page}}}{m p_a B} \sum_{j=1}^m \|(\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \\
&\quad + \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}}\right)^2 + (1-p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \\
&\leq \frac{2(1-p_a)L_i^2}{p_a} \|x^{t+1} - x^t\|^2 + \frac{1-p_{\text{page}}}{m p_a B} \sum_{j=1}^m \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)\|^2 \\
&\quad + \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}}\right)^2 + (1-p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \\
&\leq \left(\frac{2(1-p_a)L_i^2}{p_a} + \frac{(1-p_{\text{page}})L_{\max}^2}{p_a B} \right) \|x^{t+1} - x^t\|^2 \\
&\quad + \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}}\right)^2 + (1-p_{\text{page}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2,
\end{aligned}$$

574 where we used Assumption 4. Finally, we prove the last inequality:

$$\begin{aligned}
&\mathbb{E}_B \left[\mathbb{E}_{p_{\text{page}}} \left[\|k_i^{t+1}\|^2 \right] \right] \\
&= p_{\text{page}} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) \right\|^2 \\
&\quad + (1-p_{\text{page}}) \mathbb{E}_B \left[\left\| \frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) \right\|^2 \right] \\
&\stackrel{(17)}{=} p_{\text{page}} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{page}}} (h_i^t - \nabla f_i(x^t)) \right\|^2 \\
&\quad + (1-p_{\text{page}}) \mathbb{E}_B \left[\left\| \frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\
&\quad + (1-p_{\text{page}}) \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\
&\stackrel{(16)}{\leq} 2p_{\text{page}} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \frac{2b^2}{p_{\text{page}}} \|h_i^t - \nabla f_i(x^t)\|^2 \\
&\quad + (1-p_{\text{page}}) \mathbb{E}_B \left[\left\| \frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right] \\
&\quad + (1-p_{\text{page}}) \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\
&\leq 2 \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \frac{2b^2}{p_{\text{page}}} \|h_i^t - \nabla f_i(x^t)\|^2
\end{aligned}$$

$$+ (1 - p_{\text{page}}) \mathbb{E}_B \left[\left\| \frac{1}{B} \sum_{j \in I_i^t} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\|^2 \right].$$

575 Using the independence of elements in the mini-batch, we have

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_{\text{page}}} \left[\|k_i^{t+1}\|^2 \right] \right] \\ & \leq 2 \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \frac{2b^2}{p_{\text{page}}} \|h_i^t - \nabla f_i(x^t)\|^2 \\ & \quad + \frac{1 - p_{\text{page}}}{B^2} \mathbb{E}_B \left[\sum_{j \in I_i^t} \|(\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\ & = 2 \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \frac{2b^2}{p_{\text{page}}} \|h_i^t - \nabla f_i(x^t)\|^2 \\ & \quad + \frac{1 - p_{\text{page}}}{Bm} \sum_{j=1}^m \|(\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \\ & \leq 2 \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \frac{2b^2}{p_{\text{page}}} \|h_i^t - \nabla f_i(x^t)\|^2 \\ & \quad + \frac{1 - p_{\text{page}}}{Bm} \sum_{j=1}^m \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)\|^2 \end{aligned}$$

576 It is left to consider Assumptions 3 and 4 to get

$$\begin{aligned} & \mathbb{E}_B \left[\mathbb{E}_{p_{\text{page}}} \left[\|k_i^{t+1}\|^2 \right] \right] \\ & \leq \left(2L_i^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \|x^{t+1} - x^t\|^2 + \frac{2b^2}{p_{\text{page}}} \|h_i^t - \nabla f_i(x^t)\|^2. \end{aligned}$$

577

□

578 **Theorem 3.** Suppose that Assumptions 1, 2, 3, 4, 7, and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{p_{\text{page}}p_a}{2-p_a}$,
579 probability $p_{\text{page}} \in (0, 1]$,

$$\begin{aligned} \gamma \leq & \left(L + \left[\frac{48\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \right. \right. \\ & \left. \left. + \frac{16}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \right]^{1/2} \right)^{-1} \end{aligned}$$

580 and $g_i^0 = h_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP-PAGE) then $\mathbb{E} \left[\|\nabla f(\hat{x}^T)\|^2 \right] \leq$
581 $\frac{2\Delta_0}{\gamma^T}$.

582 *Proof.* Let us fix constants $\nu, \rho \in [0, \infty)$ that we will define later. Considering Lemma 6, Lemma 8,
583 and the law of total expectation, we obtain

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & \quad + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& = \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\mathbb{E}_B \left[\mathbb{E}_{p_{\text{page}}} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right] \right] \right] \\
& + \nu \mathbb{E} \left[\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right] \right] \\
& + \rho \mathbb{E} \left[\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{page}}} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\left(2\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \|x^{t+1} - x^t\|^2 + \frac{2b^2}{p_{\text{page}}} \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \nu \mathbb{E} \left(\left(\frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} + \frac{(1-p_{\text{page}})L_{\max}^2}{np_a B} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{page}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \|h^t - \nabla f(x^t)\|^2 \right) \\
& + \rho \mathbb{E} \left(\left(\frac{2(1-p_a)\hat{L}^2}{p_a} + \frac{(1-p_{\text{page}})L_{\max}^2}{p_a B} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right)
\end{aligned}$$

584 After rearranging the terms, we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^{t+1} - h^{t+1}\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} \left[\|\nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(2\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \right. \\
& \quad \left. - \nu \left(\frac{2(p_a - p_{\text{aa}})\hat{L}^2}{np_a^2} + \frac{(1-p_{\text{page}})L_{\max}^2}{np_a B} \right) - \rho \left(\frac{2(1-p_a)\hat{L}^2}{p_a} + \frac{(1-p_{\text{page}})L_{\max}^2}{p_a B} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(\gamma + \nu \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\nu(p_a - p_{\text{aa}})b^2}{np_a^2 p_{\text{page}}} \right. \\
& \quad \left. + \rho \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

Due to $b = \frac{p_{\text{page}} p_a}{2-p_a} \leq p_{\text{page}}$, one can show that $\left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \leq 1-b$. Thus, if we take $\nu = \frac{\gamma}{b}$, then

$$\left(\gamma + \nu \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \leq \gamma + \nu(1-b) = \nu,$$

585 therefore

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{\text{aa}})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{\text{aa}})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(2\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \right. \\
& \quad \left. - \frac{\gamma}{b} \left(\frac{2(p_a - p_{\text{aa}})\hat{L}^2}{np_a^2} + \frac{(1-p_{\text{page}})L_{\max}^2}{np_a B} \right) - \rho \left(\frac{2(1-p_a)\hat{L}^2}{p_a} + \frac{(1-p_{\text{page}})L_{\max}^2}{p_a B} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\gamma(p_a - p_{\text{aa}})b}{np_a^2 p_{\text{page}}} \right. \\
& \quad \left. + \rho \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

Next, with the choice of $b = \frac{p_{\text{page}} p_a}{2-p_a}$, we ensure that

$$\left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \leq 1-b.$$

If we take $\rho = \frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\gamma(p_a - p_{\text{aa}})}{np_a^2 p_{\text{page}}}$, then

$$\left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\gamma(p_a - p_{\text{aa}})b}{np_a^2 p_{\text{page}}} + \rho \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \leq \rho,$$

586 therefore

$$\begin{aligned}
& \mathbb{E}[f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
& \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(2\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right)\right) \\
& - \frac{\gamma}{bnp_a} \left(2\left(1 - \frac{p_{aa}}{p_a}\right)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right) \\
& - \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^3 p_{\text{page}}} + \frac{2\gamma\left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2 p_{\text{page}}}\right) \left(2(1-p_a)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^3 p_{\text{page}}} + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right].
\end{aligned}$$

Let us simplify the inequality. First, due to $b \geq \frac{p_{\text{page}} p_a}{2}$, we have

$$\frac{\gamma}{bnp_a} \left(2\left(1 - \frac{p_{aa}}{p_a}\right)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right) \leq \frac{4\gamma}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a}\right)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right).$$

587 Second, due to $b \leq p_a p_{\text{page}}$ and $p_{aa} \leq p_a^2$, we get

$$\begin{aligned}
& \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^3 p_{\text{page}}} + \frac{2\gamma\left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2 p_{\text{page}}}\right) \left(2(1-p_a)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right) \\
& \leq \left(\frac{8\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma\left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2 p_{\text{page}}}\right) \left(2\left(1 - \frac{p_{aa}}{p_a}\right)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right) \\
& \leq \frac{16\gamma\omega(2\omega+1)}{np_a^2} \left(\left(1 - \frac{p_{aa}}{p_a}\right)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right) \\
& + \frac{4\gamma\left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a}\right)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right) \\
& \leq \frac{16\gamma\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right) \\
& + \frac{4\gamma}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a}\right)\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B}\right).
\end{aligned}$$

588 Combining all bounds together, we obtain the following simplified inequality:

$$\begin{aligned}
& \mathbb{E}[f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
& \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{24\gamma\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \right. \\
& \quad \left. - \frac{8\gamma}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a}\right) \hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

589 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

590 It is left to apply Lemma 3 with

$$\begin{aligned}
\Psi^t &= \frac{(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{1}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right]
\end{aligned}$$

591 to conclude the proof. \square

592 **Corollary 1.** Let the assumptions from Theorem 3 hold and $p_{\text{page}} = B/(m+B)$. Then DASHA-PP-PAGE
593 needs

$$\begin{aligned}
T &:= \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) \right. \right. \\
& \quad \left. \left. + \frac{1}{p_a} \sqrt{\frac{m}{n}} \left(\frac{\mathbb{1}_{p_a} \hat{L}}{\sqrt{B}} + \frac{L_{\max}}{B} \right) \right] \right)
\end{aligned} \tag{9}$$

594 communication rounds to get an ε -solution and the expected number of gradient calculations per
595 node equals $\mathcal{O}(m + BT)$.

596 *Proof.* In the view of Theorem 3, it is enough to do

$$T := \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \sqrt{\frac{\omega^2}{np_a^2} \left(\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right)} + \frac{1}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a}\right) \hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \right] \right)$$

597 steps to get ε -solution. Using the choice of p_{mega} and the definition of $\mathbb{1}_{p_a}$, we can get (9).

598 Note that the expected number of gradients calculations at each communication round equals $p_{\text{mega}}m +$
599 $(1 - p_{\text{mega}})B = \frac{2mB}{m+B} \leq 2B$. \square

600 **Corollary 2.** Suppose that assumptions of Corollary 1 hold, $B \leq \min \left\{ \frac{1}{p_a} \sqrt{\frac{m}{n}}, \frac{L_{\max}^2}{\frac{1}{p_a} \hat{L}^2} \right\}$ ⁶, and we
 601 use the unbiased compressor RandK with $K = \Theta(Bd/\sqrt{m})$. Then the communication complexity of
 602 Algorithm 1 is

$$\mathcal{O} \left(d + \frac{L_{\max} \Delta_0 d}{p_a \varepsilon \sqrt{n}} \right), \quad (10)$$

603 and the expected number of gradient calculations per node equals

$$\mathcal{O} \left(m + \frac{L_{\max} \Delta_0 \sqrt{m}}{p_a \varepsilon \sqrt{n}} \right). \quad (11)$$

604 *Proof.* The communication complexity equals

$$\mathcal{O}(d + KT) = \mathcal{O} \left(d + \frac{\Delta_0}{\varepsilon} \left[KL + K \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + K \frac{1}{p_a} \sqrt{\frac{m}{n}} \left(\frac{\mathbb{1}_{p_a} \hat{L}}{\sqrt{B}} + \frac{L_{\max}}{B} \right) \right] \right).$$

605 Since $B \leq \frac{L_{\max}^2}{\frac{1}{p_a} \hat{L}^2}$, we have $\frac{\mathbb{1}_{p_a} \hat{L}}{\sqrt{B}} + \frac{L_{\max}}{B} \leq \frac{2L_{\max}}{B}$ and

$$\mathcal{O}(d + KT) = \mathcal{O} \left(d + \frac{\Delta_0}{\varepsilon} \left[KL + K \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + K \frac{1}{p_a} \sqrt{\frac{m}{n}} \frac{L_{\max}}{B} \right] \right).$$

606 Note that $K = \Theta \left(\frac{Bd}{\sqrt{m}} \right) = \mathcal{O} \left(\frac{d}{p_a \sqrt{n}} \right)$ and $\omega + 1 = \frac{d}{K}$ due to Theorem 6, thus

$$\begin{aligned} \mathcal{O}(d + KT) &= \mathcal{O} \left(d + \frac{\Delta_0}{\varepsilon} \left[\frac{d}{p_a \sqrt{n}} L + \frac{d}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + \frac{d}{p_a \sqrt{n}} L_{\max} \right] \right) \\ &= \mathcal{O} \left(d + \frac{L_{\max} \Delta_0 d}{p_a \varepsilon \sqrt{n}} \right). \end{aligned}$$

607 Using the same reasoning, the expected number of gradient calculations per node equals

$$\begin{aligned} \mathcal{O}(m + BT) &= \mathcal{O} \left(m + \frac{\Delta_0}{\varepsilon} \left[BL + B \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + B \frac{1}{p_a} \sqrt{\frac{m}{n}} \left(\frac{\mathbb{1}_{p_a} \hat{L}}{\sqrt{B}} + \frac{L_{\max}}{B} \right) \right] \right) \\ &= \mathcal{O} \left(m + \frac{\Delta_0}{\varepsilon} \left[BL + B \frac{d}{K p_a \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + B \frac{1}{p_a} \sqrt{\frac{m}{n}} \frac{L_{\max}}{B} \right] \right) \\ &= \mathcal{O} \left(m + \frac{\Delta_0}{\varepsilon} \left[\frac{1}{p_a} \sqrt{\frac{m}{n}} L + \frac{\sqrt{m}}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + \frac{1}{p_a} \sqrt{\frac{m}{n}} L_{\max} \right] \right) \\ &= \mathcal{O} \left(m + \frac{L_{\max} \Delta_0 \sqrt{m}}{p_a \varepsilon \sqrt{n}} \right). \end{aligned}$$

608 □

609 D.5 Proof for DASHA-PP-FINITE-MVR

610 **Lemma 9.** Suppose that Assumptions 3, 4, and 8 hold. For h_i^{t+1} , h_{ij}^{t+1} and k_i^{t+1} from Algorithm 1
 611 (DASHA-PP-FINITE-MVR) we have

1.

$$\begin{aligned} &\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\ &\leq \left(\frac{2L_{\max}^2}{np_a B} + \frac{2(p_a - p_{aa}) \hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \end{aligned}$$

⁶If $\mathbb{1}_{p_a} = 0$, then $\frac{L_{\max}^2}{\frac{1}{p_a} \hat{L}^2} = +\infty$

$$\begin{aligned}
& + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \frac{2b^2}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

2.

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \\
& \leq \left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)L_i^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \\
& + \frac{2b^2}{p_a B m} \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 + \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n].
\end{aligned}$$

3.

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{2 \left(1 - \frac{p_a B}{m} \right) L_{\max}^2}{\frac{p_a B}{m}} \|x^{t+1} - x^t\|^2 \\
& + \left(\frac{2 \left(1 - \frac{p_a B}{m} \right) b^2}{\frac{p_a B}{m}} + (1-b)^2 \right) \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2, \quad \forall i \in [n], \forall j \in [m].
\end{aligned}$$

4.

$$\begin{aligned}
& \mathbb{E}_B \left[\|k_i^{t+1}\|^2 \right] \\
& \leq \left(\frac{2L_{\max}^2}{B} + 2L_i^2 \right) \|x^{t+1} - x^t\|^2 \\
& + \frac{2b^2}{B m} \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 + 2b^2 \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n].
\end{aligned}$$

612 *Proof.* We start by proving the first inequality. Note that

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} [h_i^{t+1}] \right] \\
& = p_a \left(h_i^t + \frac{1}{p_a} \mathbb{E}_B [k_i^{t+1}] \right) + (1-p_a)h_i^t \\
& = h_i^t + \frac{1}{m} \sum_{j=1}^m \frac{B}{m} \cdot \frac{m}{B} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))) + \left(1 - \frac{B}{m} \right) \cdot 0 \\
& = \nabla f_i(x^{t+1}) + (1-b)(h_i^t - \nabla f_i(x^t)),
\end{aligned}$$

613 thus

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& \stackrel{(17)}{=} \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \mathbb{E}_B [\mathbb{E}_{p_a} [h^{t+1}]]\|^2 \right] \right] + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

614 We can use Lemma 1 with $r_i = h_i^t$ and $s_i = k_i^{t+1}$ to obtain

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_B \left[\|k_i^{t+1} - \mathbb{E}_B [k_i^{t+1}]\|^2 \right] + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\mathbb{E}_B [k_i^{t+1}]\|^2
\end{aligned}$$

$$\begin{aligned}
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \\
& = \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_B \left[\left\| \frac{1}{m} \sum_{j=1}^m k_{ij}^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))) \right\|^2 \right] \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

615 Next, we again use Lemma 1 with $r_i = 0$, $s_i = \nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))$,

616 $p_a = \frac{B}{m}$, and $p_{aa} = \frac{B(B-1)}{m(m-1)}$:

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{1}{n^2 p_a} \sum_{i=1}^n \left(\frac{m-B}{Bm(m-1)} \sum_{j=1}^m \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))\|^2 \right) \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \\
& \leq \frac{1}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))\|^2 \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \\
& \stackrel{(16)}{\leq} \frac{2}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)\|^2 + \frac{2b^2}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \\
& + \frac{2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

617 Due to Assumptions 3 and 4, we have

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& \leq \left(\frac{2L_{\max}^2}{np_a B} + \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \\
& + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \frac{2b^2}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

618 Let us get the bound for the second inequality:

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \\
& \stackrel{(17)}{=} \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - (\nabla f_i(x^{t+1}) + (1-b)(h_i^t - \nabla f_i(x^t)))\|^2 \right] \right] \\
& + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2
\end{aligned}$$

$$\begin{aligned}
&= p_a \mathbb{E}_B \left[\left\| h_i^t + \frac{1}{p_a} k_i^{t+1} - (\nabla f_i(x^{t+1}) + (1-b)(h_i^t - \nabla f_i(x^t))) \right\|^2 \right] \\
&\quad + (1-p_a) \left\| h_i^t - (\nabla f_i(x^{t+1}) + (1-b)(h_i^t - \nabla f_i(x^t))) \right\|^2 \\
&\quad + (1-b)^2 \left\| h_i^t - \nabla f_i(x^t) \right\|^2 \\
&\stackrel{(17)}{=} \frac{1}{p_a} \mathbb{E}_B \left[\left\| k_i^{t+1} - \mathbb{E}_B[k_i^{t+1}] \right\|^2 \right] \\
&\quad + \frac{1-p_a}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)) \right\|^2 \\
&\quad + (1-b)^2 \left\| h_i^t - \nabla f_i(x^t) \right\|^2.
\end{aligned}$$

619 Let us use Lemma 1 with $r_i = 0$, $s_i = \nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))$, $p_a = \frac{B}{m}$, and
620 $p_{aa} = \frac{B(B-1)}{m(m-1)}$:

$$\begin{aligned}
&\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_i^{t+1} - \nabla f_i(x^{t+1}) \right\|^2 \right] \right] \\
&\leq \frac{1}{p_a} \left(\frac{m-B}{Bm(m-1)} \sum_{j=1}^m \left\| \nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t)) \right\|^2 \right) \\
&\quad + \frac{1-p_a}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)) \right\|^2 \\
&\quad + (1-b)^2 \left\| h_i^t - \nabla f_i(x^t) \right\|^2 \\
&\leq \frac{1}{p_a B m} \sum_{j=1}^m \left\| \nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t)) \right\|^2 \\
&\quad + \frac{1-p_a}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)) \right\|^2 \\
&\quad + (1-b)^2 \left\| h_i^t - \nabla f_i(x^t) \right\|^2 \\
&\stackrel{(16)}{\leq} \frac{2}{p_a B m} \sum_{j=1}^m \left\| \nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) \right\|^2 + \frac{2(1-p_a)}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) \right\|^2 \\
&\quad + \frac{2b^2}{p_a B m} \sum_{j=1}^m \left\| h_{ij}^t - \nabla f_{ij}(x^t) \right\|^2 + \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \left\| h_i^t - \nabla f_i(x^t) \right\|^2 \\
&\leq \left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)L_i^2}{p_a} \right) \left\| x^{t+1} - x^t \right\|^2 \\
&\quad + \frac{2b^2}{p_a B m} \sum_{j=1}^m \left\| h_{ij}^t - \nabla f_{ij}(x^t) \right\|^2 + \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \left\| h_i^t - \nabla f_i(x^t) \right\|^2,
\end{aligned}$$

621 where we used Assumptions 3 and 4. We continue the proof by considering

622 $\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1}) \right\|^2 \right] \right]$:

$$\begin{aligned}
&\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1}) \right\|^2 \right] \right] \\
&\stackrel{(17)}{=} \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\left\| h_{ij}^{t+1} - (\nabla f_{ij}(x^{t+1}) + (1-b)(h_{ij}^t - \nabla f_{ij}(x^t))) \right\|^2 \right] \right] \\
&\quad + (1-b)^2 \left\| h_{ij}^t - \nabla f_{ij}(x^t) \right\|^2 \\
&= \frac{p_a B}{m} \mathbb{E}_B \left[\left\| h_{ij}^t + \frac{m}{B p_a} (\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))) - (\nabla f_{ij}(x^{t+1}) + (1-b)(h_{ij}^t - \nabla f_{ij}(x^t))) \right\|^2 \right] \\
&\quad + \left(1 - \frac{p_a B}{m} \right) \left\| h_{ij}^t - (\nabla f_{ij}(x^{t+1}) + (1-b)(h_{ij}^t - \nabla f_{ij}(x^t))) \right\|^2
\end{aligned}$$

$$\begin{aligned}
& + (1-b)^2 \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \\
& = \frac{\left(1 - \frac{p_a B}{m}\right)^2}{\frac{p_a B}{m}} \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))\|^2 \\
& \quad + \left(1 - \frac{p_a B}{m}\right) \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))\|^2 \\
& \quad + (1-b)^2 \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \\
& = \frac{\left(1 - \frac{p_a B}{m}\right)}{\frac{p_a B}{m}} \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))\|^2 \\
& \quad + (1-b)^2 \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \\
& \stackrel{(16)}{\leq} \frac{2\left(1 - \frac{p_a B}{m}\right)}{\frac{p_a B}{m}} \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)\|^2 + \left(\frac{2\left(1 - \frac{p_a B}{m}\right)b^2}{\frac{p_a B}{m}} + (1-b)^2\right) \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2.
\end{aligned}$$

623 It is left to consider Assumption 4:

$$\begin{aligned}
& \mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{2\left(1 - \frac{p_a B}{m}\right) L_{\max}^2}{\frac{p_a B}{m}} \|x^{t+1} - x^t\|^2 + \left(\frac{2\left(1 - \frac{p_a B}{m}\right)b^2}{\frac{p_a B}{m}} + (1-b)^2\right) \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2.
\end{aligned}$$

624 Finally, we obtain the bound for the last inequality of the lemma:

$$\begin{aligned}
& \mathbb{E}_B \left[\|k_i^{t+1}\|^2 \right] \\
& \stackrel{(17)}{=} \mathbb{E}_B \left[\|k_i^{t+1} - \mathbb{E}_B[k_i^{t+1}]\|^2 \right] \\
& \quad + \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2.
\end{aligned}$$

625 Using Lemma 1, we get

$$\begin{aligned}
& \mathbb{E}_B \left[\|k_i^{t+1}\|^2 \right] \\
& \leq \frac{m-B}{Bm(m-1)} \sum_{j=1}^m \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))\|^2 \\
& \quad + \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& \leq \frac{1}{Bm} \sum_{j=1}^m \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t) - b(h_{ij}^t - \nabla f_{ij}(x^t))\|^2 \\
& \quad + \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& \stackrel{(16)}{\leq} \frac{2}{Bm} \sum_{j=1}^m \|\nabla f_{ij}(x^{t+1}) - \nabla f_{ij}(x^t)\|^2 + 2 \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\
& \quad + \frac{2b^2}{Bm} \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 + 2b^2 \|h_i^t - \nabla f_i(x^t)\|^2 \\
& \leq \left(\frac{2L_{\max}^2}{B} + 2L_i^2 \right) \|x^{t+1} - x^t\|^2 \\
& \quad + \frac{2b^2}{Bm} \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 + 2b^2 \|h_i^t - \nabla f_i(x^t)\|^2,
\end{aligned}$$

626 where we used Assumptions 3 and 4. □

Theorem 7. Suppose that Assumptions 1, 2, 3, 4, 7, and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{\frac{p_a B}{m}}{2 - \frac{p_a B}{m}}$,

$$\gamma \leq \left(L + \sqrt{\frac{148\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{L_{\max}^2}{B} \right) + \frac{72m}{np_a^2 B} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{L_{\max}^2}{B} \right)} \right)^{-1},$$

627 $g_i^0 = h_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ and $h_{ij}^0 = \nabla f_{ij}(x^0)$ for all $i \in [n], j \in [m]$ in Algorithm 1
 628 (DASHA-PP-FINITE-MVR) then $\mathbb{E} [\|\nabla f(\hat{x}^T)\|^2] \leq \frac{2\Delta_0}{\gamma T}$.

629 *Proof.* Let us fix constants $\nu, \rho, \delta \in [0, \infty)$ that we will define later. Considering Lemma 6, Lemma 9,
 630 and the law of total expectation, we obtain

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & + \delta \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\ & + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ & + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right] \\ & + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & + \delta \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\ & = \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\ & + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ & + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\mathbb{E}_B \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right] \right] \\ & + \nu \mathbb{E} \left[\mathbb{E}_B \left[\mathbb{E}_{p_a} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] \right] \right] \\ & + \rho \mathbb{E} \left[\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \\ & + \delta \mathbb{E} \left[\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \right] \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \|x^{t+1} - x^t\|^2 + \frac{2b^2}{Bmn} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 + \frac{2b^2}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \nu \mathbb{E} \left(\left(\frac{2L_{\max}^2}{np_a B} + \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \frac{2b^2}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right. \\
& \quad \left. + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \right) \\
& + \rho \mathbb{E} \left(\left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \frac{2b^2}{p_a B n m} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 + \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right) \\
& + \delta \mathbb{E} \left(\frac{2 \left(1 - \frac{p_a B}{m}\right) L_{\max}^2}{\frac{p_a B}{m}} \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \left(\frac{2 \left(1 - \frac{p_a B}{m}\right) b^2}{\frac{p_a B}{m}} + (1-b)^2 \right) \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right).
\end{aligned}$$

Due to $b = \frac{\frac{p_a B}{m}}{2 - \frac{p_a B}{m}} \leq \frac{p_a}{2 - p_a}$, we have

$$\left(\frac{2 \left(1 - \frac{p_a B}{m}\right) b^2}{\frac{p_a B}{m}} + (1-b)^2 \right) \leq 1 - b$$

and

$$\left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \leq 1 - b.$$

631 Moreover, we consider that $1 - \frac{p_a B}{m} \leq 1$, therefore

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \delta \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \|x^{t+1} - x^t\|^2 + \frac{2b^2}{Bmn} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 + \frac{2b^2}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \nu \mathbb{E} \left(\left(\frac{2L_{\max}^2}{np_a B} + \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \frac{2b^2}{n^2 p_a B m} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \\
& \quad \left. + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \right) \\
& + \rho \mathbb{E} \left(\left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad + \frac{2b^2}{p_a B n m} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 + (1-b) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \left. \right) \\
& + \delta \mathbb{E} \left(\frac{2mL_{\max}^2}{p_a B} \|x^{t+1} - x^t\|^2 + (1-b) \frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right).
\end{aligned}$$

632 After rearranging the terms, we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \delta \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \right. \\
& \quad \left. - \nu \left(\frac{2L_{\max}^2}{np_a B} + \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \right) - \rho \left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) - \delta \frac{2mL_{\max}^2}{p_a B} \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + (\gamma + \nu(1-b)^2) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\nu(p_a - p_{aa})b^2}{np_a^2} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{2\nu b^2}{np_a B} + \frac{2\rho b^2}{p_a B} + \delta(1-b) \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right].
\end{aligned}$$

633 Thus, if we take $\nu = \frac{\gamma}{b}$, then $\gamma + \nu(1-b)^2 \leq \nu$ and

$$\mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right]$$

$$\begin{aligned}
& + \frac{\gamma}{b} \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \delta \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\
\leq & \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \right. \\
& \quad \left. - \left(\frac{2\gamma L_{\max}^2}{bnp_a B} + \frac{2\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} \right) - \rho \left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) - \delta \frac{2mL_{\max}^2}{p_a B} \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})b}{np_a^2} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{2\gamma b}{np_a B} + \frac{2\rho b^2}{p_a B} + \delta(1-b) \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right].
\end{aligned}$$

Next, if we take $\rho = \frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2}$, then

$$\left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})b}{np_a^2} + \rho(1-b) \right) = \rho,$$

634 therefore

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \delta \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\
\leq & \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \right. \\
& \quad - \left(\frac{2\gamma L_{\max}^2}{bnp_a B} + \frac{2\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} \right) - \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \\
& \quad \left. - \delta \frac{2mL_{\max}^2}{p_a B} \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{2\gamma b}{np_a B} + \frac{16b^3\gamma\omega(2\omega+1)}{np_a^3 B} + \frac{4b^2\gamma(p_a - p_{aa})}{nBp_a^3} + \delta(1-b) \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right].
\end{aligned}$$

635 Due to $b \leq p_a$ and $\frac{p_a - p_{aa}}{p_a} \leq 1$, we have

$$\begin{aligned}
& \frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{2\gamma b}{np_a B} + \frac{16b^3\gamma\omega(2\omega+1)}{np_a^3 B} + \frac{4b^2\gamma(p_a - p_{aa})}{nBp_a^3} \\
& \leq \frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{2\gamma b}{np_a B} + \frac{16b^2\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{4\gamma b}{np_a B} \\
& = \frac{24b^2\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma b}{np_a B}.
\end{aligned}$$

636 Let us take $\delta = \frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B}$. Thus

$$\left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{2\gamma b}{np_a B} + \frac{16b^3\gamma\omega(2\omega+1)}{np_a^3 B} + \frac{4b^2\gamma(p_a - p_{aa})}{nBp_a^3} + \delta(1-b) \right) \leq \delta$$

637 and

$$\begin{aligned}
& \mathbb{E}[f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \right. \\
& \quad \left. - \left(\frac{2\gamma L_{\max}^2}{bnp_a B} + \frac{2\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} \right) - \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \right. \\
& \quad \left. - \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \frac{2mL_{\max}^2}{p_a B} \right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right].
\end{aligned}$$

638 Let us simplify the term near $\mathbb{E} \left[\|x^{t+1} - x^t\|^2 \right]$. Due to $b \leq p_a$, $\frac{p_a - p_{aa}}{p_a} \leq 1$, and $1 - p_a \leq 1$, we
639 have

$$\begin{aligned}
& \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \\
& + \left(\frac{2\gamma L_{\max}^2}{bnp_a B} + \frac{2\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} \right) \\
& + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \frac{2mL_{\max}^2}{p_a B} \\
& \leq \frac{12\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \\
& + \left(\frac{6\gamma L_{\max}^2}{bnp_a B} + \frac{6\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} \right) \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \frac{2mL_{\max}^2}{p_a B}
\end{aligned}$$

640 Considering that $b \leq \frac{p_a B}{m}$ and $b \geq \frac{p_a B}{2m}$, we obtain

$$\begin{aligned}
& \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) \\
& + \left(\frac{2\gamma L_{\max}^2}{bnp_a B} + \frac{2\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} \right) \\
& + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \left(\frac{2L_{\max}^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \frac{2mL_{\max}^2}{p_a B} \\
& \leq \frac{36\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) + \left(\frac{18\gamma L_{\max}^2}{bnp_a B} + \frac{6\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} \right) \\
& \leq \frac{36\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) + \left(\frac{36m\gamma L_{\max}^2}{np_a^2 B^2} + \frac{12m\gamma(p_a - p_{aa})\hat{L}^2}{Bnp_a^3} \right).
\end{aligned}$$

641 All in all, we have

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{36\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2L_{\max}^2}{B} + 2\hat{L}^2 \right) - \left(\frac{36m\gamma L_{\max}^2}{np_a^2 B^2} + \frac{12m\gamma(p_a - p_{aa})\hat{L}^2}{Bnp_a^3} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right].
\end{aligned}$$

642 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^{t+1} - \nabla f_{ij}(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{24b\gamma\omega(2\omega+1)}{np_a^2 B} + \frac{6\gamma}{np_a B} \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right].
\end{aligned}$$

643 It is left to apply Lemma 3 with

$$\begin{aligned}
\Psi^t &= \frac{(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{1}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b\omega(2\omega+1)}{np_a^2} + \frac{2(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{24b\omega(2\omega+1)}{np_a^2 B} + \frac{6}{np_a B} \right) \mathbb{E} \left[\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \|h_{ij}^t - \nabla f_{ij}(x^t)\|^2 \right]
\end{aligned}$$

644 to conclude the proof. \square

645 D.6 Proof for DASHA-PP-MVR

646 Let us denote $\nabla f_i(x^{t+1}; \xi_i^{t+1}) := \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^{t+1}; \xi_{ij}^{t+1})$.

647 **Lemma 10.** Suppose that Assumptions 3, 5, 6 and 8 hold. For h_i^{t+1} and k_i^{t+1} from Algorithm 1
648 (DASHA-PP-MVR) we have

1.

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{2b^2\sigma^2}{np_a B} + \left(\frac{2(1-b)^2 L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa}) \hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \\
& \quad + \frac{2(p_a - p_{aa}) b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

2.

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{2b^2\sigma^2}{p_a B} + \left(\frac{2(1-b)^2 L_\sigma^2}{p_a B} + \frac{2(1-p_a) L_i^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \\
& \quad + \left(\frac{2(1-p_a) b^2}{p_a} + (1-b)^2 \right) \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n].
\end{aligned}$$

3.

$$\mathbb{E}_k \left[\|k_i^{t+1}\|^2 \right] \leq \frac{2b^2\sigma^2}{B} + \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 2L_i^2 \right) \|x^{t+1} - x^t\|^2 + 2b^2 \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n].$$

649 *Proof.* First, let us proof the bound for $\mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right]$:

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& = \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \mathbb{E}_k [\mathbb{E}_{p_a} [h^{t+1}]]\|^2 \right] \right] + \|\mathbb{E}_k [\mathbb{E}_{p_a} [h^{t+1}]] - \nabla f(x^{t+1})\|^2.
\end{aligned}$$

650 Using

$$\mathbb{E}_k [\mathbb{E}_{p_a} [h_i^{t+1}]] = h_i^t + \mathbb{E}_k [k_i^{t+1}] = h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))$$

651 and (17), we have

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& = \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \mathbb{E}_k [\mathbb{E}_{p_a} [h^{t+1}]]\|^2 \right] \right] + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

652 We can use Lemma 1 with $r_i = h_i^t$ and $s_i = k_i^{t+1}$ to obtain

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|k_i^{t+1} - \mathbb{E}_k [k_i^{t+1}]\|^2 \right] + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\mathbb{E}_k [k_i^{t+1}]\|^2 + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \\
& = \frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - b(h_i^t - \nabla f_i(x^t; \xi_i^{t+1})) \right. \\
& \quad \left. - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)))\|^2 \right] \\
& \quad + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& \quad + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \\
& \stackrel{(16)}{\leq} \frac{2}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|b(\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^{t+1}))\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{2}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|(1-b)(\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)))\|^2 \right] \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \\
& = \frac{2b^2}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \frac{2(1-b)^2}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2. \\
& = \frac{2b^2}{n^2 p_a B^2} \sum_{i=1}^n \sum_{j=1}^B \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \frac{2(1-b)^2}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

653 In the last equality, we use the independence of elements in the mini-batches. Due to Assumption 5,
654 we get

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{2b^2 \sigma^2}{n p_a B} \\
& + \frac{2(1-b)^2}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
& + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \\
& \stackrel{(16)}{\leq} \frac{2b^2 \sigma^2}{n p_a B} \\
& + \frac{2(1-b)^2}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
& + \frac{2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2. \\
& = \frac{2b^2 \sigma^2}{n p_a B} \\
& + \frac{2(1-b)^2}{n^2 p_a B^2} \sum_{i=1}^n \sum_{j=1}^B \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \nabla f_i(x^t; \xi_{ij}^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \\
& + (1-b)^2 \|h^t - \nabla f(x^t)\|^2,
\end{aligned}$$

655 where we use the independence of elements in the mini-batches. Using Assumptions 3 and 6, we
656 obtain

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{2b^2 \sigma^2}{np_a B} + \left(\frac{2(1-b)^2 L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa}) \widehat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \\
& \quad + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + (1-b)^2 \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

657 Now, we prove the second inequality:

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \\
& = \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - \mathbb{E}_k [\mathbb{E}_{p_a} [h_i^{t+1}]]\|^2 \right] \right] \\
& \quad + \|\mathbb{E}_k [\mathbb{E}_{p_a} [h_i^{t+1}]] - \nabla f_i(x^{t+1})\|^2 \\
& = \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - (h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)))\|^2 \right] \right] \\
& \quad + \|h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)) - \nabla f_i(x^{t+1})\|^2 \\
& = \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - (h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)))\|^2 \right] \right] \\
& \quad + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2 \\
& = p_a \mathbb{E}_k \left[\left\| h_i^t + \frac{1}{p_a} k_i^{t+1} - (h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))) \right\|^2 \right] \\
& \quad + (1-p_a) \|h_i^t - (h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)))\|^2 \\
& \quad + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2 \\
& = p_a \mathbb{E}_k \left[\left\| \frac{1}{p_a} k_i^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))) \right\|^2 \right] \\
& \quad + (1-p_a) \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& \quad + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2 \\
& \stackrel{(17)}{=} \frac{1}{p_a} \mathbb{E}_k \left[\left\| k_i^{t+1} - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))) \right\|^2 \right] \\
& \quad + \frac{(1-p_a)^2}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& \quad + (1-p_a) \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& \quad + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2 \\
& = \frac{1}{p_a} \mathbb{E}_k \left[\left\| \nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - b(h_i^t - \nabla f_i(x^t; \xi_i^{t+1})) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))) \right\|^2 \right] \\
& \quad + \frac{1-p_a}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& \quad + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2 \\
& = \frac{1}{p_a} \mathbb{E}_k \left[\left\| b(\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^{t+1})) + (1-b)(\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))) \right\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1-p_a}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2 \\
& \stackrel{(16)}{\leq} \frac{2b^2}{p_a} \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \frac{2(1-b)^2}{p_a} \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
& + \frac{1-p_a}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2.
\end{aligned}$$

658 Considering the independence of elements in the mini-batch, we obtain

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \\
& = \frac{2b^2}{p_a B^2} \sum_{j=1}^B \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \frac{2(1-b)^2}{p_a B^2} \sum_{j=1}^B \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \nabla f_i(x^t; \xi_{ij}^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
& + \frac{1-p_a}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& + (1-b)^2 \|h_i^t - \nabla f_i(x^t)\|^2. \\
& \stackrel{(16)}{\leq} \frac{2b^2}{p_a B^2} \sum_{j=1}^B \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \nabla f_i(x^{t+1})\|^2 \right] \\
& + \frac{2(1-b)^2}{p_a B^2} \sum_{j=1}^B \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \nabla f_i(x^t; \xi_{ij}^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
& + \frac{2(1-p_a)}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \|h_i^t - \nabla f_i(x^t)\|^2
\end{aligned}$$

659 Next, we use Assumptions 3, 6, 5, to get

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \\
& \leq \frac{2b^2\sigma^2}{p_a B} + \left(\frac{2(1-b)^2 L_\sigma^2}{p_a B} + \frac{2(1-p_a) L_i^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \\
& + \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \|h_i^t - \nabla f_i(x^t)\|^2.
\end{aligned}$$

660 It is left to prove the bound for $\mathbb{E}_k \left[\|k_i^{t+1}\|^2 \right]$:

$$\begin{aligned}
& \mathbb{E}_k \left[\|k_i^{t+1}\|^2 \right] \\
& = \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - b(h_i^t - \nabla f_i(x^t; \xi_i^{t+1}))\|^2 \right] \\
& \stackrel{(17)}{=} \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - b(h_i^t - \nabla f_i(x^t; \xi_i^{t+1})) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t)))\|^2 \right] \\
& + \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& = \mathbb{E}_k \left[\|b(\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^{t+1})) + (1-b)(\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)))\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - b(h_i^t - \nabla f_i(x^t))\|^2 \\
& \stackrel{(16)}{\leq} 2b^2 \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^{t+1})\|^2 \right] \\
& + 2(1-b)^2 \mathbb{E}_k \left[\|\nabla f_i(x^{t+1}; \xi_i^{t+1}) - \nabla f_i(x^t; \xi_i^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))\|^2 \right] \\
& + 2 \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 + 2b^2 \|h_i^t - \nabla f_i(x^t)\|^2.
\end{aligned}$$

661 Using Assumptions 3, 6, 5 and the independence of elements in the mini-batch, we get

$$\begin{aligned}
& \mathbb{E}_k \left[\|k_i^{t+1}\|^2 \right] \\
& \leq \frac{2b^2 \sigma^2}{B} + \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 2L_i^2 \right) \|x^{t+1} - x^t\|^2 + 2b^2 \|h_i^t - \nabla f_i(x^t)\|^2.
\end{aligned}$$

662

□

663 **Theorem 4.** Suppose that Assumptions 1, 2, 3, 5, 6, 7 and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$,
664 $b \in \left(0, \frac{p_a}{2-p_a}\right]$,

$$\begin{aligned}
\gamma \leq & \left(L + \left[\frac{48\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{(1-b)^2 L_\sigma^2}{B} \right) \right. \right. \\
& \left. \left. + \frac{12}{np_a b} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1-b)^2 L_\sigma^2}{B} \right) \right]^{1/2} \right)^{-1},
\end{aligned}$$

665 and $g_i^0 = h_i^0$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP-MVR). Then

$$\begin{aligned}
\mathbb{E} \left[\|\nabla f(\hat{x}^T)\|^2 \right] & \leq \frac{1}{T} \left[\frac{2\Delta_0}{\gamma} + \frac{2}{b} \|h^0 - \nabla f(x^0)\|^2 \right. \\
& + \left(\frac{32b\omega(2\omega+1)}{np_a^2} + \frac{4 \left(1 - \frac{p_{aa}}{p_a} \right)}{np_a} \right) \left(\frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right) \\
& \left. + \left(\frac{48b^2\omega(2\omega+1)}{p_a^2} + \frac{12b}{p_a} \right) \frac{\sigma^2}{nB} \right].
\end{aligned}$$

666 *Proof.* Let us fix constants $\nu, \rho \in [0, \infty)$ that we will define later. Considering Lemma 6, Lemma 10,
667 and the law of total expectation, we obtain

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& = \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\mathbb{E}_k \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right] \right] \\
& + \nu \mathbb{E} \left[\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right] \\
& + \rho \mathbb{E} \left[\mathbb{E}_B \left[\mathbb{E}_{p_a} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \\
\leq & \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{4\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\frac{2b^2\sigma^2}{B} + \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\hat{L}^2 \right) \|x^{t+1} - x^t\|^2 + 2b^2 \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \nu \mathbb{E} \left(\frac{2b^2\sigma^2}{np_a B} + \left(\frac{2(1-b)^2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \right) \\
& + \rho \mathbb{E} \left(\frac{2b^2\sigma^2}{p_a B} + \left(\frac{2(1-b)^2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right).
\end{aligned}$$

668 After rearranging the terms, we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^{t+1} - h^{t+1}\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
\leq & \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} \left[\|\nabla f(x^t)\|^2 \right] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\hat{L}^2 \right) \right. \\
& \quad \left. - \nu \left(\frac{2(1-b)^2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \right) - \rho \left(\frac{2(1-b)^2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \right) \mathbb{E} \left[\|x^{t+1} - x^t\|^2 \right] \\
& + \left(\gamma + \nu(1-b)^2 \right) \mathbb{E} \left[\|h^t - \nabla f(x^t)\|^2 \right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\nu(p_a - p_{aa})b^2}{np_a^2} + \rho \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \nu \frac{2b^2}{np_a} + \rho \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

669 By taking $\nu = \frac{\gamma}{b}$, one can show that $(\gamma + \nu(1-b)^2) \leq \nu$, and

$$\begin{aligned}
& \mathbb{E}[f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
& \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 2\hat{L}^2\right)\right. \\
& \quad \left. - \frac{\gamma}{b} \left(\frac{2(1-b)^2 L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2}\right) - \rho \left(\frac{2(1-b)^2 L_\sigma^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a}\right)\right] \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})b}{np_a^2} + \rho \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2\right)\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma b}{np_a} + \rho \frac{2b^2}{p_a}\right) \frac{\sigma^2}{B}.
\end{aligned}$$

670 Note that $b \leq \frac{p_a}{2-p_a}$, thus

$$\begin{aligned}
& \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})b}{np_a^2} + \rho \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2\right)\right) \\
& \leq \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})b}{np_a^2} + \rho(1-b)\right).
\end{aligned}$$

671 And if we take $\rho = \frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2}$, then

$$\left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})b}{np_a^2} + \rho(1-b)\right) \leq \rho,$$

672 and

$$\begin{aligned}
& \mathbb{E}[f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
& \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 2\hat{L}^2\right)\right. \\
& \quad \left. - \frac{\gamma}{np_a b} \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 2\left(1 - \frac{p_{aa}}{p_a}\right)\hat{L}^2\right)\right)
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^3} + \frac{2\gamma(1-\frac{p_{aa}}{p_a})}{np_a^2} \right) \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2(1-p_a)\widehat{L}^2 \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma b}{np_a} + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

673 Let us simplify the inequality. First, due to $b \leq p_a$ and $(1-p_a) \leq (1-\frac{p_{aa}}{p_a})$, we have

$$\begin{aligned}
& \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^3} + \frac{2\gamma(1-\frac{p_{aa}}{p_a})}{np_a^2} \right) \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2(1-p_a)\widehat{L}^2 \right) \\
& = \frac{8b\gamma\omega(2\omega+1)}{np_a^3} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2(1-p_a)\widehat{L}^2 \right) \\
& \quad + \frac{2\gamma(1-\frac{p_{aa}}{p_a})}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2(1-p_a)\widehat{L}^2 \right) \\
& \leq \frac{8\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\widehat{L}^2 \right) \\
& \quad + \frac{2\gamma}{np_ab} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\left(1-\frac{p_{aa}}{p_a}\right)\widehat{L}^2 \right),
\end{aligned}$$

674 therefore

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& \quad + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& \quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{12\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\widehat{L}^2 \right) \right. \\
& \quad \left. - \frac{3\gamma}{np_ab} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\left(1-\frac{p_{aa}}{p_a}\right)\widehat{L}^2 \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& \quad + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& \quad + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma b}{np_a} + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B} \\
& = \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& \quad + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& \quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{24\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{(1-b)^2L_\sigma^2}{B} + \widehat{L}^2 \right) \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{6\gamma}{np_a b} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}^2 \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{8b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma b}{np_a} + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

675 Also, we can simplify the last term:

$$\begin{aligned}
& \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{2b^2}{p_a} \\
& = \frac{16b^3\gamma\omega(2\omega+1)}{np_a^3} + \frac{4b^2\gamma \left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2} \\
& \leq \frac{16b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{4b\gamma}{np_a},
\end{aligned}$$

676 thus

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{24\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
& \quad \left. - \frac{6\gamma}{np_a b} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}^2 \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{24b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{6\gamma b}{np_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

677 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right]
\end{aligned}$$

$$+ \left(\frac{24b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{6\gamma b}{np_a} \right) \frac{\sigma^2}{B}.$$

678 It is left to apply Lemma 3 with

$$\begin{aligned} \Psi^t &= \frac{(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ &+ \frac{1}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{8b\omega(2\omega+1)}{np_a^2} + \frac{2(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \end{aligned}$$

679 and $C = \left(\frac{24b^2\omega(2\omega+1)}{p_a^2} + \frac{6b}{p_a} \right) \frac{\sigma^2}{nB}$ to conclude the proof. \square

680 **Corollary 3.** Suppose that assumptions from Theorem 4 hold, momentum $b =$
681 $\Theta \left(\min \left\{ \frac{p_a}{\omega} \sqrt{\frac{n\varepsilon B}{\sigma^2}}, \frac{p_a n \varepsilon B}{\sigma^2} \right\} \right)$, $\frac{\sigma^2}{n\varepsilon B} \geq 1$, and $h_i^0 = \frac{1}{B_{\text{init}}} \sum_{k=1}^{B_{\text{init}}} \nabla f_i(x^0; \xi_{ik}^0)$ for all $i \in [n]$,
682 and batch size $B_{\text{init}} = \Theta \left(\frac{\sqrt{p_a B}}{b} \right)$, then Algorithm 1 (DASHA-PP-MVR) needs

$$\begin{aligned} T &:= \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right. \right. \\ &\quad \left. \left. + \frac{\sigma}{p_a \sqrt{\varepsilon n}} \left(\frac{\mathbb{1}_{p_a} \hat{L}}{\sqrt{B}} + \frac{L_\sigma}{B} \right) \right] + \frac{\sigma^2}{\sqrt{p_a n \varepsilon B}} \right) \end{aligned}$$

683 communication rounds to get an ε -solution and the number of stochastic gradient calculations per
684 node equals $\mathcal{O}(B_{\text{init}} + BT)$.

685 *Proof.* Using the result from Theorem 4, we have

$$\begin{aligned} &\mathbb{E} [\|\nabla f(\hat{x}^T)\|^2] \\ &\leq \frac{1}{T} \left[2\Delta_0 \left(L + \sqrt{\frac{48\omega(2\omega+1)}{np_a^2}} \left(\hat{L}^2 + \frac{(1-b)^2 L_\sigma^2}{B} \right) + \frac{12}{np_a b} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1-b)^2 L_\sigma^2}{B} \right) \right) \right. \\ &\quad \left. + \frac{2}{b} \|h^0 - \nabla f(x^0)\|^2 + \left(\frac{32b\omega(2\omega+1)}{np_a^2} + \frac{4 \left(1 - \frac{p_{aa}}{p_a} \right)}{np_a} \right) \left(\frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right) \right] \\ &\quad + \left(\frac{48b^2\omega(2\omega+1)}{p_a^2} + \frac{12b}{p_a} \right) \frac{\sigma^2}{nB} \end{aligned}$$

686 We choose b to ensure $\left(\frac{48b^2\omega(2\omega+1)}{p_a^2} + \frac{12b}{p_a} \right) \frac{\sigma^2}{nB} = \Theta(\varepsilon)$. Note that $\frac{1}{b} =$

687 $\Theta \left(\max \left\{ \frac{\omega}{p_a} \sqrt{\frac{\sigma^2}{n\varepsilon B}}, \frac{\sigma^2}{p_a n \varepsilon B} \right\} \right) \leq \Theta \left(\max \left\{ \frac{\omega^2}{p_a}, \frac{\sigma^2}{p_a n \varepsilon B} \right\} \right)$, thus

$$\begin{aligned} &\mathbb{E} [\|\nabla f(\hat{x}^T)\|^2] \\ &= \mathcal{O} \left(\frac{1}{T} \left[\Delta_0 \left(L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \left(\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right) \right. \right. \\ &\quad \left. \left. + \frac{1}{b} \|h^0 - \nabla f(x^0)\|^2 + \left(\frac{b\omega^2}{np_a^2} + \frac{1}{np_a} \right) \left(\frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right) \right] + \varepsilon \right), \end{aligned}$$

688 where $\mathbb{1}_{p_a} = \sqrt{1 - \frac{p_{aa}}{p_a}}$. It enough to take the following T to get ε -solution.

$$T = \mathcal{O} \left(\frac{1}{\varepsilon} \left[\Delta_0 \left(L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \left(\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right) \right. \right. \\ \left. \left. + \frac{1}{b} \|h^0 - \nabla f(x^0)\|^2 + \left(\frac{b\omega^2}{np_a^2} + \frac{1}{np_a} \right) \left(\frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right) \right] \right).$$

689 Let us bound the norms:

$$\begin{aligned} \mathbb{E} \left[\|h^0 - \nabla f(x^0)\|^2 \right] &= \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \frac{1}{B_{\text{init}}} \sum_{k=1}^{B_{\text{init}}} \nabla f_i(x^0; \xi_{ik}^0) - \nabla f(x^0) \right\|^2 \right] \\ &= \frac{1}{n^2 B_{\text{init}}^2} \sum_{i=1}^n \sum_{k=1}^{B_{\text{init}}} \mathbb{E} \left[\|\nabla f_i(x^0; \xi_{ik}^0) - \nabla f_i(x^0)\|^2 \right] \\ &\leq \frac{\sigma^2}{n B_{\text{init}}}. \end{aligned}$$

690 Using the same reasoning, one can get $\frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\|h_i^0 - \nabla f_i(x^0)\|^2 \right] \leq \frac{\sigma^2}{B_{\text{init}}}$. Combining all inequalities, we have

$$T = \mathcal{O} \left(\frac{1}{\varepsilon} \left[\Delta_0 \left(L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \left(\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right) \right. \right. \\ \left. \left. + \frac{\sigma^2}{bn B_{\text{init}}} + \frac{b\omega^2 \sigma^2}{np_a^2 B_{\text{init}}} + \frac{\sigma^2}{np_a B_{\text{init}}} \right] \right).$$

692 Using the choice of B_{init} and b , we obtain

$$\begin{aligned} T &= \mathcal{O} \left(\frac{1}{\varepsilon} \left[\Delta_0 \left(L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \left(\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right) \right. \right. \\ &\quad \left. \left. + \frac{\sigma^2}{\sqrt{p_a} n B} + \frac{b^2 \omega^2 \sigma^2}{np_a^{5/2} B} + \frac{b \sigma^2}{p_a^{3/2} n B} \right] \right) \\ &= \mathcal{O} \left(\frac{1}{\varepsilon} \left[\Delta_0 \left(L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \left(\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right) \right. \right. \\ &\quad \left. \left. + \frac{\sigma^2}{\sqrt{p_a} n B} + \frac{\varepsilon}{\sqrt{p_a}} \right] \right) \end{aligned}$$

$$= \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \left(\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right] + \frac{\sigma^2}{\sqrt{p_a} n \varepsilon B} + \frac{1}{\sqrt{p_a}} \right).$$

693 Using $\frac{\sigma^2}{n \varepsilon B} \geq 1$, we can conclude the proof of the inequality. The number of stochastic gradients that
 694 each node calculates equals $B_{\text{init}} + 2BT = \mathcal{O}(B_{\text{init}} + BT)$. \square

695 **Corollary 4.** Suppose that assumptions of Corollary 3 hold, batch size $B \leq \min \left\{ \frac{\sigma}{p_a \sqrt{\varepsilon n}}, \frac{L_\sigma^2}{\mathbb{1}_{p_a}^2 \hat{L}^2} \right\}$,
 696 we take RandK compressors with $K = \Theta \left(\frac{Bd\sqrt{\varepsilon n}}{\sigma} \right)$. Then the communication complexity equals

$$\mathcal{O} \left(\frac{d\sigma}{\sqrt{p_a} \sqrt{n \varepsilon}} + \frac{L_\sigma \Delta_0 d}{p_a \sqrt{n \varepsilon}} \right), \quad (12)$$

697 and the expected number of stochastic gradient calculations per node equals

$$\mathcal{O} \left(\frac{\sigma^2}{\sqrt{p_a} n \varepsilon} + \frac{L_\sigma \Delta_0 \sigma}{p_a \varepsilon^{3/2} n} \right). \quad (13)$$

698 *Proof.* The communication complexity equals

$$\mathcal{O}(d + KT) = \mathcal{O} \left(d + \frac{\Delta_0}{\varepsilon} \left[KL + K \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + K \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \left(\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right] + K \frac{\sigma^2}{\sqrt{p_a} n \varepsilon B} \right).$$

699 Due to $B \leq \frac{L_\sigma^2}{\mathbb{1}_{p_a}^2 \hat{L}^2}$, we have $\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \leq \frac{2L_\sigma}{\sqrt{B}}$ and

$$\mathcal{O}(d + KT) = \mathcal{O} \left(d + \frac{\Delta_0}{\varepsilon} \left[KL + K \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + K \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \frac{L_\sigma}{\sqrt{B}} \right] + K \frac{\sigma^2}{\sqrt{p_a} n \varepsilon B} \right).$$

700 From Theorem 6, we have $\omega + 1 = \frac{d}{K}$. Since $K = \Theta \left(\frac{Bd\sqrt{\varepsilon n}}{\sigma} \right) = \mathcal{O} \left(\frac{d}{p_a \sqrt{n}} \right)$, the communication
 701 complexity equals

$$\begin{aligned} \mathcal{O}(d + KT) &= \mathcal{O} \left(d + \frac{\Delta_0}{\varepsilon} \left[\frac{d}{p_a \sqrt{n}} L + \frac{d}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{d}{p_a \sqrt{n}} L_\sigma \right] + \frac{d\sigma}{\sqrt{p_a} \sqrt{n \varepsilon}} \right) \\ &= \mathcal{O} \left(\frac{d\sigma}{\sqrt{p_a} \sqrt{n \varepsilon}} + \frac{L_\sigma \Delta_0 d}{p_a \sqrt{n \varepsilon}} \right) \end{aligned}$$

702 And the expected number of stochastic gradient calculations per node equals

$$\begin{aligned} &\mathcal{O}(B_{\text{init}} + BT) \\ &= \mathcal{O} \left(\frac{\sigma^2}{\sqrt{p_a} n \varepsilon} + \frac{B\omega}{\sqrt{p_a}} \sqrt{\frac{\sigma^2}{n \varepsilon B}} + \frac{\Delta_0}{\varepsilon} \left[BL + B \frac{\omega}{p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + B \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \left(\mathbb{1}_{p_a} \hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right] + B \frac{\sigma^2}{\sqrt{p_a} n \varepsilon B} \right) \\ &= \mathcal{O} \left(\frac{\sigma^2}{\sqrt{p_a} n \varepsilon} + \frac{Bd}{K \sqrt{p_a}} \sqrt{\frac{\sigma^2}{n \varepsilon B}} + \frac{\Delta_0}{\varepsilon} \left[BL + B \frac{d}{K p_a \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + B \sqrt{\frac{\sigma^2}{p_a^2 \varepsilon n^2 B}} \frac{L_\sigma}{\sqrt{B}} \right] + \frac{\sigma^2}{\sqrt{p_a} n \varepsilon} \right) \end{aligned}$$

$$\begin{aligned}
&= \mathcal{O} \left(\frac{\sigma^2}{\sqrt{p_a} n \varepsilon} + \frac{\sigma^2}{\sqrt{p_a} n \varepsilon \sqrt{B}} + \frac{\Delta_0}{\varepsilon} \left[\frac{\sigma}{p_a \sqrt{\varepsilon} n} L + \frac{\sigma}{p_a \sqrt{\varepsilon} n} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma}{p_a \sqrt{\varepsilon} n} L_\sigma \right] \right) \\
&= \mathcal{O} \left(\frac{\sigma^2}{\sqrt{p_a} n \varepsilon} + \frac{L_\sigma \Delta_0 \sigma}{p_a \varepsilon^{3/2} n} \right).
\end{aligned}$$

703

□

E Analysis of DASHA-PP under Polyak-Łojasiewicz Condition

In this section, we provide the theoretical convergence rates of DASHA-PP under Polyak-Łojasiewicz Condition.

Assumption 9. The function f satisfy (Polyak-Łojasiewicz) PL-condition:

$$\|\nabla f(x)\|^2 \geq 2\mu(f(x) - f^*), \quad \forall x \in \mathbb{R}, \quad (31)$$

where $f^* = \inf_{x \in \mathbb{R}^d} f(x) > -\infty$.

Under Polyak-Łojasiewicz condition, a (random) point \hat{x} is ε -solution, if $\mathbb{E}[f(\hat{x})] - f^* \leq \varepsilon$.

We now provide the convergence rates of DASHA-PP under PL-condition.

E.1 Gradient Setting

Theorem 8. Suppose that Assumption 1, 2, 3, 7, 8 and 9 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{p_a}{2-p_a}$,

$$\gamma \leq \min \left\{ \left(L + \sqrt{\frac{200\omega(2\omega+1)}{np_a^2} + \frac{48}{np_a^2} \left(1 - \frac{p_{aa}}{p_a}\right) \hat{L}} \right)^{-1}, \frac{a}{4\mu} \right\},$$

and $h_i^0 = g_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP), then $\mathbb{E}[f(x^T)] - f^* \leq (1 - \gamma\mu)^T \Delta_0$.

Let us provide bounds up to logarithmic factors and use $\tilde{\mathcal{O}}(\cdot)$ notation. The provided theorem states that to get ε -solution DASHA-PP have to run

$$\tilde{\mathcal{O}} \left(\frac{\omega+1}{p_a} + \frac{L}{\mu} + \frac{\omega \hat{L}}{p_a \mu \sqrt{n}} + \frac{\hat{L}}{p_a \mu \sqrt{n}} \right),$$

communication rounds. The method DASHA from (Tyurin and Richtárik, 2023), have to run

$$\tilde{\mathcal{O}} \left(\omega + \frac{L}{\mu} + \frac{\omega \hat{L}}{\mu \sqrt{n}} \right),$$

communication rounds to get ε -solution. The difference is the same as in the general nonconvex case (see Section 6.1). Up to Lipschitz constants factors, we get the degeneration up to $1/p_a$ factor due to the partial participation.

E.2 Finite-Sum Setting

Theorem 9. Suppose that Assumption 1, 2, 3, 7, 4, 8, and 9 hold. Let us take $a = \frac{p_a}{2\omega+1}$, probability $p_{page} = \frac{B}{m+B}$, $b = \frac{p_{page} p_a}{2-p_a}$,

$$\gamma \leq \min \left\{ \left(L + \sqrt{\frac{200\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{(1-p_{page})L_{\max}^2}{B} \right) + \frac{48}{np_a^2 p_{page}} \left(\left(1 - \frac{p_{aa}}{p_a}\right) \hat{L}^2 + \frac{(1-p_{page})L_{\max}^2}{B} \right)} \right)^{-1}, \frac{a}{2\mu}, \frac{b}{2\mu} \right\},$$

and $h_i^0 = g_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP-PAGE), then $\mathbb{E}[f(x^T)] - f^* \leq (1 - \gamma\mu)^T \Delta_0$.

The provided theorem states that to get ε -solution DASHA-PP have to run

$$\tilde{\mathcal{O}} \left(\frac{\omega+1}{p_a} + \frac{m}{p_a B} + \frac{L}{\mu} + \frac{\omega}{p_a \mu \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + \frac{\sqrt{m}}{p_a \mu \sqrt{n} B} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) \right),$$

communication rounds. The method DASHA-PAGE from (Tyurin and Richtárik, 2023), have to run

$$\tilde{\mathcal{O}} \left(\omega + \frac{m}{B} + \frac{L}{\mu} + \frac{\omega}{\mu \sqrt{n}} \left(\hat{L} + \frac{L_{\max}}{\sqrt{B}} \right) + \frac{\sqrt{m}}{\mu \sqrt{n} B} \left(\frac{L_{\max}}{\sqrt{B}} \right) \right),$$

communication rounds to get ε -solution. We can guarantee the degeneration up to $1/p_a$ factor due to the partial participation only if $B = \mathcal{O}\left(\frac{L_{\max}^2}{L^2}\right)$. The same conclusion we have in Section 6.2.

727 E.3 Stochastic Setting

Theorem 10. Suppose that Assumption 1, 2, 3, 7, 5, 6, 8 and 9 hold. Let us take $a = \frac{p_a}{2\omega+1}$,
 $b \in \left(0, \frac{p_a}{2-p_a}\right]$,

$$\gamma \leq \min \left\{ \left(L + \sqrt{\frac{200\omega(2\omega+1)}{np_a^2} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \hat{L}^2 \right)} + \frac{40}{np_a b} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \left(1 - \frac{p_{aa}}{p_a}\right) \hat{L}^2 \right) \right)^{-1}, \frac{a}{2\mu}, \frac{b}{2\mu} \right\},$$

728 and $h_i^0 = g_i^0$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP-MVR), then

$$\begin{aligned} & \mathbb{E} [f(x^T) - f^*] \\ & \leq (1 - \gamma\mu)^T \left(\Delta_0 + \frac{2\gamma}{b} \|h^0 - \nabla f(x^0)\|^2 + \left(\frac{40\gamma b\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right) \\ & \quad + \frac{1}{\mu} \left(\frac{100b^2\omega(2\omega+1)}{p_a^2} + \frac{20b}{p_a} \right) \frac{\sigma^2}{nB}. \end{aligned}$$

729 The provided theorems states that to get ε -solution DASHA-PP have to run

$$\tilde{\mathcal{O}} \left(\underbrace{\frac{\omega+1}{p_a} + \frac{\omega}{p_a} \sqrt{\frac{\sigma^2}{\mu n \varepsilon B}}}_{\mathcal{P}_2} + \frac{\sigma^2}{p_a \mu n \varepsilon B} + \frac{L}{\mu} + \frac{\omega}{p_a \mu \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \underbrace{\frac{\sigma}{p_a n \mu^{3/2} \sqrt{\varepsilon B}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right)}_{\mathcal{P}_1} \right) \quad (32)$$

730 communication rounds. We take $b = \Theta \left(\min \left\{ \frac{p_a}{\omega} \sqrt{\frac{\mu n \varepsilon B}{\sigma^2}}, \frac{p_a \mu n \varepsilon B}{\sigma^2} \right\} \right) \geq$
 731 $\Theta \left(\min \left\{ \frac{p_a}{\omega^2}, \frac{p_a \mu n \varepsilon B}{\sigma^2} \right\} \right).$

732 The method DASHA-SYNC-MVR from (Tyurin and Richtárik, 2023), have to run

$$\tilde{\mathcal{O}} \left(\omega + \frac{\sigma^2}{\mu n \varepsilon B} + \frac{L}{\mu} + \frac{\omega}{\mu \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma}{n \mu^{3/2} \sqrt{\varepsilon B}} \left(\frac{L_\sigma}{\sqrt{B}} \right) \right) \quad (33)$$

733 communication rounds to get ε -solution⁷.

734 In the stochastic setting, the comparison is a little bit more complicated. As in the finite-sum setting,
 735 we have to take $B = \mathcal{O} \left(\frac{L_\sigma^2}{\hat{L}^2} \right)$ to guarantee the degeneration up to $1/p_a$ of the term \mathcal{P}_1 from (32).

736 However, DASHA-PP-MVR has also suboptimal term \mathcal{P}_2 . This suboptimality is tightly connected with
 737 the suboptimality of B_{init} in the general nonconvex case, which we discuss in Section 6.3, and it also
 738 appears in the analysis of DASHA-MVR (Tyurin and Richtárik, 2023). Let us provide the counterpart
 739 of Corollary 4. The corollary reveals that we can escape regimes when \mathcal{P}_2 is the bottleneck by
 740 choosing the parameters of the compressors.

741 **Corollary 5.** Suppose that assumptions of Theorem 10 hold, batch size $B \leq \min \left\{ \frac{\sigma}{p_a \sqrt{\mu \varepsilon n}}, \frac{L_\sigma^2}{\hat{L}^2} \right\}$,
 742 we take RandK compressors with $K = \Theta \left(\frac{B d \sqrt{\mu \varepsilon n}}{\sigma} \right)$. Then the communication complexity equals

$$\tilde{\mathcal{O}} \left(\frac{d\sigma}{p_a \sqrt{\mu \varepsilon n}} + \frac{dL_\sigma}{p_a \mu \sqrt{n}} \right),$$

743 and the expected number of stochastic gradient calculations per node equals

$$\tilde{\mathcal{O}} \left(\frac{\sigma^2}{p_a \mu n \varepsilon} + \frac{\sigma L_\sigma}{p_a n \mu^{3/2} \sqrt{\varepsilon}} \right).$$

744 Up to Lipschitz constants, DASHA-PP-MVR has the state-of-the-art oracle complexity under PL-
 745 condition (see (Li et al., 2021a)). Moreover, DASHA-PP-MVR has the state-of-the-art communication
 746 complexity of DASHA for a small enough μ .

⁷For simplicity, we omitted $\frac{d}{\zeta_C}$ term from the complexity in the stochastic setting, where ζ_C is defined in Definition 12. For instance, for the RandK compressor (see Definition 5 and Theorem 6), $\zeta_C = K$ and $\frac{d}{\zeta_C} = \Theta(\omega)$.

747 E.4 Proofs of Theorems

748 The following proofs almost repeat the proofs from Section D. And one of the main changes is that
749 instead of Lemma 3, we use the following lemma.

750 E.4.1 Standard Lemma under Polyak-Łojasiewicz Condition

751 **Lemma 11.** *Suppose that Assumptions 1 and 9 hold and*

$$\mathbb{E} [f(x^{t+1})] + \gamma \Psi^{t+1} \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] + (1 - \gamma\mu)\gamma\Psi^t + \gamma C,$$

752 *where Ψ^t is a sequence of numbers, $\Psi^t \geq 0$ for all $t \in [T]$, constant $C \geq 0$, constant $\mu > 0$, and*
753 *constant $\gamma \in (0, 1/\mu)$. Then*

$$\mathbb{E} [f(x^T) - f^*] \leq (1 - \gamma\mu)^T ((f(x^0) - f^*) + \gamma\Psi^0) + \frac{C}{\mu}. \quad (34)$$

754 *Proof.* We subtract f^* and use PL-condition (31) to get

$$\begin{aligned} \mathbb{E} [f(x^{t+1}) - f^*] + \gamma\Psi^{t+1} &\leq \mathbb{E} [f(x^t) - f^*] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] + \gamma\Psi^t + \gamma C \\ &\leq (1 - \gamma\mu) \mathbb{E} [f(x^t) - f^*] + (1 - \gamma\mu)\gamma\Psi^t + \gamma C \\ &= (1 - \gamma\mu) (\mathbb{E} [f(x^t) - f^*] + \gamma\Psi^t) + \gamma C. \end{aligned}$$

755 Unrolling the inequality, we have

$$\begin{aligned} \mathbb{E} [f(x^{t+1}) - f^*] + \gamma\Psi^{t+1} &\leq (1 - \gamma\mu)^{t+1} ((f(x^0) - f^*) + \gamma\Psi^0) + \gamma C \sum_{i=0}^t (1 - \gamma\mu)^i \\ &\leq (1 - \gamma\mu)^{t+1} ((f(x^0) - f^*) + \gamma\Psi^0) + \frac{C}{\mu}. \end{aligned}$$

756 It is left to note that $\Psi^t \geq 0$ for all $t \in [T]$. □

757 E.4.2 Generic Lemma

758 We now provide the counterpart of Lemma 6.

759 **Lemma 12.** *Suppose that Assumptions 2, 7, 8 and 9 hold and let us take $a = \frac{p_a}{2\omega+1}$, then*

$$\begin{aligned} &\mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ &\leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\ &\quad + (1 - \gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ &\quad + \frac{10\gamma(2\omega+1)\omega}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right]. \end{aligned}$$

760 *Proof.* Let us fix some constants $\kappa, \eta \in [0, \infty)$ that we will define later. Using the same reasoning as
761 in Lemma 6, we can get

$$\begin{aligned} &\mathbb{E} [f(x^{t+1})] \\ &\quad + \kappa \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ &\leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \end{aligned}$$

$$\begin{aligned}
& + \left(\gamma + \kappa (1 - a)^2 \right) \mathbb{E} \left[\|g^t - h^t\|^2 \right] \\
& + \left(\frac{\kappa a^2 ((2\omega + 1) p_a - p_{aa})}{np_a^2} + \eta \left(\frac{a^2 (2\omega + 1 - p_a)}{p_a} + (1 - a)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(\frac{2\kappa\omega}{np_a} + \frac{2\eta\omega}{p_a} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right].
\end{aligned}$$

762 Let us take $\kappa = \frac{2\gamma}{a}$. One can show that $\gamma + \kappa (1 - a)^2 \leq (1 - \frac{a}{2}) \kappa$, and thus

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] \\
& + \frac{2\gamma}{a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \left(1 - \frac{a}{2} \right) \frac{2\gamma}{a} \mathbb{E} [\|g^t - h^t\|^2] \\
& + \left(\frac{2\gamma a ((2\omega + 1) p_a - p_{aa})}{np_a^2} + \eta \left(\frac{a^2 (2\omega + 1 - p_a)}{p_a} + (1 - a)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(\frac{4\gamma\omega}{anp_a} + \frac{2\eta\omega}{p_a} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right].
\end{aligned}$$

763 Considering the choice of a , one can show that $\left(\frac{a^2 (2\omega + 1 - p_a)}{p_a} + (1 - a)^2 \right) \leq 1 - a$. If we take
764 $\eta = \frac{4\gamma((2\omega+1)p_a-p_{aa})}{np_a^2}$, then $\left(\frac{2\gamma a ((2\omega + 1) p_a - p_{aa})}{np_a^2} + \eta \left(\frac{a^2 (2\omega + 1 - p_a)}{p_a} + (1 - a)^2 \right) \right) \leq (1 - \frac{a}{2}) \eta$ and

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] \\
& + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega + 1) p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \left(1 - \frac{a}{2} \right) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \left(1 - \frac{a}{2} \right) \frac{4\gamma((2\omega + 1) p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(\frac{2\gamma(2\omega + 1)\omega}{np_a^2} + \frac{8\gamma((2\omega + 1) p_a - p_{aa})\omega}{np_a^3} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right] \\
& \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\
& + \left(1 - \frac{a}{2} \right) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \left(1 - \frac{a}{2} \right) \frac{4\gamma((2\omega + 1) p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{10\gamma(2\omega + 1)\omega}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right].
\end{aligned}$$

765 It is left to consider that $\gamma \leq \frac{a}{2\mu}$, and therefore $1 - \frac{a}{2} \leq 1 - \gamma\mu$. □

766 **E.4.3 Proof for DASHA-PP under PL-condition**

Theorem 8. Suppose that Assumption 1, 2, 3, 7, 8 and 9 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{p_a}{2-p_a}$,

$$\gamma \leq \min \left\{ \left(L + \sqrt{\frac{200\omega(2\omega+1)}{np_a^2} + \frac{48}{np_a^2} \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}} \right)^{-1}, \frac{a}{4\mu} \right\},$$

767 and $h_i^0 = g_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP), then $\mathbb{E}[f(x^T)] - f^* \leq$
 768 $(1 - \gamma\mu)^T \Delta_0$.

769 *Proof.* Let us fix constants $\nu, \rho \in [0, \infty)$ that we will define later. Considering Lemma 12, Lemma 7,
 770 and the law of total expectation, we obtain

$$\begin{aligned} & \mathbb{E}[f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\ & + \nu \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\ & \leq \mathbb{E}\left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2}\right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2\right] \\ & + (1 - \gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\ & + \frac{10\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E}\left[2\widehat{L}^2 \|x^{t+1} - x^t\|^2 + 2b^2 \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right] \\ & + \nu \mathbb{E}\left[\frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \|x^{t+1} - x^t\|^2 + \frac{2b^2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + (1-b)^2 \|h^t - \nabla f(x^t)\|^2\right] \\ & + \rho \mathbb{E}\left[\frac{2(1-p_a)\widehat{L}^2}{p_a} \|x^{t+1} - x^t\|^2 + \left(\frac{2b^2(1-p_a)}{p_a} + (1-b)^2\right) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right]. \end{aligned}$$

771 After rearranging the terms, we get

$$\begin{aligned} & \mathbb{E}[f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\ & + \nu \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\ & \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\ & + (1 - \gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\ & - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{20\gamma\omega(2\omega+1)\widehat{L}^2}{np_a^2} - \nu \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} - \rho \frac{2(1-p_a)\widehat{L}^2}{p_a}\right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\ & + (\gamma + \nu(1-b)^2) \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] \\ & + \left(\frac{20b^2\gamma\omega(2\omega+1)}{np_a^2} + \nu \frac{2b^2(p_a - p_{aa})}{np_a^2} + \rho \left(\frac{2b^2(1-p_a)}{p_a} + (1-b)^2\right)\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right]. \end{aligned}$$

772 By taking $\nu = \frac{2\gamma}{b}$, one can show that $(\gamma + \nu(1-b)^2) \leq (1 - \frac{b}{2})\nu$, and

$$\mathbb{E}[f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right]$$

$$\begin{aligned}
& + \frac{2\gamma}{b} \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
\leq & \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{20\gamma\omega(2\omega + 1)\hat{L}^2}{np_a^2} - \frac{4\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} - \rho \frac{2(1 - p_a)\hat{L}^2}{p_a} \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{b}{2} \right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{20b^2\gamma\omega(2\omega + 1)}{np_a^2} + \frac{4\gamma b(p_a - p_{aa})}{np_a^2} + \rho \left(\frac{2b^2(1 - p_a)}{p_a} + (1 - b)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

773 Note that $b = \frac{p_a}{2 - p_a}$, thus

$$\begin{aligned}
& \left(\frac{20b^2\gamma\omega(2\omega + 1)}{np_a^2} + \frac{4\gamma b(p_a - p_{aa})}{np_a^2} + \rho \left(\frac{2b^2(1 - p_a)}{p_a} + (1 - b)^2 \right) \right) \\
& \leq \left(\frac{20b^2\gamma\omega(2\omega + 1)}{np_a^2} + \frac{4\gamma b(p_a - p_{aa})}{np_a^2} + \rho(1 - b) \right).
\end{aligned}$$

774 And if we take $\rho = \frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}$, then

$$\left(\frac{20b^2\gamma\omega(2\omega + 1)}{np_a^2} + \frac{4\gamma b(p_a - p_{aa})}{np_a^2} + \rho(1 - b) \right) \leq \left(1 - \frac{b}{2} \right) \rho,$$

775 and

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
\leq & \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{20\gamma\omega(2\omega + 1)\hat{L}^2}{np_a^2} - \frac{4\gamma(p_a - p_{aa})\hat{L}^2}{bnp_a^2} \right. \\
& \quad \left. - \frac{80b\gamma\omega(2\omega + 1)(1 - p_a)\hat{L}^2}{np_a^3} - \frac{16\gamma(p_a - p_{aa})(1 - p_a)\hat{L}^2}{np_a^3} \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{b}{2} \right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2} \right) \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

776 Due to $\frac{p_a}{2} \leq b \leq p_a$, we have

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
&\quad + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{100\gamma\omega(2\omega + 1)\hat{L}^2}{np_a^2} - \frac{24\gamma(p_a - p_{aa})\hat{L}^2}{np_a^3}\right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
&\quad + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right].
\end{aligned}$$

777 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned}
&\mathbb{E}[f(x^{t+1})] + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
&\quad + \frac{2\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
&\leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
&\quad + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right].
\end{aligned}$$

778 Note that $\gamma \leq \frac{a}{4\mu} \leq \frac{p_a}{4\mu} \leq \frac{b}{2\mu}$, thus $1 - \frac{b}{2} \leq 1 - \gamma\mu$ and

$$\begin{aligned}
&\mathbb{E}[f(x^{t+1})] + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
&\quad + \frac{2\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
&\leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
&\quad + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad + (1 - \gamma\mu) \frac{2\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + (1 - \gamma\mu) \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right].
\end{aligned}$$

779 In the view of Lemma 11 with

$$\begin{aligned}
\Psi^t &= \frac{2(2\omega + 1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{4((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad + \frac{2}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(\frac{40b\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right],
\end{aligned}$$

780 we can conclude the proof of the theorem. \square

781 E.4.4 Proof for DASHA-PP-PAGE under PL-condition

Theorem 9. Suppose that Assumption 1, 2, 3, 7, 4, 8, and 9 hold. Let us take $a = \frac{p_a}{2\omega + 1}$, probability

$$p_{\text{page}} = \frac{B}{m+B}, b = \frac{p_{\text{page}} p_a}{2-p_a},$$

$$\gamma \leq \min \left\{ \left(L + \sqrt{\frac{200\omega(2\omega + 1)}{np_a^2} \left(\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right)} + \frac{48}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \right)^{-1}, \frac{a}{2\mu}, \frac{b}{2\mu} \right\},$$

782 and $h_i^0 = g_i^0 = \nabla f_i(x^0)$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP-PAGE), then $\mathbb{E}[f(x^T)] - f^* \leq$
 783 $(1 - \gamma\mu)^T \Delta_0$.

784 *Proof.* Let us fix constants $\nu, \rho \in [0, \infty)$ that we will define later. Considering Lemma 12, Lemma 8,
 785 and the law of total expectation, we obtain

$$\begin{aligned}
 & \mathbb{E}[f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
 & + \nu \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
 & \leq \mathbb{E}\left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2}\right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2\right] \\
 & + (1 - \gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
 & + \frac{10\gamma(2\omega+1)\omega}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2\right] \\
 & + \nu \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
 & \leq \mathbb{E}\left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2}\right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2\right] \\
 & + (1 - \gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
 & + \frac{10\gamma(2\omega+1)\omega}{np_a^2} \mathbb{E}\left[\left(2\hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B}\right) \|x^{t+1} - x^t\|^2 + \frac{2b^2}{p_{\text{page}}} \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right] \\
 & + \nu \mathbb{E}\left[\left(\frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} + \frac{(1 - p_{\text{page}})L_{\max}^2}{np_a B}\right) \|x^{t+1} - x^t\|^2\right. \\
 & \quad \left. + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{page}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}}\right)^2 + (1 - p_{\text{page}})\right) \|h^t - \nabla f(x^t)\|^2\right] \\
 & + \rho \mathbb{E}\left[\left(\frac{2(1 - p_a)\hat{L}^2}{p_a} + \frac{(1 - p_{\text{page}})L_{\max}^2}{p_a B}\right) \|x^{t+1} - x^t\|^2\right. \\
 & \quad \left. + \left(\frac{2(1 - p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}}\right)^2 + (1 - p_{\text{page}})\right) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right].
 \end{aligned}$$

786 After rearranging the terms, we get

$$\begin{aligned}
 & \mathbb{E}[f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
 & + \nu \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
 & \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
 & + (1 - \gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right]
 \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{10\gamma\omega(2\omega+1)}{np_a^2} \left(2\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \right. \\
& \quad \left. - \nu \left(\frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} + \frac{(1-p_{\text{page}})L_{\max}^2}{np_a B} \right) - \rho \left(\frac{2(1-p_a)\hat{L}^2}{p_a} + \frac{(1-p_{\text{page}})L_{\max}^2}{p_a B} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(\gamma + \nu \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{20b^2\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{2\nu(p_a - p_{aa})b^2}{np_a^2 p_{\text{page}}} \right. \\
& \quad \left. + \rho \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

Due to $b = \frac{p_{\text{page}} p_a}{2-p_a} \leq p_{\text{page}}$, one can show that $\left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \leq 1-b$. Thus, if we take $\nu = \frac{2\gamma}{b}$, then

$$\left(\gamma + \nu \left(p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \leq \gamma + \nu(1-b) = \left(1 - \frac{b}{2} \right) \nu,$$

787 therefore

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{10\gamma\omega(2\omega+1)}{np_a^2} \left(2\hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) \right. \\
& \quad \left. - \frac{2\gamma}{bn p_a} \left(2 \left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1-p_{\text{page}})L_{\max}^2}{B} \right) - \rho \left(\frac{2(1-p_a)\hat{L}^2}{p_a} + \frac{(1-p_{\text{page}})L_{\max}^2}{p_a B} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{b}{2} \right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{20b^2\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{4\gamma(p_a - p_{aa})b}{np_a^2 p_{\text{page}}} \right. \\
& \quad \left. + \rho \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

Next, with the choice of $b = \frac{p_{\text{page}} p_a}{2-p_a}$, we ensure that

$$\left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \leq 1-b.$$

If we take $\rho = \frac{40b^2\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}}$, then

$$\left(\frac{20b^2\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{4\gamma(p_a - p_{aa})b}{np_a^2 p_{\text{page}}} + \rho \left(\frac{2(1-p_a)b^2}{p_a p_{\text{page}}} + p_{\text{page}} \left(1 - \frac{b}{p_{\text{page}}} \right)^2 + (1-p_{\text{page}}) \right) \right) \leq \left(1 - \frac{b}{2} \right) \rho,$$

788 therefore

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1 - \gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{10\gamma\omega(2\omega+1)}{np_a^2} \left(2\hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \right. \\
& \quad \left. - \frac{2\gamma}{bnp_a} \left(2 \left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \right. \\
& \quad \left. - \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^3 p_{\text{page}}} + \frac{8\gamma \left(1 - \frac{p_{aa}}{p_a} \right)}{np_a^2 p_{\text{page}}} \right) \left(2(1 - p_a) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{b}{2} \right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2} \right) \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

Let us simplify the inequality. First, due to $b \geq \frac{p_{\text{page}} p_a}{2}$, we have

$$\frac{2\gamma}{bnp_a} \left(2 \left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \leq \frac{8\gamma}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right).$$

789 Second, due to $b \leq p_a p_{\text{page}}$ and $p_{aa} \leq p_a^2$, we get

$$\begin{aligned}
& \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^3 p_{\text{page}}} + \frac{8\gamma \left(1 - \frac{p_{aa}}{p_a} \right)}{np_a^2 p_{\text{page}}} \right) \left(2(1 - p_a) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \\
& \leq \left(\frac{40\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma \left(1 - \frac{p_{aa}}{p_a} \right)}{np_a^2 p_{\text{page}}} \right) \left(2 \left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \\
& \leq \frac{80\gamma\omega(2\omega+1)}{np_a^2} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \\
& \quad + \frac{16\gamma \left(1 - \frac{p_{aa}}{p_a} \right)}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \\
& \leq \frac{80\gamma\omega(2\omega+1)}{np_a^2} \left(\hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \\
& \quad + \frac{16\gamma}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right).
\end{aligned}$$

790 Combining all bounds together, we obtain the following inequality:

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2]
\end{aligned}$$

$$\begin{aligned}
& + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{100\gamma\omega(2\omega + 1)}{np_a^2} \left(\hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \right. \\
& \quad \left. - \frac{24\gamma}{np_a^2 p_{\text{page}}} \left(\left(1 - \frac{p_{aa}}{p_a}\right) \hat{L}^2 + \frac{(1 - p_{\text{page}})L_{\max}^2}{B} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

791 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

792 Note that $\gamma \leq \frac{b}{2\mu}$, thus $1 - \frac{b}{2} \leq 1 - \gamma\mu$ and

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{4\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + (1 - \gamma\mu) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + (1 - \gamma\mu) \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2 p_{\text{page}}} + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right].
\end{aligned}$$

793 It is left to apply Lemma 11 with

$$\begin{aligned}
\Psi^t & = \frac{2(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{4((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{2}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(\frac{40b\omega(2\omega + 1)}{np_a^2 p_{\text{page}}} + \frac{8(p_a - p_{aa})}{np_a^2 p_{\text{page}}} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right]
\end{aligned}$$

794 to conclude the proof. \square

795 **E.4.5 Proof for DASHA-PP-MVR under PL-condition**

Theorem 10. Suppose that Assumption 1, 2, 3, 7, 5, 6, 8 and 9 hold. Let us take $a = \frac{p_a}{2\omega+1}$,
 $b \in \left(0, \frac{p_a}{2-p_a}\right]$,

$$\gamma \leq \min \left\{ \left(L + \sqrt{\frac{200\omega(2\omega+1)}{np_a^2} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \widehat{L}^2 \right) + \frac{40}{np_a b} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}^2 \right)} \right)^{-1}, \frac{a}{2\mu}, \frac{b}{2\mu} \right\},$$

796 and $h_i^0 = g_i^0$ for all $i \in [n]$ in Algorithm 1 (DASHA-PP-MVR), then

$$\begin{aligned} & \mathbb{E} [f(x^T) - f^*] \\ & \leq (1-\gamma\mu)^T \left(\Delta_0 + \frac{2\gamma}{b} \|h^0 - \nabla f(x^0)\|^2 + \left(\frac{40\gamma b\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right) \\ & \quad + \frac{1}{\mu} \left(\frac{100b^2\omega(2\omega+1)}{p_a^2} + \frac{20b}{p_a} \right) \frac{\sigma^2}{nB}. \end{aligned}$$

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\ & \quad + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ & \quad + \frac{10\gamma(2\omega+1)\omega}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right]. \end{aligned}$$

797 *Proof.* Let us fix constants $\nu, \rho \in [0, \infty)$ that we will define later. Considering Lemma 12, Lemma 10,
 798 and the law of total expectation, we obtain

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & \quad + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\ & \quad + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ & \quad + \frac{10\gamma(2\omega+1)\omega}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|k_i^{t+1}\|^2 \right] \\ & \quad + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma \|h^t - \nabla f(x^t)\|^2 \right] \\ & \quad + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{10\gamma\omega(2\omega+1)}{np_a^2} \mathbb{E} \left[\frac{2b^2\sigma^2}{B} + \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\widehat{L}^2 \right) \|x^{t+1} - x^t\|^2 + 2b^2 \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \nu \mathbb{E} \left(\frac{2b^2\sigma^2}{np_a B} + \left(\frac{2(1-b)^2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + (1-b)^2 \|h^t - \nabla f(x^t)\|^2 \right) \\
& + \rho \mathbb{E} \left(\frac{2b^2\sigma^2}{p_a B} + \left(\frac{2(1-b)^2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \right. \\
& \quad \left. + \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right).
\end{aligned}$$

799 After rearranging the terms, we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{10\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\widehat{L}^2 \right) \right. \\
& \quad \left. - \nu \left(\frac{2(1-b)^2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) - \rho \left(\frac{2(1-b)^2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + (\gamma + \nu(1-b)^2) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{20b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{2\nu(p_a - p_{aa})b^2}{np_a^2} + \rho \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{20b^2\gamma\omega(2\omega+1)}{np_a^2} + \nu \frac{2b^2}{np_a} + \rho \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

800 By taking $\nu = \frac{2\gamma}{b}$, one can show that $(\gamma + \nu(1-b)^2) \leq (1 - \frac{b}{2})\nu$, and

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{10\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\widehat{L}^2 \right) \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{2\gamma}{b} \left(\frac{2(1-b)^2 L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa}) \hat{L}^2}{np_a^2} \right) - \rho \left(\frac{2(1-b)^2 L_\sigma^2}{p_a B} + \frac{2(1-p_a) \hat{L}^2}{p_a} \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{20b^2 \gamma \omega (2\omega + 1)}{np_a^2} + \frac{4\gamma (p_a - p_{aa}) b}{np_a^2} + \rho \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{20b^2 \gamma \omega (2\omega + 1)}{np_a^2} + \frac{4\gamma b}{np_a} + \rho \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

801 Note that $b \leq \frac{p_a}{2-p_a}$, thus

$$\begin{aligned}
& \left(\frac{20b^2 \gamma \omega (2\omega + 1)}{np_a^2} + \frac{4\gamma (p_a - p_{aa}) b}{np_a^2} + \rho \left(\frac{2(1-p_a)b^2}{p_a} + (1-b)^2 \right) \right) \\
& \leq \left(\frac{20b^2 \gamma \omega (2\omega + 1)}{np_a^2} + \frac{4\gamma (p_a - p_{aa}) b}{np_a^2} + \rho (1-b) \right).
\end{aligned}$$

802 And if we take $\rho = \frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a-p_{aa})}{np_a^2}$, then

$$\left(\frac{20b^2 \gamma \omega (2\omega + 1)}{np_a^2} + \frac{4\gamma (p_a - p_{aa}) b}{np_a^2} + \rho (1-b) \right) \leq \rho,$$

803 and

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a-p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{10\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 2\hat{L}^2 \right) \right. \\
& \quad \left. - \frac{2\gamma}{np_a b} \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 2 \left(1 - \frac{p_{aa}}{p_a}\right) \hat{L}^2 \right) \right. \\
& \quad \left. - \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^3} + \frac{8\gamma \left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2} \right) \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 2(1-p_a) \hat{L}^2 \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a-p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{20b^2 \gamma \omega (2\omega + 1)}{np_a^2} + \frac{4\gamma b}{np_a} + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a-p_{aa})}{np_a^2} \right) \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

804 Let us simplify the inequality. First, due to $b \leq p_a$ and $(1-p_a) \leq \left(1 - \frac{p_{aa}}{p_a}\right)$, we have

$$\left(\frac{40b\gamma\omega(2\omega+1)}{np_a^3} + \frac{2\gamma \left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2} \right) \left(\frac{2(1-b)^2 L_\sigma^2}{B} + 8(1-p_a) \hat{L}^2 \right)$$

$$\begin{aligned}
&= \frac{40b\gamma\omega(2\omega+1)}{np_a^3} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2(1-p_a)\widehat{L}^2 \right) \\
&\quad + \frac{8\gamma \left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2(1-p_a)\widehat{L}^2 \right) \\
&\leq \frac{40\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\widehat{L}^2 \right) \\
&\quad + \frac{8\gamma}{np_ab} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2 \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}^2 \right),
\end{aligned}$$

805 therefore

$$\begin{aligned}
&\mathbb{E}[f(x^{t+1})] + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\
&\quad + \frac{2\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\
&\leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
&\quad + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{50\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2\widehat{L}^2 \right) \right. \\
&\quad \left. - \frac{10\gamma}{np_ab} \left(\frac{2(1-b)^2L_\sigma^2}{B} + 2 \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}^2 \right) \right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
&\quad + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right] \\
&\quad + \left(\frac{20b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{4\gamma b}{np_a} + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B} \\
&\leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
&\quad + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{100\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{(1-b)^2L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
&\quad \left. - \frac{20\gamma}{np_ab} \left(\frac{(1-b)^2L_\sigma^2}{B} + \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}^2 \right) \right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
&\quad + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right] \\
&\quad + \left(\frac{20b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{4\gamma b}{np_a} + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{2b^2}{p_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

806 Also, we can simplify the last term:

$$\begin{aligned}
&\left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \frac{2b^2}{p_a} \\
&= \frac{80b^3\gamma\omega(2\omega+1)}{np_a^3} + \frac{16b^2\gamma \left(1 - \frac{p_{aa}}{p_a}\right)}{np_a^2}
\end{aligned}$$

$$\leq \frac{80b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{16b\gamma}{np_a},$$

807 thus

$$\begin{aligned} & \mathbb{E}[f(x^{t+1})] + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\ & + \frac{2\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\ & \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\ & + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\ & - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{100\gamma\omega(2\omega+1)}{np_a^2} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \widehat{L}^2\right)\right. \\ & \quad \left. - \frac{20\gamma}{np_a b} \left(\frac{(1-b)^2 L_\sigma^2}{B} + \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}^2\right)\right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\ & + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right] \\ & + \left(\frac{100b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{20\gamma b}{np_a}\right) \frac{\sigma^2}{B}. \end{aligned}$$

808 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned} & \mathbb{E}[f(x^{t+1})] + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\ & + \frac{2\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\ & \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\ & + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \\ & + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2\right] \\ & + \left(\frac{100b^2\gamma\omega(2\omega+1)}{np_a^2} + \frac{20\gamma b}{np_a}\right) \frac{\sigma^2}{B}. \end{aligned}$$

809 Note that $\gamma \leq \frac{b}{2\mu}$, thus $1 - \frac{b}{2} \leq 1 - \gamma\mu$ and

$$\begin{aligned} & \mathbb{E}[f(x^{t+1})] + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2\right] \\ & + \frac{2\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \left(\frac{40b\gamma\omega(2\omega+1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2}\right) \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2\right] \\ & \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\ & + (1-\gamma\mu) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + (1-\gamma\mu) \frac{4\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2\right] \end{aligned}$$

$$\begin{aligned}
& + (1 - \gamma\mu) \frac{2\gamma}{b} \mathbb{E} \left[\|h^t - \nabla f(x^t)\|^2 \right] + (1 - \gamma\mu) \left(\frac{40b\gamma\omega(2\omega + 1)}{np_a^2} + \frac{8\gamma(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{100b^2\gamma\omega(2\omega + 1)}{np_a^2} + \frac{20\gamma b}{np_a} \right) \frac{\sigma^2}{B}.
\end{aligned}$$

810 It is left to apply Lemma 11 with

$$\begin{aligned}
\Psi^t &= \frac{2(2\omega + 1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{4((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
&+ \frac{2}{b} \mathbb{E} \left[\|h^t - \nabla f(x^t)\|^2 \right] + \left(\frac{40b\omega(2\omega + 1)}{np_a^2} + \frac{8(p_a - p_{aa})}{np_a^2} \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right]
\end{aligned}$$

811 and $C = \left(\frac{100b^2\omega(2\omega+1)}{p_a^2} + \frac{20b}{p_a} \right) \frac{\sigma^2}{nB}$ to conclude the proof. \square

812 **Corollary 5.** Suppose that assumptions of Theorem 10 hold, batch size $B \leq \min \left\{ \frac{\sigma}{p_a\sqrt{\mu\varepsilon n}}, \frac{L_\sigma^2}{L^2} \right\}$,
813 we take RandK compressors with $K = \Theta \left(\frac{Bd\sqrt{\mu\varepsilon n}}{\sigma} \right)$. Then the communication complexity equals

$$\tilde{\mathcal{O}} \left(\frac{d\sigma}{p_a\sqrt{\mu\varepsilon n}} + \frac{dL_\sigma}{p_a\mu\sqrt{n}} \right),$$

814 and the expected number of stochastic gradient calculations per node equals

$$\tilde{\mathcal{O}} \left(\frac{\sigma^2}{p_a\mu n\varepsilon} + \frac{\sigma L_\sigma}{p_a n\mu^{3/2}\sqrt{\varepsilon}} \right).$$

815 *Proof.* In the view of Theorem 10, DASHA-PP have to run

$$\tilde{\mathcal{O}} \left(\frac{\omega + 1}{p_a} + \frac{\omega}{p_a} \sqrt{\frac{\sigma^2}{\mu n\varepsilon B}} + \frac{\sigma^2}{p_a\mu n\varepsilon B} + \frac{L}{\mu} + \frac{\omega}{p_a\mu\sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma}{p_a n\mu^{3/2}\sqrt{\varepsilon B}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right)$$

816 communication rounds in the stochastic settings to get ε -solution. Note that $K = \mathcal{O} \left(\frac{d}{p_a\sqrt{n}} \right)$.

817 Moreover, we can skip the initialization procedure and initialize h_i^0 and g_i^0 , for instance, with zeros
818 because the initialization error is under a logarithm. Considering Theorem 6, the communication
819 complexity equals

$$\begin{aligned}
& \tilde{\mathcal{O}} \left(K \frac{\omega + 1}{p_a} + K \frac{\omega}{p_a} \sqrt{\frac{\sigma^2}{\mu n\varepsilon B}} + K \frac{\sigma^2}{p_a\mu n\varepsilon B} + K \frac{L}{\mu} + K \frac{\omega}{p_a\mu\sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + K \frac{\sigma}{p_a n\mu^{3/2}\sqrt{\varepsilon B}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right) \\
&= \tilde{\mathcal{O}} \left(K \frac{\omega + 1}{p_a} + K \frac{\omega}{p_a} \sqrt{\frac{\sigma^2}{\mu n\varepsilon B}} + K \frac{\sigma^2}{p_a\mu n\varepsilon B} + K \frac{L}{\mu} + K \frac{\omega}{p_a\mu\sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + K \frac{\sigma L_\sigma}{p_a n\mu^{3/2}\sqrt{\varepsilon B}} \right) \\
&= \tilde{\mathcal{O}} \left(\frac{d}{p_a} + \frac{d}{p_a} \sqrt{\frac{\sigma^2}{\mu n\varepsilon B}} + \frac{K\sigma^2}{p_a\mu n\varepsilon B} + \frac{dL}{p_a\mu\sqrt{n}} + \frac{d}{p_a\mu\sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{K\sigma L_\sigma}{p_a n\mu^{3/2}\sqrt{\varepsilon B}} \right) \\
&= \tilde{\mathcal{O}} \left(\frac{d}{p_a} + \frac{d\sigma}{p_a\sqrt{\mu n\varepsilon B}} + \frac{d\sigma}{p_a\sqrt{\mu\varepsilon n}} + \frac{dL}{p_a\mu\sqrt{n}} + \frac{d}{p_a\mu\sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{dL_\sigma}{p_a\mu\sqrt{n}} \right) \\
&= \tilde{\mathcal{O}} \left(\frac{d\sigma}{p_a\sqrt{\mu\varepsilon n}} + \frac{dL_\sigma}{p_a\mu\sqrt{n}} \right).
\end{aligned}$$

820 The expected number of stochastic gradient calculations per node equals

$$\tilde{\mathcal{O}} \left(B \frac{\omega + 1}{p_a} + B \frac{\omega}{p_a} \sqrt{\frac{\sigma^2}{\mu n\varepsilon B}} + B \frac{\sigma^2}{p_a\mu n\varepsilon B} + B \frac{L}{\mu} + B \frac{\omega}{p_a\mu\sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + B \frac{\sigma}{p_a n\mu^{3/2}\sqrt{\varepsilon B}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) \right)$$

$$\begin{aligned}
&= \tilde{\mathcal{O}} \left(B \frac{\omega + 1}{p_a} + B \frac{\omega}{p_a} \sqrt{\frac{\sigma^2}{\mu n \varepsilon B}} + B \frac{\sigma^2}{p_a \mu n \varepsilon B} + B \frac{L}{\mu} + B \frac{\omega}{p_a \mu \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + B \frac{\sigma}{p_a n \mu^{3/2} \sqrt{\varepsilon B}} \left(\frac{L_\sigma}{\sqrt{B}} \right) \right) \\
&= \tilde{\mathcal{O}} \left(\frac{Bd}{K p_a} + \frac{Bd}{K p_a} \sqrt{\frac{\sigma^2}{\mu n \varepsilon B}} + \frac{\sigma^2}{p_a \mu n \varepsilon} + B \frac{L}{\mu} + \frac{Bd}{K p_a \mu \sqrt{n}} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma L_\sigma}{p_a n \mu^{3/2} \sqrt{\varepsilon}} \right) \\
&= \tilde{\mathcal{O}} \left(\frac{\sigma}{p_a \sqrt{\mu \varepsilon n}} + \frac{\sigma^2}{p_a \mu \varepsilon n \sqrt{B}} + \frac{\sigma^2}{p_a \mu n \varepsilon} + \frac{\sigma L}{p_a \mu^{3/2} \sqrt{\varepsilon} n} + \frac{\sigma}{p_a \mu^{3/2} \sqrt{\varepsilon} n} \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma L_\sigma}{p_a n \mu^{3/2} \sqrt{\varepsilon}} \right) \\
&= \tilde{\mathcal{O}} \left(\frac{\sigma^2}{p_a \mu n \varepsilon} + \frac{\sigma L_\sigma}{p_a n \mu^{3/2} \sqrt{\varepsilon}} \right).
\end{aligned}$$

821

□

822 F Description of DASHA-PP-SYNC-MVR

823 By analogy to (Tyurin and Richtárik, 2023), we provide a “synchronized” version of the algorithm.
 824 With a small probability, participating nodes calculate and send a mega batch without compression.
 825 This helps us to resolve the suboptimality of DASHA-PP-MVR w.r.t. ω . Note that this suboptimality is
 826 not a problem. We show in Corollary 4 that DASHA-PP-MVR can have the optimal oracle complexity
 827 and SOTA communication complexity with the particular choices of parameters of the compressors.

Algorithm 8 DASHA-PP-SYNC-MVR

```

1: Input: starting point  $x^0 \in \mathbb{R}^d$ , stepsize  $\gamma > 0$ , momentum  $a \in (0, 1]$ , momentum  $b \in$ 
   (0, 1], probability  $p_{\text{mega}} \in (0, 1]$ , batch size  $B'$  and  $B$ , probability  $p_a \in (0, 1]$  that a node is
   participating(a), number of iterations  $T \geq 1$ .
2: Initialize  $g_i^0, h_i^0$  on the nodes and  $g^0 = \frac{1}{n} \sum_{i=1}^n g_i^0$  on the server
3: for  $t = 0, 1, \dots, T - 1$  do
4:    $x^{t+1} = x^t - \gamma g^t$ 
5:    $c^{t+1} = \begin{cases} 1, & \text{with probability } p_{\text{mega}}, \\ 0, & \text{with probability } 1 - p_{\text{mega}} \end{cases}$ 
6:   Broadcast  $x^{t+1}, x^t$  to all participating(a) nodes
7:   for  $i = 1, \dots, n$  in parallel do
8:     if  $i^{\text{th}}$  node is participating(a) then
9:       if  $c^{t+1} = 1$  then
10:        Generate i.i.d. samples  $\{\xi_{ik}^{t+1}\}_{k=1}^{B'}$  of size  $B'$  from  $\mathcal{D}_i$ .
11:         $k_i^{t+1} = \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) - \frac{b}{p_{\text{mega}}} \left( h_i^t - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) \right)$ 
12:         $m_i^{t+1} = \frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t)$ 
13:       else
14:        Generate i.i.d. samples  $\{\xi_{ij}^{t+1}\}_{j=1}^B$  of size  $B$  from  $\mathcal{D}_i$ .
15:         $k_i^{t+1} = \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^t; \xi_{ij}^{t+1})$ 
16:         $m_i^{t+1} = C_i \left( \frac{1}{p_a} k_i^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right)$ 
17:       end if
18:        $h_i^{t+1} = h_i^t + \frac{1}{p_a} k_i^{t+1}$ 
19:        $g_i^{t+1} = g_i^t + m_i^{t+1}$ 
20:       Send  $m_i^{t+1}$  to the server
21:     else
22:        $h_i^{t+1} = h_i^t$ 
23:        $m_i^{t+1} = 0$ 
24:        $g_i^{t+1} = g_i^t$ 
25:     end if
26:   end for
27:    $g^{t+1} = g^t + \frac{1}{n} \sum_{i=1}^n m_i^{t+1}$ 
28: end for
29: Output:  $\hat{x}^T$  chosen uniformly at random from  $\{x^t\}_{k=0}^{T-1}$ 
   (a): For the formal description see Section 2.2.

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828 In the following theorem, we provide the convergence rate of DASHA-PP-SYNC-MVR.

Theorem 11. Suppose that Assumptions 1, 2, 3, 5, 6, 7 and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$,
 $b = \frac{p_{\text{mega}} p_a}{2-p_a}$, probability $p_{\text{mega}} \in (0, 1]$, batch size $B' \geq B \geq 1$

$$\gamma \leq \left(L + \sqrt{\frac{8(2\omega+1)\omega}{np_a^2} \left(\hat{L}^2 + \frac{L_\sigma^2}{B} \right) + \frac{16}{np_{\text{mega}} p_a^2} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{L_\sigma^2}{B} \right)} \right)^{-1},$$

829 and $h_i^0 = g_i^0$ for all $i \in [n]$ in Algorithm 8. Then

$$\begin{aligned} \mathbb{E} \left[\|\nabla f(\hat{x}^T)\|^2 \right] &\leq \frac{1}{T} \left[\frac{2\Delta_0}{\gamma} + \frac{4}{p_{\text{mega}} p_a} \|h^0 - \nabla f(x^0)\|^2 + \frac{4 \left(1 - \frac{p_{aa}}{p_a}\right)}{n p_{\text{mega}} p_a} \frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right] \\ &\quad + \frac{12\sigma^2}{nB'}. \end{aligned}$$

830 First, we introduce the expected density of compressors (Gorbunov et al., 2021; Tyurin and Richtárik, 2023).

832 **Definition 12.** The expected density of the compressor \mathcal{C}_i is $\zeta_{\mathcal{C}_i} := \sup_{x \in \mathbb{R}^d} \mathbb{E} [\|\mathcal{C}_i(x)\|_0]$, where
833 $\|x\|_0$ is the number of nonzero components of $x \in \mathbb{R}^d$. Let $\zeta_{\mathcal{C}} = \max_{i \in [n]} \zeta_{\mathcal{C}_i}$.

834 Note that $\zeta_{\mathcal{C}}$ is finite and $\zeta_{\mathcal{C}} \leq d$.

835 In the next corollary, we choose particular algorithm parameters to reveal the communication and
836 oracle complexity.

Corollary 6. Suppose that assumptions from Theorem 11 hold, probability $p_{\text{mega}} = \min \left\{ \frac{\zeta_{\mathcal{C}}}{d}, \frac{n\varepsilon B}{\sigma^2} \right\}$,
batch size $B' = \Theta \left(\frac{\sigma^2}{n\varepsilon} \right)$, and $h_i^0 = g_i^0 = \frac{1}{B_{\text{init}}} \sum_{k=1}^{B_{\text{init}}} \nabla f_i(x^0; \xi_{ik}^0)$ for all $i \in [n]$, initial batch size
 $B_{\text{init}} = \Theta \left(\frac{B}{p_{\text{mega}} \sqrt{p_a}} \right) = \Theta \left(\max \left\{ \frac{Bd}{\sqrt{p_a} \zeta_{\mathcal{C}}}, \frac{\sigma^2}{\sqrt{p_a} n\varepsilon} \right\} \right)$, then DASHA-PP-SYNC-MVR needs

$$T := \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \left(\frac{\omega}{p_a \sqrt{n}} + \sqrt{\frac{d}{p_a^2 \zeta_{\mathcal{C}} n}} \right) \left(\hat{L} + \frac{L_{\sigma}}{\sqrt{B}} \right) + \frac{\sigma}{p_a \sqrt{\varepsilon} n} \left(\frac{\hat{L}}{\sqrt{B}} + \frac{L_{\sigma}}{B} \right) \right] + \frac{\sigma^2}{\sqrt{p_a} n \varepsilon B} \right).$$

837 communication rounds to get an ε -solution, the expected communication complexity is equal to
838 $\mathcal{O}(d + \zeta_{\mathcal{C}} T)$, and the expected number of stochastic gradient calculations per node equals $\mathcal{O}(B_{\text{init}} +$
839 $BT)$, where $\zeta_{\mathcal{C}}$ is the expected density from Definition 12.

840 The main improvement of Corollary 6 over Corollary 3 is the size of the initial batch size B_{init} .
841 However, Corollary 4 reveals that we can avoid regimes when DASHA-PP-MVR is suboptimal.

842 We also provide a theorem under PL-condition (see Assumption 9).

Theorem 13. Suppose that Assumptions 1, 2, 3, 5, 6, 7, 8 and 9 hold. Let us take $a = \frac{p_a}{2\omega+1}$,
 $b = \frac{p_{\text{mega}} p_a}{2-p_a}$, probability $p_{\text{mega}} \in (0, 1]$, batch size $B' \geq B \geq 1$,

$$\gamma \leq \min \left\{ \left(L + \sqrt{\frac{16(2\omega+1)\omega}{np_a^2} \left(\frac{L_{\sigma}^2}{B} + \hat{L}^2 \right) + \left(\frac{48L_{\sigma}^2}{np_{\text{mega}} p_a^2 B} + \frac{24 \left(1 - \frac{p_{aa}}{p_a}\right) \hat{L}^2}{np_{\text{mega}} p_a^2} \right)} \right)^{-1}, \frac{a}{2\mu}, \frac{b}{2\mu} \right\},$$

843 and $h_i^0 = g_i^0$ for all $i \in [n]$ in Algorithm 8. Then

$$\begin{aligned} &\mathbb{E} [f(x^T) - f^*] \\ &\leq (1 - \gamma\mu)^T \left(\Delta_0 + \frac{2\gamma}{b} \|h^0 - \nabla f(x^0)\|^2 + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right) + \frac{20\sigma^2}{\mu n B'}. \end{aligned}$$

844 Let us provide bounds up to logarithmic factors and use $\tilde{\mathcal{O}}(\cdot)$ notation.

Corollary 7. Suppose that assumptions from Theorem 13 hold, probability $p_{\text{mega}} =$
 $\min \left\{ \frac{\zeta_{\mathcal{C}}}{d}, \frac{\mu n \varepsilon B}{\sigma^2} \right\}$, batch size $B' = \Theta \left(\frac{\sigma^2}{\mu n \varepsilon} \right)$ then DASHA-PP-SYNC-MVR needs

$$T := \tilde{\mathcal{O}} \left(\frac{\omega+1}{p_a} + \frac{d}{p_a \zeta_{\mathcal{C}}} + \frac{\sigma^2}{p_a \mu n \varepsilon B} + \frac{L}{\mu} + \frac{\omega}{p_a \mu \sqrt{n}} \left(\frac{L_{\sigma}}{\sqrt{B}} + \hat{L} \right) + \left(\frac{\sqrt{d}}{p_a \mu \sqrt{\zeta_{\mathcal{C}} n}} + \frac{\sigma}{p_a \mu^{3/2} \sqrt{\varepsilon B}} \right) \left(\frac{L_{\sigma}}{\sqrt{B}} + \hat{L} \right) \right).$$

communication rounds to get an ε -solution, the expected communication complexity is equal to $\tilde{\mathcal{O}}(\zeta_c T)$, and the expected number of stochastic gradient calculations per node equals $\tilde{\mathcal{O}}(BT)$, where ζ_c is the expected density from Definition 12.

The proof of this corollary almost repeats the proof of Corollary 6. Note that we can skip the initialization procedure and initialize h_i^0 and g_i^0 , for instance, with zeros because the initialization error is under a logarithm.

Let us assume that $\frac{d}{\zeta_c} = \Theta(\omega)$ (holds for the RandK compressor), then the convergence rate of DASHA-PP-SYNC-MVR is

$$\tilde{\mathcal{O}}\left(\frac{\omega+1}{p_a} + \frac{\sigma^2}{p_a \mu n \varepsilon B} + \frac{L}{\mu} + \frac{\omega}{p_a \mu \sqrt{n}} \left(\frac{L_\sigma}{\sqrt{B}} + \hat{L}\right) + \frac{\sigma}{p_a n \mu^{3/2} \sqrt{\varepsilon B}} \left(\frac{L_\sigma}{\sqrt{B}} + \hat{L}\right)\right). \quad (35)$$

Comparing (35) with the rate of DASHA-PP-MVR (32), one can see that DASHA-PP-SYNC-MVR improves the suboptimal term \mathcal{P}_2 from (32). However, Corollary 5 reveals that we can escape these suboptimal regimes by choosing the parameter K of RandK compressors in a particular way.

E.1 Proof for DASHA-PP-SYNC-MVR

In this section, we provide the proof of the convergence rate for DASHA-PP-SYNC-MVR. There are four different sources of randomness in Algorithm 8: the first one from random samples ξ_i^{t+1} , the second one from compressors $\{\mathcal{C}_i\}_{i=1}^n$, the third one from availability of nodes, and the fourth one from c^{t+1} . We define $\mathbb{E}_k[\cdot]$, $\mathbb{E}_c[\cdot]$, $\mathbb{E}_{p_a}[\cdot]$ and $\mathbb{E}_{p_{\text{mega}}}[\cdot]$ to be conditional expectations w.r.t. ξ_i^{t+1} , $\{\mathcal{C}_i\}_{i=1}^n$, availability, and c^{t+1} , accordingly, conditioned on all previous randomness. Moreover, we define $\mathbb{E}_{t+1}[\cdot]$ to be a conditional expectation w.r.t. all randomness in iteration $t+1$ conditioned on all previous randomness.

Let us denote

$$\begin{aligned} k_{i,1}^{t+1} &:= \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) - \frac{b}{p_{\text{mega}}} \left(h_i^t - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) \right), \\ k_{i,2}^{t+1} &:= \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^t; \xi_{ij}^{t+1}), \\ h_{i,1}^{t+1} &:= \begin{cases} h_i^t + \frac{1}{p_a} k_{i,1}^{t+1}, & i^{\text{th}} \text{ node is participating,} \\ h_i^t, & \text{otherwise,} \end{cases} \\ h_{i,2}^{t+1} &:= \begin{cases} h_i^t + \frac{1}{p_a} k_{i,2}^{t+1}, & i^{\text{th}} \text{ node is participating,} \\ h_i^t, & \text{otherwise,} \end{cases} \\ g_{i,1}^{t+1} &:= \begin{cases} g_i^t + \frac{1}{p_a} k_{i,1}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t), & i^{\text{th}} \text{ node is participating,} \\ g_i^t, & \text{otherwise,} \end{cases} \\ g_{i,2}^{t+1} &:= \begin{cases} g_i^t + \mathcal{C}_i \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right), & i^{\text{th}} \text{ node is participating,} \\ g_i^t, & \text{otherwise,} \end{cases} \end{aligned}$$

$h_1^{t+1} := \frac{1}{n} \sum_{i=1}^n h_{i,1}^{t+1}$, $h_2^{t+1} := \frac{1}{n} \sum_{i=1}^n h_{i,2}^{t+1}$, $g_1^{t+1} := \frac{1}{n} \sum_{i=1}^n g_{i,1}^{t+1}$, and $g_2^{t+1} := \frac{1}{n} \sum_{i=1}^n g_{i,2}^{t+1}$. Note, that

$$h^{t+1} = \begin{cases} h_1^{t+1}, & c^{t+1} = 1, \\ h_2^{t+1}, & c^{t+1} = 0, \end{cases}$$

and

$$g^{t+1} = \begin{cases} g_1^{t+1}, & c^{t+1} = 1, \\ g_2^{t+1}, & c^{t+1} = 0 \end{cases}$$

First, we will prove two lemmas.

869 **Lemma 13.** Suppose that Assumptions 3, 5, 7 and 8 hold and let us consider sequences $\{g_i^{t+1}\}_{i=1}^n$
 870 and $\{h_i^{t+1}\}_{i=1}^n$ from Algorithm 8, then

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \right] \\ & \leq \frac{2(1-p_{\text{mega}})\omega}{n^2 p_a} \sum_{i=1}^n \|k_{i,2}^{t+1}\|^2 + \left(\frac{(p_a - p_{aa})a^2}{n^2 p_a^2} + \frac{2(1-p_{\text{mega}})a^2\omega}{n^2 p_a} \right) \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \\ & \quad + (1-a)^2 \|g^t - h^t\|^2, \end{aligned}$$

871 and

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|g_i^{t+1} - h_i^{t+1}\|^2 \right] \right] \right] \\ & \leq \frac{2(1-p_{\text{mega}})\omega}{p_a} \|k_{i,2}^{t+1}\|^2 + \left(\frac{(1-p_a)a^2}{p_a} + \frac{2(1-p_{\text{mega}})a^2\omega}{p_a} \right) \|g_i^t - h_i^t\|^2 \\ & \quad + (1-a)^2 \|g_i^t - h_i^t\|^2, \quad \forall i \in [n]. \end{aligned}$$

872 *Proof.* First, we get the bound for $\mathbb{E}_{t+1} [\|g^{t+1} - h^{t+1}\|^2]$:

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \right] \\ & = p_{\text{mega}} \mathbb{E}_{p_a} \left[\|g_1^{t+1} - h_1^{t+1}\|^2 \right] + (1-p_{\text{mega}}) \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g_2^{t+1} - h_2^{t+1}\|^2 \right] \right]. \end{aligned}$$

873 Using

$$\mathbb{E}_{p_a} [g_{i,1}^{t+1} - h_{i,1}^{t+1}] = g_i^t + k_{i,1}^{t+1} - a(g_i^t - h_i^t) - h_i^t - k_{i,1}^{t+1} = (1-a)(g_i^t - h_i^t)$$

874 and

$$\mathbb{E}_{\mathcal{C}} [\mathbb{E}_{p_a} [g_{i,2}^{t+1} - h_{i,2}^{t+1}]] = g_i^t + k_{i,2}^{t+1} - a(g_i^t - h_i^t) - h_i^t - k_{i,2}^{t+1} = (1-a)(g_i^t - h_i^t),$$

875 we have

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \right] \\ & \stackrel{(17)}{=} p_{\text{mega}} \mathbb{E}_{p_a} \left[\|g_1^{t+1} - h_1^{t+1} - \mathbb{E}_{p_a} [g_1^{t+1} - h_1^{t+1}]\|^2 \right] \\ & \quad + (1-p_{\text{mega}}) \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\|g_2^{t+1} - h_2^{t+1} - \mathbb{E}_{p_a} [g_2^{t+1} - h_2^{t+1}]\|^2 \right] \right] \\ & \quad + (1-a)^2 \|g^t - h^t\|^2. \end{aligned}$$

876 We can use Lemma 1 two times with i) $r_i = g_i^t - h_i^t$ and $s_i = -a(g_i^t - h_i^t)$ and ii) $r_i = g_i^t - h_i^t$ and

877 $s_i = p_a \mathcal{C}_i \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - k_{i,2}^{t+1}$, to obtain

$$\begin{aligned} & \mathbb{E}_{\mathcal{C}} \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \right] \\ & \leq \frac{p_{\text{mega}} a^2 (p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \\ & \quad + (1-p_{\text{mega}}) \left(\frac{1}{n^2 p_a} \sum_{i=1}^n \mathbb{E}_{\mathcal{C}} \left[\left\| p_a \mathcal{C}_i \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - (k_{i,2}^{t+1} - a(g_i^t - h_i^t)) \right\|^2 \right] \right) \\ & \quad + (1-p_{\text{mega}}) \left(\frac{a^2 (p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right) \\ & \quad + (1-a)^2 \|g^t - h^t\|^2 \\ & = \frac{a^2 (p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \end{aligned}$$

$$\begin{aligned}
& + (1 - p_{\text{mega}}) \left(\frac{p_a}{n^2} \sum_{i=1}^n \mathbb{E}_C \left[\left\| \mathcal{C}_i \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) \right\|^2 \right] \right) \\
& + (1 - a)^2 \|g^t - h^t\|^2 \\
& \leq \frac{a^2 (p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \\
& + \frac{(1 - p_{\text{mega}}) p_a \omega}{n^2} \sum_{i=1}^n \left\| \frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right\|^2 \\
& + (1 - a)^2 \|g^t - h^t\|^2 \\
& = \frac{a^2 (p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \\
& + \frac{(1 - p_{\text{mega}}) \omega}{n^2 p_a} \sum_{i=1}^n \|k_{i,2}^{t+1} - a (g_i^t - h_i^t)\|^2 \\
& + (1 - a)^2 \|g^t - h^t\|^2.
\end{aligned}$$

878 In the last inequality, we use Assumption 7. Next, using (16), we have

$$\begin{aligned}
& \mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|g^{t+1} - h^{t+1}\|^2 \right] \right] \right] \\
& \leq \frac{2(1 - p_{\text{mega}}) \omega}{n^2 p_a} \sum_{i=1}^n \|k_{i,2}^{t+1}\|^2 + \left(\frac{(p_a - p_{aa}) a^2}{n^2 p_a^2} + \frac{2(1 - p_{\text{mega}}) \omega a^2}{n^2 p_a} \right) \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \\
& + (1 - a)^2 \|g^t - h^t\|^2.
\end{aligned}$$

879 The second inequality can be proved almost in the same way:

$$\begin{aligned}
& \mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|g_i^{t+1} - h_i^{t+1}\|^2 \right] \right] \right] \\
& = p_{\text{mega}} \mathbb{E}_{p_a} \left[\|g_{i,1}^{t+1} - h_{i,1}^{t+1}\|^2 \right] + (1 - p_{\text{mega}}) \mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\|g_{i,2}^{t+1} - h_{i,2}^{t+1}\|^2 \right] \right] \\
& \stackrel{(17)}{=} p_{\text{mega}} \mathbb{E}_{p_a} \left[\|g_{i,1}^{t+1} - h_{i,1}^{t+1} - (1 - a)(g_i^t - h_i^t)\|^2 \right] + (1 - p_{\text{mega}}) \mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\|g_{i,2}^{t+1} - h_{i,2}^{t+1}\|^2 \right] \right] \\
& + p_{\text{mega}} (1 - a)^2 \|g_i^t - h_i^t\|^2 \\
& = \frac{p_{\text{mega}} (1 - p_a) a^2}{p_a} \|g_i^t - h_i^t\|^2 + (1 - p_{\text{mega}}) \mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\|g_{i,2}^{t+1} - h_{i,2}^{t+1}\|^2 \right] \right] \\
& + p_{\text{mega}} (1 - a)^2 \|g_i^t - h_i^t\|^2 \\
& \stackrel{(17)}{=} \frac{p_{\text{mega}} (1 - p_a) a^2}{p_a} \|g_i^t - h_i^t\|^2 + (1 - p_{\text{mega}}) \mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\|g_{i,2}^{t+1} - h_{i,2}^{t+1} - (1 - a)(g_i^t - h_i^t)\|^2 \right] \right] \\
& + (1 - a)^2 \|g_i^t - h_i^t\|^2 \\
& = \frac{p_{\text{mega}} (1 - p_a) a^2}{p_a} \|g_i^t - h_i^t\|^2 \\
& + (1 - p_{\text{mega}}) p_a \mathbb{E}_C \left[\left\| g_i^t + \mathcal{C}_i \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - \left(h_i^t + \frac{1}{p_a} k_{i,2}^{t+1} \right) - (1 - a)(g_i^t - h_i^t) \right\|^2 \right] \\
& + (1 - p_{\text{mega}}) (1 - p_a) \|g_i^t - h_i^t - (1 - a)(g_i^t - h_i^t)\|^2 \\
& + (1 - a)^2 \|g_i^t - h_i^t\|^2 \\
& = \frac{p_{\text{mega}} (1 - p_a) a^2}{p_a} \|g_i^t - h_i^t\|^2 \\
& + (1 - p_{\text{mega}}) p_a \mathbb{E}_C \left[\left\| \mathcal{C}_i \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - \left(\frac{1}{p_a} k_{i,2}^{t+1} - a (g_i^t - h_i^t) \right) \right\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + (1 - p_{\text{mega}}) (1 - p_a) a^2 \|g_i^t - h_i^t\|^2 \\
& + (1 - a)^2 \|g_i^t - h_i^t\|^2 \\
& \stackrel{(17)}{=} \left(\frac{p_{\text{mega}}(1 - p_a)a^2}{p_a} + \frac{(1 - p_{\text{mega}})(1 - p_a)a^2}{p_a} \right) \|g_i^t - h_i^t\|^2 \\
& + (1 - p_{\text{mega}}) p_a \text{Ec} \left[\left\| \mathcal{C}_i \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) \right\|^2 \right] \\
& + (1 - a)^2 \|g_i^t - h_i^t\|^2 \\
& = \frac{(1 - p_a)a^2}{p_a} \|g_i^t - h_i^t\|^2 \\
& + (1 - p_{\text{mega}}) p_a \text{Ec} \left[\left\| \mathcal{C}_i \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) - \left(\frac{1}{p_a} k_{i,2}^{t+1} - \frac{a}{p_a} (g_i^t - h_i^t) \right) \right\|^2 \right] \\
& + (1 - a)^2 \|g_i^t - h_i^t\|^2 \\
& \leq \frac{(1 - p_a)a^2}{p_a} \|g_i^t - h_i^t\|^2 \\
& + \frac{(1 - p_{\text{mega}})\omega}{p_a} \|k_{i,2}^{t+1} - a(g_i^t - h_i^t)\|^2 \\
& + (1 - a)^2 \|g_i^t - h_i^t\|^2 \\
& \stackrel{(16)}{\leq} \frac{2(1 - p_{\text{mega}})\omega}{p_a} \|k_{i,2}^{t+1}\|^2 + \left(\frac{(1 - p_a)a^2}{p_a} + \frac{2(1 - p_{\text{mega}})a^2\omega}{p_a} \right) \|g_i^t - h_i^t\|^2 \\
& + (1 - a)^2 \|g_i^t - h_i^t\|^2.
\end{aligned}$$

880

□

881 **Lemma 14.** Suppose that Assumptions 3, 5, 6 and 8 hold and let us consider sequence $\{h_i^{t+1}\}_{i=1}^n$
882 from Algorithm 8, then

$$\begin{aligned}
& \text{E}_k \left[\text{E}_{p_a} \left[\text{E}_{p_{\text{mega}}} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right] \\
& \leq \frac{2b^2\sigma^2}{np_{\text{mega}}p_a B'} + \left(\frac{2p_{\text{mega}}L_\sigma^2}{np_a B'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + \frac{(1 - p_{\text{mega}})L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\hat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \\
& + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{mega}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h^t - \nabla f(x^t)\|^2,
\end{aligned}$$

883

$$\begin{aligned}
& \text{E}_k \left[\text{E}_{p_a} \left[\text{E}_{p_{\text{mega}}} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \\
& \leq \frac{2b^2\sigma^2}{p_a p_{\text{mega}} B'} + \left(\frac{2p_{\text{mega}}L_\sigma^2}{p_a B'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + \frac{(1 - p_{\text{mega}})L_\sigma^2}{p_a B} + \frac{2(1 - p_a)L_i^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \\
& + \frac{2(1 - p_a)b^2}{p_{\text{mega}}p_a} \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2, \quad \forall i \in [n],
\end{aligned}$$

884 and

$$\text{E}_k \left[\|k_{i,2}^{t+1}\|^2 \right] \leq \left(\frac{L_\sigma^2}{B} + L_i^2 \right) \|x^{t+1} - x^t\|^2, \quad \forall i \in [n],$$

885 *Proof.* First, we prove the bound for $\text{E}_k \left[\text{E}_{p_a} \left[\text{E}_{p_{\text{mega}}} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right]$. Using

$$\text{E}_k \left[\text{E}_{p_a} \left[h_{i,1}^{t+1} \right] \right]$$

$$\begin{aligned}
&= h_i^t + E_k \left[\frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) - \frac{b}{p_{\text{mega}}} \left(h_i^t - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) \right) \right] \\
&= h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{mega}}} (h_i^t - \nabla f_i(x^t))
\end{aligned}$$

886 and

$$\begin{aligned}
&E_k [E_{p_a} [h_{i,2}^{t+1}]] \\
&= h_i^t + E_k \left[\frac{1}{B} \sum_{j=1}^B \nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^t; \xi_{ij}^{t+1}) \right] \\
&= h_i^t + \nabla f_i(x^{t+1}) - \nabla f_i(x^t),
\end{aligned}$$

887 we have

$$\begin{aligned}
&E_k [E_{p_a} [E_{p_{\text{mega}}} [\|h^{t+1} - \nabla f(x^{t+1})\|^2]]] \\
&= p_{\text{mega}} E_k [E_{p_a} [\|h_1^{t+1} - \nabla f(x^{t+1})\|^2]] + (1 - p_{\text{mega}}) E_k [E_{p_a} [\|h_2^{t+1} - \nabla f(x^{t+1})\|^2]] \\
&\stackrel{(17)}{=} p_{\text{mega}} E_k [E_{p_a} [\|h_1^{t+1} - E_k [E_{p_a} [h_1^{t+1}]]\|^2]] + (1 - p_{\text{mega}}) E_k [E_{p_a} [\|h_2^{t+1} - E_k [E_{p_a} [h_2^{t+1}]]\|^2]] \\
&\quad + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h^t - \nabla f(x^t)\|^2.
\end{aligned}$$

888 We can use Lemma 1 two times with i) $r_i = h_i^t$ and $s_i = k_{i,1}^{t+1}$ and ii) $r_i = h_i^t$ and $s_i = k_{i,2}^{t+1}$, to
889 obtain

$$\begin{aligned}
&E_k [E_{p_a} [E_{p_{\text{mega}}} [\|h^{t+1} - \nabla f(x^{t+1})\|^2]]] \\
&\leq p_{\text{mega}} \left(\frac{1}{n^2 p_a} \sum_{i=1}^n E_k [\|k_{i,1}^{t+1} - E_k [k_{i,1}^{t+1}]\|^2] + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{mega}}} (h_i^t - \nabla f_i(x^t)) \right\|^2 \right) \\
&\quad + (1 - p_{\text{mega}}) \left(\frac{1}{n^2 p_a} \sum_{i=1}^n E_k [\|k_{i,2}^{t+1} - E_k [k_{i,2}^{t+1}]\|^2] + \frac{p_a - p_{aa}}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \right) \\
&\quad + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h^t - \nabla f(x^t)\|^2 \\
&\stackrel{(16)}{\leq} \frac{p_{\text{mega}}}{n^2 p_a} \sum_{i=1}^n E_k [\|k_{i,1}^{t+1} - E_k [k_{i,1}^{t+1}]\|^2] \\
&\quad + \frac{1 - p_{\text{mega}}}{n^2 p_a} \sum_{i=1}^n E_k [\|k_{i,2}^{t+1} - E_k [k_{i,2}^{t+1}]\|^2] \\
&\quad + \frac{2(p_a - p_{aa})}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\
&\quad + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{mega}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h^t - \nabla f(x^t)\|^2.
\end{aligned} \tag{36}$$

890 Let us consider $E_k [\|k_{i,1}^{t+1} - E_k [k_{i,1}^{t+1}]\|^2]$.

$$\begin{aligned}
&E_k [\|k_{i,1}^{t+1} - E_k [k_{i,1}^{t+1}]\|^2] \\
&= E_k \left[\left\| \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) - \frac{b}{p_{\text{mega}}} \left(h_i^t - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) \right) \right\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& - \left(\nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{mega}}} (h_i^t - \nabla f_i(x^t)) \right) \Big\| ^2 \Big] \\
& = \mathbb{E}_k \left[\left\| \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) + \frac{b}{p_{\text{mega}}} \left(\frac{1}{B'} \sum_{k=1}^{B'} \nabla f_i(x^t; \xi_{ik}^{t+1}) \right) \right. \right. \\
& \quad \left. \left. - \left(\nabla f_i(x^{t+1}) - \nabla f_i(x^t) + \frac{b}{p_{\text{mega}}} (\nabla f_i(x^t)) \right) \right\| ^2 \right] \\
& = \frac{1}{B'^2} \sum_{k=1}^{B'} \mathbb{E}_k \left[\left\| \frac{b}{p_{\text{mega}}} (\nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \nabla f_i(x^{t+1})) \right. \right. \\
& \quad \left. \left. + \left(1 - \frac{b}{p_{\text{mega}}} \right) (\nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \nabla f_i(x^t; \xi_{ik}^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t))) \right\| ^2 \right],
\end{aligned}$$

891 where we used independence of the mini-batch samples. Using (16), we get

$$\begin{aligned}
& \mathbb{E}_k \left[\left\| k_{i,1}^{t+1} - \mathbb{E}_k [k_{i,1}^{t+1}] \right\| ^2 \right] \\
& \leq \frac{2b^2}{B'^2 p_{\text{mega}}^2} \sum_{k=1}^{B'} \mathbb{E}_k \left[\left\| \nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \nabla f_i(x^{t+1}) \right\| ^2 \right] \\
& \quad + \frac{2}{B'^2} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 \sum_{k=1}^{B'} \mathbb{E}_k \left[\left\| \nabla f_i(x^{t+1}; \xi_{ik}^{t+1}) - \nabla f_i(x^t; \xi_{ik}^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\| ^2 \right].
\end{aligned}$$

892 Due to Assumptions 5 and 6, we have

$$\mathbb{E}_k \left[\left\| k_{i,1}^{t+1} - \mathbb{E}_k [k_{i,1}^{t+1}] \right\| ^2 \right] \leq \frac{2b^2 \sigma^2}{B' p_{\text{mega}}^2} + \frac{2L_\sigma^2}{B'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 \|x^{t+1} - x^t\|^2. \quad (37)$$

893 Next, we estimate the bound for $\mathbb{E}_k \left[\left\| k_{i,2}^{t+1} - \mathbb{E}_k [k_{i,2}^{t+1}] \right\| ^2 \right]$.

$$\begin{aligned}
& \mathbb{E}_k \left[\left\| k_{i,2}^{t+1} - \mathbb{E}_k [k_{i,2}^{t+1}] \right\| ^2 \right] \\
& = \mathbb{E}_k \left[\left\| \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \frac{1}{B} \sum_{j=1}^B \nabla f_i(x^t; \xi_{ij}^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\| ^2 \right] \\
& = \frac{1}{B^2} \sum_{j=1}^B \mathbb{E}_k \left[\left\| \nabla f_i(x^{t+1}; \xi_{ij}^{t+1}) - \nabla f_i(x^t; \xi_{ij}^{t+1}) - (\nabla f_i(x^{t+1}) - \nabla f_i(x^t)) \right\| ^2 \right].
\end{aligned}$$

894 Due to Assumptions 6, we have

$$\mathbb{E}_k \left[\left\| k_{i,2}^{t+1} - \mathbb{E}_k [k_{i,2}^{t+1}] \right\| ^2 \right] \leq \frac{L_\sigma^2}{B} \|x^{t+1} - x^t\|^2. \quad (38)$$

895 Plugging (37) and (38) into (36), we obtain

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right] \\
& \leq \frac{p_{\text{mega}}}{np_a} \left(\frac{2b^2 \sigma^2}{B' p_{\text{mega}}^2} + \frac{2L_\sigma^2}{B'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 \|x^{t+1} - x^t\|^2 \right) \\
& \quad + \frac{(1 - p_{\text{mega}}) L_\sigma^2}{np_a B} \|x^{t+1} - x^t\|^2 \\
& \quad + \frac{2(p_a - p_{\text{aa}})}{n^2 p_a^2} \sum_{i=1}^n \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2
\end{aligned}$$

$$+ \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{mega}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h^t - \nabla f(x^t)\|^2.$$

896 Using Assumption 3, we get

$$\begin{aligned} & \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] \right] \right] \\ & \leq \frac{2b^2 \sigma^2}{n p_{\text{mega}} p_a B'} + \left(\frac{2p_{\text{mega}} L_\sigma^2}{n p_a B'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + \frac{(1 - p_{\text{mega}}) L_\sigma^2}{n p_a B} + \frac{2(p_a - p_{aa}) \hat{L}^2}{n p_a^2} \right) \|x^{t+1} - x^t\|^2 \\ & \quad + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{mega}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h^t - \nabla f(x^t)\|^2. \end{aligned}$$

897 Using almost the same derivations, we can prove the second inequality:

$$\begin{aligned} & \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \\ & = p_{\text{mega}} \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_{i,1}^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] + (1 - p_{\text{mega}}) \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_{i,2}^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \\ & \stackrel{(17)}{=} p_{\text{mega}} \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_{i,1}^{t+1} - \mathbb{E}_k [h_{i,1}^{t+1}] \|^2 \right] \right] + (1 - p_{\text{mega}}) \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\|h_{i,2}^{t+1} - \mathbb{E}_k [h_{i,2}^{t+1}] \|^2 \right] \right] \\ & \quad + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \\ & = p_{\text{mega}} p_a \mathbb{E}_k \left[\left\| h_i^t + \frac{1}{p_a} k_{i,1}^{t+1} - (h_i^t + \mathbb{E}_k [k_{i,1}^{t+1}]) \right\|^2 \right] \\ & \quad + p_{\text{mega}} (1 - p_a) \|h_i^t - (h_i^t + \mathbb{E}_k [k_{i,1}^{t+1}])\|^2 \\ & \quad + (1 - p_{\text{mega}}) p_a \mathbb{E}_k \left[\left\| h_i^t + \frac{1}{p_a} k_{i,2}^{t+1} - (h_i^t + \mathbb{E}_k [k_{i,2}^{t+1}]) \right\|^2 \right] \\ & \quad + (1 - p_{\text{mega}}) (1 - p_a) \|h_i^t - (h_i^t + \mathbb{E}_k [k_{i,2}^{t+1}])\|^2 \\ & \quad + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \\ & = p_{\text{mega}} p_a \mathbb{E}_k \left[\left\| \frac{1}{p_a} k_{i,1}^{t+1} - \mathbb{E}_k [k_{i,1}^{t+1}] \right\|^2 \right] \\ & \quad + p_{\text{mega}} (1 - p_a) \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{mega}}} (h_i^t - \nabla f_i(x^t)) \right\|^2 \\ & \quad + (1 - p_{\text{mega}}) p_a \mathbb{E}_k \left[\left\| \frac{1}{p_a} k_{i,2}^{t+1} - \mathbb{E}_k [k_{i,2}^{t+1}] \right\|^2 \right] \\ & \quad + (1 - p_{\text{mega}}) (1 - p_a) \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\ & \quad + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \\ & \stackrel{(17)}{=} \frac{p_{\text{mega}}}{p_a} \mathbb{E}_k \left[\|k_{i,1}^{t+1} - \mathbb{E}_k [k_{i,1}^{t+1}]\|^2 \right] \\ & \quad + \frac{(1 - p_{\text{mega}})}{p_a} \mathbb{E}_k \left[\|k_{i,2}^{t+1} - \mathbb{E}_k [k_{i,2}^{t+1}]\|^2 \right] \\ & \quad + \frac{p_{\text{mega}} (1 - p_a)}{p_a} \left\| \nabla f_i(x^{t+1}) - \nabla f_i(x^t) - \frac{b}{p_{\text{mega}}} (h_i^t - \nabla f_i(x^t)) \right\|^2 \\ & \quad + \frac{(1 - p_{\text{mega}}) (1 - p_a)}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \end{aligned}$$

$$\begin{aligned}
& + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2 \\
& \stackrel{(16)}{\leq} \frac{p_{\text{mega}}}{p_a} \mathbb{E}_k \left[\|k_{i,1}^{t+1} - \mathbb{E}_k[k_{i,1}^{t+1}]\|^2 \right] \\
& + \frac{(1 - p_{\text{mega}})}{p_a} \mathbb{E}_k \left[\|k_{i,2}^{t+1} - \mathbb{E}_k[k_{i,2}^{t+1}]\|^2 \right] \\
& + \frac{2(1 - p_a)}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\
& + \frac{2(1 - p_a)b^2}{p_{\text{mega}}p_a} \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2.
\end{aligned}$$

898 Using (37) and (38), we get

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \\
& \leq \frac{2b^2\sigma^2}{p_ap_{\text{mega}}B'} + \frac{2p_{\text{mega}}L_\sigma^2}{p_aB'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 \|x^{t+1} - x^t\|^2 \\
& + \frac{(1 - p_{\text{mega}})L_\sigma^2}{p_aB} \|x^{t+1} - x^t\|^2 \\
& + \frac{2(1 - p_a)}{p_a} \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\
& + \frac{2(1 - p_a)b^2}{p_{\text{mega}}p_a} \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2.
\end{aligned}$$

899 Next, due to Assumption 3, we obtain

$$\begin{aligned}
& \mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \\
& \leq \frac{2b^2\sigma^2}{p_ap_{\text{mega}}B'} + \left(\frac{2p_{\text{mega}}L_\sigma^2}{p_aB'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + \frac{(1 - p_{\text{mega}})L_\sigma^2}{p_aB} + \frac{2(1 - p_a)L_i^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \\
& + \frac{2(1 - p_a)b^2}{p_{\text{mega}}p_a} \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1 - p_{\text{mega}}) \right) \|h_i^t - \nabla f_i(x^t)\|^2.
\end{aligned}$$

900 The third inequality can be proved with the help of (38) and Assumption 3.

$$\begin{aligned}
& \mathbb{E}_k \left[\|k_{i,2}^{t+1}\|^2 \right] \\
& \stackrel{(17)}{=} \mathbb{E}_k \left[\|k_{i,2}^{t+1} - \mathbb{E}_k[k_{i,2}^{t+1}]\|^2 \right] + \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\
& \leq \frac{L_\sigma^2}{B} \|x^{t+1} - x^t\|^2 + \|\nabla f_i(x^{t+1}) - \nabla f_i(x^t)\|^2 \\
& \leq \left(\frac{L_\sigma^2}{B} + L_i^2 \right) \|x^{t+1} - x^t\|^2.
\end{aligned}$$

901

□

Theorem 11. Suppose that Assumptions 1, 2, 3, 5, 6, 7 and 8 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{p_{\text{mega}}p_a}{2-p_a}$, probability $p_{\text{mega}} \in (0, 1]$, batch size $B' \geq B \geq 1$

$$\gamma \leq \left(L + \sqrt{\frac{8(2\omega+1)\omega}{np_a^2} \left(\hat{L}^2 + \frac{L_\sigma^2}{B} \right) + \frac{16}{np_{\text{mega}}p_a^2} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{L_\sigma^2}{B} \right)} \right)^{-1},$$

902 and $h_i^0 = g_i^0$ for all $i \in [n]$ in Algorithm 8. Then

$$\mathbb{E} \left[\|\nabla f(\hat{x}^T)\|^2 \right] \leq \frac{1}{T} \left[\frac{2\Delta_0}{\gamma} + \frac{4}{p_{\text{mega}}p_a} \|h^0 - \nabla f(x^0)\|^2 + \frac{4 \left(1 - \frac{p_{aa}}{p_a} \right)}{np_{\text{mega}}p_a} \frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right]$$

$$+ \frac{12\sigma^2}{nB'}.$$

903 *Proof.* Due to Lemma 2 and the update step from Line 4 in Algorithm 8, we have

$$\begin{aligned} & \mathbb{E}_{t+1} [f(x^{t+1})] \\ & \leq \mathbb{E}_{t+1} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \frac{\gamma}{2} \|g^t - \nabla f(x^t)\|^2 \right] \\ & = \mathbb{E}_{t+1} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \frac{\gamma}{2} \|g^t - h^t + h^t - \nabla f(x^t)\|^2 \right] \\ & \stackrel{(17)}{\leq} \mathbb{E}_{t+1} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma (\|g^t - h^t\|^2 + \|h^t - \nabla f(x^t)\|^2) \right]. \end{aligned}$$

904 Let us fix constants $\kappa, \eta, \nu, \rho \in [0, \infty)$ that we will define later. Considering Lemma 13, Lemma 14,
905 and the law of total expectation, we obtain

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \kappa \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & \quad + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma (\|g^t - h^t\|^2 + \|h^t - \nabla f(x^t)\|^2) \right] \\ & \quad + \kappa \mathbb{E} \left[\mathbb{E}_k \left[\mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} [\|g^{t+1} - h^{t+1}\|^2] \right] \right] \right] \right] \\ & \quad + \eta \mathbb{E} \left[\mathbb{E}_k \left[\mathbb{E}_C \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \right] \right] \right] \right] \\ & \quad + \nu \mathbb{E} \left[\mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] \right] \right] \right] \\ & \quad + \rho \mathbb{E} \left[\mathbb{E}_k \left[\mathbb{E}_{p_a} \left[\mathbb{E}_{p_{\text{mega}}} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \right] \right] \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma (\|g^t - h^t\|^2 + \|h^t - \nabla f(x^t)\|^2) \right] \\ & \quad + \kappa \mathbb{E} \left(\frac{2(1-p_{\text{mega}})\omega}{np_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \|x^{t+1} - x^t\|^2 \right. \\ & \quad \left. + \left(\frac{(p_a - p_{aa})a^2}{n^2 p_a^2} + \frac{2(1-p_{\text{mega}})a^2\omega}{n^2 p_a} \right) \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2 \right) \\ & \quad + \eta \mathbb{E} \left(\frac{2(1-p_{\text{mega}})\omega}{p_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \|x^{t+1} - x^t\|^2 \right. \\ & \quad \left. + \left(\frac{(1-p_a)a^2}{p_a} + \frac{2(1-p_{\text{mega}})a^2\omega}{p_a} \right) \frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2 \right) \\ & \quad + \nu \mathbb{E} \left(\frac{2b^2\sigma^2}{np_{\text{mega}}p_a B'} + \left(\frac{2p_{\text{mega}}L_\sigma^2}{np_a B'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + \frac{(1-p_{\text{mega}})L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \right. \\ & \quad \left. + \frac{2(p_a - p_{aa})b^2}{n^2 p_a^2 p_{\text{mega}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + (1-p_{\text{mega}}) \right) \|h^t - \nabla f(x^t)\|^2 \right) \\ & \quad + \rho \mathbb{E} \left(\frac{2b^2\sigma^2}{p_a p_{\text{mega}} B'} + \left(\frac{2p_{\text{mega}}L_\sigma^2}{p_a B'} \left(1 - \frac{b}{p_{\text{mega}}} \right)^2 + \frac{(1-p_{\text{mega}})L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \right) \end{aligned}$$

$$+ \frac{2(1-p_a)b^2}{np_{\text{mega}}p_a} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + \left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}}\right)^2 + (1-p_{\text{mega}}) \right) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \Bigg).$$

Let us simplify the last inequality. Since $B' \geq B$ and $b = \frac{p_{\text{mega}}p_a}{2-p_a} \leq p_{\text{mega}}$, we have $1 - p_{\text{mega}} \leq 1$,

$$\frac{2p_{\text{mega}}L_\sigma^2}{p_aB'} \left(1 - \frac{b}{p_{\text{mega}}}\right)^2 \leq \frac{2p_{\text{mega}}L_\sigma^2}{p_aB},$$

$$\left(p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}}\right)^2 + (1-p_{\text{mega}}) \right) \leq 1 - b,$$

and

$$\left(\frac{2(1-p_a)b^2}{p_{\text{mega}}p_a} + p_{\text{mega}} \left(1 - \frac{b}{p_{\text{mega}}}\right)^2 + (1-p_{\text{mega}}) \right) \leq 1 - b.$$

906 Thus

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \kappa \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & \leq \mathbb{E} \left[f(x^t) - \frac{\gamma}{2} \|\nabla f(x^t)\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2} \right) \|x^{t+1} - x^t\|^2 + \gamma (\|g^t - h^t\|^2 + \|h^t - \nabla f(x^t)\|^2) \right] \\ & + \kappa \mathbb{E} \left(\frac{2\omega}{np_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \|x^{t+1} - x^t\|^2 \right. \\ & \quad \left. + \frac{((2\omega + 1)p_a - p_{aa})a^2}{n^2p_a^2} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2 \right) \\ & + \eta \mathbb{E} \left(\frac{2\omega}{p_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \|x^{t+1} - x^t\|^2 \right. \\ & \quad \left. + \frac{(2\omega + 1 - p_a)a^2}{p_a} \frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 + (1-a)^2 \|g^t - h^t\|^2 \right) \\ & + \nu \mathbb{E} \left(\frac{2b^2\sigma^2}{np_{\text{mega}}p_aB'} + \left(\frac{2L_\sigma^2}{np_aB} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) \|x^{t+1} - x^t\|^2 \right. \\ & \quad \left. + \frac{2(p_a - p_{aa})b^2}{n^2p_a^2p_{\text{mega}}} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 + (1-b) \|h^t - \nabla f(x^t)\|^2 \right) \\ & + \rho \mathbb{E} \left(\frac{2b^2\sigma^2}{p_ap_{\text{mega}}B'} + \left(\frac{2L_\sigma^2}{p_aB} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \|x^{t+1} - x^t\|^2 \right. \\ & \quad \left. + (1-b) \frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right). \end{aligned}$$

907 After rearranging the terms, we get

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \kappa \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{2\kappa\omega}{np_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) - \frac{2\eta\omega}{p_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
&\quad \left. - \nu \left(\frac{2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
&\quad + (\gamma + \kappa(1-a)^2) \mathbb{E} [\|g^t - h^t\|^2] \\
&\quad + \left(\kappa \frac{((2\omega+1)p_a - p_{aa})a^2}{np_a^2} + \eta \left(\frac{(2\omega+1-p_a)a^2}{p_a} + (1-a)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
&\quad + (\gamma + \nu(1-b)) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
&\quad + \left(\nu \frac{2(p_a - p_{aa})b^2}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
&\quad + \left(\frac{2\nu b^2}{np_{\text{mega}} p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'}.
\end{aligned}$$

908 Let us take $\kappa = \frac{\gamma}{a}$, thus $\gamma + \kappa(1-a)^2 \leq \kappa$ and

$$\begin{aligned}
&\mathbb{E} [f(x^{t+1})] + \frac{\gamma}{a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
&\quad + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
&\leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{2\gamma\omega}{anp_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) - \frac{2\eta\omega}{p_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
&\quad \left. - \nu \left(\frac{2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
&\quad + \frac{\gamma}{a} \mathbb{E} [\|g^t - h^t\|^2] \\
&\quad + \left(\frac{\gamma((2\omega+1)p_a - p_{aa})a}{np_a^2} + \eta \left(\frac{(2\omega+1-p_a)a^2}{p_a} + (1-a)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
&\quad + (\gamma + \nu(1-b)) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
&\quad + \left(\nu \frac{2(p_a - p_{aa})b^2}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
&\quad + \left(\frac{2\nu b^2}{np_{\text{mega}} p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'}.
\end{aligned}$$

909 Next, since $a = \frac{p_a}{2\omega+1}$, we have $\left(\frac{(2\omega+1-p_a)a^2}{p_a} + (1-a)^2 \right) \leq 1-a$. We the choice $\eta =$

910 $\frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2}$, we guarantee $\frac{\gamma((2\omega+1)p_a - p_{aa})a}{np_a^2} + \eta \left(\frac{(2\omega+1-p_a)a^2}{p_a} + (1-a)^2 \right) \leq \eta$ and

$$\begin{aligned}
&\mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
&\quad + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{2\gamma(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) - \frac{2\gamma((2\omega+1)p_a - p_{aa})\omega}{np_a^3} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
&\quad \left. - \nu \left(\frac{2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
&\quad + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
&\quad + (\gamma + \nu(1-b)) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
&\quad + \left(\nu \frac{2(p_a - p_{aa})b^2}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
&\quad + \left(\frac{2\nu b^2}{np_{\text{mega}} p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'} \\
&\leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
&\quad \left. - \nu \left(\frac{2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
&\quad + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
&\quad + (\gamma + \nu(1-b)) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
&\quad + \left(\nu \frac{2(p_a - p_{aa})b^2}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
&\quad + \left(\frac{2\nu b^2}{np_{\text{mega}} p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'},
\end{aligned}$$

911 where simplified the term using $p_{aa} \geq 0$. Let us take $\nu = \frac{\gamma}{b}$ to obtain

$$\begin{aligned}
&\mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
&\quad + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
&\leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
&\quad - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
&\quad \left. - \left(\frac{2\gamma L_\sigma^2}{bn p_a B} + \frac{2\gamma(p_a - p_{aa})\widehat{L}^2}{bn p_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
&\quad + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
&\quad + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2]
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{2\gamma(p_a - p_{aa})b}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{2\gamma b}{np_{\text{mega}} p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'}.
\end{aligned}$$

912 Next, we take $\rho = \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}}$, thus

$$\begin{aligned}
& \mathbb{E}[f(x^{t+1})] + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma(2\omega + 1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \hat{L}^2 \right) \right. \\
& \quad \left. - \left(\frac{2\gamma L_\sigma^2}{bn p_a B} + \frac{2\gamma(p_a - p_{aa})\hat{L}^2}{bn p_a^2} \right) - \left(\frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \right) \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\hat{L}^2}{p_a} \right) \right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{2\gamma b}{np_{\text{mega}} p_a} + \frac{4\gamma(p_a - p_{aa})b^2}{np_a^3 p_{\text{mega}}} \right) \frac{\sigma^2}{B'}.
\end{aligned}$$

913 Since $\frac{p_{\text{mega}} p_a}{2} \leq b \leq p_{\text{mega}} p_a$ and $1 - p_a \leq 1 - \frac{p_{aa}}{p_a} \leq 1$, we get

$$\begin{aligned}
& \mathbb{E}[f(x^{t+1})] + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma(2\omega + 1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \hat{L}^2 \right) \right. \\
& \quad \left. - \left(\frac{4\gamma L_\sigma^2}{np_{\text{mega}} p_a^2 B} + \frac{4\gamma(p_a - p_{aa})\hat{L}^2}{np_{\text{mega}} p_a^3} \right) - \left(\frac{4\gamma L_\sigma^2}{np_{\text{mega}} p_a^2 B} + \frac{4\gamma(1-p_a)\hat{L}^2}{np_{\text{mega}} p_a^2} \right) \right) \mathbb{E}[\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma(2\omega + 1)}{p_a} \mathbb{E}[\|g^t - h^t\|^2] + \frac{\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E}[\|h^t - \nabla f(x^t)\|^2] + \frac{2\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \frac{6\gamma\sigma^2}{nB'} \\
& \leq \mathbb{E}[f(x^t)] - \frac{\gamma}{2} \mathbb{E}[\|\nabla f(x^t)\|^2]
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \hat{L}^2 \right) - \left(\frac{8\gamma L_\sigma^2}{np_{\text{mega}} p_a^2 B} + \frac{8\gamma \left(1 - \frac{p_{\text{aa}}}{p_a}\right) \hat{L}^2}{np_{\text{mega}} p_a^2} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{\text{aa}})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \frac{2\gamma(p_a - p_{\text{aa}})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \frac{6\gamma\sigma^2}{nB'}.
\end{aligned}$$

914 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{\gamma((2\omega+1)p_a - p_{\text{aa}})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \frac{2\gamma(p_a - p_{\text{aa}})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \frac{\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{\gamma((2\omega+1)p_a - p_{\text{aa}})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \frac{\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \frac{2\gamma(p_a - p_{\text{aa}})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \frac{6\gamma\sigma^2}{nB'}.
\end{aligned}$$

915 It is left to apply Lemma 3 with

$$\begin{aligned}
\Psi^t &= \frac{(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \frac{((2\omega+1)p_a - p_{\text{aa}})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
&+ \frac{1}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \frac{2 \left(1 - \frac{p_{\text{aa}}}{p_a}\right)}{np_a p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right]
\end{aligned}$$

916 and $C = \frac{6\sigma^2}{nB'}$ to conclude the proof. \square

Corollary 6. Suppose that assumptions from Theorem 11 hold, probability $p_{\text{mega}} = \min \left\{ \frac{\zeta_c}{d}, \frac{n\varepsilon B}{\sigma^2} \right\}$, batch size $B' = \Theta \left(\frac{\sigma^2}{n\varepsilon} \right)$, and $h_i^0 = g_i^0 = \frac{1}{B_{\text{init}}} \sum_{k=1}^{B_{\text{init}}} \nabla f_i(x^0; \xi_{ik}^0)$ for all $i \in [n]$, initial batch size $B_{\text{init}} = \Theta \left(\frac{B}{p_{\text{mega}} \sqrt{p_a}} \right) = \Theta \left(\max \left\{ \frac{Bd}{\sqrt{p_a} \zeta_c}, \frac{\sigma^2}{\sqrt{p_a} n \varepsilon} \right\} \right)$, then DASHA-PP-SYNC-MVR needs

$$T := \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \left(\frac{\omega}{p_a \sqrt{n}} + \sqrt{\frac{d}{p_a^2 \zeta_c n}} \right) \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma}{p_a \sqrt{\varepsilon} n} \left(\frac{\hat{L}}{\sqrt{B}} + \frac{L_\sigma}{B} \right) \right] + \frac{\sigma^2}{\sqrt{p_a} n \varepsilon B} \right).$$

917 communication rounds to get an ε -solution, the expected communication complexity is equal to
918 $\mathcal{O}(d + \zeta_c T)$, and the expected number of stochastic gradient calculations per node equals $\mathcal{O}(B_{\text{init}} +$
919 $BT)$, where ζ_c is the expected density from Definition 12.

920 *Proof.* Due to the choice of B' , we have

$$\begin{aligned} \mathbb{E} \left[\|\nabla f(\hat{x}^T)\|^2 \right] &\leq \frac{1}{T} \left[2\Delta_0 \left(L + \sqrt{\frac{8(2\omega+1)\omega}{np_a^2} \left(\hat{L}^2 + \frac{L_\sigma^2}{B} \right) + \frac{16}{np_{\text{mega}}p_a^2} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{L_\sigma^2}{B} \right)} \right) \right. \\ &\quad \left. + \frac{4}{p_{\text{mega}}p_a} \|h^0 - \nabla f(x^0)\|^2 + \frac{4 \left(1 - \frac{p_{aa}}{p_a} \right)}{np_{\text{mega}}p_a} \frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right] \\ &\quad + \frac{2\varepsilon}{3}. \end{aligned}$$

921 Using

$$\mathbb{E} \left[\|h^0 - \nabla f(x^0)\|^2 \right] = \mathbb{E} \left[\left\| \frac{1}{n} \sum_{i=1}^n \frac{1}{B_{\text{init}}} \sum_{k=1}^{B_{\text{init}}} \nabla f_i(x^0; \xi_{ik}^0) - \nabla f(x^0) \right\|^2 \right] \leq \frac{\sigma^2}{nB_{\text{init}}}$$

922 and

$$\frac{1}{n^2} \sum_{i=1}^n \mathbb{E} \left[\|h_i^0 - \nabla f_i(x^0)\|^2 \right] = \frac{1}{n^2} \sum_{i=1}^n \mathbb{E} \left[\left\| \frac{1}{B_{\text{init}}} \sum_{k=1}^{B_{\text{init}}} \nabla f_i(x^0; \xi_{ik}^0) - \nabla f_i(x^0) \right\|^2 \right] \leq \frac{\sigma^2}{nB_{\text{init}}},$$

923 we have

$$\begin{aligned} \mathbb{E} \left[\|\nabla f(\hat{x}^T)\|^2 \right] &\leq \frac{1}{T} \left[2\Delta_0 \left(L + \sqrt{\frac{8(2\omega+1)\omega}{np_a^2} \left(\hat{L}^2 + \frac{L_\sigma^2}{B} \right) + \frac{16}{np_{\text{mega}}p_a^2} \left(\left(1 - \frac{p_{aa}}{p_a} \right) \hat{L}^2 + \frac{L_\sigma^2}{B} \right)} \right) \right. \\ &\quad \left. + \frac{8\sigma^2}{np_{\text{mega}}p_a B_{\text{init}}} \right] \\ &\quad + \frac{2\varepsilon}{3}. \end{aligned}$$

924 Therefore, we can take the following T to get ε -solution.

$$T = \mathcal{O} \left(\frac{1}{\varepsilon} \left[\Delta_0 \left(L + \sqrt{\frac{\omega^2}{np_a^2} \left(\hat{L}^2 + \frac{L_\sigma^2}{B} \right) + \frac{1}{np_{\text{mega}}p_a^2} \left(\hat{L}^2 + \frac{L_\sigma^2}{B} \right)} \right) + \frac{\sigma^2}{np_{\text{mega}}p_a B_{\text{init}}} \right] \right)$$

925 Considering the choice of p_{mega} and B_{init} , we obtain

$$\begin{aligned} T &= \mathcal{O} \left(\frac{1}{\varepsilon} \left[\Delta_0 \left(L + \left(\frac{\omega}{p_a \sqrt{n}} + \sqrt{\frac{d}{p_a^2 \zeta_C n}} \right) \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma}{p_a \sqrt{\varepsilon n}} \left(\frac{\hat{L}}{\sqrt{B}} + \frac{L_\sigma}{B} \right) \right) + \frac{\sigma^2}{np_{\text{mega}}p_a B_{\text{init}}} \right] \right) \\ &= \mathcal{O} \left(\frac{\Delta_0}{\varepsilon} \left[L + \left(\frac{\omega}{p_a \sqrt{n}} + \sqrt{\frac{d}{p_a^2 \zeta_C n}} \right) \left(\hat{L} + \frac{L_\sigma}{\sqrt{B}} \right) + \frac{\sigma}{p_a \sqrt{\varepsilon n}} \left(\frac{\hat{L}}{\sqrt{B}} + \frac{L_\sigma}{B} \right) \right] + \frac{\sigma^2}{\sqrt{p_a} n \varepsilon B} \right). \end{aligned}$$

926 The expected communication complexity equals $\mathcal{O}(d + p_{\text{mega}}d + (1 - p_{\text{mega}})\zeta_C) =$
 927 $\mathcal{O}(d + \zeta_C)$ and the expected number of stochastic gradient calculations per node equals
 928 $\mathcal{O}(B_{\text{init}} + p_{\text{mega}}B' + (1 - p_{\text{mega}})B) = \mathcal{O}(B_{\text{init}} + B)$. \square

Theorem 13. Suppose that Assumptions 1, 2, 3, 5, 6, 7, 8 and 9 hold. Let us take $a = \frac{p_a}{2\omega+1}$, $b = \frac{p_{\text{mega}}p_a}{2-p_a}$, probability $p_{\text{mega}} \in (0, 1]$, batch size $B' \geq B \geq 1$,

$$\gamma \leq \min \left\{ \left(L + \sqrt{\frac{16(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) + \left(\frac{48L_\sigma^2}{np_{\text{mega}}p_a^2B} + \frac{24 \left(1 - \frac{p_{\text{aa}}}{p_a} \right) \widehat{L}^2}{np_{\text{mega}}p_a^2} \right)} \right)^{-1}, \frac{a}{2\mu}, \frac{b}{2\mu} \right\},$$

929 and $h_i^0 = g_i^0$ for all $i \in [n]$ in Algorithm 8. Then

$$\begin{aligned} & \mathbb{E} [f(x^T) - f^*] \\ & \leq (1 - \gamma\mu)^T \left(\Delta_0 + \frac{2\gamma}{b} \|h^0 - \nabla f(x^0)\|^2 + \frac{8\gamma(p_a - p_{\text{aa}})}{np_a^2 p_{\text{mega}}} \frac{1}{n} \sum_{i=1}^n \|h_i^0 - \nabla f_i(x^0)\|^2 \right) + \frac{20\sigma^2}{\mu n B'}. \end{aligned}$$

930 *Proof.* Let us fix constants $\kappa, \eta, \nu, \rho \in [0, \infty)$ that we will define later. As in the proof of Theorem 11,
931 we can get

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \kappa \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\ & - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{2\kappa\omega}{np_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) - \frac{2\eta\omega}{p_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\ & \quad \left. - \nu \left(\frac{2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{\text{aa}})\widehat{L}^2}{np_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\ & + (\gamma + \kappa(1-a)^2) \mathbb{E} [\|g^t - h^t\|^2] \\ & + \left(\kappa \frac{((2\omega+1)p_a - p_{\text{aa}})a^2}{np_a^2} + \eta \left(\frac{(2\omega+1-p_a)a^2}{p_a} + (1-a)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ & + (\gamma + \nu(1-b)) \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\ & + \left(\nu \frac{2(p_a - p_{\text{aa}})b^2}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\ & + \left(\frac{2\nu b^2}{np_{\text{mega}}p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'}. \end{aligned}$$

932 Let us take $\kappa = \frac{2\gamma}{a}$, thus $\gamma + \kappa(1-a)^2 \leq (1 - \frac{a}{2})\kappa$ and

$$\begin{aligned} & \mathbb{E} [f(x^{t+1})] + \frac{2\gamma}{a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \eta \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\ & + \nu \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\ & \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\ & - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{4\gamma\omega}{anp_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) - \frac{2\eta\omega}{p_a} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \end{aligned}$$

$$\begin{aligned}
& -\nu \left(\frac{2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \mathbb{E} \left[\|x^{t+1} - x^t\|^2 \right] \\
& + \left(1 - \frac{a}{2} \right) \frac{2\gamma}{a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] \\
& + \left(\frac{2\gamma((2\omega+1)p_a - p_{aa})a}{np_a^2} + \eta \left(\frac{(2\omega+1-p_a)a^2}{p_a} + (1-a)^2 \right) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + (\gamma + \nu(1-b)) \mathbb{E} \left[\|h^t - \nabla f(x^t)\|^2 \right] \\
& + \left(\nu \frac{2(p_a - p_{aa})b^2}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{2\nu b^2}{np_{\text{mega}} p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'}.
\end{aligned}$$

933 Next, since $a = \frac{p_a}{2\omega+1}$, we have $\left(\frac{(2\omega+1-p_a)a^2}{p_a} + (1-a)^2 \right) \leq 1-a$. We the choice $\eta =$
934 $\frac{2\gamma((2\omega+1)p_a - p_{aa})}{np_a^2}$, we guarantee $\frac{\gamma((2\omega+1)p_a - p_{aa})a}{np_a^2} + \eta \left(\frac{(2\omega+1-p_a)a^2}{p_a} + (1-a)^2 \right) \leq (1-\frac{a}{2})\eta$ and

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^{t+1} - h^{t+1}\|^2 \right] + \frac{2\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \nu \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} \left[\|\nabla f(x^t)\|^2 \right] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{8\gamma(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
& \quad \left. - \nu \left(\frac{2L_\sigma^2}{np_a B} + \frac{2(p_a - p_{aa})\widehat{L}^2}{np_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} \left[\|x^{t+1} - x^t\|^2 \right] \\
& + \left(1 - \frac{a}{2} \right) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] \\
& + \left(1 - \frac{a}{2} \right) \frac{2\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + (\gamma + \nu(1-b)) \mathbb{E} \left[\|h^t - \nabla f(x^t)\|^2 \right] \\
& + \left(\nu \frac{2(p_a - p_{aa})b^2}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{2\nu b^2}{np_{\text{mega}} p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'},
\end{aligned}$$

935 where simplified the term using $p_{aa} \geq 0$. Let us take $\nu = \frac{2\gamma}{b}$ to obtain

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} \left[\|g^{t+1} - h^{t+1}\|^2 \right] + \frac{2\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} \left[\|h^{t+1} - \nabla f(x^{t+1})\|^2 \right] + \rho \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} \left[\|\nabla f(x^t)\|^2 \right]
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{8\gamma(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
& \quad \left. - \left(\frac{4\gamma L_\sigma^2}{bn p_a B} + \frac{4\gamma(p_a - p_{aa})\widehat{L}^2}{bn p_a^2} \right) - \rho \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{a}{2} \right) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \left(1 - \frac{a}{2} \right) \frac{2\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(1 - \frac{b}{2} \right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] \\
& + \left(\frac{4\gamma(p_a - p_{aa})b}{np_a^2 p_{\text{mega}}} + \rho(1-b) \right) \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{4\gamma b}{np_{\text{mega}} p_a} + \frac{2\rho b^2}{p_a p_{\text{mega}}} \right) \frac{\sigma^2}{B'},
\end{aligned}$$

936 Next, we take $\rho = \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}}$, thus

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{2\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{8\gamma(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
& \quad \left. - \left(\frac{4\gamma L_\sigma^2}{bn p_a B} + \frac{4\gamma(p_a - p_{aa})\widehat{L}^2}{bn p_a^2} \right) - \left(\frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \right) \left(\frac{2L_\sigma^2}{p_a B} + \frac{2(1-p_a)\widehat{L}^2}{p_a} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{a}{2} \right) \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \left(1 - \frac{a}{2} \right) \frac{2\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(1 - \frac{b}{2} \right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2} \right) \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \left(\frac{4\gamma b}{np_{\text{mega}} p_a} + \frac{16\gamma(p_a - p_{aa})b^2}{np_a^3 p_{\text{mega}}^2} \right) \frac{\sigma^2}{B'},
\end{aligned}$$

937 Since $\frac{p_{\text{mega}} p_a}{2} \leq b \leq p_{\text{mega}} p_a$ and $1 - p_a \leq 1 - \frac{p_{aa}}{p_a} \leq 1$, we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega+1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{2\gamma((2\omega+1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{8\gamma(2\omega+1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) \right. \\
& \quad \left. - \left(\frac{8\gamma L_\sigma^2}{np_{\text{mega}} p_a^2 B} + \frac{8\gamma(p_a - p_{aa})\widehat{L}^2}{np_{\text{mega}} p_a^3} \right) - \left(\frac{16\gamma L_\sigma^2}{np_{\text{mega}} p_a^2 B} + \frac{16\gamma(1-p_a)\widehat{L}^2}{np_{\text{mega}} p_a^2} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2]
\end{aligned}$$

$$\begin{aligned}
& + \left(1 - \frac{a}{2}\right) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \left(1 - \frac{a}{2}\right) \frac{2\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \frac{20\gamma\sigma^2}{nB'} \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& - \left(\frac{1}{2\gamma} - \frac{L}{2} - \frac{8\gamma(2\omega + 1)\omega}{np_a^2} \left(\frac{L_\sigma^2}{B} + \widehat{L}^2 \right) - \left(\frac{24\gamma L_\sigma^2}{np_{\text{mega}} p_a^2 B} + \frac{24\gamma \left(1 - \frac{p_{aa}}{p_a}\right) \widehat{L}^2}{np_{\text{mega}} p_a^2} \right) \right) \mathbb{E} [\|x^{t+1} - x^t\|^2] \\
& + \left(1 - \frac{a}{2}\right) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \left(1 - \frac{a}{2}\right) \frac{2\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \frac{20\gamma\sigma^2}{nB'}.
\end{aligned}$$

938 Using Lemma 4 and the assumption about γ , we get

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{2\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + \left(1 - \frac{a}{2}\right) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + \left(1 - \frac{a}{2}\right) \frac{2\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + \left(1 - \frac{b}{2}\right) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + \left(1 - \frac{b}{2}\right) \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \frac{20\gamma\sigma^2}{nB'}.
\end{aligned}$$

939 Due to $\gamma \leq \frac{a}{2\mu}$ and $\gamma \leq \frac{b}{2\mu}$, we have

$$\begin{aligned}
& \mathbb{E} [f(x^{t+1})] + \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^{t+1} - h^{t+1}\|^2] + \frac{2\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^{t+1} - h_i^{t+1}\|^2 \right] \\
& + \frac{2\gamma}{b} \mathbb{E} [\|h^{t+1} - \nabla f(x^{t+1})\|^2] + \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^{t+1} - \nabla f_i(x^{t+1})\|^2 \right] \\
& \leq \mathbb{E} [f(x^t)] - \frac{\gamma}{2} \mathbb{E} [\|\nabla f(x^t)\|^2] \\
& + (1 - \gamma\mu) \frac{2\gamma(2\omega + 1)}{p_a} \mathbb{E} [\|g^t - h^t\|^2] + (1 - \gamma\mu) \frac{2\gamma((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\
& + (1 - \gamma\mu) \frac{2\gamma}{b} \mathbb{E} [\|h^t - \nabla f(x^t)\|^2] + (1 - \gamma\mu) \frac{8\gamma(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right] \\
& + \frac{20\gamma\sigma^2}{nB'}.
\end{aligned}$$

940 It is left to apply Lemma 11 with

$$\begin{aligned}\Psi^t &= \frac{2(2\omega + 1)}{p_a} \mathbb{E} \left[\|g^t - h^t\|^2 \right] + \frac{2((2\omega + 1)p_a - p_{aa})}{np_a^2} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|g_i^t - h_i^t\|^2 \right] \\ &+ \frac{2}{b} \mathbb{E} \left[\|h^t - \nabla f(x^t)\|^2 \right] + \frac{8(p_a - p_{aa})}{np_a^2 p_{\text{mega}}} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|h_i^t - \nabla f_i(x^t)\|^2 \right]\end{aligned}$$

941 and $C = \frac{20\sigma^2}{nB'}$ to conclude the proof. □