Physics Enrichment Program HKUST
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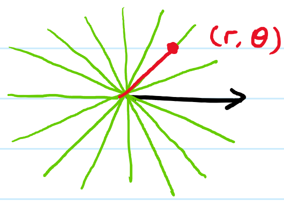
2D coordinates

Rectangular (Cartesian) Coordinate

Polar Coordinate



From (x,y) to (r, θ)

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

From (r, θ) to (x,y)

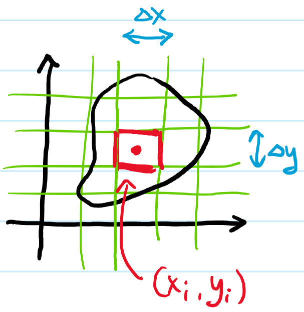
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

What about double integral on an area represented by polar coor.?

$$\iint f(x,y)\,dx\,dy \iff \iint f(r,\theta)\,\underline{r}\,dr\,d\theta$$

An extra r

In x/y coor.

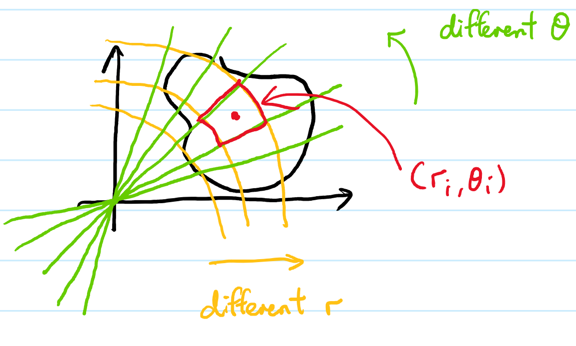


$$\iint f(x,y)\,dx\,dy \sim \sum_i \underline{f(x_i, y_i)}\,\underline{\Delta x \Delta y}$$

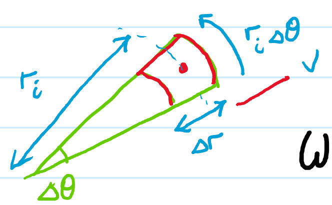
"Weight" assign to the point (x_i, y_i)

Every grid are of size $\Delta x \Delta y$

In polar coor.



Each grid's area depends on its r coordinate



When $\Delta\theta$ very small,

area $\simeq$ height x width

$\simeq \underline{\Delta r \times r_i\Delta\theta}$

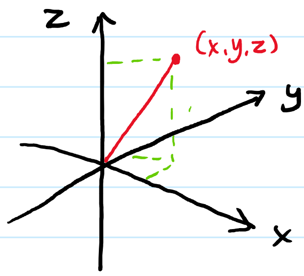
$\Rightarrow$ Integral becomes $\sum f(r_i, \theta_i) \cdot r_i \cdot \Delta r \Delta \theta$

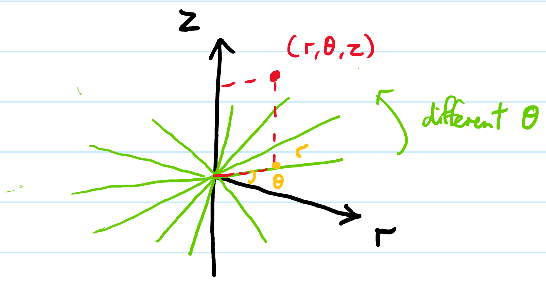
$$= \iint f(r, \theta)\, r\, dr\, d\theta$$

3D coordinates

Other than retangular coor, we also have $\begin{cases} \text{Cylindrical Coor.} \\ \text{Spherical Coor.} \end{cases}$

① Cylindrical Coordinate = Polar coordinate + z axis





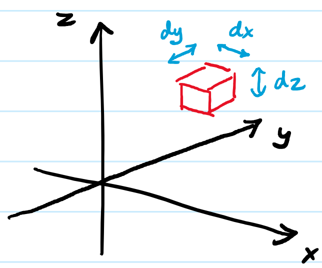
From (x,y,z) to (r,θ,z)

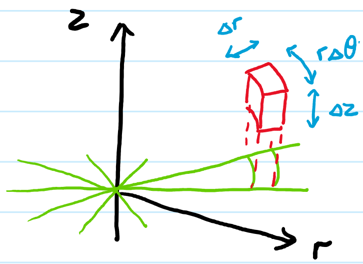
$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z \end{cases}$$

From (r,θ,z) to (x,y,z)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Unit cell of volume defined in cylindrical coordinate





$$\iiint f(x,y,z)\, dx\, dy\, dz \iff \iiint f(r,\theta,z)\, r\, dr\, d\theta\, dz$$

E.g. Solid Cylinder with density distribution $\rho(r, \theta, z) = r^2$

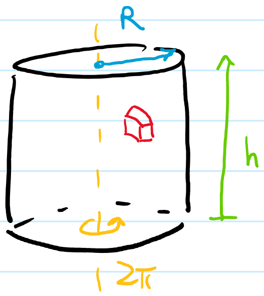
Dimension : Radius $= R$, Height $= h$.

$\because$ The mass of each volume cell $= \rho(r,\theta,z) \cdot r\,dr\,d\theta\,dz$

density × volume

So total mass $= \iiint \rho(r,\theta,z)\, r\,dr\,d\theta\,dz$

Find upper/lower bound for each dimension :



Range of r : From $r = 0$ to $r = R$

Range of z : From $z = 0$ to $z = h$

Range of θ : From $\theta = 0$ to $\theta = 2\pi$

density $\rho(r,\theta,z)$

$$\therefore \text{Total mass} = \int_{z=0}^{z=h} \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} r^2 \cdot r\,d\theta\,dr\,dz$$

$$= \int_{z=0}^{z=h} \int_{r=0}^{r=R} \boxed{\int_{\theta=0}^{\theta=2\pi} r^3\,d\theta}\, dr\,dz$$

$$= \int_{z=0}^{z=h} \int_{r=0}^{r=R} \Big[r^3\theta \Big]_{\theta=0}^{\theta=2\pi} dr\,dz$$

$$= \int_{z=0}^{z=h} \boxed{\int_{r=0}^{r=R} 2\pi r^3\,dr}\, dz$$

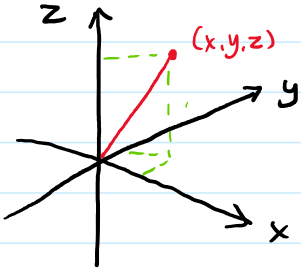
$$= \int_{z=0}^{z=h} \left[\frac{2\pi r^4}{4} \right]_{r=0}^{r=R} dz$$

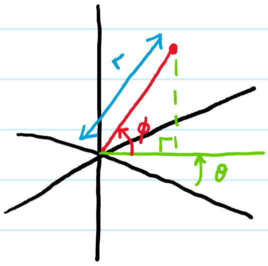
$$= \int_{z=0}^{z=h} \frac{\pi R^4}{2}\, dz$$

$$= \frac{\pi R^4 h}{2}$$

③ Spherical Coordinate = Latitude + Longitude + Altitude

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\phi \qquad\qquad\qquad \theta \qquad\qquad\qquad r$





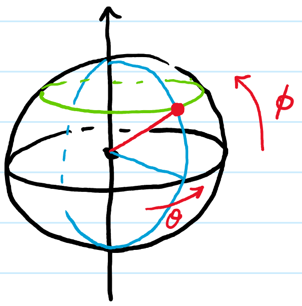
From (x,y,z) to (r,θ,ϕ)

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \phi = \tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right) \end{cases}$$

From (r,θ,ϕ) to (x,y,z)

$$\begin{cases} x = r\boxed{\cos\phi}\cos\theta \\ y = r\boxed{\cos\phi}\sin\theta \\ z = r\boxed{\sin\phi} \end{cases}$$

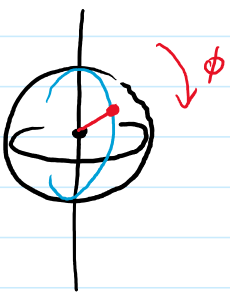
"Spherical" coordinate because const. r surface is a sphere



ϕ = Latitude, range : $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$

θ = Longitude, range : $0 \le \theta \le 2\pi$

☆ **Warning** ☆ : This is the definition appears in most Physics textbook



In Math book ϕ is defined starting from north pole

So the range is $0 \le \phi \le \pi$

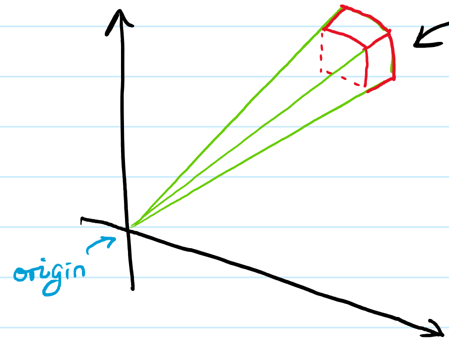
because having a 0 is easier for calculation

And $(x,y,z) \leftrightarrow (r,\theta,\phi)$ conversion becomes

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \\ \boxed{\phi = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)} \end{cases}$$

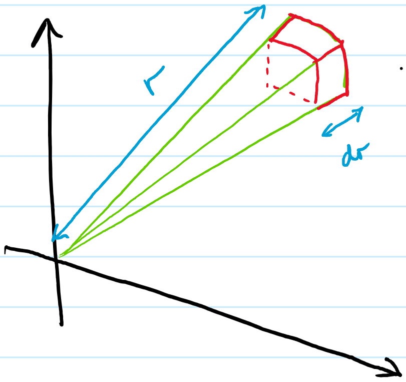
$$\begin{cases} x = r\boxed{\sin\phi}\cos\theta \\ y = r\boxed{\sin\phi}\sin\theta \\ z = r\boxed{\cos\phi} \end{cases}$$

Unit cell of volume defined in spherical coordinate

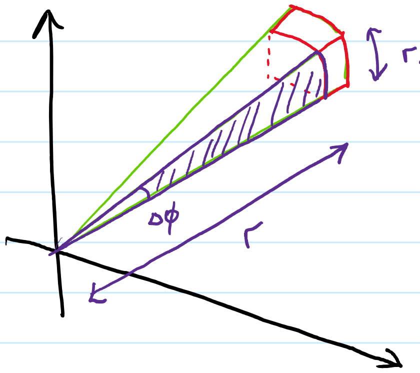


The unit cell = skin of a watermelon cone

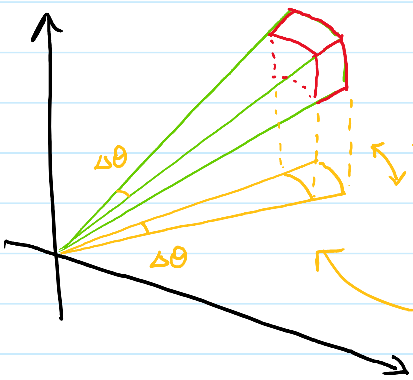
The side along r direction



The side along ϕ direction



The side along θ direction

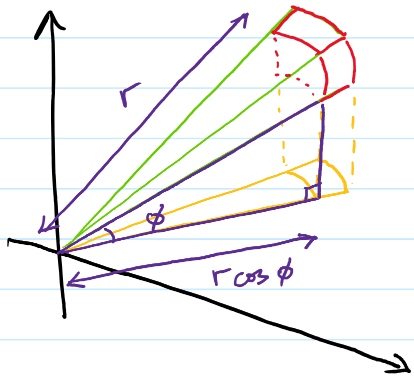


$r\cos\phi\,\Delta\theta$ in phy. text

or

$r\sin\phi\,\Delta\theta$ in math text

projection on equator plane



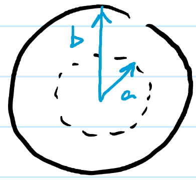
$\Rightarrow$ Unit cell has a dimension $(dr)\times(r\,d\phi)\times(r\sin\phi\,d\theta)$

$\therefore$ Volume integral written as $\iiint f(r,\theta,\phi)\,\underline{r^2\sin\phi}\,dr\,d\theta\,d\phi$

(Usually adopt Math. text notation)

E.g. Hollow thick sphere with radius range $r = a$ to $r = b$

Density of mass $= \rho(r, \theta, \phi) = r^4$



$\Rightarrow$ Total mass $= \iiint r^4 \cdot r^2 \sin\phi \, dr \, d\theta \, d\phi$

Find upper / lower bound for each dimension :

Range of r : From $r = a$ to $r = b$

Range of θ : From $\theta = 0$ to $\theta = 2\pi$

Range of ϕ : From $\phi = 0$ to $\phi = \pi$ (math text rotation)

$$\therefore \text{Total mass} = \int_{r=a}^{r=b} \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} r^4 \cdot r^2 \sin\phi \, d\phi \, d\theta \, dr$$

$$= \int_{r=a}^{r=b} \int_{\theta=0}^{\theta=2\pi} \boxed{\int_{\phi=0}^{\phi=\pi} r^6 \sin\phi \, d\phi} \, d\theta \, dr$$

$$= \int_{r=a}^{r=b} \int_{\theta=0}^{\theta=2\pi} \left[-r^6 \cos\phi \right]_{\phi=0}^{\phi=\pi} d\theta \, dr$$

$$= \int_{r=a}^{r=b} \boxed{\int_{\theta=0}^{\theta=2\pi} 2r^6 \, d\theta} \, dr$$

$$= \int_{r=a}^{r=b} \left[2r^6 \theta \right]_{\theta=0}^{\theta=2\pi} dr$$

$$= \int_{r=a}^{r=b} 4\pi r^6 \, dr$$

$$= \frac{4}{7}\pi \left(b^7 - a^7 \right)$$