Screening on a metal surface

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June 4, 2024
version 1.0

1 General concept of screening on a metal surface

Let us consider a charge ($+Q$) in a free space, as shown in Fig. 1(a). The charge causes a potential (ϕ) expressed by the following formula:

$$\phi(\boldsymbol{r}) = \frac{kQ}{|\boldsymbol{r}|}, k = \frac{1}{4\pi\epsilon_0}, \tag{1}$$

where $\boldsymbol{r}$ represents the position relative to the charge. The relationship between the potential and electric field ($\boldsymbol{E}$) is expressed as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}). \tag{2}$$

The potential and corresponding electric field distribution is shown in Fig. 1(a). The charge induces an inhomogeneous potential (blue curves). If a charge q is located at $\boldsymbol{r}$, the charge has the potential energy of $q\phi(\boldsymbol{r})$ and feels a force directed to the electric force line.

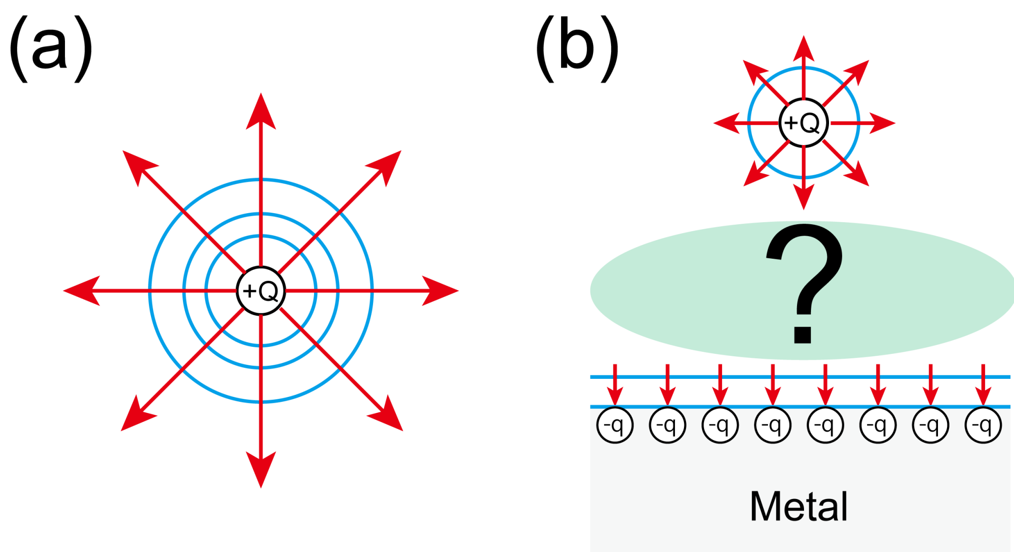

Figure 1: (a) Potential and electric force line around a single charge. (b) Potential and electric force line when a charge is located near the metal surface. Screening charges ($-q$) are induced at the metal surface to cancel out the electric force from the external charge, resulting in no electric force or potential in the metal. The red arrows and blue curves represent the electric force line and constant potential contour, respectively.

Then, let us consider a charge near the metal surface, as shown in Fig. 1(b). The external charge induces a potential and corresponding electric field. The free electrons in a metal are repelled (or attracted, depending on the charge polarity), resulting in the appearance of net charge near the metal surface. The redistribution of the electron in the metal continues until there is no field in the metal. Finally, the surface charge cancels out the electric field (and potential) induced by the external charge, resulting in no field and constant potential in the metal. This phenomenon is called the screening effect, and the induced charge is the screening charge.

In this document, we will consider the screeening effect based on fundamental electromagnetics: How is the potential distribution (and electric field) at the outside of metal, and how much charge is induced at the metal surface?

2 Mirror image method

Firstly, we will consider the electric field and potential at the outside of the metal. There is a technical way to calculate the electric field and potential distribution called a mirror image method.

As explained in the previous section, the potential in the metal is uniform ($\phi_{metal} = 0$). The in-plane component of the electric field induces a potential difference, so the electric field at the surface should have the perpendicular component only. The net electric field and potential by the external and screening charges should satisfy these two conditions at the surface. These conditions can be satisfied by considering a counter charge, as shown in Fig. 2(b). For an external charge $+Q$ at the outside of metal at $(0, a, 0)$, we consider a counter charge of $-Q$ at the mirror position of the surface plane $(0, -a, 0)$. The potential by these two charges at (x, y, z) is calculated as follows:

$$\begin{aligned} \phi(x, y, z) &= kQ\left(\frac{1}{\sqrt{x^2+(y-a)^2+z^2}} - \frac{1}{\sqrt{x^2+(y+a)^2+z^2}}\right) \\ &= kQ\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \end{aligned} \tag{3}$$

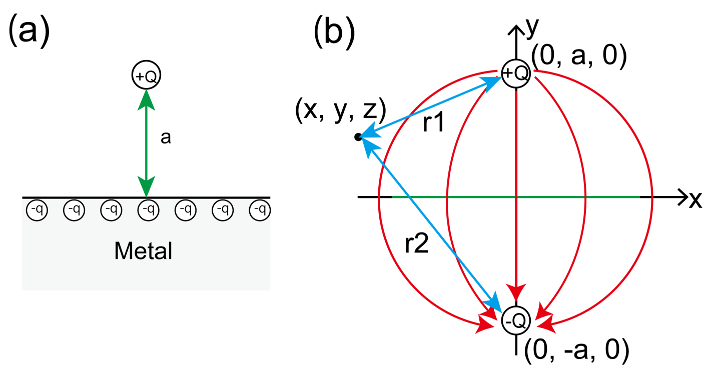

Figure 2: (a) The real charge distribution near the metal surface with an external charge. (b) Image charge method. A counter charge is located at the mirror position (y=0, represented by a green line) with respect to the metal surface. The red arrow represents the electric force line. The distance from the arbitrary position to the external charge (+Q) and image charge (-Q) is expressed as r_1 and r_2, respectively.

The electric field is calculated as follows:

$$\begin{aligned} \boldsymbol{E}(x,y,z) &= (E_x(x,y,z), E_y(x,y,z), E_z(x,y,z)) \\ &= -\nabla\phi(x,y,z), \end{aligned} \tag{4}$$

$$E_x(x,y,z) = kQx\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right), \tag{5}$$

$$E_y(x,y,z) = kQ\left(\frac{y-a}{r_1^3} - \frac{y+a}{r_2^3}\right), \tag{6}$$

$$E_z(x,y,z) = kQz\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right). \tag{7}$$

The potential at the metal surface is calculated by considering $r_1 = r_2$ (or $y = 0$) in eq. 3, as follows:

$$\phi(x,0,z) = 0 \tag{8}$$

In addition, the parallel component of the electric field at the surface is calculated as follows:

$$E_x(x,0,z) = kQx\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) = 0, \tag{9}$$

$$E_z(x,0,z) = kQz\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) = 0. \tag{10}$$

These results indicate that the boundary conditions at the metal surface are satisfied. The image charge method offers a convenient way to calculate the electric property near the metal surface without considering the detailed distribution of screening charge.

However, we should note that the image charge is just a calculational technique, and such an image charge does not exist. In reality, the screening charge is induced near the surface, which cancels out the charge at the outside, shown in Fig. 2(a). Therefore, also note that the potential and electric field calculated by the image charge method are valid only outside the metal.

3 Electric field by a plane charge

Before discussing the distribution of the screening charge, let us see the general properties of a plane charge, where charges are distributed in a plane with the density of σ, as shown in Fig. 3. We will use Gauss's law to

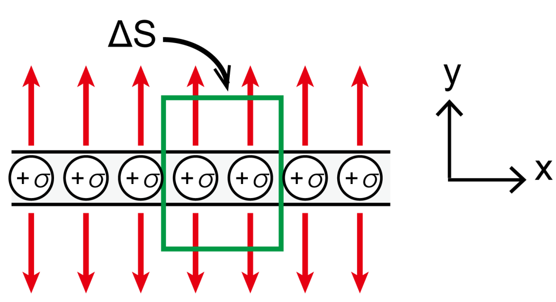

Figure 3: A schematic for a plane charge. The charges are distributed in a plane at $y = 0$ with the density of σ. The plane charge induces an electric field perpendicular to the plane, as red arrows indicate. The green rectangle represents the cuboid for performing the integration. The size of the face parallel to the x-axis is ΔS. The other face does not affect the integration.

calculate the electric field induced by the plane charge. The integration form of the Gauss's law is expressed as follows:

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q}{\epsilon_0}. \tag{11}$$

We will apply Gauss's law for a plane charge. The surface integral is performed in a cuboid containing a part of the plane charge, as shown in Fig. 3. As the electric field is perpendicular to the plane, the surface integral only has the values at the face parallel to the plane. If we assume that the area of the face is ΔS, Gauss's law yields the relationship between the electric field by the plane charge (E_p) and its density as follows:

$$\begin{aligned} \Delta S \cdot E_p - \Delta S \cdot (-E_p) &= \frac{\sigma \Delta S}{\epsilon_0} \\ \therefore E_p &= \frac{\sigma}{2\epsilon_0}. \end{aligned} \tag{12}$$

Equation 12 represents the electric field induced by the plane charge with the density of σ.

4 Screening charge

In eq. 12, we calculated the electric field induced by the plane charge. We will apply it on the screening charge induced at the metal surface. We assume an external electric field of E_{ex} near the metal surface. A screening charges are induced at the surface, forming a plane charge. The screening charge induces an electric field of E_p. This situation is represented in Fig. 4. These two fields should cancel out each other in the metal, so $E_p = E_{ex}$. On the outside, two fields are directed in the same direction. Thus, the total field (E_{total}) at the outside is

$$\begin{aligned} E_{total} &= E_p + E_{ex} \\ &= 2E_p \\ &= \frac{\sigma}{\epsilon_0}. \end{aligned} \tag{13}$$

We used the relation in eq. 12. Equation 13 represents the relationship between the screening charge density and the total electric field at the outside of the metal.

Let us go back to the situation where a charge of +Q is located at the outside. The electric field near the

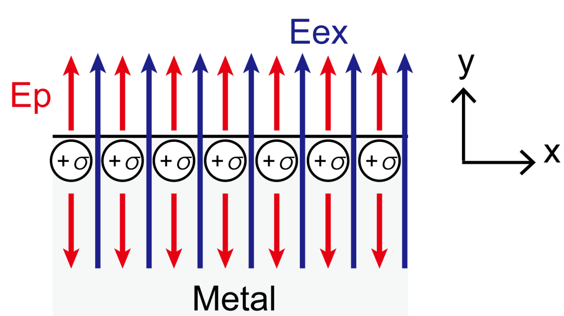

Figure 4: The schematic illustration of the electric field near the metal surface. The red arrows represent the electric field induced by the screening charge (density of σ), while the blue ones are the external field.

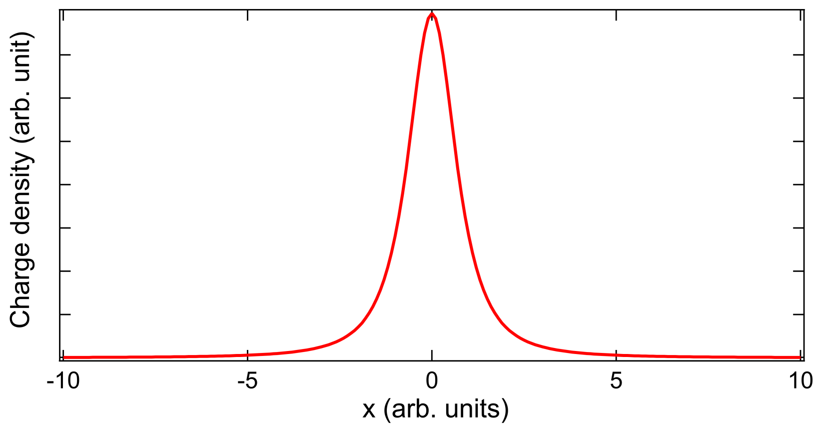

Figure 5: Distribution of screening charge at the surface for a single external charge.

surface by the external charge +Q is calculated by assuming $y = 0$ in eq. 6.[1]

$$\begin{aligned} E_y(x,0,z) &= kQ\left(\frac{-a}{r_1^3} - \frac{a}{r_2^3}\right) \\ &= -2kQ\frac{a}{(x^2+a^2+z^2)^{3/2}}. \end{aligned} \tag{14}$$

Equation 14 represents the E_{total} for the single external charge. Thus, eq. 13 and 14 yield the relationship between the external charge and screening charge density as follows:

$$\begin{aligned} \frac{\sigma(x,0,z)}{\epsilon_0} &= -2kQ\frac{a}{(x^2+a^2+z^2)^{3/2}} \\ \therefore \sigma(x,0,z) &= -\frac{Q}{2\pi}\frac{a}{(x^2+a^2+z^2)^{3/2}} \end{aligned} \tag{15}$$

Figure 5 represents the distribution of screening charge expressed in eq.15. The screening charge is the highest at $x = 0$, and decays by a factor of 3 with the distance to the charge.

Let us integrate the screening charge on all surfaces. The calculation becomes possible by considering the polar axis in the surface ($r^2 = x^2 + z^2$).

$$\begin{aligned} Q_{total} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(x,0,z)dxdz \\ &= \int_0^{\infty}\int_0^{2\pi}\sigma(r,\theta)rdrd\theta \\ &= \int_0^{\infty}\int_0^{2\pi} -\frac{Q}{2\pi}\frac{a}{(a^2+r^2)^{3/2}}rdrd\theta \\ &= -Q. \end{aligned} \tag{16}$$

It indicates that the total charge induced at the metal surface is the same as the external charge, except for the sign.

5 Screening of dipole charge

Let us consider a dipole of (+Q)-(-Q) located outside the metal surface. We assume that the charge +Q is located at $(0, a, 0)$ and -Q at $(0, a + d, 0)$. The charges +Q and -Q induce a screening charge of σ_- and σ_+,

[1] Remind that the x and z components are zero.

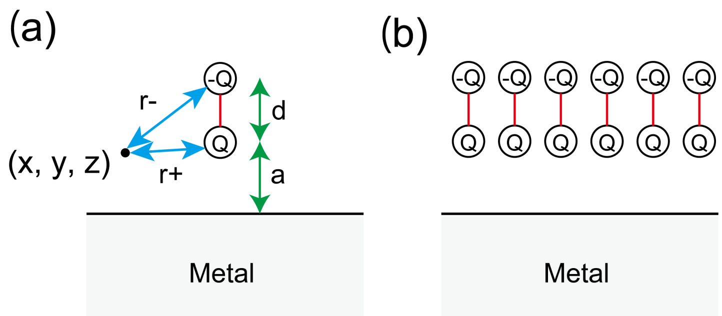

Figure 6: (a) A dipole located near the metal. (b) Many dipoles located near the metal, forming a layer.

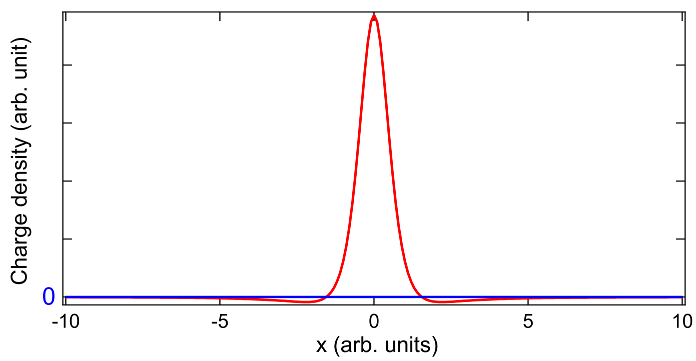

Figure 7: Distribution of screening charge at the surface for an external dipole. The blue line represents the zero net charge.

respectively, which are expressed as follows:

$$\sigma_-(x,0,z) = -\frac{Q}{2\pi}\frac{a}{{r_+}^3}, \tag{17}$$

$$\sigma_+(x,0,z) = \frac{Q}{2\pi}\frac{a+d}{{r_-}^3}. \tag{18}$$

Here, $r_+ = \sqrt{x^2+a^2+z^2}$, and $r_- = \sqrt{x^2+(a+d)^2+z^2}$, those are the distance to the two charges. Therefore, the net charge induced at the surface is expressed as follows:

$$\begin{aligned} \sigma_{net} &= \sigma_+ + \sigma_- \\ &= \frac{Q}{2\pi}\left(-\frac{a}{{r_+}^3} + \frac{a+d}{{r_-}^3}\right) \end{aligned} \tag{19}$$

An example of the charge distribution is shown in Fig. 7. At $x = 0$, where it is closest to the dipole, a negative charge is induced because the positive charge is much closer to the surface. However, the net charge is reversed at a large x.

Finally, let us consider a situation where the dipoles form a continuous layer on the surface, as shown in

Fig. 6(b). The net charge can be calculated by integrating the contribution from all the dipoles:

$$\begin{aligned}
\sigma_{total}(x,0,z) &= \int\int \sigma_{net}(x,0,z)dxdz \\
&= \int\int \sigma_{+}dxdz + \int\int \sigma_{-}dxdz \\
&= Q - Q \\
&= 0.
\end{aligned} \tag{20}$$

As we calculated in eq. 16, the net charge induced by external charge $+Q$ is $-Q$, which is irrelevant to the position of the external charge. Therefore, in the case of the external dipole, the positive and negative screening charges cancel out each other if they are integrated for the whole surface. It is a natural result because the dipole layer is identical to the capacitor: as you may have learned in class, there is no electric field at the outside of the capacitor. Because no field is induced, there is no need for the screeening charge to be induced.