These are a collection of problems that I encountered during Stat 134 so naturally some will be a bit tough. (*) means the problem is somewhat difficult. (**) means the problem is difficult. (***) means the problem is very difficult. (****) means I haven't solved it myself (it could be easy or difficult). Come to my office hours to check solutions.

Chess Club

Suppose there are 14 juniors and 16 seniors in a chess club.
(a) How many ways are there to arrange all members of the club in a line?

$30!$

(b) How many ways are there to arrange 5 members of the club in a line?

$\frac{30!}{25!}$

(c) How many ways are there to arrange 5 members of the club in a line if there must be a senior at the beginning of the line and at the end of the line?

$(16)(28)(27)(26)(15)$

Now suppose the club has to send 4 members to the state tournament.
(d) How many different ways are there to send a group of 4 students to the tournament.

$\binom{30}{4}$

(e) If the club must send 2 juniors and 2 seniors to the tournament, how many possible groupings are there?

$\binom{14}{2}\binom{16}{2}$

(f) If the club must send either all juniors or all seniors, how many possible groupings are there?

$\binom{14}{4} + \binom{16}{4}$

Best Out of 7 (*)

Team A and Team B play each other in a best out of 7 tournament (the team that wins 4 games first wins the tournament). How many possible sequences are there for team A to win?

$1 + \binom{4}{3} + \binom{5}{3} + \binom{6}{3} = 35$

Equally Likely Dice

Suppose Alice rolls a fair 12 sided die and Bob rolls a fair 8 sided die. Find the probability that:
(a) Alice's roll is larger than Bob's roll

$\frac{5}{8}$

(b) Alice's roll is the same as Bob's roll

$\frac{1}{12}$

(c) Alice's roll is smaller than Bob's roll

$\frac{7}{24}$

Duplicate Faces (**)

Roll a fair 9-sided die until you get a face that you have seen before. Let X be the number of trials until there is a duplicate. Example: 1,2,3,7,1 then $x = 5$.
Find $\mathbb{P}(X = x)$. (Hint: consider the range of X first.)

$\mathbb{P}(X = x) = \frac{(\frac{9!}{(10-x)!})(x-1)}{9^x}$

Tournament

Suppose Team A has a .8 probability of winning a game against Team B (so them Team B has .2 probability of winning a game against Team A). They play in a best 5 out of 9 tournament so that a team wins the tournament if they win 5 games first. No games will result in a tie (one team must win). Find the probability that:
(a) Team A wins in 7 games [EDIT: Added "in"]

$(.8)^4(.2)^2\binom{6}{2}$

(b) Team A wins the tournament given the tournament ended in 7 games

$\frac{(.8)^4(.2)^2\binom{6}{2}}{(.8)^4(.2)^2\binom{6}{2}+(.8)^2(.2)^4\binom{6}{2}}$

(c) Team A wins the tournament given the tournament ended in 5 games

$\frac{(.8)^5}{(.8)^5+(.2)^5}$

(d) Are the number of games played and which team wins the tournament independent? (You may need a calculator to verify answer.)

No.

This is a true story

Gerd Gigerenzer found that 95 out of 100 American doctors get this problem wrong. Today, we will be smarter (well, better probabalists at the very least) than 95% of doctors in America.
"Suppose 0.008 of woman age 40 to 50 who participate in routine screening have breast cancer. There is a test, which has a 0.90 probability of testing positive for a woman with breast cancer and a 0.07 probability of testing positive for a woman without breast cancer. A woman in this age group had a positive mammography in a routine screening. Find the probability that she actually has breast cancer."
(a) Draw a tree diagram. (Show your unconditional, conditional, and joint probabilities.)

Draw your own tree diagram.

(b) Find the probability of a misdiagnosis in either direction.

$(.992)(.07) + (.008)(.10)$

(c) Finish the problem.

$\frac{(.008)(.90)}{(.008)(.90)+(.992)(.07)}$

Sequences of Events

Roll a fair six-sided die 20 times. Find the probability of:
(a) Getting at least 19 ones

$\mathbb{P}(X \geq 19) = \binom{20}{1}(\frac{1}{6})^{19}(\frac{5}{6}) + (\frac{1}{6})^{20}$

(b) They are not all the same

$1 - (\frac{1}{6})^{20}\binom{6}{1}$

(c) Getting both ones and twos and no other numbers

$(\frac{2}{6})^{20} - 2(\frac{1}{6})^{20}$

(d) (*) Getting both ones and twos

$1 - (2(\frac{5}{6})^{20} + (\frac{4}{6})^{20})$

Zebra (*)

Pick letters with replacement from the following $\{Z, E, B, R, A\}$ until you get the letter Z. Let X be the number of trials when you stop. Say $X = 10$. Find the probability of drawing 4 A's.
(Warning: this problem can easily be phrased as "Find the probability of drawing 4 A's given $X = 10$." This, however, is not a conditional probability problem. X is some number in the naturals; $\mathbb{P}(X = 10)$ doesn't make sense.)

$(\frac{1}{4})^4(\frac{3}{4})^5\binom{9}{4}$ (For those of you I emailed solutions to, there was an error. This solution is correct.)

Alice and Bob Revisited

Alice and Bob take turns rolling a fair six sided die. They keep playing this game until someone gets a 6, and that person is declared the loser. Alice goes first.
(a) Find the probability that they play more than 10 games.

$(\frac{5}{6})^{20}$

(b) (*) Find the probability that Bob wins the game. (Just for fun: once you solve this the long way, look up Craps Principle and solve it instantly).

$\frac{6}{11}$

(c) (****) Prove that if you roll a fair n-sided die in this case, there is no value of n such that this game is fair.

Inclusion/Exclusion Hard Mode (**)

Draw 8 cards from a standard deck of 52 cards. Let X be the number of four of a kinds that you have. Find $\mathbb{P}(X = x)$. (For motivation: let A_i be the event that the ith set has been collected. Hint: go from specific to general in all cases.)

$$\mathbb{P}(X=2) = \binom{13}{2}\left(\frac{\binom{4}{4}\binom{4}{4}}{\binom{52}{8}}\right)$$

$$\mathbb{P}(X=1) = \binom{13}{1}\left(\frac{\binom{4}{4}\binom{48}{4} - \binom{12}{1}\binom{4}{4}}{\binom{52}{8}}\right)$$

$$\mathbb{P}(X=0) = 1 - 13\frac{\binom{4}{4}\binom{48}{4}}{\binom{52}{8}} + \frac{\binom{13}{2}}{\binom{52}{8}}$$

3 Little Pigs (***)

Suppose there are n rooms in a circle and each of the 3 Little Pigs check into different rooms. Let X be the number of adjacent pairs. That is,
$X = 0$: All 3 pigs are more than 1 room away from each other.
$X = 1$: 2 pigs are in rooms that are next to each other, but not next to the 3rd pig's room.
$X = 2$: All pigs are in rooms that are next to each other.
Find $\mathbb{P}(X = x)$.

$$\mathbb{P}(X=0) = 1 - n\left(\frac{\binom{2}{2}\binom{n-2}{1}}{\binom{n}{3}}\right) + n\left(\frac{\binom{3}{3}\binom{n-3}{0}}{\binom{n}{3}}\right)$$

$$\mathbb{P}(X=1) = n\left(\frac{\binom{2}{2}\binom{2}{0}\binom{n-4}{1}}{\binom{n}{3}}\right)$$

$$\mathbb{P}(X=2) = n\left(\frac{1}{\binom{n}{3}}\right)$$

Party

Suppose there are n people at a party and all of them put their keys in a box. At the end of the party, the keys are returned randomly to each person. Find the probability that:
(a) All n people got their keys back

$\frac{1}{n!}$

(b) (*) Only $n-1$ people got their keys back

0

(c) (*) *For fun:* Let $X_n :=$ the number of people who got their keys back from the group of n people. Compute $\mathbb{E}[X_n]$ and $Var(X_n)$. What is the distribution of $X := \lim_{n\to\infty} X_n$?

Solution: $\mathbb{E}[X_n] = Var(X_n) = 1$, $X \sim poisson(1)$

Bags and Coins

Suppose a bag contains three types of coins:
(i) 6 HH coins,
(ii) 3 HT (fair) coins, (EDIT: Originally said 4)
(iii) 1 TT coin.
Randomly select a coin from the bag and flip it twice. Find the probability that:
(a) The first toss lands heads

$\frac{3}{4}$

(b) The second toss lands heads

$\frac{3}{4}$

(c) The second toss lands heads given the first toss lands heads

$\frac{(\frac{6}{10})(1^2)+(\frac{3}{10})(\frac{1}{2})^2}{(\frac{3}{4})}$

Poker Hands

Google a list of 5 card poker hands and find the probabilities of all of them without looking at the answers.

http://www.math.hawaii.edu/ ramsey/Probability/PokerHands.html

Proofs by Story

After going through Piazza posts this morning, I noticed that a lot of people had difficulty with combinatorial proofs. So I'll give two examples and tips behind them, and a few extra practice problems.

Example 1: Use a combinatorial argument to prove

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$$

Scratch Work: We are given the task to show something about choosing n objects from $2n = n + n$ things on the LHS. This suggests that we should first begin with two sets of n elements in each. A very suggestive hint for the RHS is that $\binom{n}{k} = \binom{n}{n-k}$, and that $n + n = 2n$ and $k + (n-k) = k$. Currently around me, I have a book on algebraic topology and algorithms, so I will make a story out of these. ☺

Proof: Suppose we are at the Cal Student Store and they happen to have n equal amounts of textbooks for Math 142 and CS 170. Thus there are $2n$ books we can choose from and we want to see how many ways there are of picking n books from this pile of books. That is simply, $\binom{2n}{n}$. Notice however, that among these $2n$ books, there are k books for Math 142 (for $k \in \{0, \dots, n\}$) and $n-k$ books for CS 170. So among theses $2n$ books, we have $\binom{n}{k}$ ways to pick k books from the n Math 142 books, and $\binom{n}{n-k}$ ways to pick $n-k$ books from the n CS 170 books. We know that $\binom{n}{n-k} = \binom{n}{k}$, so in fact picking k books for Math 142 and $n-k$ books for CS 170 amounts to $\binom{n}{k}^2$. Now we want to range through combinations of picking n books and this basically just depends on the range of k. k goes from 0 to n so what we actually have is $\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \sum_{i=0}^{n} \binom{n}{i}^2$. ∎

Example 2: Use a combinatorial argument to prove

$$n^2 = (n-1)^2 + 2(n-1) + 1$$

Scratch Work: Here we are given a more interesting problem that does not make use of any combinations/permutations or whatever. Being the case that we have n^2 on the LHS, we just consider two sets of equal size and all the combinations between them. On the RHS, given that we have the sum of three terms each with power 0, 1, and 2, we are motivated to consider cases. For some reason, the other week I bought 60 red pens and 60 blue pens so I'll make a story out of these. ☺

Proof: Suppose I have n distinct red pens $r_1, \dots, r_n$ and n distinct blue pens $b_1, \dots, b_n$. The total way of picking two pens, one blue and one red, is obviously n^2 (consider how many pairs (r_l, b_k) we have for $l, k \in \{0, \dots, n\}$). Now let's fix a subscript, say, $i = 1$. We want to see how many ways we can get pairs that include r_1, b_1, both, or none since that will cover all n^2 pens. Obviously, there is only one pair that will include both, namely (r_1, b_1). Now we want all combinations that have either r_1 or b_1 but not both. Fix r_1 and we have $n-1$ blue pens we can consider (since we are not counting b_1). Thus we have $n-1$ choices for seeing the total number of pens with r_1 fixed and not including b_1. A similar argument will show that there are also $n-1$ ways when we fix b_1. Finally we want to see how many ways there are of picking pens that are not b_1 or r_1. There are $n-1$ pens to choose from in either case so we have $(n-1)^2$. Thus we have $(n-1)^2 + 2(n-1) + 1$ ways. ∎

Remarks: - If you are not convinced by the RHS argument, draw out a 4x4 or 5x5 table and see how the argument covers all the cases.
- We can also make a story by making the set of pens the same but making how we split them distinct. That is, suppose we have two sets of 60 colored pens. We want to show how to distribute these pens into two drawers. (An example of such an approach is in Sahai's Spring 2014 Midterm 3 Problem 4).

More Proofs by Story

(a) Use a combinatorial argument to prove

$$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$$

(Hint: emulate an argument from Example 1 except with a set of m things and another set of n things, then consider cases such as in Example 2)

Proof: Let there be m lions and n tigers at a zoo. We want to pick 2 of them to put in a separate cage. We can either pick two lions, i.e. $\binom{m}{2}$, or two tigers, i.e. $\binom{n}{2}$, or one of each, i.e. mn. ■

(b) Use a combinatorial argument to prove

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Proof: Suppose we want to pick k students to be officers for MUSA (or USA or HKN or UPE or whatever), i.e. $\binom{n}{k}$. We also want to pick a president out of those officers, i.e. k. So we have $k\binom{n}{k}$ ways to do this. Alternatively suppose we pick a president first (who will still be an officer). There are n students so n ways to pick a president. Now we have $n-1$ students left and need $k-1$ more students to be officers. That is simply just $\binom{n-1}{k-1}$. So we have $n\binom{n-1}{k-1}$. ■

(c) (*) Use a combinatorial argument to prove

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Solution: See Discrete Mathematics by Rosen section 6.4 for solutions to (c)-(f)

(d) Use a combinatorial argument to prove Pascal's Identity:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Solution: See Discrete Mathematics by Rosen section 6.4 for solutions to (c)-(f)

(e) (*) Use a combinatorial argument to prove the Binomial Theorem:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Solution: See Discrete Mathematics by Rosen section 6.4 for solutions to (c)-(f)

(f) (*) Use a combinatorial argument to prove Vandermonde's Identity:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Solution: See Discrete Mathematics by Rosen section 6.4 for solutions to (c)-(f)