# Probability!

- You have n marbles in a bag, including a golden marble. You will draw k marbles out of the bag, without replacement. What is the probability that you will draw the golden marble?

There are several ways to solve this question.

- Naively, P(first marble drawn is golden marble) + P(marble 1 not GM $\cap$ (marble 2 GM | marble 1 not GM)) + P(1, 2 not GM $\cap$ (3 is GM | 1, 2 not GM)) ...

- Slightly better: 1 - P(not getting the golden marble)
- P(not getting the GM) = P(1st not GM)*P(2nd not GM)*...
- (Independent events, again)

- You can separate the marbles into the k marbles that get picked, and the n - k marbles that don’t get picked. Then, each marble can have equal chance of being the golden marble.
- Alternatively, you can think of P(marble 1 is golden marble), P(2 is GM), …, P(k is GM)
- Since you have disjoint events, you can add probabilities!

- Now, instead of one golden marbles in the bag, there are g golden marbles in the bag, where g <= n - k (if g > n - k, then the probability of getting the golden marble would be 1). Drawing without replacement, what’s the probability that you would get a golden marble? (You stop drawing when you draw a golden marble.)

- ~~4 people get in an elevator on the ground floor. There are 10 other floors in this building, and each person independently and uniformly chooses a floor to get off on (getting on and off on the same floor would be silly). What is the probability that the elevator stops on the 7th floor?~~
- ~~Does not stop on even floors?~~
- ~~No two people go to the same floor?~~
- ~~The probability that 2 people arrive on the 7th floor?~~

- ~~You send people up the elevator until someone has gotten off on every floor. What is the expected number of people who will be sent up the elevator?~~

# Expectation

- In some contexts, expectation matches our intuition for "average value."
- Consider this bet
  - 0.5 probability of winning \$5
  - 0.5 probability of winning \$15
- The expected value of this bet, \$10, is around what a reasonable person might pay to make the bet.
- Not always the case…

# St. Petersburg Lottery

- A casino in St. Petersburg offers the following bet: a fair coin is tossed until a head appears, and the payout, which starts at $1, is doubled for every toss made.
- Payout is $2^t$, where $t$ = the total number of tails.
- How much would you pay to enter the bet?
- Calculate the expected value for this bet.

$$
\begin{aligned}
E(B) &= \sum_{a \in \mathcal{A}} a \times Pr[B = a] \\
&= 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + 8 \cdot \frac{1}{16} + \cdots \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \sum_{k=1}^{\infty} \frac{1}{2} = \infty\,.
\end{aligned}
$$

- How can we make sense of this?
- Consider the following modification: the casino no longer has unlimited money. It is backed by the Bill & Melinda Gates Foundation, and only has $\$2^{36}$, or around \$70 billion, in its reserves.
- If the outcome of the bet is higher, the player is given all the money in the reserves, but no more.
- Calculate the new expected value.

$$
\begin{aligned}
E(B) = 1 \cdot \frac{1}{2^1} + 2 \cdot \frac{1}{2^{2^2}} + \cdots + 2^{36} \cdot \frac{1}{2^{37}} + \\
2^{36} \left( \frac{1}{2^{38}} + \frac{1}{2^{39}} + \frac{1}{2^{40}} + \cdots \right) \\
= 37 \cdot \frac{1}{2} + 2^{36} \left( \frac{1}{2^{37}} \right) \\
= 19.
\end{aligned}
$$

The infinite expected value from before was due to the infinite extremely high, extremely unlikely terms. Capping the sum at any finite value results in a much more modest expected value.