CHAPTER 1: SIGNALS & SYSTEMS

Signal: set of data/information

- Mostly functions of time, but applies to other independent variables

Systems: entity that processes a set of signals (inputs) to yield another set of signals (outputs)

- Can be hardware or software

1.1 Size of Signal

[Def] Number that indicates largeness/strength of entity

- Measure includes amplitude & duration

1.1-1 Signal Energy

- must be finite for a meaningful measure of signal size
- necessary condition
  - $amplitude \to 0$ as $|t| \to \infty$, else Eq. (1) doesn't converge

Area under signal *x(t)* as a measure of signal size:

- accounts for amplitude & duration
- squared to ensure positive & negatives don't cancel

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad (1.1.1\text{-}1)$$

simplifies for real-valued signal $x(t)$ to $E_x = \int_{-\infty}^{\infty} x^2(t) dt$.

Notes:

- signal energy depends on signal AND load:
  - energy dissipated in normalized load of 1 ohm resistor if voltage x(t) were applied across resistor
  - indicative of energy capacity of signal, not actual energy; thus conservation of energy doesn't apply

1.1-2 Signal Power

- time average of the energy

When $amplitude$ does NOT $\to 0$ as $|t| \to \infty$, signal energy is infinite. Thus signal power is a more meaningful measure, if it exists:

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \qquad (1.1.2\text{-}1)$$

simplifies for real-valued signal $x(t)$ to $P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$.

- mean-square value of $|x(t)|$
- $\sqrt{P_x}$ is the root-mean-square value of $x(t)$

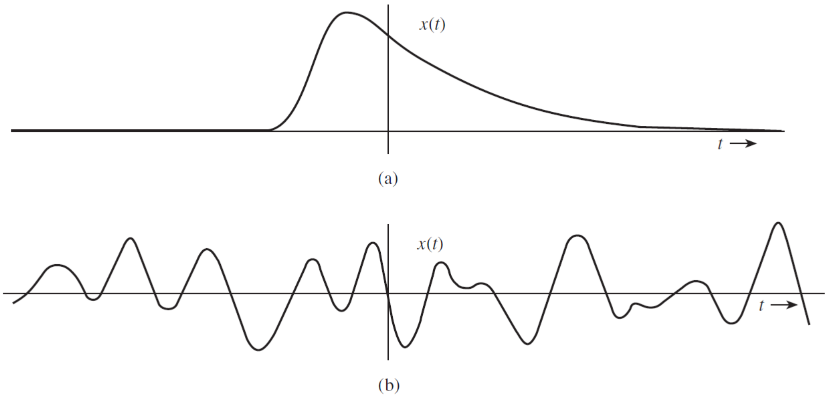


Figure 1.1 Examples of signals: **(a)** a signal with finite energy and **(b)** a signal with finite power.

Example: Determining Power & RMS Values

a) sinusoid: $x(t) = C \cdot cos(w_0 t + \theta)$

Sinusoid, with amplitude C. Period = $T_0 = 2\pi/w0$.

$$\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} C^2 cos^2(w_0 t + \theta)dt \\
&= \lim_{T\to\infty} \frac{C^2}{2T} \int_{-T/2}^{T/2} 1 + cos^2(2w_0 t + 2\theta)dt \\
&= \lim_{T\to\infty} \frac{C^2}{2T} \int_{-T/2}^{T/2} dt + \lim_{T\to\infty} \frac{C^2}{2T} \int_{-T/2}^{T/2} cos^2(2w_0 t + 2\theta)dt \\
&= \frac{C^2}{2} + 0 \\
&= \frac{C^2}{2}
\end{aligned}$$

Note: $RMS = \frac{C}{\sqrt{2}}$. While $w_0 \neq 0$, frequency doesn't affect power. If $w_0 = 0$, $P_x = C^2$.

b) sinusoidal sum: $x(t) = C_1 \cdot cos(w_1 t + \theta_1) + C_2 \cdot cos(w_2 t + \theta_2)$, $w_1 \neq w_2$

$$\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [C_1 cos(w_1 t + \theta_1) + C_2 cos(w_2 t + \theta_2)]^2 dt \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} C_1^2 cos^2(w_1 t + \theta_1)\, dt + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} C_2^2 cos^2(w_2 t + \theta_2)\, dt \\
&+ \lim_{T\to\infty} \frac{2C_1C_2}{T} \int_{-T/2}^{T/2} cos(w_1 t + \theta_1) cos(w_2 t + \theta_2)\, dt \\
&= \frac{C_1^2}{2} + \frac{C_2^2}{2}
\end{aligned}$$

$RMS = \sqrt{(C_1^2 + C_2^2)/2}$

If $w_1 = w_2$:

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [C_1 cos(w_1 t + \theta_1) + C_2 cos(w_2 t + \theta_2)]^2 dt \\ &= TBD \\ &= [C_1^2 + C_2^2 + 2C_1 C_2 cos(\theta_1 - \theta_2)]/2 \end{aligned}$$

General Case: sum of any n sinusoids with distinct frequencies

$x(t) = C_0 + \sum_{n=1}^{\infty} C_n cos(w_n t + \theta_n)$

$P_x = C_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} C_n^2$

c) complex: $x(t) = D \cdot e^{jw_0 t}$

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |De^{iw_0 t}|^2 dt \\ &= |D|^2 \end{aligned}$$

$RMS = |D|$

1.2 Signal Operations

1.2-1 Time Shifting

$\exists \phi$ s.t. signals in $\phi(t)$ are a T second delay of $x(t)$. Thus $\phi(t+T) = x(t)$ and $\phi(t) = x(t-T)$.

1.2-2 Time Scaling

$\exists \phi$ s.t. signals in $\phi(t)$ are compressed by factors of a w.r.t. $x(t)$. Thus $\phi(\frac{t}{2}) = x(t)$ and $\phi(t) = x(2t)$.

1.2-3 Time Reversal

$\exists \phi$ s.t. signals in $\phi(t)$ are reflections of $x(t)$ across the vertical axis. Thus $\phi(t) = x(-t)$.

1.3 Classification of Signals

1.3-1 Continuous-Time & Discrete-Time

signal specified for a continuum of values versus signal specified for discrete values

- nature of signal along time

1.3-2 Analog & Digital

- nature of signal along amplitude

analog signal: signal whose amplitude can take on any value in continuous range (amplitude can take on infinite values)

digital signal: signal whose amplitude takes on only a finite number of values

1.3-3 Periodic & Aperiodic

Signal $x(t)$ is periodic if for some positive constant T_0, $x(t) = x(t + T_0)\ \forall\ T$.

$$\int_a^{a+T_0} x(t)\, dt = \int_b^{b+T_0} x(t)\, dt$$

everlasting signal: $-\infty < t < \infty$. True everlasting signals cannot be generated in practice.

causal signal: $x(t) = 0, t < 0$

anti-causal signal: $x(t) = 0\ \forall\ t \geq 0$

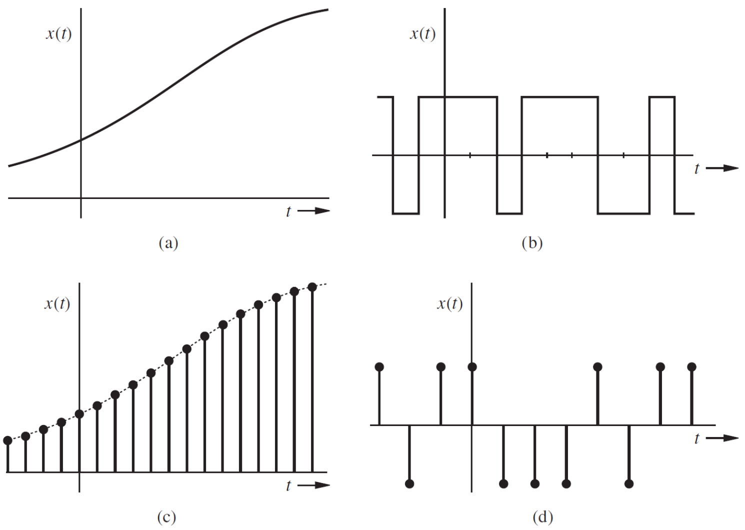


Figure 1.11 Examples of signals: **(a)** analog, continuous time; **(b)** digital, continuous time; **(c)** analog, discrete time; and **(d)** digital, discrete time.

Else, aperiodic.

1.3-4 Energy & Power

- cannot be both energy and power, but can be neither

energy signal: signal with finite energy

power signal: signal with finite, non-zero power (infinite energy); power is time average of energy (over an infinitely large [time] interval, else will not approach limit)

ramp signal: neither energy nor power signal.

e^{-at} is neither energy nor power signal for any $a \in \mathbb{R}$. However, if $a \notin \mathbb{R}$, it's a power signal with $P_x = 1$ regardless of the value of a.

1.3-5 Deterministic & Probabilistic

deterministic signal: a signal whose physical description is known completely in math/graphical form

random signal: a signal whose values cannot be predicted precisely but are known only in terms of probabilistic description (beyond scope of BME252)

1.4 Useful Signal Models

1.4-1 Unit Step Function $u(t)$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad (1.4.1\text{-}1)$$

Multiply by $u(t)$ to obtain a signal that starts at $t = 0$.

1.4-2 Unit Impulse Function $\delta(t)$

Dirac Delta:

$$\delta(t) = 0,\ t \neq 0,\ \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad (1.4.2\text{-}1)$$

Rectangular Pulse: width $\epsilon \to 0$, height $1/\epsilon \to \infty$. Undefined at $t = 0$.

e.g. 1.20(a) $\quad ae^{-at}u(t)$

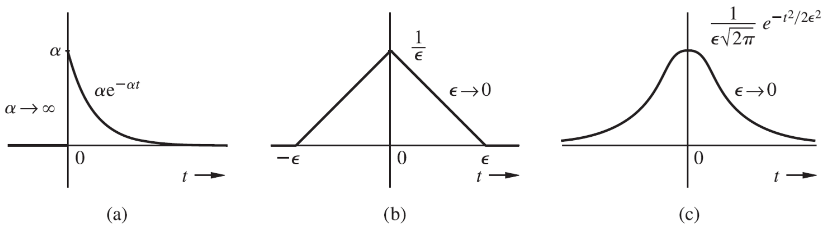


Figure 1.20 Other possible approximations to a unit impulse.

as $a \to \infty$, pulse height $\to \infty$, width/duration $\to \infty$. Yet area under curve is unity regardless of the value of a because

$$\int_0^{\infty} ae^{-at}\, dt = 1$$

Exact pulse function cannot be generated in practice, only approached. From 1.4-2, impulse function $k\delta(t) = 0 \ \forall\, t \neq 0$ has area k.

Multiplication of a Function by an Impulse

$\exists \phi(t)$ s.t. it's continuous at $t = 0$. Since impulse has non-zero value only at $t = 0$, and the value of $\phi(t)$ at $t = 0$ is $\phi(0)$:

$$\phi(t)\delta(t) = \phi(0)\delta(t)$$

Result: an impulse @ $t = 0$, has strength $\phi(0)$.

Generalization:

$$\phi(t)\delta(t - T) = \phi(T)\delta(t - T) \qquad (1.4.2\text{-}2)$$

Sampling Property of Unit Impulse Function

$$\int_{-\infty}^{\infty} \phi(t)\delta(t - T) = \phi(T)\int_{-\infty}^{\infty} \delta(t)\, dt = \phi(T) \qquad (1.4.2\text{-}3)$$

*area under the product of a function with impulse $\delta(t - T)$ is equal to the value of that function at the instant at which the unit impulse is located

Unit Impulse as Generalized Function

generalized function: defined by its effect on other functions instead of by its value at every instant of time

Impulse function is defined in terms of its effects on test function $\phi(t)$.

Unit impulse function: a function for which the area under its product with a function $\phi(t)$ is equal to the value of function $\phi(t)$ in the instant at which the impulse is located

- assumes $\phi(t)$ is continuous at location of impulse

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{du(t)}{dt}\phi(t)dt &= u(t)\phi(t)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\dot{\phi}(t)dt \\ &= \phi(\infty) - 0 - \int_{0}^{\infty} \dot{\phi}(t)dt \\ &= \phi(\infty) - \phi(t)|_{0}^{\infty} \\ &= \phi(0) \\ \frac{du(t)}{dt} &= \delta(t) \\ \int_{-\infty}^{t} \delta(\tau)d\tau &= u(t) \end{aligned} \tag{1.4.2-4}$$

1.4-3 Exponential Function e^{st}

$\exists$ complex number s s.t. $s = \sigma + jw$, therefore

$$e^{st} = e^{(\sigma+jw)t} = e^{\sigma t}e^{jwt} = e^{\sigma t}(cos(wt) + j\sin(wt))$$

Since $s^* = \sigma - jw$, then

$$e^{s^*t} = e^{(\sigma-jw)t} = e^{\sigma t}e^{-jwt} = e^{\sigma t}(cos(wt) - j\sin(wt))$$

and

$$e^{\sigma t}\cos wt = \frac{1}{2}(e^{st} + e^{s^*t}) \tag{1.4.3-1}$$

[Euler's formula] e^{st} is a generalization of e^{jwt}, where frequency variable jw is generalized to complex variable $s = \sigma + jw$. Class of functions expressed in terms of e^{st}:

CLASSES	FUNCTIONS	CONDITIONS
Constant	$k = ke^{0t}$	$s = 0$
Monotonic Exponential	$e^{\sigma t}$	$w = 0, s = \sigma$
Sinusoid	$\cos wt$	$\sigma = 0, s = \pm jw$
Exponentially Varying Sinusoid	$e^{\sigma t}\cos wt$	$s = \sigma \pm jw$

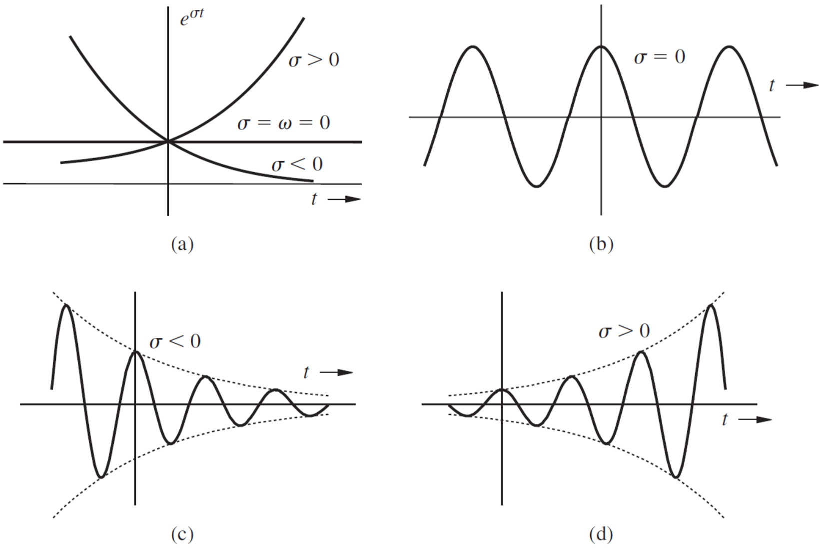


Figure 1.21 Sinusoids of complex frequency $\sigma + j\omega$.

About e^{st}:

w: frequency of oscillation

σ: rate of increase/decrease of amplitude

Signals' complex frequencies lie on **real** axis (σ axis, where $w = 0$).

If signals' frequencies lie on **imaginary** axis (w axis, where $\sigma = 0$), $e^{\sigma t} = 1$. Signals are conventional sinusoids with constant amplitude.

Figure 1.22 Complex frequency plane.

1.5 Even & Odd Functions

[def] even function X_e: symmetrical about vertical axis

[def] odd function X_o: antisymmetrical about vertical axis

$$X_e(t) = X_e(-t)$$
$$X_o(t) = -X_o(-t)$$

1.5-1 Properties of Even & Odd Functions

$$X_e \times X_o = X_o'$$
$$X_o \times X_o = X_e$$
$$X_e \times X_e = X_e'$$

Area:

$$\int_{-a}^{a} X_e(t)dt = 2\int_{0}^{a} X_e(t)dt$$
$$\int_{-a}^{a} X_o(t)dt = 0$$

*every signal can be expressed as a sum of even & odd functions

Modification for Complex Signals

- can be decomposed into even & odd components OR conjugate symmetries
- conjugate symmetric if $x_{cs}(t) = x^*(-t)$

  Real part is even, imaginary part is odd, thus even signal
- conjugate-antisymmetric if $x_{ca}(t) = -x^*(-t)$

  Real part is odd, imaginary part is even, thus odd signal

$$x(t) = x_{cs}(t) + x_{ca}(t) \text{ where}$$
$$x_{cs}(t) = \frac{x(t) + x^*(-t)}{2}$$
$$x_{ca}(t) = \frac{x(t) - x^*(-t)}{2}$$

1.6 Systems

Study of systems consists of mathematical modeling, analysis design.

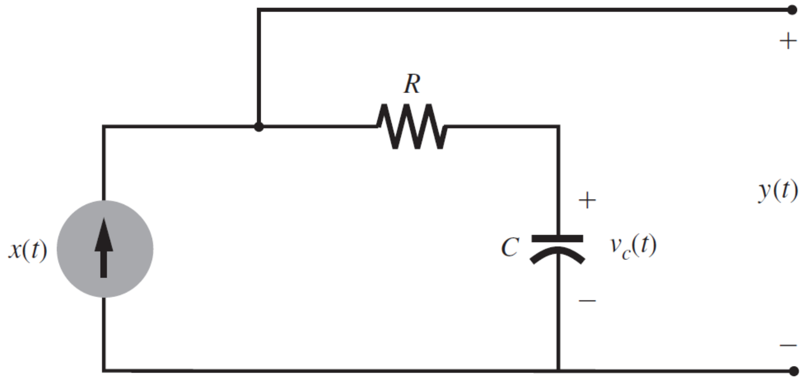
Consider RC circuit with current source $x(t)$ as input. Output is given by:

$$\begin{aligned} y(t) &= Rx(t) + \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau \\ &= Rx(t) + \frac{1}{C}\int_{-\infty}^{0} x(\tau)d\tau + \frac{1}{C}\int_{0}^{t} x(\tau)d\tau \\ &= Rx(t) + v_C(0) + \frac{1}{C}\int_{0}^{t} x(\tau)d\tau \qquad\qquad t \geq 0 \end{aligned}$$

Generalized form*:

$$y(t) = v_C(t_0) + Rx(t) + \frac{1}{C}\int_{t_0}^{t} x(\tau)d\tau \qquad t \geq t_0 \qquad (1.6\text{-}1)$$

*$v_C(t_0)$ is initial capacitator voltage



1.7 Classification of Systems

Categories of systems*:

1. linear & nonlinear
2. constant-parameter & time-varying-parameter
3. instantaneous (memoryless) & dynamic
4. causal & noncausal
5. continuous-time & discrete-time
6. analog & digital
7. invertible & non-invertible
8. stable & unstable

*other classifications, such as deterministic & probabilistic, are not in this course

1.7-1 Linear & Nonlinear Systems

superposition property:

- additivity property:

$$x_1 \to y_1 \land x_2 \to y_2 \implies x_1 + x_2 \to y_1 + y_2$$

- homogeneity (scaling property):

$$\text{if } x \to y, \text{then } \forall \text{ real and imaginary } k, kx \to ky$$

Thus,

$$x_1 \to y_1 \land x_2 \to y_2 \implies k_1x_1 + k_2x_2 \to k_1y_1 + k_2y_2 \;\forall\, k_1, k_2 \quad (1.7\text{-}1)$$

Response of a Linear System

context: **single input, single output** (SISO) systems

linear system output for $t \geq 0$:

- results from 2 independent causes:
  - initial conditions of system (system state) at $t = 0$
  - input $x(t)$ for $t \geq 0$
- must be sum of 2 components:
  - zero-input response (ZIR) resulting from initial response at $t = 0$ with input $x(t) = 0$ for $t \geq 0$.
  - zero-state response (ZSR) resulting from input $x(t)$ for $t \geq 0$ when initial conditions are assumed to be 0.
- both ZIR & ZSR must obey superposition w.r.t. their causes

Linear System Response:

$$\text{total response} = \text{ZIR} + \text{ZSR} \tag{1.7-2}$$

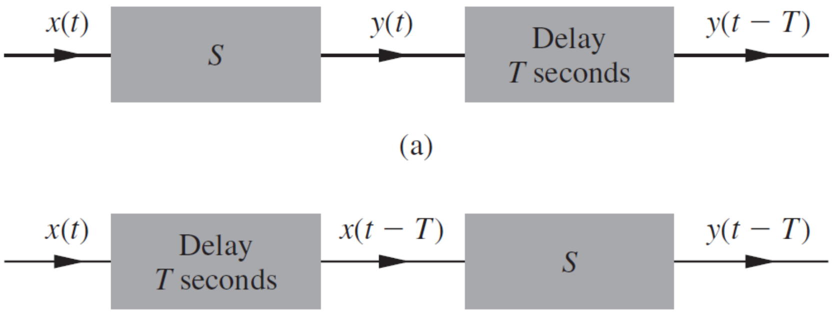
decomposition property: permits separation of an output into components resulting from initial conditions & from the input

$$y(t) = \underbrace{v_C(t_0)}_{ZIR} + \underbrace{Rx(t) + \frac{1}{C}\int_{t_0}^{t} x(\tau)d\tau}_{ZSR}$$

1.7-2 Time-Invariant & Time-Varying Systems

time-invariant: constant parameter

time invariance property:



1.7-3 Instantaneous & Dynamic Systems

A system is **instantaneous** if its output at any t depends, at most, on the strength of its inputs at same instant t.

*derivatives are NOT instantaneous: slope cannot be determined from a single point. Infinitesimally small memory must exist; see fundamental theorem of calculus.

17-4 Causal & Noncausal Systems

causal: output at any instant t_0 depends only on value of input $x(t)$ for $t \leq t_0$
.

*noncausal systems

- are realizable when independent variable is not time (e.g. space)
- are realizable with time delay
- provides upper bound performance for causal systems

1.7-5 Continuous-Time & Discrete-Time Systems

continuous-time signals: signals defined over a continuous range of time

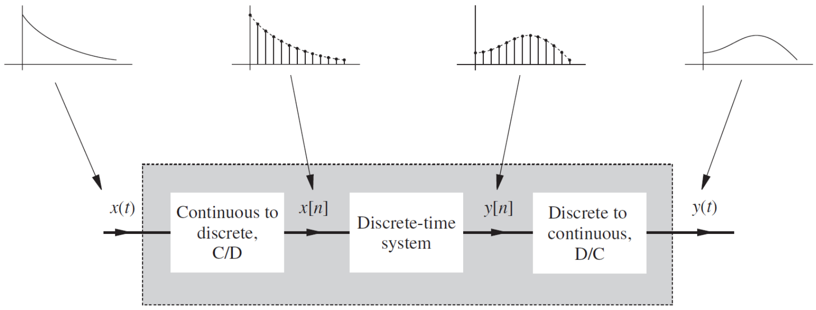
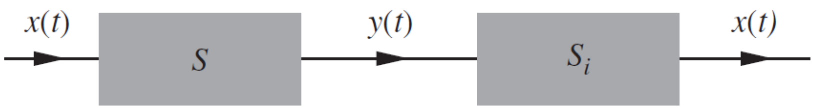


Figure 1.32 Processing continuous-time signals by discrete-time systems.

1.7-6 Analog & Digital Systems

e.g. digital computer => digital & discrete system

1.7-7 Invertible & Noninvertible Systems



e.g. noninvertable: $y(t) = |x(t)|$, $y(t) = tx(t)$

1.7-8 Stable & Unstable Systems

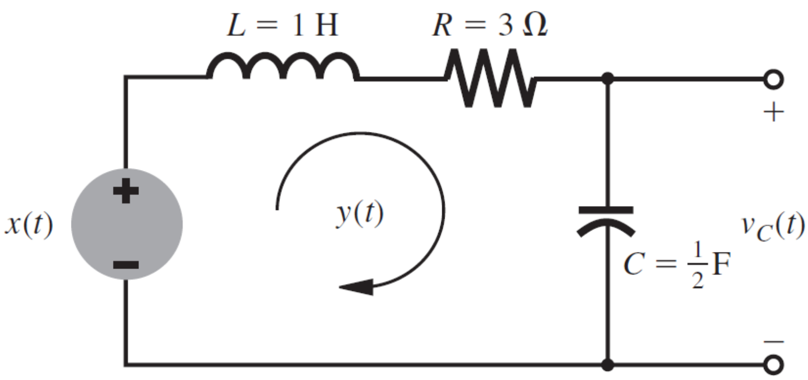
External Stability: every bounded input applied at input terminal results in a bounded output (BIBO: bounded-input/bounded-output)

1.8 System Model: Input-Output Description

system varieties: electrical, mechanical, hydraulic, acoustic, electromechanical, chemical, social, political, economic, biological, etc.

1.8-1 Electrical Systems

e.g. Series RLC circuit



Kirchhoff's voltage law around loop:

$$v_L(t) + v_R(t) + v_C(t) = x(t)$$

$$v_L(t) + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = x(t)$$

$$\frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

Differentiating both sides:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} \tag{1.8.1-1}$$

Expressed using compact notation:

$$\frac{dy(t)}{dt} \equiv Dy(t),\ \frac{d^2y(t)}{dt^2} \equiv D^2y(t),\ \ldots,\ \frac{d^Ny(t)}{dt^N} \equiv D^Ny(t)$$

$$(D^2 + 3D + 2)y(t) = Dx(t) \tag{1.8.1-2}$$

Integral operator expressed as inverse of differential operator:

$$\int_{-\infty}^{t} y(\tau)d\tau \equiv \frac{1}{D}y(t)$$
$$\frac{d}{dt}\left[\int_{-\infty}^{t} y(\tau)d\tau\right] = y(t)$$

1.8-2 Mechanical Systems

planer motion => translational (rectilinear) motion & rotational (torsional) motion

Translational Systems

basic elements: ideal masses, linear springs, dashpots (viscous damping)

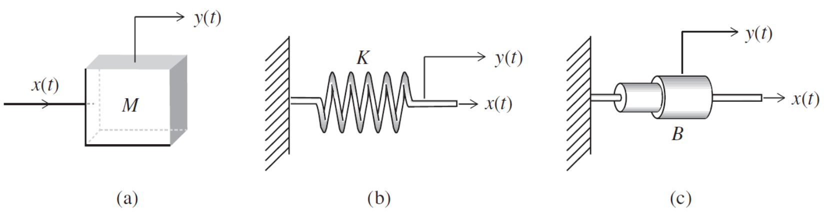


Figure 1.36 Some elements in translational mechanical systems.

Newton's law of motion (a):

$$x(t) = M\ddot{y}(t) = M\frac{d^2y(t)}{dt^2} = MD^2y(t)$$

Linear spring (K is stiffness) (b):

$$x(t) = Ky(t)$$

Linear Dashpot (B is damping coefficient) (c):

$$x(t) = B\dot{y}(t) = B\frac{dy(t)}{dt} = BDy(t)$$

Rotational Systems

motion about an axis; variables:

- torque
- angular position
- angular velocity
- angular acceleration
- torsional springs

- torsional dashpots

$$
\begin{aligned}
torque = J\ddot{\theta}(t) = J\frac{d^2\theta(t)}{dt^2} = JD^2\theta(t) = JD^2\theta(t) \\
= K\theta(t)B\dot{\theta}(t) = BD\theta(t)
\end{aligned}
$$

1.8-3 Electromechanical Systems

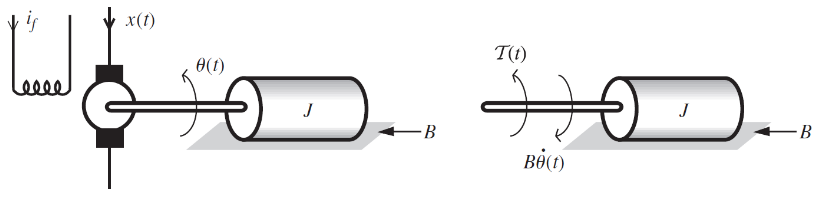


Figure 1.40 Armature-controlled dc motor.

$$
\begin{aligned}
J\ddot{\theta}(t) &= \tau(t) - B\dot{\theta}(t) \\
(JD^2 + BD)\theta(t) &= \tau(t) \\
(JD^2 + BD)\theta(t) &= K_T x(t) \\
J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} &= K_T x(t) \qquad (1.8.3\text{-}1)
\end{aligned}
$$