CHAPTER 3: TIME-DOMAIN ANALYSIS OF DISCRETE-TIME SYSTEMS

linear, time-invariant, discrete-time (LTID)

3.1 Introduction

discrete-time signal: sequence of numbers

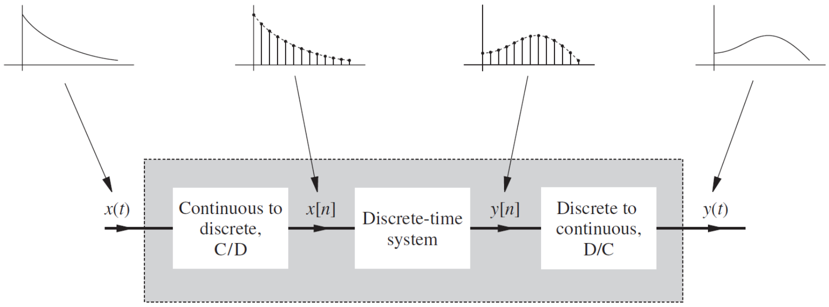


Figure 3.2 Processing a continuous-time signal by means of a discrete-time system.

continuous-time exponential $x(t) = e^{-t}$ sampled every $T = 0.1$ seconds results in a discrete-time signal $x(nT)$:

$$x(nT) = e^{-nT} = e^{-0.1n}$$

3.1-1 Size of a Discrete-Time Signal

Size measured by energy:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad (3.1.1\text{-}1)$$

if E_x is finite, signal is **energy signal**. Else, measured by signal power:

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} |x[n]|^2 \qquad (3.1.1\text{-}2)$$

*2N+1 samples in interval from -N to N

3.2 Signal Operations

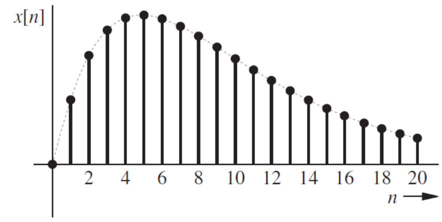
Shifting (by M units)

$$x_s[n] = x[n-M]$$

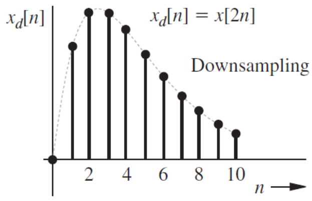
Time Reversal

$$x_r[n] = x[-n]$$

Sampling Rate Alteration: Downsampling, Upsampling, Interpolation



(a)



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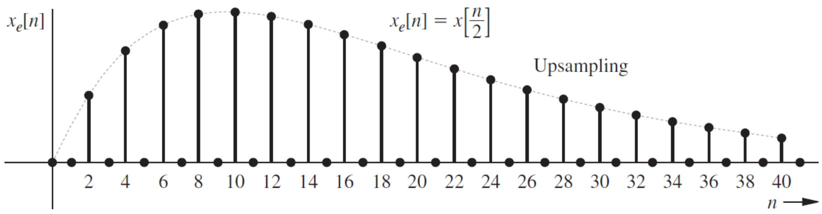

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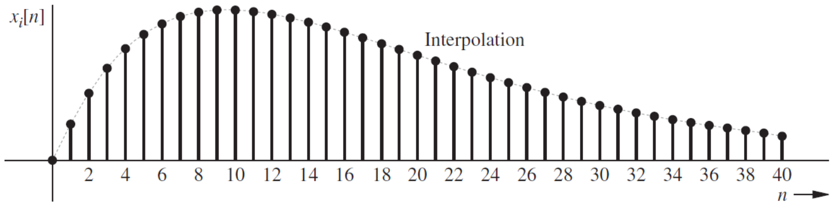

Downsampling: Compression by factor M

$$x_d[n] = x[Mn], M \in \mathbb{N}^+$$

Interpolated signal

$$x_e[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{otherwise} \end{cases}$$

Upsampling: L times that of $x[n]$: general sequence:

$$x_e[n] = x[0], \underbrace{0, 0, \dots, 0, 0}_{L-1\ zeros}, x[1], \underbrace{0, 0, \dots, 0, 0}_{L-1\ zeros}, x[2], \underbrace{0, 0, \dots, 0, 0}_{L-1\ zeros}, \dots$$

3.3 Discrete-Time Signal Models

3.3-1 Discrete-Time Impulse Function $\delta[n]$

Unit impulse sequence: Kronecker delta

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{3.3.1-1}$$

3.3-2 Discrete-Time Unit Step Function $u[n]$

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

3.3-3 Discrete-Time Exponential γ^n

continuous-time exponential $e^{\lambda t}$ can be alternatively expressed as:

$$e^{\lambda t} = \gamma^t \qquad (\gamma = e^{\lambda} \text{ or } \lambda = \ln \gamma)$$

discrete:

$$\gamma^n = e^{\lambda n} \qquad (\gamma = e^{\lambda} \text{ or } \lambda = \ln \gamma)$$

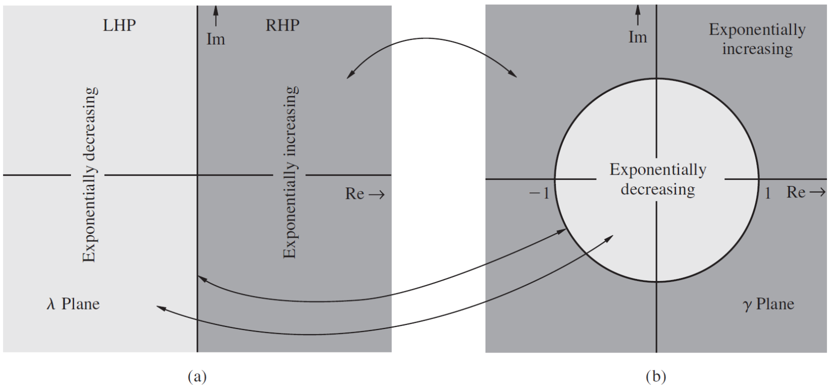

Figure 3.8 The λ plane, the γ plane, and their mapping.

e.g. signal $e^{\lambda n}$ where λ lies on left half-plane ($\lambda = a + jb, a < 0$), exponential decay

$$\gamma = e^{\lambda} = e^{a+jb} = e^a e^{jb}$$
$$|\gamma| = |e^a||e^{jb}| = e^a, \text{ because } |e^{jb}| = 1$$

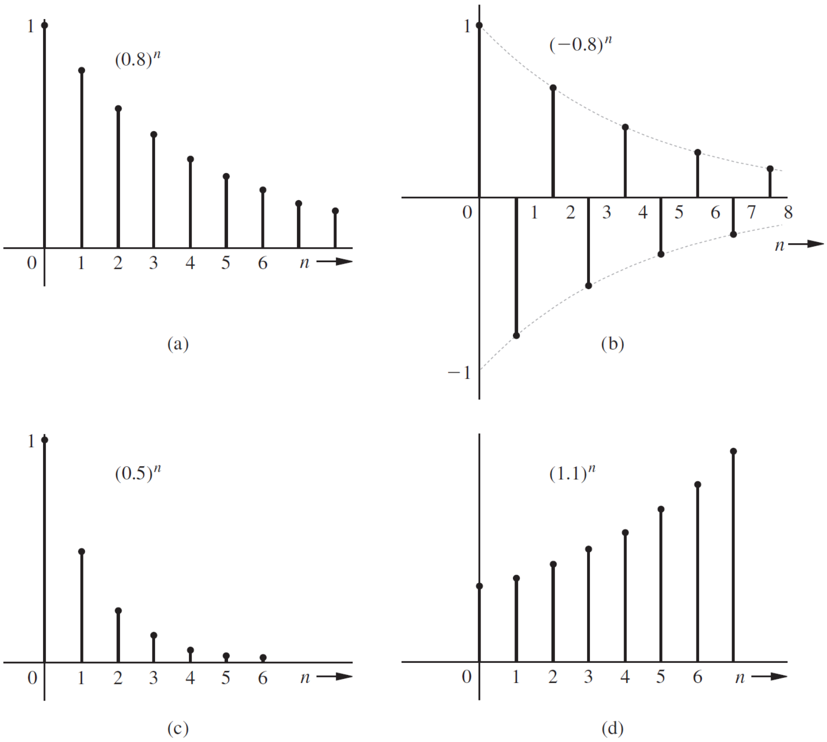

Figure 3.9 Discrete-time exponentials γ^n.

3.3-4 Discrete-Time Sinusoid $cos(\Omega n + \theta)$

General discrete-time sinusoid:

$$C\cos(\Omega n + \theta) = C\cos(2\pi F n + \theta) \text{ where } F = \Omega/2\pi$$

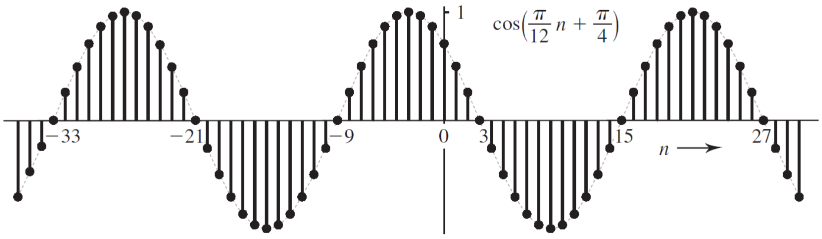

Figure 3.11 A discrete-time sinusoid $\cos\left(\frac{\pi}{12}n + \frac{\pi}{4}\right)$.

Sampled Continuous-Time Sinusoid Yields a Discrete-Time Sinusoid

A continuous-time sinusoid, $\cos wt$, sampled every T seconds yields a discrete-time sinusoid. Sample signal $x[n]$:

$$x[n] = \cos \omega nT = \cos \Omega n \text{ where } \Omega = \omega T$$

3.3-5 Discrete-Time Complex Exponential $e^{j\Omega n}$

$$\begin{aligned} e^{j\Omega n} &= (\cos \Omega n + j \sin \Omega n) \\ e^{-j\Omega n} &= (\cos \Omega n - j \sin \Omega n) \end{aligned}$$

For $r = 1$ and $\theta = n\Omega$,

$$e^{j\Omega n} = re^{j\theta}$$

3.4 Examples of Discrete-Time Systems

Savings Account

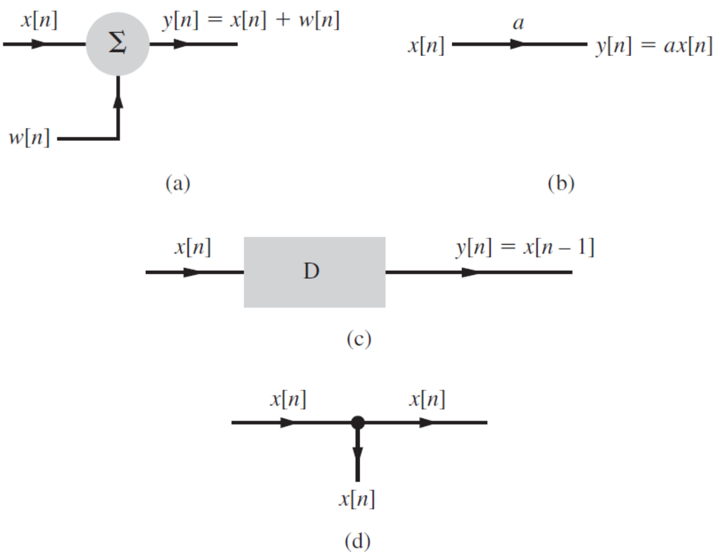

Figure 3.13 Schematic representations of basic operations on sequences.

x[n] = deposit made at the nth discrete instant
y[n] = account balance at the n^{th} instant computed immediately after receipt of the n^{th} deposit x[n]
r = interest per dollar per period T

$$\begin{aligned} y[n] &= y[n-1] + ry[n-1] + x[n] \\ &= (1 + r)y[n-1] + x[n] \end{aligned}$$

Delayed form:

$$y[n] - ay[n-1] = x[n], \ a = 1 + r \tag{3.4-1}$$

Advanced form:

$$y[n+1] - ay[n] = x[n+1] \tag{3.4-2}$$

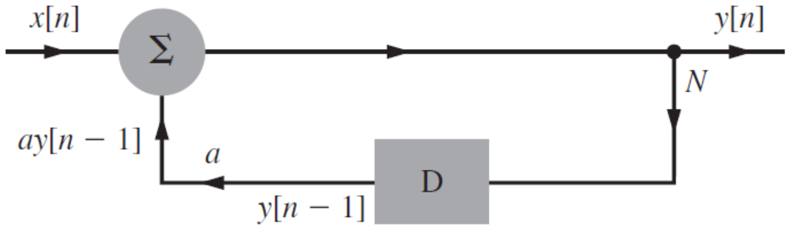

Figure 3.14 Realization of the savings account system.

- addition
- scalar multiplication
- delay
- pickoff node (node N): provides multiple copies of a signal at input

Sales Estimate

y[n]: new books sold by publisher

x[n]: students enrolled in n^{th} semester

book life: 3 semesters

1/4 students resell texts

$$y[n] + \frac{1}{4}y[n-1] + \frac{1}{16}y[n-2] = x[n] \tag{3.4-3}$$

$$y[n+2] + \frac{1}{4}y[n+1] + \frac{1}{16}y[n] = x[n+2] \tag{3.4-4}$$

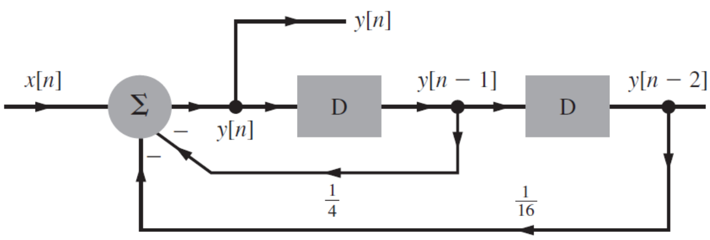

Figure 3.15 Realization of the system representing sales estimate in Ex. 3.7.

Digital Differentiator

- used in audio system with input signal bandwidth below 20kHz
- output $y(t)$ is derivative of input $x(t)$

x[n], y[n]: samples of signals $x(t)$ and $y(t)$, T seconds apart.

$$\begin{aligned} x[n] &= x(nT) \\ y[n] &= y(nT) \end{aligned} \tag{3.4-5}$$

$@t = nT$:

$$y(t) = \frac{dx(t)}{dt}$$

$$y(nT) = \left.\frac{dx(t)}{dt}\right|_{t=nT}$$

$$= \lim_{T\to 0} \frac{1}{T}[x(nT) - x[(n-1)T]]$$

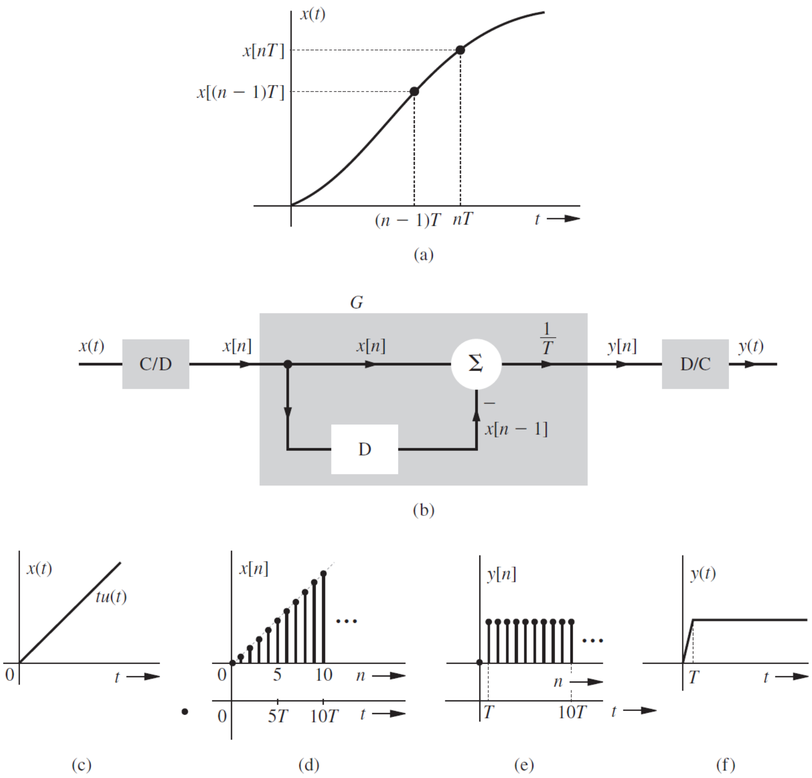

Figure 3.16 Digital differentiator and its realization.

backward difference system:

$$y[n] = \lim_{T\to 0} \frac{1}{T}[x[n] - x[n-1]]$$

but

- in practice, $T \neq 0$
- T is sufficiently small

non-recursive form:

$$y[n] = \frac{1}{T}[x[n] - x[n-1]] \qquad (3.4\text{-}6)$$

$$y[n] = \frac{1}{T}[x[n+1] - x[n]] \qquad (3.4\text{-}7)$$

Digital Integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

$$y(nT) = \lim_{T\to 0} \sum_{k=-\infty}^{n} x(kT)T$$

$$y[n] = \lim_{T\to 0} T \sum_{k=-\infty}^{n} x[k]$$

accumulator system: non-recursive form

$$y[n] = T \sum_{k=-\infty}^{n} x[k] \qquad (3.4\text{-}8)$$

recursive form:

$$y[n] - y[n-1] = Tx[n] \qquad (3.4\text{-}9)$$

Recursive & Non-Recursive forms of Difference Equation

Recursive:

- 3.4-1
- 3.4-3
- 3.4-9

Non-Recursive:

- 3.4-6
- 3.4-8

Kinship of Difference Equations to Differential Equations

- differential equation can be approximated by a difference equation of the same order

1st ODE:

$$\frac{dy(t)}{dt} + cy(t) = x(t)$$
$$\lim_{T \to 0} \frac{y[n] - y[n-1]}{T} + cy[n] = x[n]$$

Assuming non-zero, but very small T:

$$y[n] = \alpha y[n-1] = \beta x[n] \qquad (3.4\text{-}10)$$

where

$$\alpha = \frac{-1}{1 + cT}$$
$$\beta = \frac{T}{1 + cT}$$

advance form:

$$y[n+1] + \alpha y[n] = \beta x[n+1] \qquad (3.4\text{-}11)$$

Order of Difference Equation

difference equations:

- 3.1
- 3.3
- 3.7
- 3.9
- 3.11

1st ODEs:

- 3.1
- 3.7
- 3.9
- 3.11

Analog, Digital, Continuous-Time & Discrete-Time Systems

- digital filters => discrete-time systems
- analog filters => continuous-time system

Advantages of Digital Signal Processing

1. **Precision & Stability**: can tolerate considerable variation, hence less sensitive in component parameter
2. Easily **Duplicated & Fully Integrated**: doesn't require factory adjustments, complex systems placed on VLSI (very-large-scale-integrated) circuits
3. **Flexible**: easily alterable characteristics
4. Greater **Variety of Filters**
5. Easy & Cheap **Storage**: can also search/select information from cloud
6. Low Error Rates & High **Fidelity**: privacy & sophistication
7. **Simultaneously** serve a number of inputs: time-shared
8. Reliable **Reproduction** without Deterioration

3.4-1 Classification of Discrete-Time Systems

Linearity & Time Invariance

- linearity => continuous-time systems

- time/shift-invariant => discrete-time systems
    - systems whose parameter doesn't change with time

Causal & Noncausal Systems

- **causal** if output at any instant $n = k$ depends only on value of input $x[n]$ for $n \leq k$. Output depends only on past & present values of input.

Invertible & Noninvertible Systems

- **invertible** if $\exists$ inverse system S_i s.t. the cascade of S and S_i results in an identity system (output is identical to input)

Stable & Unstable Systems

- internal, OR
- external: bound input results in bounded output

Memoryless Systems & Systems with Memory

- **memoryless** if it responds at an instant n depending on input at the same time instant

DT System Properties

1. **linear**: requires homogeneity & additivity
2. **time-invariant**: shift in input => shift in output
3. **causal**: doesn't depend on future values
4. **invertible**: every input generates unique output
5. **BIBO-stable**: bound input results in bounded output
6. **memoryless**: depend on strength of current input