Epipolar Geometry with Fundamental Matrix

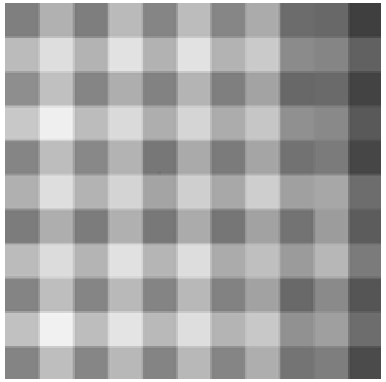
CS484 Introduction to Computer Vision

Homework 5 supplementary slides

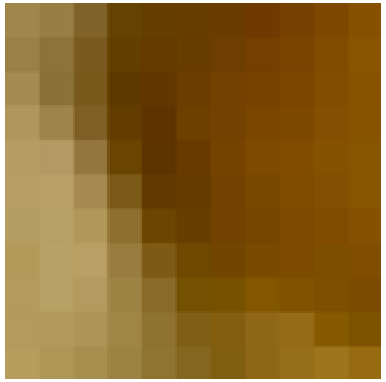
VISUAL COMPUTING Lab

Filter Demosaic

- Demosaic the raw image using the following three methods
  1. Down-sampling
  2. Linear interpolation
  3. Bicubic interpolation



Bayer pattern (RGGB)



Demosaic image

Raw image

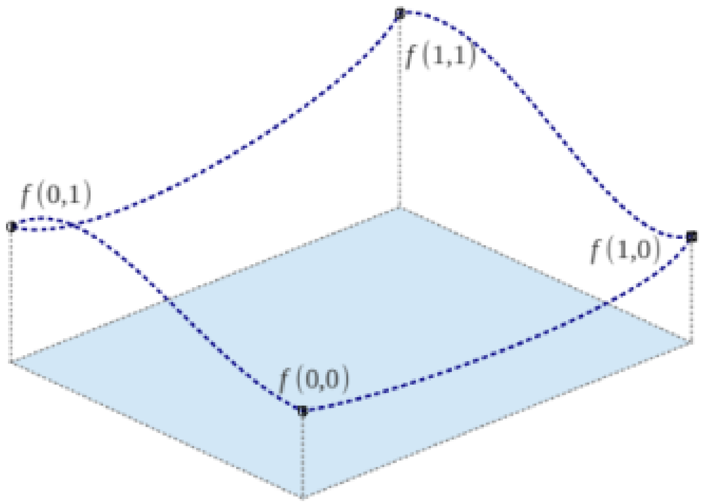
Color image (reference)

Filter Demosaic

- Bicubic interpolation

$$f(x,y)=\sum_{i=0}^{3}\sum_{j=0}^{3}a_{ij}x^i y^j \quad x,y\in[0,1]\times[0,1]$$

- 16 unknown coefficients a_{ij}a
- 16 known equations
  - $f(0,1), f(0,1), f(1,0), f(1,1)$
  - $f_x(0,1), f_x(0,1), f_x(1,0), f_x(1,1)$
  - $f_y(0,1), f_y(0,1), f_y(1,0), f_y(1,1)$
  - $f_{xy}(0,1), f_{xy}(0,1), f_{xy}(1,0), f_{xy}(1,1)$



- Difference approximation

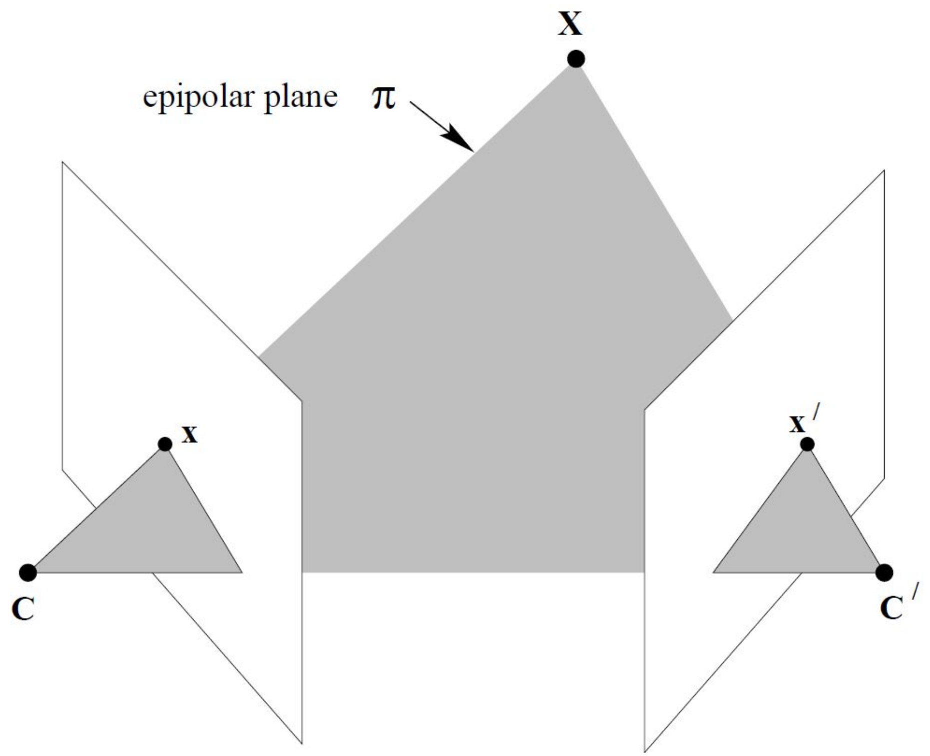
$$f_x(x,y)=\left[f(x+1,y)-f(x-1,y)\right]/2$$
$$f_y(x,y)=\left[f(x,y+1)-f(x,y-1)\right]/2$$
$$f_{xy}(x,y)=\left[f(x+1,y+1)-f(x-1,y-1)+f(x+1,y-1)-f(x-1,y+1)\right]/4$$

VISUAL COMPUTING Lab

Epipolar geometry

World coordinate $\mathbf{X}$ projects to image coordinate $\mathbf{x}$ and $\mathbf{x}'$

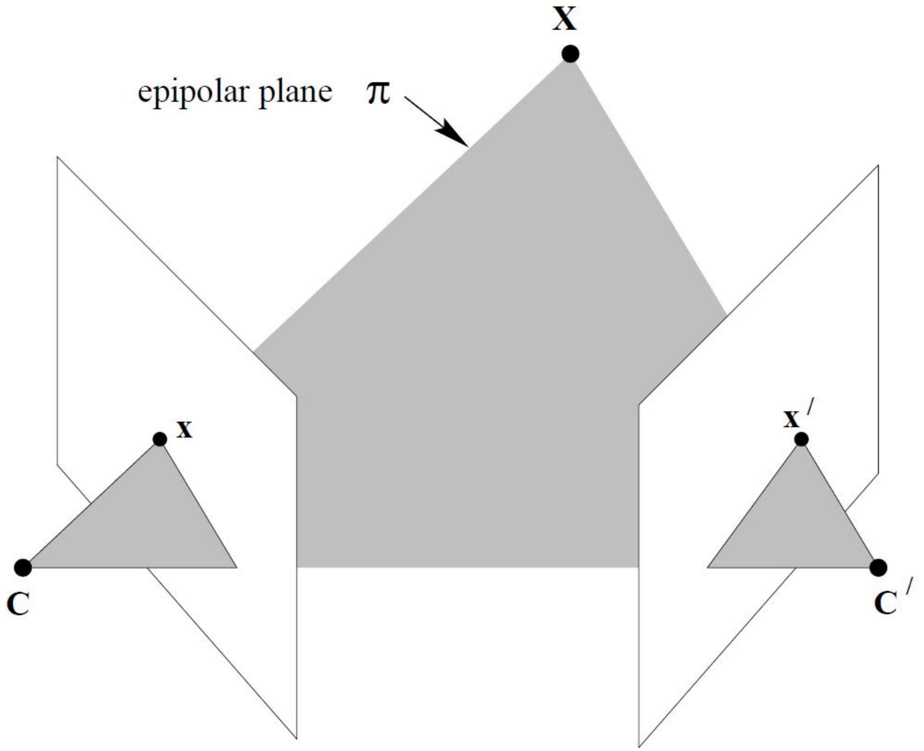
What is the relation between $\mathbf{x}$ and $\mathbf{x}'$?



Epipolar geometry

The camera centers $\mathbf{C}$ and $\mathbf{C}'$, a 3D point $\mathbf{X}$, and its image $\mathbf{x}$ and $\mathbf{x}'$ lie in a commom plane π.

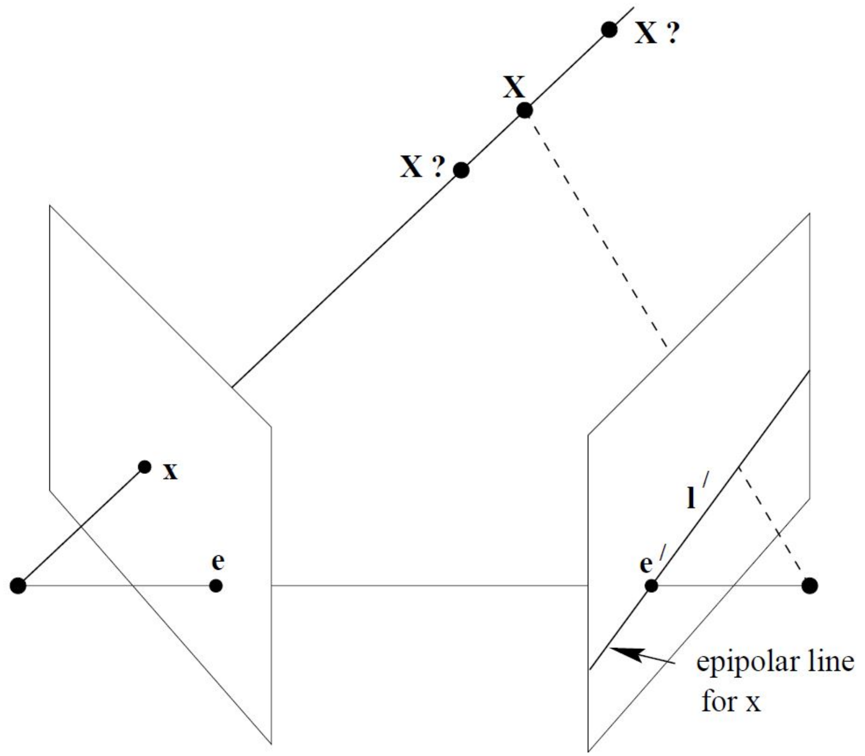
The plane π is **epipolar plane**.



Epipolar geometry

World coordinate $\mathbf{X}$ projects to image coordinate $\mathbf{x}$, but It can't distinguish with dots on the ray from $\mathbf{C}$ to $\mathbf{X}$.

The projection of the ray from $\mathbf{C}$ to $\mathbf{X}$ on the image plane 2 is the line $\mathbf{l}'$.



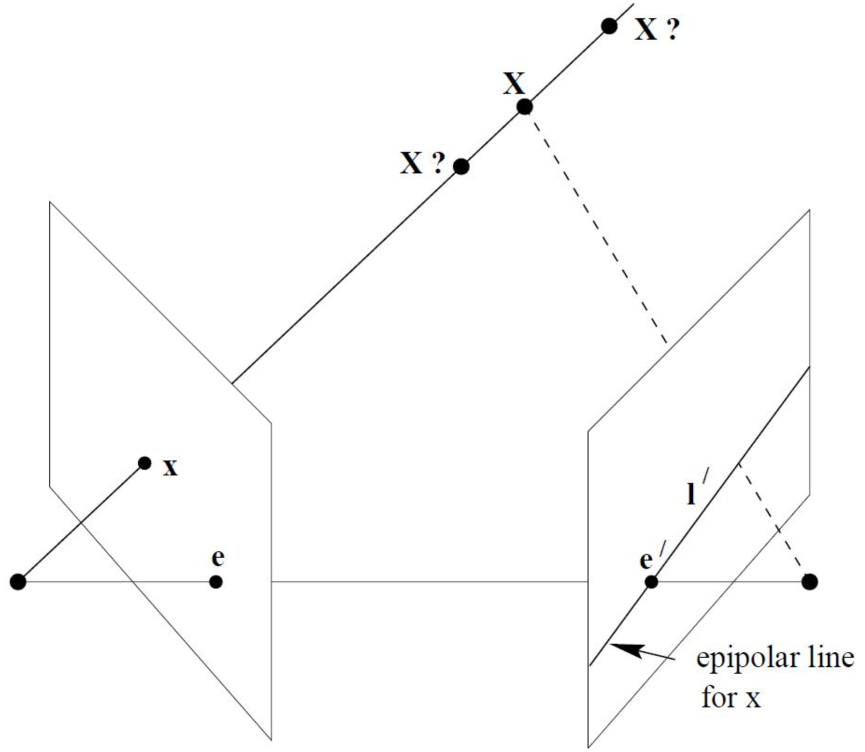
VISUAL COMPUTING Lab

Epipolar geometry

The line $\mathbf{l}'$ is the **epipolar line**.

The projection of $\mathbf{X}$ should be on the line $\mathbf{l}'$.

It is also the intersection of epipolar plane and image plane.



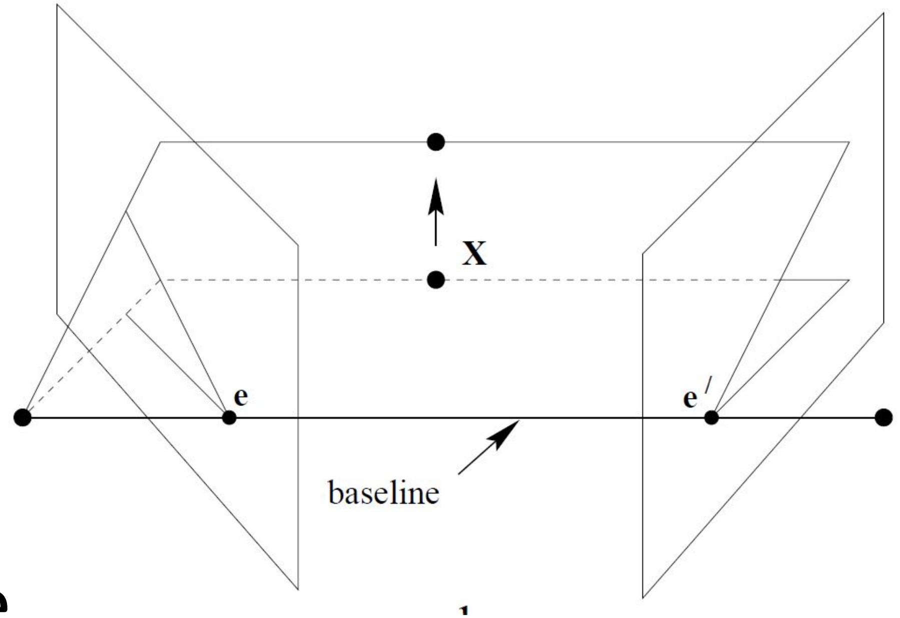
Epipolar geometry

Intersection of the epipolar planes is **baseline**.

C projects to $\mathbf{e}'$, that every epipolar line cross. The point $\mathbf{e}'$ is **epipole**.

The epipole is the intersection of the baseline and the image plane.

Every epipolar line intersect on the epipole.



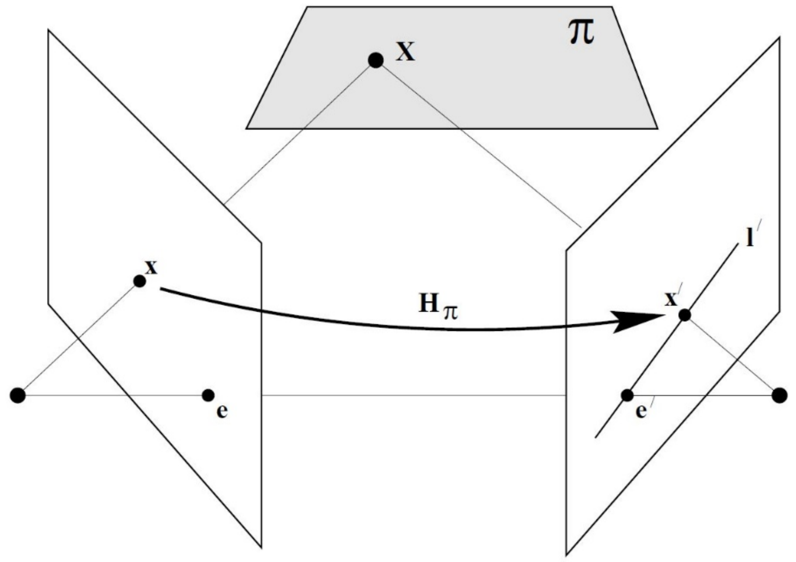
Fundamental matrix

We want to know the relation between $\mathbf{x}$ and $\mathbf{l}'$.

The line $\mathbf{l}'$ can be represented by

$$a'x' + b'y' + c' = 0$$

$\mathbf{l}'$ can be define as $\mathbf{l}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ $\quad \mathbf{l}'^T\mathbf{x}' = \mathbf{x}'^T\mathbf{l}' = 0$



The scale of $\mathbf{l}'$ can be changed.

Fundamental matrix

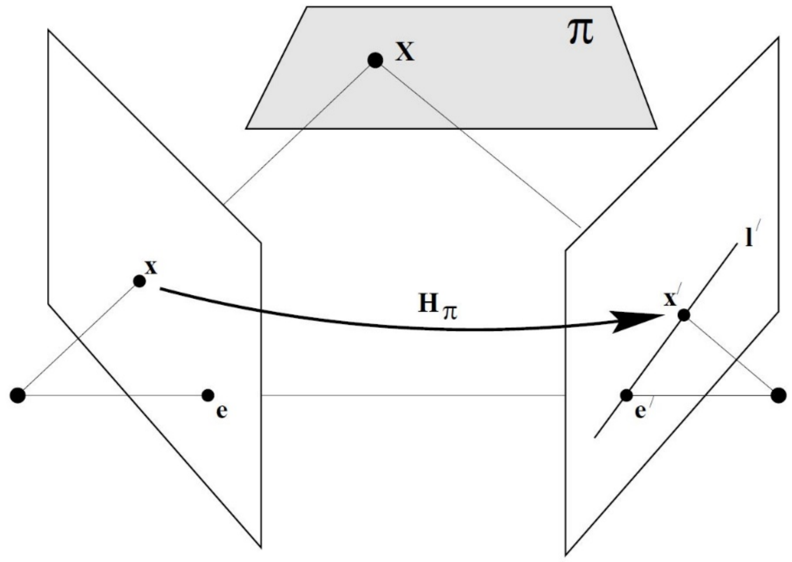
$\mathbf{l}'$ pass through $\mathbf{x}'$ and $\mathbf{e}'$.

$\mathbf{l}'$ is perpendicular to both $\mathbf{x}'$and $\mathbf{e}'$

$\mathbf{l}'$ can be written as $\mathbf{l}' = \mathbf{e}' \times \mathbf{x}'$

Cross product can be represent by multiplication with a skew-symmetric matrix

$$[\mathbf{e}']_\times = \begin{bmatrix} 0 & -e'_3 & e'_2 \\ e'_3 & 0 & -e'_1 \\ -e'_2 & e'_1 & 0 \end{bmatrix} \qquad \mathbf{e}' \times \mathbf{x}' = [\mathbf{e}']_\times \mathbf{x}'$$



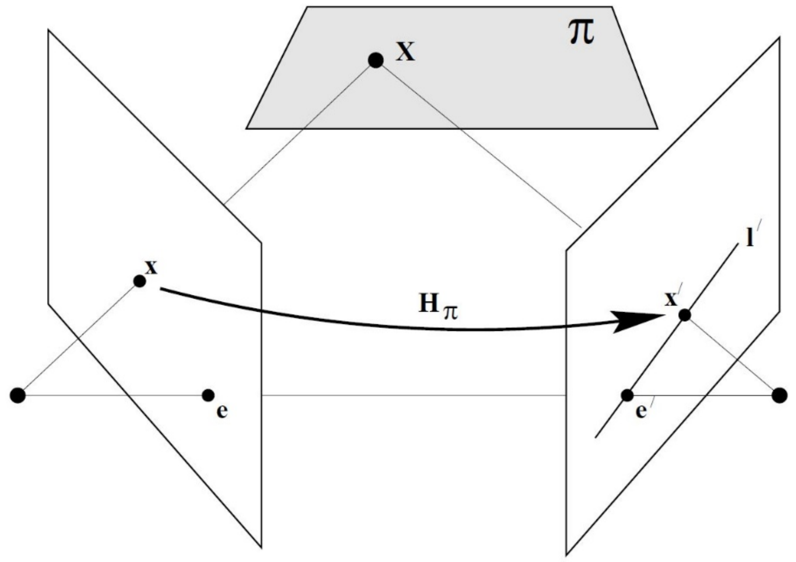
Fundamental matrix

$\mathbf{x}$ can project to **any** plane π. The projected point is $\mathbf{X}$.

The transformation from a 2D plane to another 2D plane is homography.

Homography can be represented by 3x3 non-singular matrix.

Again, $\mathbf{X}$ can project to the image plane. The projected point is $\mathbf{H}_{\pi}\mathbf{x}$, where $\mathbf{H}_{\pi}$ is the homography from the image plane through plane π to another image plane.



Fundamental matrix

$\mathbf{H}_{\pi}\mathbf{x}$ should be on the epipolar line $\mathbf{l}'$ **whether $\mathbf{H}_{\pi}\mathbf{x}$ is not same with $\mathbf{x}'$.**

Then, $\mathbf{l}'$ can be written as

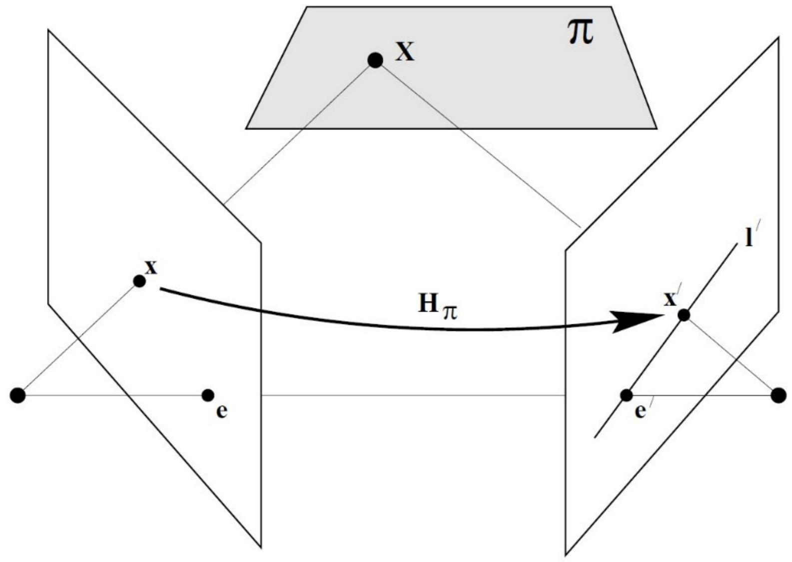
$$\mathbf{l}' = \mathbf{e}' \times \mathbf{H}_{\pi}\mathbf{x} = [\mathbf{e}']_{\times}\mathbf{H}_{\pi}\mathbf{x}$$

The fundamental matrix F is

$$\mathbf{F} = [\mathbf{e}']_{\times}\mathbf{H}_{\pi}$$

The relation between $\mathbf{x}$ and $\mathbf{x}'$ is

$$\mathbf{x}'^{T}\mathbf{F}\mathbf{x} = \mathbf{x}'^{T}\mathbf{l}' = 0$$



The properties of fundamental matrix

- $\mathtt{F}$ is a rank 2 homogeneous matrix with 7 degrees of freedom.
- **Point correspondence**: If $\mathbf{x}$ and $\mathbf{x}'$ are corresponding image points, then $\mathbf{x}'^{\mathsf{T}}\mathtt{F}\mathbf{x} = 0$.
- **Epipolar lines**:
  - ◇ $\mathbf{l}' = \mathtt{F}\mathbf{x}$ is the epipolar line corresponding to $\mathbf{x}$.
  - ◇ $\mathbf{l} = \mathtt{F}^{\mathsf{T}}\mathbf{x}'$ is the epipolar line corresponding to $\mathbf{x}'$.
- **Epipoles**:
  - ◇ $\mathtt{F}\mathbf{e} = \mathbf{0}$.
  - ◇ $\mathtt{F}^{\mathsf{T}}\mathbf{e}' = \mathbf{0}$.
- **Computation from camera matrices** $\mathtt{P}, \mathtt{P}'$:
  - ◇ General cameras,
    $\mathtt{F} = [\mathbf{e}']_{\times}\mathtt{P}'\mathtt{P}^{+}$, where $\mathtt{P}^{+}$ is the pseudo-inverse of $\mathtt{P}$, and $\mathbf{e}' = \mathtt{P}'\mathbf{C}$, with $\mathtt{P}\mathbf{C} = \mathbf{0}$.
  - ◇ Canonical cameras, $\mathtt{P} = [\mathtt{I} \mid \mathbf{0}]$, $\mathtt{P}' = [\mathtt{M} \mid \mathbf{m}]$,
    $\mathtt{F} = [\mathbf{e}']_{\times}\mathtt{M} = \mathtt{M}^{-\mathsf{T}}[\mathbf{e}]_{\times}$, where $\mathbf{e}' = \mathbf{m}$ and $\mathbf{e} = \mathtt{M}^{-1}\mathbf{m}$.
  - ◇ Cameras not at infinity $\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}]$, $\mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}]$,
    $\mathtt{F} = \mathtt{K}'^{-\mathsf{T}}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1} = [\mathtt{K}'\mathbf{t}]_{\times}\mathtt{K}'\mathtt{R}\mathtt{K}^{-1} = \mathtt{K}'^{-\mathsf{T}}\mathtt{R}\mathtt{K}^{\mathsf{T}}[\mathtt{K}\mathtt{R}^{\mathsf{T}}\mathbf{t}]_{\times}$.

Eight-point algorithm

We want to get a fundamental matrix from two images in different view.

In the images, there are matching points.

If there are m correspondences, they satisfy

$$\mathbf{x}_i'^T \mathbf{F} \mathbf{x}_i = 0 \quad i = 1, \cdots, m$$ where $$\mathbf{x}_i = \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix} \quad \mathbf{x}_i' = \begin{bmatrix} x_i' \\ y_i' \\ 1 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} f_{11} & f_{21} & f_{31} \\ f_{12} & f_{22} & f_{32} \\ f_{13} & f_{23} & f_{33} \end{bmatrix}$$

Eight-point algorithm

It can be represented by 9 unknown linear system.

$$\mathbf{Af} = 0$$

where

$$\mathbf{A} = \begin{bmatrix} xx' & xy' & x & yx' & yy' & y & x' & y' & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ xx' & xy' & x & yx' & yy' & y & x' & y' & 1 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix}$$

Eight-point algorithm

The only nonzero solution of $\mathbf{Ax} = 0$ can exist if $rank(\mathbf{A}) = 9 - 1 = 8$

Each correspondence make one equation (a row of $\boldsymbol{A}$)

It need eight points!

Eight-point algorithm: Implementation

1. $\mathbf{f} \leftarrow$ the eigenvector of $\mathbf{A}^T\mathbf{A}$ corresponding the smallest eigenvalue.
2. $\mathbf{F}$(3×3 fundamental matrix) $\leftarrow$ reshape($\mathbf{f}$)
3. $\mathbf{F} = \mathbf{USV}^{\mathrm{T}}$ where $\mathbf{U}$, $\mathbf{S}$, $\mathbf{V}$ are a singular value decomposition for $\mathbf{F}$.
4. Make the minimum singular value for $\mathbf{S}$ become zero. (the diagonal entries of $\mathbf{S}$ are the singular values for $\mathbf{S}$.)
5. $\mathbf{F} \leftarrow \mathbf{USV}^T$ by using modified $\mathbf{S}$ at 4.

Eight-point algorithm: Proof

If there are more than 8 correspondences, we should get an approximation.

$$\min_{\mathbf{f}} \|\mathbf{Af}\|^2 \quad \text{subject to} \quad \|\mathbf{f}\|^2 = 1$$

$$\mathbf{g}(\mathbf{f}) = \|\mathbf{Af}\|^2 = (\mathbf{Af})^T(\mathbf{Af}) = \mathbf{f}^T\mathbf{A}^T\mathbf{Af}$$

$$\mathbf{h}(\mathbf{f}) = 1 - \|\mathbf{f}\|^2 = 1 - \mathbf{f}^T\mathbf{f}$$

Eight-point algorithm: Proof

Make the Lagrangian of the optimization.

$$L(\mathbf{f},\lambda)=\mathbf{g}(\mathbf{f})-\lambda\mathbf{h}(\mathbf{f})=\mathbf{f}^T\mathbf{A}^T\mathbf{A}\mathbf{f}-\lambda\left(1-\mathbf{f}^T\mathbf{f}\right)$$

$$\min_{\mathbf{f}}\|\mathbf{A}\mathbf{f}\|^2 \quad s.t.\ \|\mathbf{f}\|^2=1 \quad \longrightarrow \quad \min_{\mathbf{f}} L(\mathbf{f},\lambda)$$

Eight-point algorithm: Proof

Take derivatives of the Lagrangian.

$$\partial_{\mathbf{f}} L(\mathbf{f}, \lambda) = \mathbf{A}^T \mathbf{A}\mathbf{f} - \lambda \mathbf{f} = 0$$

$$\partial_{\lambda} L(\mathbf{f}, \lambda) = 1 - \mathbf{f}^T \mathbf{f} = 0$$

$\mathbf{f}$ is normalized eigenvector of $\mathbf{A}^T \mathbf{A}$

Eight-point algorithm: Proof

Let $\mathbf{e}_\lambda$ is an eigenvector with eigenvalue λ.

$$\mathbf{g}(\mathbf{e}_\lambda) = \mathbf{e}_\lambda{}^T \mathbf{A}^T \mathbf{A} \mathbf{e}_\lambda = \mathbf{e}_\lambda{}^T \lambda \mathbf{e}_\lambda = \lambda$$

The eigenvector with the smallest eigenvalue is the result.

Eight-point algorithm: Proof

Is the result F have rank 2?

It is not guaranteed.

We should reduce the dimension by singular value decompostion.

- Get SVD of F $\mathbf{F} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

- Set the smallest singular values to 0

$$\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Recompute F $\hat{\mathbf{F}} = \mathbf{U}\hat{\boldsymbol{\Sigma}}\mathbf{V}^T$

Rectification

- There is two images J and J'.
- Matching points $\mathbf{x_i}$ and $\mathbf{x_i'}$ have the relation

$$\mathbf{x}_i'^T \mathbf{F} \mathbf{x}_i = 0$$

Rectification

- Make epipoles positioned at $[\infty, 0,1]^T$ or $[-\infty, 0,1]^T$
- Where are the positions of epipoles?

$$\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \qquad \mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \quad \mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \mathbf{e}'^T\mathbf{F} &= \mathbf{e}'^T\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \\ &= \mathbf{e}'^T \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{V}^T \\ &= \begin{bmatrix} \sigma_1\mathbf{e}'^T\mathbf{u}_1 & \sigma_2\mathbf{e}'^T\mathbf{u}_2 & 0 \end{bmatrix} \mathbf{V}^T \end{aligned}$$

Rectification

- By the property of the epipole,

$$\mathbf{e}'^T\mathbf{F} = \begin{bmatrix}\sigma_1\mathbf{e}'^T\mathbf{u}_1 & \sigma_2\mathbf{e}'^T\mathbf{u}_2 & 0\end{bmatrix}\mathbf{V}^T = 0$$

- V is non-singular matrix, so

$$\sigma_1\mathbf{e}'^T\mathbf{u}_1 = 0 \qquad \sigma_2\mathbf{e}'^T\mathbf{u}_2 = 0$$

$$\mathbf{e}' = \mathbf{u}_3$$

($\mathbf{e}'$ is perpendicular to both $\mathbf{u_1}$, $\mathbf{u_2}$,
and $\{\mathbf{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is an orthonormal basis for $\mathbb{R}^3$)

Rectification

- Translate to make principal point positioned at the image center.

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & -w/2 \\ 0 & 1 & -h/2 \\ 0 & 0 & 1 \end{bmatrix}$$

- Rotate to make y-coordinate of the epipole become zero.

$$\mathbf{RTe}' = \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix}$$

Rectification

- Make epipoles positioned at $[\infty, 0,1]^T$ or $[-\infty, 0,1]^T$.

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/k & 0 & 1 \end{bmatrix} \qquad \mathbf{GRTe}' = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{RTe}' = \begin{bmatrix} k \\ 0 \\ 1 \end{bmatrix}$$

- The homography for rectification of an image J' is

$$\mathbf{H}' = \mathbf{GRT}$$

Rectification

- You do not have to implement the homography matrix for the first camera, $\mathbf{H}$. The code for rectifying the first camera is already given.