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On the Complexity of Control of Selected Multiwinner Elections

Bachelor Thesis

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Personal Pronouns

In the chapters of this thesis, personal pronouns will be used when referring to candidates and voters. Regarding this, the approach of Chalkiadakis et al. [6] will be followed, which promotes an interleaved, (semi-) random usage of "he" and "she", since using one of these alone is inappropriate, whereas using "it" is just wrong and unacceptable. This was also done in *Economics and Computation* [39].

Keywords

Computational Social Choice; Multiwinner Elections; Voting Theory; Electoral Control; Single Non-Transferable Voting; Bloc Voting; Chamberlin-Courant Voting; Constructive Control; Destructive Control; Replacement Control; Computational Complexity; Computational Resistance

Abstract

Electoral theory is a central focus area of the rapidly growing field of computational social choice [33, 5] and provides a core framework for collective decision making [23], by concentrating on aggregating the preferences of individuals [4]; or more generally, voters. In practice, an election is characterized by a ballot response and the winner determination by the way these ballots are tallied, whereas they are formalized by social choice functions which map preference profiles to the outcome of the corresponding election [4]. This bachelor thesis deals with electoral control, which has been a prominent research interest as early as 1992 [1]. It is the malicious [28] act of when the chairman conducts structural changes in the election in order to influence its outcome to his own interest [2, P. 291]. More precisely, this thesis examines the complexity-theoretic aspects of controlling *multwinner* elections - a key variant of elections that has not been studied much in the past years [12], which aims to elect a fixed-sized set of candidates, also known as a "committee", instead of single candidates [33], as is the case in single-winner elections. This thesis considers three key voting rules, namely Single Non-Transferable Voting (SNTV), Bloc Voting and the Chamberlin-Courant Rule (CCR), providing new results about the first two, for the control scenarios of adding, deleting and replacing candidates. For Bloc Voting, the control scenario of replacing votes is also considered. Chapter 3 will show that all these control problems of both of these voting rules are **NP**-hard. There are three main foundations, which this thesis is based on. The seminal paper of Bartholdi, Tovey and Trick [1], the groundbreaking paper of Hemaspaandra, Hemaspaandra and Rothe [23] and the PhD Thesis of Andrea Loreggia [28]. The constructions used in the proofs within this thesis are all inspired by those used in the aforementioned papers with proper modifications to meet their individual requirements. After presenting new results regarding control by adding and deleting candidates for both SNTV and Bloc Voting, a very practical notion will be utilized to directly conclude further new results regarding replacement control (control by replacing either candidates or votes [30]). Independently, another formal proof will show one last new result regarding Bloc Voting for the scenario of replacing votes.

In conclusion, computational complexity provides a shield that protects voting systems from strategic actions, such as control [29] [2, P. 301], which lead to socially undesirable outcomes [32]. This thesis has recognized the significance of examining this, and successfully provided new results for multwinner elections, particularly regarding replacement control, contributing in filling the gap in the literature, with the aim to encourage a further analysis, considering other and more complex voting rules.

Chapter 1

Introduction

1.1 Computational Social Choice at a Glance

Computational Social Choice; hereinafter referred to as COMSOC, is a rapidly growing, young and interdisciplinary research field located at the intersection between computer science and social choice theory which covers topics from social and political sciences as well as economics [40, P. 1]. It has proven itself to be a core component in the research and development of multi-agent systems and artificial intelligence [16] while dealing with the computational aspects of collective decision making [5] and being closely related to the field of algorithmic game theory [40, P. 1]. All this allows COMSOC to apply the methods of computer science, such as algorithm design, complexity analysis, etc. on the mechanisms of Social Choice Theory, like voting rules and fair division methods [40, P. 8]. Moreover, COMSOC has great capabilities in transferring concepts of computer science to social choice theory, as is the case in multi-agent systems, network design, or the development of ranking algorithms [40, P. 8]. It encompasses a wide range of topics including Game Theory (cooperative and non-cooperative games), Voting Theory and Judgment (preference and judgment aggregation) as well as Fair Division [40], just to name a few.

In this thesis, the focus is going be on voting theory, i.e elections (formally defined in 2.2), and the complexity of controlling them, in particular. Although intuitively one might instantly think about political elections (presidential, parliamentary, etc.) when one speaks about voting, there are actually many other scenarios in which communities might need to run elections, or in simpler words, select an option (or a group of) among several others [21, 12]. These scenarios range over a wide spectrum and include for instance electing a leader of a group of people or an organisation (CEO of a company, captain of a sports team, etc...), or selecting a time and a place for a group of friends for their weekend-meeting, or when judges or referees filter the finalists of a competition based on their performance [21]. Notice how each of these situations *may* require a different voting rule[1] [21]. On a slightly more complex level, voting, as mentioned in [23], has been introduced as a decision-making mechanism not only in different computational settings such as planning [14, 15] and collaborative filtering [37], but in several large-scale computer settings, including web-page rank aggregation problem and the related spam reduction and similarity-search problems as well [11, 18].

1.2 History of Electoral Control

Electoral control (formally defined later in 2.3.2) was first introduced and studied in 1992 by Bartholdi, Tovey and Trick in their famous seminal paper *"How Hard is it to Control an Election?"* [1] in which they considered the constructive variant. They explored this for Plurality and Condorcet voting rules and the *standard control types* of adding, deleting as well as partitioning of either candidates or voters. Later in 2007, Hemaspaandra, Hemaspaandra and Rothe extended these results with their destructive counterparts in their paper *"Anyone but him"* [23], where in addition to the aforementioned voting rules, Approval Voting was also considered and studied in terms of its both variants. Since then, several other voting systems have also been studied in terms of the complexity of their control such as Borda by Russel in 2007 [41], Elkind at al., in 2011 [13] and later by Naveling and Rothe in 2017 [34], k-Approval by Lin in 2011 [26] and Maximin by Faliszewski et al. in 2011 [19] and later by Maushagen and Rothe in 2016 [31].

However, the complexity of control by *replacing* either candidates or votes was first studied by Loreggia et al. in 2014 and 2015 [27, 30] according to [17], followed by Erdélyi et al. in 2018 [17] and Naveling et al. in 2020 [36]. In 2021, Erdélyi et al. [16] complemented these results by solving remaining open cases for Copeland$^{\alpha}$, Maximin, k-Veto and Plurality and Veto with Runoff, among others. To the best of available information, Meier et al. [32] were the first ones to have ever explicitly addressed the complexity of control of *multiwinner* elections, as they investigated this in 2008 for the standard control types considering the following voting rules: Single Non-Transferable Voting, Bloc Voting, Approval-Voting and Cumulative-Voting. Subsequently, Y. Yang investigated the complexity of manipulation and control [42] as well as parameterized complexity of control [43] of several approval-based multiwinner voting rules in 2023.

[1]The terms "voting rules" and "voting systems" are used interchangeably as done in Meier et al. [32].

1.3 Motivation and Contribution

Even though one can consider some multiwinner elections to be generalizations of their single-winner variants, they have been studied much less [12] and only very recently. Faliszewski et al. [20] provided an axiomatic classification and hierarchy for these before they investigated the challenges that are paired with them [21]. As one can observe from the previous section, almost all earlier works that have studied the complexity of electoral control (those mentioned in section 1.2 as well as others), were concentrated on single-winner elections. Another aspect this bachelor's thesis considers is the fact that most papers that studied the complexity of control were concentrated on the constructive and destructive cases of the standard control types (as mentioned above), and it was until 2014 and 2015 when these cases were studied for the scenarios of *replacing* either candidates or votes, as mentioned above. Bearing all this in mind, one can only conclude that constructive and destructive control of *multiwinner* elections by *replacing* either candidates or votes might not have been much of an interest in the past years, but rather a gap in the literature which this thesis aims to fill. This would hopefully pave the road towards further investigation and exploration of multiwinner electoral control.

1.4 Structure of this Thesis

Chapter 2, serves as a foundational framework for this thesis. It starts with a glance at complexity theory, highlighting the complexity classes **P** and **NP**, as well as the notions of reducibility and hardness. Then, it introduces elections and formally defines them, with a focus on the multiwinner voting rules. Finally, it illustrates the common types of strategic behaviour in elections where the focus lies on electoral control. Chapter 3 is the core of this thesis. It starts with a summary of the key findings, followed by a thorough investigation of the complexity of control of Single Non-Transferable Voting (SNTV) and Bloc Voting. Using the notion of *Insensitive to Bottom-ranked Candidates (IBC)* helps proving the resistance of replacement control of voting rules satisfying this. First, it will be shown that these two voting rules are IBC, then resistant to constructive control by deleting candidates and destructive control by adding candidates. These results allow to directly conclude the resistance to constructive and destructive replacement control respectively. Independently, the resistance of Bloc Voting to constructive control by replacing votes will also be shown. Finally, Chapter 4 will conclude the important results that were presented in this thesis and suggest possible future and related work.

Chapter 2

Preliminaries

2.1 Complexity Theory

Since this thesis studies the *complexity-theoretic* aspects of controlling multiwinner elections, it is imperative to establish a solid foundation in complexity theory and the related subtopics. According to Rothe et al. [40, P. 9], for more than a half of a century, complexity theory; a branch of theoretical computer science that was evolved from the theory of computation, was dedicated to the study of the complexity of computation of problems. It deals with those that are algorithmically solvable and searches deeper about the computational cost, solving them requires [40, P. 9]. One of its key tasks is classifying problems based on their computational complexity [38, P. 54]. To put this in perspective, problems are sorted based on the computational cost required by their solutions with respect to some complexity measure (time, for instance), and those with roughly similar ones are then grouped in so called *complexity classes* [39, P. 20]. Considering time as a complexity measure, there are two pivotal complexity classes:

- Deterministic **P**olynomial Time: (**P**)
- **N**on-Deterministic **P**olynomial Time: (**NP**)

Assuming reader's familiarity with the foundations of theoretical computer science and particularly, the concept of (non)deterministic Turing machines, it is said that the complexity class **P** includes problems that can be solved by a deterministic Turing machine in polynomial time, which is considered to be efficient [39, P. 20]. This is because a polynomial function, for instance, $p(n) = n^3 + 5n + 8$ grows usually more moderately, in contrast to an exponential function [39, P. 20] such as $e(n) = 3^n$ as seen in Figure 2.1.

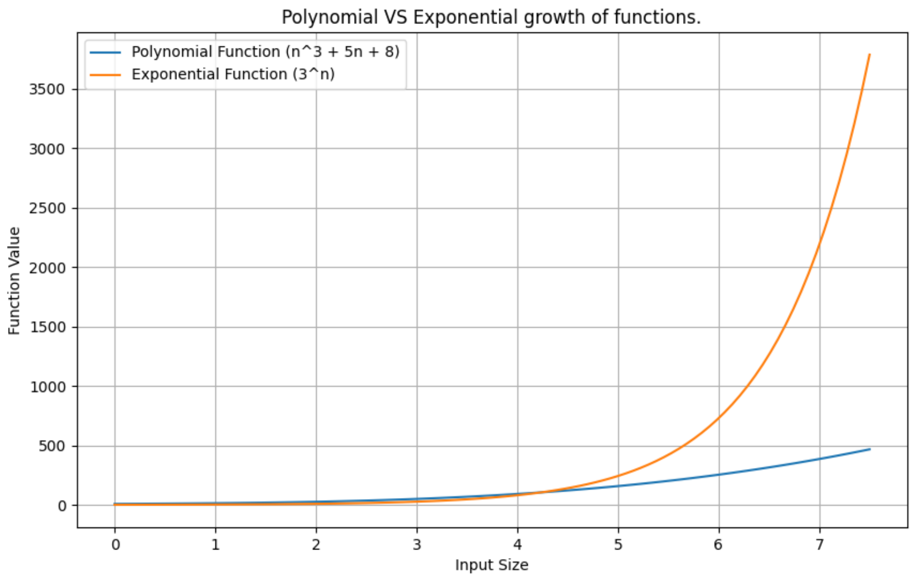

Figure 2.1: Polynomial VS Exponential growth of functions.

On the other hand, the complexity class **NP** includes problems that can be solved by nondeterministic Turing machines in polynomial time, which is viewed to be not efficient [39, P. 20, 21], due to the nondeterminism of the Turing machine. To put this in perspective [39, P. 21]: While attempting to simulate an **NP** algorithm deterministically, the deterministic algorithm would require an exponential time. Meaning, there can be up to $2^{p(n)}$ paths in the computation tree, if the runtime of an **NP** algorithm running an input of size n is bounded by a polynomial $p(n)$ and the **NP** algorithm is assumed to branch in each inner vertex of its computation tree, on any input, into at most two successor vertices. Verifying that there is really no accepting computation path is complete only after checking the very last path without success. [39, P. 21]
At this point, it is worth to mention that **P** =? **NP** is still the most important open problem in computer science, one of the seven *millennium problems* and a million-dollar question that is yet to be answered [39, P. 21].

While one can show an *upper bound* (given in the $\mathcal{O}$ notation) on the time complexity of a problem simply by providing a specific suitable algorithm that solves this problem in time within the time allowed by this upper bound [39, P. 22], things are slightly different and more complicated when it comes to showing a *lower bound*. In order to show that a problem is in **P**, one can simply provide a polynomial-time algorithm that solves the corresponding problem in time $p(n)$, for inputs of size n, where p is a polynomial and the problem is then considered efficiently solvable [39, P. 21]. However, to prove that a problem is in **NP**, another approach [39, P. 27, 28] is followed: One compares the complexity of a given problem with that of other problems in a given complexity class and tries to show that it is at least as hard to solve the given problem as it is to solve any of the other problems in the class. If succeeded, the considered problem is *hard* for the entire class, and one can speak of a problem's *lower bound*, as the related complexity class provides a lower bound for the problem at hand. This is because solving any problem in the class would be no harder than solving this problem. So if the latter belongs to this class, it is said to be *complete* for it. [39, P. 27, 28]

Reducibility and Hardness [39, P. 28]

The concept of *reducibility* can be helpful when it comes to comparing two given problems in terms of their complexity. It is a notion, on which above notions of hardness and completeness are based. Intuitively, a reduction of a decision problem A to a decision problem B means that all instances of A can be efficiently transformed into instances of B such that the original instances are yes-instances of A if and only if their transformations are yes-instances of B. In Chapter 3 of this thesis, the *polynomial-time many-one reducibility* among all other types will be used, because it allows different instances of A to be mapped to one and the same instance of B by the transformation. The following definition and lemma from *Economics and Computation* [39, P. 28, 29] will be used to demonstrate this better.

Definition 2.1. An *alphabet* is a finite, nonempty set of characters or symbols. Σ^* denotes the set of all strings over the alphabet Σ. A total function $f : \Sigma^* \to \Sigma^*$ is said to be *polynomial-time computable* if there is a an algorithm that, given any string $x \in \Sigma^*$, computes the function value $f(x)$ in polynomial time.
Let FP denote the class of all polynomial-time computable functions. Moreover, let A and B be two given decision problems encoded over the same alphabet Σ, meaning $(A, B \subseteq \Sigma^*)$ and $\mathcal{C}$ a complexity class. It is said that:

1. A is *polynomial-time many-one reducible* to B ($\leq_m^p$) if there is a function $f \in FP$ such that for each $x \in \Sigma^* : x \in A \iff f(x) \in B$.

2. B is $\leq_m^p$*-hard* for $\mathcal{C}$ (hereinafter: $\mathcal{C}$*-hard*) if $A \leq_m^p B$ for each set $A \in \mathcal{C}$.

3. B is $\leq_m^p$*-complete* in $\mathcal{C}$ (hereinafter: $\mathcal{C}$*-complete*) if B is $\mathcal{C}$*-hard* and $B \in \mathcal{C}$.

4. $\mathcal{C}$ is closed under the $\leq_m^p$-*reducibility* (hereinafter: $\leq_m^p$-*closed*) if for any two problems A and B, it follows from $A \leq_m^p B$ and $B \in \mathcal{C}$ that A is in $\mathcal{C}$.

Lemma 2.1.

1. ***P** and **NP** are $\leq_m^p$ closed.*

2. *If $A \leq_m^p B$ and B is in **P**, then A is in **P**.*

3. *If $A \leq_m^p B$ and A is **NP**-hard, then B is **NP**-hard.*

Assuming that $\mathbf{P} \neq \mathbf{NP}$, the proofs of **NP**-hardness in Chapter 3 of the problems demonstrated in this thesis are going to make use of the notion of *polynomial-time many-one reduction*, the aforementioned properties as well as the following established [22] NP-hard problems: *Hitting Set* and *Exact Cover by Three Sets*.

	Hitting-Set
Given:	A set $B = \{b_1, b_2, \cdots, b_m\}$, a family $\mathcal{S} = \{S_1, S_2, \cdots, S_n\}$ of subsets $S_i \subseteq B$ and a positive integer k.
Question:	Is there a subset $B' \subseteq B, \lvert B' \rvert \leq k$, such that each $S_i \in \mathcal{S}$ is *hit* by B', in other words: $S_i \cap B' \neq \emptyset$ for all $S_i \in \mathcal{S}$?

	Exact-Cover-By-Three-Sets (X3C)
Given:	A set $B = \{b_1, \cdots, b_m\}$, $m = 3k, k \geq 1$ and a family $\mathcal{S} = \{S_1, \cdots, S_n\}$ of subsets $S_i \subseteq B$ with $\lvert S_i \rvert = 3$, for each i, $1 \leq i \leq n$.
Question:	Is there a subfamily $\mathcal{S}' \subseteq \mathcal{S}$ such that every element of B appears in exactly one subset of $\mathcal{S}$?

2.2 Elections

This section is going to deal with elections in general. Starting with a formal definition, it distinguishes between the two main types of elections, namely single-winner and multiwinner, backing each with proper examples. For this, refer to *"Preference Aggregation by Voting"* in *Economics and Computation* [2].

Definition 2.2. A *voting procedure*, or simply, an *election* $E = (C, V)$ is a pair, where $C = \{c_1, c_2, \cdots, c_m\}$ is the set of candidates whereas $V = (v_1, v_2, \cdots, v_n)$ is the list of votes, by which voters are represented. This list is also referred to as the preference profile, since these votes express the preferences of their corresponding voters over the candidates in C. Note that V is defined as a list, instead of a set like C, given the fact that preferences could be mutual between distinctive voters. [2, P. 198]

How votes are represented differs from one voting rule to another, but most of them require preference lists, in which voters rank the candidates in decreasing order corresponding to the degree of preference using the "$\succ$" symbol [2, P. 199]. In the chapters of this thesis this symbol will be omitted, and "$a \succ b \succ c$", for instance, will simply be referred to as "$a\ b\ c$". Mathematically speaking, a preference profile is expressed by a (strict) linear order on C, meaning the "$\succ$" has the following properties [2, P. 199]:

- *Connected*: For each two distinct candidates $a, b \in C,\ a \succ b$ or $b \succ a$
- *Transitive*: For each three candidates $a, b, c \in C,\ a \succ b \wedge b \succ c \implies a \succ c$
- *Asymmetric*: For each two candidates $a, b \in C,\ a \succ b \implies$ ($b \succ a$ does not hold).

For a candidate set $C = \{c_1, c_2, \cdots, c_m\}$, the notation $|C|$ refers to the cardinality of the set C. Moreover, in a preference profile, using the symbol of a candidate set C means that candidates in C follow an arbitrary order, whereas the notation $\overrightarrow{C}$ denotes that these follow an ascending (lexicographic) order. This means the ranking looks like this: $c_1 \succ c_2 \succ \cdots \succ c_m$. Similar notations have been used in [35] and [28].

Definition 2.3. A *voting system* or a *voting rule* $\mathcal{R}$ is what defines how candidate(s) win an election [2, P. 198]. Formally, this is given by a mapping

$$f : \{(C, V) \mid (C, V) \text{ is an election}\} \rightarrow 2^C$$

also known as a *social choice correspondence* [2, P. 198]. Moreover, a *social choice function*

$$f : \{(C, V) \mid (C, V) \text{ is an election}\} \rightarrow C$$

maps a given election to a single winner [2, P. 198, 199].

For an election $E = (C, V)$, $f(E) \subseteq C$ refers to the set of winners of E, which in some cases might also be empty [2, P. 198].

While there are many types of voting rules, such as Approval and Condorcet-consistent methods [28], in this thesis, the focus is going to be on scoring rules.

Definition 2.4. *Scoring rules* are a notable group of voting rules [32]. They differ from each other by the scoring vector they use to grant each candidate points based on the ranked position by its corresponding voter(s) [28]. The sum of the points gives the total score of the candidate, and the one(s) with the highest score win [27, 28]. Formally, such vector, for m candidates for instance, has the form: $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$, satisfying $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$, where each α_i is a nonnegative integer [2, P. 200]. So in this case, the candidate taking the i-th position in the preference list of a voter, receives a_i points from this voter [2, P. 200].

2.2.1 Single-Winner Elections

This subsection is going to demonstrate simple examples of some well-known single-winner voting rules, especially since the multiwinner ones that are going to be examined in terms of complexity of control are generalizations [12] of these or are closely related to each of them. The following are three of the most popular scoring rules.

- *Plurality*: Plurality is one of the most popular voting rules and has the scoring vector of $\alpha = (1, 0, \cdots, 0)$ [2, P. 200]. In other words, a candidate $c \in C$ gets one point each time she is ranked at the top in the preference profile [9]. This score is referred to as the *plurality score* [9] of candidate c and in this thesis, this is denoted by $pl(c)$.

- *Borda:* Borda-count, or Borda score is a very famous scoring protocol, which rewards the candidate in the i-th position in the preference list $m - i$ points, such that the candidate $c \in C = \{c_1, \cdots, c_m\}$ who is ranked last gets 0 points, whereas the candidate who is ranked first gets $m - 1$ points [2, P. 201]. Hence, the scoring vector of Borda is: $\alpha = (m - 1, m - 2, \cdots, 0)$ [2, P. 201].

- *k-Approval:* Another important and well-known scoring protocol is k-Approval. For a positive integer k, $k \leq m$ for a candidate set C, where $|C| = m$, k-Approval has the scoring vector of $\alpha = (\underbrace{1, \cdots, 1}_{k}, 0, \cdots, 0)$, meaning it rewards a candidate $c \in C$ one point for each time she is ranked in the top k positions [2, P. 201]. The notation *score*(c) is used when referring to the k-Approval score of candidate c, otherwise "$k - Approval\_score(c)$" if it is not clear from the context. It is clear at this point that $1-$Approval is Plurality [2, P. 201].

Unique VS Nonunique-winner: [2, P. 198]

For an election $E = (C, V)$, in some settings, there arises the question whether a candidate $c \in C$ is a winner (while it is fine if there are other winners too). Here, c is referred to as the *nonunique-winner*. While in other settings, the question is if c is the one and only winner while referring to her as the *unique winner*.

While certainly numerous other noteworthy and significant scoring protocols exist, this thesis is focused on multiwinner elections. Therefore, any further discussion about single-winner elections would fall outside the scope of this work; something which will be refrained from.

2.2.2 Multiwinner Elections

"Multiwinner elections are even more ubiquitous than single-winner ones, but much less studied". [12]

Definition 2.5. A multiwinner election is a triple given as $E = (C, V, q)$, where similarly to single-winner elections, C and V stand for the set of candidates and the list of votes respectively, whereas $q,\ 1 \leq q \leq |C| = m$ refers to the target committee size (see below).

Definition 2.6. For a given multiwinner election $E = (C, V, q)$, a *committee selection rule* (or, a multiwinner rule) $\mathcal{R}$ is a function that returns a non-empty set $\mathcal{R}(E, q)$ of q-element subsets of C, which are referred to as committees. [12]. This is denoted by: $\mathcal{R}(E, q) = W = \{W_1, W_2, \cdots, W_n\}$ where $|W_i| = q,\ \forall\ 1 \leq i \leq n$.

Since for $q = 1$ one would be back to the case of single-winner elections, and for $q = |C|$ the solution is trivial, this thesis is going to consider target committee size values of only $1 < q < |C|$.

Just like in [21], the parallel-universe tie-breaking model [8] is assumed, where a voting rule outputs all the committees that could end up winning for some way of resolving ties that occur while executing the rule.

Definition 2.7. In a multiwinner election $E = (C, V, q)$, where $W = \{W_1, W_2, \cdots, W_s\}$ is the set of all possible winning committees with $|W_i| = q$, a candidate $c \in C$ is considered to be a:

- **Unique winner:** If she is a member of every possible winning committee, i.e $\forall W_i \in W,\ c \in W_i$. So for a target committee size q, there can be at most q unique winners in a single multiwinner election.

- **Nonunique-winner:** If she is among all possible winners, and there exists a tie-breaking rule that might elect a committee that she is not a member of. Formally, if $c \in \bigcup_{1 \leq i \leq n} W_i \wedge \exists i,\ 1 \leq i \leq n$ such that $c \notin W_i$.

- **Unique loser:** If there is no winning committee $W_i \in W$ such that $c \in W_i$, i.e. $\forall W_i \in W,\ c \notin W_i$.

- **Nonunique-loser:** A nonunique-loser is equivalent to a nonunique-winner. The difference comes from the perspective, from where c is looked at.

Having established a formal framework for multiwinner elections, the attention can now be drawn to defining the voting rules that will be investigated within the scope of this thesis.

- *Single Non-Transferable Voting (SNTV):* SNTV is the multiwinner variant of Plurality, for it returns the q candidates with the highest plurality scores [12]. Hence, one might also think of it as q-Plurality [12]. Again, for a candidate $c \in C$, the notation $pl(c)$ is used while referring to her plurality score.

- *Bloc Voting:* Bloc Voting is the multiwinner variant of k-Approval voting [12]. Since q is used while referring to the target committee size, it is said that Bloc Voting returns the q candidates with the highest q-Approval scores [12]. Doing this ensures that the unique winning committee consists of the candidates ranked in top q positions by all the voters when all their votes rank the candidates similarly, (whereas for s-Approval with $s \neq q$ this is not the case) [12].

- *Chamberlin-Courant:* Introduced by John R. Chamberlin and Paul N. Courant [7], the Chamberlin-Courant Rule (CCR) aims explicitly at proportional representation [12] and combines it with the Borda Rule [9]. It was designed with the objective to pursue finding a diverse committee such that its members accurately reflect the whole electorate [9]. In other words, given the target committee size and the set of all possible committees, the CCR selects the committee(s) that maximize(s) the *representativeness value*, which for a given committee, is the sum of the Borda weights (a.k.a Borda scores as defined in 2.2.1) of candidates that better represent their voters in this committee [9]. This thesis considers the *utilitarian* variant of CCR as it was introduced by Chamberlin and Courant themselves in 1983 [7]. Under this rule, one fixes a scoring vector of length m, and defines each vector's score for a given committee as the score that is assigned to the most preferred candidate in that committee, with the aim of finding a committee that maximizes the joint scores of all voters [9]. Formally, let r_{ix} be the rank of the candidate x in voter i's ranking and $w(r_{ix}) = m - r_{ix}$ the corresponding Borda weight [9]. $N_x(C, \pi)$ denotes the set of voters for which the representative in committee C is candidate x for profile π, i.e for the corresponding profile π, x is the most preferred candidate in the committee C for all voters in $N_x(C, \pi)$ and is given by the formula [9] :

$$\alpha(C, \pi) = \sum_{x \in C} \sum_{i \in N_x(C,\pi)} w(r_{ix})$$

For a candidate set of C, where $|C| \geq 3$ and a target committee size of $q = m - 1$, Kamwa and Merlin [24] showed in 2014, that the CCR is equivalent to SNTV [9]. Furthermore, it is noteworthy that the problem of winner determination of CCR belongs to the class of **NP**-hard problems [3, 9].

The following example shows how voting rules in Section 2.2.2 elect their committees.

Example 2.1. Let $E = (C, V, q)$ be a multiwinner election, where $C = \{a, b, c, d, e\}$ is the set of candidates, and V is the list of votes that represent the preferences of their corresponding voters, distributed as can be seen in Table 2.1. For a target committee size of $q = 3$, this example is going to illustrate how the three above-mentioned multiwinner voting systems choose their winning committees, in order to highlight the differences between each of them.

Voter group	Number of Votes	Form
v_1	21	$a\ b\ c\ d\ e$
v_2	13	$d\ e\ b\ c\ a$
v_3	8	$b\ e\ c\ a\ d$
v_4	5	$e\ c\ d\ b\ a$

Table 2.1: Voter groups of Example 2.1

Now, the plurality and q-Approval scores of each of the candidates in C, as well as the $\binom{5}{3} = 10$ representativeness values of all 3-element committees in the preference profile of E can be calculated.

$pl(a) = 21$	$pl(c) = 0$	$pl(e) = 5$
$pl(b) = 8$	$pl(d) = 13$	

Table 2.2: Plurality scores of candidates in Example 2.1

$score(a) = 21$	$score(c) = 21 + 8 + 5 = 34$	$score(e) = 13 + 8 + 5 = 26$
$score(b) = 21 + 13 + 8 = 42$	$score(d) = 13 + 5 = 18$	

Table 2.3: q-Approval scores of candidates in Example 2.1

$\alpha(\{a, b, c\}, \pi) = 21(4) + 13(2) + 8(4) + 5(3) = 157$	
$\alpha(\{a, b, d\}, \pi) = 21(4) + 13(4) + 8(4) + 5(2) = 178$	
$\alpha(\{a, b, e\}, \pi) = 175$	$\alpha(\{a, c, d\}, \pi) = 167$
$\alpha(\{a, c, e\}, \pi) = 167$	$\alpha(\{a, d, e\}, \pi) = 180$
$\alpha(\{b, c, d\}, \pi) = 162$	$\alpha(\{b, c, e\}, \pi) = 154$
$\alpha(\{b, d, e\}, \pi) = 167$	$\alpha(\{c, d, e\}, \pi) = 138$

Table 2.4: Representativeness values of committees in Example 2.1

From Table 2.2 it can clearly be seen that the committee $\{a, d, b\}$ is the SNTV winner, whereas the committee $\{b, c, e\}$ is the winning committee in Bloc Voting, per Table 2.3. The CCR, on the other hand, elects the committee $\{a, d, e\}$, according to Table 2.4.

As seen in Example 2.1, different voting rules lead to (potentially) different winning committee(s). This highlights the importance of selecting the most appropriate voting system with respect to its applicability, be it a parliamentary election, or a shortlisting process for university positions, or even the movie selection for an airline's in-flight enterta inment system on long-distance flights [12].

Nevertheless, there are some cases, in which a party or an entity aims to achieve a specific outcome from the election [28], regardless of the voting rule and its applicable scenario. This is commonly known as strategic or tactical behavior [28] which will be investigated in the following section. It is however noteworthy that such actions are usually considered malicious [28], so in order to avoid any socially undesirable outcomes from elections, it is crucial to refrain from voting rules that allow these [32].

2.3 Strategic Behavior in Elections

One of the interesting properties of elections is the *strategy proofness*.

Definition 2.8. [2, P. 238] A voting system is *strategy proof* if no voter can benefit from reporting an untruthful vote. In practice, this means that there is no voter who has a full knowledge of other voters' preferences and is therefore able to achieve a better outcome in the election by reporting insincere vote over the candidates.

2.3.1 Manipulation and Bribery

- **Manipulation:** A voting rule is manipulable if it does not satisfy strategy proofness [2, P. 238]. It is the scenario of when some voters taking part in the election, also called *manipulators*, misreport their preferences, in order to improve the election result in their favor [42]. In order to avoid socially undesirable outcomes from elections, it is crucial to refrain from voting rules whose manipulation is possible [32]. Regrettably, the celebrated Gibbard-Satterthwaite theorem insists that, every voting rule (even a multiwinner one [10]) with three or more candidates, that is not a dictatorship (i.e. the outcome of the election does not depend on a single voter [2, P. 232]) is manipulable [32].

- **Bribery:** Bribery is another way to influence the outcome of an election. One speaks of bribery when there is an external actor, called *the briber*, who conducts changes in the preference lists [2, P. 317, 318] in order to achieve a certain outcome in the election, in most cases, with limited resources or budget.

2.3.2 Control

Electoral control is the third way an entity can influence the outcome of an election for their own benefit. In contrast to manipulation and bribery, the entity or party who interferes in the election is called the *election chair* or *the chairman* and conducts structural changes in the election, depending on the desired outcome [2, P. 291]. For instance, he may decide which voters can vote or which candidates can be considered [28]. It is important not to forget to mention that, similarly to [1], in this thesis, it is also assumed that the chairman has full information about the preferences of the voters and is confident in their sincerity when casting their votes.

Constructive VS Destructive Control

A control action is:

- **Constructive:** If the chairman's goal is to turn a distinguished candidate, who is initially not a winner, to a winner [2, P. 301]. In this thesis, a unique winner.
- **Destructive:** If the chairman's goal is to prevent a distinguished candidate from winning the election, i.e. turning a winner into a non-winner [2, P. 301]. In this thesis, a unique loser.

Immunity, susceptibility, resistance and vulnerability

It is so far unknown, whether a theorem equivalent to Gibbard-Satterthwaite's one exists for the different electoral control types (Definition 2.10), which is why there might be some cases, in which the chairman's goal in controlling the election might not be reached [2, P. 300, 301].

Definition 2.9. Let $\mathcal{CT}$ be a control type and $E = (C, V, q)$ a multiwinner election:

- **Immunity:** A voting system is *immune* to $\mathcal{CT}$ if it is impossible for the chairman to make a candidate $c \in C$ a unique winner in the constructive case [1] or a unique loser in the destructive case.
- **Susceptibility:** A voting rule is susceptible to $\mathcal{CT}$ if it is not immune to the control type $\mathcal{CT}$ [1].

Moreover, every voting rule that is susceptible to $\mathcal{CT}$ is [1, 23]:

- **Vulnerable to $\mathcal{CT}$:** If the control problem corresponding to $\mathcal{CT}$ can be solved in polynomial time.
- **Resistant to $\mathcal{CT}$:** If the control problem corresponding to $\mathcal{CT}$ is **NP**-hard.

Furthermore, the following control scenarios[1] will be considered in this thesis.

Definition 2.10. Let $\mathcal{R}$ be a voting system. Each control scenario has its *constructive* variant which is due to Bartholdi et al. in 1992 [1], as well as its *destructive* variant, which was initiated by Hemaspaandra et al. in 2007 [23]. Replacement control of both variants, however, is due to Loreggia et al. [27, 30, 28].

[1] Note that these are just some of many control scenarios that exist. Defining the rest falls outside the scope of this thesis.

- **Control by Adding Candidates (CCAC/ DCAC): [2, P. 293]**

$\mathcal{E}$-Constructive (Destructive) Control by Adding Candidates	
Given:	Two sets C and D of candidates where $C \cap D = \emptyset$, a list of votes over $C \cup D$ and a distinguished candidate $c \in C$. In the case of adding *limited* number of candidates, there is also a positive integer given where $k \leq \lvert D \rvert$
Question:	Is there a subset $D' \subseteq D$ (with $\lvert D' \rvert \leq k$ in case of adding limited number of candidates) such that c is a unique winner (loser) of the election $(C \cup D', V, q)$?

- **Control by Deleting Candidates (CCDC/ DCDC): [2, P. 294]**

$\mathcal{E}$-Constructive (Destructive) Control by Deleting Candidates	
Given:	A set of candidates C, a list of votes V over C, a distinguished candidate $c \in C$ and a nonnegative integer $k \leq \lvert C \rvert$
Question:	Is it possible to delete at most k candidates from C such that c is a unique winner (loser) of the election? Meaning is there a $C' \subseteq C$ with $\lvert C \setminus C' \rvert \leq k$ such that c is a unique winner (loser) of the election (C', V, q)

- **Control by Replacing Candidates (CCRC/ DCRC): [28]**

$\mathcal{E}$-{Constructive, Destructive} Control by Replacing Candidates	
Given:	A collection of votes V over $C_1 \cup C_2$ with $C_1 \cap C_2 = \emptyset$, a distinguished candidate $c \in C_1$ and a positive integer k
Question:	**(Constructive):** Are there subsets $A \subseteq C_2$ and $D \subseteq C_1$ such that $\lvert A \rvert = \lvert D \rvert \leq k$ and c is a unique winner of the election $((C_1 \setminus D) \cup A, V, q)$?
Question:	**(Destructive):** Are there subsets $A \subseteq C_2$ and $D \subseteq C_1$ such that $\lvert A \rvert = \lvert D \rvert \leq k$ and $c \in (C_1 \setminus D)$ is a unique loser of the election $((C_1 \setminus D) \cup A, V, q)$?

- **Control by Replacing Votes (CCRV/ DCRV): [28]**

$\mathcal{E}$-Constructive (Destructive) Control by Replacing Votes	
Given:	Two collection of votes V_1, V_2 with $V_1 \cap V_2 = \emptyset$, over C, a distinguished candidate $c \in C$ and a positive integer k
Question:	Are there subsets $A \subseteq V_2$ and $D \subseteq V_1$ such that $\|A\| = \|D\| \leq k$ and c is a unique winner (loser) of the election $(C, (V_1 \setminus D) \cup A, q)$?

To prove resistance of constructive or destructive control by replacing candidates, this thesis will make use of the helpful notion of *Insensitive to Bottom-ranked Candidates (IBC)*, which was defined by J. Lang et. al in [25, P. 20] as follows:

Definition 2.11. A voting rule $\mathcal{R}$ is *IBC* if its set of winners does not change after adding or deleting a subset of candidates at the bottom of the preference profile.

And the following two proven theorems from Loreggia et al. [28].

Theorem 2.1. *Every voting rule that is IBC and resistant to CCDC is also resistant to CCRC.*

Theorem 2.2. *Every voting rule that is IBC and resistant to DCAC or DCDC is also resistant to DCRC.*

Chapter 3

Complexity of Control

3.1 Overview of Results

The final results of this thesis are summarized in Table 3.1, where “R” stands for *resistant*, indicating that the corresponding control problem is **NP**-hard.

Problem	SNTV	Bloc Voting
CCDC	R	R
DCAC	R	R
CCRC	R	R
DCRC	R	R
CCRV	?	R*

* For target committee size values of $q > 2$

Table 3.1: Summary of the results.

3.2 Single Non-Transferable Voting

This chapter will start by STNV. To prove its resistance of constructive and destructive control by replacing candidates, Theorems 2.1 and 2.2 will be considered. To do this, STNV's insensitivity to bottom-ranked candidates will be shown first.

Lemma 3.1. *SNTV is insensitive to bottom-ranked candidates.*

Proof. This is clear from the definition, as SNTV has the scoring vector $(1, 0, \cdots, 0)$. Adding or deleting candidate(s) at the bottom of the preference profile has no influence on the winning committees. Formally, let t, r be large enough positive integers, such that

$r > t$ and $C = \{a_1, \cdots, a_m\}$ a set of candidates. Moreover let $E = (C \cup \{a_{m+1}\}, V, q)$ be a multiwinner election, where V is the list of votes which looks like the following:

Number of Votes	Form
r	$a_1 \cdots a_{m+1}$
$\vdots$	$\vdots$
r	$a_q \cdots a_{m+1}$
t	$a_{q+1} \cdots a_{m+1}$

Where "$\cdots$" mean the remaining candidates of C follow in arbitrary order.
Based on the votes in V, the committee $\{a_1, \cdots, a_q\}$ wins the election. Adding the candidate a_{m+1} at the bottom of the preference profile, or removing him from there if he ever existed has no effect whatsoever on the winning committee. □

Now, one can show that SNTV is resistant to CCDC.

Theorem 3.1. *SNTV is resistant to constructive control by deleting candidates.*

Proof. For this proof, a similar approach as in [1] and [23] will be followed. The authors in [1] showed that Plurality is resistant to CCDC. Here, a similar construction will be used alongside a reduction from the Exact-Cover by 3-Sets problem (see Chapter 2.1), which can be executed in polynomial time.
Given a set $B = \{b_1, \cdots, b_m\}$, with $m = 3k$ and a family $\mathcal{S} = \{S_1, \cdots, S_n\}$ of three element subsets of B. Let b_i^1, b_i^2, b_i^3 denote the elements of S_i. Without loss of generality, assume $k \geq 5$ and construct a multiwinner election $E = (C, V, q)$ as the following:

- **Candidates:** $C = \{c, w\} \cup A \cup B \cup D$ where c is the distinguished candidate, $A = \{a_1, \cdots, a_q\}$ and $D = \{d_1, \cdots, d_{m/3}\}$. Moreover, for each i, $1 \leq i \leq n$ there are s_i candidates corresponding to the $S_i \in \mathcal{S}$.

- **Votes:** Which are divided into voter groups as can be seen in Table 3.2.

Now, based on those votes, the plurality score of each of the candidates can be calculated.

- $pl(s_i) = 4$

- $pl(w) = m/3 - 1$

- $pl(a_1) = m/3$

Voter group	Number of Votes	Form
v_1	1 for each $i,\ 1 \leq i \leq n$	$s_i\ c\ \cdots$
v_2	1 for each $i,\ 1 \leq i \leq n$	$s_i\ b_i^1\ D\ \cdots$ $s_i\ b_i^2\ D\ \cdots$ $s_i\ b_i^3\ D\ \cdots$
v_3	$m/3 - 1$	$w\ D\ \cdots$
v_4	$m/3$	$a_1\ D\ \cdots$
v_5	$m/3 - 2$ for each $r,\ 2 \leq r \leq q$	$a_r\ D\ \cdots$
v_6	$m/3 - 2$ for each $j,\ 1 \leq j \leq m$	$b_j\ D\ \cdots$

Table 3.2: Voter groups for CCDC in SNTV

- $pl(a_r) = m/3 - 2$ for $2 \leq r \leq q$
- $pl(b_j) = m/3 - 2$
- In particular: $pl(c) = 0$

Since there are at least q candidates who have a higher plurality score than the distinguished candidate's, he can never be a member of any winning committee. Thus, c loses the election.

To prove the theorem, the following claim will be proved.

Claim 3.1. *If the above X3C instance has a solution $S' \subseteq S$, then candidate c can be made a unique winner of the election $E = (C \setminus S', V, q)$ by deleting at most $m/3$ candidates.*

Proof. Assume there is a "yes" instance $S' \subseteq \mathcal{S}$, a.k.a. an exact 3-cover for the above X3C problem. Delete the candidates s_i corresponding to the S_i where $S_i \in S'$. Since the exact cover has a cardinality of $m/3$, the plurality score of the candidates will be altered as the following:

- There are $n - m/3$ many remaining s_i candidates for $i,\ 1 \leq i \leq n$ who keep their score, hence: $pl'(s_i) = pl(s_i) = 4$
- $pl'(w) = pl(w) = m/3 - 1$

- $pl'(a_1) = pl(a_1) = m/3$
- $pl'(a_r) = pl(a_r) = m/3 - 2$
- $pl'(b_j) = m/3 - 2 + 1 = m/3 - 1$, after receiving one vote each in voter group v_2.
- And particularly: $pl'(c) = m/3$ after $|S'| = m/3$ many s_i candidates were removed from the election.

It can now be noticed that candidates c and a_1 each have the highest plurality score among the rest of the candidates. Since this thesis considers target committee size values of $2 \leq q \leq |C|$, candidate c (just like a_1) is now a member of every winning committee of the election, i.e a unique winner. □

To complete the proof of Theorem 3.1 the following claim needs to be proved too:

Claim 3.2. *Let $D \subseteq C$. If c is a unique winner of the election $E = (D, V, q)$, by deleting at most $m/3$ candidates, then there exist sets $\mathcal{X}$ and $S' \subseteq \mathcal{S}$ such that:*

1. *$\mathcal{X} = \{s_i \mid s_i$ corresponds to $S_i \in S'\}$*
2. *S' is an exact 3-cover for the X3C instance $(\mathcal{S}, B)$*
3. *$\mathcal{X}$ and S' have the cardinality of exactly $m/3$.*
4. *$D = C \setminus \mathcal{X}$*

Proof. For this proof, assume there exists a subset $\mathcal{X}$ of no more than $m/3$ candidates, whose deletion would lead to candidate c's victory in the above election. Based on the above construction, c would hope to get votes from voter group v_1, because his position in the other groups is lower than $m/3$. That being said, all deletions must be from candidates s_i and in that way, c would get *no more* than $m/3$ votes. However, c should also receive *no less* than $m/3$ votes, since that would tie him with w and the b_j's, which will result in him being a nonunique winner for target committee size values of two, three and four. Getting less than $m/3$ votes might also tie c with candidates a_r and not be sufficient to win the election at all. Thus, c must receive *exactly* $m/3$ votes, which can only be achieved by deleting those candidates s_i. This, so far, proves the points *1*, *3* and *4*.

Moreover, the sets $S_i \in S'$ which the candidates $s_i \in \mathcal{X}$ correspond to, must comprise an exact 3-cover for the X3C instance. To show this, one assumes initially otherwise. After deleting the candidates and since $|S_i| = 3, \forall\ 1 \leq i \leq n$, there would be some b_j who receives two (instead of one) additional votes, making her plurality score $m/3 - 2 + 2 = m/3$. This would tie her with c and a_1, which leads to c losing the title of a unique winner, as he might not be included in a winning committee for a target committee size of $q = 2$. This means that the initial assumption was wrong and the sets S_i that the deleted s_i candidates correspond to, must form an exact 3-cover for the instance $(\mathcal{S}, B)$. This proves point *2* and completes the proof. □

To complete the proof of Theorem 3.1, this one last claim needs to be proved.

Claim 3.3. *The X3C instance in Theorem 3.1 has a solution if and only if constructive control can be achieved by deleting no more than $m/3$ candidates from the election.*

Proof. The implication from left to right side of the proof follows from Claim 3.1, whereas the implication from right to left follows from Claim 3.2. □

At this point one can be confident to say the proof of Theorem 3.1 follows directly from Claim 3.3 and Lemma 2.1. □

To put this in perspective, the following example is going to demonstrate how this explicitly works.

Example 3.1. To illustrate the scenario of CCDC in SNTV, one considers an X3C instance as the following:

- A set $B = \{b_1, \cdots, b_m\}$ where $m = 15 = 3(5)$
- A family $\mathcal{S}$ with $|\mathcal{S}| = n = 15$ of three element subsets of B,
  - such that $\mathcal{S} = \{\{b_1, b_2, b_3\}, \{b_2, b_3, b_4\}, \{b_1, b_2, b_5\}, \{b_2, b_5, b_6\}, \{b_1, b_5, b_6\}, \{b_3, b_4, b_7\}, \{b_4, b_7, b_8\}, \{b_3, b_4, b_9\}, \{b_4, b_9, b_{10}\}, \{b_3, b_9, b_{10}\}, \{b_5, b_6, b_{11}\}, \{b_6, b_{11}, b_{12}\}, \{b_9, b_{10}, b_{12}\}, \{b_{13}, b_{14}, b_{15}\}, \{b_9, b_{13}, b_{15}\}\}$
- The "yes" instance $S' \subseteq \mathcal{S}$ of the X3C Problem would be:
  - $S' = \{\{b_1, b_2, b_3\}, \{b_4, b_7, b_8\}, \{b_5, b_6, b_{11}\}, \{b_9, b_{10}, b_{12}\}, \{b_{13}, b_{14}, b_{15}\}\}$

Based on this X3C instance, construct a multiwinner Election $E = (C, V, q)$ for a target committee size of $q = 4$ where:

- **Candidates:** $C = \{c, w\} \cup B \ \cup A$, with $A = \{a_1, \cdots, a_4\}, D = \{d_1, \cdots, d_5\}$ and the distinguished candidate being c.
- **Votes:** V, Which are divided into the following voter groups

Voter group	Number of Votes	Form
v_1	1 for each $i,\ 1 \leq i \leq n$	$s_1\ c\ \cdots$ $\vdots$ $s_{15}\ c\ \cdots$
v_2	1 for each $i,\ 1 \leq i \leq n$	$s_1\ b_1\ D\ \cdots$ $s_1\ b_2\ D\ \cdots$ $s_1\ b_3\ D\ \cdots$ $s_2\ b_2\ D\ \cdots$ $s_2\ b_3\ D\ \cdots$ $s_2\ b_4\ D\ \cdots$ $\vdots$ $s_{15}\ b_9\ D\ \cdots$ $s_{15}\ b_{13}\ D\ \cdots$ $s_{15}\ b_{15}\ D\ \cdots$
v_3	$5/3 - 1 = 4$	$w\ D\ \cdots$
v_4	$m/3 = 5$	$a_1\ D\ \cdots$
v_5	$m/3 - 2 = 3$	$a_2\ D\ \cdots$ $a_3\ D\ \cdots$ $a_4\ D\ \cdots$
v_6	$m/3 - 2 = 3$ for each $j,\ 1 \leq j \leq m$	$b_1\ D\ \cdots$ $b_2\ D\ \cdots$ $\vdots$ $b_{15}\ D\ \cdots$

Table 3.3: Voter groups for CCDC in SNTV - an Example

Based on the votes in Table 3.3, the initial plurality scores of the candidates can be calculated.

- $pl(s_i) = 4$ for each $i,\ 1 \leq i \leq 15$ (one vote from v_1 and three votes from v_2).
- $pl(w) = m/3 - 1 = 4$
- $pl(a_1) = m/3 = 5$ and $pl(a_2) = pl(a_3) = pl(a_4) = m/3 - 2 = 3$
- $pl(b_j) = m/3 - 2 = 3$
- In particular: $pl(c) = 0$

Notice how there are at least $q = 4$ candidates with higher plurality scores than the distinguished candidate c, thus he can never be a member of any winning committee. This means c is a unique loser of the initial election.

Now, delete candidates s_i that correspond to the S_i in the above X3C "yes" instance. These candidates are: $s_1, s_7, s_{11}, s_{13}, s_{14}$ and deleting them from the election will alter the plurality scores of the rest of the candidates as the following:

- $pl(c) = 5$ after gaining one vote for every deletion of the candidates $s_1, s_7, s_{11}, s_{13}, s_{14}$ in voter group v_1.
- $pl(b_j) = 3 + 1 = 4$ after getting one additional vote in voter group v_2 for every deletion of the candidates $s_1, s_7, s_{11}, s_{13}, s_{14}$.
- $pl(s_i) = 4$ for $\forall\ i \in \{1, \cdots, 15\} \setminus \{1, 7, 11, 13, 14\}$
- Voter groups v_3, v_4, v_5, v_6 remain unchanged and so does the plurality score of the candidates w, a_1, a_2, a_3 and a_4.

It is clear at this point that candidates c and a_1 each have five votes, the highest among the rest of the candidates. Since a target committee size of four is considered and $4 > 2$, candidate c (just as a_1) is now a member of every possible winning committee of size four, i.e a unique winner.

It can now be easily concluded, that SNTV is also resistant to CCRC.

Corollary 3.1. *SNTV is resistant to constructive control by replacing candidates.*

Proof. Follows directly from Theorem 2.1 as SNTV is IBC according to Lemma 3.1 and resistant to CCDC per Theorem 3.1. □

Now, a similar approach will be followed to show that SNTV is resistant to DCRC.

Theorem 3.2. *SNTV is resistant to destructive control by adding candidates.*

Proof. For this proof, a similar approach as in [23] with a few changes will be followed. For this, a multiwinner election from a Hitting Set instance will be constructed.
Given a triple $(B, \mathcal{S}, k)$, where $B = \{b_1, b_2, \cdots, b_m\}$ is a set, $\mathcal{S} = \{S_1, S_2, \cdots, S_n\}$ is a family of subsets S_i of B and $k \leq m$ is a positive integer, let $E = (C, V, q)$ be a multiwinner election with:

- **Candidates:** $C = \{c\} \cup A \cup B$
  Where $A = \{a_1, a_2, \cdots, a_q\}$ and $B = \{b_1, b_2, \cdots, b_m\}$.

- **Votes:** V, divided into the following voter groups:

Voter group	Number of Votes	Form
v_1	$2(m-k)+2n(k+1)+3q$	$c\ a_1\ \cdots$
v_2	$2(k+1)$ for each $i,\ 1 \leq i \leq n$	$S_i\ c\ \cdots$
v_3	$2n(k+1)+3(q+1)$ for each $r, 1 \leq r \leq q$	$a_r\ c\ \cdots$
v_4	2 for each $j,\ 1 \leq j \leq m$ for each $r, 1 \leq r \leq q$	$b_j\ a_r\ \cdots$

Table 3.4: Voter groups for DCAC in SNTV

Where "$\cdots$" denotes the remaining candidates in C follow in some arbitrary order.

It is important to note here that in voter groups v_2 and v_4, the candidates $b_j \in B$ are initially not registered. Hence, the sets $S_1, \cdots, S_n$ are initially empty sets and candidates c and a_r are ranked first in v_3 and v_4 respectively.

Now, based on the votes in Table 3.4, the plurality score of each candidate will be calculated.

- $pl(c) = 2(m-k)+2n(k+1)+3q+2n(k+1) = 2(m-k)+4n(k+1)+3q$
- $pl(a_r) = 2m+2n(k+1)+3(q+1)$ for $r,\ 1 \leq r \leq q$

Based on the above plurality scores, it can clearly be seen that the candidate c is initially a unique winner of the election $(C \setminus B, V, q)$, which is the election (C, V, q) without registering any candidates from the set B.

Now, it can be showed that SNTV is resistant to destructive control by adding i.e registering candidates $B' \subseteq B$ where $|B| \leq k$.

Claim 3.4. *If B' is a hitting set of size k, then candidate c is not in any winning committee of the election $(B' \cup \{c\} \cup A, V)$.*

Proof. If B' is a hitting set of $\mathcal{S}$ of size k, then registering the candidates of B' would cause the plurality scores to change as the following:

- $pl'(c) = 2(m-k) + 2n(k+1) + 3q$ after losing $2n(k+1)$ votes from v_2.
- $pl'(a_r) = 2(m - |B'|) + 2n(k+1) + 3(q+1)$ since each a_r loses votes wherever a $b_j \in B'$ was added in voter group v_4.
- $pl(b_j) \leq 2q + 2n(k+1)$, since she may not necessarily be the first element in the S_i in voter group v_2. Nevertheless, every b_j has at least $2q$ votes.

Notice now that $pl'(c) < pl(a_r)$ for $r,\ 1 \leq r \leq q$, i.e. there are q candidates who have higher a plurality score than c, so any winning committee of size q would consist of candidates in A only. This means c can never be a member of any winning committee. Thus, c is now made a unique loser of the election $(B' \cup \{c\} \cup A, V, q)$. □

To complete the proof of Theorem 3.2, the following claim needs to be proved.

Claim 3.5. *Let $D \subseteq B \cup A$. If c is not in any winning committee of the election $(D \cup \{c\}, V, q)$, then there exists a set $B' \subseteq B$ such that:*

1. *$D = B' \cup A$*
2. *For the election $(B' \cup A \cup \{c\}, V, q)$, the winning committees consist of candidates in A only.*
3. *B' is a hitting set of $\mathcal{S}$ of size less or equal to k.*

Proof. Let $D \subseteq B \cup A$ and assume that c is a unique loser (not a member of any winning committee) of the election $(D \cup \{c\}, V, q)$. This means that there are at least q candidates who have higher plurality scores than her. Let $B' \subseteq B$ be a set such that $D = B' \cup A$ then $D \cup \{c\} = B' \cup A \cup \{c\}$ and winning committees consist of candidates in A only. This actually proves the first two points. For the third one, recall the following in the election $(B' \cup A \cup \{c\}, V, q)$:

- $pl(b_j) \leq 2q + 2n(k+1)$
- $pl'(a_r) = 2(m - |B'|) + 2n(k+1) + 3(q+1)$
- And particularly: $pl'(c) = 2(m-k) + 2n(k+1) + 3q + 2(k+1)l$

Where l is the number of sets in $\mathcal{S}$ that might not have been hit by B'. Meaning, there exists a set $S_i \in \mathcal{S}$ such that $S_i \cap B' = \emptyset$, which would imply the existence of $2(k+1)$ additional votes in the voter group v_2 of the form "$S_i\ c\ \cdots$" where $S_i = \emptyset$. These additional votes would then go to candidate c. To prove the third point of the claim and in order to guarantee that c is a unique loser, $l = 0$ needs to hold. From the second point of the claim, which was just proved, recall that there are at least q candidates who have higher plurality scores than c. Based on the plurality scores above, notice that these candidates should be $a_r \in A$, and that $pl'(c) < pl'(a_r)$ holds only when $l = 0$. However:

$$\begin{aligned}
2(m-k) + 2n(k+1) + 3q + 2(k+1)l &< 2(m-|B'|) + 3(q+1) + 2n(k+1) \qquad (3.1)\\
2(m-k) + 3q + 2(k+1)l &< 2(m-|B'|) + 3q + 3\\
2m - 2k + 2(k+1)l &< 2m - 2|B'| + 3\\
2k - 2(k+1)l &> 2|B'| - 3\\
\text{and for the smallest } l \text{ such that } l \neq 0 :&\\
2k - 2k - 2 &> 2|B'| - 3\\
-2 + 3 &> 2|B'|\\
\frac{1}{2} &> |B'| \ \lightning
\end{aligned}$$

This means the only possible value for l such that the equation (3.1) is valid is zero, which implies that B' is a hitting set of size at most k, proving the third point. □

To complete the proof, the following claim will be proved to show that the initial construction yields a polynomial-time-many-one reduction from the Hitting Set problem to the problem of destructive control by adding candidates for SNTV.

Claim 3.6. *$\mathcal{S}$ has a hitting set of size less than or equal to k if and only if destructive control by adding candidates can be executed for the election with qualified candidates $\{c\} \cup A$, spoiler candidates B, distinguished candidate c, a preference profile V and a target committee size of q.*

Proof. Assuming $\mathcal{S}$ has a hitting set of size less or equal to k, then since $k \leq m$, $\mathcal{S}$ has a hitting set of size k. Hence, the implication from left to right follows from Claim 3.4 whereas the one from right to left follows from Claim 3.5. □

At this point one can be confident to say the proof of Lemma 3.2 follows directly from Claim 3.6 and Lemma 2.1. □

To put this in perspective, the following example will be demonstrating the proof with concrete numbers.

Example 3.2. First, construct a Hitting Set instance with a set $B = \{b_1, b_2, \cdots, b_m\}$ and a set $\mathcal{S} = \{S_1, S_2, \cdots, S_n\}$, with $m = n = 5$ and a positive integer $k = 2$.

So for $B = \{b_1, b_2, b_3, b_4, b_5\}$, let $S_1 = \{b_1, b_2\}$, $S_2 = \{b_3, b_4\}$, $S_3 = \{b_1, b_4\}$, $S_4 = \{b_2, b_3\}$ and $S_5 = \{b_2, b_4\}$. Clearly, $B' = \{b_2, b_4\} \subset B$ is a hitting set of B with $|B'| \leq k = 2$.

Based on the above, a multiwinner election $E = (C, V, q)$ will be constructed, with a target committee size of $q = 5$ and:

- **Candidates:** $C = \{c\} \cup A \cup B$, where $A = \{a_1, a_2, \cdots, a_5\}$
- **Votes:** Which are divided into the following voter groups:

Voter group	Number of Votes	Form
v_1	$2(3) + 2(5)(3) + 3(5) = 51$	$c\ a_1\ \cdots$
v_2	$2(3) = 6$	$b_1\ b_2\ c\ \cdots$
		$b_3\ b_4\ c\ \cdots$
		$b_1\ b_4\ c\ \cdots$
		$b_2\ b_3\ c\ \cdots$
		$b_2\ b_4\ c\ \cdots$
v_3	$2(5)(3) + 3(6) = 48$	$a_1\ c\ \cdots$
		$\vdots\ c\ \cdots$
		$a_5\ c\ \cdots$
v_4	2	$b_1\ a_1\ \cdots$
	2	$b_1\ a_2\ \cdots$
	$\vdots$	$b_1\ \vdots\ \cdots$
	2	$b_1\ a_5\ \cdots$
	2	$b_2\ a_1\ \cdots$
	$\vdots$	$\vdots\ \vdots\ \cdots$
	2	$b_5\ a_5\ \cdots$

Table 3.5: Voter groups for DCAC in SNTV - an Example

Note again that candidates $b_1, \cdots, b_5$ in v_2 and v_4 are mentioned initially as placeholders, and are at this point of the election still not registered. Now, based on votes in Table 3.5, the plurality scores of the candidates are given as the following.

- $pl(c) = 51 + 30 = 81$ (51 votes from v_1 and a total of 30 from v_3).
- $pl(a_1) = pl(a_2) = \cdots = pl(a_5) = 58$ (two votes from each of the five b_j, and 48 votes from v_3)

Clearly, candidate c has the highest plurality score among the rest of the candidates, hence she is a member of every winning committee. In this case, these committees are $W = \{c, a_x, a_y, a_z, a_u\}$ where $x, y, z, u \in \{1, \cdots, 5\}$.

The candidates in B' can now be registered, and the fact that $B' \subseteq B$ is a hitting set, alters the election and its results as the following.

- Candidate c will lose her $2n(k+1) = 2(5)(3) = 30$ votes as a result of adding $S_1, \cdots, S_n$ in v_2. She will then have 51 votes.
- Candidates $a_1, \cdots, a_5$ will each lose $2(2) = 4$ votes as a result of adding candidates from B'. This will leave each of them with $2(m - |B'|) + 48 = 2(3) + 48 = 54$ votes.
- Candidates b_j will each get $2q = 10$ votes from v_4 and candidates b_2 and b_4 who form a hitting set, will each get 6 additional votes from v_2 for every time they were ranked in the first place in the random order within the corresponding S_i. In this case, candidate b_2 will get $2(6) = 12$ and candidate b_4 will get 0 additional votes.

To sum up, the new plurality scores of the candidates are as follows:

- $pl'(c) = 51$
- $pl'(a_1) = \cdots = pl'(a_5) = 54$
- $pl(b_2) = 10 + 12 = 22$
- $pl(b_4) = 10$

Candidate c might not be the candidate with the lowest plurality score, but nevertheless, she loses the election, since there are q candidates besides her, who have higher plurality scores. This means, there can never exist a winning committee which she happens to be a member of. Candidate c loses the election as a unique loser.

It can now be easily concluded that SNTV is also resistant to DCRC.

Corollary 3.2. *SNTV is resistant to destructive control by replacing candidates.*

Proof. Follows directly from Theorem 2.2 as SNTV is IBC according to Lemma 3.1 and resistant to DCAC per Theorem 3.2. $\square$

3.3 Bloc Voting

In this section a similar strategy as in Section 3.2 will be followed in order to show that Bloc Voting is resistant to constructive and destructive control by replacing candidates. For that, the start is going to be by showing that Bloc Voting is also IBC.

Lemma 3.2. *Bloc Voting is insensitive to bottom-ranked candidates.*

Proof. Just like SNTV, this is clear from the definition, as for a target committee size of q, Bloc Voting considers the q candidates having the highest q-Approval scores, which in return has the scoring vector of $\alpha = (\underbrace{1, \cdots, 1}_{k}, 0, \cdots, 0)$. Adding or deleting candidate(s) at the bottom of the profile has no influence on the winning committees, because target committee size values of only $2 \leq q \leq m-1$ are considered. Hence the addition or deletion would take place at the $m+1$-th position.

Formally, let t be a large enough positive integer, $C = \{a_1, \cdots, a_m\}$ a set of candidates and $E = (C \cup \{a_{m+1}\}, V, q)$, a multiwinner election, where V is the list of votes that looks like the following:

Number of Votes	Form
t	$a_1 \ \cdots \ a_q \ a_{q+1} \ \cdots \ a_m \ a_{m+1}$

Where "$\cdots$" mean the remaining candidates of C follow in arbitrary order.
For the target committee size q, the committee $\{a_1, \cdots, a_q\}$ is the winner of the election. Adding a candidate a_{m+1} at the bottom of the election or deleting him if he ever existed has no effect whatsoever on the winning committee. □

Theorem 3.3. *Bloc Voting is resistant to constructive control by deleting candidates.*

Proof. For this proof, a similar approach as in Theorem 3.1 will be followed to prove the resistance of Bloc Voting to CCAC. Again, the approach of [1] will be used, by applying a reduction from the Exact-Cover by 3-Sets problem, which can be executed in polynomial time.
Given a set $B = \{b_1, \cdots, b_m\}$ with $m = 3k$ and a family $\mathcal{S} = \{S_1, \cdots, S_n\}$ of three element subsets of B. Let b_i^1, b_i^2, b_i^3 denote the elements of S_i. Without loss of generality, assume $k \geq 5$ and construct a multiwinner election $E = (C, V, q)$ as the following:

- **Candidates:** For each $i, 1 \leq i \leq n$ there are s_i candidates corresponding to the $S_i \in \mathcal{S}$ and a set A_i of candidates a_i. There are also E_j sets of candidates e_j for each j, $1 \leq j \leq m$ and $|A_i| = |E_j| = q - 1$. The notation $a_i^t \in A_i$ and $e_j^t \in E_j$ refers to the t-th element of the sets A_i and E_j respectively. The distinguished candidate is c and there are two other set of candidates $D = \{d_1, \cdots, d_q\}, G = \{g_1, \cdots, g_{q-1}\}$ and $H = \{h_1, \cdots, h_{m/3}\}$. So overall, the set of candidates is: $C = \{c\} \cup A_i \cup B \cup D \cup E_j \cup G \cup H \cup \left(\bigcup s_i\right)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

- **Votes:** Which are divided into voter groups as can be seen the following table.

Voter group	Number of Votes	Form
v_1	1 for each i, $1 \leq i \leq n$	$s_i \overrightarrow{A_i} c \cdots$
v_2	$m/3 - 1$	$\overrightarrow{D} H \cdots$
v_3	1	$d_1 \overrightarrow{G} H \cdots$
v_4	1 for each i, $1 \leq i \leq n$	$s_i \overrightarrow{A_i} b_i^1 H \cdots$ $s_i \overrightarrow{A_i} b_i^2 H \cdots$ $s_i \overrightarrow{A_i} b_i^3 H \cdots$
v_5	$m/3 - 2$ for each j, $1 \leq j \leq m$	$b_j \overrightarrow{E_j} H \cdots$

Table 3.6: Voter groups for CCDC in Bloc Voting

Based on these votes, the q-Approval scores of the candidates can now be calculated.

- $score(s_i) = 4$
- $score(a_i^t) = 4$ for $1 \leq t \leq q - 1$
- $score(d_1) = m/3$ and $score(d_t) = m/3 - 1$ for $2 \leq t \leq q$
- $score(b_j) = m/3 - 2$
- $score(e_j^t) = m/3 - 2$ for $1 \leq t \leq q - 1$
- $score(g_t) = 1$ for $1 \leq t \leq q - 1$ and $score(h_t) = 0$ for $1 \leq t \leq m/3$

- Particularly: $score(c) = 0$

Notice again that there are at least q candidates who have a higher q-Approval score than the one of the distinguished candidate c, hence she can never be a member of any winning committee. Thus, c loses the election.

Now, one can show that c can be made a winner if the above X3C has a solution by deleting at most $m/3$ candidates.

Claim 3.7. *If the above X3C instance has a solution $S' \subseteq S$, then the candidate c can be made a unique winner of the election $E = (C \setminus S', V, q)$ by deleting at most $m/3$ candidates.*

Proof. Assume there is a "yes" instance $S' \subseteq S$, a.k.a an exact 3-cover for the above X3C problem. Similar to the proof in Claim 3.1, delete the candidates s_i corresponding to the S_i in the cover, i.e. $S_i \in S'$. Since the exact cover has the cardinality of exactly $|S'| = m/3$, the q-Approval scores of the candidates change according to the following.

- The distinguished candidate c gets $m/3$ votes in total from voter group v_1.
- Candidates b_j will each get one additional vote from voter group v_4 due to the deletion of candidates s_i, making their score: $score'(b_j) = m/3 - 1$.
- There are $n - m/3$ candidates s_i who keep their positions as well as their four votes since they are not deleted.
- Candidates in D and G also keep their votes.
- Candidates in sets A_i, E_i as well as H also keep their votes and scores.

Notice that candidates c and d_1 now both have the highest q-Approval scores. Hence c is now made a unique winner of the Election. □

To complete the proof, the following claim needs to be proved.

Claim 3.8. *Let $D \subseteq C$. If c is a unique winner of the election $E = (D, V, q)$ by deleting at most $m/3$ candidates, then there exist sets $\mathcal{X}$ and $S' \subseteq S$ such that:*

1. $\mathcal{X} = \{s_i \mid s_i$ *corresponds to* $S_i \in S'\}$
2. *S' is an exact 3-cover for the X3C instance* $(\mathcal{S}, B)$

3. $\mathcal{X}$ *and* S' *have the cardinality of exactly* $m/3$.

4. $D = C \setminus \mathcal{X}$

Proof. This proof is similar to the proof of Claim 3.2. To start, assume again, there exists a subset $\mathcal{X}$ of no more than $m/3$ candidates whose deletion would lead to candidate c's victory in the above election. Based on the above construction, c would also hope to get votes from voter group v_1, because her position in the other groups is lower than $m/3$. That being said, all deletions must be from candidates s_i and in that way, c would get *no more* than $m/3$ votes. This would have no effect on the q-Approval scores of the first $q-1$ candidates in A_i. However, c should also receive *no less* than $m/3$ votes, since that would tie her with candidates in $D \setminus \{d_1\}$ and the b_j candidates, which will result in her not being a unique winner anymore, or even not be sufficient to win the election at all. This is because in the first case, candidate d_1 would have $m/3$ votes, and there would be more than $q-1$ candidates with $m/3-1$ votes, and in the second case, there would be more than q candidates who have higher q-Approval scores than c's. Thus, c must receive *exactly* $m/3$ votes, which can only be achieved by deleting those candidates s_i. This so far proves the points *1*, *3* and *4*.
Moreover, the sets $S_i \in S'$ which the candidates $s_i \in \mathcal{X}$ correspond to, must comprise an exact 3-cover for the X3C instance. To show this, one assumes initially otherwise. After deleting the candidates and since $|S_i| = 3, \forall\, 1 \leq i \leq n$ there would be some b_j who receives two (instead of one) additional votes making his plurality score $m/3 - 2 + 2 = m/3$. This would tie him with c and d_1, which leads to c losing the title of a unique winner, as she might not be included in a winning committee for a target committee size of $q = 2$. This means that the initial assumption was wrong and the sets S_i that the deleted s_i candidates correspond to, must form an exact 3-cover for the instance $(\mathcal{S}, B)$. This proves point *2* and completes the proof. □

To complete the proof of Theorem 3.3, one claims the following.

Claim 3.9. *The X3C instance in Theorem 3.3 has a solution if and only if constructive control can be achieved by deleting no more than* $m/3$ *candidates from the election.*

Proof. The implication from left to right follows from Claim 3.7, whereas the implication from right to left follows from Claim 3.8. □

At this point one can be confident to say the proof of Theorem 3.3 follows directly from Claim 3.9 and Lemma 2.1. □

It can now be easily concluded that Bloc Voting is also resistant to CCRC.

Corollary 3.3. *Bloc Voting is resistant to constructive control by replacing candidates.*

Proof. Follows directly from Theorem 2.1 as Bloc Voting is IBC according to Lemma 3.2 and resistant to CCDC per Theorem 3.3. □

Now, a similar approach will be followed to show that Bloc Voting is resistant to DCRC.

Theorem 3.4. *Bloc Voting is resistant to destructive control by adding candidates.*

Proof. For this proof, a similar approach as in [23] and Theorem 3.2 will again be followed. For that, a multiwinner election will be constructed from a Hitting Set instance as the following.

Given a triple $(B, \mathcal{S}, k)$, where $B = \{b_1, b_2, \cdots, b_m\}$ is a set, $\mathcal{S} = \{S_1, S_2, \cdots, S_n\}$ is a family of subsets S_i of B and $k \leq m$ is a positive integer, construct the following multiwinner election $E = (C, V, q)$, where:

- **Candidates:** $C = \{c\} \cup A \cup B \cup D_i \cup G$, for i, $1 \leq i \leq n$. Where $A = \{a_1, \cdots, a_{q-1}\}$, $D_i = \{d_i^1, \cdots, d_i^{q-1}\}$ and $G = \{g_1, \cdots, g_q\}$
- **Votes:** V, divided into the voter groups demonstrated in the following table.

Voter group	Number of Votes	Form
v_1	$2(m-k) + 2n(k+1) + 3$	$c \ \overrightarrow{A} \ \cdots$
v_2	$2(k+1)$ for each i, $1 \leq i \leq n$	$\overrightarrow{D_i} \ S_i \ c \ \cdots$
v_3	$2n(k+1) + 4$	$\overrightarrow{G} \ \cdots$
v_4	2 for each j, $1 \leq j \leq m$	$b_j \ \overrightarrow{G} \ \cdots$

Table 3.7: Voter groups for DCAC in Bloc Voting

Where S_i denotes that the elements of S_i follow in some arbitrary order. Note again, that in voter groups v_2 and v_4, the candidates $b_j \in B$ are initially not registered. Hence, the sets S_i, for $i, 1 \leq i \leq n$ are initially empty sets and the distinguished candidate c is ranked in the q-th position in v_2 and candidates in G are ranked first in v_4. Based on the votes in Table 3.7, the q-Approval scores of the candidates can now be calculated.

- $score(c) = 2(m-k) + 2n(k+1) + 3 + 2n(k+1) = 2(m-k) + 4n(k+1) + 3$
- $score(a_t) = 2(m-k) + 2n(k+1) + 3$ for $t,\ 1 \leq t \leq q-1$
- $score(d_i^t) = 2(k+1),\ \forall d_i^t \in D_i$
- $score(g_t) = 2m + 2n(k+1) + 4,\ \forall g_t \in G$

Notice that candidate c has the highest q-Approval score. This makes him a unique winner of the election $E = (C \setminus B, V, q)$, that is, without registering candidates B.

Claim 3.10. *If B' is a hitting set instance of size k, then candidate c is a unique loser of the election $E = ((C \setminus B) \cup B', V, q)$, i.e. not a member of any winning committee.*

Proof. If B' is a hitting set of $\mathcal{S}$ size k, then registering candidates from B' would alter the q-Approval scores of the candidates as the following:

- $score'(c) = 2(m-k) + 2n(k+1) + 3$ after losing $2n(k+1)$ votes in v_2, since $\forall i : |S_i| > 1$, which will cause c to be pushed beyond the first q positions.
- $score'(g_q) = 2(m - |B'|) + 2n(k+1) + 4$, after registering one b_j in v_4.
- $score(b_j) \leq 2 + 2n(k+1)$, as it may not always necessarily be the case that a particular b_j is ranked in the q-th positions in v_2. Nevertheless, every b_j has at least two votes in all cases.
- Candidates in A keep their previous votes.
- Candidates in D_i keep their $2(k+1)$ votes.
- The first $q-1$ candidates in G keep their $2m + 2n(k+1) + 4$ votes as well.

The fact that $score'(c) < score(g_q) < score(g_1) \leq \cdots \leq score(g_{q-1})$, means there are q candidates who have q-Approval scores higher than candidate c at all times. Thus, he can never be a member of the winning committee which consists of the candidates in G. This means that c is now made a unique loser of the election. $\square$

To complete the proof, the following claim will be proved:

Claim 3.11. *Let $D \subseteq B \cup A \cup D_i \cup G$. If c is not a member of any winning committee of the election $(D \cup \{c\}, V, q)$, then there exists a set $B' \subseteq B$ such that:*

1. $D = B' \cup A \cup D_i \cup G$

2. *For the election* $(B' \cup A \cup D_i \cup G \cup \{c\}, V, q)$ *the winning committees consist of only* G.

3. $B' \subseteq B$ *is a hitting set of* $\mathcal{S}$ *of size of less or equal to* k.

Proof. Let $D \subseteq B \cup A \cup D_i \cup G$ and assume that c is not a member of any winning committee, i.e. a unique loser of the election $(D \cup \{c\}, V, q)$. This means there are at least q candidates who definitely have higher q-approval scores than c. Let $B' \subseteq B$ be a set such that $D = B' \cup A \cup D_i \cup G$, then $D \cup \{c\} = B' \cup A \cup D_i \cup G \cup \{c\}$ and the candidates who have higher q-Approval scores than c have to be the candidates in G, making them the winning committee. This proves the points *1* and *2*. For *3*, recall from the construction above that after registering candidates in B' that:

- $score(c) = 2(m-k) + 2n(k+1) + 3 + 2(k+1)l$, where l is the number of sets in $\mathcal{S}$ that might not have been hit by B'.
- $score(g^q) = 2(m - |B'|) + 2n(k+1) + 4$
- $score(g_t) = 2m + 2n(k+1) + 4$ for $t, 1 \leq t \leq q-1$
- $score(b_j) \leq 2 + 2n(k+1)$
- $score(a_t) = 2(m-k) + 2n(k+1) + 3,\ \forall t,\ 1 \leq t \leq q-1$
- $score(d_i^t) = 2(k+1), \forall d_i^t \in D_i$

Based on the above scores, if the q candidates in the winning committee are the candidates in G, then $score(c) < score(g), \forall g \in G$. However, in order for $score(c) < score(g_q)$ to hold, $l = 0$ should hold too. Note that:

$$2(m-k) + 2n(k+1) + 3 + 2(k+1)l < 2(m - |B'|) + 2n(k+1) + 4$$
$$2m - 2k + 2(k+1)l < 2m - 2|B'| + 1$$
$$2k - 2(k+1)l > 2|B'| - 1$$

and for the smallest l such that $l \neq 0$:

$$2k - 2k - 2 > 2|B'| - 1$$
$$-1/2 > |B'| \ \lightning$$

This means the only possible value of l such that $score(c) < score(g_q)$ holds is zero. This means all sets in $\mathcal{S}$ are hit by B', hence, $B' \subseteq B$ is a hitting set of $\mathcal{S}$ of size less and equal to k, which proves the third point. $\square$

To complete the proof of the theorem, one can show that the initial construction yields a polynomial-time-many-one reduction from the Hitting Set problem to the problem of destructive control by adding candidates for Bloc Voting.

Claim 3.12. *$\mathcal{S}$ has a hitting set of size less than or equal to k if and only if destructive control by adding candidates can be executed for the election with qualified candidates $A \cup D_i \cup G$, for i, $1 \leq i \leq n$, spoiler candidates B, distinguished candidate c, a preference profile V, and a target committee size of q.*

Proof. Assuming $\mathcal{S}$ has a hitting set of size less than or equal to k. But since $k \leq m$, $\mathcal{S}$ has a hitting set of size k. Thus, the implication from left to right implies from Claim 3.10, whereas the implication from right to left implies from Claim 3.11. □

At this point, one can certainly say that the proof follows directly from Claim 3.12 and Lemma 2.1. □

Next, it can be easily concluded that Bloc Voting is also resistant to DCRC.

Corollary 3.4. *Bloc Voting is resistant to destructive control by replacing candidates.*

Proof. Follows directly from Theorem 2.2 as Bloc Voting is IBC according to Lemma 3.2 and resistant to DCAC per Theorem 3.4. □

It will now be shown that Bloc Voting is also resistant when it comes to replacing votes.

Theorem 3.5. *Bloc Voting is resistant to constructive control by replacing votes.*

Proof. For this proof, refer to Loreggia's PhD Thesis [28], where it was shown that for $2 < k < m - 3$, k-Approval is resistant to CCRV. Here, a similar approach with minor changes will be followed, in order to show that for values $2 < q < m$, Bloc Voting is also resistant to CCRV. Assuming that $m > 3$, a reduction from the X3C problem will be used, which can be executed in polynomial time. Let $(B, \mathcal{S}, k)$ be a triple representing an X3C instance, where $B = \{b_1, \cdots, b_{3k}\}$ for $k \geq 1$ and $\mathcal{S} = \{S_1, \cdots, S_n\}$, with $S_j = \{b_j^1, b_j^2, b_j^3\}, b_j^t \in B$. For an arbitrary X3C instance, construct a multiwinner election $E = (C, V, q)$, as the following:

- **Candidates:** $C = \{c\} \cup B \cup A_i \cup D_j \cup E \cup G$ for $1 \leq i \leq n$, and $1 \leq j \leq m$ where $A_i = \{a_i^1, \cdots, a_i^{q-3}\}, D_j = \{d_j^1, \cdots, d_j^{q-1}\}$, $E = \{e_1, \cdots, e_q\}$ and $G = \{g_1, \cdots, g_{q-1}\}$.

- **Votes:** V, representing the list of *registered* votes, which are divided into voter groups as can be seen in Table 3.8.
- **Votes:** $U = (u_1, \cdots, u_k)$ representing the initially unregistered additional votes, which the chairman may add. These votes have the form: "$c\ \overrightarrow{G} \cdots$"

Voter group	Number of Votes	Form
v_1	1 for each i, $1 \leq i \leq n$	$S_i\ \overrightarrow{A_i} \cdots$
v_2	$n - l_j$ for each j, $1 \leq j \leq m$	$b_j\ \overrightarrow{D_j}\ \cdots$
v_3	$n - k$	$c\ \overrightarrow{E}\ \cdots$
v_4	k	$\overrightarrow{E}\ \cdots$
v_5	0	$c\ \overrightarrow{G}\ \cdots$

Table 3.8: Voter groups for CCRV in Bloc Voting

Where l_j is the number of subsets S_i of $\mathcal{S}$ where b_j occurs.

Based on the votes in Table 3.8, the q-Approval scores of the candidates can now be calculated.

- $score(b_j) = n$. That is l_j votes from voter group v_1 and $n - l_j$ votes from v_2.
- $score(a_i^t) = 1$ for $1 \leq t \leq q - 3$
- $score(d_j^t) = n - l_j$ for $1 \leq t \leq q - 1$
- $score(c) = n - k$
- $score(e_q) = k$
- $score(e_t) = n$, for $1 \leq t \leq q - 1$, since the first $q - 1$ candidates in E get $n - k$ votes from v_3 and k additional votes from v_4.
- $score(g_t) = 0,\ \forall t,\ 1 \leq t \leq q - 1$

It can clearly be observed, that there are more than q candidates who have higher q-Approval scores than c, hence she can never be a member of any winning committee. This means c loses the election initially. To complete the proof, the following claim will be proved.

Claim 3.13. *The above X3C instance has a solution if and only if constructive control can be executed by replacing at most k votes.*

Proof. Assume the above X3C instance has a solution $S' = \{S_{i1}, S_{i2}, \cdots, S_{ik}\}$. The chairman can then make c win the election by replacing all the votes in U with votes $V' = \{v_{i_1}, \cdots, v_{i_k}\}$ corresponding to the elements in the X3C solution. This would alter the q-Approval score of the candidates as the following:

- $score(b_j) = n - 1$, since each b_j would lose one vote as a result of removing $V' = \{v_{i1}, \cdots, v_{ik}\}$ corresponding to the elements in the X3C solution, as each b_j occurs there only one time.
- $score(c) = n$ after gaining the k additional votes from U.
- $score(g_t) = k$, for $1 \leq t \leq q - 1$
- The scores of candidates in the sets A_i, D_j as well as E remain unchanged.

It can now be observed that c alongside $q - 1$ many candidates from the set E have q-Approval scores of n, the highest among the rest of the candidates. Since there are no more than q candidates who satisfy this, the winning committee is $W = \{c, e_1, \cdots, e_{q-1}\}$, which includes c. So c is now a unique winner of the election $E = (C, (V \setminus V') \cup U, q)$.

To complete the proof, assume that the problem of replacement control has a solution and the candidate c can be made a winner by replacing at most k candidates in the election $E = (C, V, q)$. Apparently, this can only be achieved by deleting k votes, in a way that candidates b_j lose at least one point. In the election E, there would be the following scores:

- $score(c) = n$
- $score(b_i) = n - k_i$ where k_i is the number of $S_{i_j} \in S'$ in which b_i occurs.
- $score(e_t) = n$ for $1 \leq t \leq q - 1$
- $score(d_i^t) = n - l_j$ for $1 \leq t \leq q - 1$

- $score(g_t) = k$, for $1 \leq t \leq q-1$

Since c is made a unique winner, and there are already $q-1$ candidates who have the same score as her, $score(c) > score(b_i)$ must be satisfied. This means that $\forall i: \ k_i > 0$ should also hold. As a matter of fact, k_i has to satisfy $k_i = 1$ for all i, to be more specific. To prove this, assume there exists an i such that $k_i > 1$, and c is a unique winner. By the pigeon-hole argument, there would be some other k_j where $k_j = 0$ for $j \neq i$, contradicting with the earlier requirement that $\forall i: \ k_i > 0$. This means the initial assumption was wrong and $k_i = 1$, which implies that each b_i occurs exactly one time in each $S_{i_j} \in S'$. This means that S' is a solution for the X3C problem. □

Complementing the construction with Claim 3.13 and its proof together with Lemma 2.1 completes the proof of the theorem. □

3.4 Chamberlin-Courant

Recognizing the significance of the CCR as a multiwinner voting system has led to including it in this thesis. However, due to its complexity, especially within the timeframe dedicated for this thesis, and in adherence to the stipulated examination regulations that limit the number of its pages, it was not possible to include any investigation of the complexity of control of CCR. Nevertheless, it is believed that the problems of CCDC, DCAC, CCRC as well as DCRC are **NP**-hard. This is due to the fact that the problem of winner determination in CCR alone is **NP**-hard according to Betzler et al. [3], as mentioned in [9]. Another reason to believe these problems are **NP**-hard is the agreement between CCR and SNTV which was studied by Diss et al. [9], where they have also mentioned that according to [24], the CCR is equivalent to SNTV when the target committee size is given by $q = |C| - 1$ and the candidate set C has the cardinality of three or more. However it is essential to note that these are not formal proofs, but rather speculations that require further and deep investigation.

Chapter 4

Conclusion

This section summarizes the most fundamental results that have been presented in this thesis. Chapter 2 has established a solid foundation of the most important definitions and theorems that have been utilised to complete proofs of the theorem. Chapter 3 has concentrated on the complexity of control of two of the most popular multiwinner voting systems; Single Non-Transferable voting and Bloc Voting. It was showed that SNTV is resistant to constructive control by deleting and replacing candidates and destructive control by adding and replacing candidates, and that Bloc Voting is resistant to constructive control by deleting and replacing candidates, destructive control by adding and replacing candidates as well as to constructive control by replacing votes. The new results of control problems by adding and deleting candidates as well as replacing votes were achieved by providing formal proofs, whereas those of replacing candidates were direct conclusions, thanks to the practical notion of IBC.

The problem of CCRV for SNTV remains open, as it was not possible within the given time limit for this thesis, to determine whether this is in **P** or in **NP**. This explains the "?" in Table 3.1. One reason to believe that SNTV is vulnerable to this type of control is because Plurality (its single-winner variant) is vulnerable to CCRV as shown in [28]. Another reason is that SNTV is shown to be vulnerable to control by adding and deleting voters as well, according to Meier et al. [32]. However, as no efficient and correct algorithm could be found that solves it, the speculations grew that SNTV might actually be *resistant* to CCRV.

Future Work

One of the key aims of this thesis was drawing the attention to the fact that the complexity of replacement control of multiwinner elections was not much of an interest in the past few years. This thesis is hoped to be a starting point that encourages further investigation of this interesting and significant topic. Open cases from this thesis include the study of CCRV of SNTV, DCRC of both SNTV and Bloc Voting, as well as all control scenarios concerning the Chamberlin-Courant Rule. These results could not be accommodated in this thesis due to their complexity with respect to the limited available time and the given exam regulations controlling the length of the thesis. Given its huge significance, it is particularly encouraged to consider CCR in any future work about complexity of control of mutltiwinner elections.

List of Tables

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Ehrenwörtliche Erklärung

Hiermit versichere ich, die vorliegende Bachelorarbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

Alle Stellen, die aus den Quellen entnommen wurden, sind als solche gekennzeichnet.

Diese Arbeit hat in dieser oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

Düsseldorf, 15.01.2024

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