The generalized Dicke model with broken parity symmetry

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1 Introduction

The Dicke model was first introduced by Dicke in 1957[?] to accurately describe light matter interaction on a quantum mechanical level. In its simplest form it describes a quantum mechanical Spin of length j, a single bosonic mode as well as a coupling term. There is a whole group of models closely related to the Dicke model, such as the Rabi model[?] which depicts the special case of $j = \frac{1}{2}$, or the Jaynes Cummings[?] or Tavis Cummings models [?], which apply the rotating wave approximation with $j = \frac{1}{2}$ or arbitrary spin, respectively. The Dicke and related models gained interest of physicists for numerous reasons and can be applied in many different fields of research, including atom physics[?], optomechanics [?] and quantum computing[?]. Since it has a well defined and well understood classical limit, it is also of interest for reseaching the transition between classcal and quantum mechanics [?], especially in the context of chaos[?]. The Dicke model also gained interest because it displays a quantum phase transition. Despite the discovery of the phase transition by Dicke himself in 19XX[?] it is still a subject of research interest in various fields, ranging from its recent experimental realization in a Bose Einstein Condensate or the discussion wheather the nature of the phase transition is quantum at all [?], or just an afterimage of the classical limit, where the minimum energy shows a bifurcation at a crittical coupling. Without disregarding the work of Someone et al. [?], I will still call it a quantum phase transition, as the majority of the scientific community. Many areas of parameter space of the Dicke model have been subject of study before[?-?], yet few have included a parity breaking term, as is studied in this thesis. It will be subject of this thesis, weather breaking the parity has any fundamental effect on the spectrum and eigenstates, and dynamical properties of the Dicke model. The parity breaking term is caused by an angle α which shifts the constant magnetic field in the $x - z$-plain away from the z axis. Another angle is used to determine the ratio of the rotating and counterrotating terms of the coupling, making it possible to continuously shift between rotating wave approximation and the Standard Dicke model.

In the cause of this thesis I will use several different methods to qualify or quantify chaos in a quantum mechanical system, yet it remains to be said that we still have not universally agreed what the definite characteristic property of a chaotic quantum mechanical system is [?].

2 Basics

The Standard Dicke hamiltonian reads

$$H = \hbar \left[\omega_0 \cos\alpha J_z + \omega a^\dagger a + \frac{g}{\sqrt{j}} J_x (a^\dagger + a) \right] \quad . \tag{1}$$

It describes a single bosonic mode of frequency ω with creation and annihilation operators $a^\dagger$ and a, and a spin length j. The spin operators J_i follow the standard spin algebra $[J_k, J_l] = i \sum_m \epsilon_{klm} J_m$. For small j, these operators may describe a single atom or molecule, for larger j it can describe an ensemble of N spin $\frac{1}{2}$ paricles of which only the subspace of maximum total spin $j = \frac{N}{2}$ is concidered. This assumption is justified, because the ground state of the Spin ensemble, disregarding the coupling, is in this subspace and no operators that change the total spinare present.

By rewriting the Spin operators J_x and J_y in spin rising and lowering operators $J_\pm = J_x \pm i J_y$, one can identify resonant and antiresonant terms in the coupling. The resonant terms aJ_+ and $a^\dagger J_-$ conserve the number of excitations $N_{\text{exc}} = a^\dagger a + J_z + j$ in the system. The antiresonant terms aJ_- and $a^\dagger J_+$ do not. Thereby it is an often used and justified method to neglect the antiresonant terms of the coupling - the rotating wave approximation [?]. Not only does the existence of a second constant of motion beside the energy have significant influence on the dynamics of the system, but the breakdown of the Hilbertspace into infinitely many, independent subspaces of dimension $2j + 1$, makes the numerical handling of the problem a lot easier. Nonetheless there are many situations in which the rotating wave approximation is not applicable[?]. To study the influence of this approximation we will consider a generalized form of the Standard Dicke Hamiltonian by using the angle δ to set the ratio of resonant and antiresonant terms. The coupling term then reads $\frac{g}{\sqrt{j}} \left(\cos(\delta)(aJ_+ + a^\dagger J_-) + \sin(\delta)(a^\dagger J_+ + aJ_-) \right)$, which can shift from rotating wave approximation $\delta = 0$ to the Standard coupling $\delta = \pi/4$ to a purely antiresonant coupling $\delta = \pi/2$. Neglecting the rotating wave approximation, what remains is the parity symmetry $P = \exp(i\pi N_{\text{exc}})$, which can be interpreted as whether the number of excitations is even or odd. One can see, that the antiresonant terms only ever create or annihilate two excitations, so the parity remains conserved. Whether this has a significant influence on the dynamics of the system is subject of this thesis, but one way or the other the Hilbertspace separates into two independent subspaces. Their dimension is infinite but after applying a suitable truncation of the bosonic Hilbertspace, exploiting the parity can lead to significantly less numeric efforts. A second geralization is now applied in order to break the parity. For this a term that creates or annihilates one excitation is needed. We use a second angle α to shift the constant magnetic field, that splits the energy levels of the Spin from the z axis into the $x - z$ plain, so that the generalized Dicke Hamiltonian considered in this thesis reads

$$\hat{H} = \hbar \left[\omega_0 (\cos\alpha J_z + \sin\alpha J_x) + \omega a^\dagger a + \frac{g}{\sqrt{j}} \left(\cos(\delta)(aJ_+ + a^\dagger J_-) + \sin(\delta)(a^\dagger J_+ + aJ_-) \right) \right] \quad . \tag{2}$$

One of the important qualties of the Dicke Model is, that it has a well defined classical limit[?]. There are different limits to be considered. The thermodynamic limit describes the Dicke model in the background of N particles in, e.g.

an optical cavity, the limit being $N \to \infty$ as is the usual thermodynamic limit in solid state or statistical physics[?]. It is to be noted, that the system remains quantum mechanical. The classical limit yields very similar results, but applies a limit, where the Spin length becomes infinite, resulting in infinitely many energy levels and possible orientations of the spin -as a classical angular momentum- , while $\hbar$ is sent to zero in a way, that $\hbar j = s$ remains constant. This limit is usually referred to as the classical limit, sometimes called the classical spin limit, because it is also possible to apply a limit, that makes the oscillator classical, the classical oscillator limit [? Fehske et al und noch jemand, vielleicht?]. Here the frequency, i.e. the energy spacing, of the oscillator is sent to zero, so that the oscillator can be described by a classical continuous harmonic oscillator, while the Spin remains quantum mechanical. This limit will only briefly be covered in this thesis (if at all, as of 26.06.), because the classical spin limit yields enough classical and, more importantly, analytically approachable results. It is worth noting, that even in the classical spin limit, henceforth classical limit, the bosonic part of the Hamiltonian becomes a classical oscillator, since the energy spacing vanishes as $\hbar$ approaches zero.

2.1 Properties of the Spectrum and Eigenstates

The energy spectrum of a quantum system reveals much of its properties. It can also give information of chaotic or regular dynamics [?] as will be shown in the following section. One of the most prominent features of quantum chaos is avoided crossing, i.e. no energy degeneracies. This roots from the structure of the spectrum and eigenstates of the system. If there are certain conserved quantities, be it an observable like the number of exciations as in the Tavis-Cummings model, or a symmetry like the parity, as in the Standard Dicke model, the spectrum is separated into different subsets, which are independet of each other. In such a case degeneracies between different subsets are allowed, because of their independence, in other words the quantum states differ in some quantum number. If no such conserved quantity exists, the systems eigenvalues will avoid degeneracies, because —? The mapping of nontrivial (i.e. not just counting them) quantum numbers to eigenvalues is a key aspect of integrable dynamics, the inability is a sign of nonintegrability. The mechanism of avoided crossing is especially usefull, since it can be observed whether or not an underlying conserved quantity is known or not. If degenerate or even nearly degenerate states exist, one can assume, that the is a conserved quantity and the system may follow regular dynamics. If on the other hand there are no arbitrarily small energy differences, there seams to be no other invariant beside the energy, and the system may behave chaotically, at least in the classical limit. In classical mechanics a criterion for regular or integrable dynamics is the number of invariants in involution. Every invariant restricts the phase space accesible to any given initial values to a torus, so if there are enough invariants to be found, the dynamics become regular or integrable. It is important to notice here, that according to Noethers Theorem [? source necessary?] only continuous symmetries create invariants. Thus the parity, although spliting the Hilbert space, does not necessary have a great influence on the dynamics, at least in or close to the classical limit. What remains to be studied is whether or how breaking the parity influences the dynamics in case of small spins, because the derived criterion of conserved quantities can not necessarily be converted to the small spin case.

2.2 chaos and integrability in quantum mechanics

It is important to distinguish the different terms that appear, when discussing dynamics of a quantum system. There is integrable vs. nonintegrable, which describes the number of conserved quantities and as a result, whether or not the trajectory of an arbitrary classical phase space-point can be determinded analytically (at least theoretically). Then there is regular vs. chaotic, which in classical mechanics describes whether two different initial conditions can have exponentially diverging trajectories. It is a lot more difficult to make global statements about chaocity in a given system, since the dynamics can (and in many cases do) depend on the initial conditions. The easiest example is the mathematic pendulum. While most of its phase space follows regular dynamics, the separatrix is a line of chaotic points, most prominently the overhead pendulum, where slightly diffenent initial conditions determine whether it moves left or right. That makes it a prime example of a chaotic point in Phase space, while the trajectory for any other point, not on the separatrix, is well known

and the dynamics are regular. Even this simple example shows that regular or chaotic dynamics are not to be determined for a system, but rather for system and initial conditions.

It is a important question of quantum mechanics, if and how these fundamental concepts can be converted to quantum physics. While the criterion for regularity can easily be translated into quantum mechanics by searching for quantum invariants, that alone can not surfice. In quantum mechanics one can find arbitrarily many invariants by time averaging, i.e. taking any operator and crossing out nondiagonal elements in the energy basis. The number of invariants can not be the key, but the concept itself is difficult to convert, since any time evolition in quantum mechanics is unitary and by decomposing any initial state in energy eigenstates the exact time evolution is (theoretically) known for any system and initial state. On the other hand, the concept of regularity can be understood in a quantum mechanical context, at least an a qualitative level, but there is no quantitative measure for it, as for example a Lyapunov exponent in classical mechanics.

An established method to distinguish regular and chaotic dynamics are Poincaré sections. The idea is to make a twodimensional section of the trajectory by chosing a set of contitions to be fulfilled. If the resulting set of points in the twodimensional picture is limited to a onedimensional subspace, i.e. a line or a set of distinguished points, the dynamics are regular, if it covers the while plain or an area it means chaotic dynamics, because the trajectory covers the whole energy shell. In the quantum mechanical system, there is an equvalent measure, that projects a state onto a coherent state, the parameters of which are defined by the classical Poincaré condition.

$$Q_{\mathrm{Poinc}}(\vartheta,\phi) = |\langle \alpha_{\mathrm{Poinc}}(E), z|\Psi\rangle|^2 = |\sum_m \tilde{\beta}_m \langle z(\vartheta,\phi)|m\rangle|^2 \tag{3}$$

$$\text{mit } \tilde{\beta}_m = \sum_n \beta_{n,m}\langle \alpha_{\mathrm{Poinc}}(E)|n\rangle \tag{4}$$

While this projection shows remarkable resemblance for larger spin, it is not applicable for small spin, because the coherent states become to widely spread and nothing can be read from the data. And even for large spin it can only be a qualitative measure and be viewed in comparison to the classical Poincaré section[**EP09**].

A different and more extensive, albeit still qualitative method is the Peres lattice [**AP84**].

3 Time Evolution

To study the time evolution of a quantum system numerically it is necessary to find an efficient way to implement the time evolution operator. Using this operator directly instead of expanding the initial state into the energy eigenstates is advisable, so that one need not diagonalize the full hamiltonian. It has been shown that the Chebyshev expansion is a very usefull approach to this problem, because the resulting expansion coefficients, which are given by Bessel functions, degrease controllably for high orders, and the operator polynomials are more stable than a power expansion or established other approaches. Thus it is usefull especially for large timesteps.

In every bosonic system a truncation is needed, when calculating the time evolution. The truncation is deemed sufficient, if a larger maximum state does not change the results of the calculation significanty, i.e. only on a level of numeric uncertainty.

Here it is very important to be cautious, because two truncations are applied, one for the time evolution operator and one for the bosonic Hilbert space.