Challenge Problem 5

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Download all python codes from

```
https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/ChallengeProblem5/Codes
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and latex-tikz codes from

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https://github.com/ka−raja−babu/Matrix−Theory/
    tree/main/ChallengeProblem5
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1 Challenge Question 5

Express the axis of a parabola in terms of $\mathbf{V}, \mathbf{u}, f$ in general .

2 Solution

Lemma 2.1. *Axis of any conic is given by*

$$\boxed{-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \mathbf{x} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right)} \tag{2.0.1}$$

Note:-This will give the minor axis in case of conics having two axes.

Proof. The general equation of a conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.2}$$

which can be written in matrix form as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

where,

$$\mathbf{V} = \begin{pmatrix}\mathbf{v_1} & \mathbf{v_2}\end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.5}$$

Axis of a conic can be written as :

$$\mathbf{A}\mathbf{x} = B \tag{2.0.6}$$

where,$\mathbf{A}$ is the direction vector of the axis and B is a constant.

Axis of a conic must satisfies two conditions:

1) It must divide the conic into two symmetrical parts.
2) It must pass through the vertex.

By condition 1:

$$\mathbf{A} = -\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \tag{2.0.7}$$

By condition 2:

$$B = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) \tag{2.0.8}$$

where,$\mathbf{e_1}$ and $\mathbf{e_2}$ are standard basis vector such that

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.9}$$

Hence,the axis of a parabola and the minor axis of an ellipse and a hyperbola ,is given by

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \mathbf{x} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) \tag{2.0.10}$$

□

Lemma 2.2. *Major axis of any conic having two axes is given by*

$$\boxed{\mathbf{n}^T\mathbf{x} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right)} \tag{2.0.11}$$

where,

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \mathbf{n} = 0 \tag{2.0.12}$$

$$\mathbf{n}^T\mathbf{c} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) \tag{2.0.13}$$

Proof. Major axis of any conic must satisfies two conditions:

1) It must be perpendicular to the minor axis.
2) It must passes through the vertex of conic.

By using condition 1:

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \mathbf{n} = 0 \tag{2.0.14}$$

By using condition 2:

$$\mathbf{n}^T\mathbf{c} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) \tag{2.0.15}$$

Hence,the major axis is given by

$$\mathbf{n}^T\mathbf{x} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) \tag{2.0.16}$$

where $\mathbf{n}$ is the normal vector passing through the vertex. □

3 Examples

1) Parabola

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \tag{3.0.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ \frac{-101}{2} \end{pmatrix} \tag{3.0.3}$$

$$f = 19 \tag{3.0.4}$$

$\therefore$

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T = \begin{pmatrix} -21 & 28 \end{pmatrix} \tag{3.0.5}$$

Now,

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \tag{3.0.6}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-29}{25} \\ \frac{22}{25} \end{pmatrix} \tag{3.0.7}$$

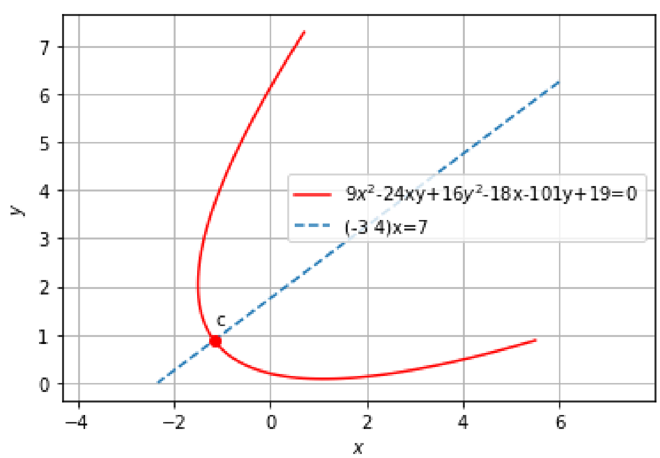

Fig. 3.1: $9x^2$-24xy+16y^2-18x-101y+19=0

So,

$$\mathbf{Vc} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \begin{pmatrix} \frac{-29}{25} \\ \frac{22}{25} \end{pmatrix} = \begin{pmatrix} -21 \\ 28 \end{pmatrix} \tag{3.0.8}$$

and,

$$\mathbf{e_2}^T \left(\mathbf{Vc}\right) - \mathbf{e_1}^T \left(\mathbf{Vc}\right) = 28 + 21 = 49 \tag{3.0.9}$$

Hence,the axis is given by

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \mathbf{x} = \mathbf{e_2}^T \left(\mathbf{Vc}\right) - \mathbf{e_1}^T \left(\mathbf{Vc}\right) \tag{3.0.10}$$

$$\implies \begin{pmatrix} -21 & 28 \end{pmatrix} \mathbf{x} = 49 \tag{3.0.11}$$

$$\implies \boxed{\begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = 7} \tag{3.0.12}$$

2) Parabola

$$y^2 - 4x + 2y + 4 = 0 \tag{3.0.13}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.14}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{3.0.15}$$

$$f = 4 \tag{3.0.16}$$

$\therefore$

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{3.0.17}$$

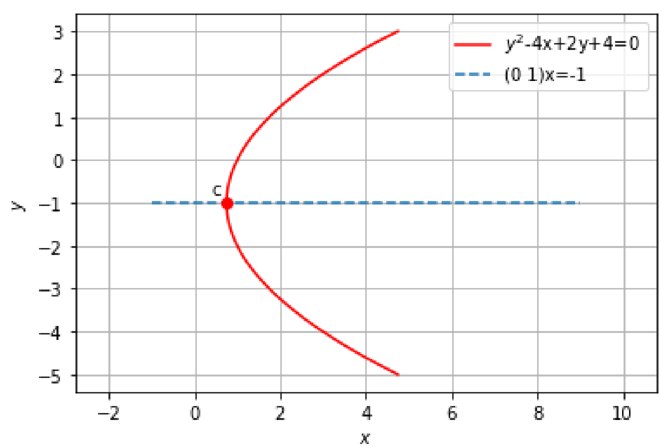

Fig. 3.2: y^2-4x+2y+4=0

Now,

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \tag{3.0.18}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \tag{3.0.19}$$

So,

$$\mathbf{Vc} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \tag{3.0.20}$$

and,

$$\mathbf{e_2}^T(\mathbf{Vc}) - \mathbf{e_1}^T(\mathbf{Vc}) = -1 - 0 = -1 \quad (3.0.21)$$

Hence,the axis is given by

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \mathbf{x} = \mathbf{e_2}^T(\mathbf{Vc}) - \mathbf{e_1}^T(\mathbf{Vc}) \quad (3.0.22)$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -1} \quad (3.0.23)$$

3) Parabola

$$y^2 = 8x \quad (3.0.24)$$

$$\implies y^2 - 8x = 0 \quad (3.0.25)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.26)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \quad (3.0.27)$$

$$f = 0 \quad (3.0.28)$$

$\therefore$

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad (3.0.29)$$

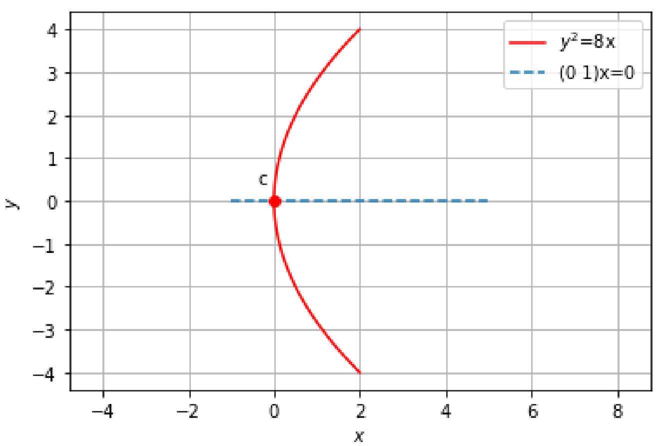

Fig. 3.3: y^2=8x

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.0.30)$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.31)$$

So,

$$\mathbf{Vc} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.32)$$

and,

$$\mathbf{e_2}^T(\mathbf{Vc}) - \mathbf{e_1}^T(\mathbf{Vc}) = 0 \quad (3.0.33)$$

Hence,the axis is given by

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \mathbf{x} = \mathbf{e_2}^T(\mathbf{Vc}) - \mathbf{e_1}^T(\mathbf{Vc}) \quad (3.0.34)$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0} \quad (3.0.35)$$

4) Ellipse

$$x^2 + xy + y^2 = 100 \quad (3.0.36)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (3.0.37)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.38)$$

$$f = -100 \quad (3.0.39)$$

$\therefore$

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T = \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.40)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \quad (3.0.41)$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.42)$$

So,

$$\mathbf{Vc} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.43)$$

and,

$$\mathbf{e_2}^T(\mathbf{Vc}) - \mathbf{e_1}^T(\mathbf{Vc}) = 0 \quad (3.0.44)$$

Hence,the minor axis is given by

$$-\left(\mathbf{v_1} - \mathbf{v_2}\right)^T \mathbf{x} = \mathbf{e_2}^T(\mathbf{Vc}) - \mathbf{e_1}^T(\mathbf{Vc}) \quad (3.0.45)$$

$$\implies \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = 0 \quad (3.0.46)$$

$$\implies \boxed{\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 0} \quad (3.0.47)$$

The normal vector $\mathbf{n}$ is given by

$$-\left(\mathbf{v_1}-\mathbf{v_2}\right)^T \mathbf{n} = 0 \tag{3.0.48}$$

$$\implies \begin{pmatrix}\frac{-1}{2} & \frac{1}{2}\end{pmatrix}\mathbf{n} = 0 \tag{3.0.49}$$

$$\implies \mathbf{n} = \begin{pmatrix}1\\1\end{pmatrix} \tag{3.0.50}$$

Hence,the major axis is given by

$$\mathbf{n}^T\mathbf{x} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) \tag{3.0.51}$$

$$\implies \boxed{\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 0} \tag{3.0.52}$$

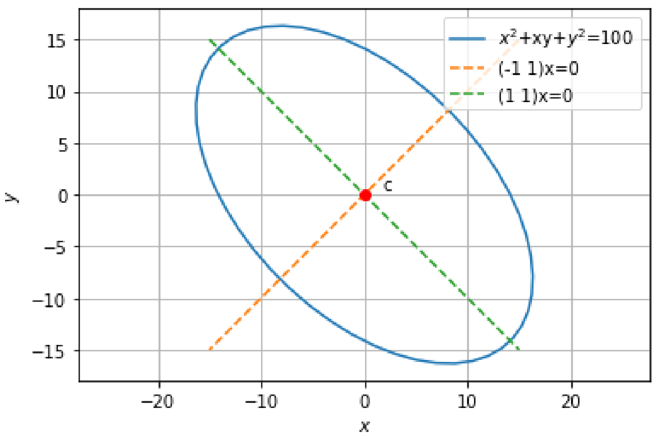

Fig. 3.4: x^2+xy+y^2=100

5) Hyperbola

$$xy - 3y + 2 = 0 \tag{3.0.53}$$

Here,

$$\mathbf{V} = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \tag{3.0.54}$$

$$\mathbf{u} = \frac{-3}{2}\begin{pmatrix}0\\1\end{pmatrix} \tag{3.0.55}$$

$$f = 2 \tag{3.0.56}$$

$\therefore$

$$-\left(\mathbf{v_1}-\mathbf{v_2}\right)^T = \begin{pmatrix}\frac{1}{2} & \frac{-1}{2}\end{pmatrix} \tag{3.0.57}$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \tag{3.0.58}$$

$$\implies \mathbf{c} = \begin{pmatrix}3\\0\end{pmatrix} \tag{3.0.59}$$

So,

$$\mathbf{Vc} = \frac{1}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}0\\\frac{3}{2}\end{pmatrix} \tag{3.0.60}$$

and,

$$\mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) = \frac{3}{2} \tag{3.0.61}$$

Hence,the minor axis is given by

$$-\left(\mathbf{v_1}-\mathbf{v_2}\right)^T \mathbf{x} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) \tag{3.0.62}$$

$$\implies \begin{pmatrix}\frac{1}{2} & \frac{-1}{2}\end{pmatrix}\mathbf{x} = \frac{3}{2} \tag{3.0.63}$$

$$\implies \boxed{\begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x} = 3} \tag{3.0.64}$$

The normal vector $\mathbf{n}$ is given by

$$-\left(\mathbf{v_1}-\mathbf{v_2}\right)^T \mathbf{n} = 0 \tag{3.0.65}$$

$$\implies \begin{pmatrix}\frac{1}{2} & \frac{-1}{2}\end{pmatrix}\mathbf{n} = 0 \tag{3.0.66}$$

$$\implies \mathbf{n} = \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\end{pmatrix} \tag{3.0.67}$$

Hence,the major axis is given by

$$\mathbf{n}^T\mathbf{x} = \mathbf{e_2}^T\left(\mathbf{Vc}\right) - \mathbf{e_1}^T\left(\mathbf{Vc}\right) \tag{3.0.68}$$

$$\implies \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\end{pmatrix}\mathbf{x} = \frac{3}{2} \tag{3.0.69}$$

$$\implies \boxed{\begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3} \tag{3.0.70}$$

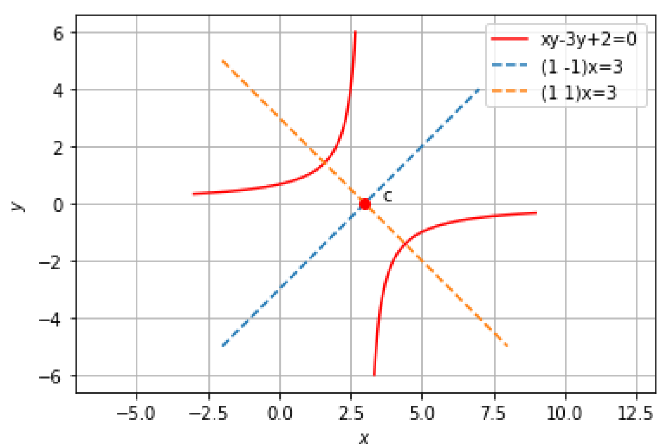

Fig. 3.5: xy-3y+2=0