

Challenge Problem 5

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Download all python codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem5/Codes>

and latex-tikz codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem5>

1 CHALLENGE QUESTION 5

Express the axis of a parabola in terms of $\mathbf{V}, \mathbf{u}, f$ in general .

2 SOLUTION

The general equation of a conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

which can be written in matrix form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

where,

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.4)$$

Axis of a conic can be written as :

$$\mathbf{A} \mathbf{x} = B \quad (2.0.5)$$

where, \mathbf{A} is the direction vector of the axis and B is a constant.

Axis of a conic must satisfies two conditions:

- 1) It must divide the conic into two symmetrical parts.
- 2) It must pass through the vertex.

By condition 1:

$$\mathbf{A} = -(\mathbf{v}_1 - \mathbf{v}_2)^T \quad (2.0.6)$$

By condition 2:

$$B = (0 \ 1) \mathbf{V} \mathbf{c} - (1 \ 0) \mathbf{V} \mathbf{c} \quad (2.0.7)$$

Hence, the axis of a parabola and the minor axis of an ellipse and a hyperbola, is given by

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{x} = (0 \ 1) \mathbf{V} \mathbf{c} - (1 \ 0) \mathbf{V} \mathbf{c} \quad (2.0.8)$$

And, the major axis of an ellipse and a hyperbola is given by:

$$\mathbf{n}^T \mathbf{x} = (0 \ 1) \mathbf{V} \mathbf{c} - (1 \ 0) \mathbf{V} \mathbf{c} \quad (2.0.9)$$

where, \mathbf{n} is the normal vector passing through vertex \mathbf{c} such that it satisfies these two conditions:

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{n} = 0 \quad (2.0.10)$$

$$\mathbf{n}^T \mathbf{c} = 0 \quad (2.0.11)$$

3 EXAMPLES

1) Parabola

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \quad (3.0.1)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (3.0.2)$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -101/2 \end{pmatrix} \quad (3.0.3)$$

$$f = 19 \quad (3.0.4)$$

\therefore

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T = (-21 \ 28) \quad (3.0.5)$$

Now,

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \quad (3.0.6)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} -29/25 \\ 25/25 \end{pmatrix} \quad (3.0.7)$$

So,

$$\mathbf{V} \mathbf{c} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \begin{pmatrix} -29/25 \\ 25/25 \end{pmatrix} = \begin{pmatrix} -21 \\ 28 \end{pmatrix} \quad (3.0.8)$$

and,

$$(0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} = 28 + 21 = 49 \quad (3.0.9)$$

Hence, the axis is given by

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{x} = (0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} \quad (3.0.10)$$

$$\Rightarrow (-21 \ 28) \mathbf{x} = 49 \quad (3.0.11)$$

$$\Rightarrow \boxed{(-3 \ 4) \mathbf{x} = 7} \quad (3.0.12)$$

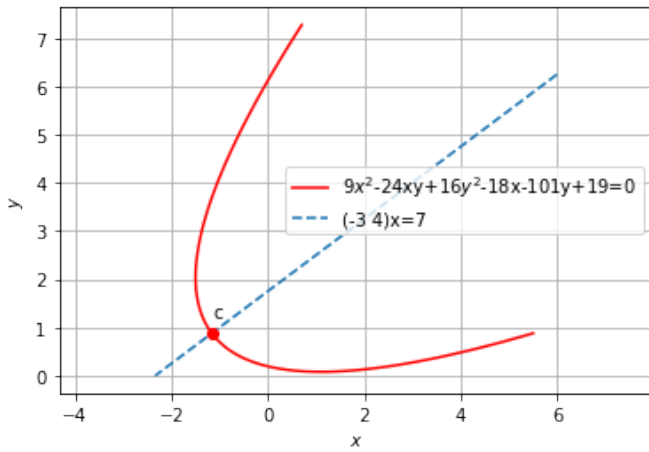


Fig. 3.1: $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

2) Parabola

$$y^2 - 4x + 2y + 4 = 0 \quad (3.0.13)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.14)$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3.0.15)$$

$$f = 4 \quad (3.0.16)$$

\therefore

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T = (0 \ 1) \quad (3.0.17)$$

Now,

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \quad (3.0.18)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \quad (3.0.19)$$

So,

$$\mathbf{Vc} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad (3.0.20)$$

and,

$$(0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} = -1 - 0 = -1 \quad (3.0.21)$$

Hence, the axis is given by

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{x} = (0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} \quad (3.0.22)$$

$$\Rightarrow \boxed{(0 \ 1) \mathbf{x} = -1} \quad (3.0.23)$$

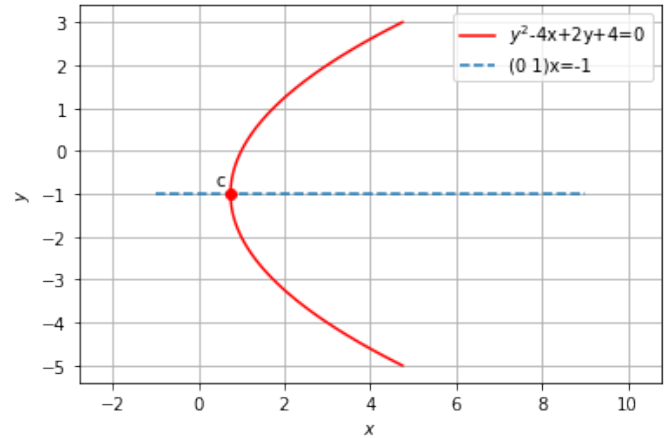


Fig. 3.2: $y^2 - 4x + 2y + 4 = 0$

3) Parabola

$$y^2 = 8x \quad (3.0.24)$$

$$\Rightarrow y^2 - 8x = 0 \quad (3.0.25)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.26)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \quad (3.0.27)$$

$$f = 0 \quad (3.0.28)$$

\therefore

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T = (0 \ 1) \quad (3.0.29)$$

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.0.30)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.31)$$

So,

$$\mathbf{Vc} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.32)$$

and,

$$(0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} = 0 \quad (3.0.33)$$

Hence, the axis is given by

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{x} = (0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} \quad (3.0.34)$$

$$\Rightarrow \boxed{(0 \ 1) \mathbf{x} = 0} \quad (3.0.35)$$

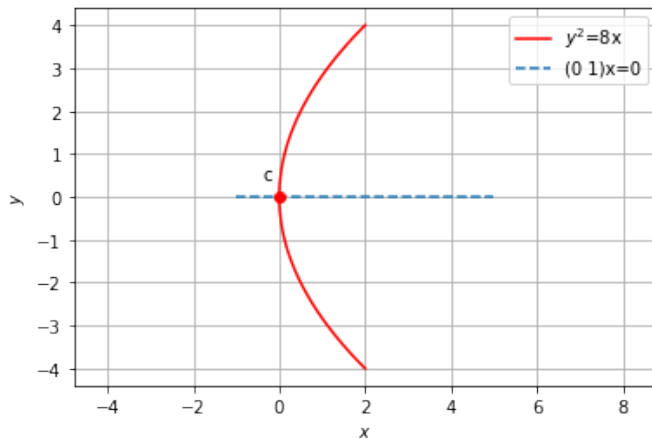


Fig. 3.3: $y^2=8x$

4) Ellipse

$$x^2 + xy + y^2 = 100 \quad (3.0.36)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (3.0.37)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.38)$$

$$f = -100 \quad (3.0.39)$$

\therefore

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.40)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1} \mathbf{u} \quad (3.0.41)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.42)$$

So,

$$\mathbf{Vc} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.43)$$

and,

$$(0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} = 0 \quad (3.0.44)$$

Hence, the minor axis is given by

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{x} = (0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} \quad (3.0.45)$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = 0 \quad (3.0.46)$$

$$\Rightarrow \boxed{(-1 \ 1) \mathbf{x} = 0} \quad (3.0.47)$$

The normal vector \mathbf{n} is given by

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{n} = 0 \quad (3.0.48)$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{n} = 0 \quad (3.0.49)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.0.50)$$

Hence, the major axis is given by

$$\mathbf{n}^T \mathbf{x} = (0 \ 1) \mathbf{Vc} - (1 \ 0) \mathbf{Vc} \quad (3.0.51)$$

$$\Rightarrow \boxed{(1 \ 1) \mathbf{x} = 0} \quad (3.0.52)$$

5) Hyperbola

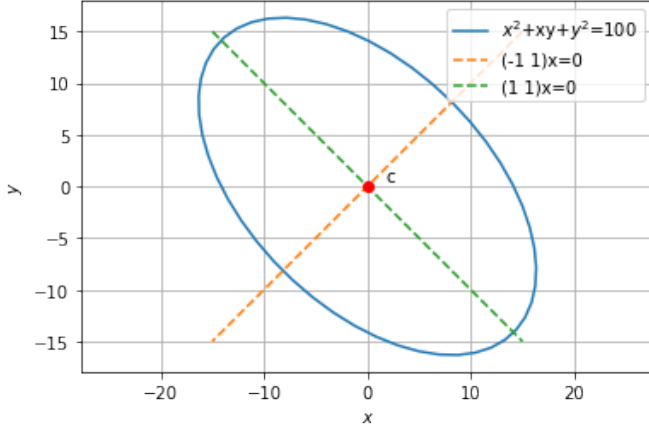
$$xy - 3y + 2 = 0 \quad (3.0.53)$$

Here,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.54)$$

$$\mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.55)$$

$$f = 2 \quad (3.0.56)$$

Fig. 3.4: $x^2+xy+y^2=100$

\therefore

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad (3.0.57)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \quad (3.0.58)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.0.59)$$

So,

$$\mathbf{V}\mathbf{c} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{2} \end{pmatrix} \quad (3.0.60)$$

and,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{V}\mathbf{c} - \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{V}\mathbf{c} = \frac{3}{2} \quad (3.0.61)$$

Hence, the minor axis is given by

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{x} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{V}\mathbf{c} - \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{V}\mathbf{c} \quad (3.0.62)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{3}{2} \quad (3.0.63)$$

$$\Rightarrow \boxed{(1 \quad -1) \mathbf{x} = 3} \quad (3.0.64)$$

The normal vector \mathbf{n} is given by

$$-(\mathbf{v}_1 - \mathbf{v}_2)^T \mathbf{n} = 0 \quad (3.0.65)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \mathbf{n} = 0 \quad (3.0.66)$$

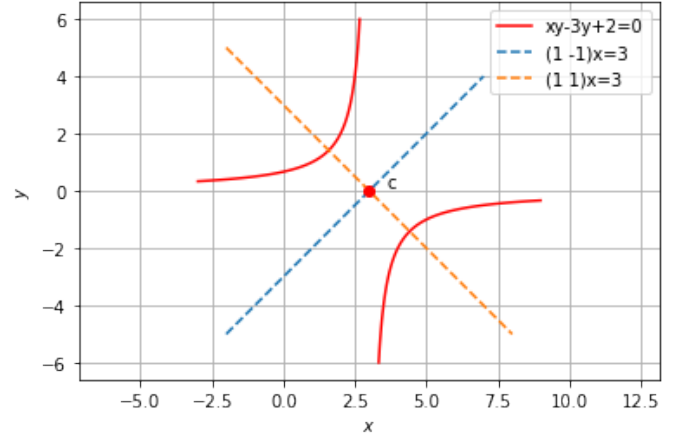
$$\Rightarrow \mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \quad (3.0.67)$$

Hence, the major axis is given by

$$\mathbf{n}^T \mathbf{x} = \begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{V}\mathbf{c} - \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{V}\mathbf{c} \quad (3.0.68)$$

$$\Rightarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{3}{2} \quad (3.0.69)$$

$$\Rightarrow \boxed{(1 \quad 1) \mathbf{x} = 3} \quad (3.0.70)$$

Fig. 3.5: $xy-3y+2=0$