

# Challenge Problem 5

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Download all python codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem5/Codes>

and latex-tikz codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem5>

## 1 CHALLENGE QUESTION 5

Express the axis of a parabola in terms of  $\mathbf{V}, \mathbf{u}, f$  in general .

## 2 SOLUTION

**Lemma 2.1.** *Axis of any conic is given by*

$$\mathbf{p}^T \begin{pmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (2.0.1)$$

where,  $\mathbf{c}$  is the vertex of conic and  $\mathbf{p}$  is the eigen vector of  $\mathbf{V}$  having smaller eigen value.

*Proof.* The general equation of a conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.2)$$

which can be written in matrix form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.5)$$

Let the eigen vector of  $\mathbf{V}$  having smaller eigen value be  $\mathbf{p}$  .

Now, normal vector of  $\mathbf{p}$  having same magnitude, is given by

$$\mathbf{p}_n^T \mathbf{p} = 0 \quad (2.0.6)$$

$$\Rightarrow \mathbf{p}_n^T = \mathbf{p}^T \begin{pmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix} \quad (\because \mathbf{p}^T \begin{pmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix} \mathbf{p} = 0) \quad (2.0.7)$$

where  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are standard basis vector such that

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.8)$$

According to the principal axis theorem,

- 1) Each eigen vector of  $\mathbf{V}$  is parallel to either major axis or minor axis.
- 2) Normal vector of each eigen vector of  $\mathbf{V}$  is normal to either major axis or minor axis.
- 3) Axes pass through the vertex  $\mathbf{c}$  of the conic.

$\therefore$  Axis is given by

$$\mathbf{p}_n^T (\mathbf{x} - \mathbf{c}) = 0 \quad (2.0.9)$$

$$\Rightarrow \mathbf{p}^T \begin{pmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (2.0.10)$$

□

## 3 EXAMPLES

- 1) Parabola

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \quad (3.0.1)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (3.0.2)$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -\frac{101}{2} \end{pmatrix} \quad (3.0.3)$$

$$f = 19 \quad (3.0.4)$$

Now,

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \quad (3.0.5)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-29}{25} \\ \frac{25}{25} \\ \frac{25}{25} \end{pmatrix} \quad (3.0.6)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x + \frac{29}{25} \\ y - \frac{25}{25} \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 25x + 29 \\ 25y - 22 \end{pmatrix} \quad (3.0.7)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.8)$$

$$\Rightarrow \begin{vmatrix} 9 - \lambda & -12 \\ -12 & 16 - \lambda \end{vmatrix} = 0 \quad (3.0.9)$$

$$\Rightarrow \lambda^2 - 25\lambda = 0 \quad (3.0.10)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 25 \quad (3.0.11)$$

For  $\lambda_1 = 0$ ,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (3.0.12)$$

$$\Rightarrow \mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 0 \end{pmatrix} \quad (3.0.13)$$

$$\Rightarrow \mathbf{p}_1 = \frac{3}{5} \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \quad (3.0.14)$$

Similarly for  $\lambda_2 = 25$ ,

$$\mathbf{p}_2 = \frac{4}{5} \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (3.0.15)$$

$\because \lambda_1 < \lambda_2$

Hence, the axis using  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1^T (-\mathbf{e}_2 \quad \mathbf{e}_1) (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.16)$$

$$\Rightarrow \frac{3}{5} \begin{pmatrix} \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{1}{25} \begin{pmatrix} 25x + 29 \\ 25y - 22 \end{pmatrix} = 0 \quad (3.0.17)$$

$$\Rightarrow -3x + 4y = 7 \quad (3.0.18)$$

$$\Rightarrow \boxed{(-3 \quad 4) \mathbf{x} = 7} \quad (3.0.19)$$

2) Parabola

$$y^2 - 4x + 2y + 4 = 0 \quad (3.0.20)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.21)$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3.0.22)$$

$$f = 4 \quad (3.0.23)$$

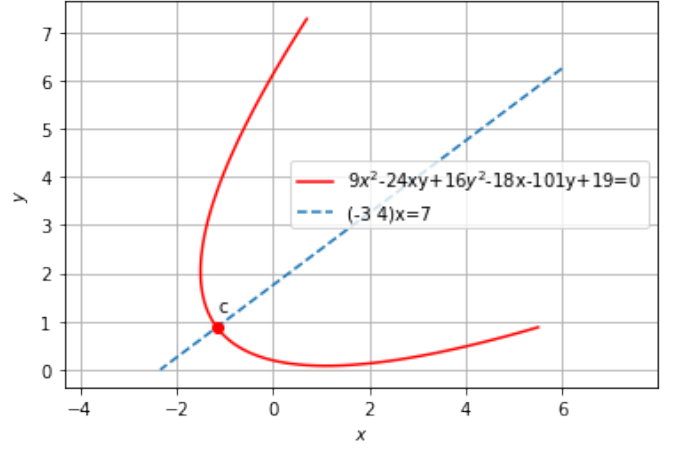


Fig. 3.1:  $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

Now,

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \quad (3.0.24)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \quad (3.0.25)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} \quad (3.0.26)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.27)$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \quad (3.0.28)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 1 \quad (3.0.29)$$

For  $\lambda_1 = 0$ ,

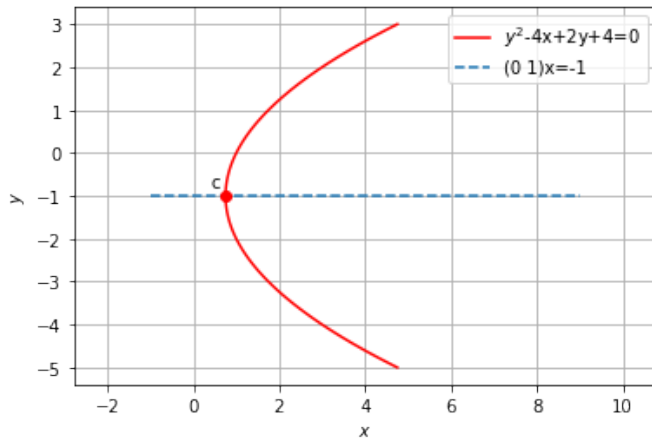
$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.30)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.31)$$

Similarly for  $\lambda_2 = 1$ ,

$$\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.32)$$

$\because \lambda_1 < \lambda_2$

Fig. 3.2:  $y^2 - 4x + 2y + 4 = 0$ 

Hence, the axis using  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1^T \begin{pmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.33)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} = 0 \quad (3.0.34)$$

$$\Rightarrow y + 1 = 0 \quad (3.0.35)$$

$$\Rightarrow \boxed{(0 \ 1)\mathbf{x} = -1} \quad (3.0.36)$$

### 3) Parabola

$$y^2 = 8x \quad (3.0.37)$$

$$\Rightarrow y^2 - 8x = 0 \quad (3.0.38)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.39)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \quad (3.0.40)$$

$$f = 0 \quad (3.0.41)$$

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.0.42)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.43)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.0.44)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.45)$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \quad (3.0.46)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 1 \quad (3.0.47)$$

For  $\lambda_1 = 0$ ,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.48)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.49)$$

Similarly for  $\lambda_2 = 1$ ,

$$\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.50)$$

$\because \lambda_1 < \lambda_2$

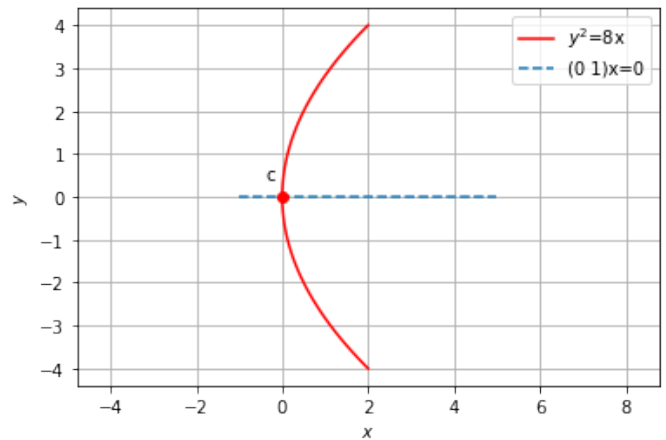
Hence, the axis using  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1^T \begin{pmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.51)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad (3.0.52)$$

$$\Rightarrow y = 0 \quad (3.0.53)$$

$$\Rightarrow \boxed{(0 \ 1)\mathbf{x} = 0} \quad (3.0.54)$$

Fig. 3.3:  $y^2 = 8x$ 

### 4) Ellipse

$$x^2 + xy + y^2 = 100 \quad (3.0.55)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (3.0.56)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.57)$$

$$f = -100 \quad (3.0.58)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \quad (3.0.59)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.60)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.0.61)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.62)$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & \frac{1}{2} \\ \frac{1}{2} & 1 - \lambda \end{vmatrix} = 0 \quad (3.0.63)$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (3.0.64)$$

$$\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \quad (3.0.65)$$

For  $\lambda_1 = \frac{1}{2}$ ,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.66)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.0.67)$$

Similarly for  $\lambda_2 = \frac{3}{2}$ ,

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.0.68)$$

$\because \lambda_1 < \lambda_2$

Hence, the major axis using  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1^T (-\mathbf{e}_2 \quad \mathbf{e}_1) (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.69)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad (3.0.70)$$

$$\Rightarrow x + y = 0 \quad (3.0.71)$$

$$\Rightarrow \boxed{(1 \quad 1)\mathbf{x} = 0} \quad (3.0.72)$$

and the minor axis using  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2^T (-\mathbf{e}_2 \quad \mathbf{e}_1) (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.73)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad (3.0.74)$$

$$\Rightarrow -x + y = 0 \quad (3.0.75)$$

$$\Rightarrow \boxed{(-1 \quad 1)\mathbf{x} = 0} \quad (3.0.76)$$

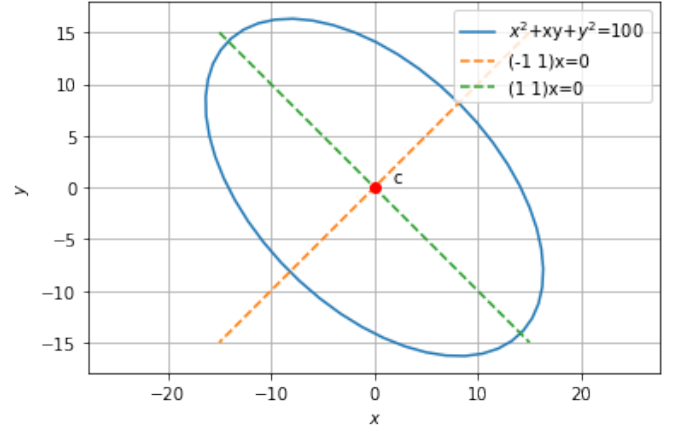


Fig. 3.4:  $x^2 + xy + y^2 = 100$

## 5) Hyperbola

$$xy - 3y + 2 = 0 \quad (3.0.77)$$

Here,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.78)$$

$$\mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.79)$$

$$f = 2 \quad (3.0.80)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \quad (3.0.81)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.0.82)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - 3 \\ y \end{pmatrix} \quad (3.0.83)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.84)$$

$$\Rightarrow \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \quad (3.0.85)$$

$$\Rightarrow \lambda^2 - \frac{1}{4} = 0 \quad (3.0.86)$$

$$\Rightarrow \lambda_1 = \frac{-1}{2}, \lambda_2 = \frac{1}{2} \quad (3.0.87)$$

For  $\lambda_1 = \frac{-1}{2}$ ,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.88)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.0.89)$$

Similarly for  $\lambda_2 = \frac{1}{2}$ ,

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.0.90)$$

and the minor axis using  $\mathbf{p}_2$  is given by

$$\mathbf{p}_2^T \begin{pmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.95)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x-3 \\ y \end{pmatrix} = 0 \quad (3.0.96)$$

$$\Rightarrow x - y = 3 \quad (3.0.97)$$

$$\Rightarrow \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3} \quad (3.0.98)$$

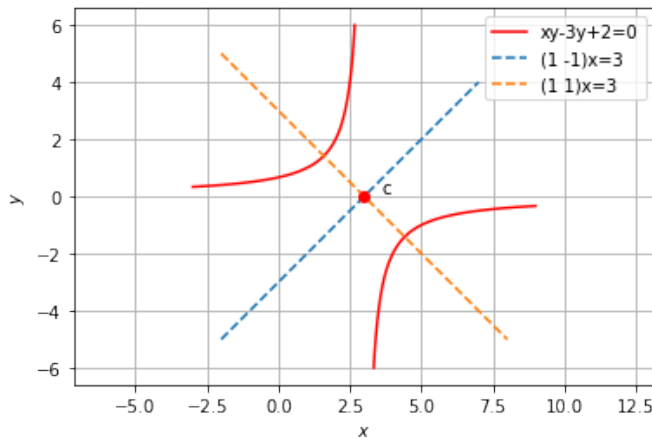


Fig. 3.5:  $xy - 3y + 2 = 0$

$\because \lambda_1 < \lambda_2$

Hence, the major axis using  $\mathbf{p}_1$  is given by

$$\mathbf{p}_1^T \begin{pmatrix} -\mathbf{e}_2 & \mathbf{e}_1 \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.91)$$

$$\Rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x-3 \\ y \end{pmatrix} = 0 \quad (3.0.92)$$

$$\Rightarrow x + y = 3 \quad (3.0.93)$$

$$\Rightarrow \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3} \quad (3.0.94)$$