

Challenge Problem 5

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Download all python codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem5/Codes>

and latex-tikz codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem5>

1 CHALLENGE QUESTION 5

Express the axis of a parabola in terms of $\mathbf{V}, \mathbf{u}, f$ in general .

2 SOLUTION

Lemma 2.1. *Axis of any conic is given by*

$$\frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}} \quad (2.0.1)$$

where, \mathbf{c} is the vertex of conic and \mathbf{p} is the eigen vector of \mathbf{V} having smaller eigen value.

Proof. The general equation of a conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.2)$$

which can be written in matrix form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.5)$$

Let the vertex be

$$\mathbf{c} = \begin{pmatrix} x_c \\ y_c \end{pmatrix} \quad (2.0.6)$$

and, the origin be

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.7)$$

and, the eigen vector of \mathbf{V} having smaller eigen value be

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad (2.0.8)$$

Now, according to the principal axis theorem,

- 1) Each eigen vector of \mathbf{V} is parallel to either major axis or minor axis.
- 2) Eigen vectors of \mathbf{V} are orthogonal to each other.
- 3) Principal axes pass through the vertex \mathbf{c} .

Now, direction vector of eigen vector \mathbf{p} is

$$\mathbf{m}_1 = \mathbf{p} - \mathbf{O} \quad (2.0.9)$$

$$\Rightarrow \mathbf{m}_1 = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \quad (2.0.10)$$

$$\Rightarrow \mathbf{m}_1 = p_1 \begin{pmatrix} 1 \\ \frac{p_2}{p_1} \end{pmatrix} \quad (2.0.11)$$

$$\Rightarrow \mathbf{m}_1 = \begin{pmatrix} 1 \\ \frac{p_2}{p_1} \end{pmatrix} \quad (\because k\mathbf{m} \equiv \mathbf{m}) \quad (2.0.12)$$

$$\Rightarrow \mathbf{m}_1 = \begin{pmatrix} 1 \\ \frac{\mathbf{e}_2^T \mathbf{p}}{\mathbf{e}_1^T \mathbf{p}} \end{pmatrix} \quad (2.0.13)$$

and, direction vector of axis is

$$\mathbf{m}_2 = \mathbf{x} - \mathbf{c} \quad (2.0.14)$$

$$\Rightarrow \mathbf{m}_2 = \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \quad (2.0.15)$$

$$\Rightarrow \mathbf{m}_2 = (x - x_c) \begin{pmatrix} 1 \\ \frac{y - y_c}{x - x_c} \end{pmatrix} \quad (2.0.16)$$

$$\Rightarrow \mathbf{m}_2 = \begin{pmatrix} 1 \\ \frac{y - y_c}{x - x_c} \end{pmatrix} \quad (\because k\mathbf{m} \equiv \mathbf{m}) \quad (2.0.17)$$

$$\Rightarrow \mathbf{m}_2 = \begin{pmatrix} 1 \\ \frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})} \end{pmatrix} \quad (2.0.18)$$

\therefore Eigen vector is parallel to axis.

∴ Axis is given by

$$\mathbf{m}_1 = \mathbf{m}_2 \quad (2.0.19)$$

$$\Rightarrow \begin{pmatrix} 1 \\ \frac{\mathbf{e}_2^T \mathbf{p}}{\mathbf{e}_1^T \mathbf{p}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})} \end{pmatrix} \quad (2.0.20)$$

$$\Rightarrow \frac{\mathbf{e}_2^T \mathbf{p}}{\mathbf{e}_1^T \mathbf{p}} = \frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})} \quad (2.0.21)$$

$$\Rightarrow \frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}} \quad (2.0.22)$$

□

3 EXAMPLES

1) Parabola

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \quad (3.0.1)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (3.0.2)$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -\frac{101}{2} \end{pmatrix} \quad (3.0.3)$$

$$f = 19 \quad (3.0.4)$$

Now,

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \quad (3.0.5)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} -\frac{29}{25} \\ \frac{25}{25} \\ \frac{22}{25} \end{pmatrix} \quad (3.0.6)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x + \frac{29}{25} \\ y - \frac{22}{25} \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 25x + 29 \\ 25y - 22 \end{pmatrix} \quad (3.0.7)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.8)$$

$$\Rightarrow \begin{vmatrix} 9 - \lambda & -12 \\ -12 & 16 - \lambda \end{vmatrix} = 0 \quad (3.0.9)$$

$$\Rightarrow \lambda^2 - 25\lambda = 0 \quad (3.0.10)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 25 \quad (3.0.11)$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (3.0.12)$$

$$\Rightarrow \mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 0 \end{pmatrix} \quad (3.0.13)$$

$$\Rightarrow \mathbf{p}_1 = \frac{3}{5} \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \quad (3.0.14)$$

Similarly for $\lambda_2 = 25$,

$$\mathbf{p}_2 = \frac{4}{5} \begin{pmatrix} -\frac{3}{4} \\ 1 \end{pmatrix} \quad (3.0.15)$$

∵ $\lambda_1 < \lambda_2$

Hence, the axis using \mathbf{p}_1 is given by

$$\frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}_1} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}_1} \quad (3.0.16)$$

$$\Rightarrow \frac{y - \frac{22}{25}}{\frac{3}{5}} = \frac{x + \frac{29}{25}}{\frac{4}{5}} \quad (3.0.17)$$

$$\Rightarrow -75x + 100y = 175 \quad (3.0.18)$$

$$\Rightarrow \boxed{(-3 \ 4) \mathbf{x} = 7} \quad (3.0.19)$$

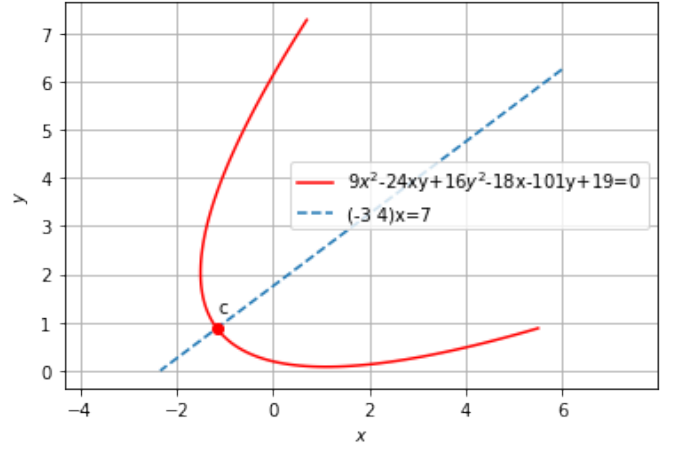


Fig. 3.1: $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

2) Parabola

$$y^2 - 4x + 2y + 4 = 0 \quad (3.0.20)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.21)$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3.0.22)$$

$$f = 4 \quad (3.0.23)$$

Now,

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \quad (3.0.24)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \quad (3.0.25)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} \quad (3.0.26)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.27)$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \quad (3.0.28)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 1 \quad (3.0.29)$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.30)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.31)$$

Similarly for $\lambda_2 = 1$,

$$\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.32)$$

Hence, the axis using \mathbf{p}_1 is given by

$$\frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}_1} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}_1} \quad (3.0.33)$$

$$\Rightarrow \frac{y + 1}{0} = \frac{x - \frac{3}{4}}{1} \quad (3.0.34)$$

$$\Rightarrow y + 1 = 0 \quad (3.0.35)$$

$$\Rightarrow \boxed{(0 \ 1) \mathbf{x} = -1} \quad (3.0.36)$$

3) Parabola

$$y^2 = 8x \quad (3.0.37)$$

$$\Rightarrow y^2 - 8x = 0 \quad (3.0.38)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.39)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \quad (3.0.40)$$

$$f = 0 \quad (3.0.41)$$

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.0.42)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.43)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.0.44)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.45)$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \quad (3.0.46)$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 1 \quad (3.0.47)$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.48)$$

$$\Rightarrow \mathbf{p}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.0.49)$$

Similarly for $\lambda_2 = 1$,

$$\mathbf{p}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.50)$$

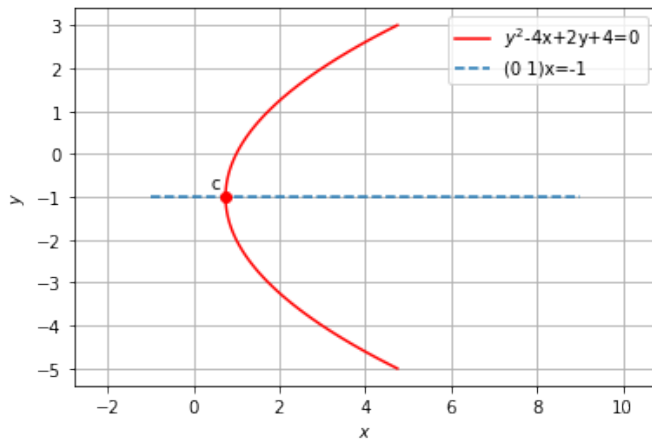


Fig. 3.2: $y^2 - 4x + 2y + 4 = 0$

$\therefore \lambda_1 < \lambda_2$

$$\because \lambda_1 < \lambda_2$$

Hence, the axis using \mathbf{p}_1 is given by

$$\frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}_1} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}_1} \quad (3.0.51)$$

$$\Rightarrow \frac{y}{0} = \frac{x}{1} \quad (3.0.52)$$

$$\Rightarrow y = 0 \quad (3.0.53)$$

$$\Rightarrow \boxed{(0 \ 1)\mathbf{x} = 0} \quad (3.0.54)$$

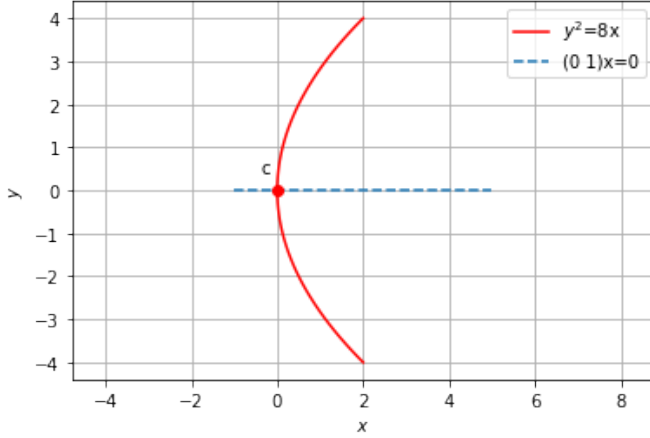


Fig. 3.3: $y^2=8x$

4) Ellipse

$$x^2 + xy + y^2 = 100 \quad (3.0.55)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (3.0.56)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.57)$$

$$f = -100 \quad (3.0.58)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \quad (3.0.59)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.60)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.0.61)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.62)$$

$$\Rightarrow \begin{vmatrix} 1 - \lambda & \frac{1}{2} \\ \frac{1}{2} & 1 - \lambda \end{vmatrix} = 0 \quad (3.0.63)$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (3.0.64)$$

$$\Rightarrow \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \quad (3.0.65)$$

For $\lambda_1 = \frac{1}{2}$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.66)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.0.67)$$

Similarly for $\lambda_2 = \frac{3}{2}$,

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.0.68)$$

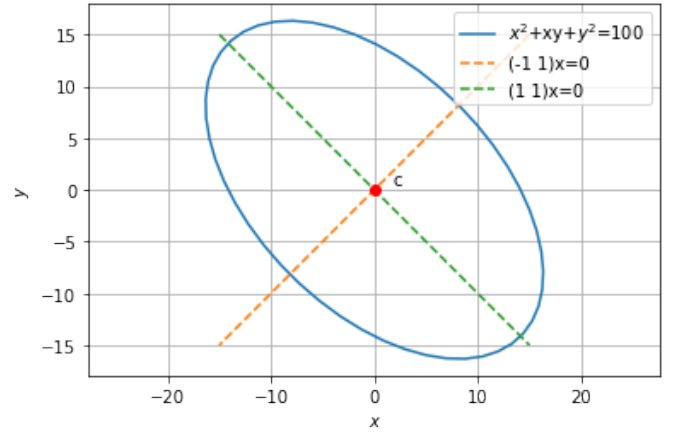


Fig. 3.4: $x^2+xy+y^2=100$

$$\because \lambda_1 < \lambda_2$$

Hence, the major axis using \mathbf{p}_1 is given by

$$\frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}_1} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}_1} \quad (3.0.69)$$

$$\Rightarrow \frac{y}{\frac{1}{\sqrt{2}}} = \frac{x}{\frac{-1}{\sqrt{2}}} \quad (3.0.70)$$

$$\Rightarrow y = -x \quad (3.0.71)$$

$$\Rightarrow \boxed{(1 \ 1)\mathbf{x} = 0} \quad (3.0.72)$$

And, the minor axis using \mathbf{p}_2 is given by

$$\frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}_2} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}_2} \quad (3.0.73)$$

$$\Rightarrow \frac{y}{\frac{1}{\sqrt{2}}} = \frac{x}{\frac{1}{\sqrt{2}}} \quad (3.0.74)$$

$$\Rightarrow y = x \quad (3.0.75)$$

$$\Rightarrow \boxed{(-1 \ 1) \mathbf{x} = 0} \quad (3.0.76)$$

5) Hyperbola

$$xy - 3y + 2 = 0 \quad (3.0.77)$$

Here,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.78)$$

$$\mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.79)$$

$$f = 2 \quad (3.0.80)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1} \mathbf{u} \quad (3.0.81)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.0.82)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - 3 \\ y \end{pmatrix} \quad (3.0.83)$$

Now,

$$|\mathbf{V} - \lambda \mathbf{I}| = 0 \quad (3.0.84)$$

$$\Rightarrow \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \quad (3.0.85)$$

$$\Rightarrow \lambda^2 - \frac{1}{4} = 0 \quad (3.0.86)$$

$$\Rightarrow \lambda_1 = \frac{-1}{2}, \lambda_2 = \frac{1}{2} \quad (3.0.87)$$

For $\lambda_1 = \frac{-1}{2}$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.88)$$

$$\Rightarrow \mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.0.89)$$

Similarly for $\lambda_2 = \frac{1}{2}$,

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.0.90)$$

$\therefore \lambda_1 < \lambda_2$

Hence, the major axis using \mathbf{p}_1 is given by

$$\frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}_1} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}_1} \quad (3.0.91)$$

$$\Rightarrow \frac{y}{\frac{1}{\sqrt{2}}} = \frac{x - 3}{\frac{-1}{\sqrt{2}}} \quad (3.0.92)$$

$$\Rightarrow x + y = 3 \quad (3.0.93)$$

$$\Rightarrow \boxed{(1 \ 1) \mathbf{x} = 3} \quad (3.0.94)$$

And, the minor axis using \mathbf{p}_2 is given by

$$\frac{\mathbf{e}_2^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_2^T \mathbf{p}_2} = \frac{\mathbf{e}_1^T (\mathbf{x} - \mathbf{c})}{\mathbf{e}_1^T \mathbf{p}_2} \quad (3.0.95)$$

$$\Rightarrow \frac{y}{\frac{1}{\sqrt{2}}} = \frac{x - 3}{\frac{1}{\sqrt{2}}} \quad (3.0.96)$$

$$\Rightarrow -x + y = 3 \quad (3.0.97)$$

$$\Rightarrow \boxed{(-1 \ 1) \mathbf{x} = 3} \quad (3.0.98)$$

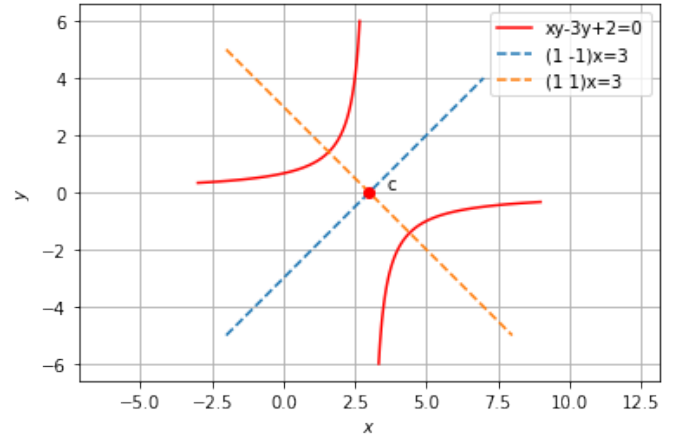


Fig. 3.5: $xy - 3y + 2 = 0$