Challenge Problem 5

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Download all python codes from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem5/Codes

and latex-tikz codes from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem5

1 Challenge Question 5

Express the axis of a parabola in terms of $\mathbf{V}, \mathbf{u}, f$ in general .

2 Solution

Lemma 2.1. *Axis of any conic is given by*

$$\boxed{(\mathbf{e_2} - \mathbf{e_1})^T \ [\mathbf{V}(\mathbf{x} - \mathbf{c})] = 0} \tag{2.0.1}$$

Note:-This will give the minor axis in case of conics having two axes.

Proof. The general equation of a conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.2}$$

which can be written in matrix form as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.5}$$

Axis of a conic can be written as :

$$\mathbf{A}\mathbf{x} = B \tag{2.0.6}$$

where,$\mathbf{A}$ is the direction vector of the axis and B is a constant.

Axis of a conic must satisfies two conditions:

1) It must divide the conic into two symmetrical parts.
2) It must pass through the vertex.

By condition 1:

$$\mathbf{A} = (\mathbf{e_2} - \mathbf{e_1})^T \mathbf{V} \tag{2.0.7}$$

By condition 2:

$$B = (\mathbf{e_2} - \mathbf{e_1})^T (\mathbf{V}\mathbf{c}) \tag{2.0.8}$$

where,$\mathbf{e_1}$ and $\mathbf{e_2}$ are standard basis vector such that

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.9}$$

Hence,the axis of a parabola and the minor axis of an ellipse and a hyperbola ,is given by

$$\left[(\mathbf{e_2} - \mathbf{e_1})^T \mathbf{V}\right]\mathbf{x} = (\mathbf{e_2} - \mathbf{e_1})^T (\mathbf{V}\mathbf{c}) \tag{2.0.10}$$

$$\implies (\mathbf{e_2} - \mathbf{e_1})^T \ [\mathbf{V}(\mathbf{x} - \mathbf{c})] = 0 \tag{2.0.11}$$

□

Lemma 2.2. *Major axis of any conic having two axes is given by*

$$\boxed{\mathbf{n}^T\mathbf{x} = (\mathbf{e_2} - \mathbf{e_1})^T (\mathbf{V}\mathbf{c})} \tag{2.0.12}$$

where,

$$\left[(\mathbf{e_2} - \mathbf{e_1})^T \mathbf{V}\right]\mathbf{n} = 0 \tag{2.0.13}$$

$$\mathbf{n}^T\mathbf{c} = (\mathbf{e_2} - \mathbf{e_1})^T (\mathbf{V}\mathbf{c}) \tag{2.0.14}$$

Proof. Major axis of any conic must satisfies two conditions:

1) It must be perpendicular to the minor axis.
2) It must passes through the vertex of conic.

By using condition 1:

$$\left[(\mathbf{e_2} - \mathbf{e_1})^T \mathbf{V}\right]\mathbf{n} = 0 \tag{2.0.15}$$

By using condition 2:

$$\mathbf{n}^T\mathbf{c} = (\mathbf{e_2} - \mathbf{e_1})^T (\mathbf{V}\mathbf{c}) \tag{2.0.16}$$

Hence,the major axis is given by

$$\mathbf{n}^T\mathbf{x} = (\mathbf{e_2} - \mathbf{e_1})^T (\mathbf{V}\mathbf{c}) \tag{2.0.17}$$

where $\mathbf{n}$ is the normal vector passing through the vertex. □

3 Examples

1) Parabola

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \tag{3.0.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ \frac{-101}{2} \end{pmatrix} \tag{3.0.3}$$

$$f = 19 \tag{3.0.4}$$

Now,

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \tag{3.0.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-29}{25} \\ \frac{22}{25} \end{pmatrix} \tag{3.0.6}$$

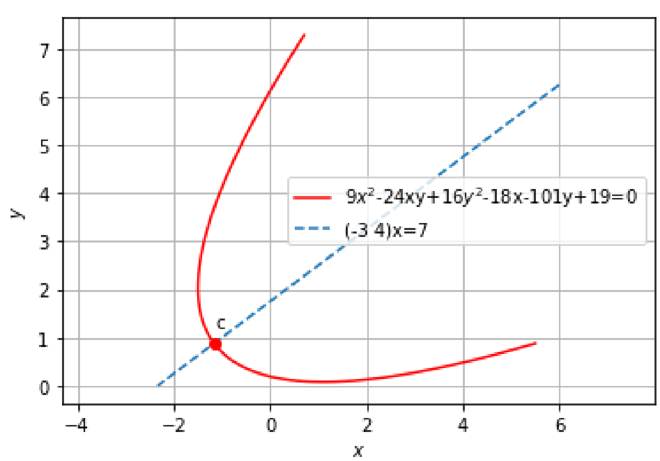

Fig. 3.1: $9x^2$-24xy+16y^2-18x-101y+19=0

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x + \frac{29}{25} \\ y - \frac{22}{25} \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 25x + 29 \\ 25y - 22 \end{pmatrix} \tag{3.0.7}$$

and,

$$\mathbf{V}(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} 9x - 12y + 21 \\ -12x + 16y - 28 \end{pmatrix} \tag{3.0.8}$$

Hence,the axis is given by

$$(\mathbf{e_2} - \mathbf{e_1})^T \left[\mathbf{V}(\mathbf{x} - \mathbf{c})\right] = 0 \tag{3.0.9}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 9x - 12y + 21 \\ -12x + 16y - 28 \end{pmatrix} = 0 \tag{3.0.10}$$

$$\implies -21x + 28y = 49 \tag{3.0.11}$$

$$\implies \boxed{\begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = 7} \tag{3.0.12}$$

2) Parabola

$$y^2 - 4x + 2y + 4 = 0 \tag{3.0.13}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.14}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{3.0.15}$$

$$f = 4 \tag{3.0.16}$$

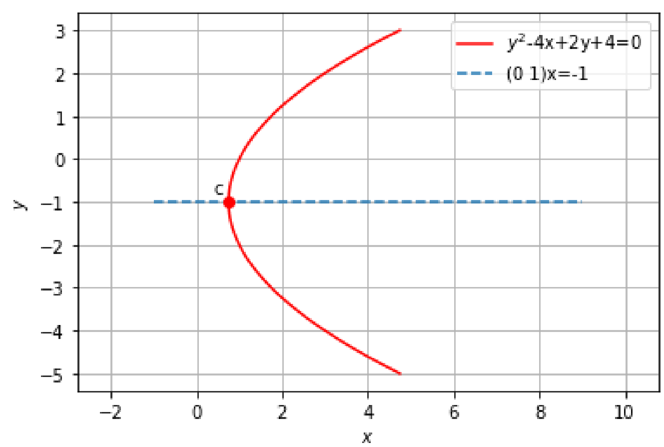

Fig. 3.2: y^2-4x+2y+4=0

Now,

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \tag{3.0.17}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \tag{3.0.18}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} \tag{3.0.19}$$

and,

$$\mathbf{V}(\mathbf{x}-\mathbf{c}) = \begin{pmatrix} 0 \\ y+1 \end{pmatrix} \tag{3.0.20}$$

Hence,the axis is given by

$$(\mathbf{e_2}-\mathbf{e_1})^T \ [\mathbf{V}(\mathbf{x}-\mathbf{c})] = 0 \tag{3.0.21}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ y+1 \end{pmatrix} = 0 \tag{3.0.22}$$

$$\implies y+1 = 0 \tag{3.0.23}$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = -1} \tag{3.0.24}$$

3) Parabola

$$y^2 = 8x \tag{3.0.25}$$

$$\implies y^2 - 8x = 0 \tag{3.0.26}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.27}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{3.0.28}$$

$$f = 0 \tag{3.0.29}$$

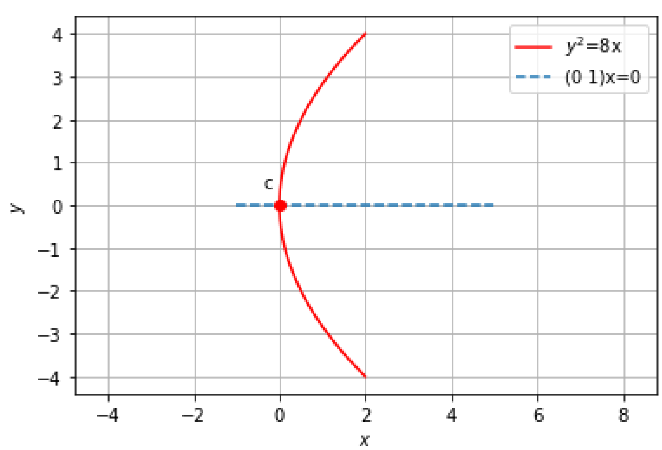

Fig. 3.3: y^2=8x

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.30}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.31}$$

So,

$$(\mathbf{x}-\mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.0.32}$$

and,

$$\mathbf{V}(\mathbf{x}-\mathbf{c}) = \begin{pmatrix} 0 \\ y \end{pmatrix} \tag{3.0.33}$$

Hence,the axis is given by

$$(\mathbf{e_2}-\mathbf{e_1})^T \ [\mathbf{V}(\mathbf{x}-\mathbf{c})] = 0 \tag{3.0.34}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = 0 \tag{3.0.35}$$

$$\implies y = 0 \tag{3.0.36}$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix}\mathbf{x} = 0} \tag{3.0.37}$$

4) Ellipse

$$x^2 + xy + y^2 = 100 \tag{3.0.38}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \tag{3.0.39}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.40}$$

$$f = -100 \tag{3.0.41}$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \tag{3.0.42}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.43}$$

So,

$$(\mathbf{x}-\mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.0.44}$$

and,

$$\mathbf{V}(\mathbf{x}-\mathbf{c}) = \begin{pmatrix} x+\frac{y}{2} \\ \frac{x}{2}+y \end{pmatrix} \tag{3.0.45}$$

Hence,the minor axis is given by

$$(\mathbf{e_2}-\mathbf{e_1})^T \ [\mathbf{V}(\mathbf{x}-\mathbf{c})] = 0 \tag{3.0.46}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} x+\frac{y}{2} \\ \frac{x}{2}+y \end{pmatrix} = 0 \tag{3.0.47}$$

$$\implies -x + y = 0 \tag{3.0.48}$$

$$\implies \boxed{\begin{pmatrix} -1 & 1 \end{pmatrix}\mathbf{x} = 0} \tag{3.0.49}$$

The normal vector $\mathbf{n}$ is given by

$$\left[(\mathbf{e_2} - \mathbf{e_1})^T \mathbf{V}\right] \mathbf{n} = 0 \tag{3.0.50}$$

$$\implies \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{n} = 0 \tag{3.0.51}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.0.52}$$

Hence,the major axis is given by

$$\mathbf{n}^T \mathbf{x} = (\mathbf{e_2} - \mathbf{e_1})^T (\mathbf{Vc}) \tag{3.0.53}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0} \tag{3.0.54}$$

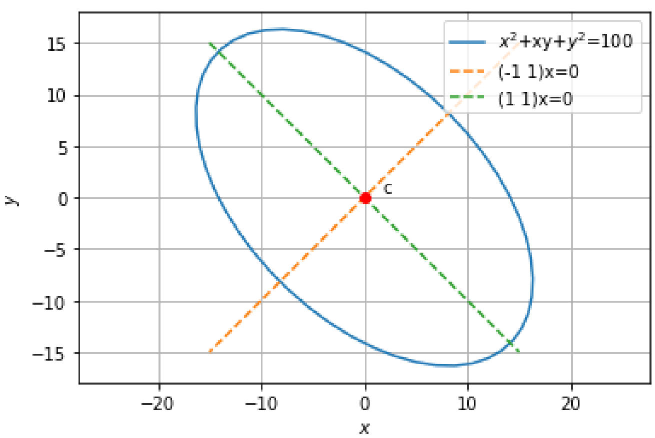

Fig. 3.4: x^2+xy+y^2=100

5) Hyperbola

$$xy - 3y + 2 = 0 \tag{3.0.55}$$

Here,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.0.56}$$

$$\mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.0.57}$$

$$f = 2 \tag{3.0.58}$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1} \mathbf{u} \tag{3.0.59}$$

$$\implies \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.0.60}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - 3 \\ y \end{pmatrix} \tag{3.0.61}$$

and,

$$\mathbf{V}(\mathbf{x} - \mathbf{c}) = \frac{1}{2} \begin{pmatrix} y \\ x - 3 \end{pmatrix} \tag{3.0.62}$$

Hence,the minor axis is given by

$$(\mathbf{e_2} - \mathbf{e_1})^T \left[\mathbf{V}(\mathbf{x} - \mathbf{c})\right] = 0 \tag{3.0.63}$$

$$\implies \begin{pmatrix} -1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} y \\ x - 3 \end{pmatrix} = 0 \tag{3.0.64}$$

$$\implies x - y = 3 \tag{3.0.65}$$

$$\implies \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3} \tag{3.0.66}$$

The normal vector $\mathbf{n}$ is given by

$$\left[(\mathbf{e_2} - \mathbf{e_1})^T \mathbf{V}\right] \mathbf{n} = 0 \tag{3.0.67}$$

$$\implies \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \mathbf{n} = 0 \tag{3.0.68}$$

$$\implies \mathbf{n} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \tag{3.0.69}$$

Hence,the major axis is given by

$$\mathbf{n}^T \mathbf{x} = (\mathbf{e_2} - \mathbf{e_1})^T (\mathbf{Vc}) \tag{3.0.70}$$

$$\implies \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} = \frac{3}{2} \tag{3.0.71}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3} \tag{3.0.72}$$

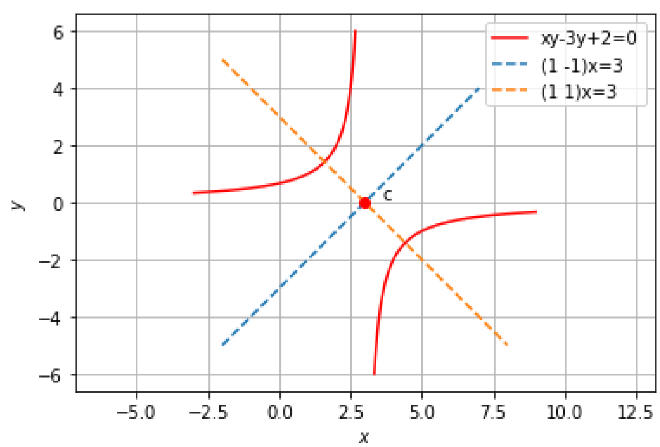

Fig. 3.5: xy-3y+2=0