

Challenge Problem 5

K.A. Raja Babu

Download all python codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem5/Codes>

and latex-tikz codes from

<https://github.com/ka-raja-babu/Matrix-Theory/tree/main/ChallengeProblem5>

1 CHALLENGE QUESTION 5

Express the axis of a parabola in terms of $\mathbf{V}, \mathbf{u}, f$ in general .

2 SOLUTION

Lemma 2.1. *Axis of any conic is given by*

$$(\mathbf{e}_2 \pm \mathbf{e}_1)^T \mathbf{V} (\mathbf{x} - \mathbf{c}) = 0 \quad (2.0.1)$$

Proof. The general equation of a conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.2)$$

which can be written in matrix form as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.3)$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \quad (2.0.5)$$

Axis of a conic can be written as :

$$\mathbf{A} \mathbf{x} = B \quad (2.0.6)$$

where, \mathbf{A} is the direction vector of the axis and B is a constant.

Axis of a conic must satisfies two conditions:

- 1) It must divide the conic into two symmetrical parts.
- 2) It must pass through the vertex.

By condition 1:

$$\mathbf{A} = (\mathbf{e}_2 \pm \mathbf{e}_1)^T \mathbf{V} \quad (2.0.7)$$

By condition 2:

$$B = (\mathbf{e}_2 \pm \mathbf{e}_1)^T \mathbf{V} \mathbf{c} \quad (2.0.8)$$

where, \mathbf{e}_1 and \mathbf{e}_2 are standard basis vector such that

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (2.0.9)$$

Hence, the axis of a parabola is given by

$$(\mathbf{e}_2 \pm \mathbf{e}_1)^T \mathbf{V} \mathbf{x} = (\mathbf{e}_2 \pm \mathbf{e}_1)^T \mathbf{V} \mathbf{c} \quad (2.0.10)$$

$$\implies (\mathbf{e}_2 \pm \mathbf{e}_1)^T \mathbf{V} (\mathbf{x} - \mathbf{c}) = 0 \quad (2.0.11)$$

□

3 EXAMPLES

1) Parabola

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \quad (3.0.1)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \quad (3.0.2)$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ -\frac{101}{2} \end{pmatrix} \quad (3.0.3)$$

$$f = 19 \quad (3.0.4)$$

Now,

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \quad (3.0.5)$$

$$\implies \mathbf{c} = \begin{pmatrix} -\frac{29}{25} \\ \frac{25}{25} \\ \frac{25}{25} \end{pmatrix} \quad (3.0.6)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x + \frac{29}{25} \\ y - \frac{25}{25} \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 25x + 29 \\ 25y - 25 \end{pmatrix} \quad (3.0.7)$$

and,

$$\mathbf{V} (\mathbf{x} - \mathbf{c}) = \begin{pmatrix} 9x - 12y + 21 \\ -12x + 16y - 28 \end{pmatrix} \quad (3.0.8)$$

Hence, the axis is given by

a) Approach 1

$$(\mathbf{e}_2 - \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.9)$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 9x - 12y + 21 \\ -12x + 16y - 28 \end{pmatrix} = 0 \quad (3.0.10)$$

$$\Rightarrow -21x + 28y = 49 \quad (3.0.11)$$

$$\Rightarrow \boxed{(-3 \ 4)\mathbf{x} = 7} \quad (3.0.12)$$

b) Approach 2

$$(\mathbf{e}_2 + \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.13)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 9x - 12y + 21 \\ -12x + 16y - 28 \end{pmatrix} = 0 \quad (3.0.14)$$

$$\Rightarrow -3x + 4y = 7 \quad (3.0.15)$$

$$\Rightarrow \boxed{(-3 \ 4)\mathbf{x} = 7} \quad (3.0.16)$$

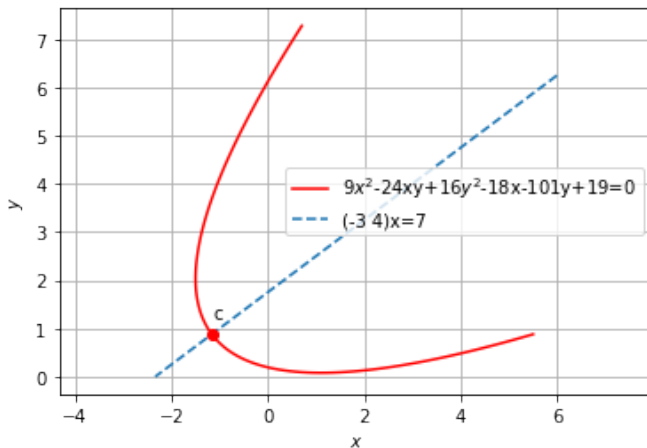


Fig. 3.1: $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$

2) Parabola

$$y^2 - 4x + 2y + 4 = 0 \quad (3.0.17)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.18)$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3.0.19)$$

$$f = 4 \quad (3.0.20)$$

Now,

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \quad (3.0.21)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \quad (3.0.22)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} \quad (3.0.23)$$

and,

$$\mathbf{V}(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} 0 \\ y + 1 \end{pmatrix} \quad (3.0.24)$$

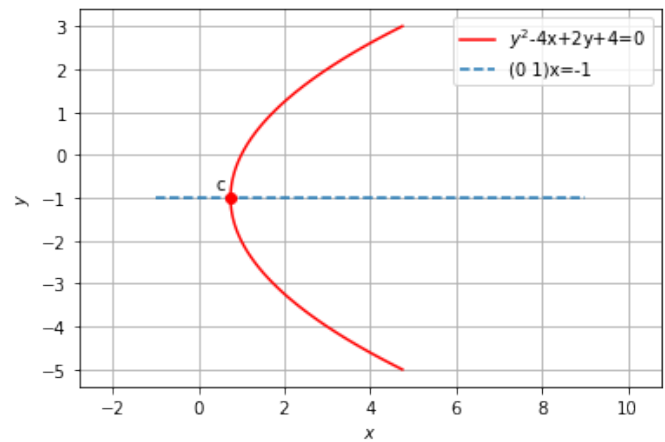


Fig. 3.2: $y^2 - 4x + 2y + 4 = 0$

Hence, the axis is given by

a) Approach 1

$$(\mathbf{e}_2 - \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.25)$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ y + 1 \end{pmatrix} = 0 \quad (3.0.26)$$

$$\Rightarrow y + 1 = 0 \quad (3.0.27)$$

$$\Rightarrow \boxed{(0 \ 1)\mathbf{x} = -1} \quad (3.0.28)$$

b) Approach 2

$$(\mathbf{e}_2 + \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.29)$$

$$\Rightarrow (1 \ 1) \begin{pmatrix} 0 \\ y+1 \end{pmatrix} = 0 \quad (3.0.30)$$

$$\Rightarrow y + 1 = 0 \quad (3.0.31)$$

$$\Rightarrow \boxed{(0 \ 1)\mathbf{x} = -1} \quad (3.0.32)$$

3) Parabola

$$y^2 = 8x \quad (3.0.33)$$

$$\Rightarrow y^2 - 8x = 0 \quad (3.0.34)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.0.35)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \quad (3.0.36)$$

$$f = 0 \quad (3.0.37)$$

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (3.0.38)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.39)$$

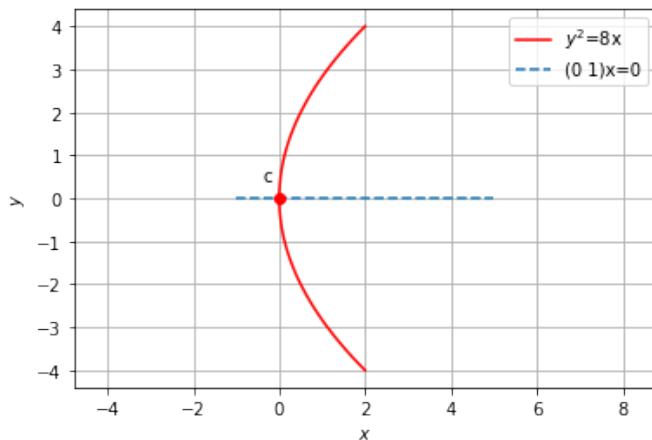


Fig. 3.3: $y^2=8x$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.0.40)$$

and,

$$\mathbf{V}(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} 0 \\ y \end{pmatrix} \quad (3.0.41)$$

Hence, the axis is given by

a) Approach 1

$$(\mathbf{e}_2 - \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.42)$$

$$\Rightarrow (-1 \ 1) \begin{pmatrix} 0 \\ y \end{pmatrix} = 0 \quad (3.0.43)$$

$$\Rightarrow y = 0 \quad (3.0.44)$$

$$\Rightarrow \boxed{(0 \ 1)\mathbf{x} = 0} \quad (3.0.45)$$

b) Approach 2

$$(\mathbf{e}_2 + \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.46)$$

$$\Rightarrow (1 \ 1) \begin{pmatrix} 0 \\ y \end{pmatrix} = 0 \quad (3.0.47)$$

$$\Rightarrow y = 0 \quad (3.0.48)$$

$$\Rightarrow \boxed{(0 \ 1)\mathbf{x} = 0} \quad (3.0.49)$$

4) Ellipse

$$x^2 + xy + y^2 = 100 \quad (3.0.50)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (3.0.51)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.52)$$

$$f = -100 \quad (3.0.53)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1} \mathbf{u} \quad (3.0.54)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.55)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.0.56)$$

and,

$$\mathbf{V}(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x + \frac{y}{2} \\ \frac{x}{2} + y \end{pmatrix} \quad (3.0.57)$$

Hence, the axes are given by

a) Minor Axis

$$(\mathbf{e}_2 - \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.58)$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} x + \frac{y}{2} \\ \frac{x}{2} + y \end{pmatrix} = 0 \quad (3.0.59)$$

$$\Rightarrow -x + y = 0 \quad (3.0.60)$$

$$\Rightarrow \boxed{\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 0} \quad (3.0.61)$$

b) Major Axis

$$(\mathbf{e}_2 + \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.62)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x + \frac{y}{2} \\ \frac{x}{2} + y \end{pmatrix} = 0 \quad (3.0.63)$$

$$\Rightarrow x + y = 0 \quad (3.0.64)$$

$$\Rightarrow \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0} \quad (3.0.65)$$

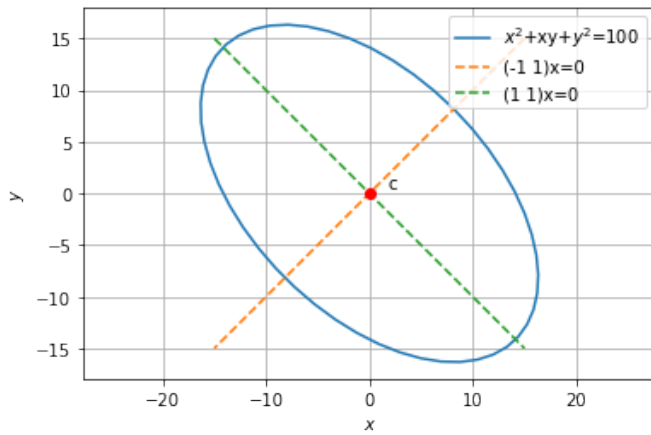


Fig. 3.4: $x^2 + xy + y^2 = 100$

5) Hyperbola

$$xy - 3y + 2 = 0 \quad (3.0.66)$$

Here,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.67)$$

$$\mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.68)$$

$$f = 2 \quad (3.0.69)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1} \mathbf{u} \quad (3.0.70)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.0.71)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - 3 \\ y \end{pmatrix} \quad (3.0.72)$$

and,

$$\mathbf{V}(\mathbf{x} - \mathbf{c}) = \frac{1}{2} \begin{pmatrix} y \\ x - 3 \end{pmatrix} \quad (3.0.73)$$

Hence, the axes are given by

a) Minor Axis

$$(\mathbf{e}_2 - \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.74)$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} y \\ x - 3 \end{pmatrix} = 0 \quad (3.0.75)$$

$$\Rightarrow x - y = 3 \quad (3.0.76)$$

$$\Rightarrow \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3} \quad (3.0.77)$$

b) Major Axis

$$(\mathbf{e}_2 + \mathbf{e}_1)^T \mathbf{V}(\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.78)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} y \\ x - 3 \end{pmatrix} = 0 \quad (3.0.79)$$

$$\Rightarrow x + y = 3 \quad (3.0.80)$$

$$\Rightarrow \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3} \quad (3.0.81)$$

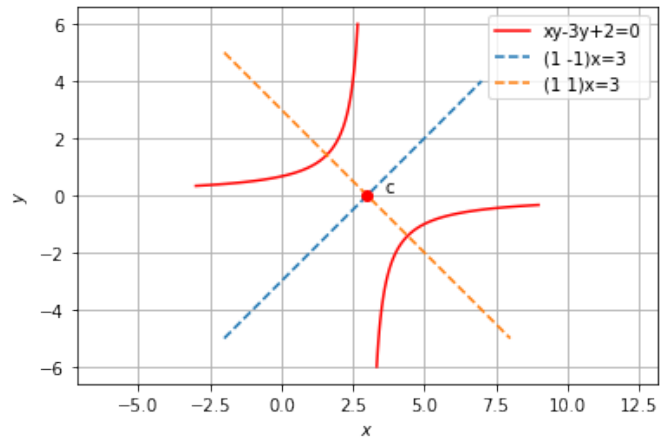


Fig. 3.5: $xy - 3y + 2 = 0$