Challenge Problem 5

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1 CHALLENGE QUESTION 5

Express the axis of a parabola in terms of $\mathbf{V}, \mathbf{u}, f$ in general .

2 SOLUTION

Lemma 2.1. *Axis of any conic is given by*

$$\boxed{\mathbf{p}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0} \qquad (2.0.1)$$

where, $\mathbf{c}$ *is the vertex of conic and* $\mathbf{p}$ *is the eigen vector of* $\mathbf{V}$ *having smaller eigen value.*

Proof. The general equation of a conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \qquad (2.0.2)$$

which can be written in matrix form as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \qquad (2.0.3)$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad (2.0.4)$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \qquad (2.0.5)$$

Let the eigen vector of $\mathbf{V}$ having smaller eigen value be

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \qquad (2.0.6)$$

$\therefore$ Normal vector of $\mathbf{p}$ is given by

$$\mathbf{p_n}^T\mathbf{p} = 0 \qquad (2.0.7)$$

$$\implies \mathbf{p_n}^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 0 \qquad (2.0.8)$$

$$\implies \mathbf{p_n} = \begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} \qquad (2.0.9)$$

$$\implies \mathbf{p_n}^T = \mathbf{p}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} \qquad (2.0.10)$$

where $\mathbf{e_1}$ and $\mathbf{e_2}$ are standard basis vector such that

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (2.0.11)$$

Now,according to the principal axis theorem,

1) Each eigen vector of $\mathbf{V}$ is parallel to either major axis or minor axis.
2) Normal vector of each eigen vector of $\mathbf{V}$ is normal to either major axis or minor axis.
3) Axes pass through the vertex $\mathbf{c}$ of the conic.

$\therefore$ Axis is given by

$$\mathbf{p_n}^T (\mathbf{x} - \mathbf{c}) = 0 \qquad (2.0.12)$$

$$\implies \boxed{\mathbf{p}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0} \qquad (2.0.13)$$

$\square$

3 EXAMPLES

1) Parabola

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \qquad (3.0.1)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \qquad (3.0.2)$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ \frac{-101}{2} \end{pmatrix} \qquad (3.0.3)$$

$$f = 19 \qquad (3.0.4)$$

Now,

$$\begin{pmatrix} -39 & -73 \\ 9 & -12 \\ -12 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -19 \\ -21 \\ 28 \end{pmatrix} \tag{3.0.5}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-29}{25} \\ \frac{22}{25} \end{pmatrix} \tag{3.0.6}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x + \frac{29}{25} \\ y - \frac{22}{25} \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 25x + 29 \\ 25y - 22 \end{pmatrix} \tag{3.0.7}$$

Now,

$$\left| \mathbf{V} - \lambda \mathbf{I} \right| = 0 \tag{3.0.8}$$

$$\implies \begin{vmatrix} 9 - \lambda & -12 \\ -12 & 16 - \lambda \end{vmatrix} = 0 \tag{3.0.9}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{3.0.10}$$

$$\implies \lambda_1 = 0, \lambda_2 = 25 \tag{3.0.11}$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{3.0.12}$$

$$\implies \mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 1 & \frac{-4}{3} \\ 0 & 0 \end{pmatrix} \tag{3.0.13}$$

$$\implies \mathbf{p_1} = \frac{3}{5} \begin{pmatrix} \frac{4}{3} \\ 1 \end{pmatrix} \tag{3.0.14}$$

Similarly for $\lambda_2 = 25$,

$$\mathbf{p_2} = \frac{4}{5} \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix} \tag{3.0.15}$$

Fig. 3.1: $9x^2$-24xy+$16y^2$-18x-101y+19=0

$\because \lambda_1 < \lambda_2$

Hence, the axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \tag{3.0.16}$$

$$\implies \frac{3}{5} \begin{pmatrix} \frac{4}{3} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{1}{25} \begin{pmatrix} 25x + 29 \\ 25y - 22 \end{pmatrix} = 0 \tag{3.0.17}$$

$$\implies -3x + 4y = 7 \tag{3.0.18}$$

$$\implies \boxed{\begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = 7} \tag{3.0.19}$$

2) Parabola

$$y^2 - 4x + 2y + 4 = 0 \tag{3.0.20}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.21}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{3.0.22}$$

$$f = 4 \tag{3.0.23}$$

Now,

$$\begin{pmatrix} -4 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} \tag{3.0.24}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{3}{4} \\ -1 \end{pmatrix} \tag{3.0.25}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} \tag{3.0.26}$$

Now,

$$\left| \mathbf{V} - \lambda \mathbf{I} \right| = 0 \tag{3.0.27}$$

$$\implies \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \tag{3.0.28}$$

$$\implies \lambda_1 = 0, \lambda_2 = 1 \tag{3.0.29}$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.30}$$

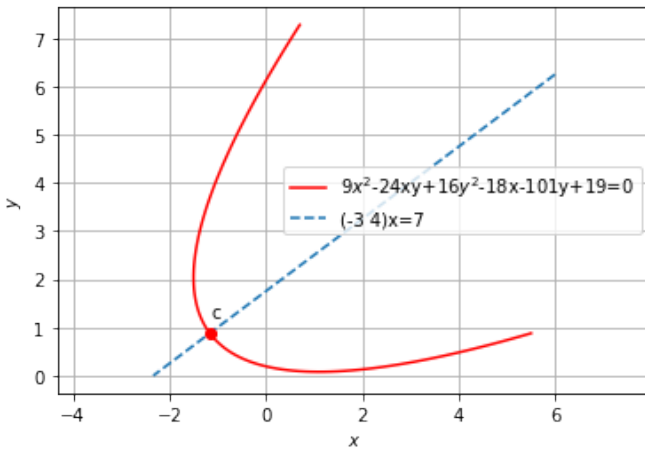$$\implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.31}$$

Similarly for $\lambda_2 = 1$,

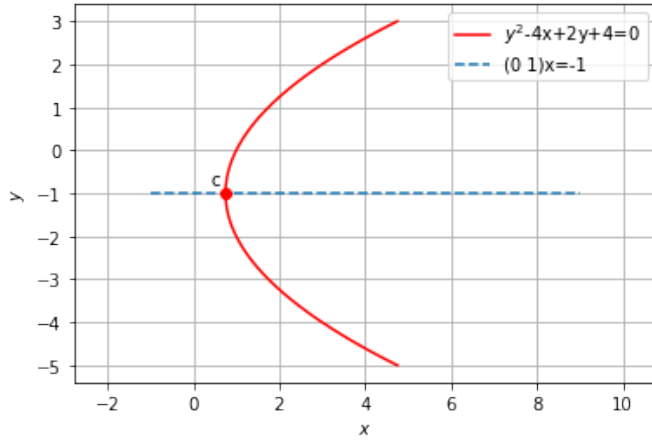$$\mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (3.0.32)$$

Fig. 3.2: y^2-4x+2y+4=0

$\because \lambda_1 < \lambda_2$

Hence,the axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \qquad (3.0.33)$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} = 0 \qquad (3.0.34)$$

$$\implies y + 1 = 0 \qquad (3.0.35)$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -1} \qquad (3.0.36)$$

3) Parabola

$$y^2 = 8x \qquad (3.0.37)$$

$$\implies y^2 - 8x = 0 \qquad (3.0.38)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (3.0.39)$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \qquad (3.0.40)$$

$$f = 0 \qquad (3.0.41)$$

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (3.0.42)$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3.0.43)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \qquad (3.0.44)$$

Now,

$$\left| \mathbf{V} - \lambda \mathbf{I} \right| = 0 \qquad (3.0.45)$$

$$\implies \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \qquad (3.0.46)$$

$$\implies \lambda_1 = 0, \lambda_2 = 1 \qquad (3.0.47)$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (3.0.48)$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (3.0.49)$$

Similarly for $\lambda_2 = 1$,

$$\mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (3.0.50)$$

$\because \lambda_1 < \lambda_2$

Hence,the axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \qquad (3.0.51)$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad (3.0.52)$$

$$\implies y = 0 \qquad (3.0.53)$$

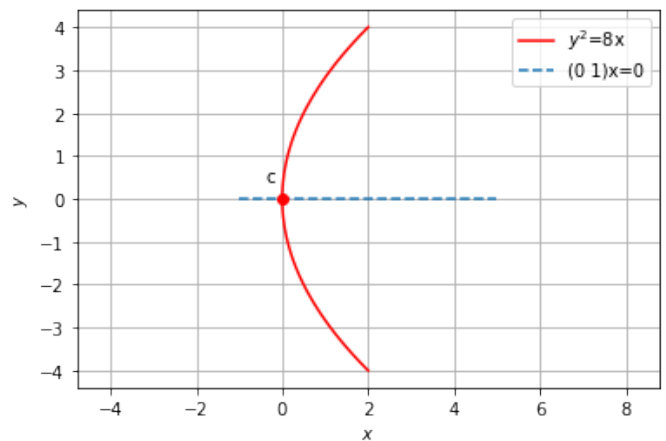$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0} \qquad (3.0.54)$$

Fig. 3.3: y^2=8x

4) Ellipse

$$x^2 + xy + y^2 = 100 \qquad (3.0.55)$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \quad (3.0.56)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.57)$$

$$f = -100 \quad (3.0.58)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \quad (3.0.59)$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.0.60)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \quad (3.0.61)$$

Now,

$$|\mathbf{V} - \lambda\mathbf{I}| = 0 \quad (3.0.62)$$

$$\implies \begin{vmatrix} 1 - \lambda & \frac{1}{2} \\ \frac{1}{2} & 1 - \lambda \end{vmatrix} = 0 \quad (3.0.63)$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \quad (3.0.64)$$

$$\implies \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \quad (3.0.65)$$

For $\lambda_1 = \frac{1}{2}$,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad (3.0.66)$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad (3.0.67)$$

Similarly for $\lambda_2 = \frac{3}{2}$,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (3.0.68)$$

$\because \lambda_1 < \lambda_2$

Hence, the major axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.69)$$

$$\implies \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad (3.0.70)$$

$$\implies x + y = 0 \quad (3.0.71)$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0} \quad (3.0.72)$$

and the minor axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.73)$$

$$\implies \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \quad (3.0.74)$$

$$\implies -x + y = 0 \quad (3.0.75)$$

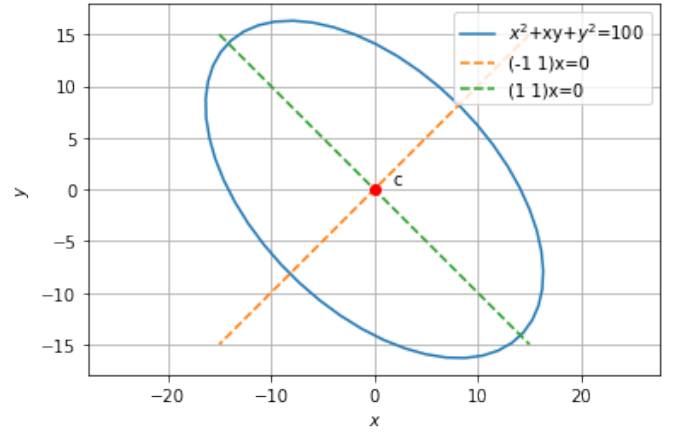$$\implies \boxed{\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 0} \quad (3.0.76)$$

Fig. 3.4: $x^2 + xy + y^2 = 100$

5) Hyperbola

$$xy - 3y + 2 = 0 \quad (3.0.77)$$

Here,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.0.78)$$

$$\mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (3.0.79)$$

$$f = 2 \quad (3.0.80)$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \quad (3.0.81)$$

$$\implies \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad (3.0.82)$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - 3 \\ y \end{pmatrix} \quad (3.0.83)$$

Now,

$$\left|\mathbf{V} - \lambda \mathbf{I}\right| = 0 \qquad (3.0.84)$$

$$\implies \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \qquad (3.0.85)$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \qquad (3.0.86)$$

$$\implies \lambda_1 = \frac{-1}{2}, \lambda_2 = \frac{1}{2} \qquad (3.0.87)$$

For $\lambda_1 = \frac{-1}{2}$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad (3.0.88)$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad (3.0.89)$$

Similarly for $\lambda_2 = \frac{1}{2}$,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (3.0.90)$$

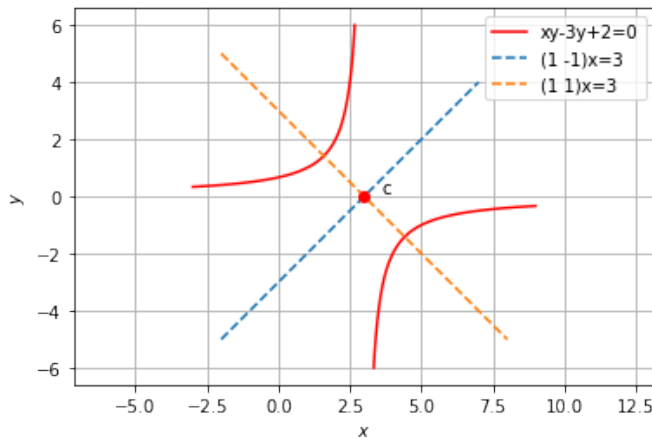Fig. 3.5: xy-3y+2=0

$\because \lambda_1 < \lambda_2$

Hence, the major axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0$$

$$(3.0.91)$$

$$\implies \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x - 3 \\ y \end{pmatrix} = 0$$

$$(3.0.92)$$

$$\implies x + y = 3$$

$$(3.0.93)$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 3} \qquad (3.0.94)$$

and the minor axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \quad (3.0.95)$$

$$\implies \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x - 3 \\ y \end{pmatrix} = 0 \quad (3.0.96)$$

$$\implies x - y = 3 \quad (3.0.97)$$

$$\implies \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = 3} \qquad (3.0.98)$$