Challenge Problem 5

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Download all python codes from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem5/Codes

and latex-tikz codes from

https://github.com/ka−raja−babu/Matrix−Theory/tree/main/ChallengeProblem5

1 Challenge Question 5

Express the axis of a parabola in terms of $\mathbf{V}, \mathbf{u}, f$ in general .

2 Solution

Lemma 2.1. *Axis of any conic is given by*

$$\boxed{\mathbf{p}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0} \tag{2.0.1}$$

where,$\mathbf{c}$ *is the vertex of conic and* $\mathbf{p}$ *is the eigen vector of* $\mathbf{V}$ *having smaller eigen value.*

Proof. The general equation of a conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.2}$$

which can be written in matrix form as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

where,

$$\mathbf{V} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u} = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.5}$$

Let the eigen vector of $\mathbf{V}$ having smaller eigen value be

$$\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{2.0.6}$$

Now,normal vector of $\mathbf{p}$ having same magnitude,is given by

$$\mathbf{p_n}^T\mathbf{p} = 0 \tag{2.0.7}$$

$$\implies \mathbf{p_n}^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = 0 \tag{2.0.8}$$

$$\implies \mathbf{p_n} = \begin{pmatrix} -p_2 \\ p_1 \end{pmatrix} \tag{2.0.9}$$

$$\implies \mathbf{p_n}^T = \mathbf{p}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} \tag{2.0.10}$$

where $\mathbf{e_1}$ and $\mathbf{e_2}$ are standard basis vector such that

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.0.11}$$

According to the principal axis theorem,

1) Each eigen vector of $\mathbf{V}$ is parallel to either major axis or minor axis.
2) Normal vector of each eigen vector of $\mathbf{V}$ is normal to either major axis or minor axis.
3) Axes pass through the vertex $\mathbf{c}$ of the conic.

$\therefore$ Axis is given by

$$\mathbf{p_n}^T (\mathbf{x} - \mathbf{c}) = 0 \tag{2.0.12}$$

$$\implies \boxed{\mathbf{p}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0} \tag{2.0.13}$$

□

3 Examples

1) Parabola

$$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0 \tag{3.0.1}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 9 & -12 \\ -12 & 16 \end{pmatrix} \tag{3.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -9 \\ \frac{-101}{2} \end{pmatrix} \tag{3.0.3}$$

$$f = 19 \tag{3.0.4}$$

Now,

$$\begin{pmatrix}-39 & -73\\ 9 & -12\\ -12 & 16\end{pmatrix}\mathbf{c} = \begin{pmatrix}-19\\ -21\\ 28\end{pmatrix} \tag{3.0.5}$$

$$\implies \mathbf{c} = \begin{pmatrix}\frac{-29}{25}\\ \frac{22}{25}\end{pmatrix} \tag{3.0.6}$$

So,

$$(\mathbf{x}-\mathbf{c}) = \begin{pmatrix}x+\frac{29}{25}\\ y-\frac{22}{25}\end{pmatrix} = \frac{1}{25}\begin{pmatrix}25x+29\\ 25y-22\end{pmatrix} \tag{3.0.7}$$

Now,

$$\left|\mathbf{V}-\lambda\mathbf{I}\right| = 0 \tag{3.0.8}$$

$$\implies \begin{vmatrix}9-\lambda & -12\\ -12 & 16-\lambda\end{vmatrix} = 0 \tag{3.0.9}$$

$$\implies \lambda^2 - 25\lambda = 0 \tag{3.0.10}$$

$$\implies \lambda_1 = 0, \lambda_2 = 25 \tag{3.0.11}$$

For $\lambda_1 = 0$,

$$\mathbf{V}-\lambda_1\mathbf{I} = \begin{pmatrix}9 & -12\\ -12 & 16\end{pmatrix} \tag{3.0.12}$$

$$\implies \mathbf{V}-\lambda_1\mathbf{I} = \begin{pmatrix}1 & \frac{-4}{3}\\ 0 & 0\end{pmatrix} \tag{3.0.13}$$

$$\implies \mathbf{p_1} = \frac{3}{5}\begin{pmatrix}\frac{4}{3}\\ 1\end{pmatrix} \tag{3.0.14}$$

Similarly for $\lambda_2 = 25$,

$$\mathbf{p_2} = \frac{4}{5}\begin{pmatrix}\frac{-3}{4}\\ 1\end{pmatrix} \tag{3.0.15}$$

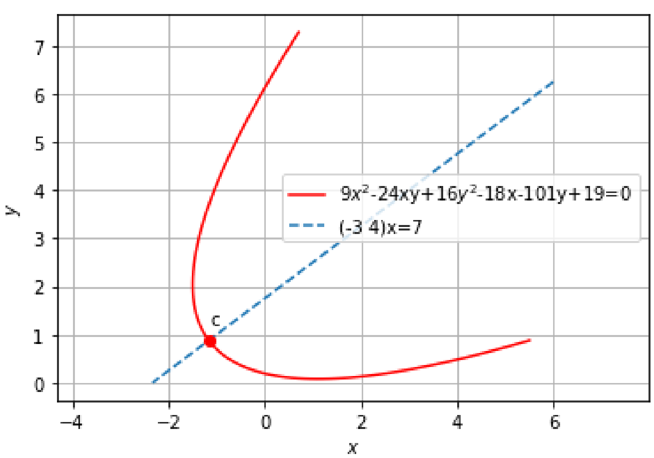

Fig. 3.1: 9x^2-24xy+16y^2-18x-101y+19=0

$\because \lambda_1 < \lambda_2$

Hence,the axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T\begin{pmatrix}-\mathbf{e_2} & \mathbf{e_1}\end{pmatrix}(\mathbf{x}-\mathbf{c}) = 0 \tag{3.0.16}$$

$$\implies \frac{3}{5}\begin{pmatrix}\frac{4}{3} & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\frac{1}{25}\begin{pmatrix}25x+29\\ 25y-22\end{pmatrix} = 0 \tag{3.0.17}$$

$$\implies -3x+4y = 7 \tag{3.0.18}$$

$$\implies \boxed{\begin{pmatrix}-3 & 4\end{pmatrix}\mathbf{x} = 7} \tag{3.0.19}$$

2) Parabola

$$y^2 - 4x + 2y + 4 = 0 \tag{3.0.20}$$

Here,

$$\mathbf{V} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{3.0.21}$$

$$\mathbf{u} = \begin{pmatrix}-2\\ 1\end{pmatrix} \tag{3.0.22}$$

$$f = 4 \tag{3.0.23}$$

Now,

$$\begin{pmatrix}-4 & 1\\ 0 & 0\\ 0 & 1\end{pmatrix}\mathbf{c} = \begin{pmatrix}-4\\ 0\\ -1\end{pmatrix} \tag{3.0.24}$$

$$\implies \mathbf{c} = \begin{pmatrix}\frac{3}{4}\\ -1\end{pmatrix} \tag{3.0.25}$$

So,

$$(\mathbf{x}-\mathbf{c}) = \begin{pmatrix}x-\frac{3}{4}\\ y+1\end{pmatrix} \tag{3.0.26}$$

Now,

$$\left|\mathbf{V}-\lambda\mathbf{I}\right| = 0 \tag{3.0.27}$$

$$\implies \begin{vmatrix}-\lambda & 0\\ 0 & 1-\lambda\end{vmatrix} = 0 \tag{3.0.28}$$

$$\implies \lambda_1 = 0, \lambda_2 = 1 \tag{3.0.29}$$

For $\lambda_1 = 0$,

$$\mathbf{V}-\lambda_1\mathbf{I} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \tag{3.0.30}$$

$$\implies \mathbf{p_1} = \begin{pmatrix}1\\ 0\end{pmatrix} \tag{3.0.31}$$

Similarly for $\lambda_2 = 1$,

$$\mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.0.32}$$

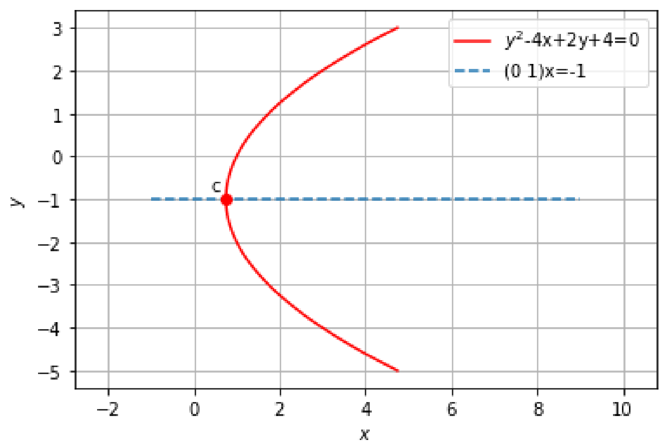

Fig. 3.2: y^2-4x+2y+4=0

$\because \lambda_1 < \lambda_2$
Hence,the axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \tag{3.0.33}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x - \frac{3}{4} \\ y + 1 \end{pmatrix} = 0 \tag{3.0.34}$$

$$\implies y + 1 = 0 \tag{3.0.35}$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = -1} \tag{3.0.36}$$

3) Parabola

$$y^2 = 8x \tag{3.0.37}$$

$$\implies y^2 - 8x = 0 \tag{3.0.38}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.39}$$

$$\mathbf{u} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} \tag{3.0.40}$$

$$f = 0 \tag{3.0.41}$$

Now,

$$\begin{pmatrix} -8 & 1 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.42}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.43}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.0.44}$$

Now,

$$\left| \mathbf{V} - \lambda \mathbf{I} \right| = 0 \tag{3.0.45}$$

$$\implies \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = 0 \tag{3.0.46}$$

$$\implies \lambda_1 = 0, \lambda_2 = 1 \tag{3.0.47}$$

For $\lambda_1 = 0$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.48}$$

$$\implies \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.0.49}$$

Similarly for $\lambda_2 = 1$,

$$\mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.0.50}$$

$\because \lambda_1 < \lambda_2$
Hence,the axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \tag{3.0.51}$$

$$\implies \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{3.0.52}$$

$$\implies y = 0 \tag{3.0.53}$$

$$\implies \boxed{\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0} \tag{3.0.54}$$

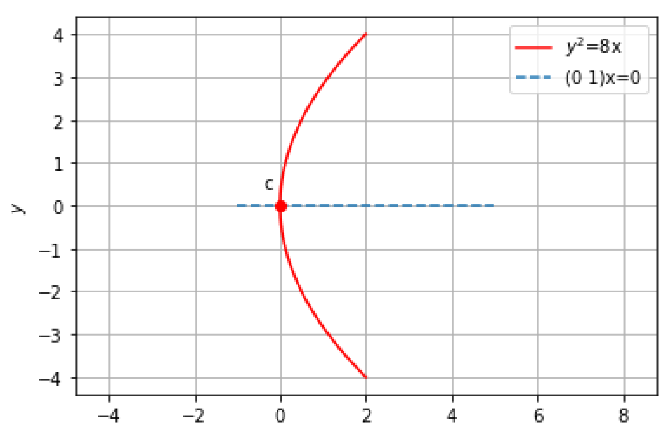

Fig. 3.3: y^2=8x

4) Ellipse

$$x^2 + xy + y^2 = 100 \tag{3.0.55}$$

Here,

$$\mathbf{V} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \tag{3.0.56}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.57}$$

$$f = -100 \tag{3.0.58}$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \tag{3.0.59}$$

$$\implies \mathbf{c} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.0.60}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.0.61}$$

Now,

$$\left|\mathbf{V} - \lambda \mathbf{I}\right| = 0 \tag{3.0.62}$$

$$\implies \begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda \end{vmatrix} = 0 \tag{3.0.63}$$

$$\implies \lambda^2 - 2\lambda + \frac{3}{4} = 0 \tag{3.0.64}$$

$$\implies \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{2} \tag{3.0.65}$$

For $\lambda_1 = \frac{1}{2}$,

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.0.66}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.0.67}$$

Similarly for $\lambda_2 = \frac{3}{2}$,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.0.68}$$

$\because \lambda_1 < \lambda_2$

Hence,the major axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \tag{3.0.69}$$

$$\implies \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{3.0.70}$$

$$\implies x + y = 0 \tag{3.0.71}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 0} \tag{3.0.72}$$

and the minor axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \tag{3.0.73}$$

$$\implies \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \tag{3.0.74}$$

$$\implies -x + y = 0 \tag{3.0.75}$$

$$\implies \boxed{\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 0} \tag{3.0.76}$$

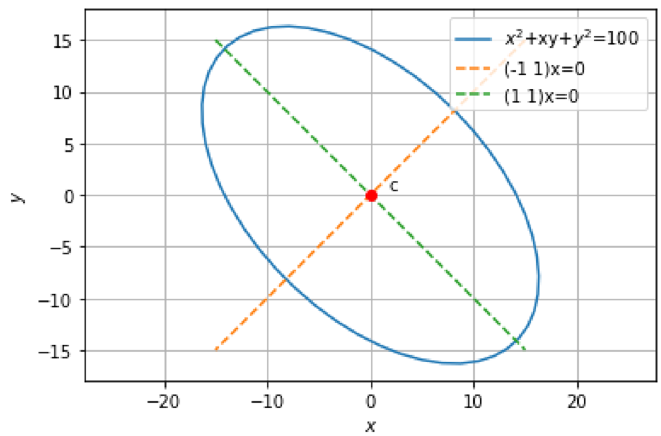

Fig. 3.4: x^2+xy+y^2=100

5) Hyperbola

$$xy - 3y + 2 = 0 \tag{3.0.77}$$

Here,

$$\mathbf{V} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.0.78}$$

$$\mathbf{u} = \frac{-3}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.0.79}$$

$$f = 2 \tag{3.0.80}$$

Now,

$$\mathbf{c} = \mathbf{V}^{-1}\mathbf{u} \tag{3.0.81}$$

$$\implies \mathbf{c} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{3.0.82}$$

So,

$$(\mathbf{x} - \mathbf{c}) = \begin{pmatrix} x - 3 \\ y \end{pmatrix} \tag{3.0.83}$$

Now,

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = 0 \tag{3.0.84}$$

$$\implies \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = 0 \tag{3.0.85}$$

$$\implies \lambda^2 - \frac{1}{4} = 0 \tag{3.0.86}$$

$$\implies \lambda_1 = \frac{-1}{2}, \lambda_2 = \frac{1}{2} \tag{3.0.87}$$

For $\lambda_1 = \frac{-1}{2}$,

$$\mathbf{V} - \lambda_1\mathbf{I} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{3.0.88}$$

$$\implies \mathbf{p_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{3.0.89}$$

Similarly for $\lambda_2 = \frac{1}{2}$,

$$\mathbf{p_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.0.90}$$

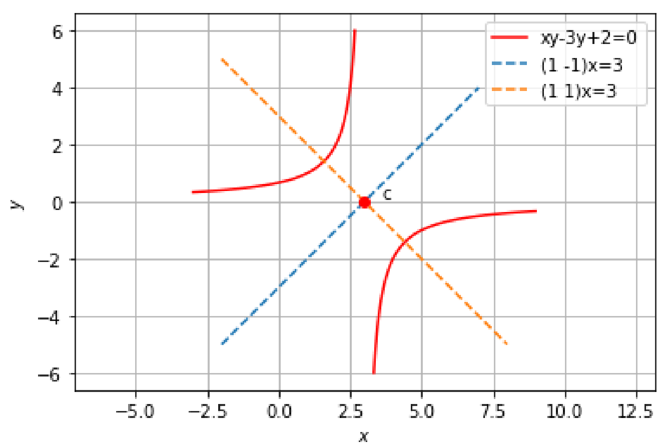

Fig. 3.5: xy-3y+2=0

$\because \lambda_1 < \lambda_2$

Hence,the major axis using $\mathbf{p_1}$ is given by

$$\mathbf{p_1}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \tag{3.0.91}$$

$$\implies \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x-3 \\ y \end{pmatrix} = 0 \tag{3.0.92}$$

$$\implies x + y = 3 \tag{3.0.93}$$

$$\implies \boxed{\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3} \tag{3.0.94}$$

and the minor axis using $\mathbf{p_2}$ is given by

$$\mathbf{p_2}^T \begin{pmatrix} -\mathbf{e_2} & \mathbf{e_1} \end{pmatrix} (\mathbf{x} - \mathbf{c}) = 0 \tag{3.0.95}$$

$$\implies \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x-3 \\ y \end{pmatrix} = 0 \tag{3.0.96}$$

$$\implies x - y = 3 \tag{3.0.97}$$

$$\implies \boxed{\begin{pmatrix} 1 & -1 \end{pmatrix}\mathbf{x} = 3} \tag{3.0.98}$$