UNIT– I

PARTIAL DIFFERENTIAL EQUATIONS

This unit covers topics that explain the formation of partial differential equations and the solutions of special types of partial differential equations.

1.1 INTRODUCTION

A partial differential equation is one which involves one or more partial derivatives. The order of the highest derivative is called the order of the equation. A partial differential equation contains more than one independent variable. But, here we shall consider partial differential equations involving one dependent variable 'z' and only two independent variables x and y so that z = f(x,y). We shall denote

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t.$$

A partial differential equation is linear if it is of the first degree in the dependent variable and its partial derivatives. If each term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be homogeneous, otherwise it is non homogeneous.

1.2 Formation of Partial Differential Equations

Partial differential equations can be obtained by the elimination of arbitrary constants or by the elimination of arbitrary functions.

By the elimination of arbitrary constants

Let us consider the function

$$\phi(x, y, z, a, b) = 0 \quad \text{-------------(1)}$$

where a & b are arbitrary constants

Differentiating equation (1) partially w.r.t x & y, we get

$$\frac{\partial \phi}{\partial x} + p\frac{\partial \phi}{\partial z} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (2)}$$

$$\frac{\partial \phi}{\partial y} + q\frac{\partial \phi}{\partial z} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Eliminating a and b from equations (1), (2) and (3), we get a partial differential equation of the first order of the form f (x,y,z, p, q) = 0

Example 1

Eliminate the arbitrary constants a & b from z = ax + by + ab

Consider $z = ax + by + ab$ ______________ (1)

Differentiating (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = a \qquad \text{i.e,} \quad p = a \qquad \text{\_\_\_\_\_\_\_\_\_\_} \quad (2)$$

$$\frac{\partial z}{\partial y} = b \qquad \text{i.e,} \quad q = b \qquad \text{\_\_\_\_\_\_\_\_} \quad (3)$$

Using (2) & (3) in (1), we get

$$z = px + qy + pq$$

which is the required partial differential equation.

Example 2

Form the partial differential equation by eliminating the arbitrary constants a and b from
$z = (x^2 + a^2)(y^2 + b^2)$

Given $z = (x^2 + a^2)(y^2 + b^2)$ ____________ (1)

Differentiating (1) partially w.r.t x & y , we get

$$p = 2x\,(y^2 + b^2)$$

$$q = 2y\,(x + a)$$

Substituting the values of p and q in (1), we get

$$4xyz = pq$$

which is the required partial differential equation.

Example 3

Find the partial differential equation of the family of spheres of radius one whose centre lie in the xy - plane.

The equation of the sphere is given by

$$(x - a)^2 + (y - b)^2 + z^2 = 1 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

Differentiating (1) partially w.r.t x & y , we get

$$2(x-a) + 2zp = 0$$
$$2(y-b) + 2zq = 0$$

From these equations we obtain

$$x-a = -zp \quad \text{\_\_\_\_\_\_\_\_\_\_} \quad (2)$$
$$y-b = -zq \quad \text{\_\_\_\_\_\_\_\_\_\_} \quad (3)$$

Using (2) and (3) in (1), we get

$$z^2p^2 + z^2q^2 + z^2 = 1$$
$$\text{or} \quad z^2(p^2 + q^2 + 1) = 1$$

Example 4

Eliminate the arbitrary constants a, b & c from

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ and form the partial differential equation.

The given equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

Differentiating (1) partially w.r.t x & y, we get

$$\frac{2x}{a^2} + \frac{2zp}{c^2} = 0$$

$$\frac{2y}{b^2} + \frac{2zq}{c^2} = 0$$

Therefore we get

$$\frac{x}{a^2} + \frac{zp}{c^2} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (2)$$

$$\frac{y}{b^2} + \frac{zq}{c^2} = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (3)$$

Again differentiating (2) partially w.r.t 'x', we set

$(1/a^2) + (1/c^2)(zr + p^2) = 0$ _______________ (4)

Multiplying (4) by x, we get

$$\frac{x}{a^2} + \frac{xz\,r}{c^2} + \frac{p^2x}{c^2} = 0$$

From (2) , we have

$$\frac{-zp}{c^2} + \frac{xzr}{c^2} + \frac{p^2x}{c^2} = 0$$

or $-zp + xzr + p^2x = 0$

By the elimination of arbitrary functions

Let u and v be any two functions of x, y, z and $\Phi(u, v) = 0$, where Φ is an arbitrary function. This relation can be expressed as

$u = f(v)$ _______________ (1)

Differentiating (1) partially w.r.t x & y and eliminating the arbitrary functions from these relations, we get a partial differential equation of the first order of the form
$f(x, y, z, p, q) = 0.$

Example 5
Obtain the partial differential equation by eliminating 'f ' from $z = (x+y) f(x^2 - y^2)$

Let us now consider the equation

$$z = (x+y)\ f(x^2 - y^2) \quad \_\_\_\_\_\_\_\_ \quad (1)$$

Differentiating (1) partially w.r.t x & y , we get

$$p = (x+y) f'(x^2 - y^2) \cdot 2x + f(x^2 - y^2)$$
$$q = (x+y) f'(x^2 - y^2) \cdot (-2y) + f(x^2 - y^2)$$

These equations can be written as

$$p - f(x^2 - y^2) = (x+y) f'(x^2 - y^2) \cdot 2x \quad \_\_\_\_\_\_\_\_ \quad (2)$$
$$q - f(x^2 - y^2) = (x+y) f'(x^2 - y^2) \cdot (-2y) \quad \_\_\_\_\_\_\_\_ \quad (3)$$

Hence, we get

$$\frac{p - f(x^2 - y^2)}{q - f(x^2 - y^2)} = -\frac{x}{y}$$

i.e, $py - yf(x^2 - y^2) = -qx + xf(x^2 - y^2)$

i.e, $py + qx = (x+y)\ f(x^2 - y^2)$

Therefore, we have by(1), $py + qx = z$

Example 6

Form the partial differential equation by eliminating the arbitrary function f from
$z = e^y f(x + y)$

Consider $z = e^y f(x+y) \quad \_\_\_\_\_\_\_\_ \quad (1)$

Differentiating (1) partially w .r. t x & y, we get

$$p = e^y f'(x + y)$$
$$q = e^y f'(x + y) + f(x + y) \cdot e^y$$

Hence, we have

$$q = p + z$$

Example 7

Form the PDE by eliminating f & Φ from $z = f(x+ay) + \Phi(x-ay)$

Consider $z = f(x+ay) + \Phi(x-ay)$ ________________ (1)

Differentiating (1) partially w.r.t x &y , we get

$p = f'(x+ay) + \Phi'(x-ay)$ ____________________ (2)

$q = f'(x+ay).a + \Phi'(x-ay)(-a)$ ________________ (3)

Differentiating (2) & (3) again partially w.r.t x & y, we get

$r = f''(x+ay) + \Phi''(x-ay)$
$t = f''(x+ay).a^2 + \Phi''(x-ay)(-a)^2$

i.e, $t = a^2\{f''(x+ay) + \Phi''(x-ay)\}$

or $t = a^2 r$

Exercises:

1. Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from the following equations.
   - (i) $z = ax + by$
   - (ii) $\dfrac{x^2+y^2}{a^2} + \dfrac{z^2}{b^2} = 1$
   - (iii) $z = ax + by + \sqrt{a^2+b^2}$
   - (iv) $ax^2 + by^2 + cz^2 = 1$
   - (v) $z = a^2x + b^2y + ab$

2. Find the PDE of the family of spheres of radius 1 having their centres lie on the xy plane{Hint: $(x-a)^2 + (y-b)^2 + z^2 = 1$}

3. Find the PDE of all spheres whose centre lie on the (i) z axis (ii) x-axis

4. Form the partial differential equations by eliminating the arbitrary functions in the following cases.
   - (i) $z = f(x+y)$
   - (ii) $z = f(x^2-y^2)$
   - (iii) $z = f(x^2+y^2+z^2)$
   - (iv) $\phi(xyz, x+y+z) = 0$

(v) $z = x + y + f(xy)$

(vi) $z = xy + f(x^2 + y^2)$

(vii) $z = f\left(\frac{xy}{z}\right)$

(viii) $F(xy + z^2, x + y + z) = 0$

(ix) $z = f(x + iy) + f(x - iy)$

(x) $z = f(x^3 + 2y) + g(x^3 - 2y)$

1.3 **SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION**

A solution or integral of a partial differential equation is a relation connecting the dependent and the independent variables which satisfies the given differential equation. A partial differential equation can result both from elimination of arbitrary constants and from elimination of arbitrary functions as explained in section 1.2. But, there is a basic difference in the two forms of solutions. A solution containing as many arbitrary constants as there are independent variables is called a complete integral. Here, the partial differential equations contain only two independent variables so that the complete integral will include two constants.A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

Singular Integral

Let $f(x,y,z,p,q) = 0$ ---------- (1)

be the partial differential equation whose complete integral is

$\phi(x,y,z,a,b) = 0$ ----------- (2)

where ‘a’ and ‘b’ are arbitrary constants.

Differentiating (2) partially w.r.t. a and b, we obtain

$$\frac{\partial\phi}{\partial a} = 0 \qquad \text{----------- (3)}$$

and

$$\frac{\partial\phi}{\partial b} = 0 \qquad \text{----------- (4)}$$

The eliminant of ‘a’ and ‘b’ from the equations (2), (3) and (4), when it exists, is called the singular integral of (1).

General Integral

In the complete integral (2), put $b = F(a)$, we get

$$\phi(x,y,z,a,F(a)) = 0 \qquad \text{---------- (5)}$$

Differentiating (2), partially w.r.t.a, we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} F'(a) = 0 \qquad \text{-------- (6)}$$

The eliminant of 'a' between (5) and (6), if it exists, is called the general integral of (1).

SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS.

The first order partial differential equation can be written as

$$f(x,y,z,p,q) = 0,$$

where $p = \partial z/\partial x$ and $q = \partial z / \partial y$. In this section, we shall solve some standard forms of equations by special methods.

Standard I : f (p,q) = 0. i.e, equations containing p and q only.

Suppose that $z = ax + by + c$ is a solution of the equation $f(p,q) = 0$, where $f(a,b) = 0$.

Solving this for b, we get $b = F(a)$.

Hence the complete integral is $z = ax + F(a)y + c$ ------------ (1)

Now, the singular integral is obtained by eliminating a & c between

$$z = ax + y\,F(a) + c$$
$$0 = x + y\,F'(a)$$
$$0 = 1.$$

The last equation being absurd, the singular integral does not exist in this case.

To obtain the general integral, let us take $c = \Phi(a)$.

Then, $z = ax + F(a)\, y + \Phi(a)$ -------------- (2)

Differentiating (2) partially w.r.t. a, we get

$$0 = x + F'(a).\, y + \Phi'(a) \qquad \text{--------------- (3)}$$

Eliminating ‘a’ between (2) and (3), we get the general integral

Example 8

Solve $pq = 2$

The given equation is of the form $f(p,q) = 0$

The solution is $z = ax + by + c$, where $ab = 2$.

Solving, $b = \dfrac{2}{a}$.

The complete integral is

$$Z = ax + \frac{2}{a}\, y + c \qquad \text{---------- (1)}$$

Differentiating (1) partially w.r.t ‘c’, we get

$$0 = 1,$$

which is absurd. Hence, there is no singular integral.

To find the general integral, put $c = \Phi(a)$ in (1), we get

$$Z = ax + \frac{2}{a}\, y + \Phi(a)$$

Differentiating partially w.r.t ‘a’, we get

$$0 = x - \frac{2}{a^2}\, y + \Phi'(a)$$

Eliminating ‘a’ between these equations gives the general integral.

Example 9

Solve pq + p +q = 0

The given equation is of the form f (p,q) = 0.

The solution is z = ax + by +c, where ab + a + b = 0.

Solving, we get

$$b = -\frac{a}{1+a}$$

Hence the complete Integral is $z = ax - \left(\frac{a}{1+a}\right) y + c$ ------ (1)

Differentiating (1) partially w.r.t. 'c', we get

$$0 = 1.$$

The above equation being absurd, there is no singular integral for the given partial differential equation.

To find the general integral, put c = Φ (a) in (1), we have

$$z = ax - \left(\frac{a}{1+a}\right) y + \Phi(a) \qquad \text{------(2)}$$

Differentiating (2) partially w.r.t a, we get

$$0 = x - \frac{1}{(1+a)^2} y + \Phi'(a) \qquad 1 \qquad \text{----- (3)}$$

Eliminating 'a' between (2) and (3) gives the general integral.

Example 10

Solve $p^2 + q^2 = npq$

The solution of this equation is z = ax + by + c, where $a^2 + b^2 = nab$.

Solving, we get

$$b = a\left(\frac{n \pm \sqrt{(n^2-4)}}{}\right)$$

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Hence the complete integral is

$$z = ax + a\left(\frac{n \pm \sqrt{n^2 - 4}}{2}\right) y + c \qquad \text{-----------}(1)$$

Differentiating (1) partially w.r.t c, we get 0 = 1, which is absurd. Therefore, there is no singular integral for the given equation.

To find the general Integral, put C = Φ (a), we get

$$z = ax + a\left(\frac{n + \sqrt{n^2 - 4}}{2}\right) y + \Phi(a)$$

Differentiating partially w.r.t 'a', we have

$$0 = x + \left(\frac{n \pm \sqrt{n^2 - 4}}{2}\right) y + \Phi'(a)$$

The eliminant of 'a' between these equations gives the general integral

Standard II : Equations of the form f (x,p,q) = 0, f (y,p,q) = 0 and f (z,p,q) = 0.
i.e, one of the variables x,y,z occurs explicitly.

(i) Let us consider the equation f (x,p,q) = 0.

Since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

or $\quad dz = p\,dx + q\,dy$

Assume that q = a.

Then the given equation takes the form f (x, p,a) = 0

Solving, we get $p = \Phi(x,a)$.

Therefore, $\quad dz = \Phi(x,a)\, dx + a\, dy$.

Integrating, $z = \int \Phi(x,a)\, dx + ay + b$ which is a complete Integral.

(ii) Let us consider the equation $f(y,p,q) = 0$.

Assume that $p = a$.

Then the equation becomes $f(y,a,q) = 0$

Solving, we get $q = \Phi(y,a)$.

Therefore, $dz = a\,dx + \Phi(y,a)\,dy$.

Integrating, $z = ax + \int\Phi(y,a)\,dy + b$, which is a complete Integral.

(iii) Let us consider the equation $f(z, p, q) = 0$.

Assume that $q = ap$.

Then the equation becomes $f(z, p, ap) = 0$

Solving, we get $p = \Phi(z,a)$. Hence $dz = \Phi(z,a)\,dx + a\,\Phi(z,a)\,dy$.

$$\text{ie, } \frac{dz}{\Phi(z,a)} = dx + a\,dy.$$

Integrating, $$\int \frac{dz}{\Phi(z,a)} = x + ay + b,$$ which is a complete Integral.

Example 11

Solve $q = xp + p^2$

Given $q = xp + p^2$ -------------(1)

This is of the form $f(x,p,q) = 0$.

Put $q = a$ in (1), we get

$$a = xp + p^2$$

$$\text{i.e, } p^2 + xp - a = 0.$$

Therefore, $$p = \frac{-x + \sqrt{(x^2 + 4a)}}{2}$$

Integrating , $$z = \int \left(\frac{-x \pm \sqrt{x^2 + 4a}}{2} \right) dx + ay + b$$

Thus, $$z = -\frac{x^2}{4} \pm \left\{ \frac{x}{4} \sqrt{(4a + x^2)} + a \sinh^{-1}\left(\frac{x}{2\sqrt{a}} \right) \right\} + ay + b$$

Example 12

Solve $q = yp^2$

This is of the form $f(y,p,q) = 0$

Then, put $p = a$.

Therfore, the given equation becomes $q = a^2y$.

Since $dz = pdx + qdy$, we have

$$dz = adx + a^2y\, dy$$

Integrating, we get $z = ax + \dfrac{a^2y^2}{2} + b$

Example 13

Solve $9(p^2z + q^2) = 4$

This is of the form $f(z,p,q) = 0$

Then, putting $q = ap$, the given equation becomes

$$9(p^2z + a^2p^2) = 4$$

Therefore, $$p = \pm \frac{2}{3(\sqrt{z + a^2})}$$

and $$q = \pm \frac{2a}{3(\sqrt{z + a^2})}$$

Since $dz = pdx + qdy$,

$$dz = \pm \frac{2}{3} \frac{1}{\sqrt{z + a^2}} dx \pm \frac{2}{3} a \frac{1}{\sqrt{z + a^2}} dy$$

Multiplying both sides by $\sqrt{z + a^2}$, we get

$$\sqrt{z + a^2}\, dz = \frac{2}{3} dx + \frac{2}{3} a\, dy,$$ which on integration gives,

$$\frac{(z+a^2)^{3/2}}{3/2} = \frac{2}{3} x + \frac{2}{3} ay + b.$$

or $$(z + a^2)^{3/2} = x + ay + b.$$

Standard III : $f_1(x,p) = f_2(y,q)$. ie, equations in which 'z' is absent and the variables are separable.

Let us assume as a trivial solution that

$$f(x,p) = g(y,q) = a \text{ (say)}.$$

Solving for p and q, we get $p = F(x,a)$ and $q = G(y,a)$.

But $$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Hence $dz = pdx + qdy = F(x,a)\, dx + G(y,a)\, dy$

Therefore, $z = \int F(x,a)\, dx + \int G(y,a)\, dy + b$, which is the complete integral of the given equation containing two constants a and b. The singular and general integrals are found in the usual way.

Example 14

Solve $pq = xy$

The given equation can be written as

$$\frac{p}{x} = \frac{y}{q} = a \text{ (say)}$$

Therefore, $\dfrac{p}{x} = a$ implies $p = ax$

and $\dfrac{y}{q} = a$ implies $q = \dfrac{y}{a}$

Since $dz = pdx + qdy$, we have

$$dz = axdx + \frac{y}{a}\,dy,$$ which on integration gives.

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + b$$

Example 15

Solve $p^2 + q^2 = x^2 + y^2$

The given equation can be written as

$$p^2 - x^2 = y^2 - q^2 = a^2 \text{ (say)}$$

$p^2 - x^2 = a^2$ implies $p = \sqrt{(a^2 + x^2)}$

and $y^2 - q^2 = a^2$ implies $q = \sqrt{(y^2 - a^2)}$

But $dz = pdx + qdy$

ie, $dz = \sqrt{a^2 + x^2}\,dx + \sqrt{y^2 - a^2}\,dy$

Integrating, we get

$$z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + b$$

Standard IV (Clairaut's form)

Equation of the type $z = px + qy + f(p,q)$ ------(1) is known as Clairaut's form.

Differentiating (1) partially w.r.t x and y, we get

$$p = a \quad \text{and} \quad q = b.$$

Therefore, the complete integral is given by

$$z = ax + by + f(a,b).$$

Example 16

Solve $z = px + qy + pq$

The given equation is in Clairaut's form.

Putting $p = a$ and $q = b$, we have

$$z = ax + by + ab \qquad \text{--------} \ (1)$$

which is the complete integral.

To find the singular integral, differentiating (1) partially w.r.t a and b, we get

$$0 = x + b$$
$$0 = y + a$$

Therefore we have, $a = -y$ and $b = -x$.

Substituting the values of a & b in (1), we get

$$z = -xy - xy + xy$$

or $\quad z + xy = 0$, which is the singular integral.

To get the general integral, put $b = \Phi(a)$ in (1).

Then $\quad z = ax + \Phi(a)y + a\,\Phi(a) \qquad \text{----------} \ (2)$

Differentiating (2) partially w.r.t a, we have

$$0 = x + \Phi'(a)\,y + a\Phi'(a) + \Phi(a) \qquad \text{----------} \ (3)$$

Eliminating 'a' between (2) and (3), we get the general integral.

Example 17

Find the complete and singular solutions of $z = px + qy + \sqrt{1 + p^2 + q^2}$

The complete integral is given by

$$z = ax + by + \sqrt{1 + a^2 + b^2} \qquad \text{-------- (1)}$$

To obtain the singular integral, differentiating (1) partially w.r.t a & b. Then,

$$0 = x + \frac{a}{\sqrt{1 + a^2 + b^2}}$$

$$0 = y + \frac{b}{\sqrt{1 + a^2 + b^2}}$$

Therefore,

$$x = \frac{-a}{\sqrt{(1 + a^2 + b^2)}} \qquad \text{-----------(2)}$$

and

$$y = \frac{-b}{\sqrt{(1 + a^2 + b^2)}} \qquad \text{---------- (3)}$$

Squaring (2) & (3) and adding, we get

$$x^2 + y^2 = \frac{a^2 + b^2}{1 + a^2 + b^2}$$

Now, $$1 - x^2 - y^2 = \frac{1}{1 + a^2 + b^2}$$

i.e, $$1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2}$$

Therefore,

$$\sqrt{(1 + a^2 + b^2)} = \frac{1}{\sqrt{1 - x^2 - y^2}} \qquad \text{-----------(4)}$$

Using (4) in (2) & (3), we get

$$x = -a\sqrt{1 - x^2 - y^2}$$

and $$y = -b\sqrt{1 - x^2 - y^2}$$

Hence, $$a = \frac{-x}{\sqrt{1-x^2-y^2}} \quad \text{and} \quad b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Substituting the values of a & b in (1) , we get

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

which on simplification gives

$$z = \sqrt{1 - x^2 - y^2}$$

or $$x^2 + y^2 + z^2 = 1,$$ which is the singular integral.

Exercises

Solve the following Equations

1. $pq = k$
2. $p + q = pq$
3. $\sqrt{p} + \sqrt{q} = x$
4. $p = y^2q^2$
5. $z = p^2 + q^2$
6. $p + q = x + y$
7. $p^2z^2 + q^2 = 1$
8. $z = px + qy - 2\sqrt{pq}$
9. $\{z - (px + qy)\}^2 = c^2 + p^2 + q^2$
10. $z = px + qy + p^2q^2$

EQUATIONS REDUCIBLE TO THE STANDARD FORMS

Sometimes, it is possible to have non – linear partial differential equations of the first order which do not belong to any of the four standard forms discussed earlier. By changing the variables suitably, we will reduce them into any one of the four standard forms.

Type (i) : Equations of the form $F(x^m p, y^n q) = 0$ (or) $F(z, x^m p, y^n q) = 0$.

Case(i) : If $m \neq 1$ and $n \neq 1$, then put $x^{1-m} = X$ and $y^{1-n} = Y$.

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = \dfrac{\partial z}{\partial X}(1-m)\, x^{-m}$

Therefore, $x^m p = \dfrac{\partial z}{\partial X}(1-m) = (1 - m)\, P$, where $P = \dfrac{\partial z}{\partial X}$

Similarly, $y^n q = (1-n)Q$, where $Q = \dfrac{\partial z}{\partial Y}$

Hence, the given equation takes the form $F(P,Q) = 0$ (or) $F(z,P,Q) = 0$.

Case(ii) : If m = 1 and n = 1, then put $\log x = X$ and $\log y = Y$.

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} = \dfrac{\partial z}{\partial X} \quad \dfrac{1}{x}$

Therefore, $xp = \dfrac{\partial z}{\partial X} = P$.

Similarly, $yq = Q$.

Example 18

Solve $x^4p^2 + y^2zq = 2z^2$

The given equation can be expressed as

$(x^2p)^2 + (y^2q)z = 2z^2$

Here $m = 2$, $n = 2$

Put $X = x^{1-m} = x^{-1}$ and $Y = y^{1-n} = y^{-1}$.

We have $x^m p = (1-m)\, P$ and $y^n q = (1-n)Q$

i.e, $x^2p = -P$ and $y^2q = -Q$.

Hence the given equation becomes

$$P^2 - Qz = 2z^2 \qquad \text{----------(1)}$$

This equation is of the form $f(z,P,Q) = 0$.

Let us take $Q = aP$.

Then equation (1) reduces to

$$P^2 - aPz = 2z^2$$

Hence, $$P = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z$$

and $$Q = a\left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z$$

Since $dz = PdX + QdY$, we have

$$dz = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z\, dX + a \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) z\, dY$$

i.e, $$\frac{dz}{z} = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) (dX + a\, dY)$$

Integrating, we get

$$\log z = \left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)(X + aY) + b$$

Therefore, $$\log z = \left(\frac{a \pm \sqrt{(a^2 + 8)}}{2}\right) \left(\frac{1}{x} + \frac{a}{y}\right) + b$$ which is the complete solution.

Example 19

Solve $x^2p^2 + y^2q^2 = z^2$

The given equation can be written as

$$(xp)^2 + (yq)^2 = z^2$$

Here $m = 1$, $n = 1$.

Put $X = \log x$ and $Y = \log y$.

Then $xp = P$ and $yq = Q$.

Hence the given equation becomes

$$P^2 + Q^2 = z^2 \quad \text{-------------(1)}$$

This equation is of the form $F(z,P,Q) = 0$.

Therefore, let us assume that $Q = aP$.

Now, equation (1) becomes,

$$P^2 + a^2 P^2 = z^2$$

Hence $\quad P = \dfrac{z}{\sqrt{(1+a^2)}}$

and $\quad Q = \dfrac{az}{\sqrt{(1+a^2)}}$

Since $dz = PdX + QdY$, we have

$$dz = \frac{z}{\sqrt{(1+a^2)}}dX + \frac{az}{\sqrt{(1+a^2)}} dY.$$

i.e, $\sqrt{(1+a^2)} \dfrac{dz}{z} = dX + a\, dY.$

Integrating, we get

$$\sqrt{(1+a^2)} \log z = X + aY + b.$$

Therefore, $\sqrt{(1+a^2)} \log z = \log x + a \log y + b$, which is the complete solution.

Type (ii) : Equations of the form $F(z^k p, z^k q) = 0$ (or) $F(x, z^k p) = G(y, z^k q)$.

Case (i) : If $k \neq -1$, put $Z = z^{k+1}$,

Now $\dfrac{\partial Z}{\partial x} = \dfrac{\partial Z}{\partial z} \dfrac{\partial z}{\partial x} = (k+1)z^k. \dfrac{\partial z}{\partial x} = (k+1) z^k p.$

Therefore, $z^k p = \dfrac{1}{k+1} \dfrac{\partial Z}{\partial x}$

Similarly, $z^k q = \frac{1}{k+1} \frac{\partial Z}{\partial y}$

Case (ii) : If k = -1, put Z = log z.

$$\text{Now, } \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = \frac{1}{z}\, p$$

$$\text{Similarly, } \quad \frac{\partial Z}{\partial y} = \frac{1}{z}\, q.$$

Example 20

Solve $z^4q^2 - z^2p = 1$

The given equation can also be written as

$$(z^2q)^2 - (z^2p) = 1$$

Here k = 2. Putting $Z = z^{k+1} = z^3$, we get

$$Z^k p = \frac{1}{k+1}\frac{\partial Z}{\partial x} \quad \text{and} \quad Z^k q = \frac{1}{k+1}\frac{\partial Z}{\partial y}$$

$$\text{i.e, } Z^2 p = \frac{1}{3}\frac{\partial Z}{\partial x} \quad \text{and} \quad Z^2 q = \frac{1}{3}\frac{\partial Z}{\partial y}$$

Hence the given equation reduces to

$$\left(\frac{Q}{3}\right)^2 - \left(\frac{P}{3}\right) = 1$$

i.e, $Q^2 - 3P - 9 = 0$,

which is of the form F(P,Q) = 0.

Hence its solution is Z = ax + by + c, where $b^2 - 3a - 9 = 0$.

Solving for b, $b = \pm\sqrt{(3a + 9)}$

Hence the complete solution is

$$Z = ax \pm \sqrt{(3a + 9)} \,.\, y + c$$

or $$z^3 = ax \pm \sqrt{(3a + 9)}\, y + c$$

Exercises

Solve the following equations.

1. $x^2p^2 + y^2p^2 = z^2$
2. $z^2 (p^2+q^2) = x^2 + y^2$
3. $z^2 (p^2x^2 + q^2) = 1$
4. $2x^4p^2 – yzq – 3z^2 = 0$
5. $p^2 + x^2y^2q^2 = x^2 z^2$
6. $x^2p + y^2q = z^2$
7. $x^2/p + y^2/q = z$
8. $z^2 (p^2 – q^2) = 1$
9. $z^2 (p^2/x^2 + q^2/y^2) = 1$
10. $p^2x + q^2y = z$.

1.4 Lagrange's Linear Equation

Equations of the form $Pp + Qq = R$ ________ (1), where P, Q and R are functions of x, y, z, are known as Lagrange's equations and are linear in 'p' and 'q'.To solve this equation, let us consider the equations u = a and v = b, where a, b are arbitrary constants and u, v are functions of x, y, z.

Since 'u ' is a constant, we have du = 0 -----------(2).

But 'u' as a function of x, y, z,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Comparing (2) and (3), we have

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (3)}$$

Similarly, $$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_ (4)}$$

By cross-multiplication, we have

$$\frac{dx}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z}} = \frac{dy}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x}} = \frac{dz}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}}$$

(or)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{———————— (5)}$$

Equations (5) represent a pair of simultaneous equations which are of the first order and of first degree.Therefore, the two solutions of (5) are u = a and v = b. Thus, $\phi(u, v) = 0$ is the required solution of (1).

Note :

To solve the Lagrange's equation,we have to form the subsidiary or auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which can be solved either by the method of grouping or by the method of multipliers.

Example 21

Find the general solution of px + qy = z**.**

Here, the subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking the first two ratios , $\dfrac{dx}{x} = \dfrac{dy}{y}$

Integrating, log x = log y + log c_1

or $\quad x = c_1 y$

i.e, $\quad c_1 = x / y$

From the last two ratios, $\quad \dfrac{dy}{y} = \dfrac{dz}{z}$

Integrating, $\log y = \log z + \log c_2$

or $\quad y = c_2 z$

i.e, $\quad c_2 = y / z$

Hence the required general solution is

$\Phi(x/y, y/z) = 0$, where Φ is arbitrary

Example 22

Solve $\quad p \tan x + q \tan y = \tan z$

The subsidiary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking the first two ratios , $\quad \dfrac{dx}{\tan x} = \dfrac{dy}{\tan y}$

ie, $\quad \cot x\, dx = \cot y\, dy$

Integrating, $\log \sin x = \log \sin y + \log c_1$

ie, $\quad \sin x = c_1 \sin y$

Therefore, $\quad c_1 = \sin x / \sin y$

Similarly, from the last two ratios, we get

$\sin y = c_2 \sin z$

i.e, $\quad c_2 = \sin y / \sin z$

Hence the general solution is

$$\Phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0, \text{ where } \Phi \text{ is arbitrary.}$$

Example 23

Solve $(y-z)\,p + (z-x)\,q = x-y$

Here the subsidiary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

Using multipliers 1,1,1,

$$\text{each ratio} = \frac{dx + dy + dz}{0}$$

Therefore, $dx + dy + dz = 0$.

Integrating, $x + y + z = c_1$ ______________ (1)

Again using multipliers x, y and z,

$$\text{each ratio} = \frac{xdx + ydy + zdz}{0}$$

Therefore, $xdx + ydy + zdz = 0$.

Integrating, $x^2/2 + y^2/2 + z^2/2 = \text{constant}$

or $x^2 + y^2 + z^2 = c_2$ ____________ (2)

Hence from (1) and (2), the general solution is

$$\Phi(x + y + z, x^2 + y^2 + z^2) = 0$$

Example 24

Find the general solution of $(mz - ny)\,p + (nx - lz)q = ly - mx$.

Here the subsidiary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Using the multipliers x, y and z, we get

$$\text{each fraction} = \frac{xdx + ydy + zdz}{0}$$

` $\therefore$ xdx + ydy + zdz = 0, which on integration gives

$$x^2/2 + y^2/2 + z^2/2 = \text{constant}$$

or $x^2 + y^2 + z^2 = c_1$ ___________ (1)

Again using the multipliers l, m and n, we have

$$\text{each fraction} = \frac{ldx + mdy + ndz}{0}$$

` $\therefore$ ldx + mdy + ndz = 0, which on integration gives

$lx + my + nz = c_2$ ___________ (2)

Hence, the required general solution is

$$\Phi(x^2 + y^2 + z^2 , \ lx + my + nz) = 0$$

Example 25

Solve $(x^2 - y^2 - z^2)\, p + 2xy\, q = 2xz.$

The subsidiary equations are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

Taking the last two ratios,

$$\frac{dx}{2xy} = \frac{dz}{2xz}$$

$$2xy \qquad 2xz$$

ie, $$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get $\log y = \log z + \log c_1$

or $y = c_1 z$

i.e, $c_1 = y/z$ ___________ (1)

Using multipliers x, y and z, we get

each fraction = $$\frac{xdx + y\,dy + zdz}{x(x^2-y^2-z^2)+2xy^2+2xz^2} = \frac{xdx + y\,dy + zdz}{x(x^2+y^2+z^2)}$$

Comparing with the last ratio, we get

$$\frac{xdx + y\,dy + zdz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz}$$

i.e, $$\frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2} = \frac{dz}{z}$$

Integrating, $\log(x^2+y^2+z^2) = \log z + \log c_2$

or $x^2+y^2+z^2 = c_2 z$

i.e, $$c_2 = \frac{x^2+y^2+z^2}{z} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} (2)$$

From (1) and (2), the general solution is $\Phi(c_1, c_2) = 0$.

i.e, $$\Phi\left((y/z), \frac{x^2+y^2+z^2}{z}\right) = 0$$

Exercises

Solve the following equations

1. $px^2 + qy^2 = z^2$
2. $pyz + qzx = xy$
3. $xp – yq = y^2 – x^2$
4. $y^2zp + x^2zq = y^2x$
5. $z (x – y) = px^2 – qy^2$
6. $(a – x) p + (b – y) q = c – z$
7. $(y^2z\, p) /x + xzq = y^2$
8. $(y^2 + z^2) p – xyq + xz = 0$
9. $x^2p + y^2q = (x + y) z$
10. $p – q = \log (x+y)$
11. $(xz + yz)p + (xz – yz)q = x^2 + y^2$
12. $(y – z)p – (2x + y)q = 2x + z$

1.5 PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS.

Homogeneous Linear Equations with constant Coefficients.

A homogeneous linear partial differential equation of the n^{th} order is of the form

$$c_0 \frac{\partial^n z}{\partial x^n} + c_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots\dots + c_n \frac{\partial^n z}{\partial y^n} = F(x,y) \qquad \text{---------} (1)$$

where $c_0, c_1, \text{---------}, c_n$ are constants and F is a function of ‘x’ and ‘y’. It is homogeneous because all its terms contain derivatives of the same order.

Equation (1) can be expressed as

$$(c_0D^n + c_1D^{n-1} D' + \dots\dots + c_n D'^n) z = F(x,y)$$

or $$f(D,D') z = F(x,y) \qquad \text{---------}(2),$$

where, $\dfrac{\partial}{\partial x} \equiv D$ and $\dfrac{\partial}{\partial y} \equiv D'$.

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely, the complementary function and the particular integral.

The complementary function is the complete solution of $f(D,D')\,z = 0$-------(3), which must contain n arbitrary functions as the degree of the polynomial $f(D,D')$. The particular integral is the particular solution of equation (2).

Finding the complementary function

Let us now consider the equation $f(D,D')\,z = F(x,y)$

The auxiliary equation of (3) is obtained by replacing D by m and D' by 1.

i.e, $c_0 m^n + c_1 m^{n-1} + \ldots + c_n = 0$ ---------(4)

Solving equation (4) for 'm', we get 'n' roots. Depending upon the nature of the roots, the Complementary function is written as given below:

Roots of the auxiliary equation	Nature of the roots	Complementary function(C.F)
$m_1, m_2, m_3 \ldots, m_n$	distinct roots	$f_1(y+m_1x)+f_2(y+m_2x) + \ldots+f_n(y+m_nx)$.
$m_1 = m_2 = m, m_3, m_4, \ldots, m_n$	two equal roots	$f_1(y+m_1x)+xf_2(y+m_1x) + f_3(y+m_3x) + \ldots+ f_n(y+m_nx)$.
$m_1 = m_2 = \ldots = m_n = m$	all equal roots	$f_1(y+mx)+xf_2(y+mx) + x^2f_3(y+mx)+\ldots + \ldots+x^{n-1} f_n(y+mx)$

Finding the particular Integral

Consider the equation $f(D,D')\,z = F(x,y)$.

Now, the P.I is given by $\dfrac{1}{f(D,D')} F(x,y)$

Case (i) : When $F(x,y) = e^{ax+by}$

$$\text{P.I} = \frac{1}{f(D,D')} e^{ax+by}$$

Replacing D by 'a' and D' by 'b', we have

$$\text{P.I} = \frac{1}{f(a,b)} e^{ax+by}, \quad \text{where } f(a,b) \neq 0.$$

$f(a,b)$

Case (ii) : When $F(x,y) = \sin(ax + by)$ (or) $\cos(ax + by)$

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

Replacing $D^2 = -a^2$, $DD'^2 = -ab$ and $D' = -b^2$, we get

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by), \text{ where } f(-a^2, -ab, -b^2) \neq 0.$$

Case (iii) : When $F(x,y) = x^m y^n$,

$$P.I = \frac{1}{f(D,D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

Case (iv) : When F(x,y) is any function of x and y.

$$P.I = \frac{1}{f(D,D')} F(x,y).$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions considering $f(D,D')$ as a function of D alone.

Then operate each partial fraction on F(x,y) in such a way that

$$\frac{1}{D - mD'} F(x,y) = \int F(x, c-mx)\, dx,$$

where c is replaced by y+mx after integration

Example 26

Solve $(D^3 - 3D^2D' + 4D'^3) z = e^{x+2y}$

The auxillary equation is $m = m^3 - 3m^2 + 4 = 0$

The roots are m = -1,2,2

Therefore the C.F is $f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$.

$$\text{P.I.} = \frac{e^{x+2y}}{D^3 - 3D^2D' + 4D'^3} \quad \text{(Replace D by 1 and } D' \text{ by 2)}$$

$$= \frac{e^{x+2y}}{1-3(1)(2)+4(2)^3}$$

$$= \frac{e^{x+2y}}{27}$$

Hence, the solution is z = C.F. + P.I

ie, $$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$$

Example 27

Solve $(D^2 - 4DD' + 4D'^2) z = \cos(x - 2y)$

The auxiliary equation is $m^2 - 4m + 4 = 0$

Solving, we get m = 2,2.

Therefore the C.F is $f_1(y+2x) + xf_2(y+2x)$.

$$\therefore \text{P.I} = \frac{1}{D^2 - 4DD' + 4D'^2} \cos(x-2y)$$

Replacing D^2 by -1, DD' by 2 and D'^2 by -4, we have

$$\text{P.I} = \frac{1}{(-1) - 4(2) + 4(-4)} \cos(x-2y)$$

$$= -\frac{\cos(x-2y)}{25}$$

$\therefore$ Solution is $z = f_1(y+2x) + xf_2(y+2x) - \dfrac{\qquad}{25}$.

Example 28

Solve $(D^2 - 2DD')\, z = x^3y + e^{5x}$

The auxiliary equation is $m^2 - 2m = 0$.

Solving, we get $m = 0,2$.
Hence the C.F is $f_1(y) + f_2(y+2x)$.

$$P.I_1 = \frac{x^3y}{D^2 - 2DD'}$$

$$= \frac{1}{D^2\left(1 - \dfrac{2D'}{D}\right)} (x^3y)$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1} (x^3y)$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \dots\dots\right)(x^3y)$$

$$= \frac{1}{D^2}\left((x^3y) + \frac{2}{D} D'(x^3y) + \frac{4}{D^2} D'^2(x^3y) + \dots\dots\right)$$

$$= \frac{1}{D^2}\left((x^3y) + \frac{2}{D}(x^3) + \frac{4}{D^2}(0) + \dots\dots\right)$$

$$P.I_1 = \frac{1}{D^2}(x^3y) + \frac{2}{D^3}(x^3)$$

$$P.I_1 = \frac{x^5y}{20} + \frac{x^6}{60}$$

$$P.I_2 = \frac{e^{5x}}{D^2 - 2DD'} \quad \text{(Replace D by 5 and } D' \text{ by 0)}$$

$$= \frac{e^{5x}}{25}$$

$$\therefore \text{Solution is } Z = f_1(y) + f_2(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60} + \frac{e^{5x}}{25}$$

Example 29

Solve $(D^2 + DD' - 6D'^2) z = y \cos x$.

The auxiliary equation is $m^2 + m - 6 = 0$.
Therefore, $m = -3, 2$.

Hence the C.F is $f_1(y-3x) + f_2(y + 2x)$.

$$P.I = \frac{y \cos x}{D^2 + DD' - 6D'^2}$$

$$= \frac{y \cos x}{(D + 3D')(D - 2D')}$$

$$= \frac{1}{(D+3D')} \frac{1}{(D - 2D')} y \cos x$$

$$= \frac{1}{(D+3D')} \int (c - 2x) \cos x \, dx, \quad \text{where } y = c - 2x$$

$$= \frac{1}{(D+3D')} \int (c - 2x) \, d(\sin x)$$

$$= \frac{1}{(D+3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)]$$

$$= \frac{1}{(D+3D')} [y \sin x - 2 \cos x)]$$

$$= \int [(c + 3x) \sin x - 2 \cos x] \, dx, \quad \text{where } y = c + 3x$$

$$= \int (c + 3x)\, d(-\cos x) - 2\int \cos x \; dx$$

$$= (c + 3x)\; (-\cos x) - (3)\; (-\sin x) - 2 \sin x$$

$$= -y \;\cos x + \sin x$$

Hence the complete solution is

$$z = f_1(y - 3x) + f_2(y + 2x) - y \cos x + \sin x$$

Example 30

Solve $r - 4s + 4t = e^{2x+y}$

Given equation is $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+y}$

i.e, $(D^2 - 4DD' + 4D'^2)\, z = e^{2x+y}$

The auxiliary equation is $m^2 - 4m + 4 = 0$.

Therefore, $m = 2,2$

Hence the C.F is $f_1(y + 2x) + x\, f_2(y + 2x)$.

$$\text{P.I.} = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2}$$

Since $D^2 - 4DD'+4D'^2 = 0$ for $D = 2$ and $D' = 1$, we have to apply the general rule.

$$\therefore \text{P.I.} = \frac{e^{2x+y}}{(D - 2D')\,(D - 2D')}$$

$$= \frac{1}{(D - 2D')}\;\frac{1}{(D - 2D')}\; e^{2x+y}$$

$$= \frac{1}{(D - 2D')} \int e^{2x+c-2x}\, dx, \quad \text{where } y = c - 2x.$$

$$= \frac{1}{(D-2D')} \int e^c \, dx$$

$$= \frac{1}{(D-2D')} \; e^c \, .x$$

$$= \frac{1}{D-2D'} \; xe^{\,y+2x}$$

$= \int xe^{c-2x+2x}\, dx$, where $y = c - 2x$.

$= \int xe^c \, dx$

$= e^c . x^2/2$

$$= \frac{x^2 e^{y+2x}}{2}$$

Hence the complete solution is

$$z = f_1(y+2x) + f_2(y+2x) + \frac{1}{2} x^2 e^{2x+y}$$

1.6 Non – Homogeneous Linear Equations

Let us consider the partial differential equation

$$f(D,D')\, z = F(x,y) \qquad \text{------- (1)}$$

If $f(D,D')$ is not homogeneous, then (1) is a non–homogeneous linear partial differential equation. Here also, the complete solution = C.F + P.I.

The methods for finding the Particular Integrals are the same as those for homogeneous linear equations.

But for finding the C.F, we have to factorize $f(D,D')$ into factors of the form $D - mD' - c$.

Consider now the equation

$$(D - mD' - c)\, z = 0 \text{ -----------(2).}$$

This equation can be expressed as

$$p - mq = cz \text{ ---------}(3),$$

which is in Lagrangian form.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz} \text{ -----------}(4)$$

The solutions of (4) are $y + mx = a$ and $z = be^{cx}$.

Taking $b = f(a)$, we get $z = e^{cx} f(y+mx)$ as the solution of (2).

Note:

1. If $(D-m_1D' - C_1)(D - m_2D'-C_2) \ldots\ldots (D - m_nD'-C_n) z = 0$ is the partial differential equation, then its complete solution is

   $z = e^{c_1 x} f_1(y+m_1x) + e^{c_2 x} f_2(y+m_2x) + \ldots\ldots + e^{c_n x} f_n(y+m_nx)$

2. In the case of repeated factors, the equation $(D-mD' - C)^n z = 0$ has a complete

   solution $z = e^{cx} f_1(y+mx) + x\, e^{cx} f_2(y+mx) + \ldots\ldots + x^{n-1} e^{cx} f_n(y+mx)$.

Example 31

Solve $(D-D'-1)(D-D'-2)z = e^{2x-y}$

Here $m_1 = 1$, $m_2 = 1$, $c_1 = 1$, $c_2 = 2$.

Therefore, the C.F is $e^x f_1(y+x) + e^{2x} f_2(y+x)$.

$$\text{P.I.} = \frac{e^{2x-y}}{(D - D' - 1)(D-D' - 2)} \quad \text{Put } D = 2,\ D' = -1.$$

$$= \frac{e^{2x-y}}{(2-(-1)-1)(2-(-1)-2)}$$

$$= \frac{e^{2x-y}}{2}$$

Hence the solution is $z = e^{x} f_1 (y+x) + e^{2x} f_2 (y+x) + \dfrac{e^{2x-y}}{2}$.

Example 32

Solve $(D^2 - DD' + D' - 1) z = \cos (x + 2y)$

The given equation can be rewritten as

$$(D-D'+1) (D-1) z = \cos (x + 2y)$$

Here $m_1 = 1, m_2 = 0, c_1 = -1, c_2 = 1$.

Therefore, the C.F $= e^{-x} f_1(y+x) + e^{x} f_2 (y)$

$$\text{P.I} = \frac{1}{(D^2 - DD' + D' - 1)} \cos (x+2y) \quad [\text{Put } D^2 = -1, DD' = -2, D'^2 = -4]$$

$$= \frac{1}{-1 - (-2) + D' - 1} \cos (x+2y)$$

$$= \frac{1}{D'} \cos (x+2y)$$

$$= \frac{\sin (x+2y)}{2}$$

Hence the solution is $z = e^{-x} f_1(y+x) e^{x} f_2(y) + \dfrac{\sin(x+2y)}{2}$.

Example 33

Solve $[(D + D' - 1) (D + 2D' - 3)] z = e^{x+2y} + 4 + 3x + 6y$

Here $m_1 = -1, m_2 = -2, c_1 = 1, c_2 = 3$.

Hence the C.F is $z = e^{x} f_1(y - x) + e^{3x} f_2(y - 2x)$.

$$P.I_1 = \frac{e^{x+2y}}{(D+D' - 1)(D + 2D' - 3)} \quad [\text{Put } D = 1, D' = 2]$$

$$= \frac{e^{x+2y}}{(1+2-1)(1+4-3)}$$

$$= \frac{e^{x+2y}}{4}$$

$$P.I_2 = \frac{1}{(D+D' - 1)(D + 2D' - 3)} (4 + 3x + 6y)$$

$$= \frac{1}{3[1-(D+D')]\left(1 - \dfrac{D+2D'}{3}\right)} (4 + 3x + 6y)$$

$$= \frac{1}{3}[1-(D+D')]^{-1}\left(1 - \frac{D+2D'}{3}\right)^{-1}(4 + 3x + 6y)$$

$$= \frac{1}{3}[1 + (D+D') + (D+D')^2 + \ldots]\left(1 + \frac{D+2D'}{3} + \frac{1}{9}(D+2D')^2 + \ldots\right] . (4 + 3x + 6y)$$

$$= \frac{1}{3}\left(1 + \frac{4}{3}D + \frac{5}{3}D' + \ldots\right)(4 + 3x + 6y)$$

$$= \frac{1}{3}\left(4 + 3x + 6y + \frac{4}{3}(3) + \frac{5}{3}(6)\right)$$

$$= x + 2y + 6$$

Hence the complete solution is

$$z = e^{x}f_1(y-x) + e^{3x} f_2(y-2x) + \frac{e^{x+2y}}{4} + x + 2y + 6.$$

Exercises

(a) Solve the following homogeneous Equations.

1. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(2x+y)$

2. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} = \sin x.\cos 2y$

3. $(D^2 + 3DD' + 2D'^2)\, z = x + y$

4. $(D^2 - DD' + 2D'^2)\, z = xy + e^{x}.\cosh y$

$$\left\{ \text{Hint: } e^{x}.\cosh y = e^{x}.\left(\frac{e^{y} + e^{-y}}{2}\right) = \frac{e^{x+y} + e^{x-y}}{2} \right\}$$

5. $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(x+2y) + e^{2x+y}$

6. $(D^2 + 4DD' - 5D'^2)\, z = 3e^{2x-y} + \sin(x-2y)$

7. $(D^2 - DD' - 30D'^2)\, z = xy + e^{6x+y}$

8. $(D^2 - 4D'^2)\, z = \cos 2x.\cos 3y$

9. $(D^2 - DD' - 2D'^2)\, z = (y-1)e^{x}$

10. $4r + 12s + 9t = e^{3x-2y}$

(b) Solve the following non – homogeneous equations.

1. $(2DD' + D'^2 - 3D')\,z = 3\cos(3x - 2y)$
2. $(D^2 + DD' + D' - 1)\,z = e^{-x}$
3. $r - s + p = x^2 + y^2$
4. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$
5. $(D^2 - D'^2 - 3D + 3D')\,z = xy + 7.$