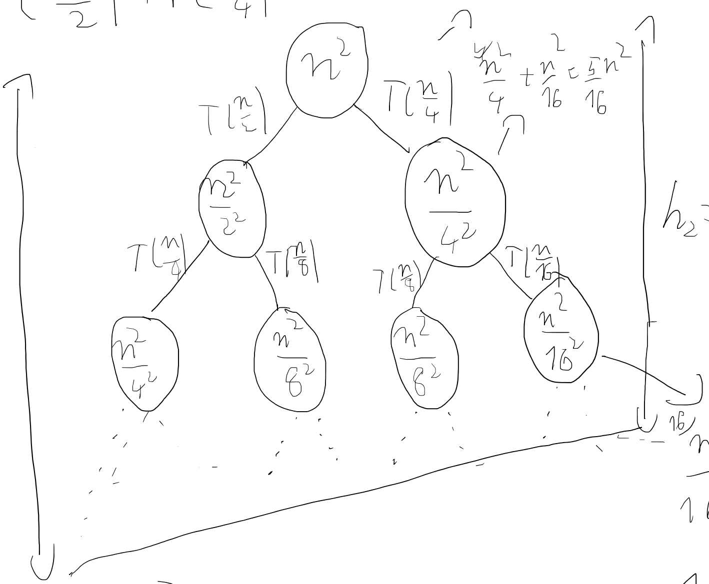
De data trecuta

Friday, 12 November 2021 10.06

$$T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + n^2$$

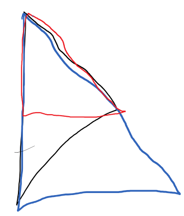


n^2

$\frac{n^2}{4} + \frac{n^2}{16} = \frac{5n^2}{16}$

$h_1 = \log_2 n$

$h_2 = \log_4 n$



$$\frac{n^2}{16} + 2\frac{n^2}{64} + \frac{n^2}{256} =$$

$$= \frac{16+8+1}{256} n^2 = \frac{25}{256} n^2 = \frac{5^2 n^2}{16^2}$$

nivel k: $\left(\frac{5}{16}\right)^k n^2$

$T_1 : h_1$, $T_2 : h_2$

$$T_2(n) \le T(n) \le T_1(n)$$

$$T_2(n) = n^2 \sum_{k=0}^{h_2=\log_4 n} \left(\frac{5}{16}\right)^k =$$

$$= n^2 \cdot \frac{1}{11}\left[16 - \frac{5^{\log_4 n+1}}{16^{\log_4 n}}\right] =$$

$$= n^2 \cdot \frac{1}{11}\left[16 - \frac{5 n^{\log_4 5}}{n^2}\right]$$

$$= n^2 \cdot \frac{16}{11} - \frac{5}{11} \cdot n^{\log_4 5 - 2}$$

$\le \log_4 4 - 2$

$$\ge n^2 \cdot \frac{16}{11} - \frac{5}{11} n^{-1} \in \Theta(n^2)$$

$$\Rightarrow T(n) \in \Theta(n^2)$$

$$T_1(n) = n^2 \sum_{k=0}^{h_1=\log n} \left(\frac{5}{16}\right)^k$$

$$= n^2 \cdot \frac{1}{11}\left[16 - \frac{5^{\log n+1}}{16^{\log n}}\right]$$

$$= n^2 \cdot \frac{1}{11}\left(16 - 5 \cdot \left(\frac{5}{16}\right)^{\log n}\right)$$

$$= n^2 \cdot \frac{1}{11}\left[16 - 5 n^{\log \frac{5}{16}}\right]$$

$$T_1(n) = \frac{16}{11} n^2 - \frac{5}{11} n^2 \cdot n^{\log \frac{5}{16}}$$

$\log 5 \le \log 8 = 3$

$$= \frac{16}{11} n^2 - \frac{5}{11} n^2 \cdot n^{\log 5 - 4}$$

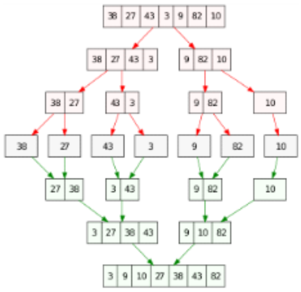
$$T_1(n) \le \frac{16}{11} n^2 - \frac{5}{11} n^2 \cdot n^{-1} \in \Theta(n^2)$$

Merge sort

Algorithm 1 Pseudocod mergesort

```
function MERGE(V_1[1..n], V_2[1..m])
    Let R[1..n + m]
    i = 1, j = 1
    while i < n and j < m do
        if V_1[i] < V_2[j] then R[i + j] = V_1[i + +]
        elseR[i + j] = V_2[j + +]
        end if
    end while
    while i < n do
        R[i + j] = V_1[i + +]
    end while
    while j < m do
        R[i + j] = V_2[j + +]
    end while
end function
```

```
function MERGESORT(V[1..n], start, end)
    if start ≥ end then return
    end if
    middle = (start + end)/2
    MergeSort(V, start, middle)
    MergeSort(V, middle, end)
    return Merge(V[start..middle], V[middle..end])
end function
```



$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Dem. prin inducție $T(n) \in \Theta(n \log n)$

Caz de bază

$n = 1 \Rightarrow \log 1 = 0 \Rightarrow 0 \leq T(1) \leq 0$, $T(1) = 1$

$n = 2 \Rightarrow 2c_1 \leq T(2) \leq 2c_2 \Rightarrow 2c_1 \leq 4 \leq 2c_2 \Rightarrow c_1 \leq 2$, $c_2 \geq 2$

$T(2) = 2T(1) + 2 = 4$

Ip. inductivă

$P\left(\frac{n}{2}\right): T\left(\frac{n}{2}\right) \in \Theta\left(\frac{n}{2}\log\frac{n}{2}\right)\ \forall n \in \{2, 4, 8, 16, \dots\} \Rightarrow \exists c_1, c_2, \exists n_0$ a.î. $\forall n \geq n_0$ $c_1 \frac{n}{2}\log\frac{n}{2} \leq T\left(\frac{n}{2}\right) \leq c_2\frac{n}{2}\log\frac{n}{2}$

Vreau să dem. $T(n) \in \Theta(n \log n) \Rightarrow \exists c_1', c_2', \exists n_0'$ a.î. $\forall n \geq n_0'$ $c_1' n \log n \leq T(n) \leq c_2' n \log n$

$c_1 \frac{n}{2}\log\frac{n}{2} \leq T\left(\frac{n}{2}\right) \leq c_2 \frac{n}{2}\log\frac{n}{2} \quad \Big| \cdot 2$

$c_1 n \log\frac{n}{2} \leq 2T\left(\frac{n}{2}\right) \leq c_2 n \log\frac{n}{2} \quad \Big| + n$

$c_1 n \log\frac{n}{2} + n \leq 2T\left(\frac{n}{2}\right) + n \leq c_2 n \log\frac{n}{2} + n$

$c_1 n \log n - c_1 n + \underbrace{n}_{\geq 0} \leq T(n) \leq c_2 n \log n - c_2 n + n$

$c_1 n \log n + n(1 - c_1) \leq T(n) \leq c_2 n \log n + n(1 - c_2) \leq c_2 n \log n$

$c_1 n \log n \leq$

$1 - c_2 \leq 0$

$c_2 \geq 1$

$1 - c_1 \geq 0$

$c_1 \leq 1$

$c_1 n \log n \leq T(n) \leq c_2 n \log n \Rightarrow T(n) \in \Theta(n \log n)$

Teorema Master

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$T(n) = aT\left(\frac{n}{b}\right) + f(n)$, a, b - ct

$a \geq 1$

$b > 1$

$f(n) \geq 0$

$\log_b a \to n^{\log_b a}$

1. $f(n) \in O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0 \Rightarrow T(n) \in \Theta(n^{\log_b a})$

2. $f(n) \in \Theta(n^{\log_b a} \log_2^k n)$, $k \geq 0 \Rightarrow T(n) \in \Theta(n^{\log_b a} \cdot \log_2^{k+1} n)$

3. $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$

şi

$af\left(\frac{n}{b}\right) \leq cf(n)$ pt. $c < 1$

pt. n suficient de mare

$\Rightarrow T(n) \in \Theta(f(n))$

Ex.

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$T(n) = \underbrace{2^n}_{\text{nu e ct.}} T\left(\frac{n}{16}\right) + \sqrt{n} \log n \Rightarrow NU$

$T(n) = \underbrace{\frac{1}{2}}_{<1} T\left(\frac{n}{4}\right) + n \Rightarrow nu$

$T(n) = 2T\left(\frac{n}{2}\right) \underbrace{- n^2}_{f(n) \le 0} \Rightarrow nu$

d. $T(n) = 16\, T\left(\frac{n}{4}\right) + n!$

$a = 16 \quad \log_b a = \log_4 16 = 2 \Rightarrow n^2$

$b = 4$

$n! \in \Omega(n^{2+\varepsilon}), \varepsilon > 0$

de ex. $\varepsilon = 1$

$n! \in \Omega(n^3)$

$a f\left(\frac{n}{b}\right) \le c f(n)$ pt. n suf. mare și $c < 1$

$16 \left(\frac{n}{4}\right)! \le c\, n!$

$c = \frac{1}{2} \Rightarrow 32 \left(\frac{n}{4}\right)! \le n!$

$n = 4 \Rightarrow 32 \cdot 1! \le 4! = 24$ F

$n = 8 \Rightarrow 32 \cdot 2! \le 8!$

$64 \le 8!$ A

$8! \ge 8 \cdot 7 \cdot 6 = 336 \ge 64$

$\Rightarrow T(n) \in \Theta(n!)$

b. $T(n) = T\left(\frac{n}{4}\right) + 1$

$a = 1 \Rightarrow \log_b a = \log_4 1 = 0 \Rightarrow n^0 = 1$

$b = 4$

1. $1 \in O(n^{0-\varepsilon}), \varepsilon > 0 \Rightarrow$ nu pot

2. $1 \in \Theta(\underbrace{n^0 \log^0 n}_{1}) \Rightarrow T(n) \in \Theta(n^0 \log^1 n)$

$T(n) \in \Theta(\log n)$

$n\sqrt{n} = n^{\frac{3}{2}}$

c. $T(n) = 4\, T\left(\frac{n}{2}\right) + n\sqrt{n}$

$a = 4 \Rightarrow \log_b a = \log_2 4 = 2 \Rightarrow n^2$

$b = 2$

$n^{\frac{3}{2}} \in O(n^{2-\varepsilon}), \varepsilon > 0 \Rightarrow T(n) \in \Theta(n^2)$

$\varepsilon = \frac{1}{4}$ $n^{1.5} \in O(n^{1.75})$ |

$T(n) = 2T(\frac{n}{2}) + \sin(n)$ $\Rightarrow$ nu pot

$h(n) \leq 0$

$T(n) = T(\frac{n}{2}) + n(2-\cos n)$ $\begin{matrix} a=1 \\ b=2 \end{matrix} \Rightarrow \log_b a = 0 \Rightarrow n^0 = 1$

3. $n(2-\cos n) \in \Omega(n^{0+\varepsilon})$

$\varepsilon = 0.1$

$a h(\frac{n}{b}) \leq c\, h(n)$ $\frac{n}{2}(2-\cos\frac{n}{2}) \leq cn(2-\cos n)$

$n = 2k\pi$, k - impar

$k\pi(2-\underbrace{\cos k\pi}_{=-1}) \leq c\,2k\pi(2-\underbrace{\cos 2k\pi}_{=1})$

$3k\pi \leq c\,2k\pi(2-1)$

$c \geq \frac{3}{2}$, c nu e $< 1 \Rightarrow$ fals $\Rightarrow$ nu pot

4.a. $T(n) = 2T(\frac{n}{2}) + n\log n$

$\begin{matrix} a=2 \\ b=2 \end{matrix} \Rightarrow \log_b a = \log_2 2 = 1 \Rightarrow n^1$

2. $n\log n \in \Theta(n^1 \cdot \log^1 n) \Rightarrow T(n) \in \Theta(n\log^2 n)$

3. $n\log n \in \Omega(n^{1+\varepsilon})$, $\varepsilon > 0$?

$\frac{n\log n}{n^{1+\varepsilon}} = \frac{\log n}{n^\varepsilon} \to 0 \Rightarrow$ nu $\in \Omega(n^{1+\varepsilon})$

4.b. $T(n) = 3T(\frac{n}{3}) + \frac{n}{\log n}$

$\begin{matrix} a=3 \\ b=3 \end{matrix} \Rightarrow \log_b a = 1 \Rightarrow n^1$

1. $\frac{n}{\log n} \in O(n^{1-\varepsilon})$, $\varepsilon > 0$

$\frac{n}{\log n \cdot n^{1-\varepsilon}} = \cancel{n}\frac{1}{\log n \cdot n^{-\varepsilon}} = \frac{n^\varepsilon}{\log n} \to \infty$

2. $\frac{n}{\log n} \in \Theta(n^1 \cdot \log^{-1} n) \Rightarrow$ nu se aplică $(k < 0)$

A useful extension of Case 2 handles all values of k:[3]

Case	Condition on $f(n)$ in relation to c_{crit}, i.e. $\log_b a$	Master Theorem bound	Notational examples
2a	When $f(n) = \Theta(n^{c_{crit}} \log^k n)$ for any $k > -1$	... then $T(n) = \Theta\left(n^{c_{crit}} \log^{k+1} n\right)$ (The bound is the splitting term, where the log is augmented by a single power.)	If $b = a^2$ and $f(n) = \Theta(n^{1/2} / \log^{1/2} n)$, then $T(n) = \Theta(n^{1/2} \log^{1/2} n)$.
2b	When $f(n) = \Theta(n^{c_{crit}} \log^k n)$ for $k = -1$	... then $T(n) = \Theta\left(n^{c_{crit}} \log \log n\right)$ (The bound is the splitting term, where the log reciprocal is replaced by an iterated log.)	If $b = a^2$ and $f(n) = \Theta(n^{1/2} / \log n)$, then $T(n) = \Theta(n^{1/2} \log \log n)$.
2c	When $f(n) = \Theta(n^{c_{crit}} \log^k n)$ for any $k < -1$	... then $T(n) = \Theta\left(n^{c_{crit}}\right)$ (The bound is the splitting term, where the log disappears.)	If $b = a^2$ and $f(n) = \Theta(n^{1/2} / \log^2 n)$, then $T(n) = \Theta(n^{1/2})$.

$\rightarrow T(n) \in \Theta(n \log \log n)$