# Metoda substitutiei

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Metoda substituției $\to$ inducție

merge sort

**Algorithm 1** Pseudocod Merge-Sort

```
function MERGE-SORT(A, p, r)
    if p ≤ r then
        q = ⌊(p + r)/2⌋
        MERGE-SORT(A, p, q)
        MERGE-SORT(A, q + 1, r)
        MERGE(A, p, q, r)
    end if
end function
```

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

$$T(n) \in \Theta(n \log n)$$

Dem. prin inducție

$T(n) = 2T\left(\frac{n}{2}\right) + n$ , $T(1) = 1$

$P(n)$ : $T(n) \in \Theta(n \log n) \iff \exists\, c_1, c_2 > 0$ a.î. $\exists\, n_0, \forall n \geq n_0$

$c_1 n \log n \leq T(n) \leq c_2 n \log n$

<u>Caz de bază</u>

$n = 1 \Rightarrow \log 1 = 0 \Rightarrow 0 \leq T(1) \leq 0$

$n = 2 \Rightarrow c_1 \cdot 2 \cdot \log 2 \leq T(2) \leq c_2 \cdot 2 \cdot \log 2$

$2c_1 \leq T(2) \leq 2c_2 \Big| \Rightarrow 2c_1 \leq 4 \leq 2c_2$

$c_1 \leq 2 \leq c_2$

$T(2) = 2T(1) + 2$

$= 2 + 2 = 4$

<u>Ip. inductivă</u>

$P\left(\frac{n}{2}\right)$ : $T\left(\frac{n}{2}\right) \in \Theta\left(\frac{n}{2} \log \frac{n}{2}\right)$ $\forall n \in \{2, 4, 8, 16, \dots\}$

$\exists\, c_1, c_2 > 0, \exists\, n_0$ a.î. $\forall n \geq n_0$

$c_1 \frac{n}{2} \log \frac{n}{2} \leq T\left(\frac{n}{2}\right) \leq c_2 \frac{n}{2} \log \frac{n}{2} \Big| \cdot 2$

$c_1 n \log \frac{n}{2} \leq 2T\left(\frac{n}{2}\right) \leq c_2 n \log \frac{n}{2} \Big| + n$

$c_1 n \log n - c_1 n + n \leq 2T\left(\frac{n}{2}\right) + n \leq c_2 n \log n - c_2 n + n$

$P(n)$ : $T(n) \in \Theta(n \log n)$

$\exists\, c_1', c_2' > 0, \exists\, n_0'$ a.î. $\forall n \geq n_0'$

$c_1' n \log n \leq T(n) \leq c_2' n \log n$

$c_1 n \log n \leq c_1 n \log n + n(1 - c_1) \leq T(n) \leq c_2 n \log n + n(1 - c_2) \leq c_2 n \log n$

$1 - c_1 \geq 0$ $\qquad$ $1 - c_2 \leq 0$

$c_1 \leq 1$ $\qquad$ $c_2 \geq 1$

$c_1 n \log n \leq T(n) \leq c_2 n \log n \Rightarrow T(n) \in \Theta(n \log n)$

# Teorema Master

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$T(n) = aT\left(\frac{n}{b}\right) + f(n)$, $a \geq 1$, $b \geq 1$ și sunt ct.

$f(n) \geq 0$, $f(n) \nearrow$

$\log_b a$

1. $f(n) \in O(n^{\log_b a - \varepsilon})$, $\varepsilon > 0 \Rightarrow T(n) \in \Theta(n^{\log_b a})$

2. $f(n) \in \Theta(n^{\log_b a} \log^k n)$, $k \geq 0 \Rightarrow T(n) \in \Theta(n^{\log_b a} \log^{k+1} n)$

3. $f(n) \in \Omega(n^{\log_b a + \varepsilon})$, $\varepsilon > 0$ $\Rightarrow T(n) \in \Theta(f(n))$

$a f\left(\frac{n}{b}\right) \leq c f(n)$ $\exists\, c < 1$ și n suf. de mare

Ex

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$T(n) = 2^n T\left(\frac{n}{2}\right) + \sqrt{n}\log n \not\Rightarrow$ nu pot

nu e ct.

$T(n) = \frac{1}{2} T\left(\frac{n}{4}\right) + 5 \not\Rightarrow$ nu pot

$a < 1$

$T(n) = 5T\left(\frac{n}{8}\right) - n! \not\Rightarrow$ nu pot ( $h(n) < 0$ )

d) $T(n) = 16\, T\left(\frac{n}{4}\right) + n!$

$a = 16,\ b = 4$

$\log_b a = \log_4 16 = 2 \to n^2$

Cazul 3

$n! \in \Omega(n^{2+\varepsilon}),\ \varepsilon > 0$

$a\, h\left(\frac{n}{b}\right) \le c\, h(n),\ c < 1$

$16 \left(\frac{n}{4}\right)! \le c\, n!$ $\quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

$c = \frac{1}{2}$ $\quad n = 4$

$16 \cdot 1! \le 4! \cdot \frac{1}{2} = 12$

$n = 8$

$16 \cdot 2! \le \frac{1}{2} \cdot 8!$ A $\Rightarrow T(n) \in \Theta(n!)$

b. $T(n) = T\left(\frac{n}{4}\right) + 1$ $\quad k = 0$

$a = 1,\ b = 4$

$\log_b a = \log_4 1 = 0$

$n^{\log_b a} = n^0 = 1$

Cazul 2

$1 \in \Theta(n^0) \Rightarrow T(n) \in \Theta(n^0 \log n)$

$(\Theta(1))$ $\quad T(n) \in \Theta(\log n)$

c. $T(n) = 4T\left(\frac{n}{2}\right) + n\sqrt{n}$

$a = 4$ $\quad \log_b a = \log_2 4 = 2 \to n^{\log_b a} = n^2$

$b = 2$

$n\sqrt{n} = n \cdot n^{\frac{1}{2}} = n^{\frac{3}{2}}$

Cazul 1: $n^{\frac{3}{2}} \in O(n^{2-\varepsilon})$, cum $\varepsilon = 0.1 \Rightarrow T(n) \in \Theta(n^2)$

$T(n) = 2T\left(\frac{n}{2}\right) + \underbrace{|\sin(n)|}_{h(n) \ge 0}$ $\not\Rightarrow$ nu se aplică

$h(n)$ nu e crescător

• $T(n) = T\left(\frac{n}{2}\right) + n(2 - \cos n)$, $n(2 - \cos n) \ge 0$ și $\nearrow$

$b=2$
$a=1$ $\Rightarrow \log_b a = 0$

cazul 3. $n(2-\cos n) \in \Omega(n^0)$

$a f(\frac{n}{b}) \le c f(n)$

$\frac{n}{2}(2-\cos\frac{n}{2}) \le c\, n(2-\cos n)$

$n = 2\pi k$, $k$ impar

$\pi k(2-\underbrace{\cos\pi k}_{=-1}) \le c 2\pi k(2-\underbrace{\cos 2\pi k}_{=1})$

$3\pi k \le 2c\pi k \Rightarrow c \ge \frac{3}{2}$, $c$ nu e $< 1 \Rightarrow$ nu se aplică

4.a. $T(n) = 2T(\frac{n}{2}) + n\log n$

$a=2$
$b=2$ $\Rightarrow \log_b a = 1 \Rightarrow n^{\log_b a} = n$

cazul 2 cu $k=1$

$n\log n \in \Theta(n^1 \log^1 n) \Rightarrow T(n) \in \Theta(n\log^2 n)$

# Fibonacci

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$T(n) = T(n-1) + T(n-2) + 1$

$P_n \; T(n) = ab^n + c$

$ab^n + c = ab^{n-1} + c + ab^{n-2} + c + 1$

$a\cancel{b}b^n + c = a(b^{n-1} + b^{n-2}) + 2c + 1$

$c = 2c + 1 \qquad b^n = b^{n-1} + b^{n-2} \;|\; : b^{n-2} \neq 0$

$c = -1 \qquad b^2 = b + 1$

$b^2 - b - 1 = 0$

$b = \frac{1 \pm \sqrt{5}}{2} = \underset{\simeq 1.61}{\varphi} \Rightarrow T(n) \in O(\varphi^n)$

$T(n) \in O(2^n)$ prin inducție

Cazul de bază

$T(1) = 1 \in O(2^1) = O(1)$

i.p. ind.

P.p. că $T(n) \in O(2^n) \rightarrow P(k-1) \quad \forall n \in \overline{1, k-1}$

Vreau să dem. $T(k) \in O(2^k) \rightarrow P(k)$

$T(k) = T(k-1) + T(k-2) + 1$

$T(k) = O(2^{k-1}) + O(2^{k-2}) + 1$

$T(k-1) \in O(2^{k-1}) \Leftrightarrow \exists c_1, \exists k_0^1$ a.î. $\forall k \geq k_0^1 \; T(k-1) \leq c_1 2^{k-1}$

$T(k-2) \in O(2^{k-2}) \Leftrightarrow \exists c_2, \exists k_0^2$ a.î. $\forall k \geq k_0^2 \; T(k-2) \leq c_2 2^{k-2} \leq c_2 2^{k-1} \leq c_1 2^{k-1}$ (iau $c_2 \leq c_1$)

$\overset{+}{\Rightarrow} T(k-1) + T(k-2)$

$T(k) \leq c_1 2^k \Rightarrow T(k) \in O(2^k)$

)

$) \le c_1 \cdot 2 \cdot 2^{k-1} = c_1 \cdot 2^k$