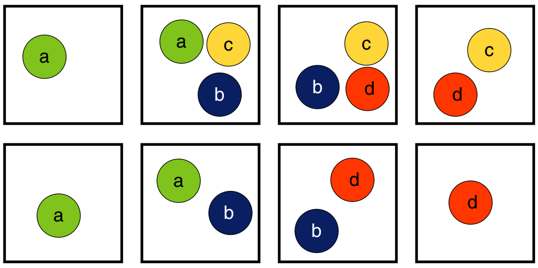
Bayesian Learning

Exercise 1 : Probability Basics (Conditional Independence)

There are eight boxes containing different colored balls as shown in the illustration below:



The balls can be green, blue, yellow, or red (also marked a, b, c, d in the figure). When picking one of the eight boxes at random, let A refer to the event "box contains a green ball," B to the event "box contains a blue ball, " C to the event "box contains a yellow ball," and D to the event "box contains a red ball." Hence, $A \cap B$ is the event "box contains both a green and a blue ball," etc.

(a) Calculate $P(A)$, $P(B)$, $P(C)$, and $P(D)$.

(b) Calculate $P(A \cap B)$, $P(A \cap C)$, $P(B \cap C)$, and $P(B \cap D)$.

(c) Check all that apply:

- [X] The events A and B are statistically independent.
- [ ] The events A and C are statistically independent.
- [ ] The events B and C are statistically independent.
- [X] The events B and D are statistically independent.

(d) Calculate $P(A \mid C)$, $P(B \mid C)$, and $P(A \cap B \mid C)$.

(e) Calculate $P(B \mid D)$, $P(C \mid D)$, and $P(B \cap C \mid D)$

(f) Check all that apply:

- [ ] The events A and B are conditionally independent given C.
- [X] The events B and C are conditionally independent given D.

Answer

ad a) $P(A) = P(B) = P(D) = \frac{4}{8} = \frac{1}{2}$; $P(C) = \frac{3}{8}$.

ad b) $P(A \cap B) = P(B \cap C) = P(B \cap D) = \frac{2}{8} = \frac{1}{4}$, $P(A \cap C) = \frac{1}{8}$.

ad c) A and B are independent because $P(A)P(B) = P(A \cap B)$ (or, equivalently, $P(A) = P(A \mid B)$). The same holds for B and D.

ad d) $P(A \mid C) = \frac{1}{3}$, $P(B \mid C) = \frac{2}{3}$, $P(A \cap B \mid C) = \frac{1}{3}$.

ad e) $P(B \mid D) = \frac{2}{4} = \frac{1}{2}$, $P(C \mid D) = \frac{2}{4} = \frac{1}{2}$, $P(B \cap C \mid D) = \frac{1}{4}$.

ad f) Since $P(B \mid D)P(C \mid D) = P(B \cap C \mid D)$, the events B and C are conditionally independent given D.

Exercise 2 : Bayes' Rule

A hospital database contains diagnoses (diseases) along with observed symptoms:

Patient	Diagnosis	Symptoms								
		S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
1	C_1	✓		✓		✓				
2	C_2		✓		✓	✓		✓		
3	C_3	✓		✓			✓		✓	
4	C_4		✓		✓	✓		✓		
5	C_3	✓		✓					✓	
6	C_5					✓				✓
7	C_3	✓		✓			✓			
8	C_2		✓					✓		

(a) Compute the prior probabilities $P(C_i)$.

$\boxed{\text{Answer}}$

$$P(C_1) = \frac{1}{8} = 0.125$$
$$P(C_2) = \frac{2}{8} = \frac{1}{4} = 0.25$$
$$P(C_3) = \frac{3}{8} = 0.375$$
$$P(C_4) = \frac{1}{8} = 0.125$$
$$P(C_5) = \frac{1}{8} = 0.125$$

(b) Compute the posterior probabilities $P(C_i \mid S_4)$ of the diagnoses C_i given symptom S_4.

$\boxed{\text{Answer}}$

First, compute $P(S_4|C_i)$, $i = 1, \ldots, 5$:

$$P(S_4|C_1) = \frac{P(S_4 \cap C_1)}{P(C_1)} = 0.0$$
$$P(S_4|C_2) = \frac{P(S_4 \cap C_2)}{P(C_2)} = 0.5$$
$$P(S_4|C_3) = \frac{P(S_4 \cap C_3)}{P(C_3)} = 0.0$$
$$P(S_4|C_4) = \frac{P(S_4 \cap C_4)}{P(C_4)} = 1.0$$
$$P(S_4|C_5) = \frac{P(S_4 \cap C_5)}{P(C_5)} = 0.0$$

Then, compute the a-posteriori probabilities $P(C_i|S_4)$, $i = 1, \dots, 5$:

$$P(C_1|S_4) = \frac{P(C_1) \cdot P(S_4|C_1)}{\sum_{j=1}^{5} P(C_j) \cdot P(S_4|C_j)} = \frac{\frac{1}{8} \cdot 0}{\frac{1}{8} \cdot 0 + \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 0} = \frac{0}{\frac{2}{8}} = 0$$

$$P(C_2|S_4) = \frac{P(C_2) \cdot P(S_4|C_2)}{\sum_{j=1}^{5} P(C_j) \cdot P(S_4|C_j)} = \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{8} \cdot 0 + \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 0} = \frac{\frac{1}{8}}{\frac{2}{8}} = \frac{1}{2}$$

$$P(C_3|S_4) = \frac{P(C_3) \cdot P(S_4|C_3)}{\sum_{j=1}^{5} P(C_j) \cdot P(S_4|C_j)} = \frac{\frac{3}{8} \cdot 0}{\frac{1}{8} \cdot 0 + \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 0} = \frac{0}{\frac{2}{8}} = 0$$

$$P(C_4|S_4) = \frac{P(C_4) \cdot P(S_4|C_4)}{\sum_{j=1}^{5} P(C_j) \cdot P(S_4|C_j)} = \frac{\frac{1}{8} \cdot 1}{\frac{1}{8} \cdot 0 + \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 0} = \frac{\frac{1}{8}}{\frac{2}{8}} = \frac{1}{2}$$

$$P(C_5|S_4) = \frac{P(C_5) \cdot P(S_4|C_5)}{\sum_{j=1}^{5} P(C_j) \cdot P(S_4|C_j)} = \frac{\frac{1}{8} \cdot 0}{\frac{1}{8} \cdot 0 + \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 0} = \frac{0}{\frac{2}{8}} = 0$$

Exercise 3 : Naïve Bayes

Given is the following dataset to classify whether a dog is dangerous or well-behaved in character:

Color	Fur	Size	Character (C)
brown	ragged	small	well-behaved
black	ragged	big	dangerous
black	smooth	big	dangerous
black	curly	small	well-behaved
white	curly	small	well-behaved
white	smooth	small	dangerous
red	ragged	big	well-behaved

(a) Determine the parameters for a Naïve Bayes classifier on this dataset.

Answer

Class priors:

$$P(\text{well-behaved}) = \frac{4}{7}$$
$$P(\text{dangerous}) = \frac{3}{7}$$

Attribute-value probabilities given class:

Attribute	Value	Class	$P(\text{Attribute} = \text{Value} \mid \text{Class})$
Color	brown	well-behaved	1.00
Color	brown	dangerous	0.00
Color	black	well-behaved	$\frac{1}{4} = 0.25$
Color	black	dangerous	$\frac{2}{3} = 0.6\bar{6}$
Color	white	well-behaved	$\frac{1}{4} = 0.25$
Color	white	dangerous	$\frac{1}{3} = 0.3\bar{3}$
Color	red	well-behaved	1.00
Color	red	dangerous	0.00
Fur	ragged	well-behaved	$\frac{1}{2} = 0.50$
Fur	ragged	dangerous	$\frac{1}{3} = 0.3\bar{3}$
Fur	smooth	well-behaved	0.00
Fur	smooth	dangerous	$\frac{2}{3} = 0.6\bar{6}$
Fur	curly	well-behaved	$\frac{1}{2} = 0.50$
Fur	curly	dangerous	0.00
Size	small	well-behaved	$\frac{3}{4} = 0.75$
Size	small	dangerous	$\frac{1}{3} = 0.3\bar{3}$
Size	big	well-behaved	$\frac{1}{4} = 0.25$
Size	big	dangerous	$\frac{2}{3} = 0.6\bar{6}$

(b) Classify the new example (Color=black, Fur=ragged, Size=small) using your Naïve Bayes classifier.

Answer

For reduced verbosity, let the following events be defined: $A_1 = (\text{Class=well-behaved})$, $A_2 = (\text{Class=dangerous})$, $B_1 = (\text{Color=black})$, $B_2 = (\text{Fur=ragged})$, $B_3 = (\text{Size=small})$.

With the Naïve Bayes assumption we have:

$$P(A_1 \mid B_1, B_2, B_3) = \frac{P(A_1) \cdot P(B_1, B_2, B_3 \mid A_1)}{P(B_1, B_2, B_3)} \overset{\text{NB}}{\approx} \frac{P(A_1) \cdot \prod_{j=1}^{3} P(B_j \mid A_1)}{\sum_{i=1}^{2} P(A_i) \prod_{j=1}^{3} P(B_j \mid A_i)}$$

and equivalently for A_2; The denominator in both cases is:

$(P(A_1) \cdot P(B_1 \mid A_1) \cdot P(B_2 \mid A_1) \cdot P(B_3 \mid A_1)) +$
$(P(A_2) \cdot P(B_1 \mid A_2) \cdot P(B_2 \mid A_2) \cdot P(B_3 \mid A_2)) = \left(\frac{4}{7} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{3}{4}\right) + \left(\frac{3}{7} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}\right) = \frac{3}{56} + \frac{2}{63} = \frac{43}{504} \approx 0.085$

Hence, we get:

$P(A_1 \mid B_1, B_2, B_3) \approx \frac{\frac{4}{7} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{3}{4}}{\frac{43}{504}} = \frac{27}{43} \approx 0.628$

and

$P(A_2 \mid B_1, B_2, B_3) \approx \frac{\frac{3}{7} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}}{\frac{17}{252}} = \frac{8}{17} \approx 0.471$

Thus, A_1 is more likely under the Naïve Bayes assumption. Note how the class probabilities do not add up to 1, since Naïve Bayes is only an approximation of the true values.