# P8160 Simulation Project - Hierarchical Logistic Model for Mulit-Center Clinical Trial

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## Introduction

*multicenter clinical trial???*

In this project, our objective is to design a simulation study that estimates the population-level probability of adverse events

We are interested in estimating the overall (population-level) probability of an adverse event, marginalizing over both X (across all patients) and the random effect distribution b (across all clinics)

## Statistical Methods

*overall description of what we did, the code in this section specifies functions*

### 1. Initialization and Function Definitions

We specified the clinic random effect (b) as b ~ $t_5$ and the patient-level covariates (X) as X ~ Gamma(2, 1).

```
# logistic function
logistic <- function(z) {
  1 / (1 + exp(-z))
}

# Set model parameters
alpha <- 0        # Intercept
beta  <- 1        # Covariate coefficient

# Define random generators for non-normal distributions
#                   of random effect b and covariate x
# Example: b ~ t_5 (heavy-tailed distribution)
r_b <- function(n) {
  rt(n, df = 5)
}

# Example: x ~ Gamma(shape=2, rate=1)
r_x <- function(n) {
  rgamma(n, shape = 2, rate = 1)
}
```

### 2. Sampling from Simple Monte Carlo (MC)

```
simpleMC <- function(N) {
  # Sample b from f(b)
```

```
  b_samp <- r_b(N)
  # Sample x from f(x)
  x_samp <- r_x(N)

  # Compute p_i = logistic(alpha + b_i + beta*x_i)
  p_values <- logistic(alpha + b_samp + beta * x_samp)

  # Simple MC estimate is the average of p_values
  est <- mean(p_values)

  return(list(est = est))
}
```

### 3. Designing Control Variate (CV)

```
# 3.1 Define a control variate g(b, x)
#     It should be correlated with the target function but have a known or easily
#     approximated expectation. As an example, we use a probit-based function.

g_function <- function(b, x) {
  # Example: replace logistic with pnorm (probit)
  z <- alpha + 0.8 * b + 0.8 * x  # 0.8 is a heuristic factor here
  return(pnorm(z))
}

# 3.2 Approximate E[g] using a large-scale MC, since we might not have a closed-form
approx_Eg <- function(N = 5e5) {
  b_samp <- r_b(N)
  x_samp <- r_x(N)
  mean(g_function(b_samp, x_samp))
}

# 3.3 Control variate MC estimate
cvMC <- function(N, Eg_true = NULL) {
  if(is.null(Eg_true)) {
    # If not provided, approximate E[g] with a bigger MC sample
    Eg_true <- approx_Eg(N = 5e5)
  }
  # Sample from original distributions
  b_samp <- r_b(N)
  x_samp <- r_x(N)

  # Target function p(b,x)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)
  # Control variate g(b,x)
  g_vals <- g_function(b_samp, x_samp)

  # (1) Simple estimates of p and g
  p_mean <- mean(p_vals)
  g_mean <- mean(g_vals)

  # (2) Optimal gamma via covariance
  cov_pg <- cov(p_vals, g_vals)
```

```
  var_g  <- var(g_vals)
  gamma_opt <- cov_pg / var_g

  # (3) Construct CV estimator
  p_cv <- p_mean - gamma_opt * (g_mean - Eg_true)

  # This returns the CV estimate, gamma, and other useful metrics
  return(list(est = p_cv))
}
```

**4. Importance Sampling (IS)**

```
# 4.1 Define the auxiliary distribution h(b,x)
#     Here we show an example of independent sampling from h(b)*h(x).
#     We'll pick t_3 for b (heavier tail) and tweak Gamma parameters for x.

# b ~ t_3
r_b_IS <- function(n) {
  rt(n, df = 3)
}
# PDF of b for t_3
d_b_IS <- function(b) {
  dt(b, df = 3)
}

# x ~ Gamma(shape=2, rate=0.5)
r_x_IS <- function(n) {
  rgamma(n, shape = 2, rate = 0.5)
}
# PDF of x for Gamma(shape=2, rate=0.5)
d_x_IS <- function(x) {
  dgamma(x, shape = 2, rate = 0.5)
}

# 4.2 Implement Importance Sampling
IS_MC <- function(N) {
  # Sample from h(b,x)
  b_samp <- r_b_IS(N)
  x_samp <- r_x_IS(N)

  # Compute original PDF values f(b), f(x)
  fb_vals <- dt(b_samp, df = 5)              # original t_5 for b
  fx_vals <- dgamma(x_samp, shape = 2, rate = 1)  # original Gamma for x

  # Compute auxiliary PDF values h(b), h(x)
  hb_vals <- d_b_IS(b_samp)
  hx_vals <- d_x_IS(x_samp)

  # Weights w_i = [f(b_i)*f(x_i)] / [h(b_i)*h(x_i)]
  w <- (fb_vals * fx_vals) / (hb_vals * hx_vals)

  # Target function p(b,x)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)
```

```r
  # IS estimate (unbiased if we average p*w)
  est <- mean(p_vals * w)

  # Rough variance estimate
  #p_var <- mean((p_vals * w)^2) - p_hat^2
  #se  <- sqrt(p_var / N)

  #return(list(est = est, se = se, w_mean = mean(w)))
  return(list(est = est))
}
```

# Results

**5. Comparing Bias, Variance, and CPU Time for Simple MC, CV, and Importance Sampling**

```r
# 5.1. Obtain a "true value" via a large-scale Simple MC
set.seed(9999)
N_large <- 5e6

res_truth <- simpleMC(N_large)
true_value <- res_truth$est
# cat("Approx. True Value =", true_value, "\n")


# 5.2. Repeat simulations for smaller N and compare bias, variance
simulate_three_methods <- function(N, Eg_true) {
  # Simple MC
  res_mc <- simpleMC(N)
  # CV
  res_cv <- cvMC(N, Eg_true = Eg_true)
  # IS
  res_is <- IS_MC(N)

  c(mc = res_mc$est, cv = res_cv$est, is = res_is$est)
}

set.seed(1234)
N_reps <- 50        # number of repeated simulations
N_small <- 1e5      # sample size for each repeated run


# Pre-calculate Eg_true for CV (to avoid doing it multiple times)
Eg_val <- approx_Eg(N=5e5)

results_matrix <- replicate(N_reps, simulate_three_methods(N_small, Eg_true = Eg_val))
# results_matrix is now 3 x N_reps, each column is one replicate

mc_est <- results_matrix["mc", ]
cv_est <- results_matrix["cv", ]
is_est <- results_matrix["is", ]

# Compute mean estimate, bias, variance
```

```r
mc_mean <- mean(mc_est)
cv_mean <- mean(cv_est)
is_mean <- mean(is_est)

mc_bias <- mc_mean - true_value
cv_bias <- cv_mean - true_value
is_bias <- is_mean - true_value

mc_var <- var(mc_est)
cv_var <- var(cv_est)
is_var <- var(is_est)

df_comparison <- data.frame(
  method   = c("MC", "CV", "IS", "True"),
  mean_est = c(mc_mean, cv_mean, is_mean, true_value),
  bias     = c(mc_bias, cv_bias, is_bias, NA),
  variance = c(mc_var, cv_var, is_var, NA)
)


# 5.3 Compare CPU time
test_time <- microbenchmark(
  MC = { simpleMC(N_small) },
  CV = { cvMC(N_small, Eg_true = Eg_val) },
  IS = { IS_MC(N_small) },
  times = 10  # repeat each method 10 times
)
```

Table 1: Estimated model mean, bias, and variance for each method using 100,000 samples; 'True' value is the MC result with 6,000,000 samples

| Method | Estimated Mean | Bias | Variance |
|---|---|---|---|
| MC | 0.7857719 | -0.000144669 | 4.420e-07 |
| CV | 0.7857767 | -0.000139931 | 8.000e-09 |
| IS | 0.7855314 | -0.000385163 | 3.016e-06 |
| True | 0.7859166 | NA | NA |

Table 2: CPU time statistics (ms) for each method

| Method | Min. | Lower Quartile | Mean | Median | Upper Quartile | Max. | # Eval |
|---|---|---|---|---|---|---|---|
| MC | 13.72758 | 13.86915 | 13.99927 | 14.00097 | 14.05886 | 14.34889 | 10 |
| CV | 16.63850 | 16.65084 | 16.86150 | 16.91703 | 16.99626 | 17.06584 | 10 |
| IS | 34.85221 | 35.14815 | 36.51008 | 35.43017 | 35.82031 | 41.48388 | 10 |

**Monte Carlo (MC)** is the fastest because it only generates random samples and computes the mean, with O(N) complexity and no additional calculations.

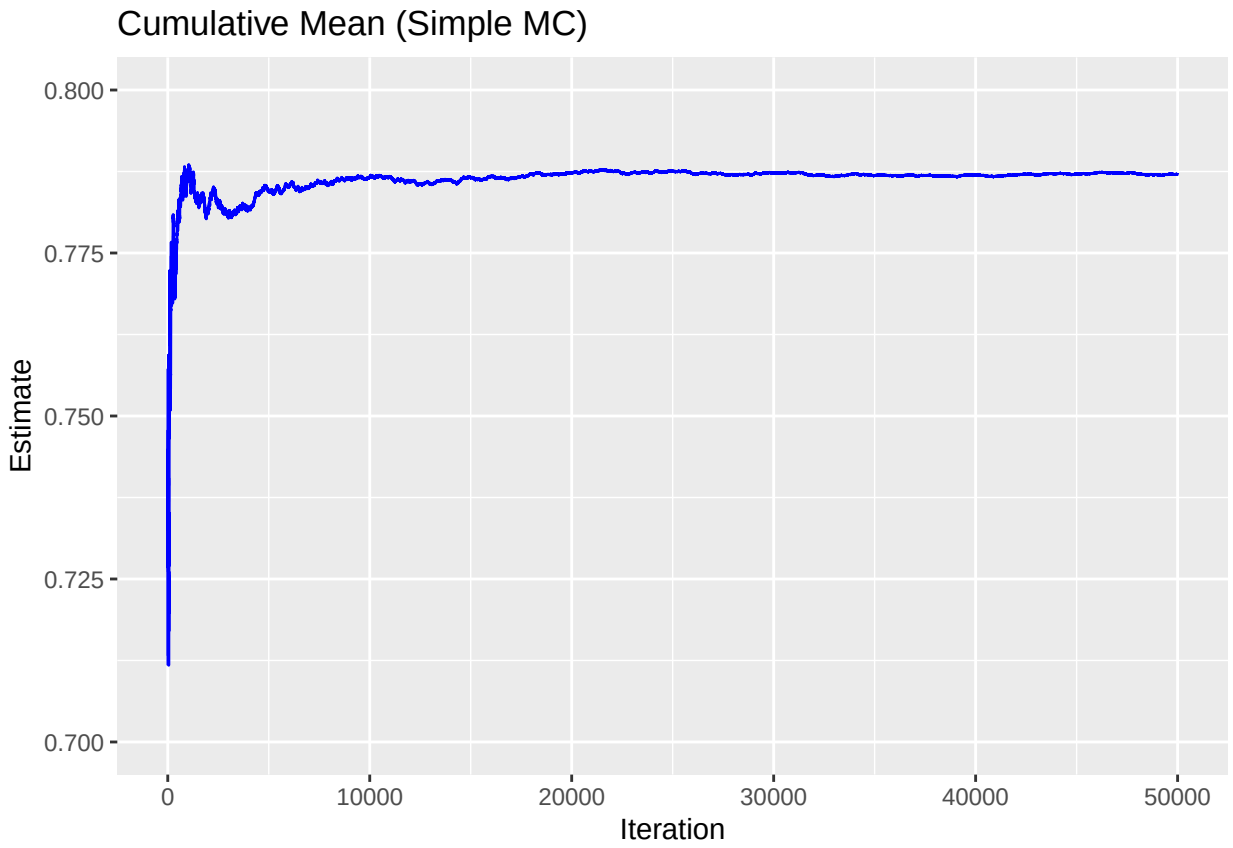
**Control Variates (CV)** is slightly slower due to the extra step of computing a regression coefficient to reduce variance, but it improves estimation accuracy.

**Importance Sampling (IS)** is the slowest because it requires computing importance weights, normalizing them, and handling probability density functions, making it computationally more expensive (O(N log N) or higher).

### 6. Cumulative Convergence Plots (Extra Credit)

#### Simple MC

```
# 6.1 Example of cumulative mean for Simple MC
cumulative_MC <- function(N) {
  b_samp <- r_b(N)
  x_samp <- r_x(N)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)
  cum_mean <- cumsum(p_vals) / seq_len(N)
  return(cum_mean)
}

# 6.2 Plot cumulative mean
set.seed(999)
N <- 5e4
cm_mc <- cumulative_MC(N)

df_mc <- data.frame(
  iter = 1:N,
  mean_est = cm_mc
)

ggplot(df_mc, aes(x = iter, y = mean_est)) +
  geom_line(color = "blue") +
  labs(title = "Cumulative Mean (Simple MC)",
       x = "Iteration",
       y = "Estimate") +
  ylim(c(0.7, 0.8))
```

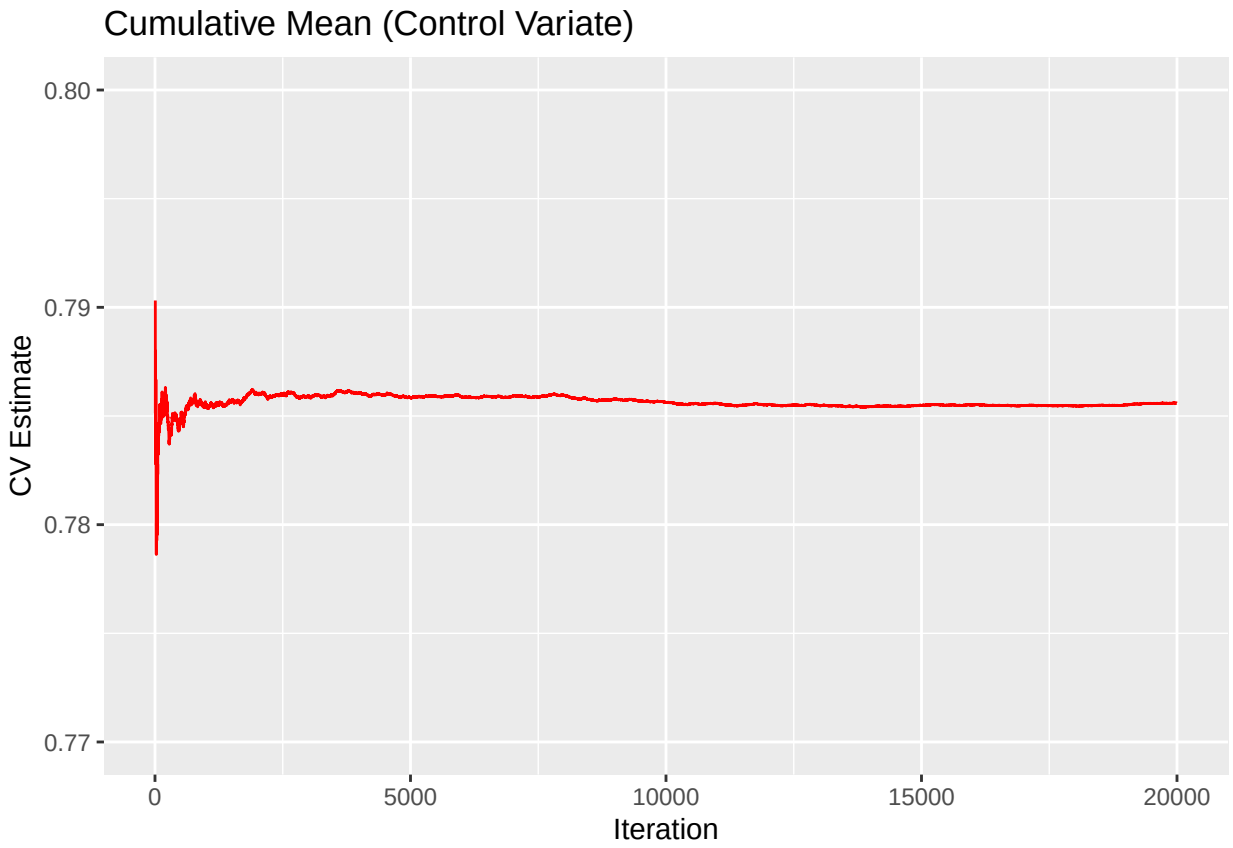



**Control Variate Cumulative Mean**

```r
# This function returns a vector of CV estimates for i from 1 to N.
cumulative_CV <- function(N, Eg_true = NULL, largeN_approx = 5e5) {
  if (is.null(Eg_true)) {
    # If E[g] is not provided, approximate with a larger MC
    Eg_true <- approx_Eg(N = largeN_approx)
  }

  # Sample from original distributions
  b_samp <- r_b(N)
  x_samp <- r_x(N)

  # Compute p(b,x) and g(b,x)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)
  g_vals <- g_function(b_samp, x_samp)

  # We will keep track of partial sums to compute means, covariance, etc.
  cum_p  <- cumsum(p_vals)
  cum_g  <- cumsum(g_vals)
  cum_pg <- cumsum(p_vals * g_vals)   # needed for Cov(p,g)
  cum_g2 <- cumsum(g_vals^2)          # needed for Var(g)

  # Vector to store CV estimates at each iteration
  pCV_cum <- numeric(N)
```

```r
  # The first data point (i=1) does not allow us to compute a covariance
  # We can simply set pCV_cum[1] to p_vals[1], or NA, etc.
  pCV_cum[1] <- p_vals[1]

  # Loop over i = 2 to N
  for (i in 2:N) {
    # partial means
    p_bar_i <- cum_p[i] / i
    g_bar_i <- cum_g[i] / i

    # Cov(p,g) = E[ p*g ] - E[p]*E[g ]
    # partial estimate of E[p*g] = cum_pg[i] / i
    pg_bar_i <- cum_pg[i] / i
    cov_pg_i <- pg_bar_i - (p_bar_i * g_bar_i)

    # Var(g) = E[g^2] - (E[g])^2
    g2_bar_i <- cum_g2[i] / i
    var_g_i  <- g2_bar_i - (g_bar_i^2)

    # Optimal gamma
    gamma_i <- cov_pg_i / var_g_i

    # CV estimate at iteration i
    pCV_cum[i] <- p_bar_i - gamma_i * (g_bar_i - Eg_true)
  }

  return(pCV_cum)
}

# Example usage and plotting:
set.seed(1234)
N <- 20000
cv_cum_values <- cumulative_CV(N)

df_cv <- data.frame(
  iter = 1:N,
  cv_est = cv_cum_values
)

library(ggplot2)
ggplot(df_cv, aes(x = iter, y = cv_est)) +
  geom_line(color = "red") +
  labs(title = "Cumulative Mean (Control Variate)",
       x = "Iteration",
       y = "CV Estimate") +
  ylim(c(0.77, 0.8))
```

```
## Warning: Removed 5 rows containing missing values or values outside the scale range
## (`geom_line()`).
```

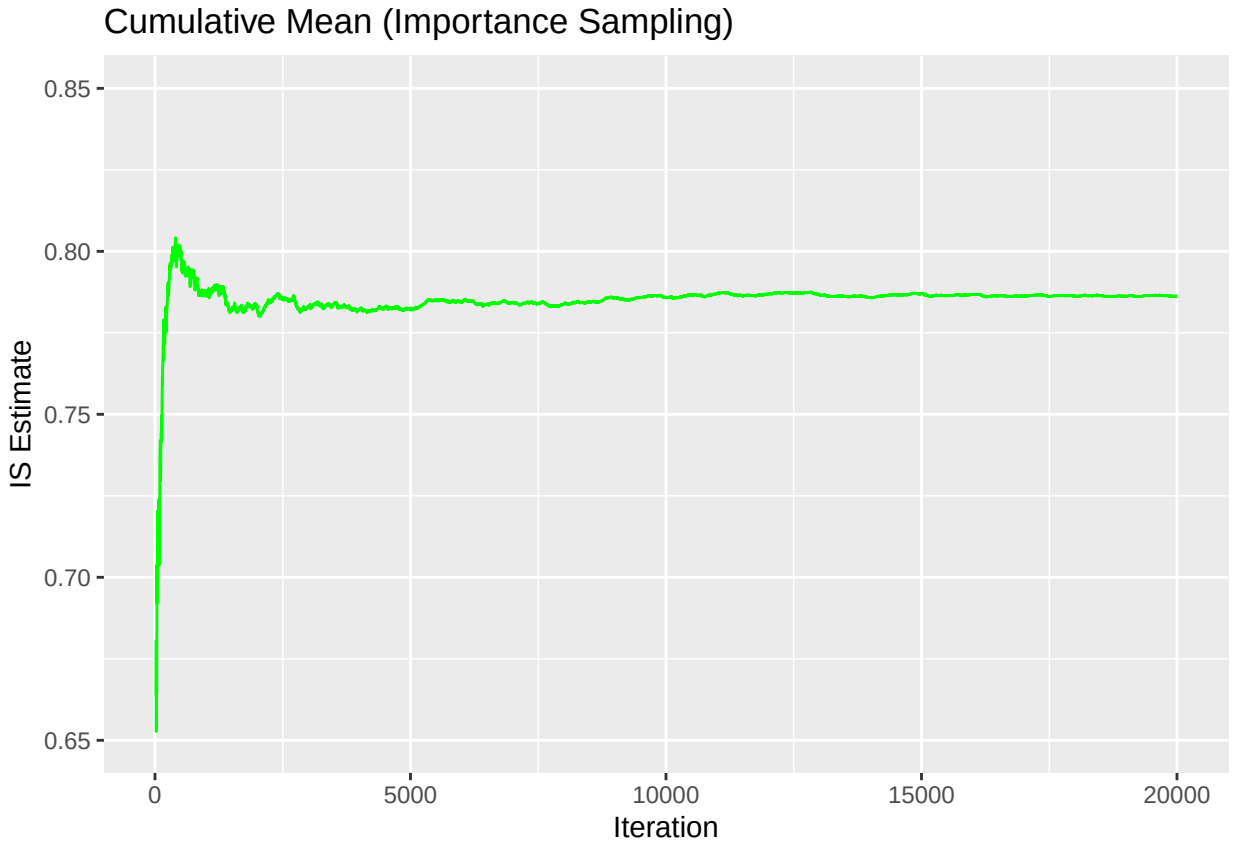



**Importance Sampling Cumulative Mean**

```
cumulative_IS <- function(N) {
  # Sample from auxiliary distribution h(b,x)
  b_samp <- r_b_IS(N)
  x_samp <- r_x_IS(N)

  # Compute original PDF f(b), f(x)
  fb_vals <- dt(b_samp, df = 5)           # original b ~ t_5
  fx_vals <- dgamma(x_samp, shape=2, rate=1)   # original x ~ Gamma(2,1)

  # Compute auxiliary PDF h(b), h(x)
  hb_vals <- d_b_IS(b_samp)
  hx_vals <- d_x_IS(x_samp)

  # Weights
  w <- (fb_vals * fx_vals) / (hb_vals * hx_vals)

  # Target function p(b,x)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)

  # Build cumulative sums
  cum_w  <- cumsum(w)
  cum_pw <- cumsum(p_vals * w)

  # IS estimate at iteration i: cum_pw[i] / cum_w[i]
```

```
  pIS_cum <- cum_pw / cum_w

  return(pIS_cum)
}

## Example usage and plotting:
set.seed(5678)
N <- 20000
is_cum_values <- cumulative_IS(N)

df_is <- data.frame(
  iter = 1:N,
  is_est = is_cum_values
)

ggplot(df_is, aes(x = iter, y = is_est)) +
  geom_line(color = "green") +
  labs(title = "Cumulative Mean (Importance Sampling)",
       x = "Iteration",
       y = "IS Estimate") +
  ylim(c(0.65, 0.85))
```

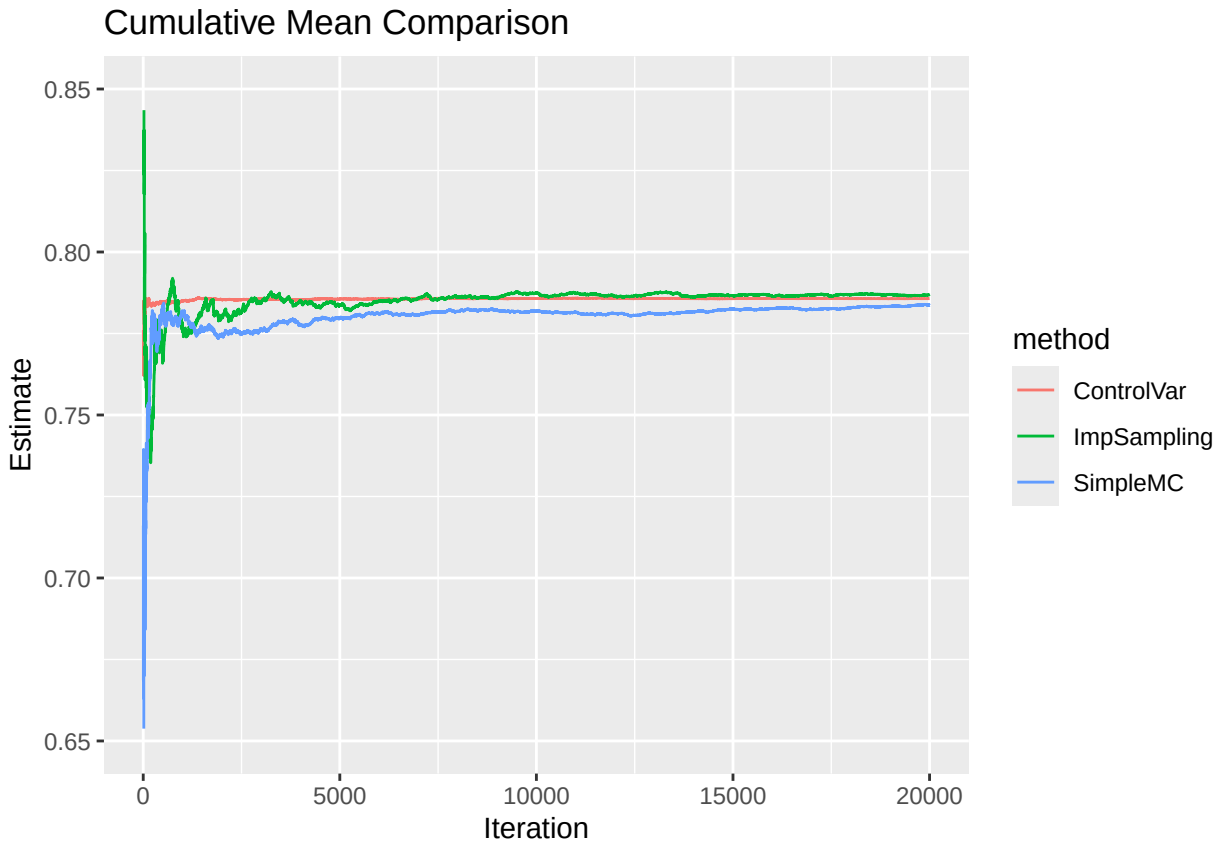
```
## Warning: Removed 20 rows containing missing values or values outside the scale range
## (`geom_line()`).
```




```
N <- 20000
set.seed(999)
```

```r
# 1. Simple MC
cm_mc <- cumulative_MC(N)
df_mc <- data.frame(iter = 1:N, method = "SimpleMC", estimate = cm_mc)

# 2. CV
cm_cv <- cumulative_CV(N)
df_cv <- data.frame(iter = 1:N, method = "ControlVar", estimate = cm_cv)

# 3. IS
cm_is <- cumulative_IS(N)
df_is <- data.frame(iter = 1:N, method = "ImpSampling", estimate = cm_is)

# Combine
df_all <- rbind(df_mc, df_cv, df_is)

ggplot(df_all, aes(x = iter, y = estimate, color = method)) +
  geom_line() +
  labs(title = "Cumulative Mean Comparison",
       x = "Iteration",
       y = "Estimate") +
  ylim(c(0.65, 0.85))
```

```
## Warning: Removed 20 rows containing missing values or values outside the scale range
## (`geom_line()`).
```




```r
  theme_bw()
```

```
## List of 136
##  $ line                            :List of 6
##   ..$ colour       : chr "black"
##   ..$ linewidth    : num 0.5
##   ..$ linetype     : num 1
##   ..$ lineend      : chr "butt"
##   ..$ arrow        : logi FALSE
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_line" "element"
##  $ rect                            :List of 5
##   ..$ fill         : chr "white"
##   ..$ colour       : chr "black"
##   ..$ linewidth    : num 0.5
##   ..$ linetype     : num 1
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_rect" "element"
##  $ text                            :List of 11
##   ..$ family       : chr ""
##   ..$ face         : chr "plain"
##   ..$ colour       : chr "black"
##   ..$ size         : num 11
##   ..$ hjust        : num 0.5
##   ..$ vjust        : num 0.5
##   ..$ angle        : num 0
##   ..$ lineheight   : num 0.9
##   ..$ margin       : 'margin' num [1:4] 0points 0points 0points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : logi FALSE
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ title                           : NULL
##  $ aspect.ratio                    : NULL
##  $ axis.title                      : NULL
##  $ axis.title.x                    :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : NULL
##   ..$ vjust        : num 1
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 2.75points 0points 0points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.title.x.top                :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : NULL
##   ..$ vjust        : num 0
```

```
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 0points 0points 2.75points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.title.x.bottom             : NULL
##  $ axis.title.y                    :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : NULL
##   ..$ vjust        : num 1
##   ..$ angle        : num 90
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 0points 2.75points 0points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.title.y.left               : NULL
##  $ axis.title.y.right              :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : NULL
##   ..$ vjust        : num 1
##   ..$ angle        : num -90
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 0points 0points 0points 2.75points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.text                       :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : chr "grey30"
##   ..$ size         : 'rel' num 0.8
##   ..$ hjust        : NULL
##   ..$ vjust        : NULL
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : NULL
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.text.x                     :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
```

```
##   ..$ size         : NULL
##   ..$ hjust        : NULL
##   ..$ vjust        : num 1
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 2.2points 0points 0points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.text.x.top                :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : NULL
##   ..$ vjust        : num 0
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 0points 0points 2.2points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.text.x.bottom             : NULL
##  $ axis.text.y                    :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : num 1
##   ..$ vjust        : NULL
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 0points 2.2points 0points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.text.y.left               : NULL
##  $ axis.text.y.right              :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : num 0
##   ..$ vjust        : NULL
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 0points 0points 0points 2.2points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
```

```
##  $ axis.text.theta                : NULL
##  $ axis.text.r                    :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : num 0.5
##   ..$ vjust        : NULL
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : 'margin' num [1:4] 0points 2.2points 0points 2.2points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ axis.ticks                     :List of 6
##   ..$ colour       : chr "grey20"
##   ..$ linewidth    : NULL
##   ..$ linetype     : NULL
##   ..$ lineend      : NULL
##   ..$ arrow        : logi FALSE
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_line" "element"
##  $ axis.ticks.x                   : NULL
##  $ axis.ticks.x.top               : NULL
##  $ axis.ticks.x.bottom            : NULL
##  $ axis.ticks.y                   : NULL
##  $ axis.ticks.y.left              : NULL
##  $ axis.ticks.y.right             : NULL
##  $ axis.ticks.theta               : NULL
##  $ axis.ticks.r                   : NULL
##  $ axis.minor.ticks.x.top         : NULL
##  $ axis.minor.ticks.x.bottom      : NULL
##  $ axis.minor.ticks.y.left        : NULL
##  $ axis.minor.ticks.y.right       : NULL
##  $ axis.minor.ticks.theta         : NULL
##  $ axis.minor.ticks.r             : NULL
##  $ axis.ticks.length              : 'simpleUnit' num 2.75points
##   ..- attr(*, "unit")= int 8
##  $ axis.ticks.length.x            : NULL
##  $ axis.ticks.length.x.top        : NULL
##  $ axis.ticks.length.x.bottom     : NULL
##  $ axis.ticks.length.y            : NULL
##  $ axis.ticks.length.y.left       : NULL
##  $ axis.ticks.length.y.right      : NULL
##  $ axis.ticks.length.theta        : NULL
##  $ axis.ticks.length.r            : NULL
##  $ axis.minor.ticks.length        : 'rel' num 0.75
##  $ axis.minor.ticks.length.x      : NULL
##  $ axis.minor.ticks.length.x.top  : NULL
##  $ axis.minor.ticks.length.x.bottom: NULL
##  $ axis.minor.ticks.length.y      : NULL
##  $ axis.minor.ticks.length.y.left : NULL
##  $ axis.minor.ticks.length.y.right : NULL
```

```
##  $ axis.minor.ticks.length.theta   : NULL
##  $ axis.minor.ticks.length.r       : NULL
##  $ axis.line                       : list()
##   ..- attr(*, "class")= chr [1:2] "element_blank" "element"
##  $ axis.line.x                     : NULL
##  $ axis.line.x.top                 : NULL
##  $ axis.line.x.bottom              : NULL
##  $ axis.line.y                     : NULL
##  $ axis.line.y.left                : NULL
##  $ axis.line.y.right               : NULL
##  $ axis.line.theta                 : NULL
##  $ axis.line.r                     : NULL
##  $ legend.background               :List of 5
##   ..$ fill         : NULL
##   ..$ colour       : logi NA
##   ..$ linewidth    : NULL
##   ..$ linetype     : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_rect" "element"
##  $ legend.margin                   : 'margin' num [1:4] 5.5points 5.5points 5.5points 5.5points
##   ..- attr(*, "unit")= int 8
##  $ legend.spacing                  : 'simpleUnit' num 11points
##   ..- attr(*, "unit")= int 8
##  $ legend.spacing.x                : NULL
##  $ legend.spacing.y                : NULL
##  $ legend.key                      : NULL
##  $ legend.key.size                 : 'simpleUnit' num 1.2lines
##   ..- attr(*, "unit")= int 3
##  $ legend.key.height               : NULL
##  $ legend.key.width                : NULL
##  $ legend.key.spacing              : 'simpleUnit' num 5.5points
##   ..- attr(*, "unit")= int 8
##  $ legend.key.spacing.x            : NULL
##  $ legend.key.spacing.y            : NULL
##  $ legend.frame                    : NULL
##  $ legend.ticks                    : NULL
##  $ legend.ticks.length             : 'rel' num 0.2
##  $ legend.axis.line                : NULL
##  $ legend.text                     :List of 11
##   ..$ family       : NULL
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : 'rel' num 0.8
##   ..$ hjust        : NULL
##   ..$ vjust        : NULL
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : NULL
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ legend.text.position            : NULL
##  $ legend.title                    :List of 11
##   ..$ family       : NULL
```

```
##   ..$ face         : NULL
##   ..$ colour       : NULL
##   ..$ size         : NULL
##   ..$ hjust        : num 0
##   ..$ vjust        : NULL
##   ..$ angle        : NULL
##   ..$ lineheight   : NULL
##   ..$ margin       : NULL
##   ..$ debug        : NULL
##   ..$ inherit.blank: logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
##  $ legend.title.position           : NULL
##  $ legend.position                 : chr "right"
##  $ legend.position.inside          : NULL
##  $ legend.direction                : NULL
##  $ legend.byrow                    : NULL
##  $ legend.justification            : chr "center"
##  $ legend.justification.top        : NULL
##  $ legend.justification.bottom     : NULL
##  $ legend.justification.left       : NULL
##  $ legend.justification.right      : NULL
##  $ legend.justification.inside     : NULL
##  $ legend.location                 : NULL
##  $ legend.box                      : NULL
##  $ legend.box.just                 : NULL
##  $ legend.box.margin               : 'margin' num [1:4] 0cm 0cm 0cm 0cm
##   ..- attr(*, "unit")= int 1
##  $ legend.box.background           : list()
##   ..- attr(*, "class")= chr [1:2] "element_blank" "element"
##  $ legend.box.spacing              : 'simpleUnit' num 11points
##   ..- attr(*, "unit")= int 8
##   [list output truncated]
##  - attr(*, "class")= chr [1:2] "theme" "gg"
##  - attr(*, "complete")= logi TRUE
##  - attr(*, "validate")= logi TRUE
```

## Discussion and Practical Implications

## Conclusion