

P8160 Simulation Project - Hierarchical Logistic Model for Multicenter Clinical Trial

Group 3A: Kate Colvin, Xuanyu Guo, Dang Lin, and Boxiang Tang

Introduction and Background

A multicenter clinical trial is a trial that enrolls participants at multiple different medical institutions. These different trial sites can have different characteristics, such as patient demographics, treatment practices, or geographic factors, that impact the outcome of trial patients. Such variability between trial sites can complicate trial outcomes, and

In this project, we designed a simulation study that estimates the population-level probability of adverse events, accounting for differences in demographic variables between patients and across clinics.

Statistical Methods

overall description of what we did, the code in this section specifies functions

1. Initialization and Function Definitions

We specified the clinic random effect (b) as $b \sim t_5$ and the patient-level covariates (X) as $X \sim \text{Gamma}(2, 1)$.

```
# logistic function
logistic <- function(z) {
  1 / (1 + exp(-z))
}

# Set model parameters
alpha <- 0      # Intercept
beta  <- 1      # Covariate coefficient

# Define random generators for non-normal distributions
#               of random effect b and covariate x
# Example: b ~ t_5 (heavy-tailed distribution)
r_b <- function(n) {
  rt(n, df = 5)
}

# Example: x ~ Gamma(shape=2, rate=1)
r_x <- function(n) {
  rgamma(n, shape = 2, rate = 1)
}
```

2. Sampling from Simple Monte Carlo (MC)

```

simpleMC <- function(N) {
  # Sample b from f(b)
  b_samp <- r_b(N)
  # Sample x from f(x)
  x_samp <- r_x(N)

  # Compute p_i = logistic(alpha + b_i + beta*x_i)
  p_values <- logistic(alpha + b_samp + beta * x_samp)

  # Simple MC estimate is the average of p_values
  est <- mean(p_values)

  return(list(est = est))
}

```

3. Designing Control Variate (CV)

```

# 3.1 Define a control variate g(b, x)
# It should be correlated with the target function but have a known or easily
# approximated expectation. As an example, we use a probit-based function.

g_function <- function(b, x) {
  # Example: replace logistic with pnorm (probit)
  z <- alpha + 0.8 * b + 0.8 * x # 0.8 is a heuristic factor here
  return(pnorm(z))
}

# 3.2 Approximate E[g] using a large-scale MC, since we might not have a closed-form
approx_Eg <- function(N = 5e5) {
  b_samp <- r_b(N)
  x_samp <- r_x(N)
  mean(g_function(b_samp, x_samp))
}

# 3.3 Control variate MC estimate
cvMC <- function(N, Eg_true = NULL) {
  if(is.null(Eg_true)) {
    # If not provided, approximate E[g] with a bigger MC sample
    Eg_true <- approx_Eg(N = 5e5)
  }
  # Sample from original distributions
  b_samp <- r_b(N)
  x_samp <- r_x(N)

  # Target function p(b,x)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)
  # Control variate g(b,x)
  g_vals <- g_function(b_samp, x_samp)

  # (1) Simple estimates of p and g
  p_mean <- mean(p_vals)
  g_mean <- mean(g_vals)
}

```

```

# (2) Optimal gamma via covariance
cov_pg <- cov(p_vals, g_vals)
var_g <- var(g_vals)
gamma_opt <- cov_pg / var_g

# (3) Construct CV estimator
p_cv <- p_mean - gamma_opt * (g_mean - Eg_true)

# This returns the CV estimate, gamma, and other useful metrics
return(list(est = p_cv))
}

```

4. Importance Sampling (IS)

```

# 4.1 Define the auxiliary distribution h(b,x)
# Here we show an example of independent sampling from h(b)*h(x).
# We'll pick t_3 for b (heavier tail) and tweak Gamma parameters for x.

# b ~ t_3
r_b_IS <- function(n) {
  rt(n, df = 3)
}

# PDF of b for t_3
d_b_IS <- function(b) {
  dt(b, df = 3)
}

# x ~ Gamma(shape=2, rate=0.5)
r_x_IS <- function(n) {
  rgamma(n, shape = 2, rate = 0.5)
}

# PDF of x for Gamma(shape=2, rate=0.5)
d_x_IS <- function(x) {
  dgamma(x, shape = 2, rate = 0.5)
}

# 4.2 Implement Importance Sampling
IS_MC <- function(N) {
  # Sample from h(b,x)
  b_samp <- r_b_IS(N)
  x_samp <- r_x_IS(N)

  # Compute original PDF values f(b), f(x)
  fb_vals <- dt(b_samp, df = 5) # original t_5 for b
  fx_vals <- dgamma(x_samp, shape = 2, rate = 1) # original Gamma for x

  # Compute auxiliary PDF values h(b), h(x)
  hb_vals <- d_b_IS(b_samp)
  hx_vals <- d_x_IS(x_samp)

  # Weights w_i = [f(b_i)*f(x_i)] / [h(b_i)*h(x_i)]
  w <- (fb_vals * fx_vals) / (hb_vals * hx_vals)
}

```

```

# Target function p(b,x)
p_vals <- logistic(alpha + b_samp + beta * x_samp)

# IS estimate (unbiased if we average p*w)
est <- mean(p_vals * w)

# Rough variance estimate
#p_var <- mean((p_vals * w)^2) - p_hat^2
#se <- sqrt(p_var / N)

#return(list(est = est, se = se, w_mean = mean(w)))
return(list(est = est))
}

```

Results

5. Comparing Bias, Variance, and CPU Time for Simple MC, CV, and Importance Sampling

```

# 5.1. Obtain a "true value" via a large-scale Simple MC
set.seed(9999)
N_large <- 5e6

res_truth <- simpleMC(N_large)
true_value <- res_truth$est
# cat("Approx. True Value =", true_value, "\n")

# 5.2. Repeat simulations for smaller N and compare bias, variance
simulate_three_methods <- function(N, Eg_true) {
  # Simple MC
  res_mc <- simpleMC(N)
  # CV
  res_cv <- cvMC(N, Eg_true = Eg_true)
  # IS
  res_is <- IS_MC(N)

  c(mc = res_mc$est, cv = res_cv$est, is = res_is$est)
}

set.seed(1234)
N_reps <- 50      # number of repeated simulations
N_small <- 1e5    # sample size for each repeated run

# Pre-calculate Eg_true for CV (to avoid doing it multiple times)
Eg_val <- approx_Eg(N=5e5)

results_matrix <- replicate(N_reps, simulate_three_methods(N_small, Eg_true = Eg_val))
# results_matrix is now 3 x N_reps, each column is one replicate

mc_est <- results_matrix["mc", ]
cv_est <- results_matrix["cv", ]
is_est <- results_matrix["is", ]

```

```

# Compute mean estimate, bias, variance
mc_mean <- mean(mc_est)
cv_mean <- mean(cv_est)
is_mean <- mean(is_est)

mc_bias <- mc_mean - true_value
cv_bias <- cv_mean - true_value
is_bias <- is_mean - true_value

mc_var <- var(mc_est)
cv_var <- var(cv_est)
is_var <- var(is_est)

df_comparison <- data.frame(
  method = c("MC", "CV", "IS", "True"),
  mean_est = c(mc_mean, cv_mean, is_mean, true_value),
  bias = c(mc_bias, cv_bias, is_bias, NA),
  variance = c(mc_var, cv_var, is_var, NA)
)

# 5.3 Compare CPU time
test_time <- microbenchmark(
  MC = { simpleMC(N_small) },
  CV = { cvMC(N_small, Eg_true = Eg_val) },
  IS = { IS_MC(N_small) },
  times = 10 # repeat each method 10 times
)

```

Table 1: Estimated model mean, bias, and variance for each method using 100,000 samples; ‘True’ value is the MC result with 6,000,000 samples

Method	Estimated Mean	Bias	Variance
MC	0.7857719	-0.000144669	4.420e-07
CV	0.7857767	-0.000139931	8.000e-09
IS	0.7855314	-0.000385163	3.016e-06
True	0.7859166	NA	NA

Table 2: CPU time statistics (ms) for each method

Method	Min.	Lower Quartile	Mean	Median	Upper Quartile	Max.	# Eval
MC	12.68774	12.80237	12.92569	12.87853	12.91557	13.58359	10
CV	15.55610	15.62403	15.74131	15.71423	15.77483	16.25925	10
IS	32.78499	32.91373	34.64433	33.10758	36.48824	38.32348	10

Monte Carlo (MC) is the fastest because it only generates random samples and computes the mean, with $O(N)$ complexity and no additional calculations.

Control Variates (CV) is slightly slower due to the extra step of computing a regression coefficient to reduce variance, but it improves estimation accuracy.

Importance Sampling (IS) is the slowest because it requires computing importance weights, normalizing them, and handling probability density functions, making it computationally more expensive ($O(N \log N)$ or higher).

6. Cumulative Convergence Plots (Extra Credit)

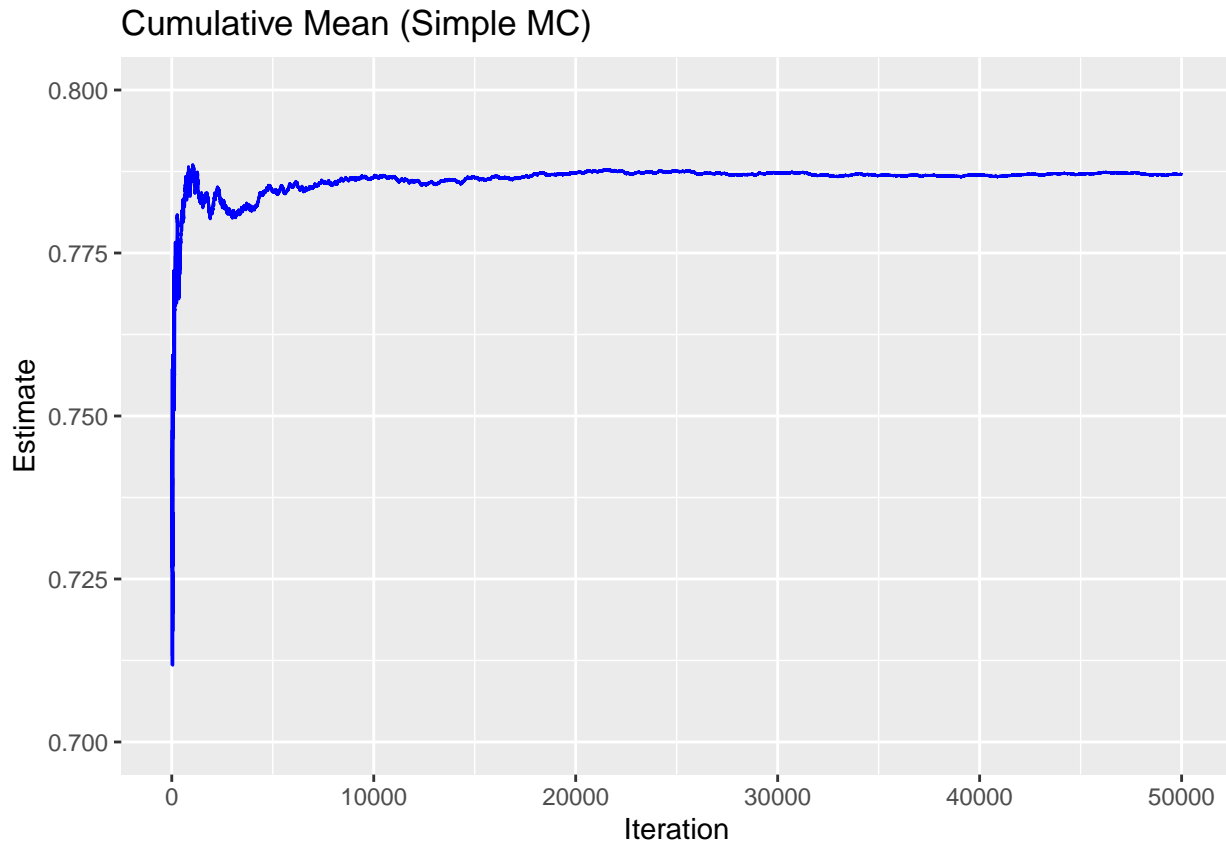
Simple MC

```
# 6.1 Example of cumulative mean for Simple MC
cumulative_MC <- function(N) {
  b_samp <- r_b(N)
  x_samp <- r_x(N)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)
  cum_mean <- cumsum(p_vals) / seq_len(N)
  return(cum_mean)
}

# 6.2 Plot cumulative mean
set.seed(999)
N <- 5e4
cm_mc <- cumulative_MC(N)

df_mc <- data.frame(
  iter = 1:N,
  mean_est = cm_mc
)

ggplot(df_mc, aes(x = iter, y = mean_est)) +
  geom_line(color = "blue") +
  labs(title = "Cumulative Mean (Simple MC)",
       x = "Iteration",
       y = "Estimate") +
  ylim(c(0.7, 0.8))
```



Control Variate Cumulative Mean

```
# This function returns a vector of CV estimates for i from 1 to N.
cumulative_CV <- function(N, Eg_true = NULL, largeN_approx = 5e5) {
  if (is.null(Eg_true)) {
    # If E[g] is not provided, approximate with a larger MC
    Eg_true <- approx_Eg(N = largeN_approx)
  }

  # Sample from original distributions
  b_samp <- r_b(N)
  x_samp <- r_x(N)

  # Compute p(b,x) and g(b,x)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)
  g_vals <- g_function(b_samp, x_samp)

  # We will keep track of partial sums to compute means, covariance, etc.
  cum_p <- cumsum(p_vals)
  cum_g <- cumsum(g_vals)
  cum_pg <- cumsum(p_vals * g_vals) # needed for Cov(p,g)
  cum_g2 <- cumsum(g_vals^2) # needed for Var(g)

  # Vector to store CV estimates at each iteration
  pCV_cum <- numeric(N)
```

```

# The first data point (i=1) does not allow us to compute a covariance
# We can simply set pCV_cum[1] to p_vals[1], or NA, etc.
pCV_cum[1] <- p_vals[1]

# Loop over i = 2 to N
for (i in 2:N) {
  # partial means
  p_bar_i <- cum_p[i] / i
  g_bar_i <- cum_g[i] / i

  # Cov(p,g) = E[ p*g ] - E[p]*E[g ]
  # partial estimate of E[p*g] = cum_pg[i] / i
  pg_bar_i <- cum_pg[i] / i
  cov_pg_i <- pg_bar_i - (p_bar_i * g_bar_i)

  # Var(g) = E[g^2] - (E[g])^2
  g2_bar_i <- cum_g2[i] / i
  var_g_i <- g2_bar_i - (g_bar_i^2)

  # Optimal gamma
  gamma_i <- cov_pg_i / var_g_i

  # CV estimate at iteration i
  pCV_cum[i] <- p_bar_i - gamma_i * (g_bar_i - Eg_true)
}

return(pCV_cum)
}

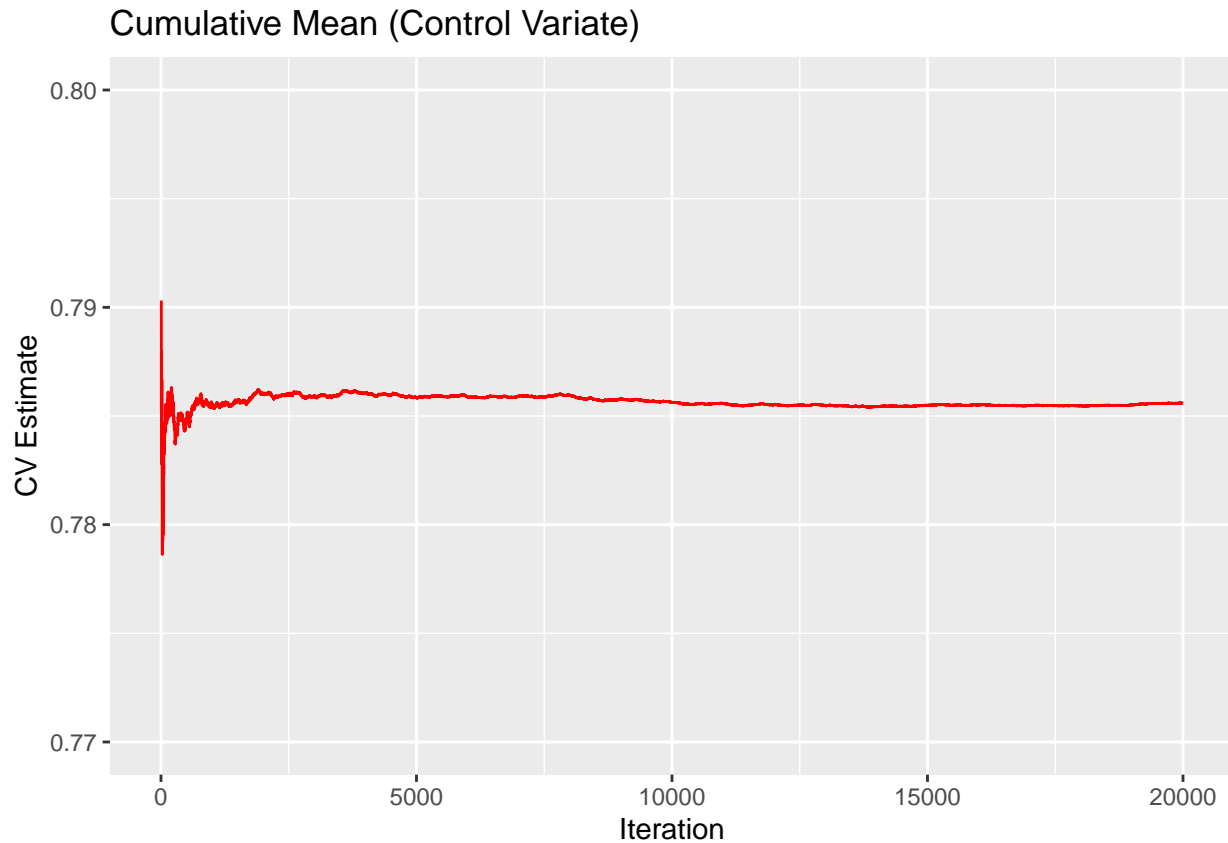
# Example usage and plotting:
set.seed(1234)
N <- 20000
cv_cum_values <- cumulative_CV(N)

df_cv <- data.frame(
  iter = 1:N,
  cv_est = cv_cum_values
)

library(ggplot2)
ggplot(df_cv, aes(x = iter, y = cv_est)) +
  geom_line(color = "red") +
  labs(title = "Cumulative Mean (Control Variate)",
       x = "Iteration",
       y = "CV Estimate") +
  ylim(c(0.77, 0.8))

## Warning: Removed 5 rows containing missing values or values outside the scale range
## (`geom_line()`).

```

Importance Sampling Cumulative Mean

```
cumulative_IS <- function(N) {
  # Sample from auxiliary distribution h(b,x)
  b_samp <- r_b_IS(N)
  x_samp <- r_x_IS(N)

  # Compute original PDF f(b), f(x)
  fb_vals <- dt(b_samp, df = 5)          # original b ~ t_5
  fx_vals <- dgamma(x_samp, shape=2, rate=1) # original x ~ Gamma(2,1)

  # Compute auxiliary PDF h(b), h(x)
  hb_vals <- d_b_IS(b_samp)
  hx_vals <- d_x_IS(x_samp)

  # Weights
  w <- (fb_vals * fx_vals) / (hb_vals * hx_vals)

  # Target function p(b,x)
  p_vals <- logistic(alpha + b_samp + beta * x_samp)

  # Build cumulative sums
  cum_w <- cumsum(w)
  cum_pw <- cumsum(p_vals * w)

  # IS estimate at iteration i: cum_pw[i] / cum_w[i]
}
```

```

pIS_cum <- cum_pw / cum_w

  return(pIS_cum)
}

## Example usage and plotting:
set.seed(5678)
N <- 20000
is_cum_values <- cumulative_IS(N)

df_is <- data.frame(
  iter = 1:N,
  is_est = is_cum_values
)

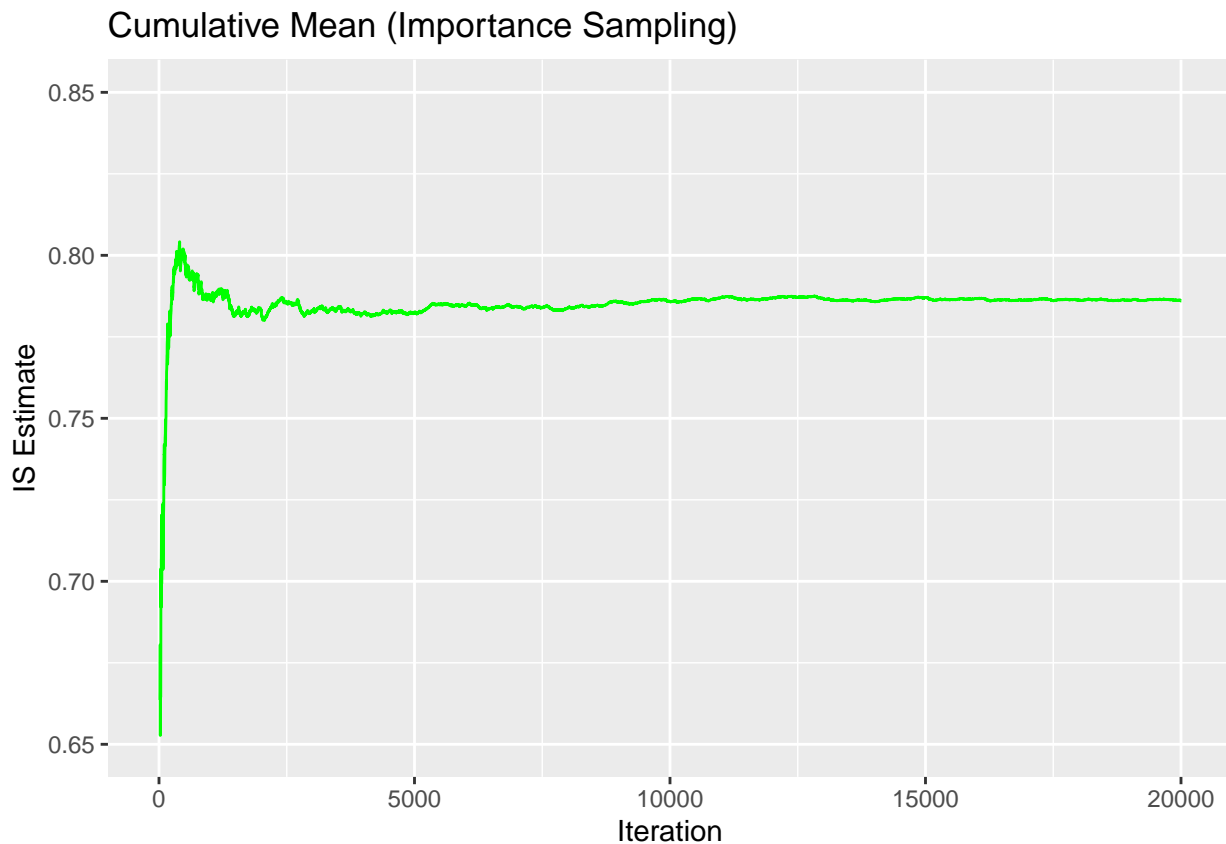
ggplot(df_is, aes(x = iter, y = is_est)) +
  geom_line(color = "green") +
  labs(title = "Cumulative Mean (Importance Sampling)",
       x = "Iteration",
       y = "IS Estimate") +
  ylim(c(0.65, 0.85))

```

```

## Warning: Removed 20 rows containing missing values or values outside the scale range
## (`geom_line()`).

```



```

N <- 20000
set.seed(999)

```

```

# 1. Simple MC
cm_mc <- cumulative_MC(N)
df_mc <- data.frame(iter = 1:N, method = "SimpleMC", estimate = cm_mc)

# 2. CV
cm_cv <- cumulative_CV(N)
df_cv <- data.frame(iter = 1:N, method = "ControlVar", estimate = cm_cv)

# 3. IS
cm_is <- cumulative_IS(N)
df_is <- data.frame(iter = 1:N, method = "ImpSampling", estimate = cm_is)

# Combine
df_all <- rbind(df_mc, df_cv, df_is)

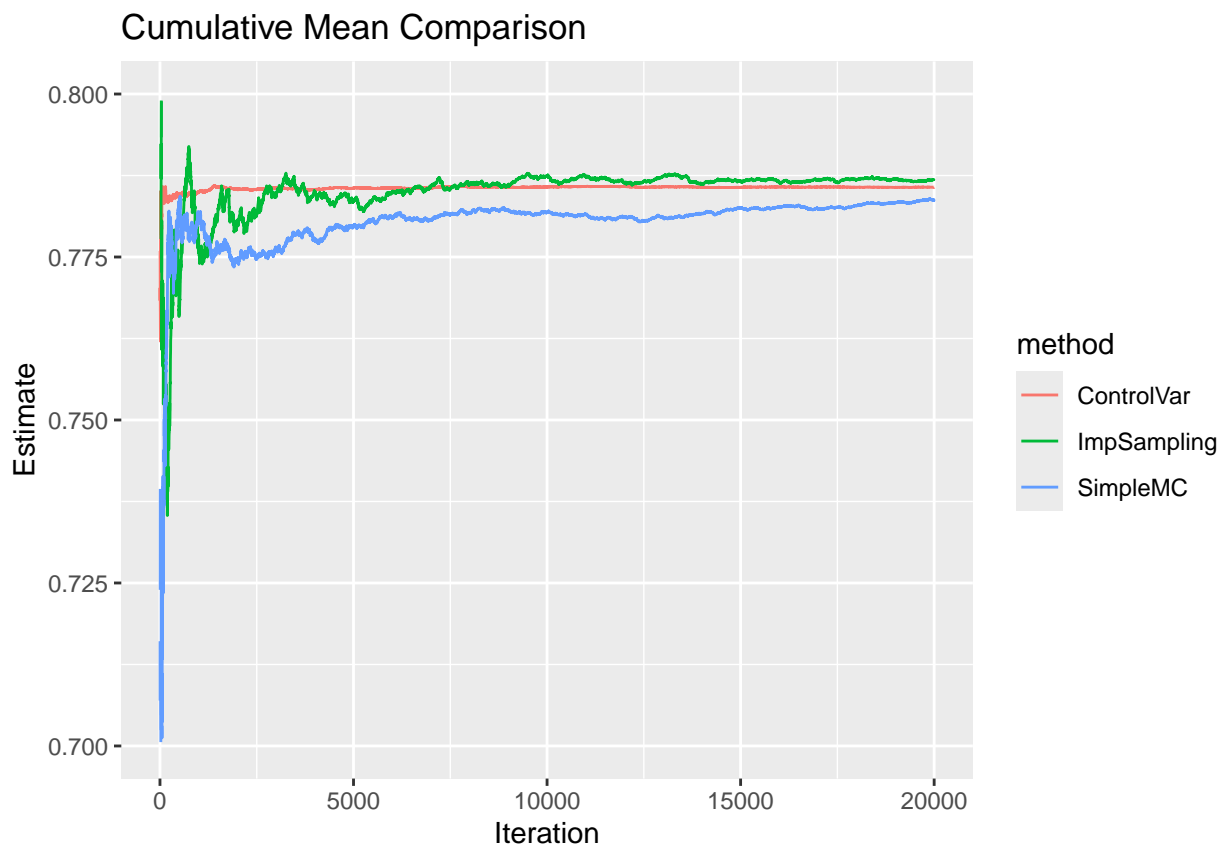
ggplot(df_all, aes(x = iter, y = estimate, color = method)) +
  geom_line() +
  labs(title = "Cumulative Mean Comparison",
       x = "Iteration",
       y = "Estimate") +
  ylim(c(0.7, 0.8))

```

```

## Warning: Removed 29 rows containing missing values or values outside the scale range
## (`geom_line()`).

```



```
theme_bw()
```

```

## List of 136
## $ line                                     :List of 6
##   ..$ colour          : chr "black"
##   ..$ linewidth       : num 0.5
##   ..$ linetype        : num 1
##   ..$ lineend         : chr "butt"
##   ..$ arrow           : logi FALSE
##   ..$ inherit.blank   : logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_line" "element"
## $ rect                                     :List of 5
##   ..$ fill            : chr "white"
##   ..$ colour          : chr "black"
##   ..$ linewidth       : num 0.5
##   ..$ linetype        : num 1
##   ..$ inherit.blank   : logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_rect" "element"
## $ text                                    :List of 11
##   ..$ family          : chr ""
##   ..$ face             : chr "plain"
##   ..$ colour          : chr "black"
##   ..$ size            : num 11
##   ..$ hjust           : num 0.5
##   ..$ vjust           : num 0.5
##   ..$ angle           : num 0
##   ..$ lineheight      : num 0.9
##   ..$ margin          : 'margin' num [1:4] 0points 0points 0points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug           : logi FALSE
##   ..$ inherit.blank   : logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ title                                     : NULL
## $ aspect.ratio         : NULL
## $ axis.title            : NULL
## $ axis.title.x          :List of 11
##   ..$ family          : NULL
##   ..$ face            : NULL
##   ..$ colour          : NULL
##   ..$ size            : NULL
##   ..$ hjust           : NULL
##   ..$ vjust           : num 1
##   ..$ angle           : NULL
##   ..$ lineheight      : NULL
##   ..$ margin          : 'margin' num [1:4] 2.75points 0points 0points 0points
##   .. ..- attr(*, "unit")= int 8
##   ..$ debug           : NULL
##   ..$ inherit.blank   : logi TRUE
##   ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.title.x.top      :List of 11
##   ..$ family          : NULL
##   ..$ face            : NULL
##   ..$ colour          : NULL
##   ..$ size            : NULL
##   ..$ hjust           : NULL
##   ..$ vjust           : num 0

```

```

## ..$ angle      : NULL
## ..$ lineheight : NULL
## ..$ margin      : 'margin' num [1:4] 0points 0points 2.75points 0points
## ..- attr(*, "unit")= int 8
## ..$ debug       : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.title.x.bottom : NULL
## $ axis.title.y        :List of 11
## ..$ family          : NULL
## ..$ face             : NULL
## ..$ colour          : NULL
## ..$ size            : NULL
## ..$ hjust           : NULL
## ..$ vjust           : num 1
## ..$ angle           : num 90
## ..$ lineheight      : NULL
## ..$ margin          : 'margin' num [1:4] 0points 2.75points 0points 0points
## ..- attr(*, "unit")= int 8
## ..$ debug           : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.title.y.left  : NULL
## $ axis.title.y.right :List of 11
## ..$ family          : NULL
## ..$ face             : NULL
## ..$ colour          : NULL
## ..$ size            : NULL
## ..$ hjust           : NULL
## ..$ vjust           : num 1
## ..$ angle           : num -90
## ..$ lineheight      : NULL
## ..$ margin          : 'margin' num [1:4] 0points 0points 0points 2.75points
## ..- attr(*, "unit")= int 8
## ..$ debug           : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.text          :List of 11
## ..$ family          : NULL
## ..$ face             : NULL
## ..$ colour          : chr "grey30"
## ..$ size            : 'rel' num 0.8
## ..$ hjust           : NULL
## ..$ vjust           : NULL
## ..$ angle           : NULL
## ..$ lineheight      : NULL
## ..$ margin          : NULL
## ..$ debug           : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.text.x        :List of 11
## ..$ family          : NULL
## ..$ face             : NULL
## ..$ colour          : NULL

```

```

## ..$ size          : NULL
## ..$ hjust         : NULL
## ..$ vjust         : num 1
## ..$ angle         : NULL
## ..$ lineheight    : NULL
## ..$ margin        : 'margin' num [1:4] 2.2points 0points 0points 0points
## .. ..- attr(*, "unit")= int 8
## ..$ debug         : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.text.x.top      :List of 11
## ..$ family         : NULL
## ..$ face           : NULL
## ..$ colour         : NULL
## ..$ size           : NULL
## ..$ hjust          : NULL
## ..$ vjust          : num 0
## ..$ angle          : NULL
## ..$ lineheight     : NULL
## ..$ margin         : 'margin' num [1:4] 0points 0points 2.2points 0points
## .. ..- attr(*, "unit")= int 8
## ..$ debug         : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.text.x.bottom   : NULL
## $ axis.text.y          :List of 11
## ..$ family         : NULL
## ..$ face           : NULL
## ..$ colour         : NULL
## ..$ size           : NULL
## ..$ hjust          : num 1
## ..$ vjust          : NULL
## ..$ angle          : NULL
## ..$ lineheight     : NULL
## ..$ margin         : 'margin' num [1:4] 0points 2.2points 0points 0points
## .. ..- attr(*, "unit")= int 8
## ..$ debug         : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.text.y.left     : NULL
## $ axis.text.y.right    :List of 11
## ..$ family         : NULL
## ..$ face           : NULL
## ..$ colour         : NULL
## ..$ size           : NULL
## ..$ hjust          : num 0
## ..$ vjust          : NULL
## ..$ angle          : NULL
## ..$ lineheight     : NULL
## ..$ margin         : 'margin' num [1:4] 0points 0points 0points 2.2points
## .. ..- attr(*, "unit")= int 8
## ..$ debug         : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"

```

```

## $ axis.text.theta : NULL
## $ axis.text.r :List of 11
## ..$ family : NULL
## ..$ face : NULL
## ..$ colour : NULL
## ..$ size : NULL
## ..$ hjust : num 0.5
## ..$ vjust : NULL
## ..$ angle : NULL
## ..$ lineheight : NULL
## ..$ margin : 'margin' num [1:4] 0points 2.2points 0points 2.2points
## ..- attr(*, "unit")= int 8
## ..$ debug : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ axis.ticks :List of 6
## ..$ colour : chr "grey20"
## ..$ linewidth : NULL
## ..$ linetype : NULL
## ..$ lineend : NULL
## ..$ arrow : logi FALSE
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_line" "element"
## $ axis.ticks.x : NULL
## $ axis.ticks.x.top : NULL
## $ axis.ticks.x.bottom : NULL
## $ axis.ticks.y : NULL
## $ axis.ticks.y.left : NULL
## $ axis.ticks.y.right : NULL
## $ axis.ticks.theta : NULL
## $ axis.ticks.r : NULL
## $ axis.minor.ticks.x.top : NULL
## $ axis.minor.ticks.x.bottom : NULL
## $ axis.minor.ticks.y.left : NULL
## $ axis.minor.ticks.y.right : NULL
## $ axis.minor.ticks.theta : NULL
## $ axis.minor.ticks.r : NULL
## $ axis.ticks.length : 'simpleUnit' num 2.75points
## ..- attr(*, "unit")= int 8
## $ axis.ticks.length.x : NULL
## $ axis.ticks.length.x.top : NULL
## $ axis.ticks.length.x.bottom : NULL
## $ axis.ticks.length.y : NULL
## $ axis.ticks.length.y.left : NULL
## $ axis.ticks.length.y.right : NULL
## $ axis.ticks.length.theta : NULL
## $ axis.ticks.length.r : NULL
## $ axis.minor.ticks.length : 'rel' num 0.75
## $ axis.minor.ticks.length.x : NULL
## $ axis.minor.ticks.length.x.top : NULL
## $ axis.minor.ticks.length.x.bottom: NULL
## $ axis.minor.ticks.length.y : NULL
## $ axis.minor.ticks.length.y.left : NULL
## $ axis.minor.ticks.length.y.right : NULL

```

```

## $ axis.minor.ticks.length.theta : NULL
## $ axis.minor.ticks.length.r : NULL
## $ axis.line : list()
## ..- attr(*, "class")= chr [1:2] "element_blank" "element"
## $ axis.line.x : NULL
## $ axis.line.x.top : NULL
## $ axis.line.x.bottom : NULL
## $ axis.line.y : NULL
## $ axis.line.y.left : NULL
## $ axis.line.y.right : NULL
## $ axis.line.theta : NULL
## $ axis.line.r : NULL
## $ legend.background :List of 5
## ..$ fill : NULL
## ..$ colour : logi NA
## ..$ linewidth : NULL
## ..$ linetype : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_rect" "element"
## $ legend.margin : 'margin' num [1:4] 5.5points 5.5points 5.5points 5.5points
## ..- attr(*, "unit")= int 8
## $ legend.spacing : 'simpleUnit' num 11points
## ..- attr(*, "unit")= int 8
## $ legend.spacing.x : NULL
## $ legend.spacing.y : NULL
## $ legend.key : NULL
## $ legend.key.size : 'simpleUnit' num 1.2lines
## ..- attr(*, "unit")= int 3
## $ legend.key.height : NULL
## $ legend.key.width : NULL
## $ legend.key.spacing : 'simpleUnit' num 5.5points
## ..- attr(*, "unit")= int 8
## $ legend.key.spacing.x : NULL
## $ legend.key.spacing.y : NULL
## $ legend.frame : NULL
## $ legend.ticks : NULL
## $ legend.ticks.length : 'rel' num 0.2
## $ legend.axis.line : NULL
## $ legend.text :List of 11
## ..$ family : NULL
## ..$ face : NULL
## ..$ colour : NULL
## ..$ size : 'rel' num 0.8
## ..$ hjust : NULL
## ..$ vjust : NULL
## ..$ angle : NULL
## ..$ lineheight : NULL
## ..$ margin : NULL
## ..$ debug : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ legend.text.position : NULL
## $ legend.title :List of 11
## ..$ family : NULL

```



```

## ..$ face          : NULL
## ..$ colour        : NULL
## ..$ size          : NULL
## ..$ hjust         : num 0
## ..$ vjust         : NULL
## ..$ angle         : NULL
## ..$ lineheight    : NULL
## ..$ margin        : NULL
## ..$ debug         : NULL
## ..$ inherit.blank: logi TRUE
## ..- attr(*, "class")= chr [1:2] "element_text" "element"
## $ legend.title.position : NULL
## $ legend.position       : chr "right"
## $ legend.position.inside : NULL
## $ legend.direction      : NULL
## $ legend.byrow          : NULL
## $ legend.justification  : chr "center"
## $ legend.justification.top : NULL
## $ legend.justification.bottom : NULL
## $ legend.justification.left : NULL
## $ legend.justification.right : NULL
## $ legend.justification.inside : NULL
## $ legend.location       : NULL
## $ legend.box            : NULL
## $ legend.box.just       : NULL
## $ legend.box.margin     : 'margin' num [1:4] 0cm 0cm 0cm 0cm
## ..- attr(*, "unit")= int 1
## $ legend.box.background : list()
## ..- attr(*, "class")= chr [1:2] "element_blank" "element"
## $ legend.box.spacing    : 'simpleUnit' num 11points
## ..- attr(*, "unit")= int 8
## [list output truncated]
## - attr(*, "class")= chr [1:2] "theme" "gg"
## - attr(*, "complete")= logi TRUE
## - attr(*, "validate")= logi TRUE

```

Discussion and Practical Implications

Conclusion