AMATEURISH PI STUNT NOTES

PERMUTATION ENTHUSIASTS

1. Problems 1.13

Problem 1.13.17 (alternate solution)

(a) Distance-preserving transformations in R^n are translations and orthogonal transformations. Orthogonal transformations in R^2 are generated by rotations and reflections, which satisfy the relations that define D_n. (Translations are irrelevant if only because they change the center of mass of the n-gon.)

(b) Since $\rho^{-1} = \rho^{n-1}$ and $\tau^{-1} = \tau$, any element of D_n can be written as a word in ρ and τ. We can use $\rho\tau = \tau\rho^{n-1}$ to move the τ's all the way to the left, combine the resulting powers, and reduce them modulo 2 and n.

(c) What is the conjugacy class K_g? Consider the cases (1) $g = \rho^j$ and (2) $g = \tau\rho^j$. In case (1), hgh^{-1} is either g (if $h = \rho^i$) or g^{-1} (if $h = \tau\rho^i$). We have $g^{-1} = g$ if $j = 0$ or $j = n/2$, so this gives us singleton classes $K_\epsilon = \{\epsilon\}$ and (if n is even) $K_{\rho^{n/2}} = \{\rho^{n/2}\}$, as well as $\lfloor (n-1)/2 \rfloor$ classes of size two, $K_g = \{g, g^{-1}\}$.

In case (2), first consider $g = \tau$. Then hgh^{-1} is $\tau\rho^{-2i}$ if $h = \rho^i$ or $\tau\rho^{2i}$ if $h = \tau\rho^i$. If n is odd, $\tau\rho^{\pm 2i}$ ranges over all n elements of the form $\tau\rho^j$, so there is just one more congruence class K_τ. If n is even, K_τ contains the $n/2$ elements of the form $\tau\rho^{2j}$, and a similar computation shows that $K_{\tau\rho}$ contains the $n/2$ elements of the form $\tau\rho^{2j+1}$.

To sum up, if n is odd there are $(n+3)/2$ classes with sizes $(1, n, ((n-1)/2) * 2)$, and if n is even there are $(n+6)/2$ classes with sizes $(1, 1, n/2, n/2, ((n-2)/2) * 2)$.

(d) Ignoring the hint, we make an intuitive guess and then check it. Let q be any n'th root of unity (primitive or not), and define

$$X^q(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X^q(\rho) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

We see that these matrices satisfy the defining relations

$$(X^q(\rho))^n = (X^q(\tau))^2 = I, \quad X^q(\rho)X^q(\tau) = X^q(\tau)(X^q(\rho))^{-1}.$$

Therefore, if we extend X^q to all of G by

$$X^q(\tau^i\rho^j) = (X^q(\tau))^i(X^q(\rho))^j,$$

we get a homomorphism, which is to say a representation. Noting that all the matrices $X^q(\tau\rho^j)$ have trace 0, it is straightforward to compute the corresponding characters χ^q: If n is odd we have

$$\chi^q = (2, 0, q + q^{-1}, q^2 + q^{-2}, \ldots, q^{(n-1)/2} + q^{-(n-1)/2}),$$

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and if n is even we have

$$\chi^q = (2, q^{n/2}, 0, 0, q + q^{-1}, q^2 + q^{-2}, \dots, q^{(n-2)/2} + q^{-(n-2)/2}).$$

We see that $\chi^q = \chi^{1/q}$, and otherwise all the χ^q are different, so we have $n/2$ (even n) or $(n+1)/2$ (odd n) inequivalent representations. We will show that most of these are irreducible by computing inner products.

For n odd, we have

$$\begin{aligned}
\langle \chi^p, \chi^q \rangle &= 1 \cdot 2 \cdot 2 + n \cdot 0 + 2 \sum_{k=1}^{(n-1)/2)} (p^k + p^{-k})(q^k + q^{-k}) \\
&= 4 + 2 \sum_{k=1}^{(n-1)/2)} ((pq)^k + (1/pq)^k + (p/q)^k + (q/p)^k) \\
&= 4 + 2 \sum_{k=1}^{(n-1)} ((pq)^k + (p/q)^k) \\
&= 2 \sum_{k=0}^{(n-1)} (pq)^k + 2 \sum_{k=0}^{(n-1)} (p/q)^k,
\end{aligned}$$

where the third equality follows because $(pq)^n = 1$, so $(1/pq)^k = (pq)^{n-k}$, and similarly for p/q and q/p. If the two characters are different, then $p \neq q$ and $p \neq 1/q$, and pq and p/q are both roots of unity, not equal to 1, whose degree divides n. Each of the two sums adds up all their (respective) powers with the same multiplicity, so each sum is 0. Thus χ^p and χ^q are orthogonal. If $p = q \neq 1$, the first sum is 0 and the second sum is n, so $\langle \chi^q, \chi^q \rangle = 2n = |D_n|$ and χ^q is irreducible. Taking $p = q = 1$, we find $\langle \chi^1, \chi^1 \rangle = 4n$, so χ^1 is reducible. Indeed, X^1 is equivalent to the direct sum of two copies of the degree-one representation with $X^{\pm}(\rho^j) = 1$ and $X^{\pm}(\tau\rho^j) = -1$ for each j. Including X^{triv}, we now have $(n-1)/2$ irreducibles of degree 2 and 2 irreducibles of degree 1, for $(n+3)/2$ in all, as desired.

I really should do the similar computation for n even, but I'm too lazy at the moment.