AMATEURISH PI STUNT NOTES

PERMUTATION ENTHUSIASTS

1. Problems 1.13

(1) (a) Consider the transposition $\pi = (i, j)$. It swaps the order of exactly $2(j-i-1)+1$ pairs of inputs. This is because there are $j-i-1$ entries strictly between i and j, and π swaps both i and j with all of these. It also swaps the order of i and j. Therefore, $\mathrm{inv}((i,j)) \equiv 1 \bmod 2$. Next we show that inv is a homomorphism to $\mathbb{Z}/2$. To do this, let $\mathrm{inv}_{i,j}(\pi)$ be 1 if π swaps the order of i and j and 0 otherwise, so $\mathrm{inv}(\pi) = \sum_{i \neq j} \mathrm{inv}_{i,j}(\pi)$. If we have two permutations π and τ, we have $\mathrm{inv}_{i,j}(\tau\pi) = \mathrm{inv}_{i,j}(\pi) + \mathrm{inv}_{\pi(i),\pi(j)}(\tau) \bmod 2$. I.e. i and j get swapped if π swaps i and j or τ swaps $\pi(i), \pi(j)$ (but not both because they get swapped back if so). Therefore

$$\begin{aligned}
\mathrm{inv}(\tau\pi) &= \sum_{i \neq j} \mathrm{inv}_{i,j}(\tau\pi) \\
&\equiv \sum_{i \neq j} \mathrm{inv}_{i,j}(\pi) + \mathrm{inv}_{\pi(i),\pi(j)}(\tau) \bmod 2 \\
&\equiv \left(\sum_{i \neq j} \mathrm{inv}_{i,j}(\pi) + \sum_{i \neq j} \mathrm{inv}_{\pi(i),\pi(j)}(\tau) \right) \bmod 2 \\
&\equiv \mathrm{inv}(\pi) + \mathrm{inv}(\tau) \bmod 2
\end{aligned}$$

Taken together, these facts imply that if π is a product of k transpositions, then $\mathrm{inv}(\pi) \equiv k \bmod 2$.

(b) We defined $\mathrm{sgn}(\pi) = (-1)^k$ when π is a product of k transpositions. It may be that π can be written as a product of k transpositions and also as a product of $\ell \neq k$ transpositions. However, we can see that $\mathrm{sgn}(\pi)$ is well defined because $\mathrm{inv}(\pi)$ is well defined, and we saw in (a) that $k \equiv \mathrm{inv}(\pi) \equiv \ell \bmod 2$. Thus $(-1)^k = (-1)^\ell$.

(2) (a) We clearly have $\epsilon \in G_s$ because $\epsilon s = s$ by an axiom of group actions. To see that G_s is closed under multiplication, let us be given $g, h \in G_s$, and we compute $(gh)s = g(hs) = gs = s$ (the second equality uses a group action axiom). Finally, note that if $gs = s$ than $s = g^{-1}s$, so G_s is closed under inverses.

(b) Define $\phi : G/G_s \to \mathcal{O}_s$ by $\phi(hG_s) = hs$. This is well-defined because if $h = hg$ for $g \in G_s$, then $(hg)s = h(g(s)) = hs$. (I'm assuming left cosets.) The map ϕ is surjective because if we are given any $h \in G$,

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then $\phi(hG_s) = hs$. The map ϕ is injective because if $\phi(hG_s) = \phi(kG_s)$, then $hs = ks$, so $k^{-1}h \in G_s$ and hence $hG_s = kG_s$.

(c) By Lagrange's Theorem, $G = [G : G_s]|G_s|$, where $[G : G_s]$ is the number of left cosets of G_s in G. Well, we just proved there are $\mathcal{O}_s$ of them, so the result follows.

(3) (a) We must show that every matrix $X(\pi)$ has exactly one 1 in each row and column, and zeros elsewhere. The definition of the permutation representation has $X(\pi)_{i,j} = \delta_{i,\pi(j)}$. Because π is a permutation, for each i there is exactly one j such that $i = \pi(j)$, and for each j there is exactly one i such that $i = \pi(j)$.

(b) A fixed point i of π has $\pi(i) = i$, so $X(\pi)_{i,i} = 1$, so there is a 1 on the diagonal of $X(\pi)$ in position (i, i) exactly when i is fixed under π. Thus $\mathrm{Tr}(X(\pi))$ is the number of fixed points of π.

(4) Since G is finite, it can be written as $\oplus_i C_{j_i}$, where C_{j_i} is a cyclic group of order j_i, say with generator g_i. By Corollary 1.6.8 (see also Problem 12), we must have $X(g) = cI$ for all $g \in G$, so X is one dimensional, and $X(g_i)$ is some j_ith root of unity. All such representations are irreducible since they are one dimensional. By example 1.2.3, there are j_i such representations (one for each root of unity).

So then the question is: how do we stick together the representations of the C_{j_i} to get representations for G, and how do we know that we have found all of them? Theorem 1.11.3 answers both questions: where X_i, Y_j are inequivalent representations of groups G,H, then $X_i \otimes Y_j$ is a complete list of representations of $H \times K$. (We extend this theorem from a product of 2 groups to our product of i groups in the obvious way.) Thus G has $\prod_i j_i$ irreducible representations.

(5) (a) Let $g \in N$ and $h \in G$. Then $X(hgh^{-1}) = X(h)X(g)X(h^{-1}) = X(h)X(h)^{-1} = I$, so $hgh^{-1} \in N$. This holds for all $g \in N$, so N is normal. A condition is: X is faithful iff $N = \{\epsilon\}$. To see one direction, suppose X is faithful. Then $I = X(\epsilon) = X(g)$ only if $g = \epsilon$; hence $N = \{\epsilon\}$. For the other direction, suppose $N = \{\epsilon\}$ and $X(g) = X(h)$ for some g, h. Then $X(gh^{-1}) = I$, so $gh^{-1} = \epsilon$, so $g = h$, and X is faithful.

(b) One direction is immediate, because if $g \in N$, then $\mathrm{Tr}(X(g)) = \mathrm{Tr}(I) = d$. For the other direction, suppose $\chi(g) = d$. TODO

(c) For one direction, suppose that $h \in \bigcap_i g_i H g_i^{-1}$. Then for all i, we have $hg_i \in g_i H$, so $hg_i H \subseteq g_i HH = g_i H$ sends each coset to itself, so $h \in N$. Conversely, if $X(h) = I$, then h sends each coset to itself, so $hg_i H \in g_i H$ for all i. Hence $h \in g_i H g_i^{-1}$ for all i.

(d) (i) Trivial: this is faithful exactly if G is trivial
(ii) Regular: always
(iii) Coset: when the intersection of the conjugates of H is trivial (see previous)
(iv) Sign for $\mathcal{S}_n$: for $\mathcal{S}_1$ and $\mathcal{S}_2$
(v) Defining for $\mathcal{S}_n$: always

(vi) Degree 1 for C_n: exactly when $X(g)$ is a primitive root of unity, for a generator g

(e) (i) Y is well-defined because if $gN = hN$, then there is $n \in N$ so that $gn = h$. Then $X(h) = X(gn) = X(g)X(n) = X(g)I = X(g)$.
It is faithful because if $Y(gN) = I$, then by definition we have $X(g) = I$, so $g \in N$, ie. the only coset that maps to I under Y is ϵN.
It is a representation: if gN is the identity in G/N, then $gN \in N$ so $g \in N$, and thus $Y(gN) = Y(N) = X(1) = I$. Multiplicativity follows from multiplicativity of X:

$$Y(gN)Y(hN) = X(g)X(h) = X(gh) = Y(ghN).$$

(ii) Whether or not a representation is irreducible depends only on the set of matrices (or endomorphisms) in the image. The image of Y is the same as the image of X. Said another way, if $X(g)(V) \subseteq V$ for some subspace V and for all g, then $Y(gN)(V) = X(g)(V) \subseteq V$ as well, and vice versa.

(iii) The representation Y is the regular representation of G/H. To see this, let us start by finding the kernel N of the coset representation. Suppose $n \in N$, so $ngH = gH$ for all g. Because H is normal, we have $ngH = Hng$, so $Hng = gH$. Thus $Hn = gHg^{-1} = H$, so $n \in H$. The entire argument runs backward, so $N = H$. Let V be the coset representation (so Y is a map $Y : G/H \to GL(V)$) and consider the map $\theta : V \to \mathbb{C}[G/H]$ defined by $\theta(gH) = gH$. This is clearly a bijection, so we just need to check it is a G/H-homomorphism. To see this, we compute $\theta(Y(gH)(hH)) = ghH$ and $gH\theta(hH) = ghH$ (by the definition of group multiplication in G/H).

(6) (a) To see that X is a representation, we just need to check that $X(gh) = X(g)X(h)$. We compute $X(gh) = Y(ghN) = Y(gNhN) = Y(gN)Y(hN) = X(g)X(h)$, where in the middle we used the multiplication in G/H.

(b) Let $g \in \ker(X)$, so $I = X(g) = Y(gN)$. Since Y is faithful, we have $g \in N$. Conversely, any $g \in N$ is in $\ker(X)$ because $X(g) = Y(gN) = I$.

(c) This is the same as (5)(e)(ii) – the irreducibility only depends on the image set of matrices, which remains the same under lifting.

(7) The block decomposition of X expresses V as the internal direct sum $W + Y$, where if we write any vector (w, y) aligned with the block form, we have $X(g)(w, y) = (A(g)w + B(g)y, C(g)y)$. The quotient map $V \to V/W$ projects to the second coordinate and is a G-homomorphism which takes the action X to C. Maschke's theorem says that V is isomorphic to a block diagonal action with the matrices A and C, and this is exactly the actions on W and V/W.

(8) (a) V satisfies the 4 properties of definition 1.3.1. Since each element of a basis $B = \{b_1, b_2 \ldots b_n\}$ belongs to V, each b_i also satisfies the properties of definition 1.3.1 (take any $v = b_i$).

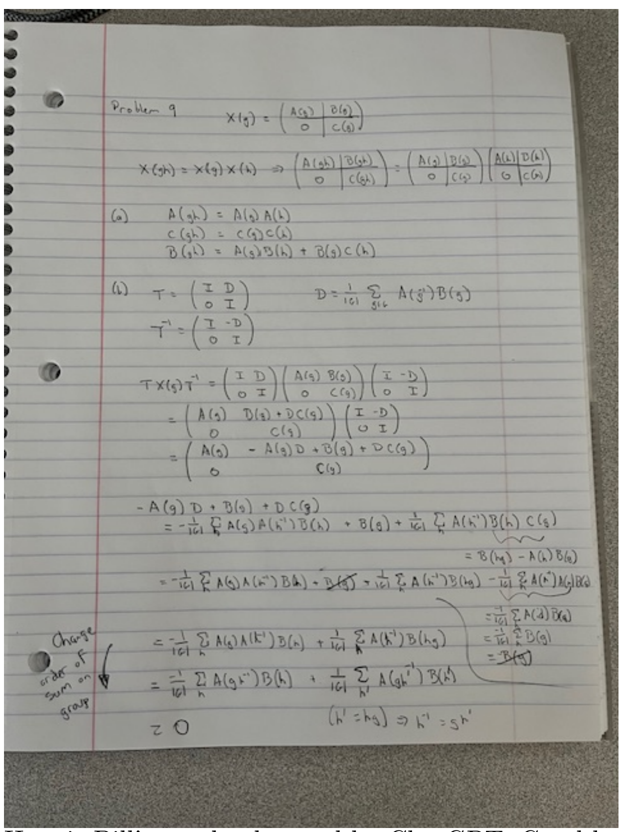
Conversely, if 1.3.1 holds for a basis $B = \{b_1, b_2, \dots, b_n\}$, write any vector v as a linear combination of the elements in B. The definition 1.3.1 properties are true for v by linearity.

(b) The map θ is linear and for all $g \in G$ and b_i in the basis $B = \{b_1, b_2, \dots, b_n\}$, we have $g\theta(b_i) = \theta(gb_i)$.
Conversely, assume we have $\theta(gb_i) = g\theta(b_i)$ and $\theta(b_i) + \theta(b_j) = \theta(b_i + b_j)$ (by definition of a homomorphism.) Write any $v \in V$ as $v = \sum_i c_i b_i$ for constants c_i. Then

$$\theta(g \sum c_i b_i) = \theta(gc_1 b_1) + \theta(gc_2 b_2) \dots \theta(gc_n b_n) = g \sum_i \theta(c_i b_i).$$

(c) Forward direction is the same as above: For all b, c in the basis, we have $\langle b, c \rangle = \langle gb, gc \rangle$ since $b, c \in V$. The converse proceeds as in part b: write v as a linear combination of basis elements, use the properties of G invariance we are assuming for the basis, and use linearity.

(9) Here's is Bill's notebook:



Here is Bill's notebook typed by ChatGPT. Good bot.
Let $g_1, g_2 \in G$ and X a reducible matrix representation:

$$X(t) = \begin{bmatrix} A(g) & B(g) \\ 0 & C(g) \end{bmatrix}$$

(a)

$$X(g_1)X(g_2) = \begin{bmatrix} A(g_1g_2) & B(g_1g_2) \\ 0 & C(g_1g_2) \end{bmatrix} = \begin{bmatrix} A(g_1) & B(g_1) \\ 0 & C(g_1) \end{bmatrix} \begin{bmatrix} A(g_2) & B(g_2) \\ 0 & C(g_2) \end{bmatrix}$$

Multiplying out the block form gives

$$A(g_1g_2) = A(g_1)A(g_2)$$
$$C(g_1g_2) = C(g_1)C(g_2)$$
$$B(g_1g_2) = A(g_1)B(g_2) + B(g_1)C(g_2)$$

(b) Let

$$T = \begin{bmatrix} I & D \\ 0 & I \end{bmatrix}, \quad D = \frac{1}{|G|}\sum_{g\in G} A(g)B(g)$$

and note

$$T^{-1} = \begin{bmatrix} I & -D \\ 0 & I \end{bmatrix}$$

.

Then:

$$TX(g)T^{-1} = \begin{bmatrix} I & D \\ 0 & I \end{bmatrix} \begin{bmatrix} A(g) & B(g) \\ 0 & C(g) \end{bmatrix} \begin{bmatrix} I & -D \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} A(g) & B(g) + DC(g) + \\ 0 & C(g) \end{bmatrix} \begin{bmatrix} I & -D \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} A(g) & -A(g)D + B(g) + DC(g) \\ 0 & C(g) \end{bmatrix} = \begin{bmatrix} A(g) & B(g) \\ 0 & C(g) \end{bmatrix}$$

Now consider $-A(g)D + B(g) + DC(g)$. We are done if we can show it is 0.

$$-A(g)D+B(g)+DC(g) = -\frac{1}{|G|}\sum_{h\in G} A(g)A(h^{-1})B(h)+B(g)+\sum_{h\in G} A(h^{-1})B(h)C(g)$$

From part (a), $B(h)C(g) = B(hg) - A(h)B(g)$.
Substituting,

$$-A(g)D+B(g)+DC(g) = -\frac{1}{|G|}\sum_{h\in G} A(g)A(h^{-1})B(h)+B(g)+\frac{1}{|G|}\sum_{h\in G} A(h^{-1})B(hg)-\frac{1}{|G|}\sum_{h\in G} A(h^{-1})A(h)B(g)$$

Consider the last summand. $A(h)A(h^{-1} = I$, so

$$-\frac{1}{|G|}\sum_{h\in G} A(h^{-1})A(h)B(g) = -\frac{1}{|G|}\sum_{h\in G} B(g) = -B(g).$$

So we have

$$-A(g)D + B(g) + DC(g) = -\frac{1}{|G|}\sum_{h\in G} A(g)A(h^{-1})B(h) + \frac{1}{|G|}\sum_{h\in G} A(h^{-1})B(hg)$$

Using the multiplicativity of A in the first summand and the fact that $gh = h'$ for some $h' \in G$ in the second (which doesn't change the sum; we're just running through all the elements of the group in a different order), we obtain:

$$-A(g)D + B(g) + DC(g) = -\frac{1}{|G|}\sum_{h\in G} A(gh^{-1})B(h) + \frac{1}{|G|}\sum_{h\in G} A(gh^{-1})B(h) = 0.$$

(10) The map $X(r) = \begin{bmatrix} 1 & \log r \\ 0 & 1 \end{bmatrix}$ satisfies

$$\begin{aligned} X(r)X(s) &= \begin{bmatrix} 1 & \log r \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & \log s \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \log(rs) \\ 0 & 1 \end{bmatrix} \\ &= X(rs), \end{aligned}$$

and we can see $X(r)\begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$.

(11) Let $H = \mathcal{S}_{n-1} \leq \mathcal{S}_n$, and let S be the set of tabloids of shape $(n-1,1)$. We want to show that $\mathbb{C}\mathcal{H} \cong \mathbb{C}\mathbf{S} \cong \mathbb{C}\{\mathbf{1},\ldots,\mathbf{n}\}$. First we need to find a transversal for H. Note that $|H| = (n-1)!$, so the index of H in $\mathcal{S}_n$ is n, so it suffices to show our n chosen cosets are pairwise disjoint. To do this, suppose $(i,n)H = (j,n)H$, so $(j,n)(i,n) \in H$. Note this product fixes n (equivalently, is in H) exactly when $i = j$. Hence the cosets are disjoint. It is also useful to observe that if $i \neq j$, we have $(i,n)(j,n) = (j,n)(i,j)$, so $(i,n)(\mathbf{j},\mathbf{n})\mathbf{H} = (\mathbf{j},\mathbf{n})\mathbf{H}$, while if $i = j$, then we compute that (i,n) interchanges the cosets ϵH and $(i,n)H$. This gives us the action on cosets.

Now define the equivalence $\theta : \mathbb{C}\mathcal{H} \to \mathbb{C}\{\mathbf{1},\ldots,\mathbf{n}\}$ by $\theta((\mathbf{i},\mathbf{n})\mathbf{H}) = \mathbf{i}$ and $\iota : \mathbb{C}\mathbf{S} \to \mathbb{C}\{\mathbf{1},\ldots,\mathbf{n}\}$ by taking a tabloid basis element to the basis element of $\mathbb{C}\{\mathbf{1},\ldots,\mathbf{n}\}$ associated with the single entry in the second level of the tabloid. Since $\{(i,n)\}_{i=1}^n$ generates $\mathcal{S}_n$, it suffices to show that these maps commute with the action of these involutions. This is immediate for ι because the image is by definition the entry in the bottom of the tabloid. For θ, we use the coset action we determined above, so $\theta((i,n)(\mathbf{j},\mathbf{n})\mathbf{H}) = \theta((\mathbf{j},\mathbf{n})\mathbf{H}) = \mathbf{j}$ if $i \neq j$, and $\theta((i,n)(\mathbf{i},\mathbf{n})\mathbf{H}) = \mathbf{n}$ and $\theta((i,n)\epsilon\mathbf{H}) = \mathbf{i}$. That is, the action of (i,n) swaps the corresponding pairs of basis elements on both sides of θ.

(12) By Corollary 1.6.8, any matrix that commutes with $X(g)$ for all g must be of the form cI. If $g \in Z_G$, then by definition $X(g)$ commutes with $X(h)$ for all $h \in G$, and the conclusion is immediate.

(13) Let G be the abelian group formed by the matrices X_i. So the map $Y(X_i) = X_i$ is a d-dimensional representation of G. By Maschke's theorem, there is a single matrix T such that TX_iT^{-1} is a block diagonal matrix of irreducible representations. It remains to show that any irreducible representation of an abelian group is 1-dimensional, which maybe we just know that, or

maybe we observe that by Corollary 1.6.8, if G is an abelian group, any image matrix in an irreducible representation must be a multiple of the identity and thus must be 1-dimensional.

(14) Suppose towards a contradiction that X is reducible, so up to isomorphism we can simultaneously write the matrices $X(g)$ in a nontrivial block form. But then $X(g)$ commutes with block diagonal matrices with blocks xI, yI, for any $x, y \in \mathbb{C}$. Many such matrices are not of the form cI, which is a contradiction.

Another solution: since only the cI commute with $X(g)$, we have $Com(X) = \{cI\}$, and thus Theorem 1.7.8 part 2 says that $m_1 = 1$ which is the only nonzero m_i in the decomposition of X. The result follows.

(15) (a) We must check that $(X\hat{\otimes}Y)(gh) = (X\hat{\otimes}Y)(g)(X\hat{\otimes}Y)(h)$. To do this, we compute:

$$\begin{aligned}(X\hat{\otimes}Y)(g)(X\hat{\otimes}Y)(h) &= (X(g) \otimes Y(g))(X(h) \otimes Y(h)) \\ &= X(g)X(h) \otimes Y(g)Y(h) \\ &= X(gh) \otimes Y(gh) \\ &= (X\hat{\otimes}Y)(gh)\end{aligned}$$

The second equality uses Lemma 1.7.7.

(b) We can compute

$$\begin{aligned}(\chi\hat{\otimes}\psi)(g) &= \operatorname{Tr}((X\hat{\otimes}Y)(g)) \\ &= \operatorname{Tr}(X(g) \otimes Y(g)) \\ &= \sum_i X(g)_{i,i} \operatorname{Tr}(Y(g)) \\ &= \sum_i X(g)_{i,i}\psi(g) \\ &= \chi(g)\psi(g)\end{aligned}$$

(c) If X and Y are both the irreducible 2-dimensional representation of $\mathcal{S}_3$, then $X\hat{\otimes}Y$ has dimension 4, but $\mathcal{S}_3$ has no 4-dimensional irreducible representations.

(d) We can check that it is irreducible by computing

$$\begin{aligned}\langle \chi\hat{\otimes}\psi, \chi\hat{\otimes}\psi \rangle &= \frac{1}{|G|}\sum_g (\chi\hat{\otimes}\psi)(g)(\chi\hat{\otimes}\psi)(g^{-1}) \\ &= \frac{1}{|G|}\sum_g \chi(g)\psi(g)\chi(g^{-1})\psi(g^{-1}) \\ &= \frac{1}{|G|}\sum_g \psi(g)\psi(g^{-1}) \\ &= \langle \psi, \psi \rangle \\ &= 1\end{aligned}$$

This relies on the fact that X is one-dimensional, so $\operatorname{Tr}(X(g^{-1})) = 1/\operatorname{Tr}(X(g))$, so $\chi(g)\chi(g^{-1}) = 1$.

(16) There are five cycle types/conjugacy classes $\epsilon, (1,2), (1,2,3), (1,2)(3,4), (1,2,3,4)$, of sizes 1, 6, 8, 3, and 6, respectively. Because there are five, we are expecting five irreducible representations. We know the trivial $\chi^{(1)}$ and sign $\chi^{(2)}$ representations are irreducible, and we know the representation $\chi^{\perp}$ orthogonal to the trivial one inside the defining representation, which we can verify is irreducible by computing its self inner product. In addition, we compute the character for $\chi^{(2)}\hat{\otimes}\chi^{\perp}$ and see it too is irreducible.

For the final irreducible, consider the normal subgroup N which is ϵ and the conjugacy class of $(1,2)(3,4)$ (This is the Klein four group $\mathbb{Z}/2 \times \mathbb{Z}/2$). In fact, we have $\mathcal{S}_4/N \cong \mathcal{S}_3$. To see this, consider the map $\phi : \mathcal{S}_3 \to \mathcal{S}_4$ defined by $\phi(\pi) = \pi N$. If $\pi N = \rho N$, then there is $n \in N$ so that $\pi = \rho n$. If $n \neq \epsilon$, then note that n, and thus ρn, does not fix 4 (here $\rho \in \mathcal{S}_3$, so if n permutes 4 away from itself, ρ cannot put it back). On the other hand π does fix 4. This contradiction implies that $n = \epsilon$, so ϕ is injective. Since $|\mathcal{S}_3| = 6$ and $|\mathcal{S}_4/N| = 6$, in fact ϕ is an isomorphism. To compute the quotient map from $\mathcal{S}_4$ to $\mathcal{S}_3$ on a permutation π which does not fix 4, we need to find $\rho n = \pi$, with $n \in N$ and ρ fixing 4.

So we can use the lifting process from problem 6 to lift each of the 3 irreducible representations of $\mathcal{S}_3$ to representations of $\mathcal{S}_4$. The trivial and sign representations lift to the trivial and sign representations, respectively, and give us nothing new. But the third irreducible $\chi^{(3)}$ does give us the final, 2 dimensional, irreducible representation $\chi^{(3)}$ of $\mathcal{S}_4$. (We can check that it is irreducible by computing its self inner product.)

	ϵ	$(1,2)$	$(1,2,3)$	$(1,2)(3,4)$	$(1,2,3,4)$
$\chi^{(1)}$ trivial	1	1	1	1	1
$\chi^{(2)}$ sign	1	-1	1	1	-1
$\chi^{\perp}$	3	1	0	-1	-1
$\chi^{(2)}\hat{\otimes}\chi^{\perp}$	3	-1	0	-1	1
$\chi^{(3)}$	2	0	-1	2	0

Problem 1.13.16 (addendum). What representations correspond to the five characters? The trivial and sign representations are of course the same as the characters themselves. We can read off the degree-3 representation $X^{\perp}$ from the block diagonal form we computed using Maschke's Theorem to get the character, and then $X^{\hat{\otimes}}(g)$ is just $\chi^{sgn}(g)X^{\perp}(g)$.

Unlike Alden, I computed the fifth character $\chi^{(3)}$ just by orthogonality with the first four, so I didn't get a corresponding representation. What is $X^{(3)}$?

It has degree 2, so we need 2-by-2 matrices. Let $q = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ be a primitive cube root of unity, with $q^2 = q^{-1} = \bar{q}$. If $g \in S_4$ is a 3-cycle, we need $M = X^{(3)}(g)$ to satisfy both $\operatorname{tr} M = \chi^{(3)}(g) = -1$, and also $M^3 = I$, whence $\det M$ is one of $\{1, q, q^2\}$. A 3-cycle is the product of two transpositions, which must have determinants ± 1, so the only possiblity is $\det M = 1$. Let the eigenvalues of M be λ_1 and λ_2. We have $\lambda_1\lambda_2 = \det M = 1$, and $\lambda_1 + \lambda_2 = \operatorname{tr} M = -1$. This gives a quadratic equation that we solve to conclude that λ_1 and λ_2 are q and q^2 in some order. The simplest matrices with those eigenvalues are

$$M_1 = \begin{pmatrix} q^2 & 0 \\ 0 & q \end{pmatrix}, \quad M_2 = \begin{pmatrix} q & 0 \\ 0 & q^2 \end{pmatrix}.$$

Noting that $M_2 = M_1^{-1}$, we would like to set things up to have $X^{(3)}(g) = M_1$ and $X^{(3)}(g^{-1}) = M_2$ for the four pairs of 3-cycles $\{g, g^{-1}\}$ in S_4.

A 3-cycle is the product of two transpositions that overlap in one element. We note that the six transpositions in S_4 come in three non-overlapping pairs, an example of which is $\{(1\,3), (2\,4)\}$. Thus we need at least three 2-by-2 matrices whose products are M_1 or M_2, each with trace 0 and square I (whence determinant ± 1). Three matrices that fit the bill are

$$T_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_b = \begin{pmatrix} 0 & q \\ q^2 & 0 \end{pmatrix}, \quad T_c = \begin{pmatrix} 0 & q^2 \\ q & 0 \end{pmatrix}.$$

Indeed, the entire representation is determined by the representations of transpositions, so we'll guess that the three matrices immediately above determine an irreducible representation whose character is $\chi^{(3)}$. It's easy to check that various things work out—for example, the traces of the non-overlapping double transpositions are 2, and the traces of the 4-cycles are 0.

To make sure (and to learn a little bit of Sage), I wrote a Jupyter notebook that computes the representations of all the elements of S_4 based on the three transposition matrices above, and then verifies the entire multiplication table.

(17) (a) We can flip (τ, order 2) rotate (ρ, order n), and playing with a shape shows that $\rho\tau = \tau\rho^{-1}$.

(b) If we have any sequence of τ and ρ, we can slide all ρ to the right using the relation $\rho\tau = \tau\rho^{-1}$, e.g. for $0 \le i \le n-1$,

$$\tau^j \rho^i = \tau^j \rho^{i-n} = \tau^j \rho^{-(n+i)} = \tau^j \rho^{-1(n-i)} = \rho^{n-i}\tau^j$$

(c) We compute:

$$(\tau^e \rho^\ell)\rho^j(\rho^{-\ell}\tau^e) = \begin{cases} \rho^j & \text{if } e = 0 \\ \rho^{-j} & \text{if } e = 1 \end{cases}$$

$$(\tau^e \rho^\ell)\tau\rho^j(\rho^{-\ell}\tau^e) = \begin{cases} \tau\rho^{2\ell-j} & \text{if } e = 0 \\ \tau\rho^{j-2\ell} & \text{if } e = 1 \end{cases}$$

These relations determine the conjugacy classes of D_n. The answer depends on whether n is odd (the issue is whether 2 is relatively prime to n, ie. whether 2 is a generator of the additive group $\mathbb{Z}/n$).

If n is even, then the conjugacy classes are

$$\{\epsilon\}, \{\rho^1, \rho^{n-1}\}, \ldots, \{\rho^{n/2-1}, \rho^{n/2+1}\}, \{\rho^{n/2}\}, \{\tau, \tau\rho^2, \ldots, \tau\rho^{n-2}\}, \{\tau\rho, \tau\rho^3, \ldots, \tau\rho^{n-1}\},$$

so there are $n/2+3$ classes total. If n is odd, the conjugacy classes are

$$\{\epsilon\}, \{\rho^1, \rho^{n-1}\}, \ldots, \{\rho^{\frac{n-1}{2}}, \rho^{\frac{n+1}{2}}\}, \{\tau, \tau\rho, \ldots, \tau\rho^{n-1}\},$$

so there are $\lfloor n/2 \rfloor + 2$ classes total.

(d) There are some simple-to-define representations X_j, which we will check are irreducible. Define X_j as follows, where ρ is mapped to a rotation

$$X_j(\rho) = \begin{bmatrix} \cos\frac{2\pi j}{n} & -\sin\frac{2\pi j}{n} \\ \sin\frac{2\pi j}{n} & \cos\frac{2\pi j}{n} \end{bmatrix}$$

and τ to a flip

$$X_j(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The representation X_1 is the "defining" representation of D_n as it's usually defined acting on a n-sided polygon in the plane. (Although this is a 2-dimensional *complex* representation!)
Let χ_j be the character of X_j. Note that

$$\chi_j(\rho^i) = \mathrm{Tr}(X_j(\rho^i)) = 2\cos\frac{2\pi ij}{n} \quad \text{and} \quad \chi_j(\tau\rho^i) = 0.$$

In particular, we can ignore $\tau\rho^i$ for all subsequent calculations of characters.
We can compute

$$\begin{aligned} \langle \chi_j, \chi_j \rangle &= \frac{1}{|D_n|} \sum_{i=0}^{n-1} \chi_j(\rho^i)\chi_j(\rho^{-i}) \\ &= \frac{1}{2n} \sum_{i=0}^{n-1} \left(2\cos\frac{2\pi ij}{n}\right)\left(2\cos\frac{-2\pi ij}{n}\right) \\ &= \frac{2}{n} \sum_{i=0}^{n-1} \cos^2\left(\frac{2\pi ij}{n}\right) \\ &= \frac{2}{n} \sum_{i=0}^{n-1} \frac{1+\cos\frac{4\pi ij}{n}}{2} \\ &= 1 + \frac{1}{n} \sum_{i=0}^{n-1} \cos\frac{4\pi ij}{n} \\ &= 1 + \delta_{j=0} \end{aligned}$$

The last equality uses the fact that the sum is zero as long as $j \neq 0$ because the sum of the kth roots of unity is zero for any $k > 1$. We conclude that X_j is irreducible for $j > 0$. This makes sense, because if $j = 0$, then the representation is diagonal and is clearly the direct sum of two other representations: the trivial representation $X^{(1)}$ and the sign representation X^τ defined by $X^\tau(\tau^e\rho^i) = (-1)^e$.
We appear to have created $n+1$ irreducible representations (the $n-1$ representations $\{X_j\}_{j=1}^{n-1}$, plus $X^{(1)}$ and X^τ), but these are not all distinct. It suffices to check which characters are the same, and it is straightforward to see that we have $\chi_j(\rho^i) = 2\cos\frac{2\pi ij}{n} = \chi_{n-j}(\rho^i)$, and these are the only pairs of characters which are the same.
That is, for n even, we have given $n/2 + 2$ irreducibles, and for n odd we have given $\lfloor n/2 \rfloor + 2$

We are missing one representation when n is even, which is given by an alternating representation $X^\rho(\tau^e\rho^i) = (-1)^i$.

Problem 1.13.17 (alternate solution)

(a) Distance-preserving transformations in R^n are translations and orthogonal transformations. Orthogonal transformations in R^2 are generated by rotations and reflections, which satisfy the relations that define D_n. (Translations are irrelevant if only because they change the center of mass of the n-gon.)

(b) Since $\rho^{-1} = \rho^{n-1}$ and $\tau^{-1} = \tau$, any element of D_n can be written as a word in ρ and τ. We can use $\rho\tau = \tau\rho^{n-1}$ to move the τ's all the way to the left, combine the resulting powers, and reduce them modulo 2 and n.

(c) What is the conjugacy class K_g? Consider the cases (1) $g = \rho^j$ and (2) $g = \tau\rho^j$. In case (1), hgh^{-1} is either g (if $h = \rho^i$) or g^{-1} (if $h = \tau\rho^i$). We have $g^{-1} = g$ if $j = 0$ or $j = n/2$, so this gives us singleton classes $K_\epsilon = \{\epsilon\}$ and (if n is even) $K_{\rho^{n/2}} = \{\rho^{n/2}\}$, as well as $\lfloor (n-1)/2 \rfloor$ classes of size two, $K_g = \{g, g^{-1}\}$.

In case (2), first consider $g = \tau$. Then hgh^{-1} is $\tau\rho^{-2i}$ if $h = \rho^i$ or $\tau\rho^{2i}$ if $h = \tau\rho^i$. If n is odd, $\tau\rho^{\pm 2i}$ ranges over all n elements of the form $\tau\rho^j$, so there is just one more congruence class K_τ. If n is even, K_τ contains the $n/2$ elements of the form $\tau\rho^{2j}$, and a similar computation shows that $K_{\tau\rho}$ contains the $n/2$ elements of the form $\tau\rho^{2j+1}$.

To sum up, if n is odd there are $(n+3)/2$ classes with sizes $(1, n, ((n-1)/2) * 2)$, and if n is even there are $(n+6)/2$ classes with sizes $(1, 1, n/2, n/2, ((n-2)/2) * 2)$.

(d) Ignoring the hint, we make an intuitive guess and then check it. Let q be any n'th root of unity (primitive or not), and define

$$X^q(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X^q(\rho) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

We see that these matrices satisfy the defining relations

$$(X^q(\rho))^n = (X^q(\tau))^2 = I, \quad X^q(\rho)X^q(\tau) = X^q(\tau)(X^q(\rho))^{-1}.$$

Therefore, if we extend X^q to all of G by

$$X^q(\tau^i\rho^j) = (X^q(\tau))^i(X^q(\rho))^j,$$

we get a homomorphism, which is to say a representation. Noting that all the matrices $X^q(\tau\rho^j)$ have trace 0, it is straightforward to compute the corresponding characters χ^q: If n is odd we have

$$\chi^q = (2, 0, q + q^{-1}, q^2 + q^{-2}, \ldots, q^{(n-1)/2} + q^{-(n-1)/2}),$$

and if n is even we have

$$\chi^q = (2, q^{n/2}, 0, 0, q + q^{-1}, q^2 + q^{-2}, \ldots, q^{(n-2)/2} + q^{-(n-2)/2}).$$

We see that $\chi^q = \chi^{1/q}$, and otherwise all the χ^q are different, so we have $n/2$ (even n) or $(n+1)/2$ (odd n) inequivalent representations. We will show that most of these are irreducible by computing inner products.

For n odd, we have

$$\begin{aligned}
\langle \chi^p, \chi^q \rangle &= 1 \cdot 2 \cdot 2 + n \cdot 0 + 2 \sum_{k=1}^{(n-1)/2} (p^k + p^{-k})(q^k + q^{-k}) \\
&= 4 + 2 \sum_{k=1}^{(n-1)/2} ((pq)^k + (1/pq)^k + (p/q)^k + (q/p)^k) \\
&= 4 + 2 \sum_{k=1}^{n-1} ((pq)^k + (p/q)^k) \\
&= 2 \sum_{k=0}^{n-1} (pq)^k + 2 \sum_{k=0}^{n-1} (p/q)^k,
\end{aligned}$$

where the third equality follows because $(pq)^n = 1$, so $(1/pq)^k = (pq)^{n-k}$, and similarly for p/q and q/p. If the two characters are different, then $p \neq q$ and $p \neq 1/q$, and pq and p/q are both roots of unity, not equal to 1, whose degree divides n. Each of the two sums adds up all their (respective) powers with the same multiplicity, so each sum is 0. Thus χ^p and χ^q are orthogonal. If $p = q \neq 1$, the first sum is 0 and the second sum is n, so $\langle \chi^q, \chi^q \rangle = 2n = |D_n|$ and χ^q is irreducible. Taking $p = q = 1$, we find $\langle \chi^1, \chi^1 \rangle = 4n$, so χ^1 is reducible. Indeed, X^1 is equivalent to the direct sum of two copies of the degree-one representation with $X^{\pm}(\rho^j) = 1$ and $X^{\pm}(\tau\rho^j) = -1$ for each j. Including X^{triv}, we now have $(n-1)/2$ irreducibles of degree 2 and 2 irreducibles of degree 1, for $(n+3)/2$ in all, as desired.

I really should do the similar computation for n even, but I'm too lazy at the moment.

(18) Suppose that s_i is a transversal for $K \subseteq H$ and u_i is a transversal for $H \subseteq G$. Then by definition we have

$$((X \uparrow_K^H) \uparrow_H^G)(g) = \left((X \uparrow_K^H)(u_i^{-1} g u_j)\right)_{i,j} = \left(\left(X(s_n^{-1} u_i^{-1} g u_j s_m)\right)_{n,m}\right)_{i,j}.$$

By Proposition 1.12.5, it suffices to show that $u_j s_i$ is a transversal for $K \subseteq G$ since the representation $X \uparrow_K^G$ does not depend on the choice of transversal. To see this, suppose we have $u_j s_i K = u_n s_m K$. Then since $s_i K, s_m K \subseteq H$, we have that the cosets $u_j H$ and $u_n H$ are not disjoint, so they coincide, so $j = n$. Multiplying by u_j^{-1}, we have $s_i K = s_m K$, so immediately we have that $i = m$. Thus $j = n$ and $i = m$, so we do have a transversal of K in G, as desired.