AMATEURISH PI STUNT NOTES

PERMUTATION ENTHUSIASTS

1. Problems 1.13

Problem 1.13.16 (alternate solution). The conjugacy classes of S_4 correspond to the cycle types, of which there are five: 1^4 (identity), $2\,1^2$ (transpositions), $2\,2$ (non-overlapping double transpositions), $3\,1$ (3-cycles), and 4 (4-cycles). Therefore we are looking for five irreducible characters. Label the conjugacy classes $K_1, \ldots, K_5$ in the order above. Simple counting arguments show that their sizes are 1, 6, 3, 8, and 6, respectively, telling us how to compute inner products of characters. The character table turns out to be as follows:

	K_1	K_2	K_3	K_4	K_5
χ^{triv}	1	1	1	1	1
χ^{sgn}	1	-1	1	1	-1
$\chi^{\perp}$	3	1	-1	0	-1
$\chi^{\otimes}$	3	-1	-1	0	1
χ^{m}	2	0	2	-1	0

The first two rows are the trivial and sign characters. To get the third character $\chi^{\perp}$, we use Maschke's theorem, following the example on page 15 of Sagan. Starting with the degree-4 standard representation, we write the matrices (for one representative of each cycle type) in the basis $\{\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4}, \mathbf{2}-\mathbf{1}, \mathbf{3}-\mathbf{1}, \mathbf{4}-\mathbf{1}\}$, which puts them in block diagonal form as the direct sum of the trivial representation and a degree-3 representation. We read off the 3-by-3 block traces $(3, 1, -1, 0, -1)$ to get $\chi^{\perp}$, and compute $\langle \chi^{\perp}, \chi^{\perp} \rangle = 24 = |S_4|$ to verify that it's irreducible.

The fourth row $\chi^{\otimes}$ comes from Problem 15(d). Taking $X = X^{sgn}$ and $Y = X^{\perp}$, and noting that X^{sgn} has degree 1, the inner tensor product $X \hat{\otimes} Y$ is just $X^{\otimes}(g) = \chi^{sgn}(g) X^{\perp}(g)$; and the problem concludes this is irreducible. The trace is just $\chi^{\otimes}(g) = \chi^{sgn}(g)\chi^{\perp}(g)$. We verify by computing inner products that it's irreducible and orthogonal to the first three characters.

We are left to find the fifth "mystery character" χ^m. If χ^m has degree d, then $\chi^m = d \cdot (1, w, x, y, z)$ for some (w, x, y, z). Orthogonality with the first four characters gives a 4-by-4 linear system that we solve to get $(w, x, y, z) = (0, 1, -1/2, 0)$; then the requirement $\langle \chi^m, \chi^m \rangle = 24$ gives $d = 2$. It is reassuring to note that $\chi^m(g) = 0$ whenever $\chi^{sgn}(g) \neq 1$, so the construction that gave us $\chi^{\otimes}$ from $\chi^{\perp}$ does not yield a sixth irreducible character from χ^m!

What representations correspond to the five characters? The trivial and sign representations are of course the same as the characters themselves. We can read off the degree-3 representation $X^{\perp}$ from the block diagonal form we computed to get the character, and then $X^{\otimes}(g)$ is just $\chi^{sgn}(g) X^{\perp}(g)$.

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What about X^m? It has degree 2, so we need 2-by-2 matrices. Let $q = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ be a primitive cube root of unity, with $q^2 = q^{-1} = \bar{q}$. If $g \in K_4$ is a 3-cycle, we need $M = X^m(g)$ to satisfy both $\operatorname{tr} M = \chi^m(g) = -1$, and also $M^3 = I$, whence $\det M$ is one of $\{1, q, q^2\}$. A 3-cycle is the product of two transpositions, which must have determinants ± 1, so the only possiblity is $\det M = 1$. Let the eigenvalues of M be λ_1 and λ_2. We have $\lambda_1 \lambda_2 = \det M = 1$, and $\lambda_1 + \lambda_2 = \operatorname{tr} M = -1$. This gives a quadratic equation that we solve to conclude that λ_1 and λ_2 are q and q^2 in some order. The simplest matrices with those eigenvalues are

$$M_1 = \begin{pmatrix} q^2 & 0 \\ 0 & q \end{pmatrix}, \quad M_2 = \begin{pmatrix} q & 0 \\ 0 & q^2 \end{pmatrix}.$$

Noting that $M_2 = M_1^{-1}$, we would like to set things up to have $X^m(g) = M_1$ and $X^m(g^{-1}) = M_2$ for four of the eight 3-cycles $g \in S_4$.

A 3-cycle is the product of two transpositions that overlap in one element. We note that the six transpositions in S_4 come in three non-overlapping pairs, an example of which is $(1\,3), (2\,4)$. Thus we need at least three 2-by-2 matrices whose products are M_1 or M_2, each with trace 2 and square I (whence determinant ± 1). Three matrices that fit the bill are

$$T_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_b = \begin{pmatrix} 0 & q \\ q^2 & 0 \end{pmatrix}, \quad T_c = \begin{pmatrix} 0 & q^2 \\ q & 0 \end{pmatrix}.$$

Indeed, the entire representation is determined by the representations of transpositions, so we'll guess that the three matrices immediately above do indeed determine an irreducible representation whose character is χ^m. It's easy to check that various things work out—for example, the traces of the non-overlapping double transpositions in K_3 are 2, and the traces of the 4-cycles in K_5 are 0.

To make sure (and to learn a little bit of Sage), I wrote a Jupyter notebook (attached) that computes the representations of all the elements of S_4 based on the three transposition matrices above, and then verifies the entire multiplication table.