AMATEURISH PI STUNT NOTES

PERMUTATION ENTHUSIASTS

1. Problems 1.13

1 (a) Consider the transposition $\pi = (i, j)$. It swaps the order of exactly $2(j-i-1)+1$ pairs of inputs. This is because there are $j-i-1$ entries strictly between i and j, and π swaps both i and j with all of these. It also swaps the order of i and j. Therefore, $\mathrm{inv}((i,j)) \equiv 1 \bmod 2$. Next we show that inv is a homomorphism to $\mathbb{Z}/2$. To do this, let $\mathrm{inv}_{i,j}(\pi)$ be 1 if π swaps the order of i and j and 0 otherwise, so $\mathrm{inv}(\pi) = \sum_{i \neq j} \mathrm{inv}_{i,j}(\pi)$. If we have two permutations π and τ, we have $\mathrm{inv}_{i,j}(\tau\pi) = \mathrm{inv}_{i,j}(\pi) + \mathrm{inv}_{\pi(i),\pi(j)}(\tau) \bmod 2$. I.e. i and j get swapped if π swaps i and j or τ swaps $\pi(i), \pi(j)$ (but not both because they get swapped back if so). Therefore

$$\begin{aligned}
\mathrm{inv}(\tau\pi) &= \sum_{i \neq j} \mathrm{inv}_{i,j}(\tau\pi) \\
&\equiv \sum_{i \neq j} \mathrm{inv}_{i,j}(\pi) + \mathrm{inv}_{\pi(i),\pi(j)}(\tau) \bmod 2 \\
&\equiv \left(\sum_{i \neq j} \mathrm{inv}_{i,j}(\pi) + \sum_{i \neq j} \mathrm{inv}_{\pi(i),\pi(j)}(\tau) \right) \bmod 2 \\
&\equiv \mathrm{inv}(\pi) + \mathrm{inv}(\tau) \bmod 2
\end{aligned}$$

Taken together, these facts imply that if π is a product of k transpositions, then $\mathrm{inv}(\pi) \equiv k \bmod 2$.

(b) We defined $\mathrm{sgn}(\pi) = (-1)^k$ when π is a product of k transpositions. It may be that π can be written as a product of k transpositions and also as a product of $\ell \neq k$ transpositions. However, we can see that $\mathrm{sgn}(\pi)$ is well defined because $\mathrm{inv}(\pi)$ is well defined, and we saw in (a) that $k \equiv \mathrm{inv}(\pi) \equiv \ell \bmod 2$. Thus $(-1)^k = (-1)^\ell$.

2 (a) We clearly have $\epsilon \in G_s$ because $\epsilon s = s$ by an axiom of group actions. To see that G_s is closed under multiplication, let us be given $g, h \in G_s$, and we compute $(gh)s = g(hs) = gs = s$ (the second equality uses a group action axiom).

(b) Define $\phi : G/G_s \to \mathcal{O}_s$ by $\phi(hG_s) = hs$. This is well-defined because if $h = hg$ for $g \in G_s$, then $(hg)s = h(g(s)) = hs$. (I'm assuming left cosets.) The map ϕ is surjective because if we are given any $h \in G$, then $\phi(hG_s) = hs$. The map ϕ is injective because if $\phi(hG_s) = \phi(kG_s)$, then $hs = ks$, so $k^{-1}h \in G_s$ and hence $hG_s = kG_s$.

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(c) We know $|G/G_s| = |G|/|G_s|$ by some isomorphism theorem, and by (b) we know $|\mathcal{O}_s| = |G/G_s|$.

3 (a) We must show that every matrix $X(\pi)$ has exactly one 1 in each row and column, and zeros elsewhere. The definition of the permutation representation has $X(\pi)_{i,j} = \delta_{i=\pi(j)}$. Because π is a permutation, for each i there is exactly one j such that $i = \pi(j)$, and for each j there is exactly one i such that $i = \pi(j)$.

(b) A fixed point i of π has $\pi(i) = i$, so $X(\pi)_{i,i} = 1$, so there is a 1 on the diagonal of $X(\pi)$ in position (i,i) exactly when i is fixed under π. Thus $\mathrm{Tr}(X(\pi))$ is the number of fixed points of π.

4 Since G is finite, it can be written as $\oplus_i C_{j_i}$, where C_{j_i} is a cyclic group of order j_i, say with generator g_i. By Corollary 1.6.8 (see also Problem 12), we must have $X(g) = cI$ for all $g \in G$, so X is one dimensional, and $X(g_i)$ is some j_ith root of unity. All such representations are irreducible since they are one dimensional.

5 (a) Let $g \in N$ and $h \in G$. Then $X(hgh^{-1}) = X(h)X(g)X(h^{-1}) = X(h)X(h)^{-1} = I$, so $hgh^{-1} \in N$. This holds for all $g \in N$, so N is normal. A condition is: X is faithful iff $N = \{\epsilon\}$. To see one direction, suppose X is faithful. Then $I = X(\epsilon) = X(g)$ only if $g = \epsilon$; hence $N = \{\epsilon\}$. For the other direction, suppose $N = \{\epsilon\}$ and $X(g) = X(h)$ for some g, h. Then $X(gh^{-1}) = I$, so $gh^{-1} = \epsilon$, so $g = h$, and X is faithful.

(b) One direction is immediate, because if $g \in N$, then $\mathrm{Tr}(X(g)) = \mathrm{Tr}(I) = d$. For the other direction, suppose $\chi(g) = d$. TODO

(c) For one direction, suppose that $h \in \bigcap_i g_i H g_i^{-1}$. Then for all i, we have $hg_i \in g_i H$, so $hg_i H \subseteq g_i HH = g_i H$ sends each coset to itself, so $h \in N$. Conversely, if $X(h) = I$, then h sends each coset to itself, so $hg_i H \in g_i H$ for all i. Hence $h \in g_i H g_i^{-1}$ for all i.

(d) (i) Trivial: this is faithful exactly if G is trivial
(ii) Regular: always
(iii) Coset: when the intersection of the conjugates of H is trivial (see previous)
(iv) Sign for $\mathcal{S}_n$: for $\mathcal{S}_1$ and $\mathcal{S}_2$
(v) Defining for $\mathcal{S}_n$: always
(vi) Degree 1 for C_n: exactly when $X(g)$ is a primitive root of unity, for a generator g

(e) (i) Y is well-defined because if $gN = hN$, then there is $n \in N$ so that $gn = h$. Then $X(h) = X(gn) = X(g)X(n) = X(g)I = X(g)$. It is faithful because if $Y(gN) = I$, then by definition we have $X(g) = I$, so $g \in N$, ie. the only coset that maps to I under Y is ϵN.
(ii) Whether or not a representation is irreducible depends only on the set of matrices (or endomorphisms) in the image. The image of Y is the same as the image of X. Said another way, if $X(g)(V) \subseteq V$ for some subspace V and for all g, then $Y(gN)(V) = X(g)(V) \subseteq V$ as well, and vice versa.

(iii) The representation Y is the regular representation of G/H. To see this, let us start by finding the kernel N of the coset representation. Suppose $n \in N$, so $ngH = gH$ for all g. Because H is normal, we have $ngH = Hng$, so $Hng = gH$. Thus $Hn = gHg^{-1} = H$, so $n \in H$. The entire argument runs backward, so $N = H$. Let V be the coset representation (so Y is a map $Y : G/H \to GL(V)$) and consider the map $\theta : V \to \mathbb{C}[G/H]$ defined by $\theta(gH) = gH$. This is clearly a bijection, so we just need to check it is a G/H-homomorphism. To see this, we compute $\theta(Y(gH)(hH)) = ghH$ and $gH\theta(hH) = ghH$ (by the definition of group multiplication in G/H).

(6) (a) To see that X is a representation, we just need to check that $X(gh) = X(g)X(h)$. We compute $X(gh) = Y(ghN) = Y(gNhN) = Y(gN)Y(hN) = X(g)X(h)$, where in the middle we used the multiplication in G/H.
(b) Let $g \ker(X)$, so $I = X(g) = Y(gN)$. Since Y is faithful, we have $gN = \epsilon N$, so $g \in N$. Conversely, any $ginN$ is in $\ker(X)$ because $X(g) = Y(gN) = Y(\epsilon N) = I$.
(c) This is the same as (5)(e)(ii) – the irreducibility only depends on the image set of matrices, which remains the same under lifting.

(7) The block decomposition of X expresses V as the internal direct sum $W + Y$, where if we write any vector (w, y) aligned with the block form, we have $X(g)(w, y) = (A(g)w + B(g)y, C(g)y)$. The quotient map $V \to V/W$ projects to the second coordinate and is a G-homomorphism which takes the action X to C. Maschke's theorem says that V is isomorphic to a block diagonal action with the matrices A and C, and this is exactly the actions on W and V/W.

(8) I'm not sure about these
(a) The action of G can be given by a matrix in the basis?
(b) The map θ is linear and for all $g \in G$ and b in the basis, we have $g\theta(b) = \theta(gb)$?
(c) For all b, c in the basis, we have $\langle b, c\rangle = \langle gb, gc\rangle$?

(10) The map $X(r) = \begin{bmatrix} 1 & \log r \\ 0 & 1 \end{bmatrix}$ satisfies

$$\begin{aligned} X(r)X(s) &= \begin{bmatrix} 1 & \log r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \log s \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \log(rs) \\ 0 & 1 \end{bmatrix} \\ &= X(rs), \end{aligned}$$

and we can see $X(r) \begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}$.

(11) Let $H = \mathcal{S}_{n-1} \subseteq \mathcal{S}_n$, and let S be the set of tabloids of shape $(n-1, 1)$. We want to show that $\mathbb{C}\mathcal{H} \cong \mathbb{C}\mathbf{S} \cong \mathbb{C}\{\mathbf{1}, \dots, \mathbf{n}\}$. First we need to find a transversal for H. Note that $|H| = (n-1)!$, so the index of H in $\mathcal{S}_n$ is n, so it suffices to show our n chosen cosets are pairwise disjoint. To do this, suppose $(i, n)H = (j, n)H$, so $(j, n)(i, n) \in H$. Note this product fixes n (equivalently, is in H) exactly when $i = j$. Hence the cosets are disjoint. It is also useful to observe that if $i \neq j$, we have $(i, n)(j, n) = (j, n)(i, j)$,

so $(i,n)(\mathbf{j},\mathbf{n})\mathbf{H} = (\mathbf{j},\mathbf{n})\mathbf{H}$, while if $i = j$, then we compute that (i,n) interchanges the cosets ϵH and $(i,n)H$. This gives us the action on cosets.

Now define the equivalence $\theta : \mathbb{C}\mathcal{H} \to \mathbb{C}\{\mathbf{1},\dots,\mathbf{n}\}$ by $\theta((\mathbf{i},\mathbf{n})\mathbf{H}) = \mathbf{i}$ and $\iota : \mathbb{C}\mathbf{S} \to \mathbb{C}\{\mathbf{1},\dots,\mathbf{n}\}$ by taking a tabloid basis element to the basis element of $\mathbb{C}\{\mathbf{1},\dots,\mathbf{n}\}$ associated with the single entry in the second level of the tabloid. Since $\{(i,n)\}_{i=1}^{n}$ generates $\mathcal{S}_n$, it suffices to show that these maps commute with the action of these involutions. This is immediate for ι because the image is by definition the entry in the bottom of the tabloid. For θ, we use the coset action we determined above, so $\theta((i,n)(\mathbf{j},\mathbf{n})\mathbf{H}) = \theta((\mathbf{j},\mathbf{n})\mathbf{H}) = \mathbf{j}$ if $i \neq j$, and $\theta((i,n)(\mathbf{i},\mathbf{n})\mathbf{H}) = \mathbf{n}$ and $\theta((i,n)\epsilon\mathbf{H}) = \mathbf{i}$. That is, the action of (i,n) swaps the corresponding pairs of basis elements on both sides of θ.

(12) By Corollary 1.6.8, any matrix that commutes with $X(g)$ for all g must be of the form cI. If $g \in Z_G$, then by definition $X(g)$ commutes with $X(h)$ for all $h \in G$, and the conclusion is immediate.

(13) Let G be the abelian group formed by the matrices X_i. So the map $Y(X_i) = X_i$ is a d-dimensional representation of G. By Maschke's theorem, there is a single matrix T such that TX_iT^{-1} is a block diagonal matrix of irreducible representations. It remains to show that any irreducible representation of an abelian group is 1-dimensional, which maybe we just know that, or maybe we observe that by Corollary 1.6.8, if G is an abelian group, any image matrix in an irreducible representation must be a multiple of the identity and thus must be 1-dimensional.

(14) Suppose towards a contradiction that X is reducible, so up to isomorphism we can simultaneously write the matrices $X(g)$ in a nontrivial block form. But then $X(g)$ commutes with block diagonal matrices with blocks xI, yI, for any $x, y \in \mathbb{C}$. Many such matrices are not of the form cI, which is a contradiction.

(15) (a) We must check that $(X\hat{\otimes}Y)(gh) = (X\hat{\otimes}Y)(g)(X\hat{\otimes}Y)(h)$. To do this, we compute:

$$\begin{aligned}(X\hat{\otimes}Y)(g)(X\hat{\otimes}Y)(h) &= (X(g)\otimes Y(g))(X(h)\otimes Y(h))\\ &= X(g)X(h)\otimes Y(g)Y(h)\\ &= X(gh)\otimes Y(gh)\\ &= (X\hat{\otimes}Y)(gh)\end{aligned}$$

The second equality uses Lemma 1.7.7.

(b) We can compute

$$\begin{aligned}(\chi\hat{\otimes}\psi)(g) &= \mathrm{Tr}((X\hat{\otimes}Y)(g))\\ &= \mathrm{Tr}(X(g)\otimes Y(g))\\ &= \sum_i X(g)_{i,i}\,\mathrm{Tr}(Y(g))\\ &= \sum_i X(g)_{i,i}\psi(g)\\ &= \chi(g)\psi(g)\end{aligned}$$

(c) If X and Y are both the irreducible 2-dimensional representation of $\mathcal{S}_3$, then $X\hat{\otimes}Y$ has dimension 4, but $\mathcal{S}_3$ has no 4-dimensional irreducible representations.

(d) We can check that it is irreducible by computing

$$\begin{aligned}\langle \chi\hat{\otimes}\psi, \chi\hat{\otimes}\psi\rangle &= \frac{1}{|G|}\sum_g (\chi\hat{\otimes}\psi)(g)(\chi\hat{\otimes}\psi)(g^{-1}) \\ &= \frac{1}{|G|}\sum_g \chi(g)\psi(g)\chi(g^{-1})\psi(g^{-1}) \\ &= \frac{1}{|G|}\sum_g \psi(g)\psi(g^{-1}) \\ &= \langle \psi, \psi\rangle \\ &= 1\end{aligned}$$

This relies on the fact that X is one-dimensional, so $\operatorname{Tr}(X(g^{-1})) = 1/\operatorname{Tr}(X(g))$, so $\chi(g)\chi(g^{-1}) = 1$.

(16) There are five cycle types/conjugacy classes $\epsilon, (1,2), (1,2,3), (1,2)(3,4), (1,2,3,4)$, of sizes 1, 6, 8, 3, and 6, respectively. Because there are five, we are expecting five irreducible representations. We know the trivial $\chi^{(1)}$ and sign $\chi^{(2)}$ representations are irreducible, and we know the representation $\chi^{\perp}$ orthogonal to the trivial one inside the defining representation, which we can verify is irreducible by computing its self inner product. In addition, we compute the character for $\chi^{(2)}\hat{\oplus}\chi^{\perp}$ and see it too is irreducible.

For the final irreducible, consider the normal subgroup N which is ϵ and the conjugacy class of $(1,2)(3,4)$ (This is the Klein four group $\mathbb{Z}/2\times\mathbb{Z}/2$). In fact, we have $\mathcal{S}_4/N \cong \mathcal{S}_3$. To see this, consider the map $\phi : \mathcal{S}_3 \to \mathcal{S}_4$ defined by $\phi(\pi) = \pi N$. If $\pi N = \rho N$, then there is $n \in N$ so that $\pi = \rho n$. If $n \neq \epsilon$, then note that n, and thus ρn, does not fix 4 (here $\rho \in \mathcal{S}_3$, so if n permutes 4 away from itself, ρ cannot put it back). On the other hand π does fix 4. This contradiction implies that $n = \epsilon$, so ϕ is injective. Since $|\mathcal{S}_3| = 6$ and $|\mathcal{S}_4/N| = 6$, in fact ϕ is an isomorphism. To compute the quotient map from $\mathcal{S}_4$ to $\mathcal{S}_3$ on a permutation π which does not fix 4, we need to find $\rho n = \pi$, with $n \in N$ and ρ fixing 4.

So we can use the lifting process from problem 6 to lift each of the 3 irreducible representations of $\mathcal{S}_3$ to representations of $\mathcal{S}_4$. The trivial and sign representations lift to the trivial and sign representations, respectively, and give us nothing new. But the third irreducible $\chi^{(3)}$ does give us the final, 2 dimensional, irreducible representation $\chi^{(3)}$ of $\mathcal{S}_4$. (We can check that it is irreducible by computing its self inner product.)

	ϵ	$(1,2)$	$(1,2,3)$	$(1,2)(3,4)$	$(1,2,3,4)$
$\chi^{(1)}$ trivial	1	1	1	1	1
$\chi^{(2)}$ sign	1	-1	1	1	-1
$\chi^{\perp}$	3	1	0	-1	-1
$\chi^{(2)}\hat{\otimes}\chi^{\perp}$	3	-1	0	-1	1
$\chi^{(3)}$	2	0	-1	2	0

(17) (a) We can flip (τ, order 2) rotate (ρ, order n), and playing with a shape shows that $\rho\tau = \tau\rho^{-1}$.

(b) If we have any sequence of τ and ρ, we can slide all ρ to the right using the relation $\rho\tau = \tau\rho^{-1}$
(c) We compute:

$$(\tau^e\rho^\ell)\rho^j(\rho^{-\ell}\tau^e) = \begin{cases} \rho^j & \text{if } e = 0 \\ \rho^{-j} & \text{if } e = 1 \end{cases}$$

$$(\tau^e\rho^\ell)\tau\rho^j(\rho^{-\ell}\tau^e) = \begin{cases} \tau\rho^{2\ell-j} & \text{if } e = 0 \\ \tau\rho^{j-2\ell} & \text{if } e = 1 \end{cases}$$

These relations determine the conjugacy classes of D_n. The answer depends on whether n is odd (the issue is whether 2 is relatively prime to n, ie. whether 2 is a generator of the additive group $\mathbb{Z}/n$).
If n is even, then the conjugacy classes are

$$\{\epsilon\}, \{\rho^1, \rho^{n-1}\}, \ldots, \{\rho^{n/2-1}, \rho^{n/2+1}\}, \{\rho^{n/2}\}, \{\tau, \tau\rho^2, \ldots, \tau\rho^{n-2}\}, \{\tau\rho, \tau\rho^3, \ldots, \tau\rho^{n-1}\},$$

so there are $n/2+3$ classes total. If n is odd, the conjugacy classes are

$$\{\epsilon\}, \{\rho^1, \rho^{n-1}\}, \ldots, \{\rho^{\frac{n-1}{2}}, \rho^{\frac{n+1}{2}}\}, \{\tau, \tau\rho, \ldots, \tau\rho^{n-1}\},$$

so there are $\lfloor n/2 \rfloor + 2$ classes total.
(d) There are some simple-to-define representations X_j, which we will check are irreducible. Define X_j as follows, where ρ is mapped to a rotation

$$X_j(\rho) = \begin{bmatrix} \cos\frac{2\pi j}{n} & -\sin\frac{2\pi j}{n} \\ \sin\frac{2\pi j}{n} & \cos\frac{2\pi j}{n} \end{bmatrix}$$

and τ to a flip

$$X_j(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The representation X_1 is the "defining" representation of D_n as it's usually defined acting on a n-sided polygon in the plane. (Although this is a 2-dimensional *complex* representation!)
Let χ_j be the character of X_j. Note that

$$\chi_j(\rho^i) = \mathrm{Tr}(X_j(\rho^i)) = 2\cos\frac{2\pi ij}{n} \quad \text{and} \quad \chi_j(\tau\rho^i) = 0.$$

In particular, we can ignore $\tau\rho^i$ for all subsequent calculations of characters.

We can compute

$$\begin{aligned}
\langle \chi_j, \chi_j \rangle &= \frac{1}{|D_n|} \sum_{i=0}^{n-1} \chi_j(\rho^i)\chi_j(\rho^{-i}) \\
&= \frac{1}{2n} \sum_{i=0}^{n-1} \left(2\cos\frac{2\pi ij}{n}\right)\left(2\cos\frac{-2\pi ij}{n}\right) \\
&= \frac{2}{n} \sum_{i=0}^{n-1} \cos^2\left(\frac{2\pi ij}{n}\right) \\
&= \frac{2}{n} \sum_{i=0}^{n-1} \frac{1+\cos\frac{4\pi ij}{n}}{2} \\
&= 1 + \frac{1}{n} \sum_{i=0}^{n-1} \cos\frac{4\pi ij}{n} \\
&= 1 + \delta_{j=0}
\end{aligned}$$

The last equality uses the fact that the sum is zero as long as $j \neq 0$ because the sum of the kth roots of unity is zero for any $k > 1$. We conclude that X_j is irreducible for $j > 0$. This makes sense, because if $j = 0$, then the representation is diagonal and is clearly the direct sum of two other representations: the trivial representation $X^{(1)}$ and the sign representation X^τ defined by $X^\tau(\tau^e\rho^i) = (-1)^e$.
We appear to have created $n+1$ irreducible representations (the $n-1$ representations $\{X_j\}_{j=1}^{n-1}$, plus $X^{(1)}$ and X^τ), but these are not all distinct. It suffices to check which characters are the same, and it is straightforward to see that we have $\chi_j(\rho^i) = 2\cos\frac{2\pi ij}{n} = \chi_{n-j}(\rho^i)$, and these are the only pairs of characters which are the same.
That is, for n even, we have given $n/2+2$ irreducibles, and for n odd we have given $\lfloor n/2 \rfloor + 2$
We are missing one representation when n is even, which is given by an alternating representation $X^\rho(\tau^e\rho^i) = (-1)^i$.

(18) Suppose that s_i is a transversal for $K \subseteq H$ and u_i is a transversal for $H \subseteq G$. Then by definition we have

$$((X\uparrow_K^H)\uparrow_H^G)(g) = \left((X\uparrow_K^H)(u_i^{-1}gu_j)\right)_{i,j} = \left(\left(X(s_n^{-1}u_i^{-1}gu_js_m)\right)_{n,m}\right)_{i,j}.$$

By Proposition 1.12.5, it suffices to show that u_js_i is a transversal for $K \subseteq G$ since the representation $X\uparrow_K^G$ does not depend on the choice of transversal. To see this, suppose we have $u_js_iK = u_ns_mK$. Then since $s_iK, s_mK \subseteq H$, we have that the cosets u_jH and u_nH are not disjoint, so they coincide, so $j = n$. Multiplying by u_j^{-1}, we have $s_iK = s_mK$, so immediately we have that $i = m$. Thus $j = n$ and $i = m$, so we do have a transversal of K in G, as desired.