AMATEURISH PI STUNT NOTES

PERMUTATION ENTHUSIASTS

1. Problems 1.13

Problem 1.13.16 (alternate solution). The conjugacy classes of S_4 correspond to the cycle types, of which there are five: 1^4 (identity), $2\,1^2$ (transpositions), $2\,2$ (non-overlapping double transpositions), $3\,1$ (3-cycles), and 4 (4-cycles). Therefore we are looking for five irreducible characters. Label the conjugacy classes $K_1, \ldots, K_5$ in the order above. Simple counting arguments show that their sizes are 1, 6, 3, 8, and 6, respectively, telling us how to compute inner products of characters. The character table turns out to be as follows:

	K_1	K_2	K_3	K_4	K_5
χ^{triv}	1	1	1	1	1
χ^{sgn}	1	-1	1	1	-1
$\chi^{\perp}$	3	1	-1	0	-1
$\chi^{\otimes}$	3	-1	-1	0	1
χ^{m}	2	0	2	-1	0

The first two rows are the trivial and sign characters. To get the third character $\chi^{\perp}$, we use Maschke's theorem, following the example on page 15 of Sagan. Starting with the degree-4 standard representation, we write the matrices (for one representative of each cycle type) in the basis $\{\mathbf{1}+\mathbf{2}+\mathbf{3}+\mathbf{4}, \mathbf{2}-\mathbf{1}, \mathbf{3}-\mathbf{1}, \mathbf{4}-\mathbf{1}\}$, which puts them in block diagonal form as the direct sum of the trivial representation and a degree-3 representation. We read off the 3-by-3 block traces $(3, 1, -1, 0, -1)$ to get $\chi^{\perp}$, and compute $\langle \chi^{\perp}, \chi^{\perp} \rangle = 24 = |S_4|$ to verify that it's irreducible.

The fourth row $\chi^{\otimes}$ comes from Problem 15(d). Taking $X = X^{sgn}$ and $Y = X^{\perp}$, and noting that X^{sgn} has degree 1, the inner tensor product $X \hat{\otimes} Y$ is just $X^{\otimes}(g) = \chi^{sgn}(g) X^{\perp}(g)$; and the problem concludes this is irreducible. The trace is just $\chi^{\otimes}(g) = \chi^{sgn}(g)\chi^{\perp}(g)$. We verify by computing inner products that it's irreducible and orthogonal to the first three characters.

We are left to find the fifth "mystery character" χ^m. If χ^m has degree d, then $\chi^m = d \cdot (1, w, x, y, z)$ for some (w, x, y, z). Orthogonality with the first four characters gives a 4-by-4 linear system that we solve to get $(w, x, y, z) = (0, 1, -1/2, 0)$; then the requirement $\langle \chi^m, \chi^m \rangle = 24$ gives $d = 2$. It is reassuring to note that $\chi^m(g) = 0$ whenever $\chi^{sgn}(g) \neq 1$, so the construction that gave us $\chi^{\otimes}$ from $\chi^{\perp}$ does not yield a sixth irreducible character from χ^m!

What representations correspond to the five characters? The trivial and sign representations are of course the same as the characters themselves. We can read off the degree-3 representation $X^{\perp}$ from the block diagonal form we computed to get the character, and then $X^{\otimes}(g)$ is just $\chi^{sgn}(g) X^{\perp}(g)$.

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What about X^m? It has degree 2, so we need 2-by-2 matrices. Let $q = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$ be a primitive cube root of unity, with $q^2 = q^{-1} = \bar{q}$. If $g \in K_4$ is a 3-cycle, we need $M = X^m(g)$ to satisfy both $\operatorname{tr} M = \chi^m(g) = -1$, and also $M^3 = I$, whence $\det M$ is one of $\{1, q, q^2\}$. A 3-cycle is the product of two transpositions, which must have determinants ± 1, so the only possiblity is $\det M = 1$. Let the eigenvalues of M be λ_1 and λ_2. We have $\lambda_1 \lambda_2 = \det M = 1$, and $\lambda_1 + \lambda_2 = \operatorname{tr} M = -1$. This gives a quadratic equation that we solve to conclude that λ_1 and λ_2 are q and q^2 in some order. The simplest matrices with those eigenvalues are

$$M_1 = \begin{pmatrix} q^2 & 0 \\ 0 & q \end{pmatrix}, \quad M_2 = \begin{pmatrix} q & 0 \\ 0 & q^2 \end{pmatrix}.$$

Noting that $M_2 = M_1^{-1}$, we would like to set things up to have $X^m(g) = M_1$ and $X^m(g^{-1}) = M_2$ for four of the eight 3-cycles $g \in S_4$.

A 3-cycle is the product of two transpositions that overlap in one element. We note that the six transpositions in S_4 come in three non-overlapping pairs, an example of which is $(1\,3), (2\,4)$. Thus we need at least three 2-by-2 matrices whose products are M_1 or M_2, each with trace 0 and square I (whence determinant ± 1). Three matrices that fit the bill are

$$T_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T_b = \begin{pmatrix} 0 & q \\ q^2 & 0 \end{pmatrix}, \quad T_c = \begin{pmatrix} 0 & q^2 \\ q & 0 \end{pmatrix}.$$

Indeed, the entire representation is determined by the representations of transpositions, so we'll guess that the three matrices immediately above determine an irreducible representation whose character is χ^m. It's easy to check that various things work out—for example, the traces of the non-overlapping double transpositions in K_3 are 2, and the traces of the 4-cycles in K_5 are 0.

To make sure (and to learn a little bit of Sage), I wrote a Jupyter notebook that computes the representations of all the elements of S_4 based on the three transposition matrices above, and then verifies the entire multiplication table.

Problem 1.13.17 (alternate solution). (a) Distance-preserving transformations in R^n are translations and orthogonal transformations. Orthogonal transformations in R^2 are generated by rotations and reflections, which satisfy the relations that define D_n. (Translations are irrelevant if only because they change the center of mass of the n-gon.)

(b) Since $\rho^{-1} = \rho^{n-1}$ and $\tau^{-1} = \tau$, any element of D_n can be written as a word in ρ and τ. We can use $\rho\tau = \tau\rho^{n-1}$ to move the τ's all the way to the left, combine the resulting powers, and reduce them modulo 2 and n.

(c) What is the conjugacy class K_g? Consider the cases (1) $g = \rho^j$ and (2) $g = \tau\rho^j$. In case (1), hgh^{-1} is either g (if $h = \rho^i$) or g^{-1} (if $h = \tau\rho^i$). We have $g^{-1} = g$ if $j = 0$ or $j = n/2$, so this gives us singleton classes $K_\epsilon = \{\epsilon\}$ and (if n is even) $K_{\rho^{n/2}} = \{\rho^{n/2}\}$, as well as $\lfloor (n-1)/2 \rfloor$ classes of size two, $K_g = \{g, g^{-1}\}$.

In case (2), first consider $g = \tau$. Then hgh^{-1} is $\tau\rho^{-2i}$ if $h = \rho^i$ or $\tau\rho^{2i}$ if $h = \tau\rho^i$. If n is odd, $\tau\rho^{\pm 2i}$ ranges over all n elements of the form $\tau\rho^j$, so there is just one more congruence class K_τ. If n is even, K_τ contains the $n/2$ elements of the form $\tau\rho^{2j}$, and a similar computation shows that $K_{\tau\rho}$ contains the $n/2$ elements of the form $\tau\rho^{2j+1}$.

To sum up, if n is odd there are $(n+3)/2$ classes with sizes $(1, n, ((n-1)/2) * 2)$, and if n is even there are $(n+6)/2$ classes with sizes $(1, 1, n/2, n/2, ((n-2)/2) * 2)$.

(d) Ignoring the hint, we make an intuitive guess and then check it. Let q be any n'th root of unity (primitive or not), and define

$$X^q(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X^q(\rho) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

We see that these matrices satisfy the defining relations

$$(X^q(\rho))^n = (X^q(\tau))^2 = I, \quad X^q(\rho)X^q(\tau) = X^q(\tau)(X^q(\rho))^{-1}.$$

Therefore, if we extend X^q to all of G by

$$X^q(\tau^i\rho^j) = (X^q(\tau))^i(X^q(\rho))^j,$$

we get a homomorphism, which is to say a representation. Noting that all the matrices $X^q(\tau\rho^j)$ have trace 0, it is straightforward to compute the corresponding characters χ^q: If n is odd we have

$$\chi^q = (2, 0, q + q^{-1}, q^2 + q^{-2}, \ldots, q^{(n-1)/2} + q^{-(n-1)/2}),$$

and if n is even we have

$$\chi^q = (2, q^{n/2}, 0, 0, q + q^{-1}, q^2 + q^{-2}, \ldots, q^{(n-2)/2} + q^{-(n-2)/2}).$$

We see that $\chi^q = \chi^{1/q}$, and otherwise all the χ^q are different, so we have $n/2$ (even n) or $(n+1)/2$ (odd n) inequivalent representations. We will show that most of these are irreducible by computing inner products.

For n odd, we have

$$\begin{aligned}
\langle \chi^p, \chi^q \rangle &= 1 \cdot 2 \cdot 2 + n \cdot 0 + 2 \sum_{k=1}^{(n-1)/2} (p^k + p^{-k})(q^k + q^{-k}) \\
&= 4 + 2 \sum_{k=1}^{(n-1)/2} ((pq)^k + (1/pq)^k + (p/q)^k + (q/p)^k) \\
&= 4 + 2 \sum_{k=1}^{n-1} ((pq)^k + (p/q)^k) \\
&= 2 \sum_{k=0}^{n-1} (pq)^k + 2 \sum_{k=0}^{n-1} (p/q)^k,
\end{aligned}$$

where the third equality follows because $(pq)^n = 1$, so $(1/pq)^k = (pq)^{n-k}$, and similarly for p/q and q/p. If the two characters are different, then $p \neq q$ and $p \neq 1/q$, and pq and p/q are both roots of unity, not equal to 1, whose degree divides n. Each of the two sums adds up all their (respective) powers with the same multiplicity, so each sum is 0. Thus χ^p and χ^q are orthogonal. If $p = q \neq 1$, the first sum is 0 and the second sum is n, so $\langle \chi^q, \chi^q \rangle = 2n = |D_n|$ and χ^q is irreducible. Taking $p = q = 1$, we find $\langle \chi^1, \chi^1 \rangle = 4n$, so χ^1 is reducible. Indeed, X^1 is equivalent to the direct sum of two copies of the degree-one representation with $X^{\pm}(\rho^j) = 1$ and $X^{\pm}(\tau\rho^j) = -1$ for each j. Including X^{triv}, we now have $(n-1)/2$ irreducibles of degree 2 and 2 irreducibles of degree 1, for $(n+3)/2$ in all, as desired.

I really should do the similar computation for n even, but I'm too lazy at the moment.