Fys4150
Project 5 Analytical solution

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Analytical solution 1D

The one-dimensional problem is written as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t},\ t > 0,\ x \in [0, L] \tag{1a}$$

$$u(x,0) = 0,\ 0 < x < L, \tag{1b}$$
$$u(0,t) = 0,\ t > 0 \tag{1c}$$
$$u(L,t) = 1,\ t > 0 \tag{1d}$$

With initial conditions at $t = 0$ and $L = 1$ is the length of the x-region of interest.

The problem consists of non-homogeneous boundary conditions, and it is not trivial to find a closed form solution. The problem can be seen as a physical problem such as a temperature gradient in a rod or flow between two infinite flat plates, where the fluid is initially at rest and the plate at $x = 1$ is given a sudden movement. From physical observations we know that as time goes, i.e. when $t \to \infty$, the problem coinsides whith its surroundings and becomes steady.

It is reasonable to introduce a solution consisting of the sum of two parts, a steady-state solution, and a transient solution which is the part that depends on the initial conditions.

We end up with the solution of the problem beeing defined as

$$u(x,t) = U(x) + V(x,t) \tag{2}$$

where $U(x)$ is the steady-state solution and $V(x,t)$ is the transient solution.

Since the steady-state problem is not changing in time, i.e. $\partial_t = 0$, it is written in a compact form as:

$$U_{xx} = 0\ ,\ x \in [0, 1] \tag{3a}$$

$$U(0) = 0 \tag{3b}$$
$$U(L) = 1 \tag{3c}$$

For the transient part of the problem we have the problem written in compact form as:

$$V_t = V_{xx}\ ,\ t > 0\ ,\ x \in [0, 1] \tag{4a}$$

$$V(x,0) = -U_s(x)\ ,\ 0 < x < 1 \tag{4b}$$
$$V(0,t) = 0\ ,\ t > 0 \tag{4c}$$
$$V(1,t) = 0\ ,\ t > 0 \tag{4d}$$

The steady state solution is solved by integrating 3a twice

$$\begin{aligned}\int \frac{\partial^2 u(x,t)}{\partial x^2} &= A \\ \int \frac{\partial u(x,t)}{\partial x} &= \int A\end{aligned}$$

and we end up with the general solution $U(x) = Ax + B$. Applying the boundary conditions at $L = 0$ and $L = 1$ we get that

$$U(x) = x \tag{5}$$

The transient problem 4a has homogeneous boundary conditions, and we can solve it by seperation of variables. We start by making the anzats that $V(x,t) = T(t)X(x)$ so that we can rewrite 4a as:

$$XT' = X''T \tag{6}$$

Reordering the equation we get the following:

$$\frac{T'}{T} = \frac{X''}{X} = k, \tag{7}$$

where k is an unknown, possible complex constant. The choosing of k can be explained by the following arguments. We are looking for a solution that is eventually going to be steady. With a positive k we would never get a steady state, as the factor $T(t)$ does not go to zero as $t \to \infty$. Choosing k as zero would give us the steady-solution which we already have. We are the then left with choosing k as a negative constant. With $k = -\lambda^2$ we ensure thath k is negative.

The term can be solved seperately as we get that the left hand side is only depentent on t and the right hand side is only depentent on x. They are also independent on each other.

For the t dependent part we get

$$\frac{T'}{T} = -\lambda^2 \tag{8}$$

$$T' = -\lambda^2 T \tag{9}$$

The equation can be solved by integration i. e.

$$\frac{1}{T}\frac{dT}{dt} = -\lambda^2 \tag{10a}$$

$$\Rightarrow \frac{1}{T} = -\lambda^2 dt \tag{10b}$$

$$\int \frac{1}{T} dT = -\int \lambda^2 dt \tag{10c}$$

$$\Rightarrow ln(T) = -\lambda^t + A \tag{10d}$$

The t dependent term is then given by

$$T = Ae^{-\lambda^2 t} \tag{11}$$

From this epression we see that the λ must be real. Otherwise the term would grow whitout bounds over time, which is not in accordance with the behaviour we are looking for.

For the x dependent term we get

$$X'' = -\lambda^2 X \tag{12a}$$

$$\Rightarrow X'' + \lambda^2 X = 0 \tag{12b}$$

Again we use an anzats, that $X = e^{\alpha x}$:

$$\alpha^2 e^{\alpha x} + \lambda^2 e^{\alpha x} = 0 \tag{13a}$$
$$\Rightarrow \alpha^2 = -\lambda^2 \tag{13b}$$
$$\Rightarrow \alpha = \pm\sqrt{-\lambda^2} \tag{13c}$$

By using that λ must be a real number, the x dependent term has the general solution

$$X = Be^{-\lambda x} + Ce^{\lambda x} \tag{14}$$

We are looking for trigonometric solution to the term, so we rewrite the expression above as:

$$X(x) = Bcos(\lambda x) + Csin(\lambda x) \tag{15}$$

Applying the boundary conditions we get that

$$X(0) = Bcos(0) + Csin(0) = 0 \tag{16}$$
$$X(1) = Bcos(\lambda) + Csin(\lambda) = 0 \tag{17}$$

From the first boundary condition wher $L = 0$ we must have that $B = 0$. We are lookong for non-trivial solutions of the problem, so for $L = 1$ we must find for which values the term $Csin(\lambda) = 0$. We know that $sin(n\pi) = 0 \; for \; n = 1, 2, \ldots$, which lets us determine that $\lambda = n\pi$.

We have then infinite solutions of the transient part on the form:

$$V_n(x,t) = A_n e^{-(n\pi)^2 t} C_n sin(n\pi x) \tag{18a}$$
$$\Rightarrow a_n e^{-(n\pi)^2 t} sin(n\pi x) \; for \; n = 1, 2, \ldots \tag{18b}$$

Where $V_n(x,t)$ is the family of particular solutions we are looking for. Our desired solution will be a certain sum of this family of particular solutions i.e. fundamental solutions:

$$\sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} sin(n\pi x) \tag{19}$$

We must add the sum above in a way that the coefficients a_n satisfies the the initial condition of the transient solution (4b). This is done by setting

$$-U(x) = \sum_{n=1}^{\infty} a_n sin(n\pi x) \tag{20}$$

To obtain an expression for the coefficients a_n we use that the functions $sin(n\pi x)$ is orthogonal to each other in the sense that

$$\int_0^1 sin(n\pi x) sin(m\pi x) dx = \begin{cases} 0 & \text{if } m \neq n \\ 1/2 & \text{if } m = n \end{cases} \tag{21}$$

Using the above statement and multiplying the particular solutions 19 by $\int_0^1 sin(m\pi x)$ we get the expression

$$-\int_0^1 U(x) sin(m\pi x) = a_n \int_0^1 sin(m\pi x) sin(n\pi x) = a_m \int_0^1 sin^2(m\pi x)$$
$$\Rightarrow \; -\int_0^1 U(x) sin(m\pi x) = \frac{a_m}{2}$$
$$\Rightarrow \; a_m = -2\int_0^1 U(x) sin(m\pi x) \tag{22}$$

solving for a_n yields

$$a_n = -2\int_0^1 x sin(m\pi x) = \left[\frac{2}{(m\pi)^2} sin(m\pi x) + \frac{2x}{m\pi} cos(m\pi x)\right]_0^1$$
$$\Rightarrow \; a_n = \frac{2}{m\pi}(-1)^m \tag{23}$$

Putting the coefficient from 23 into the fundamental solutions 19 we end up with the following solution for the transient problem:

$$V(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^k}{n\pi} e^{-(n\pi)^2 t} sin(n\pi x) \tag{24}$$

Now we get an expression for the whole problem by putting combining the steady-state solution (5) and the transient solution (24) such that we obtain

$$u(x,t) = U(x) + V(x,t) = x + 2\sum_{n=1}^{\infty} \frac{(-1)^k}{n\pi} sin(n\pi x) e^{-(n\pi)^2 t} \tag{25}$$

Analytical solution 2D

In the two-dimensional case the differential equation becomes

$$\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} = \frac{\partial u(x,y,t)}{\partial t} \ , \ t > 0 \ , \ x, y \in [0,1] \tag{26a}$$

$$u(x,y,0) = 0 \tag{26b}$$
$$u(0,y,t) = 0 \ , \ t > 0 \tag{26c}$$
$$u(1,y,t) = 0 \ , \ t > 0 \tag{26d}$$
$$u(x,0,t) = 0 \ , \ t > 0 \tag{26e}$$
$$u(x,1,t) = 1 \ , \ t > 0 \tag{26f}$$

The boundary conditions is extended in such a way that the physical problem can be that of a flow whitin a box with infinite plates in the streamwise direction (z), where the fluid is initially at rest and the plate at $y = 1$ is given a sudden movement. So we get "no slip" boundary conditions i.e. the fluid has zero movement on all boundaries except at $y = 1$.

To obtain a closed form solution for the two-dimensional we use the same argument as for the one-dimensional proplem i.e. we split the solution into two parts, consisting of a steady-state solution and a transient solution.

We end up with a solution defined as:

$$u(x,y,t) = U(x,y) + V(x,y,t) \tag{27}$$

where $U(x,y)$ is the steady-state solution and $V/x,y,t)$ is the transient solution.

The steady-solution is then written in a compact form as:

$$U_{xx} + U_{yy} = 0 \ , \ x, y \in [0,1] \tag{28a}$$

$$U(0,y) = 0 \tag{28b}$$
$$U(1,y) = 0 \tag{28c}$$
$$U(x,0) = 0 \tag{28d}$$
$$U(x,1) = 1 \tag{28e}$$

For the transient problem we have the propblem defined as:

$$V_t = V_{xx} + V_{yy} \ , \ t > 0 \ , \ x \in [0,1] \tag{29a}$$

$$\begin{aligned} V(x,0) &= -U(x,y) \ , \ x,y \in [0,1] && (29b)\\ V(0,y,t) &= 0 \ , \ t > 0 && (29c)\\ V(1,y,t) &= 0 \ , \ t > 0 && (29d)\\ V(x,0,t) &= 0 \ , \ t > 0 && (29e)\\ V(x,1,t) &= 0 \ , \ t > 0 && (29f) \end{aligned}$$

For the steady-state problem we have a 2D Laplacian equation, which we can solve by seperation of variables. We use the anzats that $U(x,y) = X(x)Y(y)$, so that we end up with:

$$YX'' + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\beta^2$$

Due to the independence of the terms on each side of the equation, we can solve the equations separately, with each equation beeing

$$\frac{X''}{X} = -\beta^2$$
$$\frac{Y''}{Y} = \beta^2$$

The sign on the right hand side in both of the above equations es determined by the fact that in the x dependent term we have homogeneous boundaries, while in the y dependent term we have non-homogeneous. The x independent term must be on a form that allows it to be homogeneous on the boundaries, while the y term must be on a form that allows it to be 0 at $y = 0$ and 1 at $y = 1$.
For the x dependent term we get

$$X'' = -\beta^2 X$$
$$\Rightarrow X'' + \beta^2 X = 0$$

which has a similar solution as that of the transient x dependent term in the one-dimensional case (14).We are again looking for a trigonometric form

$$X(x) = Acos(\beta x) + Bsin(\beta x)$$

Applying the boundary conditions we end up with the term

$$X_n(x) = B_n sin(n\pi x) \tag{30}$$

were $X_n(x)$ is the family of particular solutions.

The y dependent term can be rewritten as

$$y'' = \beta^2 Y \tag{31a}$$
$$\Rightarrow Y'' - \beta^2 Y = 0 \tag{31b}$$

We are again using an anzats so that $Y = e^{\gamma y}$. This leaves us with the term

$$\begin{aligned} \gamma^2 e^{\gamma y} - \beta^2 e^{\gamma y} &= 0 && (32a)\\ \Rightarrow \gamma^2 &= \beta^2 && (32b)\\ \Rightarrow \gamma &= \pm\beta && (32c) \end{aligned}$$

and we end up with the following expression

$$
\begin{aligned}
Y_n(y) &= C_n e^{-n\pi y} + D_n e^{n\pi y} && (33)\\
&\Rightarrow C_n cosh(n\pi y) + D_n sinh(n\pi y) && (34)
\end{aligned}
$$

We have chosen hyperbolic sines and cosines as the form of the solutions, because of the boundary at $U(x, y = 1) = 1$. We further have that $Y_n(y)$ is the family of particular solutions.

The fundamental solutions of the steady-state is the given by

$$
U(x,y) = \sum_{n=1}^{\infty} X(x)Y(y) = \sum_{n=1}^{\infty} B_n sin(n\pi x)\Big(C_n cosh(n\pi y) + D_n sinh(n\pi y)\Big) \tag{35}
$$

Applying the the boundary of $U(x, 0)$ to the equation above we get that C_n must be zero. We are then left with the expression

$$
U(x,y) = \sum_{n=1}^{\infty} c_n sin(n\pi x) sinh(n\pi y) \tag{36}
$$

Applying the last boundary $U(x, 1) = 1$ we must have that

$$
1 = \sum_{n=1}^{\infty} c_n sin(n\pi x) sinh(n\pi) \tag{37}
$$

setting $d_n = c_n sinh(n\pi)$ we get that

$$
1 = \sum_{n=1}^{\infty} d_n sin(n\pi x) \tag{38}
$$

Using again that the functions $sin(n\pi x)$ is orthogonal (21), as in the 1D case, and multiplying the above equation with $\int_0^1 sin(m\pi y)$ we get that

$$
\begin{aligned}
dn = 2\int_0^1 sin(m\pi x)dx = \frac{2}{m\pi}(1-(-1)^m && (39a)\\
\Rightarrow c_n = \frac{2}{m\pi sinh(m\pi)}(1-(-1)^m && (39b)
\end{aligned}
$$

putting the coefficients c_n back into the expression for the fundamental solutions 36 we end up with the following expression for the steady-state

$$
\begin{aligned}
U(x,y) &= \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{sin(m\pi x)sinh(m\pi y)}{\pi sinh(m\pi)}(1-(-1)^m && (40a)\\
&\Rightarrow \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{sin((2m-1)\pi x)sinh((2m-1)\pi y)}{\pi sinh((2m-1)\pi)} && (40b)
\end{aligned}
$$

For the transient problem we once again use the anzats that $V(x, y, t) = X(x)Y(y)T(t)$ in order to write the problem as

$$
\begin{aligned}
XYT' = YTX'' + XTY'' && (41a)\\
\Rightarrow \frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -k^2 && (41b)
\end{aligned}
$$

Starting by solving the t dependent term, which is solved in the same way as for the one-dimensional term (10a) and we end up with the expression

$$
T = Ae^{-k^2 t} \tag{42}
$$

for the x dependent part we have

$$\frac{X''}{X} = -\frac{Y''}{Y} - k^2 \tag{43a}$$
$$\Rightarrow \frac{X''}{X} = -\gamma^2 \tag{43b}$$

and it has the same solution form as the x dependent transient one-dimensional term 14 , as well as the steadu-state two-dimensional term 30. so we end up with

$$X(x) = Bcos(\gamma x) + Csin(\gamma x) \tag{44}$$

For the above equation to hold on the boundaries, $B = 0$ and $\gamma = n\pi$ and we end up with

$$X_n(x) = c_n sin(n\pi x) \tag{45}$$

We are left with the y dependent term, which is written as

$$\frac{Y''}{Y} = k^2 - \gamma^2 \tag{46a}$$

Solving for Y we again use the anzats $Y = e^{\alpha y}$ and obtain the term

$$\alpha^2 e^{\alpha y} + (\gamma^2 - k^2)e^{\alpha y} = 0 \tag{47a}$$
$$\alpha^2 + \gamma^2 - k^2 = 0 \tag{47b}$$
$$\Rightarrow \alpha = \pm\sqrt{k^2 - \gamma^2} \tag{47c}$$

We are looking for a solution on the form that satisfies the bounderies for the transient term, resulting in

$$Y = Dcos(\sqrt{k^2 - \gamma^2}y) + Esin(\sqrt{k^2 - \gamma^2}y) \tag{48}$$

for the above equation to satisfy the boundary conditions $Y(0) = 0$ and $Y(1) = 0$ we have that $D = 0$ and that $\sqrt{k^2 - \gamma^2} = m\pi$. The Y term is the given by

$$Y_m(y) = E_m sin(m\pi y) \tag{49}$$

we kan now define the constat $k^2 = \gamma^2 + (m\pi)^2 = (n\pi)^2 + (m\pi)^2$. Putting this back into the t dependent transient term 42 we have that

$$T = A_{nm} e^{-\pi^2(m^2+n^2)t} \tag{50}$$

Combining the x, y and t dependent terms (45, 49 and 50 respectively) we end up with the fundamental solution

$$V(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} sin(n\pi x)sin(m\pi y)e^{-\pi^2(m^2+n^2)t} \tag{51}$$

To find the last coefficient A_{nm} we use the initial condition for the transient term. We need to have that

$$-U(x, y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} sin(n\pi x)sin(m\pi y) \tag{52}$$

We can solve for the coefficient by again using the orthogonality from 21, so that when we multiply the term above with $\int_0^1 sin(m\pi x)$ and $\int_0^1 sin(n\pi y)$ we get that

$$\int_0^1\int_0^1 -U(x, y)sin(n\pi y)sin(m\pi x)dxdy = \frac{A_{mn}}{4} \tag{53a}$$
$$\Rightarrow A_{mn} = -4\int_0^1\int_0^1 U(x, y)sin(n\pi x)sin(m\pi y)dxdy \tag{53b}$$

Finally we can combine the steady-state solution and the transient solution in order to get the analytical expression

$$\begin{aligned} u(x,y,t) &= U(x,y) + V(x,y,t) \\ &= U(x,y) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn} sin(n\pi x) sin(m\pi y) e^{-\pi^2(m^2+n^2)t} \end{aligned} \tag{54}$$

Where

$$U(x,y) = \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{sin((2m-1)\pi x) sinh((2m-1)\pi y)}{\pi sinh((2m-1)\pi)} \tag{55}$$

and

$$A_{mn} = -4\int_0^1\int_0^1 U(x,y) sin(n\pi x) sin(m\pi y) dx dy \tag{56}$$