

Fys4150

Project 5 Analytical solution

Peter Killingstad and Karl Jacobsen

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Analytical solution 1D

The one-dimensional problem is written as

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t}, \quad t > 0, \quad x \in [0, L] \quad (1a)$$

$$u(x, 0) = 0, \quad 0 < x < L, \quad (1b)$$

$$u(0, t) = 0, \quad t > 0 \quad (1c)$$

$$u(L, t) = 1, \quad t > 0 \quad (1d)$$

With initial conditions at $t = 0$ and $L = 1$ is the length of the x -region of interest.

The problem consists of non-homogeneous boundary conditions, and it is not trivial to find a closed form solution. The problem can be seen as a physical problem such as a temperature gradient in a rod or flow between two infinite flat plates, where the fluid is initially at rest and the plate at $x = 1$ is given a sudden movement. From physical observations we know that as time goes, i.e. when $t \rightarrow \infty$, the problem coincides with its surroundings and becomes steady.

It is reasonable to introduce a solution consisting of the sum of two parts, a steady-state solution, and a transient solution which is the part that depends on the initial conditions.

We end up with the solution of the problem being defined as

$$u(x, t) = U(x) + V(x, t) \quad (2)$$

where $U(x)$ is the steady-state solution and $V(x, t)$ is the transient solution.

Since the steady-state problem is not changing in time, i.e. $\partial_t = 0$, it is written in a compact form as:

$$U_{xx} = 0, \quad x \in [0, 1] \quad (3a)$$

$$U(0) = 0 \quad (3b)$$

$$U(L) = 1 \quad (3c)$$

For the transient part of the problem we have the problem written in compact form as:

$$V_t = V_{xx}, \quad t > 0, \quad x \in [0, 1] \quad (4a)$$

$$V(x, 0) = -U(x), \quad 0 < x < 1 \quad (4b)$$

$$V(0, t) = 0, \quad t > 0 \quad (4c)$$

$$V(1, t) = 0, \quad t > 0 \quad (4d)$$

The steady state solution is solved by integrating 3a twice

$$\begin{aligned}\int \frac{\partial^2 u(x,t)}{\partial x^2} &= A \\ \int \frac{\partial u(x,t)}{\partial x} &= \int A\end{aligned}$$

and we end up with the general solution $U(x) = Ax + B$. Applying the boundary conditions at $L = 0$ and $L = 1$ we get that

$$U(x) = x \quad (5)$$

The transient problem 4a has homogeneous boundary conditions, and we can solve it by separation of variables. We start by making the ansatz that $V(x,t) = T(t)X(x)$ so that we can rewrite 4a as:

$$XT' = X''T \quad (6)$$

Reordering the equation we get the following:

$$\frac{T'}{T} = \frac{X''}{X} = k, \quad (7)$$

where k is an unknown, possible complex constant. The choosing of k can be explained by the following arguments. We are looking for a solution that is eventually going to be steady. With a positive k we would never get a steady state, as the factor $T(t)$ does not go to zero as $t \rightarrow \infty$. Choosing k as zero would give us the steady-solution which we already have. We are then left with choosing k as a negative constant. With $k = -\lambda^2$ we ensure that k is negative.

The term can be solved separately as we get that the left hand side is only dependent on t and the right hand side is only dependent on x . They are also independent on each other.

For the t dependent part we get

$$\frac{T'}{T} = -\lambda^2 \quad (8)$$

$$T' = -\lambda^2 T \quad (9)$$

The equation can be solved by integration i. e.

$$\frac{1}{T} \frac{dT}{dt} = -\lambda^2 \quad (10a)$$

$$\Rightarrow \frac{1}{T} = -\lambda^2 dt \quad (10b)$$

$$\int \frac{1}{T} dT = - \int \lambda^2 dt \quad (10c)$$

$$\Rightarrow \ln(T) = -\lambda^2 t + A \quad (10d)$$

The t dependent term is then given by

$$T = Ae^{-\lambda^2 t} \quad (11)$$

From this expression we see that the λ must be real. Otherwise the term would grow without bounds over time, which is not in accordance with the behaviour we are looking for.

For the x dependent term we get

$$X'' = -\lambda^2 X \quad (12a)$$

$$\Rightarrow X'' + \lambda^2 X = 0 \quad (12b)$$

Again we use an anzats, that $X = e^{\alpha x}$:

$$\alpha^2 e^{\alpha x} + \lambda^2 e^{\alpha x} = 0 \quad (13a)$$

$$\Rightarrow \alpha^2 = -\lambda^2 \quad (13b)$$

$$\Rightarrow \alpha = \pm \sqrt{-\lambda^2} \quad (13c)$$

By using that λ must be a real number, the x dependent term has the general solution

$$X = Be^{-\lambda x} + Ce^{\lambda x} \quad (14)$$

We are looking for trigonometric solution to the term, so we rewrite the expression above as:

$$X(x) = B\cos(\lambda x) + C\sin(\lambda x) \quad (15)$$

Applying the boundary conditions we get that

$$X(0) = B\cos(0) + C\sin(0) = 0 \quad (16)$$

$$X(1) = B\cos(\lambda) + C\sin(\lambda) = 0 \quad (17)$$

From the first boundary condition wher $L = 0$ we must have that $B = 0$. We are lookong for non-trivial solutions of the problem, so for $L = 1$ we must find for which values the term $C\sin(\lambda) = 0$. We know that $\sin(n\pi) = 0$ for $n = 1, 2, \dots$, which lets us determine that $\lambda = n\pi$.

We have then infinite solutions of the transient part on the form:

$$V_n(x, t) = A_n e^{-(n\pi)^2 t} C_n \sin(n\pi x) \quad (18a)$$

$$\Rightarrow a_n e^{-(n\pi)^2 t} \sin(n\pi x) \text{ for } n = 1, 2, \dots \quad (18b)$$

Where $V_n(x, t)$ is the family of particular solutions we are looking for. Our desired solution will be a certain sum of this family of particular solutions i.e. fundamental solutions:

$$\sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} \sin(n\pi x) \quad (19)$$

We must add the sum above in a way that the coefficients a_n satisfies the the initial condition of the transient solution (4b). This is done by setting

$$-U(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad (20)$$

To obtain an expression for the coefficients a_n we use that the functions $\sin(n\pi x)$ is orthogonal to each other in the sense that

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} 0 & \text{if } m \neq n \\ 1/2 & \text{if } m = n \end{cases} \quad (21)$$

Using the above statement and multiplying the particular solutions 19 by $\int_0^1 \sin(m\pi x)$ we get the expression

$$\begin{aligned} -\int_0^1 U(x) \sin(m\pi x) &= a_n \int_0^1 \sin(m\pi x) \sin(n\pi x) = a_m \int_0^1 \sin^2(m\pi x) \\ &\Rightarrow -\int_0^1 U(x) \sin(m\pi x) = \frac{a_m}{2} \\ &\Rightarrow a_m = -2 \int_0^1 U(x) \sin(m\pi x) \end{aligned} \quad (22)$$

solving for a_n yields

$$\begin{aligned} a_n = -2 \int_0^1 x \sin(m\pi x) &= \left[\frac{2}{(m\pi)^2} \sin(m\pi x) + \frac{2x}{m\pi} \cos(m\pi x) \right]_0^1 \\ &\Rightarrow a_n = \frac{2}{m\pi} (-1)^m \end{aligned} \quad (23)$$

Putting the coefficient from 23 into the fundamental solutions 19 we end up with the following solution for the transient problem:

$$V(x, t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^k}{n\pi} e^{-(n\pi)^2 t} \sin(n\pi x) \quad (24)$$

Now we get an expression for the whole problem by putting combining the steady-state solution (5) and the transient solution (24) such that we obtain

$$u(x, t) = U(x) + V(x, t) = x + 2 \sum_{n=1}^{\infty} \frac{(-1)^k}{n\pi} \sin(n\pi x) e^{-(n\pi)^2 t} \quad (25)$$

Analytical solution 2D

In the two-dimensional case the differential equation becomes

$$\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} = \frac{\partial u(x, y, t)}{\partial t}, \quad t > 0, \quad x, y \in [0, 1] \quad (26a)$$

$$u(x, y, 0) = 0, \quad x, y \in [0, 1] \quad (26b)$$

$$u(0, y, t) = 0, \quad t > 0 \quad (26c)$$

$$u(1, y, t) = 0, \quad t > 0 \quad (26d)$$

$$u(x, 0, t) = 0, \quad t > 0 \quad (26e)$$

$$u(x, 1, t) = 1, \quad t > 0 \quad (26f)$$

The boundary conditions is extended in such a way that the physical problem can be that of a flow within a box with infinite plates in the streamwise direction (z), where the fluid is initially at rest and the plate at $y = 1$ is given a sudden movement. So we get "no slip" boundary conditions i.e. the fluid has zero movement on all boundaries except at $y = 1$.

To obtain a closed form solution for the two-dimensional we use the same argument as for the one-dimensional problem i.e. we split the solution into two parts, consisting of a steady-state solution and a transient solution.

We end up with a solution defined as:

$$u(x, y, t) = U(x, y) + V(x, y, t) \quad (27)$$

where $U(x, y)$ is the steady-state solution and $V(x, y, t)$ is the transient solution.

The steady-solution is then written in a compact form as:

$$U_{xx} + U_{yy} = 0, \quad x, y \in [0, 1] \quad (28a)$$

$$U(0, y) = 0 \quad (28b)$$

$$U(1, y) = 0 \quad (28c)$$

$$U(x, 0) = 0 \quad (28d)$$

$$U(x, 1) = 1 \quad (28e)$$

For the transient problem we have the problem defined as:

$$V_t = V_{xx} + V_{yy}, \quad t > 0, \quad x \in [0, 1] \quad (29a)$$

$$V(x, 0) = -U(x, y), \quad x, y \in [0, 1] \quad (29b)$$

$$V(0, y, t) = 0, \quad t > 0 \quad (29c)$$

$$V(1, y, t) = 0, \quad t > 0 \quad (29d)$$

$$V(x, 0, t) = 0, \quad t > 0 \quad (29e)$$

$$V(x, 1, t) = 0, \quad t > 0 \quad (29f)$$

For the steady-state problem we have a 2D Laplacian equation, which we can solve by separation of variables. We use the ansatz that $U(x, y) = X(x)Y(y)$, so that we end up with:

$$\begin{aligned} YX'' + XY'' &= 0 \\ \frac{X''}{X} &= -\frac{Y''}{Y} = -\beta^2 \end{aligned}$$

Due to the independence of the terms on each side of the equation, we can solve the equations separately, with each equation being

$$\begin{aligned} \frac{X''}{X} &= -\beta^2 \\ \frac{Y''}{Y} &= \beta^2 \end{aligned}$$

The sign on the right hand side in both of the above equations is determined by the fact that in the x dependent term we have homogeneous boundaries, while in the y dependent term we have non-homogeneous. The x independent term must be on a form that allows it to be homogeneous on the boundaries, while the y term must be on a form that allows it to be 0 at $y = 0$ and 1 at $y = 1$.

For the x dependent term we get

$$\begin{aligned} X'' &= -\beta^2 X \\ \Rightarrow X'' + \beta^2 X &= 0 \end{aligned}$$

which has a similar solution as that of the transient x dependent term in the one-dimensional case (14). We are again looking for a trigonometric form

$$X(x) = A\cos(\beta x) + B\sin(\beta x)$$

Applying the boundary conditions we end up with the term

$$X_n(x) = B_n \sin(n\pi x) \quad (30)$$

where $X_n(x)$ is the family of particular solutions.

The y dependent term can be rewritten as

$$y'' = \beta^2 Y \quad (31a)$$

$$\Rightarrow Y'' - \beta^2 Y = 0 \quad (31b)$$

We are again using an ansatz so that $Y = e^{\gamma y}$. This leaves us with the term

$$\gamma^2 e^{\gamma y} - \beta^2 e^{\gamma y} = 0 \quad (32a)$$

$$\Rightarrow \gamma^2 = \beta^2 \quad (32b)$$

$$\Rightarrow \gamma = \pm\beta \quad (32c)$$

and we end up with the following expression

$$Y_n(y) = C_n e^{-n\pi y} + D_n e^{n\pi y} \quad (33)$$

$$\Rightarrow C_n \cosh(n\pi y) + D_n \sinh(n\pi y) \quad (34)$$

We have chosen hyperbolic sines and cosines as the form of the solutions, because of the boundary at $U(x, y = 1) = 1$. We further have that $Y_n(y)$ is the family of particular solutions.

The fundamental solutions of the steady-state is the given by

$$U(x, y) = \sum_{n=1}^{\infty} X(x)Y(y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) \left(C_n \cosh(n\pi y) + D_n \sinh(n\pi y) \right) \quad (35)$$

Applying the the boundary of $U(x, 0)$ to the equation above we get that C_n must be zero. We are then left with the expression

$$U(x, y) = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \sinh(n\pi y) \quad (36)$$

Applying the last boundary $U(x, 1) = 1$ we must have that

$$1 = \sum_{n=1}^{\infty} c_n \sin(n\pi x) \sinh(n\pi) \quad (37)$$

setting $d_n = c_n \sinh(n\pi)$ we get that

$$1 = \sum_{n=1}^{\infty} d_n \sin(n\pi x) \quad (38)$$

Using again that the functions $\sin(n\pi x)$ is orthogonal (21), as in the 1D case, and multiplying the above equation with $\int_0^1 \sin(m\pi y)$ we get that

$$dn = 2 \int_0^1 \sin(m\pi x) dx = \frac{2}{m\pi} (1 - (-1)^m) \quad (39a)$$

$$\Rightarrow c_n = \frac{2}{m\pi \sinh(m\pi)} (1 - (-1)^m) \quad (39b)$$

putting the coefficients c_n back into the expression for the fundamental solutions 36 we end up with the following expression for the steady-state

$$U(x, y) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\sin(m\pi x) \sinh(m\pi y)}{\pi \sinh(m\pi)} (1 - (-1)^m) \quad (40a)$$

$$\Rightarrow \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin((2m-1)\pi x) \sinh((2m-1)\pi y)}{\pi \sinh((2m-1)\pi)} \quad (40b)$$

For the transient problem we once again use the anzats that $V(x, y, t) = X(x)Y(y)T(t)$ in order to write the problem as

$$XYT' = YTX'' + XTY'' \quad (41a)$$

$$\Rightarrow \frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -k^2 \quad (41b)$$

Starting by solving the t dependent term, which is solved in the same way as for the one-dimensional term (10a) and we end up with the expression

$$T = Ae^{-k^2 t} \quad (42)$$

for the x dependent part we have

$$\frac{X''}{X} = -\frac{Y''}{Y} - k^2 \quad (43a)$$

$$\Rightarrow \frac{X''}{X} = -\gamma^2 \quad (43b)$$

and it has the same solution form as the x dependent transient one-dimensional term 14, as well as the steady-state two-dimensional term 30. so we end up with

$$X(x) = B\cos(\gamma x) + C\sin(\gamma x) \quad (44)$$

For the above equation to hold on the boundaries, $B = 0$ and $\gamma = n\pi$ and we end up with

$$X_n(x) = c_n \sin(n\pi x) \quad (45)$$

We are left with the y dependent term, which is written as

$$\frac{Y''}{Y} = k^2 - \gamma^2 \quad (46a)$$

Solving for Y we again use the ansatz $Y = e^{\alpha y}$ and obtain the term

$$\alpha^2 e^{\alpha y} + (\gamma^2 - k^2) e^{\alpha y} = 0 \quad (47a)$$

$$\alpha^2 + \gamma^2 - k^2 = 0 \quad (47b)$$

$$\Rightarrow \alpha = \pm \sqrt{k^2 - \gamma^2} \quad (47c)$$

We are looking for a solution on the form that satisfies the boundaries for the transient term, resulting in

$$Y = D\cos(\sqrt{k^2 - \gamma^2}y) + E\sin(\sqrt{k^2 - \gamma^2}y) \quad (48)$$

for the above equation to satisfy the boundary conditions $Y(0) = 0$ and $Y(1) = 0$ we have that $D = 0$ and that $\sqrt{k^2 - \gamma^2} = m\pi$. The Y term is then given by

$$Y_m(y) = E_m \sin(m\pi y) \quad (49)$$

we can now define the constant $k^2 = \gamma^2 + (m\pi)^2 = (n\pi)^2 + (m\pi)^2$. Putting this back into the t dependent transient term 42 we have that

$$T = A_{nm} e^{-\pi^2(m^2+n^2)t} \quad (50)$$

Combining the x, y and t dependent terms (45, 49 and 50 respectively) we end up with the fundamental solution

$$V(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin(n\pi x) \sin(m\pi y) e^{-\pi^2(m^2+n^2)t} \quad (51)$$

To find the last coefficient A_{nm} we use the initial condition for the transient term. We need to have that

$$-U(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin(n\pi x) \sin(m\pi y) \quad (52)$$

We can solve for the coefficient by again using the orthogonality from 21, so that when we multiply the term above with $\int_0^1 \sin(m\pi x)$ and $\int_0^1 \sin(n\pi y)$ we get that

$$\int_0^1 \int_0^1 -U(x, y) \sin(m\pi x) \sin(n\pi y) dx dy = \frac{A_{mn}}{4} \quad (53a)$$

$$\Rightarrow A_{mn} = -4 \int_0^1 \int_0^1 U(x, y) \sin(m\pi x) \sin(n\pi y) dx dy \quad (53b)$$

Finally we can combine the steady-state solution and the transient solution in order to get the analytical expression

$$\begin{aligned} u(x, y, t) &= U(x, y) + V(x, y, t) \\ &= U(x, y) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sin(n\pi x) \sin(m\pi y) e^{-\pi^2(m^2+n^2)t} \end{aligned} \quad (54)$$

Where

$$U(x, y) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin((2m-1)\pi x) \sinh((2m-1)\pi y)}{\pi \sinh((2m-1)\pi)} \quad (55)$$

and

$$A_{mn} = -4 \int_0^1 \int_0^1 U(x, y) \sin(m\pi x) \sin(n\pi y) dx dy \quad (56)$$