Fys4150
Project 5

Peter Killingstad and Karl Jacobsen

https://github.com/kaaja/fys4150

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1.1 Derivation of schemes with truncation errors

All the schemes will be derived from Taylor series expansions, and the truncation error will be related to the remainder in the Taylor-series expansions. This remainded is the error we get when truncating the series by leaving out the remainder.

1.1.1 Forward Euler

For the time derivative, we expand $u(x, t + \Delta t)$ around t

$$u(x, t + \Delta t) = u(x, t) + u_t(x, t)\Delta t + \mathcal{O}(\Delta t^2) \tag{1a}$$

$$\rightarrow u_t = \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} + \mathcal{O}(\Delta t) \tag{1b}$$

The space derivative, which is a 2nd derivative, we derive by combining two Taylor series'

$$u(x + \Delta x, t) = u(x, t) + u_x(x, t)\Delta x + \frac{u_{xx}(x, t)\Delta x^2}{2} + \frac{u_{xxx}(x, t)\Delta x^3}{6} + \mathcal{O}(\Delta x^4) \tag{2a}$$

$$u(x - \Delta x, t) = u(x, t) - u_x(x, t)\Delta x + \frac{u_{xx}(x, t)\Delta x^2}{2} - \frac{u_{xxx}(x, t)\Delta x^3}{6} + \mathcal{O}(\Delta x^4) \tag{2b}$$

Now we add (2a) and (2b) and solve for $u_{xx}(x, t)$

$$\begin{aligned}
\Big(u(x + \Delta x, t) + u(x - \Delta x, t)\Big) &= \Big(u(x, t) + u(x, t)\Big) \\
&+ \Big(u_x(x, t)\Delta x + (-u_x(x, t)\Delta x)\Big) \\
&+ \Big(\frac{u_{xx}(x, t)\Delta x^2}{2} + \frac{u_{xx}(x, t)\Delta x^2}{2}\Big) \\
&+ \Big(\frac{u_{xxx}(x, t)\Delta x^3}{6} + (-\frac{u_{xxx}(x, t)\Delta x^3}{6})\Big) \\
&+ \Big(\mathcal{O}(\Delta x^4) + \mathcal{O}(\Delta x^4)\Big) && \text{(3a)}\\
&= 2u(x, t) + u_{xx}(x, t)\Delta x^2 + \mathcal{O}(\Delta x^4) && \text{(3b)}\\
\rightarrow u_{xx}(x, t) &= \frac{u(x - \Delta x, t) - 2u(x, t) + u(x + \Delta x, t)}{\Delta x^2} + \mathcal{O}(\Delta x^2) && \text{(3c)}
\end{aligned}$$

Combining (1b) and (3c) we get the Forward Euler scheme

$$
\begin{align}
u_t(x,t) &= u_{xx}(x,t) \tag{4a}\\
\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} + \mathcal{O}(\Delta t) &= \frac{u(x-\Delta x,t)-2u(x,t)+u(x+\Delta x,t)}{\Delta x^2} + \mathcal{O}(\Delta x^2) \tag{4b}
\end{align}
$$

From (4b) we see that the scheme has a truncation error that goes like $\mathcal{O}(\Delta t)$ in time and $\mathcal{O}(\Delta x^2)$ in space.

We will now analyze the stability of the Forward Euler scheme (4b) by applying Neuman stability analyzis. From the analytical solution of the problem, we know that the particular solutions are on the form $u = e^{-(k\pi)^2 t} e^{ik\pi x}$, where k is an integer greater than one. We observe that the solutions are stable in t, meaning that the solutions do not blow up as t increaes. Based on the analytical particular solution, we make the numerical ansatz

$$
u = a_k^n e^{ik\pi x_j} \tag{5}
$$

For the numerical ansatz (5) to reproduce the characteristics of the analytical particular solution, with stability in t, we observe that $|a_k^n| < 1$ in necessary. We now plug in the ansatz (5) into the (4b) and derive an equation for $|a_k^n|$:

$$
\begin{align}
\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} &= \frac{u(x-\Delta x,t)-2u(x,t)+u(x+\Delta x,t)}{\Delta x^2} \tag{6a}\\
\frac{a_k^{n+1} e^{ik\pi(j+1)\Delta x} - a_k^n e^{ik\pi j\Delta x}}{\Delta t} &= \frac{a_k^n e^{ik\pi(j-1)\Delta x} - 2a_k^n e^{ik\pi j\Delta x} + a_k^n e^{ik\pi(j+1)\Delta x}}{\Delta x^2} \tag{6b}\\
a_k^n e^{ik\pi j\Delta x}\ \frac{a_k - 1}{\Delta t} &= a_k^n e^{ik\pi j\Delta x}\ \frac{e^{-ik\pi\Delta x} - 2 + e^{ik\pi\Delta x}}{\Delta x^2} \tag{6c}\\
\frac{a_k - 1}{\Delta t} &= \frac{e^{-ik\pi\Delta x} - 2 + e^{ik\pi\Delta x}}{\Delta x^2} \tag{6d}\\
a_k &= 1 + \frac{\Delta t}{\Delta x^2}(e^{-ik\pi\Delta x} - 2 + e^{ik\pi\Delta x}) \tag{6e}\\
&= 1 + \frac{\Delta t}{\Delta x^2}\Big(2\cos(k\pi\Delta x) - 2\Big) \tag{6f}\\
&= 1 + 2\frac{\Delta t}{\Delta x^2}\Big(\cos(k\pi\Delta x) - 1\Big) \tag{6g}\\
&= 1 + 2\frac{\Delta t}{\Delta x^2}\Big(-2\sin^2(\frac{k\pi\Delta x}{2})\Big) \tag{6h}\\
&= 1 - 4\frac{\Delta t}{\Delta x^2}\sin^2(\frac{k\pi\Delta x}{2}) \tag{6i}\\
|a_k| &= |1 - 4\frac{\Delta t}{\Delta x^2}\sin^2(\frac{k\pi\Delta x}{2})| \tag{6j}
\end{align}
$$

From (6j) we get

$$
\begin{align}
&|a_k| < 1 \text{ if } ||1 - 4\frac{\Delta t}{\Delta x^2}\sin^2(\frac{k\pi\Delta x}{2})|| < 1 \tag{7a}\\
&\to |1 - 4\frac{\Delta t}{\Delta x^2}| < 1 \to |a_k| < 1 \text{ (Since } \sin^2(k\pi\Delta x/2)_{max} = 1) \tag{7b}\\
&\to 1 - 4\frac{\Delta t}{\Delta x^2} > -1 \tag{7c}\\
&\to \frac{\Delta t}{\Delta x^2} < \frac{1}{2} \tag{7d}
\end{align}
$$

(7d) gives that the Forward Euler scheme is conditionally stable, and the condition that ensures stability.