Fys4150
Project 5 Analytical solution

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Analytical solution 1D

The one-dimensional problem is written as

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t},\ t > 0,\ x \in [0, L] \quad (1a)$$

$$u(x,0) = 0,\ 0 < x < L, \quad (1b)$$
$$u(0,t) = 0,\ t > 0 \quad (1c)$$
$$u(L,t) = 1,\ t > 0 \quad (1d)$$

With initial conditions at $t = 0$ and $L = 1$ is the length of the x-region of interest.

The problem consists of non-homogeneous boundary conditions, and it is not trivial to find a closed form solution. The problem can be seen as a physical problem such as a temperature gradient in a rod or flow between two infinite flat plates, where the fluid is initially at rest and the plate at $x = 1$ is given a sudden movement. From physical observations we know that as time goes, i.e. when $t \to \infty$, the problem coinsides whith its surroundings and becomes steady.

It is reasonable to introduce a solution consisting of the sum of two parts, a steady-state solution, and a transient solution which is the part that depends on the initial conditions.

We end up with the solution of the problem beeing defined as

$$u(x,t) = U(x) + V(x,t) \quad (2)$$

where $U(x)$ is the steady-state solution and $V(x,t)$ is the transient solution.

Since the steady-state problem is not changing in time, i.e. $\partial_t = 0$, it is written in a compact form as:

$$U_{xx} = 0\ ,\ x \in [0, 1] \quad (3a)$$

$$U(0) = 0 \quad (3b)$$
$$U(L) = 1 \quad (3c)$$

For the transient part of the problem we have the problem written in compact form as:

$$V_t = V_{xx}\ ,\ t > 0\ ,\ x \in [0, 1] \quad (4a)$$

$$V(x,0) = -U_s(x)\ ,\ 0 < x < 1 \quad (4b)$$
$$V(0,t) = 0\ ,\ t > 0 \quad (4c)$$
$$V(1,t) = 0\ ,\ t > 0 \quad (4d)$$

The steady state solution is solved by integrating 3a twice

$$\begin{aligned}\int \frac{\partial^2 u(x,t)}{\partial x^2} &= A \\ \int \frac{\partial u(x,t)}{\partial x} &= \int A\end{aligned}$$

and we end up with the general solution $U(x) = Ax + B$. Applying the boundary conditions at $L = 0$ and $L = 1$ we get that

$$U(x) = x \tag{5}$$

The transient problem 4a has homogeneous boundary conditions, and we can solve it by seperation of variables. We start by making the anzats that $V(x,t) = T(t)X(x)$ so that we can rewrite 4a as:

$$XT' = X''T \tag{6}$$

Reordering the equation we get the following:

$$\frac{T'}{T} = \frac{X''}{X} = k, \tag{7}$$

where k is an unknown, possible complex constant. The choosing of k can be explained by the following arguments. We are looking for a solution that is eventually going to be steady. With a positive k we would never get a steady state, as the factor $T(t)$ does not go to zero as $t \to \infty$. Choosing k as zero would give us the steady-solution which we already have. We are the then left with choosing k as a negative constant. With $k = -\lambda^2$ we ensure thath k is negative.

The term can be solved seperately as we get that the left hand side is only depentent on t and the right hand side is only depentent on x. They are also independent on each other.

For the t dependent part we get

$$\frac{T'}{T} = -\lambda^2 \tag{8}$$

$$T' = -\lambda^2 T \tag{9}$$

The equation can be solved by using the anzats $T = e^{\alpha t}$. We get the following expression

$$\alpha e^{\alpha t} + \lambda^2 e^{\alpha t} = 0 \tag{10a}$$

$$\alpha + \lambda^2 = 0 \tag{10b}$$

$$\Rightarrow \alpha = -\lambda^2 \tag{10c}$$

The t dependent term is then given by

$$T = Ae^{-\lambda^2 t} \tag{11}$$

From this epression we see that the λ must be real. Otherwise the term would grow whitout bounds over time, which is not in accordance with the behaviour we are looking for.

For the x dependent term we get

$$X'' = -\lambda^2 X \tag{12a}$$

$$\Rightarrow X'' + \lambda^2 X = 0 \tag{12b}$$

Again we use an anzats, that $X = e^{\alpha x}$:

$$\alpha^2 e^{\alpha x} + \lambda^2 e^{\alpha x} = 0 \tag{13a}$$

$$\Rightarrow \alpha^2 = -\lambda^2 \tag{13b}$$

$$\Rightarrow \alpha = \pm\sqrt{-\lambda^2} \tag{13c}$$

By using that λ must be a real number, the x dependent term has the general solution

$$X = Be^{-\lambda x} + Ce^{\lambda x} \tag{14}$$

We are looking for trigonometric solution to the term, so we rewrite the expression above as:

$$X(x) = Bcos(\lambda x) + Csin(\lambda x) \tag{15}$$

Applying the boundary conditions we get that

$$X(0) = Bcos(0) + Csin(0) = 0 \tag{16}$$
$$X(1) = Bcos(\lambda) + Csin(\lambda) = 0 \tag{17}$$

From the first boundary condition wher $L = 0$ we must have that $B = 0$. We are lookong for non-trivial solutions of the problem, so for $L = 1$ we must find for which values the term $Csin(\lambda) = 0$. We know that $sin(n\pi) = 0$ *for* $n = 1, 2, \ldots$, which lets us determine that $\lambda = n\pi$.

We have then infinite solutions of the transient part on the form:

$$V_n(x,t) = A_n e^{-(n\pi)^2 t} C_n sin(n\pi x) \tag{18a}$$
$$\Rightarrow a_n e^{-(n\pi)^2 t} sin(n\pi x) \ for \ n = 1, 2, \ldots \tag{18b}$$

Where $V_n(x,t)$ is the family of particular solutions we are looking for. Our desired solution will be a certain sum of this family of particular solutions i.e. fundamental solutions:

$$\sum_{n=1}^{\infty} a_n e^{-(n\pi)^2 t} sin(n\pi x) \tag{19}$$

We must add the sum above in a way that the coefficients a_n satisfies the the initial condition of the transient solution (4b). This is done by setting

$$-U(x) = \sum_{n=1}^{\infty} a_n sin(n\pi x) \tag{20}$$

To obtain an expression for the coefficients a_n we use that the functions $sin(n\pi x)$ is orthogonal to each other in the sense that

$$\int_0^1 sin(n\pi x) sin(m\pi x) dx = \begin{cases} 0 & \text{if } m \neq n \\ 1/2 & \text{if } m = n \end{cases} \tag{21}$$

Using the above statement and multiplying the particular solutions 19 by $\int_0^1 sin(m\pi x)$ we get the expression

$$\begin{aligned} -\int_0^1 U(x) sin(m\pi x) &= a_n \int_0^1 sin(m\pi x) sin(n\pi x) = a_m \int_0^1 sin^2(m\pi x) \\ &\Rightarrow -\int_0^1 U(x) sin(m\pi x) = \frac{a_m}{2} \\ &\Rightarrow a_m = -2\int_0^1 U(x) sin(m\pi x) \end{aligned} \tag{22}$$

solving for a_n yields

$$\begin{aligned} a_n = -2\int_0^1 x sin(m\pi x) &= \Big[\frac{2}{(m\pi)^2} sin(m\pi x) + \frac{2x}{m\pi} cos(m\pi x)\Big]_0^1 \\ &\Rightarrow a_n = \frac{2}{m\pi}(-1)^m \end{aligned} \tag{23}$$

Putting the coefficient from 23 into the fundamental solutions 19 we end up with the following solution for the transient problem:

$$V(x,t) = 2\sum_{n=1}^{\infty}\frac{(-1)^k}{n\pi}e^{-(n\pi)^2 t}sin(n\pi x) \tag{24}$$

Now we get an expression for the whole problem by putting combining the steady-state solution (5) and the transient solution (24) such that we obtain

$$u(x,t) = U(x) + V(x,t) = x + 2\sum_{n=1}^{\infty}\frac{(-1)^k}{n\pi}sin(n\pi x)e^{-(n\pi)^2 t} \tag{25}$$

Analytical solution 2D

In the two-dimensional case the differential equation becomes

$$\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} = \frac{\partial u(x,y,t)}{\partial t} \text{ , } t > 0 \text{ , } x,y \in [0,1] \tag{26a}$$

$$u(x,y,0) = 0 \tag{26b}$$
$$u(0,y,t) = 0 \text{ , } t > 0 \tag{26c}$$
$$u(1,y,t) = 0 \text{ , } t > 0 \tag{26d}$$
$$u(x,0,t) = 0 \text{ , } t > 0 \tag{26e}$$
$$u(x,1,t) = 1 \text{ , } t > 0 \tag{26f}$$

The boundary conditions is extended in such a way that the physical problem can be that of a flow whitin a box with infinite plates in the streamwise direction (z), where the fluid is initially at rest and the plate at $y = 1$ is given a sudden movement. So we get "no slip" boundary conditions i.e. the fluid has zero movement on all boundaries except at $y = 1$.

To obtain a closed form solution for the two-dimensional we use the same argument as for the one-dimensional proplem i.e. we split the solution into two parts, consisting of a steady-state solution and a transient solution.

We end up with a solution defined as:

$$u(x,y,t) = U(x,y) + V(x,y,t) \tag{27}$$

where $U(x,y)$ is the steady-state solution and $V/x,y,t)$ is the transient solution.

The steady-solution is then written in a compact form as:

$$U_{xx} + U_{yy} = 0 \text{ , } x,y \in [0,1] \tag{28a}$$

$$U(0,y) = 0 \tag{28b}$$
$$U(1,y) = 0 \tag{28c}$$
$$U(x,0) = 0 \tag{28d}$$
$$U(x,1) = 1 \tag{28e}$$

For the transient problem we have the propblem defined as:

$$V_t = V_{xx} \text{ , } t > 0 \text{ , } x \in [0,1] \tag{29a}$$

$$V(x,0) = -U(x,y) \text{ , } x,y \in [0,1] \tag{29b}$$
$$V(0,y,t) = 0 \text{ , } t > 0 \tag{29c}$$
$$V(1,y,t) = 0 \text{ , } t > 0 \tag{29d}$$
$$V(x,0,t) = 0 \text{ , } t > 0 \tag{29e}$$
$$V(x,1,t) = 0 \text{ , } t > 0 \tag{29f}$$

For the steady-state problem we have a 2D Laplacian equation, which we can solve by seperation of variables. We use the anzats that $U(x, y) = X(x)Y(y)$, so that we end up with:

$$YX'' + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\beta^2$$

Due to the independence of the terms on each side of the equation, we can solve the equations separately, with each equation beeing

$$\frac{X''}{X} = -\beta^2$$
$$\frac{Y''}{Y} = \beta^2$$

The sign on the right hand side in both of the above equations es determined by the fact that in the x dependent term we have homogeneous boundaries, while in the y dependent term we have non-homogeneous. The x independent term must be on a form that allows it to be homogeneous on the boundaries, while the y term must be on a form that allows it to be 0 at $y = 0$ and 1 at $y = 1$.
For the x dependent term we get

$$X'' = -\beta^2 X$$
$$\Rightarrow X'' + \beta^2 X = 0$$

which has a similar solution as that of the transient x dependent term in the one-dimensional case (14).We are again looking for a trigonometric form

$$X(x) = Acos(\beta x) + Bsin(\beta x)$$

Applying the boundary conditions we end up with the term

$$X_n(x) = B_n sin(n\pi x) \tag{30}$$

were $X_n(x)$ is the family of particular solutions.

The y dependent term can be rewritten as

$$y'' = \beta^2 Y \tag{31a}$$
$$\Rightarrow Y'' - \beta^2 Y = 0 \tag{31b}$$

We are again using an anzats so that $Y = e^{\gamma y}$. This leaves us with the term

$$\gamma^2 e^{\gamma y} - \beta^2 e^{\gamma y} = 0 \tag{32a}$$
$$\Rightarrow \gamma^2 = \beta^2 \tag{32b}$$
$$\Rightarrow \gamma = \pm\beta \tag{32c}$$

and we end up with the following expression

$$Y_n(y) = C_n e^{-n\pi y} + D_n e^{n\pi y} \tag{33}$$
$$\Rightarrow C_n cosh(n\pi y) + D_n sinh(n\pi y) \tag{34}$$

We have chosen hyperbolic sines and cosines as the form of the solutions, because of the boundary at $U(x, y = 1) = 1$. We further have that $Y_n(y)$ is the family of particular solutions.

The fundamental solutions of the steady-state is the given by

$$U(x, y) = \sum_{n=1}^{\infty} X(x)Y(y) = \sum_{n=1}^{\infty} B_n sin(n\pi x)\Big(C_n cosh(n\pi y) + D_n sinh(n\pi y)\Big) \tag{35}$$

Applying the the boundary of $U(x,0)$ to the equation above we get that C_n must be zero. We are then left with the expression

$$U(x,y) = \sum_{n=1}^{\infty} c_n sin(n\pi x) sinh(n\pi y) \tag{36}$$

Applying the last boundary $U(x,1) = 1$ we must have that

$$1 = \sum_{n=1}^{\infty} c_n sin(n\pi x) sinh(n\pi) \tag{37}$$

setting $d_n = c_n sinh(n\pi)$ we get that

$$1 = \sum_{n=1}^{\infty} d_n sin(n\pi x) \tag{38}$$

Using again that the functions $sin(n\pi x)$ is orthogonal (21), as in the 1D case, and multiplying the above equation with $\int_0^1 sin(m\pi y)$ we get that

$$dn = 2\int_0^1 sin(m\pi x)dx = \frac{2}{m\pi}(1-(-1)^m \tag{39a}$$

$$\Rightarrow c_n = \frac{2}{m\pi sinh(m\pi)}(1-(-1)^m \tag{39b}$$

putting the coefficients c_n back into the expression for the fundamental solutions 36 we end up with the following expression for the steady-state

$$U(x,y) \quad = \quad \frac{2}{\pi}\sum_{m=1}^{\infty} \frac{sin(m\pi x)sinh(m\pi y)}{\pi sinh(m\pi)}(1-(-1)^m \tag{40a}$$

$$\Rightarrow \quad \frac{4}{\pi}\sum_{m}^{\infty} \frac{sin((2m-1)\pi x)sinh((2m-1)\pi y)}{\pi sinh((2m-1)\pi)} \tag{40b}$$