Quantum Computing

Chapter 01: Introduction

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Who? Me?

- Nickname: Arm (P'/N' Arm, etc.)
- Born: Aug 1981
- Work
  - Researcher at NECTEC 2005-2024
  - Lecturer at SIIT, Thammasat University 2025-now
- Education
  - B.Eng & M.Eng in Computer Engineering, Kasetsart University, Thailand
  - Obtained Ministry of Science and Technology Scholarship of Thailand in early 2008
  - Did a PhD in Informatics (AI & Computational Linguistics) at University of Edinburgh, UK from 2008 to 2013

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Introduction

What the (quantum) fuss is all about?

- Hearsay: “Quantum computer is immensely faster than classical computer!”
  - Shor’s algorithm for factoring an integer n in $O((\log n)^3)$ time complexity, making RSA code decryption feasible in polynomial time
  - Quantum supremacy vs. quantum advantage
- Quantum computers
  - Perform computation by manipulating state superposition and entanglement in-memory with quantum-mechanical phenomena
  - Are susceptible to noise due to quantum interference
  - Can solve some polynomial-time and non-polynomial-time problems with bounded error
- The notion of classical computer refers to modern-day computers that rely on binary data

Figure 1: Quantum computer

Classical Computer

- Data: Stored as binary numbers and manipulated by electronic circuits
- Processing: Logical and arithmetic operations, data transfer, and type conversion
- Representation: Only one item out of a vast realm of all possible configurations!
- Limit: We cannot intrinsically process multiple input items in parallel

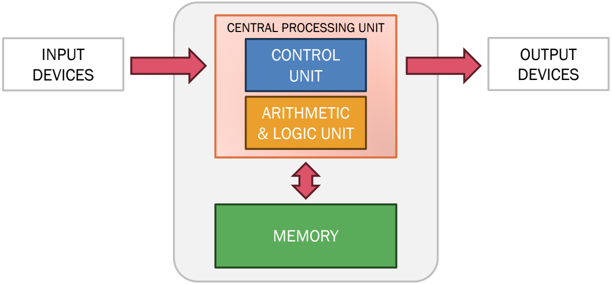


Figure 2: von Neumann architecture of classical computers

Bitstring

- Binary number is a number expressed in the base-2 numeral system: 0 and 1
- Bitstring (or binary representation) of integer N is denoted by $\mathbb{B}_L(N)$, where L is the length of such bitstring, e.g.

$$\mathbb{B}_8(33) = 00100001_2$$

- We can convert a bitstring to a decimal number by multiplying each k-th digit (from the right) with its decimal equivalent 2^{k-1} and summing these multiplicants, e.g.

$$\begin{aligned} 00100001_2 &= 1 \times 2^5 + 1 \times 2^0 \\ &= 33_{10} \end{aligned}$$

- We can also convert an integer to a bitstring by continued division as follows
  1. Find the closest exponent 2^k to N
  2. Subtract 2^k from N
  3. Repeat until the subtraction becomes zero
- Example: We convert 30_{10} to 11110_2 by

$$\begin{aligned} 30 &= 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 \\ &\quad + 1 \times 2^1 + 0 \times 2^0 \end{aligned}$$

2	30	14	6	2	0
	-16	-8	-4	-2	
	2^4	2^3	2^2	2^1	

Table 1: Conversion from 30_{10} to 11110_2

Logical Operations

- AND returns 1 iff both operands are 1

$$0011 \wedge 0101 = 0001$$

- OR returns 1 iff at least one operand is 1

$$0011 \vee 0101 = 0111$$

- NOT returns the opposite value

$$\overline{01} = 10$$

- XOR returns 1 iff only one operand is 1

$$0011 \oplus 0101 = 0110$$

- These abstract logical operations lay the foundation of data processing in classical computing, where conditional branching, jumping, and looping are predominant
- In quantum computing, these operations will be emulated by linear transformation

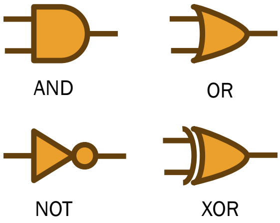


Figure 3: Logical operations

Limitations of Classical Computing

- Classical computing is linearly scalable, due to sequential logical processing of discrete states
- Parallelism in classical computing accelerates sequential processing with multiple computing cores

```python
import multiprocessing as mp

def factorial(x):
    result = 1
    for i in range(2, x+1):
        result *= i
    return result

items = [10, 20, 30, 40, 50, 60, 70, 80, 90, 100]
pool = mp.Pool(processes=4)
results = pool.map(factorial, items)
print(results)
```

- Classical parallelism struggles on looping with exponential iterations, resulting in $O(2^N)$ time complexity
- In quantum computing, multiple discrete values are represented as a superposition of bitstrings

$$\{0000, 0001, \ldots, 1111\} \quad \Rightarrow \quad \{0,1\}^4$$

  Then the quantum operations are done on this state superposition
- The quantum data processing is based on complex algebra (i.e. complex vectors, complex matrices, and measurement)

Quantum Computer

- Data: Stored as quantum states and manipulated in the memory by quantum-mechanical phonemena
- Processing: State preparation, manipulation, entanglement, correction, and measurement
- Representation: Quantum bits (or qubits) $|0\rangle$ and $|1\rangle$, which are actually complex matrices
- Advantages:
  - Exponentially faster than classical computers ... at least theoretically, thanks to quantum phase estimation
  - It can solve polynomial-time problems (P) and non-polynomial-time problems with bound errors (BQP)
  - Entanglement: This is a shared data structure with lazy evaluation. When one is evaluated, its remaining counterparts will collapse into correlated outcomes at the same time throughout the program

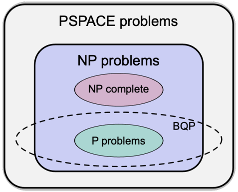


Figure 4: Problem space

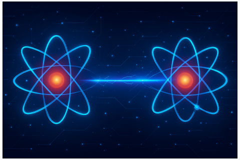


Figure 5: Entanglement

Superposition

- Light polarization: vibrating directions of electromagnetic waves
  - Light represents multiple vibration states: electricity and magnetism
  - Polaroid filters (or polarizers) blocks a specific polarization of light
  - Measuring a vibration destroys superposition

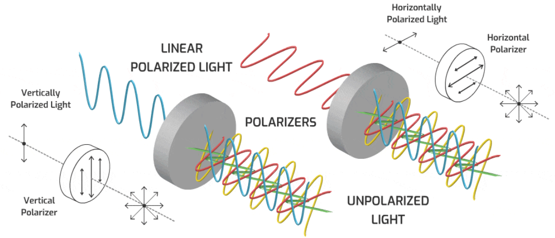


Figure 6: Light polarization[1]

[1] https://thinklucid.com/tech-briefs/polarization-explained-sony-polarized-sensor/

Quantum Entanglement

- Entanglement: is the phonemenon, where quantum state of each particle in a group cannot be described independently of the others
  - If a mixed state is factorizable, it can be decomposed into separate qubits
  - However, if it is unfactorizable, we say it is an entangled state
- Properties
  - Measuring one particle in a group will destroy the superposition, causing all of them to reveal their pure states
  - Entangled state can be used for secure data sharing, where each particle carries shared secret information

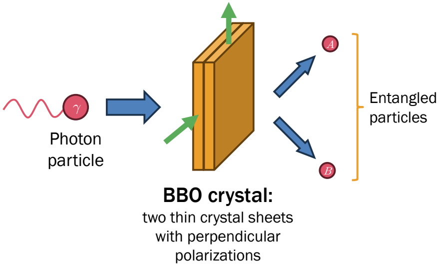


Figure 7: Entangled states are created with the SPDC process (spontaneous parametric down-conversion) using Beta-Barium Borate (BBO) crystals. In 2024, it takes 10^6 photon pumps to create 1 pair of entangled particles. Quantum entanglement is such an expensive resource!

FAQ: What Problems are Suitable for Quantum Computers?

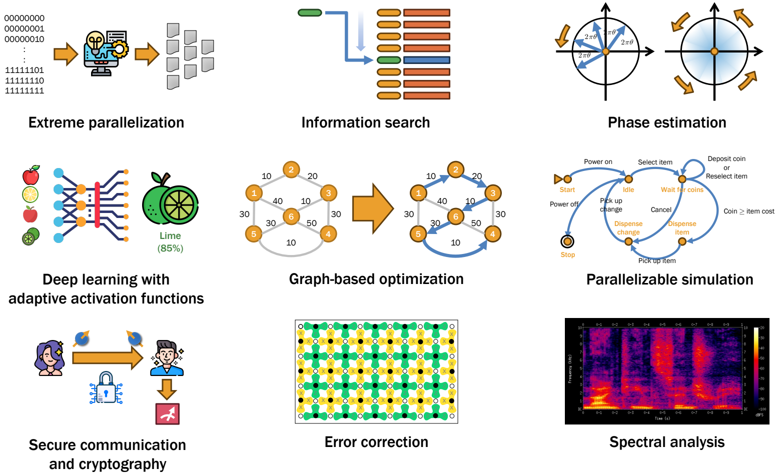


Figure 8: Quantum-suitable problems

- Superposition allows quantum computers to perform independent tasks at the same time
- Entanglement fortifies the security of data communication
- The output will be a probability distribution of possible outputs
- Warning: Quantum computers work on complex matrices and eigen-decomposition

Complex Vector

Complex Number

- Complex number is an extension of real numbers with an additional element called the imaginary part (as multiplied by the imaginary number $i = \sqrt{-1}$)
- Complex number $a + bi$ can be used to represent an arbitrary wave, whose amplitude is $A = \sqrt{a^2 + b^2}$ and whose phase is $\theta = \arctan \frac{b}{a}$

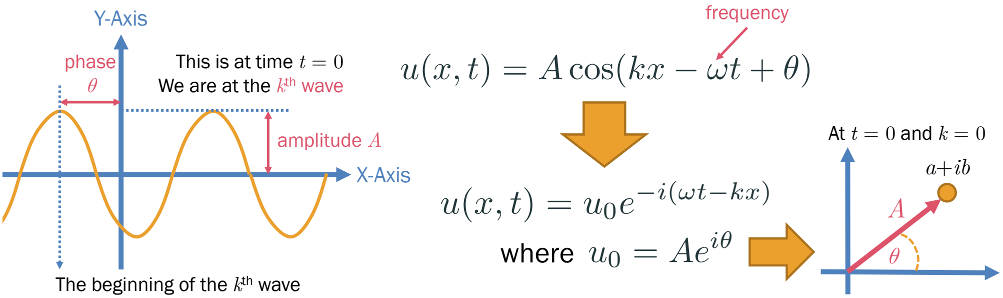


Figure 9: Wave and its complex representation

Complex Number

- Complex number is a composite of the real part and the imaginary part

$$z = a + bi$$

where $i = \sqrt{-1}$ is the imaginary number

- It can also be rewritten in the polar form

$$\begin{aligned} z &= re^{i\theta} \\ &= r\left(\cos\theta + i\sin\theta\right) \\ &= r\,\mathrm{cis}\,\theta \end{aligned}$$

where $r = \sqrt{a^2+b^2}$ is the modulus and θ is the argument of z

$$\begin{aligned} |z| &= \sqrt{a^2+b^2} \\ \arg z &= \arctan(b/a) \\ \overline{z} &= a - bi \\ z\overline{z} &= a^2+b^2 \\ z^{-1} &= \overline{z}/(a^2+b^2) \end{aligned}$$

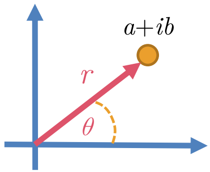


Figure 10: Complex number and polar form

Exercise 1.1: Complex Numbers

1. Let $z = a + bi$. Show that $z\overline{z} = |z|^2$.
2. Let $z = a + bi$. Show that $z^{-1} = \overline{z}/(a^2 + b^2)$.
3. Let $z_1 = a + bi$ and $z_2 = c + di$. Compute $z_1 z_2$ in terms of a, b, c, d.
4. Let $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$. Compute $z_1 z_2$ and $z_2 z_1$ in terms of $r_1, r_2, \theta_1, \theta_2$. Does complex multiplication preserve the commutative property?
5. Let $z = r \operatorname{cis}\theta$. Show that $z^n = r^n \operatorname{cis} n\theta$.
6. Let $z = r \operatorname{cis}\theta$. Show that $\overline{z} = r \operatorname{cis}(-\theta)$.
7. Let $z = r \operatorname{cis}\theta$. Show that $z\overline{z} = r^2$.
8. Show that (1) $\operatorname{cis} 0 = 1$, (2) $\operatorname{cis}\pi = -1$, (3) $\operatorname{cis}\frac{\pi}{2} = i$, (4) $\operatorname{cis}\frac{3\pi}{2} = -i$, and (5) $|\operatorname{cis}\theta| = 1$.
9. Study the following Taylor series of the exponential and trigonometric functions.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

   Show that $e^{i\theta} = \cos\theta + i\sin\theta$.

Trigonometric Values and Identities

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\pi/12$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$2-\sqrt{3}$
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$5\pi/12$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$2+\sqrt{3}$
$\pi/2$	1	0	undefined

$$\begin{aligned}\sin(\pi/2-\theta) &= \cos\theta\\ \cos(\pi/2-\theta) &= \sin\theta\end{aligned}$$

$$\begin{aligned}
\sin(\theta+\pi/2) &= -\cos\theta\\
\cos(\theta+\pi/2) &= \sin\theta\\
\sin(\theta+k\pi) &= (-1)^k\sin\theta\\
\cos(\theta+k\pi) &= (-1)^k\cos\theta\\
\sin(-\theta) &= -\sin(\theta)\\
\cos(-\theta) &= \cos(\theta)\\
\sin^2\theta+\cos^2\theta &= 1\\
\sin(\theta+\phi) &= \sin\theta\cos\phi+\cos\theta\sin\phi\\
\cos(\theta+\phi) &= \cos\theta\cos\phi-\sin\theta\sin\phi\\
\sin(\theta-\phi) &= \sin\theta\cos\phi-\cos\theta\sin\phi\\
\cos(\theta-\phi) &= \cos\theta\cos\phi+\sin\theta\sin\phi
\end{aligned}$$

Exercise 1.2: cis and $\arg$

Compute the following quantities.

1. $\operatorname{cis}(0)$
2. $\operatorname{cis}(\pi/6)$
3. $\operatorname{cis}(\pi/4)$
4. $\operatorname{cis}(\pi/3)$
5. $\operatorname{cis}(\pi/2)$
6. $\operatorname{cis}(2\pi/3)$
7. $\operatorname{cis}(3\pi/4)$
8. $\operatorname{cis}(3\pi/2)$
9. $\operatorname{cis}(4\pi/3)$
10. $\operatorname{cis}(\pi)$
11. $\arg(0)$
12. $\arg(1)$
13. $\arg(-1)$
14. $\arg(i)$
15. $\arg(-i)$
16. $\arg(1+i)$
17. $\arg(1-i)$
18. $\arg(-1-i)$
19. $\arg(3+4i)$
20. $\arg(2+i\sqrt{3})$

Complex Vector

- Complex vector is a multiplex of N waves, whose amplitudes and phases are r_k and θ_k, respectively

$$\mathbf{u} \quad = \quad \left[r_1 \operatorname{cis}\theta_1 \dots \ r_N \operatorname{cis}\theta_N\right]^\top$$

- Data point is a multiplex with amplitudes and phases

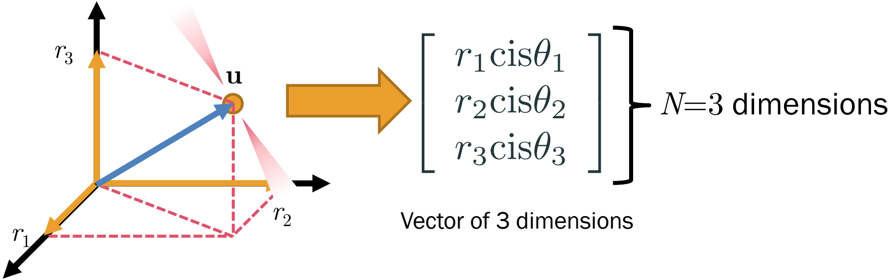


Figure 11: Complex vector as a wave multiplex

Synchronicity and Inner Product

- Inner product of wave multiplexes $\mathbf{u} = \left[r_1 \operatorname{cis}\theta_1 \dots\ r_N \operatorname{cis}\theta_N\right]^\top$ and $\mathbf{v} = \left[s_1 \operatorname{cis}\phi_1 \dots\ s_N \operatorname{cis}\phi_N\right]^\top$ is

$$\begin{aligned} \mathbf{u} \vee \mathbf{v} \quad &= \quad \mathbf{u} \cdot \overline{\mathbf{v}} \quad = \quad \mathbf{u}^\top \overline{\mathbf{v}} \\ &= \quad \sum_{k=1}^{N} r_k s_k \operatorname{cis}(\theta_k - \phi_k) \end{aligned}$$

- Phase synchronicity $\mathbf{u} \vee \mathbf{v}$ is the state of fluctuating in similar phases
  - Similar phases result in zero phase difference; i.e. real amplitude
  - Dissimilar paces result in non-zero phase difference; i.e. complex amplitude

- Norm of $\mathbf{u} = \left[u_1 \dots\ u_N\right]$ is

$$|\mathbf{u}| \quad = \quad \sqrt{\mathbf{u} \vee \mathbf{u}}$$

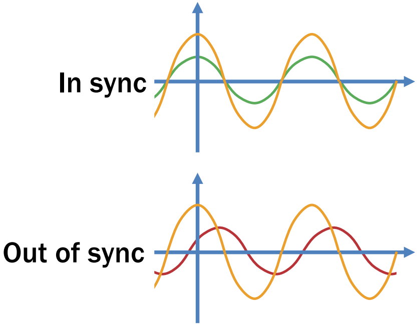


Figure 12: Phase synchronicity of two waves

Exercise 1.3: Complex Vectors and Phase Synchronicity

1. Let $\mathbf{u} = \left[u_1 \dots u_N\right]^\top$. Show that $\mathbf{u} \vee \mathbf{u} = \sum_{k=1}^{N} |u_k|^2$.
2. Let $\mathbf{u} = \left[r_1 \operatorname{cis}\theta_1 \dots r_N \operatorname{cis}\theta_N\right]^\top$. Show that $\overline{\mathbf{u}} = \left[r_1 \operatorname{cis}(-\theta_1) \dots r_N \operatorname{cis}(-\theta_N)\right]^\top$.
3. Let $\mathbf{u} = \left[r_1 \operatorname{cis}\theta_1 \dots r_N \operatorname{cis}\theta_N\right]^\top$. Show that $|\mathbf{u}| = \sqrt{\sum_{k=1}^{N} r_k^2}$.
4. Let $\mathbf{u}$ and $\mathbf{v}$ be two complex vectors of the same size N. Show that $\overline{\mathbf{u} \vee \mathbf{v}} = \mathbf{v} \vee \mathbf{u}$. Does the inner product preserve the commutative property?
5. Let $\mathbf{u} = \left[r_1 \operatorname{cis}\theta_1 \dots r_N \operatorname{cis}\theta_N\right]^\top$. Show that

$$\mathbf{u}\,(\operatorname{cis}\phi) \quad = \quad \left[r_1 \operatorname{cis}(\theta_1 + \phi) \dots r_N \operatorname{cis}(\theta_N + \phi)\right]^\top$$

6. Let $\mathbf{u} = \left[r_1 \operatorname{cis}\theta_1 \dots r_N \operatorname{cis}\theta_N\right]^\top$. Compute $|\mathbf{u}|$.
7. We can normalize a complex vector into a unit vector by:

$$\mathrm{unit}(\mathbf{u}) \quad = \quad \mathbf{u}/|\mathbf{u}|$$

   Show that $\mathrm{unit}(\mathbf{u}) \vee \mathrm{unit}(\mathbf{u}) = 1$.

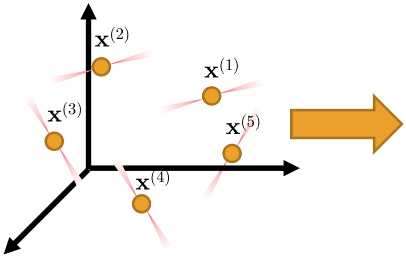
Complex Matrix

Complex Matrix

- Complex matrix is a sequence of M wave multiplexes

$$\mathbf{A} = \left[\ \mathbf{a}^{(1)} \mid \ldots \mid \mathbf{a}^{(M)}\ \right]$$

- Dataset is a sequence of wave multiplexes



$\mathbf{A}$

Matrix of dimensions 3 by 5

$$\underbrace{\begin{bmatrix} a_1^{(1)} & a_1^{(2)} & a_1^{(3)} & a_1^{(4)} & a_1^{(5)} \\ a_2^{(1)} & a_2^{(2)} & a_2^{(3)} & a_2^{(4)} & a_2^{(5)} \\ a_3^{(1)} & a_3^{(2)} & a_3^{(3)} & a_3^{(4)} & a_3^{(5)} \end{bmatrix}}_{M=5 \text{ points}} \left.\vphantom{\begin{bmatrix} a \\ a \\ a \end{bmatrix}}\right\} N=3$$

Figure 13: Complex matrix as a sequence of wave multiplexes

Hermitian Matrix

- Complex matrix is a sequence of M wave multiplexes

$$
\begin{aligned}
\mathbf{A} &= \left[\ \mathbf{a}^{(1)} \mid \dots \mid \mathbf{a}^{(M)}\ \right] \\
&= \begin{bmatrix} a_1^{(1)} & \dots & a_1^{(M)} \\ \vdots & \ddots & \vdots \\ a_N^{(1)} & \dots & a_N^{(M)} \end{bmatrix}_{N \times M}
\end{aligned}
$$

- Adjoint (conjugate transpose) of $\mathbf{A}$ is

$$
\mathbf{A}^* = \overline{\mathbf{A}}^\top
$$

- Square matrix $\mathbf{A}$ is said to be Hermitian (diagonally symmetric) if $\mathbf{A}^* = \mathbf{A}$

- $\mathbf{A}\mathbf{v}$ yields a sequence of projections of input $\mathbf{v}$ onto each basis $\mathbf{a}^{(k)}$

$$
\begin{aligned}
\mathbf{A}\mathbf{v} &= \left(\mathbf{v}^*\mathbf{A}^*\right)^* = \left(\mathbf{v}^*\mathbf{A}\right)^* \\
&= \left[\ \overline{\mathbf{v}} \vee \mathbf{a}^{(1)} \quad \dots \quad \overline{\mathbf{v}} \vee \mathbf{a}^{(M)}\ \right]^* \\
&= \left[\ \mathbf{v} \vee \overline{\mathbf{a}^{(1)}} \quad \dots \quad \mathbf{v} \vee \overline{\mathbf{a}^{(M)}}\ \right]^\top \\
&= \begin{bmatrix} \mathbf{v}^\top \mathbf{a}^{(1)} \\ \vdots \\ \mathbf{v}^\top \mathbf{a}^{(M)} \end{bmatrix}
\end{aligned}
$$

- Note that each element of $\mathbf{A}\mathbf{v}$ is equivalent to stretching and rotation of $\mathbf{v}$ by each $\mathbf{a}^{(k)}$ in linear transformation

Affine Transformation via Hermitian Matrix

- $\mathbf{AB}$ transforms each data point $\mathbf{b}^{(k)}$ onto an affine space, whose bases are specified by each column vector of Hermitian matrix $\mathbf{A}$

$$\mathbf{AB} = \left[\mathbf{Ab}^{(1)} \,\middle|\, \cdots \,\middle|\, \mathbf{Ab}^{(D)} \right]$$

- This differs from linear transformation in that each basis of the affine space is a wave superposition, performing stretching and rotation at the same time

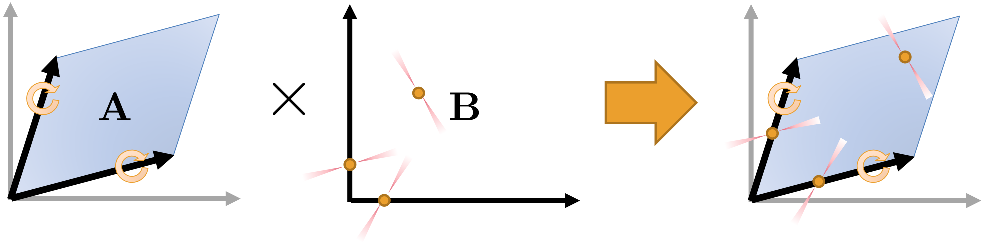


Figure 14: Affine transformation

Exercise 1.4: Complex Matrix

Check if the following matrices are Hermitian.

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2. $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$
3. $\begin{bmatrix} i & 2 \\ 1 & -i \end{bmatrix}$
4. $\begin{bmatrix} 3 & 5\operatorname{cis}\frac{\pi}{3} \\ 5\operatorname{cis}\frac{\pi}{3} & 5 \end{bmatrix}$
5. $\begin{bmatrix} 5\operatorname{cis}\frac{\pi}{3} & 3\operatorname{cis}\frac{\pi}{4} \\ 3\operatorname{cis}(-\frac{\pi}{4}) & -5\operatorname{cis}\frac{\pi}{3} \end{bmatrix}$

Compute the following matrix multiplication. Make sure the left operands are Hermitian.

6. $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
7. $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \times \begin{bmatrix} -3+4i \\ 5+6i \end{bmatrix}$
8. $\begin{bmatrix} 5\operatorname{cis}\frac{\pi}{3} & 3\operatorname{cis}\frac{\pi}{4} \\ 3\operatorname{cis}(-\frac{\pi}{4}) & -5\operatorname{cis}\frac{\pi}{3} \end{bmatrix} \times \begin{bmatrix} 4\operatorname{cis}\frac{\pi}{4} \\ 3\operatorname{cis}\frac{2\pi}{3} \end{bmatrix}$
9. $\begin{bmatrix} 5\operatorname{cis}\frac{\pi}{3} & 3\operatorname{cis}\frac{\pi}{4} \\ 3\operatorname{cis}(-\frac{\pi}{4}) & -5\operatorname{cis}\frac{\pi}{3} \end{bmatrix} \times \begin{bmatrix} \operatorname{cis}\frac{\pi}{2} \\ \operatorname{cis}\frac{\pi}{3} \end{bmatrix}$
10. $\begin{bmatrix} \operatorname{cis}\frac{\pi}{3} & \operatorname{cis}(-\frac{\pi}{4}) \\ \operatorname{cis}\frac{\pi}{4} & \operatorname{cis}\frac{3\pi}{4} \end{bmatrix} \times \begin{bmatrix} \operatorname{cis}\frac{\pi}{2} \\ \operatorname{cis}\frac{\pi}{3} \end{bmatrix}$

Exercise 1.4: Complex Matrix (cont'd)

Compute the following matrix multiplication.

11. $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -3+4i & 3-2i \\ -1 & 5+6i & 1+i \end{bmatrix}$
12. $\begin{bmatrix} \operatorname{cis}\frac{\pi}{3} & \operatorname{cis}(-\frac{\pi}{4}) \\ \operatorname{cis}\frac{\pi}{4} & \operatorname{cis}\frac{3\pi}{4} \end{bmatrix} \times \begin{bmatrix} \operatorname{cis}\frac{\pi}{2} & \operatorname{cis}\frac{\pi}{2} \\ \operatorname{cis}\frac{\pi}{3} & \operatorname{cis}\frac{\pi}{3} \end{bmatrix}$

Answer the following questions.

13. Explain why $\mathbf{A}$ has to be Hermitian so that we can compute $\mathbf{A}\mathbf{v}$, where $\mathbf{v}$ is a complex vector.
14. Check if each matrix $\mathbf{A}$ in Ex 1-5 satisfies this condition. For all pairs of column vectors $\mathbf{a}^{(j)}, \mathbf{a}^{(k)}$, $j \neq k$, in $\mathbf{A}$:

$$\mathbf{a}^{(j)} \vee \mathbf{a}^{(k)} = 0$$

15. Let $\mathbf{A} = [\mathbf{a}^{(1)} | \ldots | \mathbf{a}^{(M)}]$ be Hermitian, where each j-th element of $\mathbf{a}^{(k)}$ is

$$a_j^{(k)} = s_j^{(k)} \operatorname{cis} \phi_j^{(k)}.$$

Let $\mathbf{v} = \begin{bmatrix} v_1 \ldots v_N \end{bmatrix}^\top$, where each of its j-th element is

$$v_j = r_j \operatorname{cis} \theta_j$$

Show that each k-th element of $\mathbf{A}\mathbf{v}$ can be seen as stretching and rotation of $\mathbf{v}$ by each $\mathbf{a}^{(k)}$.

16. Explain why $\mathbf{A}\mathbf{B}$ can be seen as manipulating each data point $\mathbf{b}^{(k)}$ in $\mathbf{B}$ by stretching and rotation.

Determinant

- Determinant is the volume scaling factor of a square matrix $\mathbf{A}$ of $M \times M$ dimensions

$$\det(\mathbf{A}) = \begin{cases} a_{1,1} & \text{, if } M = 1 \\ \sum_{k=1}^{M} (-1)^{j+k} a_{j,k} \det(\mathbf{A}_{\neg j,\neg k}) & \text{, for any row } j \end{cases}$$

where minor $\mathbf{A}_{\neg j,\neg k}$ is computed by removing row j and column k from the matrix

- Example 1:

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \times 4 - 2 \times 3 = -2$$

- Example 2:

$$\det \begin{bmatrix} i & 2 \\ -i & 1 \end{bmatrix} = i \times 1 - (-i) \times 2 = 3i$$

- Example 3:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = +1 \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2 \det \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$$

Special Properties of Determinant

- For any 2×2 matrix

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- For any 3×3 matrix

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{cases} aek + bfg + cdh \\ -gec - hfa - kdb \end{cases}$$

- For any triangular or diagonal matrix

$$\det \begin{bmatrix} a_1 & \dots & \mathbf{U} \\ & \ddots & \vdots \\ \mathbf{0} & & a_M \end{bmatrix} = \det \begin{bmatrix} a_1 & & \mathbf{0} \\ \vdots & \ddots & \\ \mathbf{L} & \dots & a_M \end{bmatrix} = \det \begin{bmatrix} a_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_M \end{bmatrix} = \prod_{k=1}^{M} a_k$$

- For any square matrix

$$\det(\mathbf{A}^\top) = \det(\mathbf{A})$$

- For any block matrix, whose each $\mathbf{A}_k$ is a sub-matrix

$$\det \begin{bmatrix} \mathbf{A}_1 & & & \mathbf{0} \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{A}_N \end{bmatrix} = \prod_{k=1}^{N} \det(\mathbf{A}_k)$$

Exercise 1.5: Determinant

Find the determinants of these matrices.

1. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
2. $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
3. $\begin{bmatrix} 3+i & 0 & 0 \\ 2+4i & 3-i & 0 \\ 6+3i & 1+i & 2 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 6 & 7 \\ 0 & 0 & 0 & 8 & 9 \end{bmatrix}$

Answer the following questions.

5. Because for any square matrix $\mathbf{A}$,

$$\det(\mathbf{A}^\top) = \det(\mathbf{A})$$

now show that

$$\det(\mathbf{A}^*) = \overline{\det(\mathbf{A})}$$

6. Show that for any diagonal matrix,

$$\det \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_M \end{bmatrix} = \prod_{k=1}^{M} a_k$$

Outer Product and Wave Interaction

- Outer product of multiplexes $\mathbf{u} = \left[r_1 \operatorname{cis} \theta_1 \ldots r_N \operatorname{cis} \theta_N \right]^\top$ and $\mathbf{v} = \left[s_1 \operatorname{cis} \phi_1 \ldots s_N \operatorname{cis} \phi_N \right]^\top$ is the phase synchronicity between each pair (u_j, v_k)

$$\begin{aligned} \mathbf{u} \wedge \mathbf{v} &= \mathbf{u}\mathbf{v}^* \\ &= \begin{bmatrix} r_1 s_1 \operatorname{cis}(\theta_1 - \phi_1) & \ldots & r_1 s_N \operatorname{cis}(\theta_1 - \phi_N) \\ \vdots & \ddots & \vdots \\ r_N s_1 \operatorname{cis}(\theta_N - \phi_1) & \ldots & r_N s_N \operatorname{cis}(\theta_N - \phi_N) \end{bmatrix} \end{aligned}$$

- Differences:
  - Inner product $\mathbf{u} \vee \mathbf{v}$ reflects the synchronicity of two multiplexes, where all element-wise synchronicities are combined

$$\mathbf{u} \vee \mathbf{v} \quad = \quad \mathbf{u}^\top \overline{\mathbf{v}} \quad = \quad \sum_{k=1}^{N} u_k \overline{v}_k$$

  - Outer product $\mathbf{u} \wedge \mathbf{v}$ reflects the synchronicity of two multiplexes, where the synchronicity of each component pair is separately presented — just like a heatmap

Orthogonal Projector

- Orthogonal projector is a matrix $\mathbf{A}$ that complies with the following properties
  - Idempotency: $\mathbf{A}^2 = \mathbf{A}$
  - Self-adjunction: $\mathbf{A}^* = \mathbf{A}$
- For any unit vector $\mathbf{u}$, its self-outer product $\mathbf{u} \wedge \mathbf{u}$ is an orthogonal projector

$$\mathbf{u} \wedge \mathbf{u} \quad = \quad \mathbf{u}\mathbf{u}^* \quad = \quad \mathbf{u}\overline{\mathbf{u}}^\top$$

Note that $\mathbf{u}^*\mathbf{u} = 1$, because $|\mathbf{u}| = 1$.

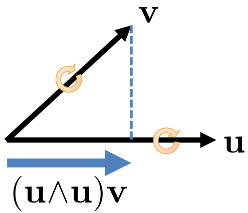


Figure 15: Orthogonal projector $\mathbf{u} \wedge \mathbf{u}$ helps project the vector $\mathbf{v}$ onto $\mathbf{u}$.

Exercise 1.6: Outer Product and Wave Interaction

Compute the following outer products.

1. $\begin{bmatrix} 1 \\ -i \end{bmatrix} \wedge \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
2. $\begin{bmatrix} 1 \\ -i \end{bmatrix} \wedge \begin{bmatrix} -3+4i \\ 5+6i \end{bmatrix}$
3. $\begin{bmatrix} 5\,\mathrm{cis}\,\frac{\pi}{3} \\ 3\,\mathrm{cis}(-\frac{\pi}{4}) \end{bmatrix} \wedge \begin{bmatrix} 4\,\mathrm{cis}\,\frac{\pi}{4} \\ 3\,\mathrm{cis}\,\frac{2\pi}{3} \end{bmatrix}$
4. $\begin{bmatrix} 5\,\mathrm{cis}\,\frac{\pi}{3} \\ 3\,\mathrm{cis}(-\frac{\pi}{4}) \end{bmatrix} \wedge \begin{bmatrix} \mathrm{cis}\,\frac{\pi}{2} \\ \mathrm{cis}\,\frac{\pi}{3} \end{bmatrix}$
5. $\begin{bmatrix} \mathrm{cis}\,\frac{\pi}{3} \\ \mathrm{cis}\,\frac{\pi}{4} \end{bmatrix} \wedge \begin{bmatrix} \mathrm{cis}\,\frac{\pi}{2} \\ \mathrm{cis}\,\frac{\pi}{3} \end{bmatrix}$

Answer the following questions.

6. Let $\mathbf{u} = \begin{bmatrix} r_1 \,\mathrm{cis}\,\theta_1 \dots r_N \,\mathrm{cis}\,\theta_N \end{bmatrix}^\top$. Compute $\mathbf{u} \wedge \mathbf{u}$.
7. Show that $(\mathbf{u} \wedge \mathbf{v})^* = \mathbf{v} \wedge \mathbf{u}$.
8. Show that for any unit vector $\mathbf{u}$, its self-outer product $\mathbf{u} \wedge \mathbf{u}$ complies with the idempotent and self-adjunct properties.
9. Show that if $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$, then

$$\begin{aligned} \mathbf{w} \wedge \mathbf{w} &= |a|^2(\mathbf{u} \wedge \mathbf{u}) + |b|^2(\mathbf{v} \wedge \mathbf{v}) \\ &\quad + a\bar{b}(\mathbf{u} \wedge \mathbf{v}) + \bar{a}b(\mathbf{v} \wedge \mathbf{u}) \end{aligned}$$

Unitary Matrix

Unitary Matrix

- Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthonormal, if they are unit vectors orthogonal to each other:

$$\mathbf{u} \vee \mathbf{v} = 0$$

- Matrix $\mathbf{A}$ is said to be orthonormal if all pairs $\mathbf{a}^{(j)}$ and $\mathbf{a}^{(k)}$ in $\mathbf{A}$ are orthonormal
- If $\mathbf{A}$ is orthonormal, the inverse of $\mathbf{A}$ is

$$\mathbf{A}^{-1} = \mathbf{A}^{*} = \overline{\mathbf{A}}^{\top}$$

- It becomes easy to find the inverse of $\mathbf{A}$; no more Gaussian elimination on $[\mathbf{A}|\mathbf{I}]$

- $\mathbf{A}$ is said to be a unitary matrix if it is Hermitian ($\mathbf{A}^{*} = \mathbf{A}$) and orthonormal ($\mathbf{A}^{-1} = \mathbf{A}^{*}$)

$$\mathbf{A}^{*}\mathbf{A} = \mathbf{A}\mathbf{A}^{*} = \mathbf{A}\mathbf{A} = \mathbf{I}$$

We can use $\mathbf{A}$ as a non-stretching rotary operator for complex vectors

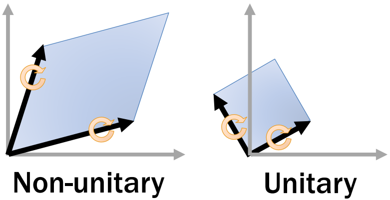


Figure 16: Non-unitary vs. unitary matrices

Exercise 1.7: Unitary Matrix

Check if these matrices are unitary (i.e. Hermitian and orthonormal).

1. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
2. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$
3. $\begin{bmatrix} i & 2 \\ 1 & -i \end{bmatrix}$
4. $\begin{bmatrix} \text{cis}\,\frac{\pi}{6} & \text{cis}\,\frac{\pi}{4} \\ \text{cis}(-\frac{\pi}{4}) & \text{cis}(-\frac{\pi}{3}) \end{bmatrix}$
5. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Answer the following questions.

6. Show that any unitary matrix of $M \times M$ dimensions can be written as

$$\mathbf{A} = \begin{bmatrix} \text{cis}\,\theta_{1,1} & \dots & \text{cis}\,\theta_{1,M} \\ \vdots & \ddots & \vdots \\ \text{cis}\,\theta_{M,1} & \dots & \text{cis}\,\theta_{M,M} \end{bmatrix}$$

7. Show that the inverse of $\mathbf{A}$ in Ex 6 is

$$\mathbf{A}^{-1} = \begin{bmatrix} \text{cis}(-\theta_{1,1}) & \dots & \text{cis}(-\theta_{M,1}) \\ \vdots & \ddots & \vdots \\ \text{cis}(-\theta_{1,M}) & \dots & \text{cis}(-\theta_{M,M}) \end{bmatrix}$$

8. From Ex 6, show that any unitary matrix can be seen as a non-stretching rotary operator.

Eigen-Decomposition

- Square matrix $\mathbf{A}$ can be factorized into an eigenvector $\mathbf{v}$ and its eigenvalue λ:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

- Interpretation:
  - Eigenvector: projectory direction in which the affine space becomes unitary
  - Eigenvalue: its distortion rate
- Matrix $\mathbf{A}$ of $M \times M$ dimensions has at most M eigen-pairs $(\lambda_k, \mathbf{v}_k)$:

$$\{(\lambda_k, \mathbf{v}_k)\}_{k=1}^{M} = \text{eigen}(\mathbf{A})$$

where $\lambda_1 > \lambda_2 > \lambda_3 > \ldots > \lambda_M$

- $\mathbf{v}_1$ is called the dominant eigenvector

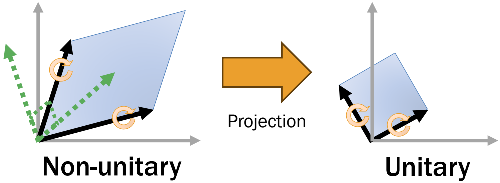


Figure 17: Eigen-decomposition

- What if the eigenvalue of $\mathbf{A}$ is zero?
  - No stretching and no rotation
  - Every transform by $\mathbf{A}$ merges to the same spot — zero vector!
  - Once that happens, we cannot recover the original data points because all information has been lost

Eigen-Decomposition (cont'd/1)

- Steps of manual calculation
  1. Compute the eigenvalues by solving

$$\det(\mathbf{A}-\lambda\mathbf{I}) \quad = \quad 0$$

     We will obtain eigenvalues $\lambda_1, \dots, \lambda_M$
  2. For each non-zero λ_k, we solve for $\mathbf{v}$:

$$(\mathbf{A}-\lambda_k\mathbf{I})\mathbf{v} \quad = \quad 0$$

     We may use row operations in Gaussian elimination here
- Example: Let's eigen-decompose

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- Step 1: Solve $\det(\mathbf{A}-\lambda\mathbf{I})=0$ for λ

$$\begin{aligned} & \det\begin{bmatrix} 0-\lambda & -i \\ i & 0-\lambda \end{bmatrix} \\ = \quad & (-\lambda)\times(-\lambda)-i\times(-i) \\ = \quad & \lambda^2-1 \quad = \quad 0 \end{aligned}$$

  The solutions for quadratic equations $ax^2+bx+c=0$ is

$$x \quad = \quad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

  Therefore, the eigenvalues are $\lambda_1=+1$ and $\lambda_2=-1$.

Eigen-Decomposition (cont'd/2)

- Steps of manual calculation
  1. Compute the eigenvalues by solving

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0$$

     We will obtain eigenvalues $\lambda_1, \ldots, \lambda_M$
  2. For each non-zero λ_k, we solve for $\mathbf{v}$:

$$(\mathbf{A} - \lambda_k \mathbf{I})\mathbf{v} = 0$$

     We may use row operations in Gaussian elimination here
- Example: Let's eigen-decompose

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

- Step 2/1: Substitute $\lambda_1 = +1$ and solve for $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v} = 0$

$$\left[\begin{array}{cc|c} -1 & -i & 0 \\ i & -1 & 0 \end{array}\right] \xrightarrow{R_2 := iR_1 + R_2} \left[\begin{array}{cc|c} -1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

  We obtain that $v_1 + iv_2 = 0$, or simply $v_2 = -iv_1$. That means $\mathbf{v} = [1 \quad -i]^\top v_1$. We choose $v_1 = 1/\sqrt{2}$ to normalize $\mathbf{v}$:

$$\mathbf{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

- Steps of manual calculation
  1. Compute the eigenvalues by solving
     $$\det(\mathbf{A} - \lambda \mathbf{I}) \quad = \quad 0$$
     We will obtain eigenvalues $\lambda_1, \ldots, \lambda_M$
  2. For each non-zero λ_k, we solve for $\mathbf{v}$:
     $$(\mathbf{A} - \lambda_k \mathbf{I})\mathbf{v} \quad = \quad \mathbf{0}$$
     We may use row operations in Gaussian elimination here
- Example: Let's eigen-decompose
  $$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$
- Step 2/2: Substitute $\lambda_2 = -1$ and solve for $(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{v} = \mathbf{0}$
  $$\left[\begin{array}{cc|c} 1 & -i & 0 \\ i & 1 & 0 \end{array}\right] \xrightarrow{R_2 := R_2 - iR_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right]$$
  We obtain that $v_1 - iv_2 = 0$, or simply $v_2 = iv_1$. That means $\mathbf{v} = [1 \;\; i]^\top v_1$. We choose $v_1 = 1/\sqrt{2}$ to normalize $\mathbf{v}$:
  $$\mathbf{v} \quad = \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Factorization of Unitary Matrix

- We can eigen-decompose a unitary matrix $\mathbf{A}$ of $M \times M$ dimensions into at most M eigen-pairs: $\{(\lambda_k, \mathbf{v}_k)\}_{k=1}^{M} = \mathrm{eigen}(\mathbf{U})$
- Let's define the change-of-basis matrix

$$P_{\mathrm{A}} = \left[\ \mathbf{v}_1 \mid \ldots \mid \mathbf{v}_M \ \right]$$

which is unitary, and the diagonal matrix of eigenvalues

$$\Lambda_{\mathrm{A}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_M \end{bmatrix}$$

which is element-wise distributive

- We can factorize matrix $\mathbf{A}$ into

$$\begin{aligned} \mathbf{A} &= P_{\mathrm{A}} \times \Lambda_{\mathrm{A}} \times (P_{\mathrm{A}})^{-1} \\ &= P_{\mathrm{A}} \times \Lambda_{\mathrm{A}} \times (P_{\mathrm{A}})^{*} \end{aligned}$$

- Advantages: Any operation applied on $\mathbf{A}$ is distributed to each diagonal element of Λ_{A} instead

$$\begin{aligned} \mathbf{A}^n &= P_{\mathrm{A}} \times \Lambda_{\mathrm{A}}^n \times (P_{\mathrm{A}})^{*} \\ \mathbf{A}^{-1} &= (P_{\mathrm{A}})^{*} \times \Lambda_{\mathrm{A}}^{-1} \times P_{\mathrm{A}} \\ \exp(\mathbf{A}) &= P_{\mathrm{A}} \times \exp(\Lambda_{\mathrm{A}}) \times (P_{\mathrm{A}})^{*} \\ \mathrm{cis}(\theta \mathbf{A}) &= P_{\mathrm{A}} \times \mathrm{cis}(\theta \Lambda_{\mathrm{A}}) \times (P_{\mathrm{A}})^{*} \end{aligned}$$

Exercise 1.8: Matrix Factorization

Factorize the following matrices. If an eigenvalue is zero, you do not have to compute its corresponding eigenvector.

1. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
3. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 0 \\ 0 & \operatorname{cis}\frac{\pi}{4} \end{bmatrix}$
6. $\begin{bmatrix} 1 & 0 \\ 0 & \operatorname{cis}\theta \end{bmatrix}$ for any θ
7. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
8. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Matrix Exponentials

Matrix Exponentials

- For a phase θ and a unitary matrix $\mathbf{A}$:

$$\operatorname{cis}(\theta\mathbf{A}) = \mathbf{I}\cos\theta + i\mathbf{A}\sin\theta$$

- Proof:
  1. Let's start with the definition

$$\operatorname{cis}(\theta\mathbf{A}) = \exp(i\theta\mathbf{A})$$

  2. By Taylor series of exponentiation

$$\begin{aligned}\exp(i\theta\mathbf{A}) &= \sum_{k=0}^{\infty}\frac{(i\theta\mathbf{A})^k}{k!}\\ &= \sum_{k=0}^{\infty}\frac{(i\theta\mathbf{A})^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{(i\theta\mathbf{A})^{2k+1}}{(2k+1)!}\\ &= \sum_{k=0}^{\infty}(-1)^k\frac{(\theta\mathbf{A})^{2k}}{(2k)!} + i\sum_{k=0}^{\infty}(-1)^k\frac{(\theta\mathbf{A})^{2k+1}}{(2k+1)!}\end{aligned}$$

- Proof (cont'd):
  3. Because $\mathbf{A}$ is unitary, $\mathbf{A}^{2k} = \mathbf{I}$

$$\begin{aligned}\exp(i\theta\mathbf{A}) &= \mathbf{I}\sum_{k=0}^{\infty}(-1)^k\frac{\theta^{2k}}{(2k)!}\\ &\quad + i\mathbf{A}\sum_{k=0}^{\infty}(-1)^k\frac{\theta^{2k+1}}{(2k+1)!}\end{aligned}$$

  4. By Taylor series of cosine and sine

$$\operatorname{cis}(\theta\mathbf{A}) = \mathbf{I}\cos\theta + i\mathbf{A}\sin\theta$$

Matrix Exponentials as Basis Rotation

- For a phase θ and a unitary matrix $\mathbf{A}$ of $M \times M$ dimensions:

$$\text{cis}(\theta\mathbf{A}) \quad = \quad \mathbf{I}\cos\theta + i\,\mathbf{A}\sin\theta$$

- If we can factorize $\mathbf{A}$ into

$$\mathbf{A} \quad = \quad P_{\mathrm{A}} \times \Lambda_{\mathrm{A}} \times (P_{\mathrm{A}})^{*}$$

then we have

$$\begin{aligned}
\text{cis}(\theta\mathbf{A}) \quad &= \quad P_{\mathrm{A}} \times \text{cis}(\theta\Lambda_{\mathrm{A}}) \times (P_{\mathrm{A}})^{*} \\
&= \quad P_{\mathrm{A}} \times \begin{bmatrix} \text{cis}(\lambda_1\theta) & & \mathsf{0} \\ & \ddots & \\ \mathsf{0} & & \text{cis}(\lambda_M\theta) \end{bmatrix} \times (P_{\mathrm{A}})^{*}
\end{aligned}$$

- Interpretation: $\text{cis}(\theta\mathbf{A})$ rotates each basis of the eigen-space P_{A} by phase θ

Exercise 1.9: Matrix Exponentials

Compute $\operatorname{cis}(\frac{\pi}{4}\mathbf{A})$ of the following matrices via $\operatorname{cis}(\theta\mathbf{A}) = \mathbf{I}\cos\theta + i\,\mathbf{A}\sin\theta$.

1. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
3. $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 0 \\ 0 & \operatorname{cis}\frac{\pi}{4} \end{bmatrix}$

Answer the following questions.

6. Show that $\mathbf{A}^{2k} = \mathbf{I}$ for any unitary matrix $\mathbf{A}$ and any positive integer $k > 0$. [Hint: $\mathbf{A}$ is Hermitian and orthonormal.]
7. Show that $\operatorname{cis}(-\theta\mathbf{A}) = \mathbf{I}\cos\theta - i\,\mathbf{A}\sin\theta$ for any unitary matrix $\mathbf{A}$.

Measurement and Reflection

- Projection: the shadow of a vector $\mathbf{v}$ on a non-zero basis vector $\mathbf{u}$

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{v} \vee \mathbf{u}}{\mathbf{u} \vee \mathbf{u}} \mathbf{u}$$

- Projection: measuring the amplitude of a wave multiplex onto the basis
- Rejection: the part of a vector $\mathbf{v}$ orthogonal to the basis vector $\mathbf{u}$

$$\mathrm{rej}_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$$

- Rejection: decomposing a wave multiplex from the basis

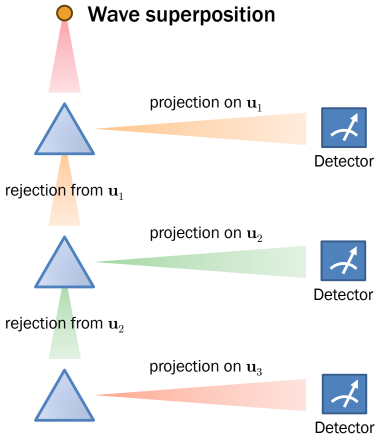


Figure 18: Beam splitting of a wave multiplex

Exercise 1.10: Measurement

Compute the projections $\mathbf{proj}_{\mathbf{u}}(\mathbf{v})$ and rejections $\mathbf{rej}_{\mathbf{u}}(\mathbf{v})$ of the following $(\mathbf{u}, \mathbf{v})$.

1. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
3. $\mathbf{u} = \begin{bmatrix} i \\ -i \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1+1i \\ 2+3i \end{bmatrix}$
4. $\mathbf{u} = \begin{bmatrix} \operatorname{cis}\frac{\pi}{3} \\ \operatorname{cis}\frac{\pi}{4} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \operatorname{cis}(-\frac{\pi}{3}) \\ \operatorname{cis}\frac{\pi}{4} \end{bmatrix}$
5. $\mathbf{u} = \begin{bmatrix} i \\ -i \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \operatorname{cis}\frac{\pi}{3} \\ \operatorname{cis}\frac{\pi}{4} \end{bmatrix}$

Answer the following questions.

6. Show that if $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors, then $\mathbf{proj}_{\mathbf{u}}(\mathbf{v}) \neq \mathbf{proj}_{\mathbf{v}}(\mathbf{u})$.
7. Let $\mathbf{U} = [\mathbf{u}_1 | \ldots | \mathbf{u}_M]$ be a unitary matrix, and $\mathbf{v}$ be a complex vector. Show that $\mathbf{proj}_{\mathbf{u}_j}(\mathbf{v})$ and $\mathbf{proj}_{\mathbf{u}_k}(\mathbf{v})$ are orthogonal to each other for all $j \neq k$.

Reflection

- Reflection: Vector $\mathbf{v}$ is flipped across the basis vector $\mathbf{u}$

$$\mathrm{refl}_{\mathbf{u}}(\mathbf{v}) \quad = \quad \mathbf{v} - 2\,\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$$

- Intuition: We have to rotate for twice the angle to get across the basis vector
- Householder reflection matrix is a reflection operator for a basis vector $\mathbf{u}$:

$$\begin{aligned} \mathrm{HR}(\mathbf{u}) &= \mathrm{I} - 2\frac{\mathbf{u} \wedge \mathbf{u}}{\mathbf{u} \vee \mathbf{u}} \\ &= \mathrm{I} - \frac{2\mathbf{u}\mathbf{u}^*}{\mathbf{u}^*\mathbf{u}} \end{aligned}$$

- $\mathrm{HR}(\mathbf{u})$ is generalized for any basis $\mathbf{u}$

- Advantages:
  - It can be used for setting a boundary, from which the vector bounces back
  - When combined with direction flipping, it can be used to amplify desired features and reduce noise

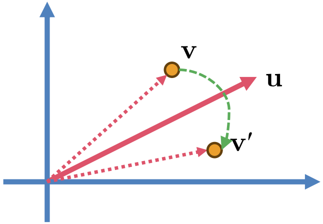


Figure 19: Reflection of $\mathbf{v}$ on the basis vector $\mathbf{u}$

Exercise 1.11: Reflection

Compute the reflection of $\mathbf{v}$ on the basis vector $\mathbf{u}$.

1. $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
2. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
3. $\mathbf{u} = \begin{bmatrix} i \\ -i \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1+1i \\ 2+3i \end{bmatrix}$
4. $\mathbf{u} = \begin{bmatrix} \operatorname{cis} \frac{\pi}{3} \\ \operatorname{cis} \frac{\pi}{4} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \operatorname{cis}(-\frac{\pi}{3}) \\ \operatorname{cis} \frac{\pi}{4} \end{bmatrix}$
5. $\mathbf{u} = \begin{bmatrix} i \\ -i \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} \operatorname{cis} \frac{\pi}{3} \\ \operatorname{cis} \frac{\pi}{4} \end{bmatrix}$

Answer the following questions.

6. Compute the Householder reflection matrix $\mathbf{HR}$ for each basis vector $\mathbf{u}$ in Ex 1-5. Then compute $\mathbf{HR}(\mathbf{u}) \times \mathbf{v}$ and compare the results with $\mathrm{refl}_{\mathbf{u}}(\mathbf{v})$.
7. Show that for any complex vectors $\mathbf{u}$

$$\mathbf{HR}(\mathbf{u}) \times \mathbf{u} \quad = \quad -\mathbf{u}$$

8. Show that for all complex vectors $\mathbf{u}, \mathbf{v}$

$$\mathbf{HR}(\mathbf{u}) \times \mathbf{v} \quad = \quad \mathbf{v} - 2\,\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$$

[Hint: $\mathbf{u}^*\mathbf{v} = (\mathbf{v}^\top \mathbf{u}^*)^\top$, and $(\mathbf{u}^*)^\top = \overline{\mathbf{u}}$.]

Conclusion

Conclusion

- Basic concepts of quantum computers
- Complex number as a wave
- Complex vector as a wave multiplex
- Complex matrix as transformation via stretching and rotation
- Inner product and outer product
- Eigen-decomposition of a square matrix
- Matrix exponentials as eigen-space rotation
- Measurement and reflection

Questions?

Target Learning Outcomes/1

- Conceptualize Quantum vs. Classical Computing
  1. Contrast the fundamental differences between classical computers and quantum computers.
  2. Identify specific problem classes suitable for quantum acceleration.
  3. Understand the qualitative roles of superposition and entanglement as data structures and processing mechanisms.
- Mastery of Complex Mathematics for Quantum States
  1. Convert between standard and polar forms of complex numbers to represent wave amplitudes and phases.
  2. Calculate the inner product and outer product of complex vectors to determine phase synchronicity and wave interactions.
  3. Manipulate complex vectors as wave superpositions and normalize them into unit vectors.

Target Learning Outcomes/2

- Proficiency in Matrix Transformations
  1. Identify and verify special types of matrices essential to quantum gates, specifically Hermitian (self-adjoint) and Unitary (orthonormal) matrices.
  2. Perform eigen-decomposition on square matrices to find eigenvalues and eigenvectors, interpreting them as distortion rates and projectory directions.
  3. Factorize unitary matrices to simplify complex operations like matrix exponentiation.
- Application of Quantum Operators
  1. Compute matrix exponentials using Taylor series expansions and interpret them as basis rotations within an eigen-space.
  2. Execute measurement operations by calculating the projection and rejection of a quantum state relative to a basis vector.
  3. Apply Householder reflection to flip vectors across a basis, understanding its role in feature amplification and noise reduction.