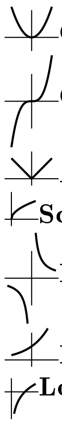
Functions

Domain Rules:
Fractions: denominator $\neq 0$
Even roots: inside ≥ 0
Odd roots: all real
Logs: inside > 0
Trig: $\sin, \cos$: all real; $\tan, \sec$: $\cos x \neq 0$; $\csc, \cot$: $\sin x \neq 0$
Composition: inner output in outer's domain

Range Rules:
$x^2, \sqrt{x}$: $[0, \infty)$
$\sqrt[3]{x}$: all real
e^x: $(0, \infty)$; $\ln x$: all real
$1/x$: all real except 0
Quadratic $ax^2 + bx + c$: $a > 0$: $[\min, \infty)$; $a < 0$: $(-\infty, \max]$
Trig: $\sin, \cos$: $[-1, 1]$; $\tan, \cot$: all real; $\sec, \csc$: $(-\infty, -1] \cup [1, \infty)$
Shifts: $f(x) + c$ shifts range by $+c$

Common Graphs:

Quadratic: x^2
Cubic: x^3
Absolute: $|x|$
Square root: $\sqrt{x}$
Reciprocal: $1/x$
Exponential: a^x
Logarithmic: $\ln(x)$

Inverse Functions:
Idea: f^{-1} reverses f ($x \to y$ becomes $y \to x$)
Exists when: f is one-to-one (horizontal line test). If not, restrict domain
Graph rule: f^{-1} is f reflected in $y = x$
Domain/Range: $f : A \to B$; $f^{-1} : B \to A$

Partial Fractions:
1. Distinct linear: $\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$
2. Repeated linear: $\frac{1}{(x-a)^n} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$
3. Irreducible quadratic: $\frac{1}{x^2+px+q} = \frac{Ax+B}{x^2+px+q}$
Example: $\int \frac{3x+5}{x^2-1} dx$
Factor: $x^2 - 1 = (x-1)(x+1)$
Decompose: $\frac{3x+5}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$
Multiply: $3x+5 = A(x+1)+B(x-1)$
Expand: $3x+5 = (A+B)x+(A-B)$
Match: $A + B = 3$, $A - B = 5 \Rightarrow A = 4$, $B = -1$

Limits: $x \to \infty$:
Exponentials: a^x $(a > 1) \to \infty$; $e^{-x} \to 0$; beat everything
Logs: $\ln x \to \infty$ (very slow); polynomials/exponentials faster
Roots: $\sqrt{x} \to \infty$ (slower than x); $1/\sqrt{x} \to 0$

Derivatives

Trig Derivatives:
$(\sin x)' = \cos x$
$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$
$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$
$(\csc x)' = -\csc x \cot x$

Inverse Trig Derivatives:
$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$
$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\tan^{-1} x)' = \frac{1}{1+x^2}$
$(\cot^{-1} x)' = -\frac{1}{1+x^2}$
$(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$
$(\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$

Exponential & Log Derivatives:
$(e^x)' = e^x$
$(a^x)' = a^x \ln a$
$(\ln x)' = \frac{1}{x}$

Logarithmic Differentiation:
When: $f(x)$ has variable in base and exponent, or complex products/quotients
Method: $f(x) = \sqrt{\frac{1+6\sin^2 x}{(1+\tan x)^2}}$
Step 1: $\ln f(x) = \frac{1}{2}\ln(1+6\sin^2 x) - 2\ln(1+\tan x)$
Step 2: $\frac{1}{f(x)} f'(x) = \frac{6\sin x \cos x}{1+6\sin^2 x} - \frac{2\sec^2 x}{1+\tan x}$
Step 3: $f'(x) = f(x)\left[\frac{6\sin x \cos x}{1+6\sin^2 x} - \frac{2\sec^2 x}{1+\tan x}\right]$

Chain Rule Example:
Differentiate: $y = \sin(3x^2 + 4x)$
Outer: $\sin(u)$, **Inner:** $u = 3x^2 + 4x$
Steps: $y' = \cos(u) \cdot u' = \cos(3x^2 + 4x)(6x + 4)$
Answer: $y' = (6x+4)\cos(3x^2+4x)$

Implicit Differentiation:
Used when: y cannot be isolated easily
Steps:
1. Differentiate both sides w.r.t. x
2. Treat y as function of $x \to$ multiply by $\frac{dy}{dx}$ when differentiating y
3. Collect $\frac{dy}{dx}$ terms
4. Solve for $\frac{dy}{dx}$

Implicit Diff Example:
Differentiate: $x^2 + y^2 = 25$
Differentiate: $2x + 2y \cdot \frac{dy}{dx} = 0$
Solve: $2y\frac{dy}{dx} = -2x$
$\frac{dy}{dx} = -\frac{x}{y}$

Applications of Derivatives

Second Derivative Test:
If $f''(c)$ exists:
$f''(c) < 0 \to$ local maximum
$f''(c) > 0 \to$ local minimum
$f''(c) = 0 \to$ inconclusive

Extrema Procedure:
Step 1: Find $f'(x)$ (differentiate)
Step 2: Find critical numbers: $f'(x) = 0$ and where $f'(x)$ undefined (in domain)
Step 3: Classify (First Derivative Test):
f' changes $+ \to -$: local max
f' changes $- \to +$: local min
No sign change: neither

Related Rates Example:
Question: Circle expanding. Radius increases at $\frac{dr}{dt} = 2$ cm/s. Find $\frac{dA}{dt}$ when $r = 5$ cm.
Solution pattern:
1. Formula: $A = \pi r^2$
2. Differentiate w.r.t. time t: $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$
3. Substitute: $r = 5$, $\frac{dr}{dt} = 2$
$\frac{dA}{dt} = 2\pi(5)(2) = 20\pi$ cm^2/s
Answer: 20π cm^2/s

Optimization Example:
Question: 80 m fencing for rectangular field along river (no fence on river side). Find max area.
Setup: x = side $\perp$ river, y = side $\parallel$ river
Total fence: $2x + y = 80 \Rightarrow y = 80 - 2x$
Area: $A = xy = x(80 - 2x) = 80x - 2x^2$
Differentiate: $A'(x) = 80 - 4x$
Set to zero: $80 - 4x = 0 \Rightarrow x = 20$
Then: $y = 80 - 2(20) = 40$
Check: $A''(x) = -4 < 0$ (max)
Answer: 20 m by 40 m

Integral

Special Case: x^{-1}:
$\int \frac{1}{x} dx = \ln|x| + C$
Warning: Do NOT apply power rule when $n = -1$

Common Integrals (Must Recognize):
$\int e^x dx = e^x + C$
$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$
$\int \cos x dx = \sin x + C$
$\int \sec^2 x dx = \tan x + C$

Trig Integrals:
$\int \cos x dx = \sin x + C$
$\int \sin x dx = -\cos x + C$
$\int \sec^2 x dx = \tan x + C$
$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$
$\int \csc x \cot x dx = -\csc x + C$

Riemann Sums:
Interval: $[a, b]$
Width: $\Delta x = \frac{b-a}{n}$
Sample points:
Left endpoints: $x_i^* = a + (i-1)\Delta x$
Right endpoints: $x_i^* = a + i\Delta x$
Left Riemann Sum: $L_n = \sum_{i=1}^n f(a + (i-1)\Delta x)\Delta x$
Right Riemann Sum: $R_n = \sum_{i=1}^n f(a + i\Delta x)\Delta x$

Net Change Theorem:
$\int_a^b$(rate of change)dt = total change over $[a, b]$

U-Substitution Example 1:
Integral: $\int \frac{x}{\sqrt{1+x^2}} dx$
Let: $u = 1 + x^2$, $du = 2xdx \Rightarrow xdx = \frac{1}{2}du$
Rewrite: $\int \frac{1}{\sqrt{u}} \cdot \frac{1}{2}du = \frac{1}{2}\int u^{-1/2}du$
Integrate: $\frac{1}{2} \cdot \frac{u^{1/2}}{1/2} = \sqrt{u} + C$
Back-substitute: $\sqrt{1 + x^2} + C$

U-Substitution Example 2:
Integral: $\int \frac{\sin x}{(1+\cos x)^2} dx$
Let: $u = 1 + \cos x$, $du = -\sin x dx$
Then: $\int \frac{\sin x}{(1+\cos x)^2} dx = -\int \frac{1}{u^2} du$
Integrate: $-\frac{u^{-1}}{-1} = u^{-1} + C$
Back-substitute: $\frac{1}{1+\cos x} + C$

Applications of Integrals

Disk Method (no hole):

Use when: region touches axis of rotation (solid goes all the way to axis)

Formula: $V = \pi \int_a^b [R(x)]^2 dx$

Meaning: Each slice forms a solid disk. Radius $R(x)$ is distance from axis to curve.

Washer Method (hole present):

Use when: region does not touch axis (empty space in middle)

Formula: $V = \pi \int_a^b [R(x)^2 - r(x)^2] dx$

Meaning: Each slice is a washer: outer circle minus inner circle. $R(x)$ = outer radius, $r(x)$ = inner radius.

Rotation Rules:

Rotate about x-axis → radii measured vertically → use $y = f(x)$

Rotate about y-axis → radii measured horizontally → often rewrite as $x = g(y)$

Always measure radius perpendicular to axis of rotation.

Average Value:

Formula: $f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$

Meaning:

- The "mean height" of graph on $[a, b]$
- Constant value whose rectangle has same area as area under f

Differential Equations

Separable DE Example 1:

Solve: $\frac{dy}{dx} = y$

Separate: $\frac{1}{y} dy = dx$

Integrate: $\ln |y| = x + C$

Exponentiate: $y = Ce^x$

Simple DE Example:

Solve: $\frac{dy}{dx} = 3x^2$

Integrate both sides: $y = x^3 + C$

Separable DE Example 2:

Solve: $\frac{dy}{dt} = 5 - y$

Separate: $\frac{dy}{5-y} = dt$

Integrate: $-\ln |5 - y| = t + C$

Rearrange: $5 - y = Ce^{-t}$

Final: $y = 5 - Ce^{-t}$

Separable DE Example 3:

Solve: $\frac{dy}{dx} = \frac{x}{y+2}$

Separate: $(y + 2)dy = xdx$

Integrate: $\frac{y^2}{2} + 2y = \frac{x^2}{2} + C$

Multiply by 2: $y^2 + 4y = x^2 + C$

Complete square: $(y+2)^2 = x^2 + K$

Final: $y = -2 \pm \sqrt{x^2 + K}$

Logistic Equation Problem:

Problem: Fish population modeled by $\frac{dP}{dt} = 0.3P(1 - \frac{P}{500})$ where $P(t)$ is number of fish after t years.

1. Parameters: Compare with $P' = rP(1 - P/K)$:

- $r = 0.3$ (growth rate)
- $K = 500$ (carrying capacity)

2. Explicit formula: Given $P(0) = 50$:

$P(t) = \frac{P_0 K e^{rt}}{(K-P_0)+P_0 e^{rt}}$

Here $P_0 = 50$, $r = 0.3$, $K = 500$:

$P(t) = \frac{50 \cdot 500 e^{0.3t}}{(500-50)+50e^{0.3t}} = \frac{25000e^{0.3t}}{450+50e^{0.3t}}$

Simplify: $P(t) = \frac{500e^{0.3t}}{9+e^{0.3t}}$

3. Population after 5 years:

$P(5) = \frac{500e^{1.5}}{9+e^{1.5}} \approx \frac{500 \cdot 4.48}{9+4.48} = \frac{2240}{13.48} \approx 166$

4. Long-term behavior:

As $t \to \infty$, $e^{0.3t} \to \infty$. In $P(t) = \frac{500e^{0.3t}}{9+e^{0.3t}}$, both numerator and denominator dominated by $e^{0.3t}$, so $\lim_{t\to\infty} P(t) = 500$. Population levels off at carrying capacity $K = 500$.

First-Order Linear DE:

Solve: $\frac{dy}{dx} - 2y = e^{3x}$

Step 1: Identify $P(x) = -2$

Step 2: Integrating factor: $\mu(x) = e^{\int(-2)dx} = e^{-2x}$

Step 3: Multiply DE by $\mu(x)$: $e^{-2x}y' - 2e^{-2x}y = e^{3x}e^{-2x}$

Right side simplifies: e^x

Left side becomes: $\frac{d}{dx}(e^{-2x}y)$

Series

Convergence & Divergence:

A series converges if: $\lim_{n\to\infty} S_n = S$ for some finite number S.

If the partial sums:

- grow without bound → diverge
- oscillate without settling → diverge

Necessary Condition for Convergence:

If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

Geometric Series:

General form: $\sum_{n=0}^{\infty} ar^n$

Convergence rule:

Converges if $|r| < 1$

Diverges if $|r| \geq 1$

Sum when convergent: $\frac{a}{1-r}$

Partial Sums:

Arithmetic sequence: $S_n = \frac{n}{2}(2a + (n - 1)d)$

where a = first term, d = common difference

Geometric sequence: $S_n = \frac{a(1-r^n)}{1-r}$

where a = first term, r = common ratio

Maclaurin Series Example:

Find first 4 non-zero terms for: $f(x) = x^2 e^{-x}$

Step 1. Known series:

$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots$

Step 2. Multiply by x^2:

$x^2 e^{-x} = x^2 \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \cdots\right)$

$= x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{6} + \frac{x^6}{24} - \cdots$

First 4 non-zero terms: $x^2 - x^3 + \frac{x^4}{2} - \frac{x^5}{6} + \cdots$

Vectors

Magnitude (length):

$|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Vector Addition:

$\vec{u} + \vec{v} = \begin{bmatrix} u_x + v_x \\ u_y + v_y \\ u_z + v_z \end{bmatrix}$

Dot Product:

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

Cross Product Magnitude:

$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

Cross Product Determinant:

$\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$

Intersection of Two Lines:

Given: $\vec{r} = \vec{a} + \lambda\vec{d}$ and $\vec{r} = \vec{c} + \mu\vec{e}$

Set equal: $\vec{a} + \lambda\vec{d} = \vec{c} + \mu\vec{e}$

Distance Between Two Points:

$|\overrightarrow{AB}| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2 + (b_z - a_z)^2}$