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Answer 1

Cube graphs have 2^n nodes. Each node is labelled with a binary string of length n. There is an edge between two vertices if their node labels differ in exactly one digit. We can see that to make a cube graph let's say Q_n we actually copy the Q_{n-1} two times, than apply the vertice rule as mentioned earlier. (NOTE: This advice was given by Faruk hoca at the lectures.) While copying, we actually copy the binary string labels too, like for making Q_3. We copy two Q_2s, both labelled with 00, 01, 11, and 10. We put zero to the first copied one, 000, 001, 011, 010. And 1 to the second copied one, 100, 101, 111, 110. Applying the rule, we can see that there are edges between 000-100, 001-101, 011-111, 010-110, since they differ in exactly one digit. So to generalize this we can make a recurrence relation. We copied Q_{n-1} two times, and each one has a_{n-1} edges, so $2a_{n-1}$. Since there are some edges, while connecting two copied graph like in the example I said before, we have to add them. Since we add 0 and 1 to each binary string, each two copied same binary string will differ in exactly one digit like 000-100 (copied ones are 00 and 00). So there will be one edge for each two copied same binary strings. By the definition I said before there are 2^n nodes in the graph. There will be $2 * 2^{n-1}$ nodes for two Q_{n-1}s. Since each two of them will construct one edge if we divide the number of nodes by two we can find newly drawed edges, which are 2^{n-1}. If we add these two we will find the recurrence relation. So,

$$a_n = 2a_{n-1} + 2^{n-1}$$

for $n \geq 2$ with the initial condition $a_1 = 1$, since two nodes, labelled with 0 and 1, will have only one edge.

Answer 2

By Table 1 in page 542 of the text book, $< 1, 2, 3, 4, 5, ... >$ will have the closed form $\frac{1}{(1-x)^2}$. We can multiply it with the constant a which is 3.

Then, $< 3, 6, 9, 12, 15, ... >$ will be of the form $\frac{3}{(1-x)^2}$, since;

$1 + 2x + 3x^2 + ...$ when we multiply with it 3 it will be like, $3 + 6x + 9x^3 + ...$, that's why we can multiply with a constant like that way, which is like using summing theorem, 2 times adding same thing to the same thing. We can use shifting theorem. If we shift it one time we have to multiply the closed form with x^1. So, we will have,

$< 0, 3, 6, 9, 12, ... >$ will be of the form $\frac{3x}{(1-x)^2}$. Finally, by using summing theorem, add

$< 1, 1, 1, ... >$ which has the form of $\dfrac{1}{1-x}$, by Table 1.
$< 0+1, 3+1, 6+1, ... > \rightarrow < 1, 4, 7, 10, 13... >$ which has the closed form of $\dfrac{3x}{(1-x)^2} + \dfrac{1}{1-x} = \dfrac{1+2x}{(1-x)^2}$.
So, the answer is

$$\frac{1+2x}{(1-x)^2}$$

Answer 3

We have to solve $a_n = a_{n-1} + 2^n$, $n \geq 1$ with $a_0 = 1$
Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ Let's write the recurrence relation with summation.
$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 2^n x^n$
By Table 1 in the page 542, $\sum_{n=0}^{\infty} a^n x^n = \dfrac{1}{1-ax}$
$\sum_{n=0}^{\infty} 2^n x^n = \dfrac{1}{1-2x}$ *where a=2*
$\sum_{n=1}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} 2^n x^n - 2^0 x^0 = \dfrac{1}{1-2x} - 1$
So put it in the main equation,
$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + \dfrac{1}{1-2x} - 1$
$\sum_{n=1}^{\infty} a_{n-1} x^n = x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} = xG(x)$ *(1)*
since $G(x) = \sum_{n=0}^{\infty} a_n x^n$ and if we write n-1 instead of n we will get the same thing which is G(x).
$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^n - a_0 x^0 = G(x) - a_0$ *(2)*
Now, let's put (1) and (2) into the equation.
$G(x) - a_0 = xG(x) + \dfrac{1}{1-2x} - 1$
$G(x) - 1 = xG(x) + \dfrac{1}{1-2x} - 1$ *since* $a_0 = 1$
$G(x)(1-x) = \dfrac{1}{1-2x}$

$G(x) = \dfrac{1}{(1-2x)(1-x)} = \dfrac{A}{1-2x} + \dfrac{B}{1-x} = \dfrac{A(1-x)}{(1-2x)(1-x)} + \dfrac{B(1-2x)}{(1-2x)(1-x)}$
$1 = A(1-x) + B(1-2x) = A + B - x(A+2B) = 1 + 0x$
$A + B = 1$ and $A + 2B = 0$ (since the coefficient of x is zero.
$A = 1 - B \rightarrow 1 - B + 2B = 0$ so $B = -1$ and $A = 2$
So, if we put A and B
$G(x) = \dfrac{2}{1-2x} - \dfrac{1}{1-x}$
So we have to generate $\dfrac{2}{1-2x}$ and $\dfrac{1}{1-x}$
$\dfrac{1}{1-2x}$ is the closed form of $< 1, 2, 4, 8, ..., 2^n, ... >$ by Table 1 in page 542,
which is $1 + ax + a^2x^2 + ... = \dfrac{1}{1-ax}$ where a=2
If we add two $\dfrac{1}{1-2x}$ them up (summing theorem) we can find $\dfrac{2}{1-2x}$
$\dfrac{1}{1-2x} + \dfrac{1}{1-2x} = \dfrac{2}{1-2x}$ is the closed form of $< 1, 2, 4, 8, ..., 2^n, ... > + < 1, 2, 4, 8, ..., 2^n, ... >$
So, $\dfrac{2}{1-2x}$ is the closed form of $< 2, 4, 8, 16, ...2^{n+1}, ... >$
By Table 1, $\dfrac{1}{1-x}$ is the closed form of $< 1, 1, 1, ..., 1^n, ... >$.
So, if we multiply $\dfrac{1}{1-x}$ with -1 and add the two function we found up we can find G(x).
$G(x) = \dfrac{2}{1-2x} - \dfrac{1}{1-x}$ is the closed form of
$< 2, 4, 8, ..., 2^{n+1}, ... > + < -1, -1, -1, ... >=< 1, 3, 7, ..., 2^{n+1} - 1, ... >$
Here, we can see that $a_0 = 1$, $a_1 = 3$, $a_2 = 7$, and $a_n = 2^{n+1} - 1$
So the answer is

$$a_n = 2^{n+1} - 1$$

Answer 4

a)

We are given that R={(a,b)|a divides b} on A={1,2,3,9,18}
So, since 1 divides every number on A, 2 divides 2 and 18, 3 divides 3, 9 and 18, 9 divides 9 and 18; we can write the relation R.
$R = \{(1,1), (1,2), (1,3), (1,9), (1,18), (2,2), (2,18), (3,3), (3,9), (3,18), (9,9), (9,18), (18,18)\}$// First, let's draw the directed graph,

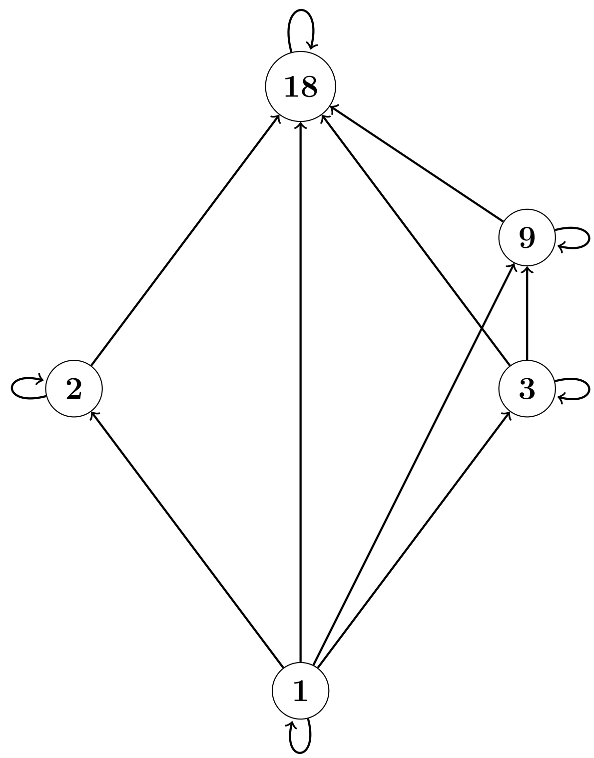

Figure 1: R in Q4, as a directed graph.

Now, we can draw the Hasse diagram of R.

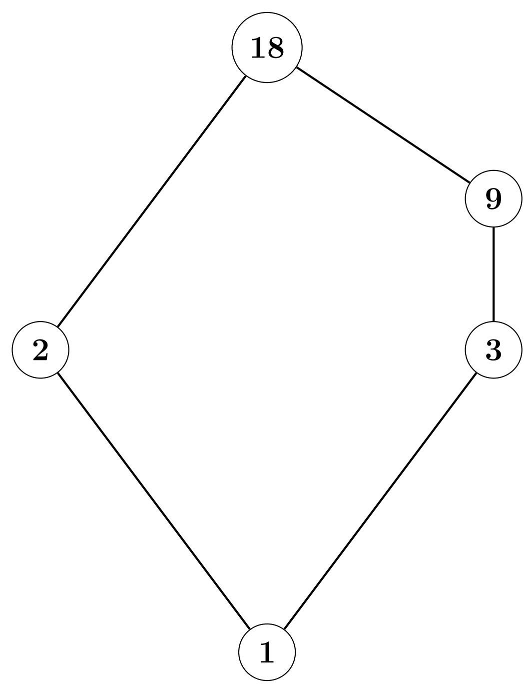

Figure 2: Hasse diagram of R.

b)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c)

To prove that (A,R) is a lattice, we have to show every pair of elements has both a least upper bound (LUB) and a greatest lower bound (GLB), we can find two of them by looking at the Hasse diagram.
for 1 and 2: LUB=2 and GLB=1;
for 1 and 3: LUB=3 and GLB=1;
for 1 and 9: LUB=9 and GLB=1;
for 1 and 18 LUB=18 and GLB=1;
for 2 and 3: LUB=18 and GLB=1;
for 2 and 9: LUB=18 and GLB=1;
for 2 and 18: LUB=18 and GLB=2;
for 3 and 9: LUB=9 and GLB=3;
for 3 and 18: LUB=18 and GLB=3;
for 9 and 18: LUB=18 and GLB=9.

d)

$R_s = R \cup R^{-1}$ *by the symmetric closure definition in the page 598 of the textbook*
Where $R^{-1} = \{(1,1),(2,1),(2,2),(3,1),(3,3),(9,1),(9,3),(9,9),(18,1),(18,2),(18,3),(18,9),(18,18)\}$
Symmetric closure is the symmetric version with least number of elements.
So,
$R_s = \{(1,1),(1,2),(1,3),(1,9),(1,18),(2,1),(2,2),(2,18),(3,1),(3,3),(3,9),(3,18),(9,1),(9,3),(9,9),$
$(9,18),(18,1),(18,2),(18,3),(18,9),(18,18)\}$
Then the matrix representation of R_s will be,

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

e)

*Integers 2 and 9 are not comparable since 2 does not divide 9, and 9 does not divide 2, i.e (2,9) or (9,2) are not int the set R.

*Integers 3 and 18 are comparable since 3 divides 18. Hence, there is a relation between them (3,18) is in the set R.

Answer 5

We will solve both a and b using matrix representation of relations.
a)
For reflexivity, all the diagonal entries of the matrix has to be 1 because of the definition of reflexivity.
For symmetry, the symmetric entries of the matrix has to be 0 to 0 or 1 to 1, because of the definition of symmetry(if (a,b) then (b,a). NOTE: the symmetric entries are the mirrored elements with respect to diagonal. Like, second row first column and first row second column. So we 2 two choices for each two non-diagonal entry (both zero or both 1). Since this matrix is n by n, there are n elements in the diagonal. There are total of n^2 entries and n of them are in diagonal. So, there are $n^2 - n$, non-diagonal entries, and each two of them can be two different entry (both zero or both one). Hence, since we have 2 choices for each two of them, there will be $2^{(\frac{n^2-n}{2})}$ different binary relations on A which are both symmetric and reflexive.
The answer is,

$$2^{(\frac{n^2-n}{2})}$$

b)
We can use a similar idea we used in part a.
For reflexivity, all the diagonal entries of the matrix has to be 1 because of the definition of reflexivity.
For antisymmetry, the symmetric entries of the matrix has to be 0 to 0, 1 to 0, or 0 to 1, because of the definition of the antisymmetry there cannot be 1 to 1. Again, the symmetric entries are the mirrored entries with respect to diagonal. So, we have 3 choices for each of two non-diagonal elements. There are n diagonal entries, so there are $n^2 - n$ non-diagonal entries. So there will be $3^{(\frac{n^2-n}{2})}$ different binary relations on A which are both antisymmetric and reflexive.
The answer is,

$$3^{(\frac{n^2-n}{2})}$$

Answer 6

I will disprove it by giving a counter example. Consider the relation $R = \{(b,a),(a,c),(c,b)\}$.
This is antisymmetric since there is no relation like (a,b) and (b,a), (a,c) and (c,a), or (c,b) and (b,c).

Transitive closure is the transitive version of this relation with least as possible elements. The definition of transitivity says that if there is a element (a,b) and (b,c) then there has to be a (a,c), so let's apply this idea.
Since there are (b,a) and (a,c) there has to be (b,c), there are (a,c) and (c,b) there has to be(a,b), there are (c,b) and (b,a) there has to be (c,a). When we add these elements to the relation, it still is not a transitive relation because there are (a,b) and (b,a) but not (a,a), let's add them too.
Since there are (a,b) and (b,a) there has to be (a,a), there are (b,c) and (c,b) there has to be (b,b), there are (c,a) and (a,c) there has to be (c,c). When we add these too the relation will be transitive.
The transitive closure of this relation is $\{(b,a),(a,c),(c,b),(b,c),(c,a),(a,b),(a,a),(b,b),(c,c)\}$
We clearly can see that this is not antisymmetric because the relation contains (b,a) and (a,b), which are symmetric elements. They do not hold the definition of the antisymmetry which is according to the textbook if there exists (x,y) and (y,x) then x=y. Here $a \neq b$, so the transitive closure of an antisymmetric relation is not always antisymmetric. I disproved it by giving a counterexample.