BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

Table of Contents

BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

THEORETICAL ANALYSIS

Basic operation is the comparison marked as (1)

Analyze B(n)

The operation executes once in each iteration regardless of the input.
So,

$$B(n) = \sum_{i=0}^{n-1} 1 = n$$

Therefore, $B(n) \in \theta(n)$.

Analyze W(n)

The operation executes once in each iteration regardless of the input.
So,

$$W(n) = \sum_{i=0}^{n-1} 1 = n$$

Therefore, $W(n) \in \theta(n)$.

Analyze A(n)

The operation executes once in each iteration regardless of the input.
So,

$$A(n) = \sum_{i=0}^{n-1} 1 = n$$

Therefore, $A(n) \in \theta(n)$.

Basic operations are the two assignments marked as (2)

Analyze B(n)

Best case input is the case that all the items of the array go to the *else* block. So, the best case happens when the array is filled with 2's because this way the basic operation does not get executed.
Therefore, $B(n) \in O(1)$.

Analyze W(n)

Same things happen in the *if* and *else if* blocks until the basic operation so these are equal and since the other case is the best case, the worst case occurs when the array is filled with 0's and 1's. The basic operation gets executed once in each iteration, so $W(n)$ can be written as

$$W(n) = \sum_{i=0}^{n-1} (\sum_{j=i}^{n-1} 1) = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$$

$$\lim_{n\to\infty} \frac{\frac{n^2+n}{2}}{n^2} = \lim_{n\to\infty} \frac{n^2+n}{2n^2} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{2}$$

Therefore, $W(n) \in \theta(n^2)$.

BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

Analyze A(n)

Let τ be the number of basic operations at step i.
So,

$$A(n) = E[\tau] = \sum_{i=0}^{n-1} E[\tau_i]$$

such that

$$E[\tau_i] = \sum_{k=0}^{2} P(k) * k$$

Sum is from 0 to 2 since these are the only possibilities.

$$E(\tau_i) = \sum_{j=i}^{n-1} (\frac{1}{3} * 1) + \sum_{j=i}^{n-1} (\frac{1}{3} * 1) + \sum_{j=i}^{n-1} (\frac{1}{3} * 0) = \sum_{j=i}^{n-1} \frac{2}{3}$$

Plug in this equation

$$A(n) = E(\tau) = \sum_{i=0}^{n-1} (\sum_{j=i}^{n-1} \frac{2}{3}) = \frac{n(n+1)}{3} = \frac{n^2+n}{3}$$

$$\lim_{n\to\infty} \frac{\frac{n^2+n}{3}}{n^2} = \lim_{n\to\infty} \frac{n^2+n}{3n^2} = \lim_{n\to\infty} (\frac{1}{3} + \frac{1}{3n}) = \frac{1}{3}$$

Therefore, $A(n) \in \theta(n^2)$.

Basic operations are the two comparisons marked as (3)

Analyze B(n)

$$B(n) = \sum_{i=0}^{n-1} min\left\{\sum_{j=i}^{n-1} (\lfloor \log_3 n \rfloor + 1), 0, \sum_{p=0}^{n-1} 1\right\} = \sum_{i=0}^{n-1} 0 = 0$$

Therefore $B(n) \in O(1)$.

Analyze W(n)

$$W(n) = \sum_{i=0}^{n-1} \max\left\{\sum_{j=i}^{n-1} (\lfloor \log_3 n \rfloor + 1),\ 0,\ \sum_{p=0}^{n-1} 1\right\} = \sum_{i=0}^{n-1} \max\{(n-i)(\lfloor \log_3 n \rfloor + 1),\ 0,\ n\} =$$

$$= \left(x = \left\lfloor \frac{n\lfloor \log_3 n \rfloor}{\lfloor \log_3 n \rfloor + 1} \right\rfloor\right) = \sum_{i=0}^{x} (n-i)(\lfloor \log_3 n \rfloor + 1) + \sum_{i=x+1}^{n-1} n = \frac{1}{2}(2n-x)(x+1)(\lfloor \log_3 n \rfloor + 1) + n(n-x-1) =$$

$$\implies \lim_{n\to\infty} \frac{\frac{1}{2}(2n-x)(x+1)(\lfloor \log_3 n \rfloor + 1) + n(n-x-1)}{n^2 \log n} =$$

$$= \lim_{n\to\infty} \frac{\left(2-\frac{x}{n}\right)\left(\frac{x}{n}+\frac{1}{n}\right)(\lfloor \log_3 n \rfloor + 1) + 2\left(\frac{\left\lfloor \frac{n}{\lfloor \log_3 n \rfloor + 1} \right\rfloor}{n} - \frac{1}{n}\right)}{2\log n} =$$

$$= \lim_{n\to\infty} \frac{\lfloor \log_3 n \rfloor + 1}{2\log n} = \lim_{n\to\infty} \frac{\log_3 n}{2\log 3 \, \log_3 n} = \frac{1}{2\log 3}$$

$$\implies W(n) \in \Theta(n^2 \log n)$$

Analyze A(n)

$$A(n) = E[\tau] = \sum_{i=0}^{n-1} E[\tau_i] = \sum_{i=0}^{n-1} \left[p(0) \sum_{j=i}^{n-1} (\lfloor \log_3 n \rfloor + 1) + p(1)0 + p(2) \left[\sum_{p=0}^{n-1} 1 \right] \right] = \frac{1}{3} \sum_{i=0}^{n-1} \left[\sum_{j=i}^{n-1} (\lfloor \log_3 n \rfloor + 1) + \sum_{p=0}^{n-1} 1 \right] =$$

$$= \frac{1}{3} \sum_{i=0}^{n-1} [(n-i)(\lfloor \log_3 n \rfloor + 1) + n - 1] = \frac{1}{6} n(n+1)(\lfloor \log_3 n \rfloor + 1) + \frac{1}{3} n(n-1)$$

$$\lim_{n\to\infty} \frac{\frac{1}{6} n(n+1)(\lfloor \log_3 n \rfloor + 1) + \frac{1}{3} n(n-1)}{n^2 \log n} = \lim_{n\to\infty} \frac{\frac{1}{6} n(n+1)(\lfloor \log_3 n \rfloor + 1)}{n^2 \log n} + \lim_{n\to\infty} \frac{\frac{1}{3} n(n-1)}{n^2 \log n} =$$

$$\lim_{n\to\infty} \frac{1}{6} \frac{n}{n} \frac{n+1}{n} \frac{\log_3 n}{\log n} + \lim_{n\to\infty} \frac{1}{3} \frac{n}{n} \frac{n-1}{n} \frac{1}{\log n} = \frac{1}{6\log 3}$$

$$\implies A(n) \in \Theta(n^2 \log n)$$

BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

Basic operations are the three assignments marked as (4)

$$\text{Let } f(x) = \sum_{t=1}^{x} \left\lfloor \frac{x}{t} \right\rfloor$$

1.

$$\sum_{t=1}^{x} \left\lfloor \frac{x}{t} \right\rfloor \leq x \sum_{t=1}^{x} \frac{1}{t} = x\left(1 + \sum_{t=2}^{x} \frac{1}{t}\right) \leq x\left(1 + \int_{1}^{x} \frac{dt}{t}\right) = x(\ln x + 1)$$

$$\implies f(x) \leq x(\ln x + 1) \ (1)$$

2.

$$\sum_{t=1}^{x} \left\lfloor \frac{x}{t} \right\rfloor \geq \sum_{t=1}^{x} \left(\frac{x}{t} - 1\right) = x \sum_{t=1}^{x} \frac{1}{t} - x \geq \left(x \int_{1}^{x+1} \frac{dt}{t}\right) - x = x\ln(x+1) - x$$

$$\implies f(x) \geq x\ln(x+1) - x \ (2)$$

From (1) & (2) $f(n) \in\sim (n \log n)$ (These are used for skipping steps in following proofs.)

Analyze B(n)

$$B(n) = \sum_{i=0}^{n-1} \min \left\{ \sum_{j=i}^{n-1} (\lfloor \log_3 n \rfloor + 1), \sum_{m=i}^{n-1} \sum_{l=m}^{n-1} \sum_{t=1}^{n} \left(\left\lfloor \frac{n-1}{t} \right\rfloor + 1 \right), \sum_{p=0}^{n-1} \sum_{j=0}^{p^2-1} 1 \right\} = \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (\lfloor \log_3 n \rfloor + 1) =$$

$$= \frac{1}{2} n(n+1)(\lfloor \log_3 n \rfloor + 1)$$

$$\lim_{n\to\infty} \frac{\frac{1}{2} n(n+1)(\lfloor \log_3 n \rfloor + 1)}{n^2 \log n} = \lim_{n\to\infty} \frac{1}{2} \frac{n}{n} \frac{n+1}{n} \frac{\log_3 n}{\log n} = \frac{1}{2 \log 3}$$

$$\implies B(n) \in \Theta(n^2 \log n)$$

Analyze W(n)

Information regarding the average case analysis is used in this part. Checking it earlier may be helpful.

BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

$$W(n) = \sum_{i=0}^{n-1} \max \left\{ \sum_{j=i}^{n-1} (\lfloor \log_3 n \rfloor + 1),\ \sum_{m=i}^{n-1} \sum_{l=m}^{n-1} \sum_{t=1}^{n} \left(\left\lfloor \frac{n-1}{t} \right\rfloor + 1 \right),\ \sum_{p=0}^{n-1} \sum_{j=0}^{p^2-1} 1 \right\} =$$

$$= \left(y = \left\lceil n + \frac{1}{2} - \sqrt{\frac{n(n-1)(2n-1)}{3(f(n-1)+n)} + \frac{1}{4}} \right\rceil \right) = \sum_{i=0}^{y-1} \frac{1}{2}(n-i)(n-i+1)(f(n-1)+n) + \sum_{i=y}^{n-1} \frac{1}{6} n(n-1)(2n-1) =$$

$$= \frac{1}{6}\left(n(n+1)(n+2) - (n-y)(n-y+1)(n-y+2)\right)(f(n-1)+n) + \frac{1}{6} n(n-1)(2n-1)(n-y) \leq$$

$$\leq \frac{1}{2} n(n+1)(\lfloor \log_3 n \rfloor + 1) + \frac{1}{6} n(n+1)(n+2)(f(n-1)+n) + \frac{1}{6} n^2(n-1)(2n-1) = 3A(n)$$

$\implies 3A(n) \geq W(n) \geq A(n)$
$\implies W(n) \in \Theta(A(n))$
$\implies W(n) \in \Theta(n^4 \log n)$

Analyze A(n)

$$A(n) = E[\tau] = \sum_{i=0}^{n-1} E[\tau_i] = \sum_{i=0}^{n-1} \left[p(0) \sum_{j=i}^{n-1} (\lfloor \log_3 n \rfloor + 1) + p(1) \sum_{m=i}^{n-1} \sum_{l=m}^{n-1} \sum_{t=1}^{n} \left(\left\lfloor \frac{n-1}{t} \right\rfloor + 1 \right) + p(2) \sum_{p=0}^{n-1} \sum_{j=0}^{p^2-1} 1 \right] =$$

$$= \frac{1}{3} \sum_{i=0}^{n-1} \left[(n-i)(\lfloor \log_3 n \rfloor + 1) + \frac{1}{2}(n-i)(n-i+1) \sum_{t=1}^{n} \left(\left\lfloor \frac{n-1}{t} \right\rfloor + 1 \right) + \frac{1}{6} n(n-1)(2n-1) \right] =$$

$$= \frac{1}{3} \left[\frac{1}{2} n(n+1)(\lfloor \log_3 n \rfloor + 1) + \frac{1}{6} n(n+1)(n+2)(f(n-1)+n) + \frac{1}{6} n^2(n-1)(2n-1) \right]$$

$$\lim_{n\to\infty} \frac{1}{3} \frac{\frac{1}{2} n(n+1)(\lfloor \log_3 n \rfloor + 1) + \frac{1}{6} n(n+1)(n+2)(f(n-1)+n) + \frac{1}{6} n^2(n-1)(2n-1)}{n^4 \log n} =$$

$$= \lim_{n\to\infty} \frac{1}{6} \frac{n}{n} \frac{n+1}{n} \frac{\log_3 n}{\log n} \frac{1}{n^2} + \lim_{n\to\infty} \frac{1}{18} \frac{n}{n} \frac{n+1}{n} \frac{n+2}{n} \frac{f(n-1)+n}{n \log n} + \lim_{n\to\infty} \frac{1}{18} \frac{n^2}{n^2} \frac{n-1}{n} \frac{2n-1}{n} \frac{1}{\log n} = \frac{1}{18}$$

$\implies A(n) \in \Theta(n^4 \log n)$

IDENTIFICATION OF BASIC OPERATION(S)

Basic operation is the operation that consumes most of the time. Hence, in our algorithm we should select the operations marked as 4 as the basic operations since they locate at the innermost loops. So, in order to execute 4, we should execute the loops that contain 1, 2 and 3 and then we can enter the loop of the 4. Therefore, 4 is the basic operation.

BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

REAL EXECUTION

Best Case

N Size	Time Elapsed
1	4.099999955542444e-06
5	3.999999989900971e-06
10	1.3200000012147939e-05
20	4.610000007687631e-05
30	0.00012890000004972535
40	0.00022509999996600527
50	0.0005964999999150677
60	0.0006704999999556094
70	0.0006936999999425097
80	0.0010416000000077474
90	0.00148449999994682
100	0.00179450000007364
110	0.0020455999999740015
120	0.0024610000000393484
130	0.0028694000000086817
140	0.0034056020001571596
150	0.0038703000000168686

Worst Case

N Size	Time Elapsed
1	1.3000000080864993e-06
5	4.5999999997548e-05
10	0.0007319999999708671
20	0.00839670000004844
30	0.043889800999977524
40	0.1364803030000985
50	0.3318517069999416
60	0.6910467159999598
70	1.331291032000081
80	2.249222351999947
90	3.685174375000088
100	5.5558656160000055
110	8.388218644999938
120	11.535699528999999
130	16.17488348100005
140	21.920207303000097
150	29.25777187000017

BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

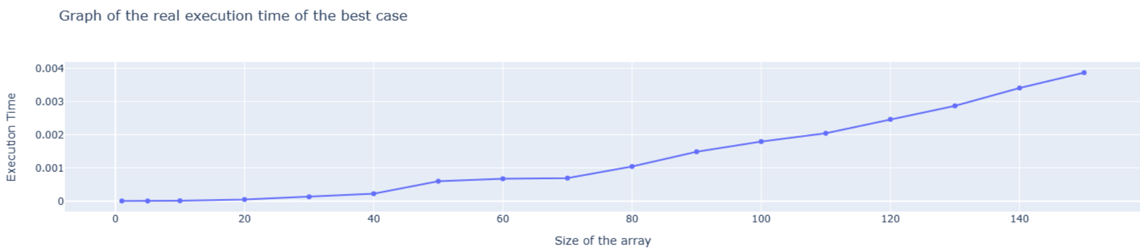
Average Case

N Size	Time Elapsed
1	6.666666839313015e-07
5	2.3841666688895202e-05
10	0.00022307500000806613
20	0.003251100083341877
30	0.018697667083349263
40	0.05484349291669824
50	0.11894803600000614
60	0.2861813151666581
70	0.48367250816667706
80	0.9451507308333286
90	1.43377829966668
100	2.2487030782499935
110	3.3483938727499947
120	4.330947497916658
130	6.31174126999997
140	8.919156217250057
150	12.986277596083331

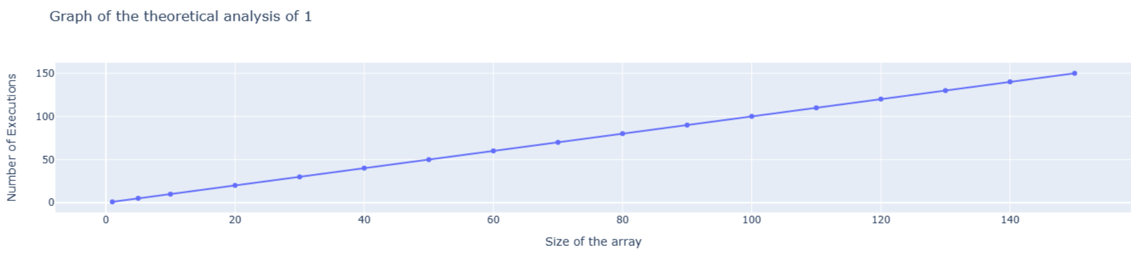
COMPARISON

Best Case

Graph of the real execution time of the algorithm

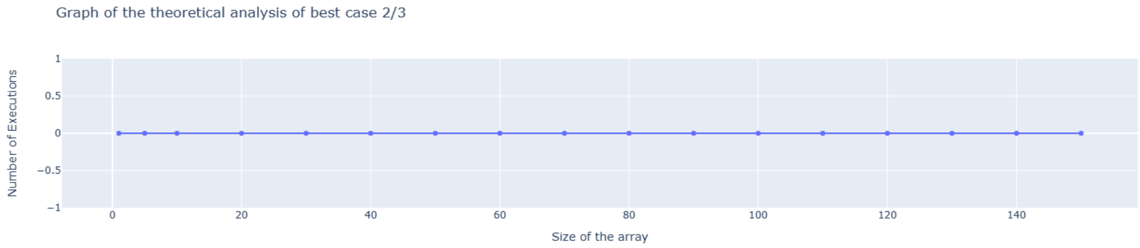


Graph of the theoretical analysis when basic operation is the operation marked as (1)

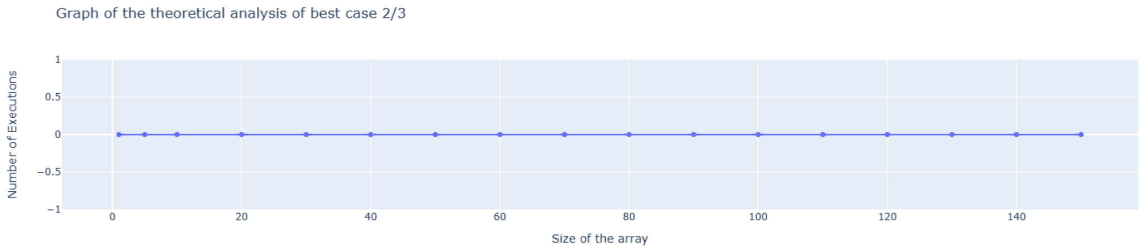


BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

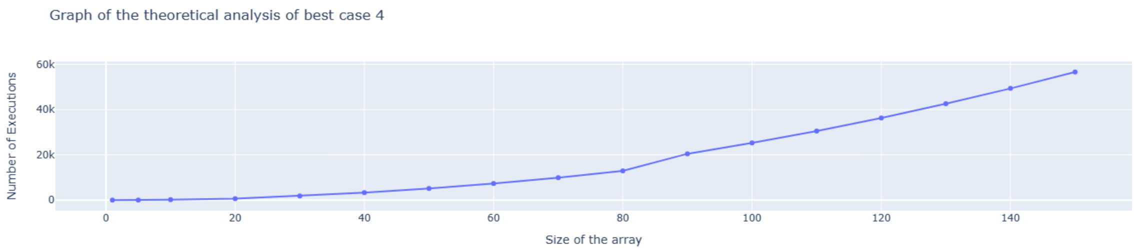
Graph of the theoretical analysis when basic operation is the operation marked as (2)



Graph of the theoretical analysis when basic operation is the operation marked as (3)



Graph of the theoretical analysis when basic operation is the operation marked as (4)



Comments

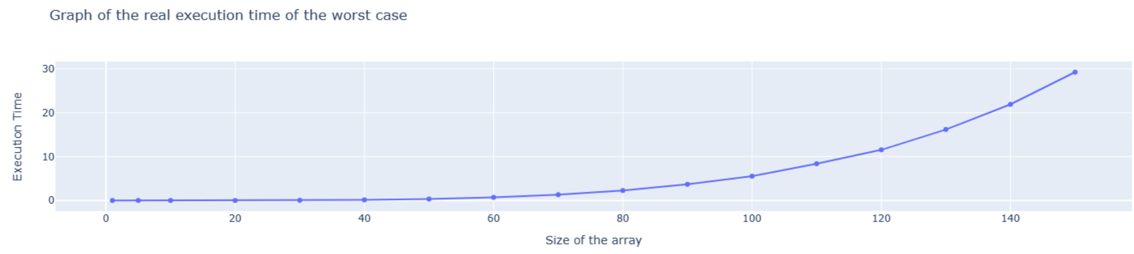
The theoretical analysis which takes 4 as the basic operation gave the most similar result to the real execution as we have already identified it as the real basic operation in an earlier part. Hence, it approves our result. Also, the theoretical analysis is successful in guessing the break that appears on the graph around the input size of 80. However, it could not guess the break on 50, so we might conclude that this break on the real execution graph could be a caused by some other process on the CPU.

1 goes on a linear way since it only traverses the list one time. 2 and 3’s results were 0 because they do not get executed. So, these operations are significantly irrelevant from the real result.

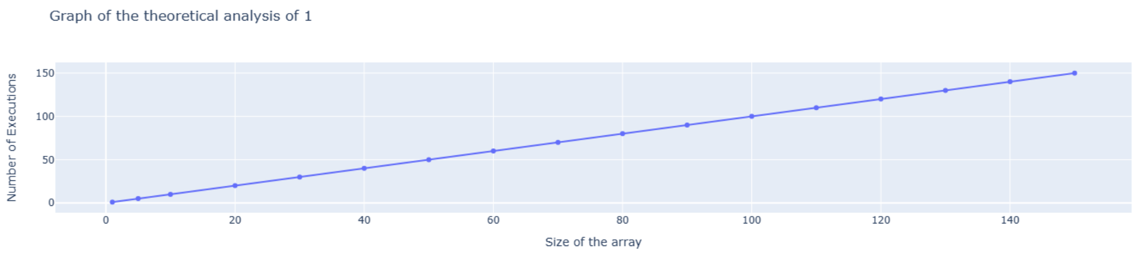
BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

Worst Case

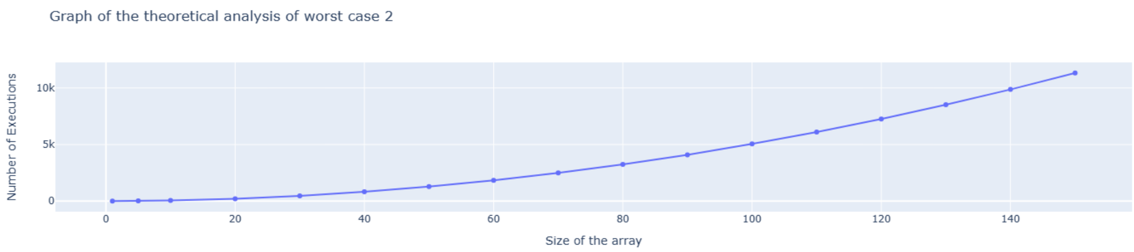
Graph of the real execution time of the algorithm



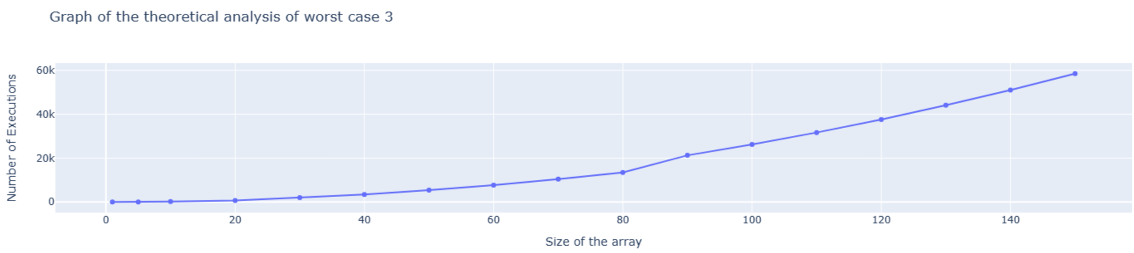
Graph of the theoretical analysis when basic operation is the operation marked as (1)



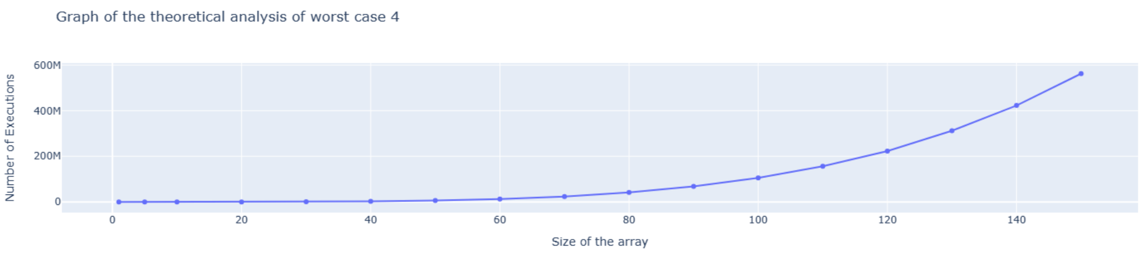
Graph of the theoretical analysis when basic operation is the operation marked as (2)



Graph of the theoretical analysis when basic operation is the operation marked as (3)



Graph of the theoretical analysis when basic operation is the operation marked as (4)



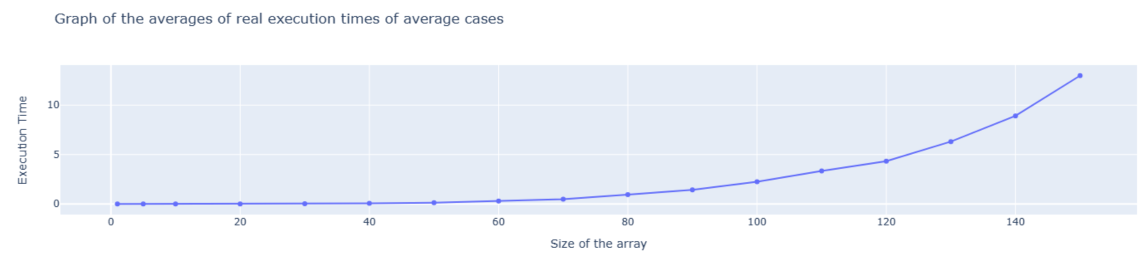
BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

Comments

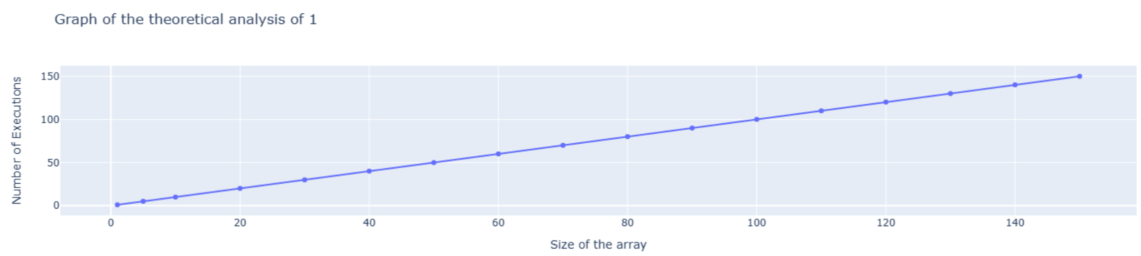
Here, we see that 4 is the closest one to the basic operation again. So, we again conclude that we guessed the basic operation correctly. In these case we see continuous improvement from 1 to 4. The reason for this is that we are working on the worst case so, most the operations get executed. 2 and 3 do not get executed 0 times unlike the best cases. Even, 3 and 4 do the same operations until some place.

Average Case

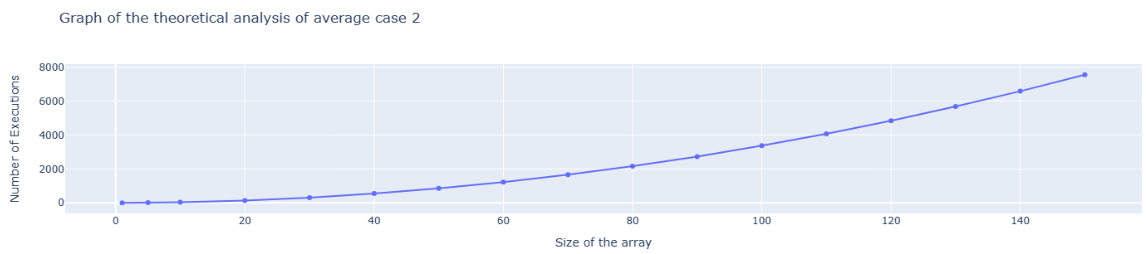
Graph of the real execution time of the algorithm



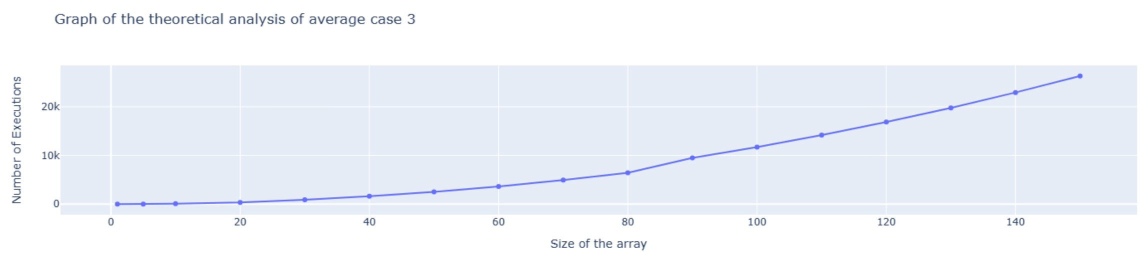
Graph of the theoretical analysis when basic operation is the operation marked as (1)



Graph of the theoretical analysis when basic operation is the operation marked as (2)

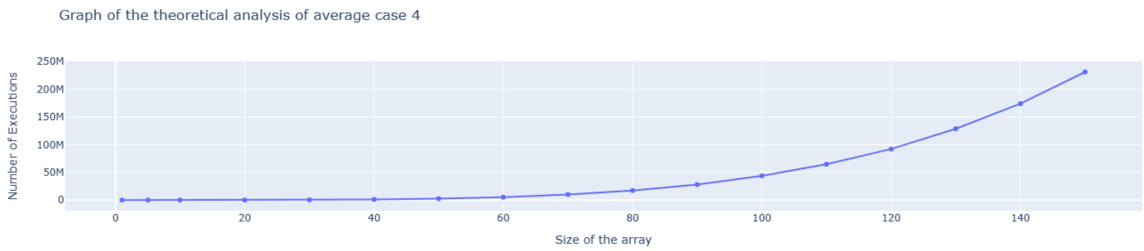


Graph of the theoretical analysis when basic operation is the operation marked as (3)



BİLGE KAAN GÜNEYLİ – 2020400051
ÖMER FARUK ERZURUMLUOĞLU – 2021400336

Graph of the theoretical analysis when basic operation is the operation marked as (4)



Comments

These graphs are very similar to the worst case graphs. This could have been guessed by observing that worst and average cases have the same complexity for all the basic operation assumptions. The only difference is the real execution time is much smaller in the average case which is an intuitive result.