First show the first part of the inequality for a non-decreasing function $f(x)$

$$\int_{0}^{n} f(x)dx \leq \sum_{k=0}^{n} f(k)$$

To prove this inequality we will use mathematical induction

Let $n = 1$

Show $\int_{0}^{1} f(x)dx \leq \sum_{k=0}^{1} f(k)$

Both of the equations represent a geometric shape, which have a bottom length of 1 and maximum height of $f(1)$. The shape which comes from the summation is the rectangle with these values.

We know that the function is non-decreasing.

So for any $n < 1, f(n) \leq f(1)$.

So the area of the area of the integral must be smaller or equal to the area of the rectangle.

Same situation applies for all the cases which have the difference of 1 between upper and lower bound.

So, we can say

$$\int_{0}^{1} f(x)dx \leq \sum_{k=0}^{1} f(k)$$

$$\int_{1}^{2} f(x)dx \leq \sum_{k=1}^{2} f(k)$$

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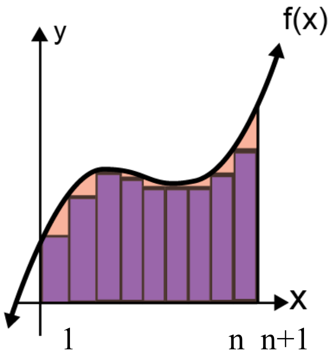
$$\int_{n-1}^{n} f(x)dx \leq \sum_{k=n-1}^{n} f(k)$$

If we sum all these inequalities up, we will reach

$$\int_{0}^{n} f(x)dx \leq \sum_{k=0}^{n} f(k)$$

For the second part of the inequality, using a figure will be appropriate.



$$\sum_{k=0}^{n} f(k) \leq \int_{1}^{n+1} f(x)dx$$

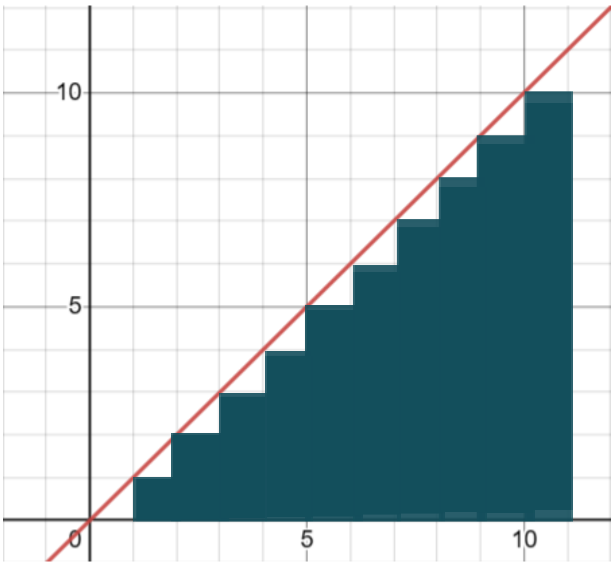
$f(x)$ is a random function (The function does not have to be non-increasing for this case, since the proof applies for all functions.).

The summation is represented by the rectangles and the integral is represented by the whole area under the curve.

So, the inequality is proven visually.

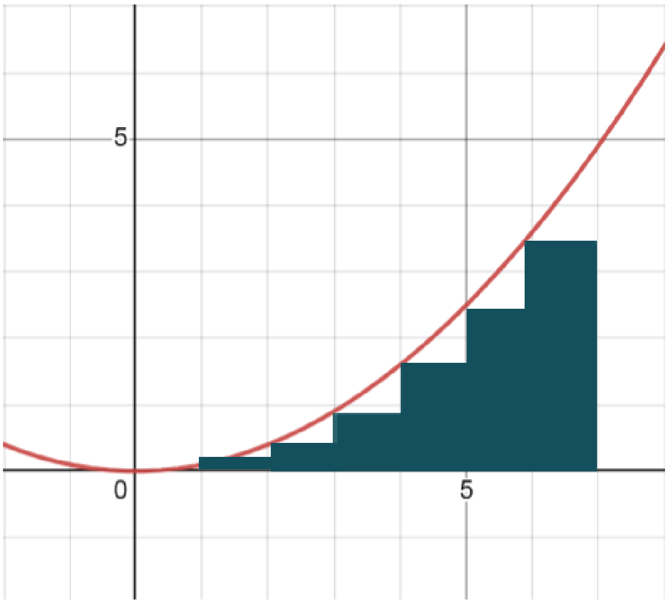
Examples:

1. $f(x) = x, \ n = 10$



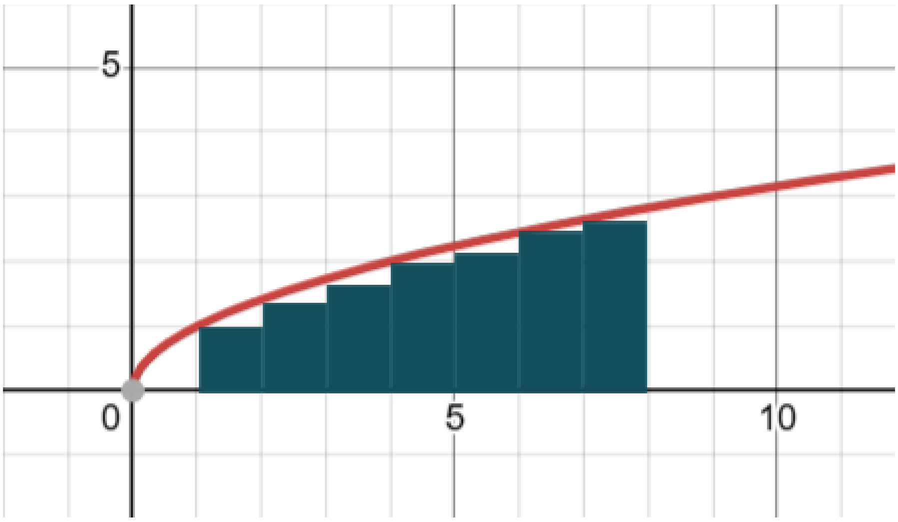
$$\int_{0}^{10} xdx = 50 < \sum_{k=0}^{10} x = 55 < \int_{1}^{11} xdx = 60$$

2. $f(x) = x^2/10,\ n = 6$ (since our domain is R^+, this function is non-increasing)



$$\int_0^6 (x^2/10)dx = 7.2 < \sum_{k=0}^{6} x^2/10 = 9.1 < \int_1^7 (x^2/10)dx = 11.4$$

3. $f(x) = \sqrt{x},\ n = 7$



$$\int_0^7 \sqrt{x}dx = 12.34684 < \sum_{k=0}^{7} \sqrt{x} = 13.47757 < \int_1^8 \sqrt{x}dx = 14.41828$$