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The Little End of the Stick

Problem

(a) If a stick is broken in 2 at random, what is the average length of the smaller piece?

(b) What is the average ratio of the smaller length to the larger?

Part (a) Solution

Clearly, the smaller piece will be no more than $\frac{1}{2}$ the length of the stick. Let S be the length of the smaller piece, which follows a continuous uniform distribution over $[0, \frac{1}{2}]$. Thus, the expected length is:

$$E[S] = \frac{0 + \frac{1}{2}}{2} = \frac{1}{4}$$

Part (b) Solution

To calculate the average ratio, we apply the definition of the expected value by integrating over the possible lengths of the smaller piece:

$$\int_0^{\frac{1}{2}} \left(\frac{x}{1-x} \right) \frac{dx}{1/2} = \ln 4 - 1 \approx 0.3863$$

The above integral is easily solved with the substitution $u = 1 - x$, giving an average ratio of about 0.3863.