

Hat Check

Problem

A hat-check girl receives n hats from n patrons, but forgets which hats belong to whom. So, she randomly disperses the hats back to the patrons.

What is the probability p_n that nobody has their hat returned to them?

Solution

A *derangement* is a permutation of a set such that no element remains in its original position. The number of derangements for a set of size n is denoted as $!n$.

For this problem, each derangement represents a dispersal of hats such that nobody receives their hat back. Thus, the proportion of derangements among all possible permutations of the n hats is the probability that nobody has their hat returned to them:

$$p_n = \frac{!n}{n!}$$

For reasonably large n ($n \geq 8$), $!n$ is very closely approximated by $!n = \frac{n!}{e}$. Applying this approximation gives the following:

$$p_n = \frac{!n}{n!} = \frac{\frac{n!}{e}}{n!} = \frac{1}{e}$$