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Craps

Problem

The game of craps, played with two die, is one of America's fastest and most popular games. Calculating the odds associated with it is an instructive exercise.

The rules are these. Only totals for the two dice count. The player throws the dice and wins at once if the total for the first throw is 7 or 11, loses at once if it is 2, 3 or 12. Any other throw is called his "point". If the first throw is a point, the player throws the dice repeatedly until he either wins by throwing his point again or loses by throwing 7. What is the player's chance to win?

Solution

Let X be a random variable representing the sum of the values on the two die, which has the range $\{2, \dots, 12\}$. By counting the relevant outcomes and dividing by the total number of outcomes in the sample space (36), we get the following probabilities for X:

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

In the range of X, the values that do not result in an immediate win or loss are $C = \{4, 5, 6, 8, 9, 10\}$. For these cases, we must calculate the probability D_x of drawing the "point" x instead of a 7 (which can be made in 6 ways). We can disregard all other values in the sample space of X because they do not result in the game terminating:

$$D_4 = \frac{3}{3+6} = \frac{1}{3}$$

$$D_5 = \frac{4}{4+6} = \frac{2}{5}$$

$$D_6 = \frac{5}{5+6} = \frac{5}{11}$$

$$D_8 = \frac{3}{3+6} = \frac{5}{11}$$

$$D_9 = \frac{4}{4+6} = \frac{2}{5}$$
$$D_{10} = \frac{3}{3+6} = \frac{1}{3}$$

This approach of only considering certain outcomes in the sample space is called the *Method of Reduced Sample Space*.

Using these probabilities, we can calculate the player's chance of winning as follows:

$$\begin{aligned} P(Win) &= P(X=7) + P(X=11) + \sum_{x \in C} P(X=x) D_x \\ &\approx 0.493 \end{aligned}$$

Thus, craps is not a fair game, although it is quite close to being so.

Alternative Solution

We can also calculate the probabilities D_x from above using the geometric distribution.

Let P_x representing the probability of the die sum not being 7 (which results in a loss) and not being the "point" x (which results in a win). Then, the probability of winning on the $(i+1)^{\text{th}}$ roll can be written as $(P_x)^i * P(X=x)$. Then, we can calculate D_x by summing this infinite geometric series for all non-negative values of i:

$$D_x = \frac{a_1}{1-r} = \frac{P(X=x)}{1-P_x}$$

As an example, we can calculate D_4 as follows:

$$D_4 = \frac{\frac{1}{12}}{1 - \left(1 - \frac{1}{12} - \frac{1}{6}\right)} = \frac{1}{3}$$