Kaan Aksoy — March 9, 2020

The Cliff-Hanger

Problem

From where he stands, one step toward a cliff would send a drunken man over the edge. He takes random steps, either toward or away from the cliff. At any step, his probability of taking a step away is $p = \frac{2}{3}$, and his probability of taking a step toward the cliff is $1 - p = \frac{1}{3}$. What is his chance of escaping the cliff?

Solution

We begin by representing the cliff as a number line, where $x = 0$ is the edge, $x = 1$ is the man's starting point, $x = 2$ is 2 steps from the edge, and so on:

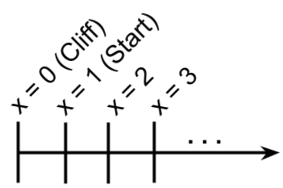

Figure 1: Number line representation of cliff

Let P_1 be the probability that the man eventually reaches $x = 0$ (the edge) starting at $x = 1$. We can write P_1 as follows:

$$P_1 = (1 - p) + pP_2$$

In the above, $(1 - p)$ represents the probability of immediately taking a step towards the edge. The latter part represents the probability of taking a step away and then eventually reaching $x = 0$. Here, as before, P_2 represents the probability that the man eventually reaches $x = 0$ starting at $x = 2$.

We can continue expanding this equation recursively. For example, P_2 can be written as $P_2 = (1 - p)P_1 + pP_3$. However, this does not lead us closer to the solution.

An important insight to observe is that $P_2 = P_1 P_1 = P_1^2$. The first P_1 is equivalent to the probability of moving from $x = 2$ to $x = 1$, and the second

P_1 is used in the traditional sense, representing the probability of moving from $x = 1$ to $x = 0$. We can plug this identity into the original equation and apply the quadratic formula to get the following:

$$\begin{aligned} P_1 &= (1-p) + pP_1^2 \\ &= \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p} \\ &= \frac{1 \pm (1-2p)}{2p} \\ &= 1, \frac{1-p}{p} \end{aligned}$$

Applying the solution $P_1 = \frac{1-p}{p}$ to $p = \frac{2}{3}$ gives a probability of $1 - P_1 = \frac{1}{2}$ of the man escaping the cliff.