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The First Ace

Problem

Shuffle an ordinary deck of 52 playing cards containing 4 aces. Then turn up cards from the top until the first ace appears. On the average, how many cards are required to produce the first ace?

Solution

Let X represent the number of cards that are turned up to produce the 1^{st} ace. For this problem, we cannot apply the *Geometric Distribution* because cards are sampled without replacement.

Instead, we begin by considering the probabilities of drawing the 1^{st} ace on the 1^{st} card, 2^{nd} card, and so on:

$$
\begin{aligned}
P(1st\ Card) &= \frac{4}{52} \\
P(2nd\ Card) &= \left(\frac{48}{52}\right)\left(\frac{4}{51}\right) \\
P(3rd\ Card) &= \left(\frac{48}{52}\right)\left(\frac{47}{51}\right)\left(\frac{4}{50}\right) \\
P(n^{th}\ card) &= 4\left[\frac{48!}{(49-x)!}\right]\left[\frac{(52-x)!}{52!}\right]
\end{aligned}
$$

Then, we can calculate the average number of cards by applying the definition of expected value:

$$E[X] = \sum_{x=1}^{52} 4x\left[\frac{48!}{(49-x)!}\right]\left[\frac{(52-x)!}{52!}\right] = \frac{53}{5} = 10.6$$

Thus, on average it will take 10.6 cards to get the 1^{st} ace.

Alternative Solution

The solution above is complex due to the unwieldy summation. Another approach is to apply the *Principle of Symmetry*, which states that n randomly placed points will divide a segment into $n+1$ pieces, each of which has the same distribution.

This problem is an application of the principle with $n = 4$, since each ace in the deck represents a division point. Then, the average length of the 5 segments (stretches of cards without an ace) is $\frac{52-4}{5} = \frac{48}{5}$. Each of these segments is immediately followed by an ace, so the expected number of cards until the 1^{st} ace is the following:

$$E[X] = \frac{48}{5} + 1 = \frac{53}{5} = 10.6$$