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Trials until First Success

Geometric Distribution

The geometric distribution describes the number of Bernoulli trials until the first success. According to the distribution, the mean number of trials until success is $\frac{1}{p}$, where p is the probability of success. From the geometric distribution, we can expect to toss a die $\frac{1}{\frac{1}{6}} = 6$ times before landing a 6.

Another Approach

Let X be the number of trials until the first success (including the success). Given a probability of success of $\frac{1}{6}$, we can write $E[X]$ as:

$$\begin{aligned} E[X] &= \sum_{i=0}^{\infty}(i+1)\left(\frac{5}{6}\right)^i\left(\frac{1}{6}\right) \\ &= \left(\frac{1}{6}\right)\left[\sum_{i=0}^{\infty} i\left(\frac{5}{6}\right)^i + \sum_{i=0}^{\infty}\left(\frac{5}{6}\right)^i\right] \end{aligned}$$

Since $|\frac{5}{6}| < 1$, the second infinite geometric series converges, and has the value $\frac{1}{1-\frac{5}{6}} = 6$. Manipulating the second series gives the following:

$$\begin{aligned} \sum_{i=0}^{\infty} i\left(\frac{5}{6}\right)^i &= \sum_{n=1}^{\infty}\sum_{m=1}^{n}\left(\frac{5}{6}\right)^n \\ &= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\left(\frac{5}{6}\right)^n \\ &= 6\sum_{m=1}^{\infty}\frac{5}{6}^m \\ &= 30 \end{aligned}$$

Substituting back into the original equation for $E[X]$, we get $E[X] = \frac{1}{6}(30+6) = 6$.