Kaan Eren Yavuz Homework-2 090200348

1-) Let S be a parametrized surface given by parametrization
$\vec{x}(u,v)=(\cos u \sin v, \sin u \sin v, \cos v)$ $0<u<2\pi$, $0<v<\pi$

a) Find coefficients of first fundamental and second fund. forms of the Surface S.

b) Find the Gaussian and Mean curvature of the Surface S.

c) Find umbilical points of S.

d) Determine hyperbolic, elliptic, and parabolic points of surface S.

e) Find the asymptotic lines.

a) $\vec{x}_u=(-\sin u \sin v, \cos u \sin v, 0)$, $\vec{x}_v=(\cos u \cos v, \sin u \cos v, -\sin v)$

$E=\sin^2 u \sin^2 v \cos^2 u \sin^2 v=\sin^2 v$, $G=\cos^2 u \cos^2 v+\sin^2 u \cos^2 v+\sin^2 v=1$

$F=-\sin u \sin v \cos u \cos v+\cos u \sin v \sin u \cos v=0$

$\vec{N}=\frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} \Rightarrow \vec{x}_u \times \vec{x}_v=\begin{vmatrix} i & j & k \\ -\sin u \sin v & \cos u \sin v & 0 \\ \cos u \cos v & \sin u \cos v & -\sin v \end{vmatrix}= \begin{matrix} i(-\cos u \sin^2 v) \\ -j(\sin^2 v \sin u) \\ +k(-(\sin^2 u \sin v \cos v-\cos^2 u \sin v \cos v)) \end{matrix}$

$\vec{x}_u \times \vec{x}_v=i(-\cos u \sin^2 v)-j(\sin^2 v \sin u)-k(\sin v \cos v)$

$|\vec{x}_u \times \vec{x}_v|=\sqrt{\cos^2 u \sin^4 v+\sin^4 v \sin^2 u+\sin^2 v \cos^2 v}=\sqrt{\sin^4 v+\sin^2 v \cos^2 v}$

$=\sqrt{\sin^2 v(\sin^2 v+\cos^2 v)}=\sin v$

$\vec{N}=(-\cos u \sin v, -\sin v \sin u, -\cos v)$

$\vec{x}_{uu}=(-\cos u \sin v, -\sin u \sin v, 0)$, $\vec{x}_{vv}=(-\cos u \sin v, -\sin u \sin v, -\cos v)$

$\vec{x}_{uv}=(-\sin u \cos v, \cos u \cos v, 0)$

$e=\langle \vec{N}, \vec{x}_{uu} \rangle=-\cos^2 u \sin^2 v-\sin^2 u \sin^2 v=-\sin^2 v$

$f=\langle \vec{N}, \vec{x}_{uv} \rangle=\cos u \sin u \cos v \sin v-\sin v \sin u \cos u \cos v=0$

$g=\langle \vec{N}, x_{vv} \rangle=-\cos^2 u \sin^2 v-\sin^2 u \sin^2 v-\cos^2 v=-1$

$-\sin^2 v-\cos^2 v=-1$

b) $K = \dfrac{eg - f^2}{EG - F^2} = \dfrac{\sin^2 v}{\sin^2 v} = 1 > 0$ $\left(K = \det(d\vec{N}_p) = \underbrace{k_1 \cdot k_2}_{\text{principal curvatures}}\right)$

Gauss curv.

$$\det(d\vec{N}_p) = \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$H = -\frac{1}{2}\operatorname{trace}(d\vec{N}_p) = -\frac{1}{2}(a_{11} + a_{22}) = -\frac{1}{2}\left[\underbrace{\frac{fF - eG}{EG - F^2}}_{a_{11}} + \underbrace{\frac{fF - gE}{EG - F^2}}_{a_{22}}\right]$

Mean curv.

$$H = \frac{1}{2}\left(\frac{gE - 2fF + eG}{EG - F^2}\right) = \frac{1}{2}\left(\frac{-\sin^2 v + 0 - \sin^2 v}{\sin^2 v}\right) = \frac{1}{2}\left(\frac{-2\sin^2 v}{\sin^2 v}\right) = -1$$

c) If $k_1 = k_2$ at the point p, then p is called an umbilical point and principal curvatures hold the equation.

$$k^2 - 2Hk + K = 0$$

$$k^2 - 2\,\frac{1}{2}\left(\frac{Eg - 2Ff + eG}{EG - F^2}\right)k + \frac{eg - f^2}{EG - F^2} = 0$$

$$(EG - F^2)k^2 - (Eg - 2Ff + eG)k + (eg - f^2) = 0$$

$$\sin^2 v\, k^2 + 2\sin^2 v\, k + \sin^2 v = 0$$

$k^2 + 2k + 1 = 0$ $\quad (k+1)^2$

$k \quad 1$

$k \quad 1$

$\Rightarrow k_1 = k_2 = -1$ principal curvatures equal each other

d) Since the Gaussian curvature K of the surface $\vec{x}(u,v)$ is positive ($K = 1 > 0$) then all points are elliptic.

e) $\mathrm{II}_p = 0 \Rightarrow e(du)^2 + 2f\,du\,dv + g(dv)^2 = 0$

$$-\sin^2 v(du)^2 - dv^2 = 0 \Rightarrow \sin^2 v(du)^2 + dv^2 = 0$$

$$\left(\frac{du}{dv}\right)^2 = -\frac{1}{\sin^2 v}$$

$$\int du = -\int \frac{dv}{\sin v}$$

$$u = -\int \csc v\, dv$$

$$u = -(-\ln|\csc v + \cot v| + C) \quad u = \ln|\csc v + \cot v| + C$$

2) $\vec{x}(t,v) = (t+v, t-v, 4tv+t+1)$

a) Show that the surface $\vec{x}$ is a ruled surface

b) Find the line of striction of the ruled surface $\vec{x}$.

c) Find the distribution parameter of $\vec{x}$.

a) $\vec{x}(t,v) = (t+v, t-v, 4tv+t+1)$

$= (t, t, t+1) + v(1, -1, 4t)$

$= \vec{\alpha}(t) + v.\vec{w}(t)$ where $\alpha(t) = (t, t, t+1)$ and

$\vec{w}(t) = (1, -1, 4t)$ Thus, $\vec{x}(t,v)$ is a ruled surface generated by the family $\{\alpha(t), \vec{w}(t)\}$

b) $\vec{\beta}(t) = \vec{\alpha}(t) - \dfrac{\langle \alpha', \vec{w}' \rangle}{\langle \vec{w}', \vec{w}' \rangle} \vec{w}(t)$

We already know $\vec{\alpha}(t)$, $\vec{w}(t)$

$\alpha'(t) = (1,1,1)$, $\vec{w}'(t) = (0,0,4)$

$v(t) = -\dfrac{(1,1,1).(0,0,4)}{|(0,0,4)|^2} = -\dfrac{4}{16} = -\dfrac{1}{4}$

$\vec{\beta}(t) = \vec{\alpha}(t) - \dfrac{\langle \vec{\alpha}', \vec{w}' \rangle}{\langle \vec{w}', \vec{w}' \rangle} \vec{w}(t) = (t, t, t+1) + \dfrac{1}{4}(1, -1, 4t)$

$\vec{\beta}(t) = \left(t + \frac{1}{4}, t - \frac{1}{4}, 2t+1\right)$

c) $\lambda = \dfrac{(\vec{\beta}', \vec{w}, \vec{w}')}{|\vec{w}|^2}$ $\quad (\vec{\beta}', \vec{w}, \vec{w}') = \begin{vmatrix} t+\frac{1}{4} & t-\frac{1}{4} & 2t+1 \\ 1 & -1 & 4t \\ 0 & 0 & 4 \end{vmatrix}$

$= \left(t+\frac{1}{4}\right)(-4) - \left(t-\frac{1}{4}\right)(4) + (2t+1)(0)$

$= -4t - 1 - 4t + 1 = -8t$

$|\vec{w}|^2 = 16$ $\quad \lambda = -\dfrac{t}{2}$

3) Let $U=\{(u,v)\in\mathbb{R}^2 \mid u>0,\ 0<v<2a\pi\},\ |a|<1\}$

Let S and $\bar{S}$ be two parametrized surfaces, respectively given by $\vec{x}: U\to S - \vec{x}(u)$, $\vec{x}(u,v)=(u\cos v, u\sin v, 0)$ and $\vec{x}^*: U\to\bar{S}=\vec{x}^*(u)$

$\vec{x}^*(u,v)=\left(au\cos\left(\frac{v}{a}\right), au\sin\left(\frac{v}{a}\right), \sqrt{1-a^2}\,u\right)$ Show that S and $\bar{S}$ are locally isometric

$\vec{x}_u=(\cos v, \sin v, 0)$, $\vec{x}_v=(-u\sin v, u\cos v, 0)$

$E=1$, $F=0$, $G=u^2\sin^2 v+u^2\cos^2 v=u^2$

$\vec{x}^*_u=\left(a\cos\left(\frac{v}{a}\right), a\sin\left(\frac{v}{a}\right), \sqrt{1-a^2}\right)$

$\vec{x}^*_v=\left(-u\sin\left(\frac{v}{a}\right), u\cos\left(\frac{v}{a}\right), 0\right)$

$E^*=a^2+1-a^2=1$, $F=0$, $G=u^2\sin^2\left(\frac{v}{a}\right)+u^2\cos^2\left(\frac{v}{a}\right)=u^2$

Since $E=E^*=1$, $F=F^*=0$, $G=G^*=u^2$ then S and $\bar{S}$ are locally isometric.

4.) Let $\vec{x}=\vec{x}(u,v)$ be a regular parametrized surface and assume that $\vec{x}$ is isothermal. Then proved that $\vec{x}_{uu}+\vec{x}_{vv}=2\lambda^2 H\vec{N}$ where $\lambda^2=\langle\vec{x}_u,\vec{x}_u\rangle=\langle\vec{x}_v,\vec{x}_v\rangle$, and H is mean curv. of the surface.

• Theorem. Let $\vec{x}=\vec{x}(u,v)$ be a parametrized surface and assume that $\vec{x}$ is isothermal.

Then $\vec{x}_{uu}+\vec{x}_{vv}=2\lambda^2\vec{H}$, $\vec{H}=H\vec{N}$

Proof: By the isothermal parametrization

$$\underbrace{\langle\vec{x}_u,\vec{x}_u\rangle}_{E}=\underbrace{\langle\vec{x}_v,\vec{x}_v\rangle}_{G}\ ; \quad \underbrace{\langle\vec{x}_u,\vec{x}_v\rangle}_{F}=0$$

$E=G$.

$$\frac{\partial}{\partial v}\langle\vec{x}_u,\vec{x}_u\rangle=\frac{\partial}{\partial u}\langle\vec{x}_v,\vec{x}_v\rangle$$

$$\langle\vec{x}_{uv},\vec{x}_u\rangle+\langle\vec{x}_u,\vec{x}_{uv}\rangle=\langle\vec{x}_{vu},x_v\rangle+\langle\vec{x}_v,\vec{x}_{vu}\rangle$$

$$2\langle\vec{x}_{uv},\vec{x}_u\rangle=2\langle\vec{x}_{vu},x_v\rangle \quad (1)$$

$$F=\langle\vec{x}_u,\vec{x}_v\rangle=0 \qquad \frac{\partial}{\partial v}\langle\vec{x}_u,\vec{x}_v\rangle=\langle\vec{x}_{uv},\vec{x}_v\rangle+\langle\vec{x}_u,\vec{x}_{vv}\rangle=0$$

$$\langle\vec{x}_{uv},\vec{x}_v\rangle=-\langle\vec{x}_u,\vec{x}_{vv}\rangle \quad (2)$$

From 1 and 2 $\Rightarrow$ $\langle\vec{x}_{uu},\vec{x}_u\rangle=\langle\vec{x}_{vv},\vec{x}_v\rangle=-\langle\vec{x}_{vv},\vec{x}_u\rangle$

$\Rightarrow\langle\vec{x}_{uu}+\vec{x}_{vv},\vec{x}_u\rangle=0 \quad (3)$

$\langle \vec{x}_u, \vec{x}_u \rangle = \langle \vec{x}_v, \vec{x}_v \rangle$

$\frac{\partial}{\partial v} \langle \vec{x}_u, \vec{x}_u \rangle = \frac{\partial}{\partial v} \langle \vec{x}_v, \vec{x}_v \rangle$

$\Rightarrow 2 \langle \vec{x}_{uv}, x_u \rangle = 2 \langle \vec{x}_{vv}, \vec{x}_v \rangle$ (4)

$F = \langle \vec{x}_u, \vec{x}_v \rangle = 0 \Rightarrow \frac{\partial}{\partial v} \langle x_u, x_v \rangle = 0 \quad \langle \vec{x}_{uv}, \vec{x}_v \rangle = -\langle \vec{x}_u, \vec{x}_{vv} \rangle$

(5)

From 4 and 5

$\Rightarrow \langle x_{uv}, x_u \rangle = \langle \vec{x}_{vv}, \vec{x}_v \rangle = -\langle x_u, x_{vv} \rangle$

$\langle \vec{x}_{uu} + \vec{x}_{vv}, \vec{x}_v \rangle = 0$ (6)

From $\left.\begin{array}{l} \langle \vec{x}_{uu} + \vec{x}_{vv}, \vec{x}_u \rangle = 0 \\ \langle \vec{x}_{uu} + \vec{x}_{vv}, \vec{x}_v \rangle = 0 \end{array}\right\}$ and $\{\vec{x}_u, \vec{x}_v\} \in T_p(S)$

tangent vectors.

$\vec{x}_{uu} + \vec{x}_{vv} = \mu \vec{N}$ for som function μ

$H = \frac{1}{2}\left(\frac{eG - 2fF + gE}{EG - F^2}\right)$ and isothermal parametrization $E = G$, $F = 0$

$H = \frac{1}{2}\left(\frac{e+g}{G}\right) \Rightarrow$; by assumption $G = \langle \vec{x}_v, \vec{x}_v \rangle = \lambda^2$

$2\lambda^2 H = e + g$ (*)

$= \langle \vec{x}_{uu}, \vec{N} \rangle + \langle \vec{x}_{vv}, \vec{N} \rangle$

$= \langle \underbrace{\vec{x}_{uu} + \vec{x}_{vv}}_{\mu \vec{N}}, \vec{N} \rangle = \mu \underbrace{\langle \vec{N}, \vec{N} \rangle}_{1}$

$2\lambda^2 H = \mu$

Then $\vec{x}_{uu} + \vec{x}_{vv} = 2\lambda^2 H \vec{N}$.

5.) Show that the catenoid given by the parametrization $\vec{x}(u,v)=(a\cosh v\cos u, a\cosh v\sin u, av)$, $0<u<2\pi$, $|v|<\infty$ is isothermal and the catenoid is minimal

A parametrized surface $\vec{x}$ is called isothermal if $E=G, F=0$ and the coordinate system (u,v) is called isothermal coordinates on S.

$x_u = (-a\cosh v \sin u, a\cosh v\cos u, 0)$

$\vec{x}_v = (a\sinh v\cos u, a\sinh v \sin u, a)$

$E = a^2\cosh^2 v\sin^2 u + a^2\cosh^2 v\cos^2 u = a^2\cosh^2 v$

$F = -a^2\cosh v\sinh v\sin u\cos u + a^2\cosh v\sinh v\cos u\sin u = 0$

$G = a^2\sinh^2 v\cos^2 u + a^2\sinh^2 v\sin^2 u + a^2 = a^2\sinh^2 v + a^2 = a^2(\sinh^2 v + 1) = a^2\cosh^2 v$

$E=G,\ F=0$ $\quad \vec{x}$ is isothermal.

$x_{uu} = (-a\cosh v\cos u, -a\cosh v\sin u, 0)$

$x_{vv} = (a\cosh v\cos u, a\cosh v\sin u, 0)$

$x_{uu} + x_{vv} = 0$

If $\vec{x}(u,v)$ is harmonic, then from the corollary $\vec{x}(u,v)$ is minimal

Corollary $\Rightarrow$ Let $\vec{x}(u,v)$ be isothermal. Then, $\vec{x}$ is minimal if and only if the components of x, y, z of $\vec{x}(u,v)$ are harmonic.