Mat 417 E Quiz 1 Kaan Eren Yavuz - 090200348

1.) Let $\vec{x}(u,v) = ((a+r\cos u)\cos v, (a+r\cos u)\sin v, r\sin u)$ $\begin{pmatrix} 0 < u < 2\pi \\ 0 < v < 2\pi \end{pmatrix}$

i) Find the area of the torus.

$$\int_0^{2\pi}\int_0^{2\pi} \sqrt{EG - F^2}\, du\, dv$$

$$\vec{x}_u = (-r\sin u \cos v, -r\sin u \sin v, r\cos u)$$

$$\vec{x}_v = (-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0)$$

$$E = \langle \vec{x}_u, \vec{x}_u \rangle = r^2$$

$$F = \langle \vec{x}_u, \vec{x}_v \rangle = 0 \qquad G = \langle \vec{x}_v, \vec{x}_v \rangle = (a+r\cos u)^2$$

$$\int_0^{2\pi}\int_0^{2\pi} \sqrt{r^2 (a+r\cos u)^2}\, du\, dv = \int_0^{2\pi}\int_0^{2\pi} r(a+r\cos u)\, du\, dv$$

$$= \int_0^{2\pi} (aru + r\sin u)\Big|_0^{2\pi} = \int_0^{2\pi} 2\pi ar\, dv = (2\pi a r v)\Big|_0^{2\pi} = 4\pi^2 ar$$

ii) Find the normal of the surface $\vec{x}$.

$$\vec{N} = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} \implies \text{We already know } \vec{x}_u \text{ and } \vec{x}_v$$

$$\vec{x}_u \times \vec{x}_v = \begin{vmatrix} i & j & k \\ -r\sin u\cos v & -r\sin u \sin v & r\cos u \\ -(a+r\cos u)\sin v & (a+r\cos u)\cos v & 0 \end{vmatrix}$$

$$= i\,(-(a+r\cos v)\, r\cos u \cos v) - j\,((a+r\cos u)\, r\cos u \sin v)$$
$$+ k\,(-((a+r\cos u)\, r\sin u \cos^2 v) - ((a+r\cos u)\, r\sin u \sin^2 v))$$

$$1$$

$$= (-(a+r\cos v)\, r\cos u\cos v,\ -(a+r\cos u)\, r\cos u \sin v,\ -(a+r\cos u)\, r\sin u)$$

$$|\vec{x}_u \times \vec{x}_v| = r(a+r\cos u)$$

$$\vec{N} = (-\cos u\cos v, -\cos u \sin v, -\sin u)$$

iii) Find the differential $d\vec{N}$ of the Gauss map $\vec{N}$.

From ii) we have

$\vec{N} = (-\cos u \cos v, -\cos u \sin v, -\sin u)$

$\vec{N}_u = (+\sin u \cos v, \sin u \sin v, -\cos u)$

$\vec{N}_v = (\cos u \sin v, -\cos u \cos v, 0)$

$\vec{N}_u = a_{11} \vec{X}_u + a_{12} \vec{X}_v$

$(\sin u \cos v, \sin u \sin v, -\cos u) = a_{11}(-r\sin u \cos v, -r\sin u \sin v, r\cos u)$

$+ a_{12}(-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0)$

$-\cos u = a_{11} r \cos u \Rightarrow a_{11} = -\frac{1}{r}$

$\sin u \cos v = +\frac{1}{r} \cancel{r} \sin u \cos v + a_{12} \cdot (-(a+r\cos u)\sin v)$

$a_{12} = 0$

$\vec{N}_v = a_{21} \vec{X}_u + a_{22} \vec{X}_v$

$(\cos u \sin v, -\cos u \cos v, 0) = a_{21}(-r\sin u \cos v, -r\sin u \sin v, r\cos u)$

$a_{22}(-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0)$

$0 = a_{21} r \cos u$ $\quad a_{21} = 0$

$\cos u \sin v = a_{22} \cdot -(a+r\cos u)\sin v$

$a_{22} = \frac{-\cos u}{a+r\cos u}$

$$d\vec{N} = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & \frac{-\cos u}{a+r\cos u} \end{pmatrix}$$

2. 1) Let $\vec{\beta} : [0, \frac{\pi}{2}] \to R^2$ be a curve given by $\vec{\beta}(t) = (u(t), v(t)) = (\ln\cot(\frac{\pi}{4} - \frac{t}{2}), \frac{\pi}{2} - t)$.
Consider the image of the curve on the unit sphere.
$\vec{X}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$.
Find the length of the curve.

$$\vec{\beta}(t) = (u(t), v(t)) = (\underbrace{\ln\cot(\tfrac{\pi}{4} - \tfrac{t}{2})}_{u}, \underbrace{\tfrac{\pi}{2} - t}_{v}) \qquad 0 \leq t \leq \frac{\pi}{2}$$

$$\vec{X}(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$$

$$S = \int_0^{\frac{\pi}{2}} \sqrt{I}\, dt, \quad I = E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\cdot\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2$$

$$\vec{X}_u = (-\sin u \sin v, \cos u \sin v, 0)$$

$$\vec{X}_v = (\cos u \cos v, \sin u \cos v, -\sin v)$$

$$E = \langle \vec{X}_u, \vec{X}_u \rangle = \sin^2 u \sin^2 v + \cos^2 u \sin^2 v = \sin^2 v$$

$$F = 0, \quad G = \cos^2 u \cos^2 v + \sin^2 u \cos^2 v + \sin^2 v = 1$$

$$u = \ln(\cot(\tfrac{\pi}{4} - \tfrac{t}{2})) \Longrightarrow \frac{du}{dt} = \frac{\frac{1}{2}\csc^2(\frac{\pi}{4} - \frac{t}{2})}{\cot(\frac{\pi}{4} - \frac{t}{2})} = \frac{1}{2}\,\frac{1}{\cos(\frac{\pi}{4} - \frac{t}{2})\sin(\frac{\pi}{4} - \frac{t}{2})}$$

$$2\cos x \sin x = \sin 2x \Longrightarrow \frac{1}{\sin 2(\frac{\pi}{4} - \frac{t}{2})} = \frac{1}{\sin(\frac{\pi}{2} - t)} = \frac{du}{dt}$$

$$v = \frac{\pi}{2} - t \Longrightarrow \frac{dv}{dt} = -1$$

$$S = \int_0^{\frac{\pi}{2}} \sqrt{\sin^2(\tfrac{\pi}{2} - t)\left(\frac{1}{\sin(\frac{\pi}{2} - t)}\right)^2 + \cancel{2\cdot 0} + (-1)^2}\; dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1 + 1}\; dt = \int_0^{\frac{\pi}{2}} \sqrt{2}\, dt = (\sqrt{2}\, t)\Big|_0^{\frac{\pi}{2}} = \frac{\sqrt{2}}{2}\pi$$

3) Let S be a regular surface and $\vec{N}$ its unit normal vector at a point $p \in S$. Let $\{w_1, w_2\}$ be an orthonormal basis for $T_p(S)$. If $d\vec{N}_p(\vec{w}_1) = 4\vec{w}_1 + 2\vec{w}_2$, $d\vec{N}_p(\vec{w}_2) = 2\vec{w}_1 + 7\vec{w}_2$, then find the principal curvatures, Gauss curvature, and mean curvature of the surface at p.

$k^2 - 2Hk + K = 0$, $k \to$ principal curvatures
$H \to$ mean curvature
$K \to$ Gauss curvature

$$d\vec{N}_p(\vec{w}) = A_{PJ}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Rightarrow d\vec{N}_p(\vec{w}) = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \end{pmatrix}$$

Gauss curv.

$K = \det(d\vec{N}_p) = 4.7 - 2.2 = 24$

$H = -\frac{1}{2}\operatorname{trace}(d\vec{N}_p) = -\frac{1}{2}(4+7) = -\frac{11}{2}$

$$k^2 - 2 \cdot -\frac{11}{2} \cdot k + 24 \Rightarrow k^2 + 11k + 24 = 0$$

$k \quad 8$
$k \quad 3$

$(k+8)(k+3) = 0$

$k_1 = -8 \quad k_2 = -3$