DT059A - Applied Optimization
Lab 1 Report

Kaan Tekin Öztekin[1]

[1]**Department of Computer and Electrical Engineering, Mid Sweden University, Sundsvall, Sweden ,**
Email: `{kaoz2500}@student.miun.se` **,**
Date: [2025-11-26]

1. Introduction

In the first lab of the Applied Optimization course, I explored the behavior of quadratic objective functions and their constraints through both analytical and numerical methods. The tasks included visualizing contours and surfaces, identifying feasible regions, finding stationary points, and classifying them using Hessian-based tests.

2. Experimental Setup

Throughout the lab, the implementations were developed in both Python (using PyCharm) and MATLAB. As this was my first experience installing and working with MATLAB, the configuration and initial adaptation required additional time.

3. Solutions - PART A

3.1. Objective Function - 1

RECAP The objective function is the function we aim to *minimize* or *maximize*. It defines the surface (or contour levels) over which the optimal point is searched. *Constraints* restrict the set of points that are feasible. They can be either inequality constraints or equality constraints:

$$f(x, y) = 2x^2 + y^2 - 2xy - 3x - 2y.$$

- Inequality constraint:

$$y - x \le 0 \quad \Longrightarrow \quad y \le x,$$

  which means that only points lying below the line $y = x$ are allowed.
- Equality constraint:

$$x^2 + y^2 = 1,$$

  which restricts the feasible set to points lying *exactly* on the unit circle.

Optimization Goal. The aim is to find a point that simultaneously:

1. satisfies all constraints, and
2. minimizes the objective function.

The feasible region becomes a specific arc on the unit circle. In this particular problem, the solution is sought on the part of the unit circle that also satisfies the condition $y \le x$. Hence, the search for the minimum of $f(x, y)$ is restricted to a specific arc on the boundary of the circle.

Visualization of the Objective Function and Constraints: Figure 1 shows the contour lines of the objective function together with the constraint boundaries $y = x$ and $x^2 + y^2 = 1$. The feasible region, defined by $y \le x$ and the unit circle, is highlighted.

The 3D surface plots in Figure 2 visualize the full shape of the objective function and the curve formed by the feasible set on this surface. These plots help identify where a minimum or maximum is likely to occur by showing the vertical variation of the function values.

From the contour and surface plots, we observe that the feasible set consists of a restricted arc on the unit circle where $y \le x$. The interior Lagrange-critical points do not lie in the feasible region, indicating that the optimum must occur on the boundary where the inequality constraint becomes active. Visual inspection of the surface suggests

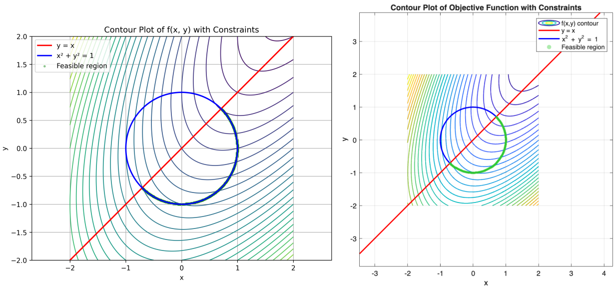


Figure 1. Comparison of contour plots generated in Python (left) and MATLAB (right).

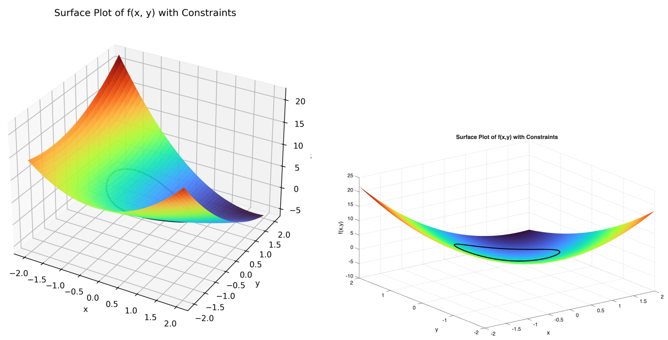


Figure 2. Comparison of surface plots generated in Python (left) and MATLAB (right).

that the minimum occurs on the upper-right part of the feasible arc, whereas the maximum occurs on the lower-left part.

Lagrange Analysis for the Equality Constraint: To incorporate the equality constraint $x^2 + y^2 = 1$, we construct the Lagrangian

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda\,(x^2 + y^2 - 1).$$

The stationarity conditions are

$$\nabla_{x,y,\lambda}\mathcal{L} = 0,$$

which results in the following system:

$$\begin{cases} (4 + 2\lambda)x - 2y - 3 = 0, \\ -2x + (2 + 2\lambda)y - 2 = 0, \\ x^2 + y^2 - 1 = 0. \end{cases}$$

Solving this system yields two equality–constrained critical points:

$$(x_1, y_1) = (-1, 0), \qquad (x_2, y_2) \approx (0.6793,\ 0.7338).$$

Both satisfy the circle constraint $x^2 + y^2 = 1$ and the Lagrange stationarity conditions. However, neither satisfies the inequality constraint $y \le x$:

$$0 \le -1 \ \ \text{(false)}, \qquad 0.7338 \le 0.6793 \ \ \text{(false)}.$$

Thus, these points are *infeasible* for the original problem. Feasible Region is given by the intersection of the unit circle with the half–plane $y \le x$. Since no interior point on the circle satisfies $\nabla f = \lambda \nabla g$ and the inequality constraint simultaneously, the optimum must occur on the boundary where the inequality becomes active:

$$y = x.$$

Solving the system

$$\begin{cases} x = y, \\ x^2 + y^2 = 1, \end{cases}$$

gives the two boundary points

$$\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right), \qquad \left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right).$$

Along the boundary $y = x$, the objective function reduces to

$$f(t,t) = t^2 - 5t, \qquad t = \pm\frac{1}{\sqrt{2}}.$$

Evaluating:

$$f\left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right) = \frac{1 - 5\sqrt{2}}{2} \approx -3.035,$$

$$f\left(-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right) = \frac{1 + 5\sqrt{2}}{2} \approx 4.035.$$

Final Result: The optimization problem subject to both constraints:

$$\boxed{\textbf{Minimum: } (x^\star, y^\star) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad f_{\min} = \frac{1 - 5\sqrt{2}}{2} \approx -3.035}$$

$$\boxed{\textbf{Maximum: } (x^\star, y^\star) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad f_{\max} = \frac{1 + 5\sqrt{2}}{2} \approx 4.035}$$

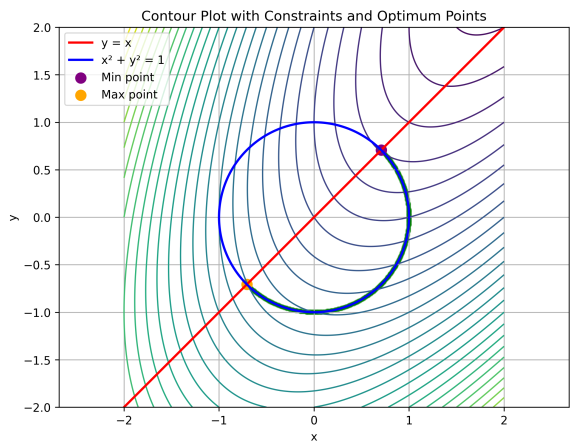


Figure 3. Illustrating the min and max point of func 1

Both extrema lie on the boundary as shown in the Figure 3 where the inequality constraint $y \le x$ is active. No Lagrange-critical point on the circle is feasible for the full constraint set.

3.2. Objective Function – 2

The second optimization problem is defined by the quadratic objective function:

$$f(x, y) = 4x^2 + 3y^2 - 5xy - 8x,$$

together with one equality constraint:

$$x + y = 4.$$

The constraint describes a straight line in the plane, and thus restricts the optimization search to points lying exactly on this line. Unlike the first problem, there is no inequality constraint and therefore no half-space restriction. The feasible set is the entire infinite line

$$y = 4 - x.$$

Since this line is unbounded, global minimization or maximization of $f(x, y)$ must be interpreted relative to the behavior of the quadratic function along this feasible set.

Visualization of the Objective Function and Constraint: Figure 4 shows the contour lines of $f(x, y)$ together with the constraint $x + y = 4$. The feasible set appears as a straight red line on the contour map.

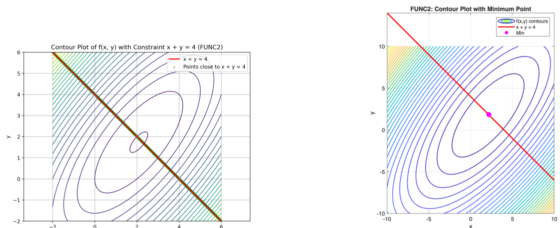


Figure 4. Contour plots in Python (left) and MATLAB (right).

The 3D surface plots in Figure 5 illustrate the shape of the objective function and the straight feasible curve traced across the surface.

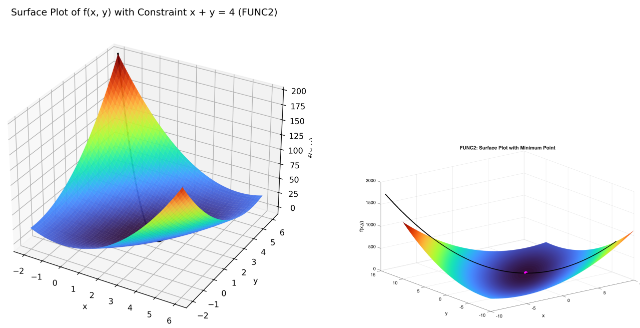


Figure 5. Surface plots in Python (left) and MATLAB (right).

Because the feasible set is the line $x + y = 4$, we eliminate y:

$$y = 4 - x.$$

Substituting into $f(x, y)$ yields a single-variable quadratic function

$$\phi(x) = f(x, 4 - x).$$

A direct expansion gives

$$\phi(x) = 6x^2 - 28x + 48.$$

This is a convex parabola (since its leading coefficient $6 > 0$), and therefore it admits a unique global minimum on the feasible set. However, since the feasible line extends to infinity, the function has *no finite global maximum*: the values of $\phi(x)$ grow unbounded as $|x| \to \infty$.

Critical Point Along the Constraint: The minimum of the convex quadratic $\phi(x)$ occurs at

$$x^\star = -\frac{b}{2a} = -\frac{-28}{2 \cdot 6} = \frac{28}{12} = \frac{7}{3},$$

(It is the formula that gives the vertex of a single-variable parabola.) with the corresponding

$$y^\star = 4 - x^\star = 4 - \frac{7}{3} = \frac{5}{3}.$$

Evaluating the objective function:

$$f_{\min} = f\left(\frac{7}{3}, \frac{5}{3}\right) = -\frac{16}{3} \approx -5.333.$$

Final Result

$$\boxed{\textbf{Minimum: } (x^\star, y^\star) = \left(\frac{7}{3}, \frac{5}{3}\right), \quad f_{\min} = -\frac{16}{3} \approx -5.333}$$

$$\boxed{\textbf{Maximum: } \text{Does not exist (unbounded above).}}$$

Figure 6 shows the numerically computed minimum point on the contour plot.

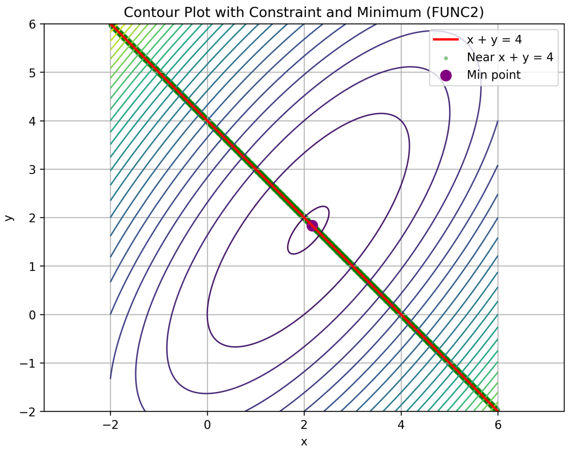


Figure 6. Minimum point of func2 on the constraint line $x + y = 4$.

3.3. Objective Function – 3

The third optimization problem is also defined by the quadratic objective function:

$$f(x, y) = 9x^2 + 13y^2 + 18xy - 4,$$

subject to the equality constraint:

$$x^2 + y^2 + 2x = 16.$$

Completing the square in x gives:

$$(x + 1)^2 + y^2 = 17,$$

which describes a circle of radius $\sqrt{17}$ centered at $(-1, 0)$. Thus the feasible set is a closed and bounded curve, ensuring the existence of both a global minimum and global maximum of the objective function on this set.

Visualization of the Objective Function and Constraint: Figure 7 shows the contour lines of $f(x, y)$ together with the constraint circle. Since the feasible region is exactly the circular boundary, optimization is restricted to this curve.

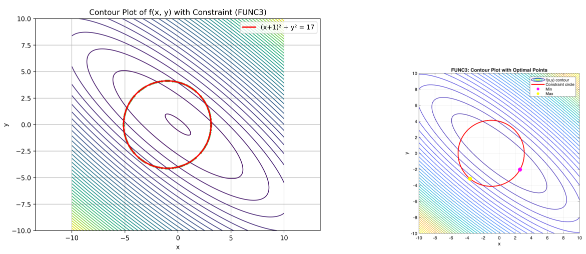


Figure 7. Contour plots in Python (left) and MATLAB (right).

The 3D surface plots in Figure 8 illustrate the behavior of the function along the feasible circular boundary.

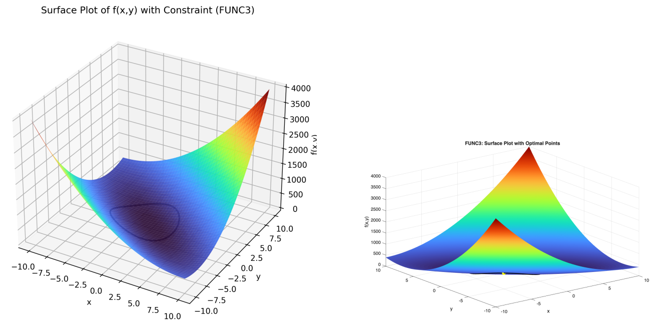


Figure 8. Surface plots in Python (left) and MATLAB (right).

To incorporate the equality constraint, we define the Lagrangian:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda\big((x + 1)^2 + y^2 - 17\big).$$

The stationarity conditions

$$\nabla_{x,y,\lambda}\mathcal{L} = 0$$

lead to the system

$$\begin{cases} (18 + 2\lambda)x + 18y - 2 + 2\lambda = 0, \\ 18x + (26 + 2\lambda)y = 0, \\ (x + 1)^2 + y^2 - 17 = 0. \end{cases}$$

Solving this nonlinear system numerically yields two Lagrange-critical points lying on the constraint circle. Both are feasible and correspond to the global minimum and global maximum of the function on the circle. An equivalent and more intuitive approach is to parametrize the constraint circle:

$$x(t) = -1 + \sqrt{17}\cos t, \qquad y(t) = \sqrt{17}\sin t,$$

Numerical computation reveals the exact locations of the extrema.

Optimal Points on the Feasible Set The numerical minimization and maximization yield:

$$(x_{\min}, y_{\min}) \approx (-3.7863, -1.1767), \qquad f_{\min} \approx -165.56,$$
$$(x_{\max}, y_{\max}) \approx (1.7863, 1.1767), \qquad f_{\max} \approx 149.56.$$

Final Result

$$\boxed{\textbf{Minimum: } (x^\star, y^\star) \approx (-3.7863, -1.1767), \qquad f_{\min} \approx -165.56}$$

$$\boxed{\textbf{Maximum: } (x^\star, y^\star) \approx (1.7863, 1.1767), \qquad f_{\max} \approx 149.56}$$

Figure 9 shows both the minimum and maximum points overlaid on the contour plot.

4. Solutions - PART B

4.1. Objective Function – 1

We consider the quadratic function

$$f(x_1, x_2) = 3x_1^2 + 2x_1x_2 + 2x_2^2 + 7.$$

To find the stationary points, we compute the gradient:

$$f_{x_1} = 6x_1 + 2x_2, \qquad f_{x_2} = 2x_1 + 4x_2.$$

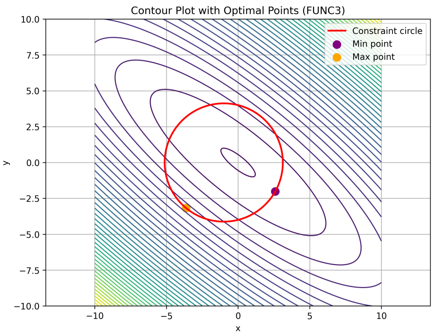


Figure 9. Minimum and maximum points on the constraint circle.

Solving the system

$$\begin{cases} 6x_1 + 2x_2 = 0, \\ 2x_1 + 4x_2 = 0, \end{cases}$$

yields a unique stationary point:

$$(x_1^*, x_2^*) = (0, 0).$$

The Hessian matrix is

$$H = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}, \qquad \det(H) = 20 > 0, \quad f_{x_1 x_1} = 6 > 0.$$

Since the Hessian is positive definite, the stationary point is a

$$\boxed{\text{Local (and global) minimum at } (0, 0)}.$$

Figures show the contour and surface plots with the stationary point marked.

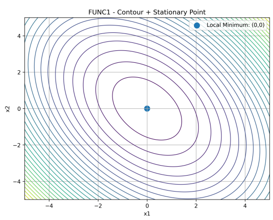


Figure 10. Contour plot with stationary point at $(0, 0)$.

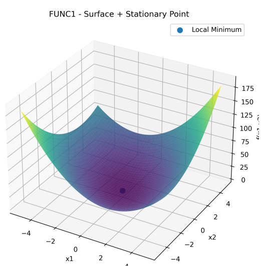


Figure 11. Surface plot with stationary point at $(0, 0)$.

4.2. Objective Function – 2

Analysis of the second quadratic function:

$$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + 3.$$

Compute the partial derivatives:

$$f_{x_1} = 2x_1 + 4x_2, \qquad f_{x_2} = 4x_1 + 2x_2.$$

Solving the system

$$\begin{cases} 2x_1 + 4x_2 = 0, \\ 4x_1 + 2x_2 = 0, \end{cases}$$

again yields a single stationary point:

$$(x_1^*, x_2^*) = (0, 0)$$

The Hessian is

$$H = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \qquad \det(H) = -12 < 0.$$

A negative determinant indicates that eigenvalues have opposite signs, hence $\boxed{\text{The stationary point } (0, 0) \text{ is a saddle point.}}$

The figures below show the contour and surface plots with the saddle point labeled.

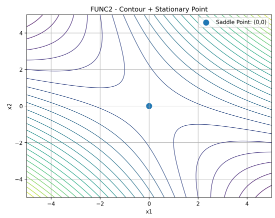


Figure 12. Contour plot with saddle point at $(0, 0)$.

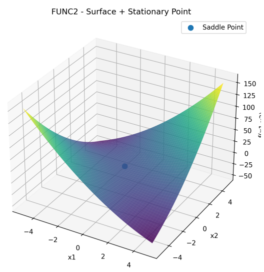
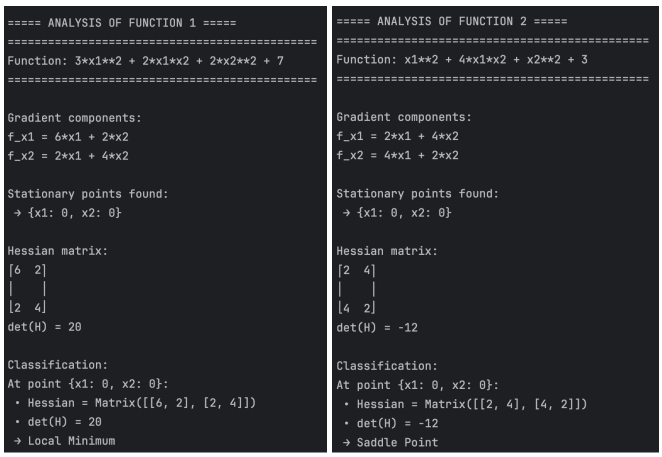


Figure 13. Surface plot with saddle point at $(0, 0)$.

In Figure 14, we can see my IDE terminal outputs for functions.



```
===== ANALYSIS OF FUNCTION 1 =====
============================================
Function: 3*x1**2 + 2*x1*x2 + 2*x2**2 + 7
============================================

Gradient components:
f_x1 = 6*x1 + 2*x2
f_x2 = 2*x1 + 4*x2

Stationary points found:
 → {x1: 0, x2: 0}

Hessian matrix:
[6  2]
|    |
[2  4]
det(H) = 20

Classification:
At point {x1: 0, x2: 0}:
 • Hessian = Matrix([[6, 2], [2, 4]])
 • det(H) = 20
 → Local Minimum
```

```
===== ANALYSIS OF FUNCTION 2 =====
============================================
Function: x1**2 + 4*x1*x2 + x2**2 + 3
============================================

Gradient components:
f_x1 = 2*x1 + 4*x2
f_x2 = 4*x1 + 2*x2

Stationary points found:
 → {x1: 0, x2: 0}

Hessian matrix:
[2  4]
|    |
[4  2]
det(H) = -12

Classification:
At point {x1: 0, x2: 0}:
 • Hessian = Matrix([[2, 4], [4, 2]])
 • det(H) = -12
 → Saddle Point
```

Figure 14. Minimum and maximum points on the constraint circle.

5. Appendix: Code Snippets

Throughout the lab, code was implemented and modified locally both in PyCharm and in MATLAB. Here is my github: https://github.com/kaanoztekin99/applied-optimization