1. **Basic Probability**

   (a) Suppose X_i the are all uniformly distributed over $(0, 1)$. Plot in MATLAB/Python and show the results of estimating the CDF using n IID random variables for $n = 10, n = 100$, and $n = 1000$. Explain what happens if n gets larger.

   (b) A fair coin is tossed 10 times. What is the probability of a run of exactly 5 heads in a row? Do not count runs of 6 or more heads in a row. Write the solution in MATLAB/Python and show the mathematical reasoning in the report.

Algorithm 1 Pseudocode for Basic Probability Problems 1(a)

```
n_values ← [10, 100, 1000]                              ▷ Define sample sizes to be used
for each n in n_values do                               ▷ Loop over each sample size
    Generate n IID random variables X_i ∼ U(0, 1)       ▷ Generate n uniform random variables
    Sort X_i in ascending order                         ▷ Order the variables to create an empirical CDF
    Plot empirical CDF using X_i and n                  ▷ Plot the empirical CDF
    Label and show the plot                             ▷ Add labels and display the plot
end for
```

Algorithm 2 Pseudocode for Basic Probability Problems 1(b)

```
p_head ← 0.5                                            ▷ Probability of getting a head
p_tail ← 0.5                                            ▷ Probability of getting a tail
                                                        ▷ Calculating probability for different scenarios
P_5_heads_middle ← p_head^5 × p_tail                    ▷ Probability of getting 5 heads followed by a tail, not at the start
P_5_heads_start ← p_head^5 × p_tail                     ▷ Probability of getting 5 heads starting from the first toss
P_5_heads_end ← p_tail × p_head^5                       ▷ Probability of getting 5 heads at the end of the sequence
                                                        ▷ Summing up all the probabilities
Total_P ← P_5_heads_middle + P_5_heads_start + P_5_heads_end
Display Total_P                                         ▷ Displaying the total probability
```

2. **Poisson Processes**
Let X be an exponential random variable of parameter λ, modeling the inter-arrival times of a Poisson arrival process. Then:

   (a) Plot the PDF f_X for $\lambda = 0.3$, $\lambda = 1$, $\lambda = 3$ using MATLAB . What do you observe? Explain the physical meaning of the parameter λ.

Algorithm 3 Pseudocode for Poisson Processes Problem 2(a)

```
lambda_values ← [0.3, 1, 3]                         ▷ Define different λ values to be used
Define x as a range of values to evaluate the PDF   ▷ Specify the range for x
for each λ in lambda_values do                      ▷ Loop over each λ value
    f_X ← λ × exp(−λ × x)                           ▷ Calculate the PDF of the exponential distribution
    Plot x vs f_X                                   ▷ Plot the PDF
    Label and show the plot                         ▷ Add labels and display the plot
end for
```

3. **Markov process**
A common example of Markov chains (and Markov processes in general) is that of queueing systems. Consider the following scenario,
A taxi stand at a busy airport. A line of taxis, which for all practical purposes can be taken to be infinitely long, is available to serve travelers. Customers wanting a taxi enter a queue and are given a taxi on a first-come, first-serve basis. Suppose it takes one unit of time (say, a minute) for the customer at the head of the queue to load himself and his luggage into a taxi. Hence, during each unit of time, one customer in the queue receives service and leaves the queue while some random number of new customers enter the end of the queue. Suppose at each time instant, the number of new customers arriving for service is described by a discrete distribution $(p_0, p_1, p_2, \cdots)$, where p_k is the probability of new customers. For such a system, the transition probability matrix of the Markov chain would look like $\mathbf{P} = \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & \\ p_0 & p_1 & & p_2 & \cdots \\ \cdots & \cdots & & \cdots & \cdots \end{bmatrix}$
The manager of the taxi stand might be interested in knowing the probability distribution of the queue length. If customers have to wait too long, they may get dissatisfied and seek other forms of transportation.

   (a) Write the MATLAB/Python code to simulate the distribution of the queue length of the taxi stand described in the example described above. For this example, take the number of arrivals per time unit, X, to be a Poisson random variable whose PMF is $P_X(k) = \frac{\lambda^k \exp(-\lambda)}{k!}$ and an average arrival rate of 0.85 arrivals per time unit. For the code, initialize the length of the simulation, $N = 10000$, $k = [0 : 10]$.

Algorithm 4 Pseudocode for Markov process Problem 3(a)

lambda $\leftarrow$ 0.85 $\triangleright$ Define the average arrival rate
$N \leftarrow 10000$ $\triangleright$ Define the length of the simulation
$k \leftarrow [0 : 10]$ $\triangleright$ Define a range for the number of arrivals
queue_length $\leftarrow$ 0 $\triangleright$ Initialize the queue length
queue_length_distribution $\leftarrow$ [ ] $\triangleright$ Initialize a list to store queue lengths
for $i = 1$ to N **do** $\triangleright$ Loop through each time unit in the simulation
    Generate a random number of arrivals using Poisson PMF $\triangleright$ $P_X(k) = (\text{lambda}^k \cdot \exp(-\text{lambda}))/k!$
    Update queue_length based on arrivals and service rate $\triangleright$ Modify the queue length
    Record queue_length in queue_length_distribution $\triangleright$ Store the current queue length
end for
Plot histogram of queue_length_distribution $\triangleright$ Visualize the distribution
Label and show the plot $\triangleright$ Add labels and display the plot