

# The Mathematics of the Stewart Platform

The Stewart Platform consists of 2 rigid frames connected by 6 variable length legs.

The Base is considered to be the reference frame work, with orthogonal axes  $x, y, z$ .

The Platform has its own orthogonal coordinates  $x', y', z'$ .

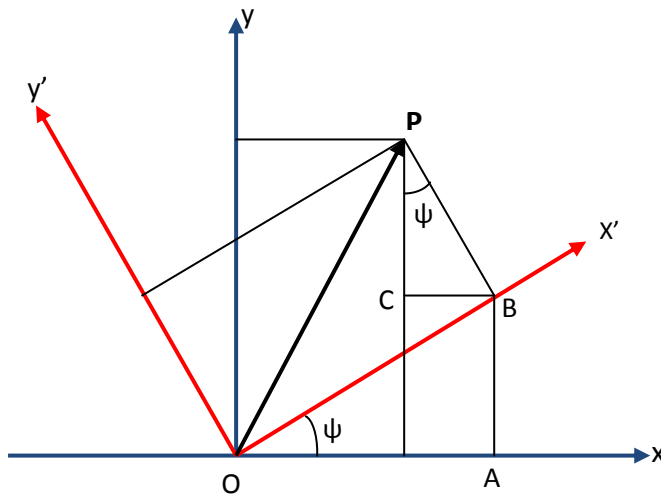
The Platform has 6 degrees of freedom with respect to the Base

The origin of the Platform coordinates can be defined by 3 translational displacements with respect to the Base, one for each axis.

Three angular displacements then define the orientation of the platform with respect to the Base. A set of Euler angles are used in the following sequence:

1. Rotate an angle  $\psi$  (yaw) around the  $z$ -axis
2. Rotate an angle  $\theta$  (pitch) around the  $y$ -axis
3. Rotate an angle  $\varphi$  (roll) around the  $x$ -axis

If we consider the first rotation  $\psi$  (yaw) around the  $z$ -axis:



$$\mathbf{P} = \mathbf{i}'x' + \mathbf{j}'y' + \mathbf{k}'z' = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

$$\begin{aligned} x &= OA - BC \\ &= x' \cos \psi - y' \sin \psi \end{aligned}$$

$$\begin{aligned} y &= AB + PC \\ &= x' \sin \psi + y' \cos \psi \end{aligned}$$

$$z = z'$$

We define the rotation matrix  $\mathbf{R}_z(\psi)$  where

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}_z(\psi) \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad \text{and} \quad \mathbf{R}_z(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly If we consider the second rotation  $\theta$  (pitch) around the  $y$ -axis we can show

$$\mathbf{R}_y(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix},$$

And for the third rotation  $\varphi$  (roll) around the x-axis:

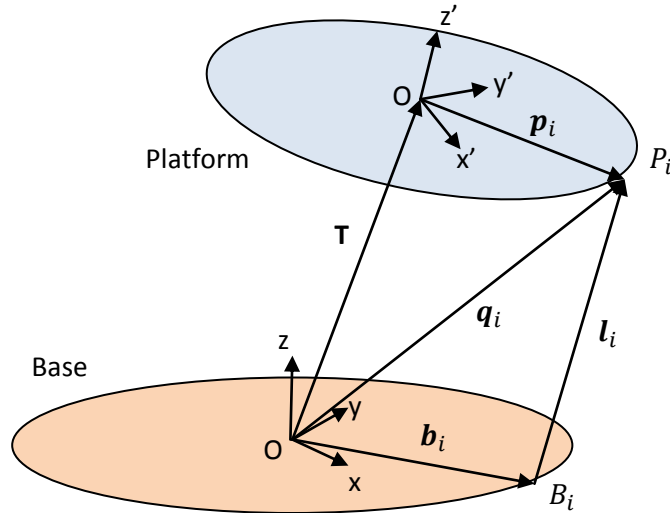
$$\mathbf{R}_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}$$

The full rotation matrix of the Platform relative to the Base is then given by:

$$\begin{aligned} {}^P\mathbf{R}_B &= \mathbf{R}_z(\psi) \cdot \mathbf{R}_y(\theta) \cdot \mathbf{R}_x(\varphi) \\ &= \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos \psi \cos \theta & -\sin \psi & \cos \psi \sin \theta \\ \sin \psi \cos \theta & \cos \psi & \sin \psi \sin \theta \\ -\sin \theta & 0 & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos \psi \cos \theta & -\sin \psi \cos \varphi + \cos \psi \sin \theta \sin \varphi & \sin \psi \sin \varphi + \cos \psi \sin \theta \cos \varphi \\ \sin \psi \cos \theta & \cos \psi \cos \varphi + \sin \psi \sin \theta \sin \varphi & -\cos \psi \sin \varphi + \sin \psi \sin \theta \cos \varphi \\ -\sin \theta & \cos \theta \sin \varphi & \cos \theta \cos \varphi \end{pmatrix} \end{aligned} \quad - (1)$$

Now consider a Stewart Platform.

For the  $i^{th}$  leg:



The coordinates  $q_i$  of the anchor point  $P_i$  with respect to the Base reference framework are given by the equation

$$q_i = T + {}^P\mathbf{R}_B \cdot p_i \quad - (2)$$

Where  $\mathbf{T}$  is the translation vector, giving the positional linear displacement of the origin of the Platform frame with respect to the Base reference framework, and  $\mathbf{p}_i$  is the vector defining the coordinates of the anchor point  $P_i$  with respect to the Platform framework.

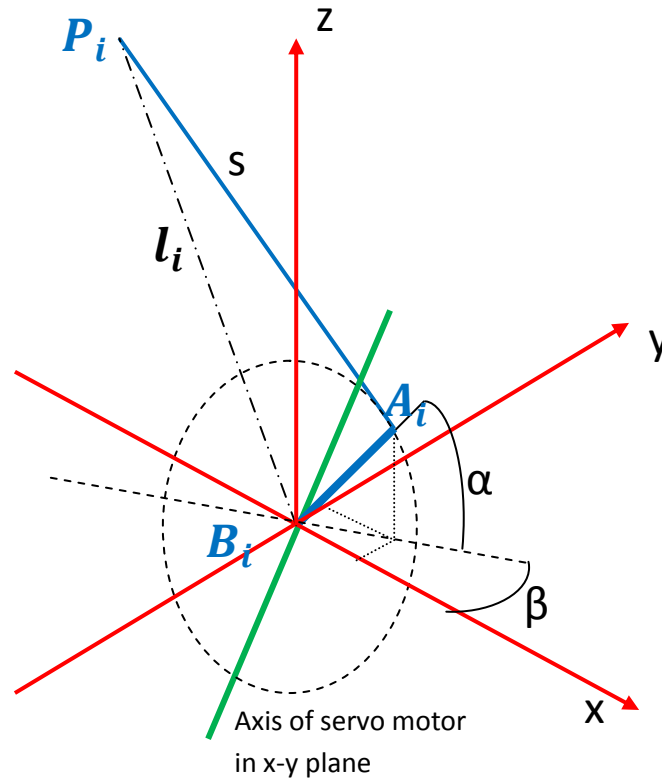
Similarly the length of the  $i^{th}$  leg is given by

$$\mathbf{l}_i = \mathbf{T} + {}^P\mathbf{R}_B \cdot \mathbf{p}_i - \mathbf{b}_i \quad - (3)$$

where  $\mathbf{b}_i$  is the vector defining the coordinates of the lower anchor point  $B_i$ . These 6 equations give the lengths of the 6 legs to achieve the desired position and attitude of the platform.

When considering the Forward Kinematics, this expression represents 18 simultaneous non-linear equations in the 6 unknowns representing the position and attitude of the platform. Much work has been done on finding the solutions to these equations; in the general case there are 40 possible solutions, although in practice many of these solutions would not be practical.

If the leg lengths are achieved via rotational servos, rather than linear servos, a further calculation is required to determine the angle of rotation of the servo. Each servo / leg combination can be represented as follows:



Where:  $a$  = length of the servo operating arm

$A_i$  are the points of the arm/leg joint on the  $i^{th}$  servo with coordinates

$$\mathbf{a} = [x_a \quad y_a \quad z_a]^T \text{ in the base framework.}$$

$B_i$  are the points of rotation of the servo arms with the coordinates

$$\mathbf{b} = [x_b \quad y_b \quad z_b]^T \text{ in the base framework.}$$

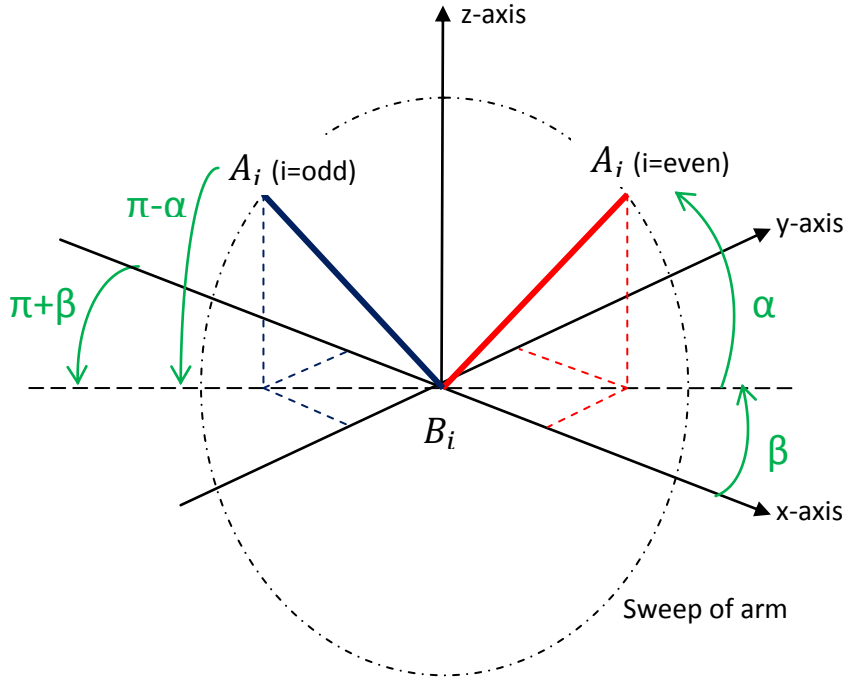
$P_i$  are the points the joints between the operating rods and the platform, with coordinates  $\mathbf{p} = [x_p \quad y_p \quad z_p]^T$  in the platform framework

$S$  = length of operating leg

$l_i$  = length of the  $i^{th}$  leg as calculated from  $l_i = \mathbf{T} + {}^P\mathbf{R}_B \cdot \mathbf{p}_i - \mathbf{b}_i$

$\alpha$  = angle of servo operating arm from horizontal

$\beta$  = angle of servo arm plane relative to the x-axis. Note that the shaft axis lies in the x-y plane where  $z = 0$



Point  $A$  is constrained to be on the servo arm, but the arrangement of the servos means that the odd and even arms are a reflection of each other. So for the even legs we have:

$$x_a = a \cos \alpha \cos \beta + x_b, \text{ and} \quad - (4)$$

$$y_a = a \cos \alpha \sin \beta + y_b, \text{ and} \quad - (5)$$

$$z_a = a \sin \alpha + z_b \quad - (6)$$

And for the odd legs we have:

$$x_a = a \cos(\pi - \alpha) \cos(\pi + \beta) + x_b, \text{ and}$$

$$y_a = a \cos(\pi - \alpha) \sin(\pi + \beta) + y_b, \text{ and}$$

$$z_a = a \sin(\pi - \alpha) + z_b$$

But  $\sin(\pi - \alpha) = \sin \alpha$ , and  $\cos(\pi - \alpha) = -\cos \alpha$

And  $\sin(\pi + \beta) = -\sin \beta$ , and  $\cos(\pi + \beta) = -\cos \beta$

Substituting these values into the equations for the odd legs, we get the same equations as (4), (5), and (6) for the even legs.

By Pythagoras we also have:

$$\begin{aligned} a^2 &= (x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2 \\ &= (x_a^2 + y_a^2 + z_a^2) + (x_b^2 + y_b^2 + z_b^2) - 2(x_a x_b + y_a y_b + z_a z_b) \end{aligned} \quad - (7)$$

$$\begin{aligned} l^2 &= (x_p - x_b)^2 + (y_p - y_b)^2 + (z_p - z_b)^2 \\ &= (x_p^2 + y_p^2 + z_p^2) + (x_b^2 + y_b^2 + z_b^2) - 2(x_p x_b + y_p y_b + z_p z_b) \end{aligned} \quad - (8)$$

$$\begin{aligned} s^2 &= (x_p - x_a)^2 + (y_p - y_a)^2 + (z_p - z_a)^2 \\ &= (x_p^2 + y_p^2 + z_p^2) + (x_a^2 + y_a^2 + z_a^2) - 2(x_p x_a + y_p y_a + z_p z_a) \end{aligned}$$

And substituting from equations (7) & (8)

$$\begin{aligned} s^2 &= l^2 - (x_b^2 + y_b^2 + z_b^2) + 2(x_p x_b + y_p y_b + z_p z_b) + a^2 - (x_b^2 + y_b^2 + z_b^2) + \\ &\quad 2(x_a x_b + y_a y_b + z_a z_b) - 2(x_p x_a + y_p y_a + z_p z_a) \end{aligned}$$

re-arranging gives

$$\begin{aligned} l^2 - (s^2 - a^2) &= 2(x_b^2 + y_b^2 + z_b^2) + 2x_a(x_p - x_b) + 2y_a(y_p - y_b) + 2z_a(z_p - z_b) - \\ &\quad 2(x_p x_b + y_p y_b + z_p z_b) \end{aligned}$$

And substituting for  $x_a, y_a, z_a$  from equations (4), (5) & (6) then gives

$$\begin{aligned} l^2 - (s^2 - a^2) &= 2(x_b^2 + y_b^2 + z_b^2) + 2(a \cos \alpha \cos \beta + x_b)(x_p - x_b) + \\ &\quad 2(a \cos \alpha \sin \beta + y_b)(y_p - y_b) + 2(a \sin \alpha + z_b)(z_p - z_b) - \\ &\quad 2(x_p x_b + y_p y_b + z_p z_b) \end{aligned}$$

$$= 2(x_b^2 + y_b^2 + z_b^2) + 2a \cos \alpha \cos \beta (x_p - x_b) + 2x_p x_b - 2x_b^2 + 2a \cos \alpha \sin \beta (y_p - y_b) + 2y_p y_b - 2y_b^2 + 2a \sin \alpha (z_p - z_b) + 2z_p z_b - 2z_b^2 - (x_p x_b + y_p y_b + z_p z_b)$$

Which reduces to

$$l^2 - (s^2 - a^2) = 2a \sin \alpha (z_p - z_b) + 2a \cos \alpha \cos \beta (x_p - x_b) + 2a \cos \alpha \sin \beta (y_p - y_b) \\ = 2a \sin \alpha (z_p - z_b) + 2a \cos \alpha [\cos \beta (x_p - x_b) + \sin \beta (y_p - y_b)]$$

Which is an equation of the form:-

$$L = M \sin \alpha + N \cos \alpha$$

Using the Trig identity for the sum of sine waves

$$a \sin x + b \cos x = c \sin(x + v)$$

$$\text{where } c = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan v = \frac{b}{a}$$

We therefore have another sine function of  $\alpha$  with a phase shift  $\delta$

$$L = \sqrt{M^2 + N^2} \sin(\alpha + \delta) \quad \text{where} \quad \delta = \tan^{-1} \frac{N}{M}$$

$$\text{Therefore} \quad \sin(\alpha + \delta) = \frac{L}{\sqrt{M^2 + N^2}}$$

$$\text{And} \quad \alpha = \sin^{-1} \frac{L}{\sqrt{M^2 + N^2}} - \tan^{-1} \frac{N}{M} \quad - (9)$$

$$\text{where } L = l^2 - (s^2 - a^2)$$

$$M = 2a(z_p - z_b)$$

$$N = 2a[\cos \beta (x_p - x_b) + \sin \beta (y_p - y_b)]$$

We now have sufficient information to calculate the lengths of the effective “legs”, and the associated angle of the servo arms, for the reverse kinematics for the platform. But to design and implement the hexapod platform we need to define a few constants in order to define the range of movement.

- 1) We need to define the “home” position of the platform. By definition this will be where the platform is at a height  $h_0$  above the base framework, and there being no other translational or rotational movement.

$$\text{i.e.} \quad \mathbf{q}_i = \mathbf{T} + {}^P\mathbf{R}_B \cdot \mathbf{p}_i$$

$$\begin{aligned}\mathbf{q}_0 &= \begin{bmatrix} 0 \\ 0 \\ h_0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} \\ &= \begin{bmatrix} x_p \\ y_p \\ h_0 + z_p \end{bmatrix}\end{aligned}$$

Similarly we will define the “home” position where the servo arms and rods are at right angles to each other.

i.e.  $l^2 = s^2 + a^2$

$$= (x_p - x_b)^2 + (y_p - y_b)^2 + (h_0 + z_p - 0)^2$$

Giving  $h_0 = \sqrt{s^2 + a^2 - (x_p - x_b)^2 - (y_p - y_b)^2} - z_p$  - (10)

and since the platform is constructed symmetrically around the z-axis, this equation will give the same result for any leg.

- 2) We can also calculate the angle  $\alpha_0$  of the servo arm at the Home position.

Using the equation  $\mathbf{l}_i = \mathbf{T} + {}^P\mathbf{R}_B \cdot \mathbf{p}_i - \mathbf{b}_i$

And remembering the symmetrical construction of the platform. The length of the legs in the “home” position is given by

$$\begin{aligned}\mathbf{l}_0 &= \begin{bmatrix} x_p \\ y_p \\ h_0 + z_p \end{bmatrix} - \begin{bmatrix} x_b \\ y_b \\ 0 \end{bmatrix} \\ l_0^2 &= (x_p - x_b)^2 + (y_p - y_b)^2 + (h_0 + z_p)^2\end{aligned}\quad - (11)$$

And the angle of the servo arm in the “home” position can be given by equation (9). Since we have symmetry, we can look at leg 2 only, where  $\beta = 0^\circ$ .

$$\alpha_0 = \sin^{-1} \frac{L_0}{\sqrt{M_0^2 + N_0^2}} - \tan^{-1} \frac{M_0}{N_0} \quad - (12)$$

where  $L_0 = l^2 - (s^2 - a^2)$

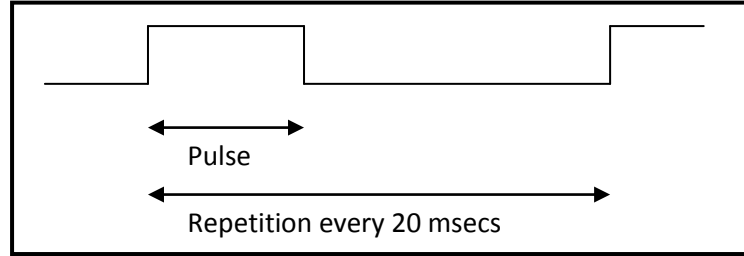
$$= s^2 + a^2 - (s^2 - a^2) = 2a^2$$

$$M_0 = 2a[\cos \beta (x_p - x_b) + \sin \beta (y_p - y_b)]$$

$$= 2a(x_p - x_b)$$

$$N_0 = 2a(h_0 + z_p)$$

The servos are mounted so that their mid point is close to the “home” position, and we will restrict their movement to  $\pm 45^\circ$ . The servos are controlled by a pulse whose duration defines the angle of the arm.



The servos used have the specification:

Neutral position (“home”) is nominally 1500μsecs

Rotation clockwise for increasing pulse width

Pulse width change for  $45^\circ$  is 400 μsecs giving a servo rate,

$$r = \frac{400}{45} \cdot \frac{360}{2\pi} = \frac{1600}{\pi} \mu\text{secs} / \text{radian}$$

From the above, the pulse widths  $W_i$  for each of the servos can be obtained by the equations:

$$W_i = W_i^0 + (\alpha_i - \alpha_0)r \text{ for even } i \text{ and} \quad -(13)$$

$$W_i = W_i^0 - (\alpha_i - \alpha_0)r \text{ for odd } i \quad - (14)$$

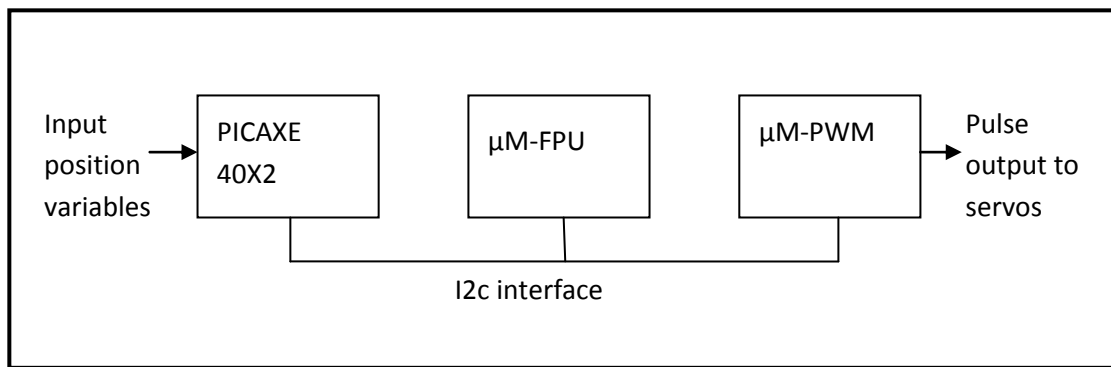
Where  $W_i^0$  is the actual pulse width required to hold the  $i^{th}$  servo in the “home” position. This is nominally 1500μsecs, but can be adjusted to compensate for differences in actual assembly.



## Controlling the Stewart Platform using RC Servos

The platform is constructed with symmetry round the z-axes, but the odd and even servos are mounted in opposition.

The circuitry used to control the platform is based on a PICAXE microcontroller with attached maths co-processor (floating point) and attached servo motor controller.



The sequence of events is as follows:

- 1) Input the positional information for the platform,  $\mathbf{b}_i$ ,  $\mathbf{p}_i$ ,  $\alpha$ ,  $\beta_i$ . These are all constants from the build of the platform.
- 2) Input the constants for the servo motors,  $W_i^0$  and  $r$
- 3) Calculate the values of  $h_0$  from equation (10), and  $\alpha_0$  from equation (12)
- 4) Input the variables for  $(x, y, z, \psi, \theta, \phi)$  for the required platform position
- 5) Calculate the rotational matrix  ${}^P\mathbf{R}_B$  from equation (1)
- 6) Calculate the effective leg lengths  $\mathbf{l}_i$  from equations (3)
- 7) Calculate the angles  $\alpha_i$  required for each servo from equation (9)
- 8) Calculate the pulse widths  $W_i$  required for each servo from equations (13) & (14)
- 9) Output the values of  $W_i$  to the μM-PWM to drive the servos
- 10) Return to step 4) to repeat the process.