問3.2

(1) $\mathrm{Ker}(\phi) = \langle 3 \rangle$ であるから

$$\phi(1) = \phi(3) = \phi(9) = \phi(11) = 1$$ であり、

$$\phi(15) = \phi(3 \cdot 5) = \phi(3) \cdot \phi(5) = \phi(5)$$

$$\phi(5) = \phi(3 \cdot 7) = \phi(3) \cdot \phi(5) = \phi(7)$$

$$\phi(7) = \phi(3 \cdot 13) = \phi(3) \cdot \phi(13) = \phi(13)$$

$$\phi(13) = \phi(3 \cdot 15) = \phi(3) \cdot \phi(15) = \phi(15)$$

となる必要があるため、

$$\phi(x) = \begin{cases} 1 & (x = 1, 3, 9, 11) \\ 5 & (x = 5, 7, 13, 15) \end{cases}$$

とすればよい

(2) $\mathrm{Im}(\phi) = \{1, 5\}$

(3) $\phi(3 \cdot 5) = \phi(15) = \phi(3)\phi(5) = \phi(7)\phi(13)$

$$\phi(3) = \underset{\phi(5)\phi(7)}{\underset{\shortparallel}{\phi(35)}} = \underset{\phi(9)\phi(11)}{\underset{\shortparallel}{\phi(99)}} = \phi$$

$\phi(5) = 5, \phi(15) = 15$

$$\phi(5) = \underset{\phi(3)\phi(7)}{\underset{\shortparallel}{\phi(21)}} = \underset{\phi(9)\phi(13)}{\underset{\shortparallel}{\phi(117)}} = \underset{\phi(13)\phi(15)}{\underset{\shortparallel}{\phi(195)}}$$

$\phi(3) = 7, \phi(7) = 3$

$\phi(13) = 9, \phi(9) = 13$

$67, 83, \underset{9 \times 11}{\underset{\shortparallel}{99}}, 115, 131, 147, 163, 179, \underset{13 \times 15}{\underset{\shortparallel}{195}}, 211$

(3) $\phi(1)$, $\phi(3)=5$, $\phi(5)=3$, $\phi(7)=7$, $\phi(9)=9$, $\phi(11)=13$, $\phi(13)=11$, $\phi(15)=15$

となる写像ϕを考える

$$\phi(1\cdot a)=\phi(a\cdot 1)=\phi(a),\ \phi(a)\phi(1)=\phi(1)\phi(a)=\phi(a),\ (a\in H_{16})$$

$$\phi(3\cdot 5)=\phi(5\cdot 3)=\phi(15)=15,\ \phi(3)\phi(5)=\phi(5)\phi(3)=15$$

$$\phi(3\cdot 7)=\phi(7\cdot 3)=\phi(5)=3,\ \phi(3)\phi(7)=\phi(7)\phi(3)=3$$

$$\phi(3\cdot 9)=\phi(9\cdot 3)=\phi(11)=13,\ \phi(3)\phi(9)=\phi(9)\phi(3)=13$$

$$\phi(3\cdot 11)=\phi(11\cdot 3)=\phi(1)=1,\ \phi(3)\phi(11)=\phi(11)\phi(3)=1$$

$$\phi(3\cdot 13)=\phi(13\cdot 3)=\phi(7)=7,\ \phi(3)\phi(13)=\phi(13)\phi(3)=7$$

$$\phi(3\cdot 15)=\phi(15\cdot 3)=\phi(13)=11,\ \phi(3)\phi(15)=\phi(15)\phi(3)=11$$

$$\phi(5\cdot 7)=\phi(7\cdot 5)=\phi(3)=5,\ \phi(5)\phi(7)=\phi(7)\phi(5)=5$$

$$\phi(5\cdot 9)=\phi(9\cdot 5)=\phi(13)=11,\ \phi(5)\phi(9)=\phi(9)\phi(5)=11$$

$$\phi(5\cdot 11)=\phi(11\cdot 5)=\phi(7)=7,\ \phi(11)\phi(5)=\phi(5)\phi(11)=7$$

$$\phi(5\cdot 13)=\phi(13\cdot 5)=\phi(1)=1,\ \phi(5)\phi(13)=\phi(13)\phi(5)=1$$

$$\phi(5\cdot 15)=\phi(15\cdot 5)=\phi(11)=13,\ \phi(5)\phi(15)=\phi(15)\phi(5)=13$$

$$\phi(7\cdot 9)=\phi(9\cdot 7)=\phi(15)=15,\ \phi(9)\phi(7)=\phi(7)\phi(9)=15$$

$$\phi(7\cdot 11)=\phi(11\cdot 7)=\phi(3)=5,\ \phi(7)\phi(11)=\phi(11)\phi(7)=5$$

$$\phi(7\cdot 13)=\phi(13\cdot 7)=\phi(11)=13,\ \phi(7)\phi(13)=\phi(13)\phi(7)=13$$

$$\phi(7\cdot 15)=\phi(15\cdot 7)=\phi(9)=9,\ \phi(7)\phi(15)=\phi(15)\phi(7)=9$$

$$\phi(9\cdot 11)=\phi(11\cdot 9)=\phi(3)=5,\ \phi(9)\phi(11)=\phi(11)\phi(9)=5$$

$$\phi(9\cdot 13)=\phi(13\cdot 9)=\phi(5)=3,\ \phi(9)\phi(13)=\phi(13)\phi(9)=3$$

$$\phi(9\cdot 15)=\phi(15\cdot 9)=\phi(7)=7,\ \phi(9)\phi(15)=\phi(15)\phi(9)=7$$

$$\phi(11\cdot 13)=\phi(13\cdot 11)=\phi(15)=15,\ \phi(11)\phi(13)=\phi(13)\phi(11)=15$$

$$\phi(11\cdot 15)=\phi(15\cdot 11)=\phi(5)=3,\ \phi(11)\phi(15)=\phi(15)\phi(11)=3$$

$$\phi(13\cdot 15)=\phi(15\cdot 13)=\phi(3)=5,\ \phi(13)\phi(15)=\phi(15)\phi(13)=5$$

$\phi(3^2)=\phi(9)=9$, $\phi(3)^2=9$, $\phi(5^2)=\phi(9)=9$, $\phi(5)^2=9$

$\phi(7^2)=\phi(1)=1$, $\phi(7)^2=1$, $\phi(9^2)=\phi(1)=1$, $\phi(9)^2=1$

$\phi(13^2)=\phi(9)=9$, $\phi(13)^2=9$, $\phi(15^2)=\phi(1)=1$, $\phi(15)^2=1$

であるからϕは準同型写像であり、

$a \neq b \Rightarrow \phi(a) \neq \phi(b)$ であるからϕは単射であり、

$G \ni {}^\forall y$ に対して ${}^\exists x \in G$ s.t. $\phi(x)=y$ が成り立つから

ϕは全射である。